



The 1:1 resonance in Hamiltonian systems

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Abstract

Two-degree-of-freedom Hamiltonian systems with an elliptic equilibrium at the origin are characterised by the frequencies of the linearisation. Considering the frequencies as parameters, the system undergoes a bifurcation when the frequencies pass through a resonance. These bifurcations are well understood for most resonances $k:l$, but not the semisimple cases 1:1 and 1:−1. A two-degree-of-freedom Hamiltonian system can be approximated to any order by an integrable normal form. The reason is that the normal form of a Hamiltonian system has an additional integral due to the normal form symmetry. The latter is intimately related to the ratio of the frequencies. For a rational frequency ratio this leads to S^1 -symmetric systems. The question we wish to address is about the co-dimension of such a system in 1:1 resonance with respect to left-right-equivalence, where the right action is S^1 -equivariant. The result is a co-dimension five unfolding of the central singularity. Two of the unfolding parameters are moduli and the remaining non-modal parameters are the ones found in the linear unfolding of this system.

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*The case $k = l = 1$ (1:1 resonance) turns out to be surprisingly complicated.
J.J. Duistermaat in [10]*

1. Introduction

One of the few available methods to study the dynamics of Hamiltonian systems is to concentrate on the equilibria. The motion itself being trivial by definition, one considers the local dynamics and linearises the vector field. A hyperbolic equilibrium, with no eigenvalues on the imaginary axis, is dynamically unstable and on a sufficiently small neighbourhood the motion is completely determined by the linearisation.

In the elliptic case the non-linear terms cannot be disposed of completely, but lead to normal forms of which one hopes that they capture the essence of the dynamics. The reasons for irremovable terms are the resonances between the eigenvalues on the imaginary axis. Excluding zero eigenvalues, the resonances of lowest order, i.e. the 1:1 and 1:−1 resonances, relate double pairs of imaginary eigenvalues.

1.1. Resonant equilibria

In the present paper we concentrate on the 1:1 resonance and study an equilibrium around which the Hamiltonian expands as

$$H(q, p) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \dots \quad (1)$$

where we omit the irrelevant constant term. The Hessian $D^2H(0)$ is positive definite and this excludes nilpotent terms. Thus a 1:1 resonance is always semisimple. It occurs persistently in

3-parameter families, cf. [19,9,10,6,18]. This is in sharp contrast with the $1:-1$ resonance where the Hessian is not definite. Then we have to distinguish a semisimple and a non-semisimple case. Unfolding the latter leads to the Hamiltonian Hopf bifurcation, which occurs persistently in 1-parameter families, cf. [29,7,15]. The semisimple $1:-1$ resonance also occurs persistently in 3-parameter families, cf. [20,18,14]. We expect its unfolding to share features of that of the $1:1$ resonance.

A comprehensive study of $k:l$ resonances, excluding the $1:\pm 1$ cases, has been made in [10]. It turns out that all higher order cases are very similar to each other. In general the unfolding co-dimension of the unfolding is two, where one parameter can be considered as a *detuning* of the resonance and the other is a *modulus*, see [33,13]. Exceptions are the resonances $1:2$ and $1:3$ with co-dimensions 1 and 3, respectively. Again one of the parameters is a detuning and in the case of $1:3$ resonance, two parameters are moduli. In all cases there is a bifurcation associated to the resonance. In general a pair of stable and unstable periodic solutions branches off from the origin. The $1:2$ and $1:3$ cases have a slightly different unfolding scenario, see [10,4,13,8]. As mentioned before the non-semisimple or nilpotent $1:-1$ resonance shows a different bifurcation (the Hamiltonian Hopf bifurcation, see [29]) and the bifurcations triggered by the semisimple $1:\pm 1$ resonances are still open.

This paper is organised as follows. In section 1.2 we state an informal version of our main theorem. Although informal it still contains the essential properties of the main theorem. Before proving our main result we review some facts on Hamiltonian systems in section 2. The system we study is in normal form and we discuss the properties we use in section 3, especially the induced S^1 -symmetry. Finally in section 4 we state and in section 5 we prove our main theorem using singularity theory for S^1 -equivariant mappings. The concluding section 6 puts our results in context. Our approach fits in the tradition of [10,29,9] and it complements [6].

1.2. Informal statement of the main theorem

In order to state our main result we need a few definitions. Here our aim is not full generality, the main theorem is formulated more precisely in section 4.2.

We study a C^∞ Hamiltonian system on \mathbb{R}^4 with standard symplectic form in the neighbourhood of an elliptic equilibrium in $1:1$ resonance. We may assume that the equilibrium is at the origin, thus the linear part of the Hamiltonian H at 0 vanishes. The matrix associated to the linearisation of the Hamiltonian vector field has coinciding pairs of eigenvalues with equal *symplectic sign*, therefore this matrix has no nilpotent part, see [16]. As a consequence the quadratic part of the Hamiltonian in the $1:1$ case has Morse index 0. This contrasts with the $1:-1$ resonance where the corresponding matrix generically does have a nilpotent part, see [29].

As a first step we apply several (symplectic) co-ordinate transformations. The first of these takes the quadratic part H_2 of H into the form presented in equation (1). Moreover, after a finite number of *normal form* transformations (see for example [29]), we may assume that a corresponding part of the Taylor expansion of H Poisson commutes with H_2 . We now make an approximation by restricting to this finite part and call it H again. The flow of H_2 generates an S^1 symmetry group and the fact that H and H_2 Poisson commute implies that H is S^1 -symmetric. The consequences of this approximation are discussed in the remarks following Theorem 1.1.

The second step is a reduction with respect to the S^1 symmetry. Restricted to the 3-sphere $\{H_2 = 1\}$, the projection mapping involved is a Hopf mapping so the *reduced phase space* is a 2-sphere. Then we apply *equivariant singularity theory* to the *map germ* (H, H_2) and find a *universal unfolding* subject to non-degeneracy conditions on the coefficients in the higher order

terms of H . By the nature of our method, we can not hope for more than local results and we exploit this fact by switching to germs, see [3,28,31]. Very briefly: a map germ is the collection of mappings equal to one another on an arbitrary small neighbourhood of a given point, say 0. Map germs are essentially determined by their Taylor expansions or even Taylor polynomials in a sense that is made more precise in section 4.1.1. In the sequel we say mapping but tacitly assume map germ.

In order to proceed we need the *generators* of the S^1 -invariant functions as co-ordinates. These are given by

$$\begin{aligned} I_1 &= \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2) \\ I_2 &= \frac{1}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2) \\ I_3 &= q_1q_2 + p_1p_2 \\ I_4 &= q_1p_2 - q_2p_1, \end{aligned}$$

see section 3.1 for more details. The generators are not independent but related by the *syzygy* $I_1^2 = I_2^2 + I_3^2 + I_4^2$. Nevertheless, H and H_2 can now be expressed as functions of I , that is $H_2(I) = I_1$ and $H(I) = H_2(I) + H_4(I) + H_6(I) + \dots + H_k(I)$. The final result is given in the next theorem.

Theorem 1.1. *A universal unfolding of the S^1 -invariant Hamiltonian*

$$H(I) = I_1 + a_1I_2^2 + a_2I_3^2 + a_3I_4^2 + b_1I_2^3 + b_2I_3^3 + b_3I_4^3$$

is given by the five parameter family ($\mu \in \mathbb{R}^5$)

$$\begin{aligned} H(I; \mu) &= I_1 + a_1I_2^2 + a_2I_3^2 + a_3I_4^2 + b_1I_2^3 + b_2I_3^3 + b_3I_4^3 \\ &\quad + \mu_1I_2 + \mu_2I_3 + \mu_3I_4 + \mu_4I_2^3 + \mu_5I_3^3 \end{aligned}$$

provided that the real coefficients a_1, a_2, a_3, b_1, b_2 and b_3 satisfy the non-degeneracy condition

$$(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) b_1 b_2 b_3 \neq 0.$$

This theorem holds for S^1 -symmetric Hamiltonian systems in 1:1 resonance. Let us make a few remarks on its scope.

Remark 1.2.

1. The unfolding terms $\mu_4I_2^3$ and $\mu_5I_3^3$ can be replaced by any pair from I_2, I_3 and I_4 .
2. The reduction of the 3-sphere defined by $\frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2) = h_2$ to the 2-sphere $I_2^2 + I_3^2 + I_4^2 = h_2^2$ is regular if $h_2 \neq 0$, so every point on the reduced phase space corresponds to an S^1 -orbit on the original phase space \mathbb{R}^4 .
3. On the reduced phase space the solution curves are defined by $(H, H_2) = (h, h_2)$. Thereby time parametrisation is lost. Solution curves consisting of a single point on the reduced phase space correspond to periodic orbits on \mathbb{R}^4 , whereas closed curves on the reduced phase space correspond to 2-tori on \mathbb{R}^4 . The former are generically isolated on S^2 , but the latter come in 1-parameter families.

4. Non- S^1 -symmetric perturbations (i.e. including non- S^1 -invariant terms in the Taylor expansion of H) do affect our result. However, normal form transformations enable us to make these perturbations as small as we wish. Nevertheless their effect is that families of 2-tori, on \mathbb{R}^4 , do not survive as such. From KAM theory one expects that these families are *Cantorised*, i.e. the 2-tori persist as a Cantor subfamily of large 2-dimensional Hausdorff measure, where the dense set of internal resonances leads to gaps in the parametrisation. Periodic orbits, as long as they are elliptic or hyperbolic, do persist, as do their bifurcations. Thus our result gives information on low periodic orbits of general Hamiltonian systems in 1:1 resonance. Homoclinic and heteroclinic connections on the reduced phase space generically do break up under non- S^1 -symmetric perturbations yielding chaotic regions familiar from Poincaré sections of for example the Hénon–Heiles system.
5. In view of the previous remark, the bifurcation diagram for the equilibrium at 0 on \mathbb{R}^4 with branches of periodic orbits is valid for general Hamiltonian systems in 1:1 resonance.

2. A few facts about Hamiltonian systems

Here we very briefly review some facts from the theory of Hamiltonian systems. We concentrate on \mathbb{R}^4 . However everywhere in the following sections \mathbb{R}^4 can be replaced by M , a C^∞ real symplectic manifold. For a thorough treatment we refer to for example [1,2].

2.1. Symplectic spaces and Hamiltonian systems

Let ω be a closed, non-degenerate skew symmetric 2-form on \mathbb{R}^4 , making (\mathbb{R}^4, ω) a *symplectic space*. Furthermore let H be a function in $C^\infty(\mathbb{R}^4, \mathbb{R})$, then the triple $(\mathbb{R}^4, \omega, H)$ is called a smooth real *Hamiltonian system*. Now let $\mathcal{X}(\mathbb{R}^4)$ be the set of smooth vector fields on \mathbb{R}^4 . The vector field $X_H \in \mathcal{X}(\mathbb{R}^4)$ satisfying

$$\omega(X_H, Y) = dH(Y)$$

for all $Y \in \mathcal{X}(\mathbb{R}^4)$, is called the *Hamiltonian vector field* of H . The vector field X_H defines the *flow* of the Hamiltonian system on \mathbb{R}^4 , we also call this the flow of H . A function f is preserved under the flow of the vector field X_H if and only if the Lie derivative of f is identically zero. Using $L_{X_H}(f) = df(X_H)$ we find that the Hamiltonian function H is preserved by the flow of X_H because

$$L_{X_H}(H) = dH(X_H) = \omega(X_H, X_H) = 0.$$

The last equality follows from the skew symmetry of ω .

2.2. Poisson brackets

Let f and g be in $C^\infty(\mathbb{R}^4, \mathbb{R})$, then we define the Poisson bracket of f and g as

$$\{f, g\} = \omega(X_f, X_g).$$

It follows from this definition that

$$\{f, g\} = L_{X_g}(f) = -L_{X_f}(g).$$

Suppose that the function f is preserved under the flow of X_H , then

$$0 = L_{X_H}(f) = \{f, H\}$$

and vice versa, so once we have the Poisson bracket we do not need the vector field X_H to determine whether f is preserved under the flow of H . Furthermore $\{f, g\} = -\{g, f\}$ so $\{H, H\} = 0$ from which again follows that H is preserved under the flow of X_H . The Poisson bracket satisfies Jacobi’s identity whence Hamiltonian vector fields form a Lie algebra; in fact we have

$$[X_f, X_g] = -X_{\{f, g\}}.$$

Thus $(C^\infty(\mathbb{R}^4), \{ \cdot, \cdot \})$ is a Lie algebra of functions.

2.3. Standard forms

Darboux’s theorem now states that there are co-ordinates such that ω becomes constant. Then by applying linear algebra we can bring ω into a standard form such that

$$\omega(\xi, \eta) = \langle \xi \mid \Omega \eta \rangle$$

for all $\xi, \eta \in \mathbb{R}^4$. Here $\langle \cdot \mid \cdot \rangle$ is the standard inner product on \mathbb{R}^4 and Ω is a linear mapping with $\Omega = -\Omega^t = -\Omega^{-1}$ which takes the standard form

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

on the standard basis $\{e_1, e_2, f_1, f_2\}$. Let us take co-ordinates $z = (q_1, q_2, p_1, p_2)$ with respect to this basis, then the Poisson bracket becomes

$$\{f, g\} = \sum_{i=1}^2 \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

Using the Poisson bracket on these co-ordinates we obtain the canonical equations of motion

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

for the Hamiltonian H . The Poisson bracket allows us to use functions instead of vector fields, which simplifies many computations.

3. Resonant Hamiltonian systems and S^1 -symmetry

On the symplectic space (\mathbb{R}^4, ω) we consider C^∞ Hamiltonian systems with an equilibrium at the origin. Furthermore suppose that the linearisation of the corresponding Hamiltonian vector field has resonant imaginary eigenvalues.

When this system has been transformed into normal form it admits an S^1 -symmetry group. Resonant eigenvalues are not generic, but when they appear in parameter families of Hamiltonian systems they are a source of bifurcations. Therefore it is useful to study unfoldings of resonant systems. Most resonances in 4-dimensional Hamiltonian systems have been studied before, see [10] and references therein. This approach has to be refined for the 1:1 and 1:−1 resonances, where the sign is the *symplectic sign*. See [29] for an extensive study of the so-called nilpotent 1:−1 resonance which in a parameter family gives rise to the *Hamiltonian Hopf bifurcation*. Our aim here is to study the 1:1 resonance. While this case has already been considered in [6], the arguments presented there are incomplete.

A resonant Hamiltonian system naturally leads to an S^1 -invariant system when passing to a normal form truncation. But we may also consider Hamiltonian systems with an externally given symplectic S^1 -action. Our results hold for such systems as well, provided that the S^1 -action satisfies the conditions in the next section.

3.1. S^1 -symmetry related to the 1:1 resonance

Since we work in the class $C^\infty(\mathbb{R}^4)$ the Hamiltonian function H has an infinite Taylor series. We now put some more structure on these functions by collecting homogeneous terms, turning $(C^\infty(\mathbb{R}^4), \{\cdot, \cdot\})$ into a graded Lie algebra. Then we expand

$$H = H_2 + H_3 + \dots + H_k + \dots$$

with $H_k \in \mathbb{R}[z]$ homogeneous of degree k . The normal form procedure acts in a very nice way on this Lie algebra, for details see [29]. The final result is that for the normal form we have $\{H_2, H_k\} = 0$ for all k and therefore $\{H_2, H\} = 0$. This means that the normal form of H is invariant under the flow of H_2 which is generated by X_{H_2} . Now we assume that the linear part X_{H_2} of the vector field X_H is in 1:1 resonance, then (the normal form of) H is S^1 -invariant with respect to the S^1 -action

$$\begin{aligned} \phi : S^1 \times \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ (\varphi, z) &\longmapsto R_\varphi z \end{aligned} \tag{2}$$

where

$$R_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

and $z = (q_1, p_1, q_2, p_2)$. The quadratic part of such a Hamiltonian system reads

$$H_2(q_1, p_1, q_2, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \frac{1}{2}(q_2^2 + p_2^2).$$

Note that this function has Morse index 0 which is intimately related to the fact that the eigenvalues of the linear part of the corresponding Hamiltonian vector field have equal symplectic sign, see [5].

Every S^1 -invariant C^∞ -function can be written as a function of so called *invariants*. This is a consequence of far more general results which we now state. We start with a theorem on *invariant polynomials*.

Theorem 3.1 (Hilbert, Schwartz). *Let Γ be a compact group which acts linearly on \mathbb{R}^n and let $\mathbb{R}[z]^\Gamma$ denote the set of Γ -invariant polynomials. Then a finite number r of polynomials $\rho_1, \dots, \rho_r \in \mathbb{R}[z]^\Gamma$ exist that generate $\mathbb{R}[z]^\Gamma$. The ρ_1, \dots, ρ_r form a Hilbert basis and are called generators. Furthermore every Γ -invariant C^∞ -function $f \in C^\infty(\mathbb{R}^n)^\Gamma$ can be written as a C^∞ -function $\hat{f} \in C^\infty(\mathbb{R}^r)$ of the r generators of $\mathbb{R}[z]^\Gamma$.*

Unfortunately the function \hat{f} need not be unique for there may be syzygies among the ρ_j .

Let us now determine the invariants of the S^1 -action associated to the 1:1 resonance. These are polynomials on the phase space and they Poisson commute with H_2 .

Lemma 3.2. *The generators in $\mathbb{R}[q, p]^{S^1}$ of the invariants of the S^1 -action associated to the 1:1 resonance are given by*

$$I_1 = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2)$$

$$I_2 = \frac{1}{2}(q_1^2 + p_1^2 - q_2^2 - p_2^2)$$

$$I_3 = q_1q_2 + p_1p_2$$

$$I_4 = q_1p_2 - q_2p_1$$

with syzygy $I_1^2 = I_2^2 + I_3^2 + I_4^2$.

For a proof we refer to [7].

Thus every S^1 -invariant C^∞ -function on \mathbb{R}^4 can be written as a C^∞ -function of the Hilbert basis $\{I_1, I_2, I_3, I_4\}$. From now on we restrict to a smaller set of functions, namely the formal series in $\mathbb{R}[[I]]$. The reasons we can do this are 1) every polynomial in I is the Taylor series of a C^∞ -function of I ; 2) we only allow for a finite number of conditions on the coefficients of a series. The latter means that we do not encounter the subtleties on infinitely flat functions, however see Remark 1.2, item 4. Moreover we are only interested in C^∞ -functions that are zero at the origin. Therefore we only consider formal series without constant terms, denoted by $\mathbb{R}[[I]]_0$.

Now a function in $\mathbb{R}[[I]]_0$ is not unique, due to the syzygy among the generators. In this respect it is worth noting that when we consider functions in $\mathbb{R}[[I]]_0$ modulo the ideal generated by $I_1^2 - (I_2^2 + I_3^2 + I_4^2)$, denoted by $\mathbb{R}[[I]]_0/\sim$, we have the following splitting, see [7]. This splitting is also not unique, but seems natural in view of the syzygy.

Lemma 3.3. $\mathbb{R}[[I_1, I_2, I_3, I_4]]_0/\sim = \mathbb{R}[[I_2, I_3, I_4]]_0 \oplus I_1\mathbb{R}[[I_2, I_3, I_4]]_0$.

When chosen in this last space the function \hat{f} in Theorem 3.1 is unique. Now that we know the generators of the invariants we can write H and H_2 as functions of these. In particular, we have $H_2(I) = I_1$.

3.2. Reduction of the S^1 -symmetry: Hamiltonian systems on S^2

We are primarily interested in the flow of H . Since the flow of H and the S^1 -action commute (H and H_2 Poisson commute), the orbits of H through an S^1 -orbit are equivalent. Therefore we wish to reduce to the orbit space \mathbb{R}^4/S^1 where points correspond to S^1 -orbits on \mathbb{R}^4 . The projection mapping

$$(q, p) \mapsto I \tag{3}$$

defined in Lemma 3.2 just does that. It allows us to reduce the dynamics of H on \mathbb{R}^4 to a 2-dimensional phase space.

The S^1 -action is generated by the vector field X_{H_2} . Now H_2 is preserved by its own flow, therefore the S^1 -action preserves I_1 which defines a 3-sphere

$$\left\{ (q, p \in \mathbb{R}^4 \mid h_2 = I_1 = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2)) \right\} .$$

As H and H_2 Poisson commute, the flow of H also preserves this 3-sphere. Because of the syzygy $I_1^2 = I_2^2 + I_3^2 + I_4^2$ the projection mapping takes the flow of H to a 2-sphere in the reduced phase space; the reduced phase space is determined by $I_1 = h_2$, $I_1^2 = I_2^2 + I_3^2 + I_4^2$. The reduced dynamics of H can simply be characterised by the level h of H . This means that an orbit of the reduced flow of H is determined by the equations

$$\begin{aligned} (H, H_2) &= (h, h_2) \\ I_2^2 + I_3^2 + I_4^2 &= h_2^2 . \end{aligned}$$

The reduced dynamics of H consists of curves on a 2-sphere. Note that in order to know the time parametrisation of these curves we still have to solve a generally difficult differential equation. But we do have a full geometric characterisation.

This leads us to the following. We consider the set of smooth S^1 -invariant mappings $C^\infty(\mathbb{R}^4, \mathbb{R}^2)^{S^1}$ of the form (H, H_2) . The reduced dynamics of H is determined once we specify its value by $(H, H_2) = (h, h_2)$. In the next section we address the question whether a polynomial H exists such that this mapping is stable in the sense of singularity theory.

Remark 3.4.

1. For a far more complete account of general *regular reduction* see for example [1,2]. More details about the 1:1 resonance can be found in [7] where the projection mapping (3) is shown to be the Hopf mapping from S^3 to S^2 .
2. Other resonances like $k:l$ give rise to a different reduced phase space, having singularities. These arise from non-trivial isotropy subgroups of the S^1 -symmetry group in these cases. They again turn up in new generators with a higher order syzygy. In 4-dimensional resonant Hamiltonian systems the situation is relatively simple, there are four generators and one syzygy. In higher dimensions both the number of generators and the number of syzygies depend on the resonance, *i.e.* on the ratios $k_1 : k_2 : \dots : k_n$, making it computationally difficult. Then the Gröbner basis algorithm is indispensable.

Table 1
Poisson bracket of the real generators I of the invariants.

$\{\cdot, \cdot\}$	I_1	I_2	I_3	I_4
I_1	0	0	0	0
I_2	0	0	$-2I_4$	$2I_3$
I_3	0	$2I_4$	0	$-2I_2$
I_4	0	$-2I_3$	$2I_2$	0

Both sides of the syzygy define a *Casimir element*, i.e. their Poisson brackets with the I vanish. A straightforward calculation yields Table 1 of Poisson brackets.

The invariants from Lemma 3.2 are sometimes called Hopf variables. Indeed, I_1 generates the S^1 -symmetry (2) and hence is an integral of motion for every Hamiltonian system with that symmetry. The Hopf mapping

$$(I_2, I_3, I_4) : S^3_{2I_1} \longrightarrow S^2_{I_1^2}$$

from the 3-sphere

$$S^3_{2I_1} = \left\{ (q, p) \in T^*\mathbb{R}^2 \mid q_1^2 + q_2^2 + p_1^2 + p_2^2 = 2I_1 \right\}$$

to the 2-sphere

$$S^2_{I_1^2} = \left\{ (I_2, I_3, I_4) \in \mathbb{R}^3 \mid I_2^2 + I_3^2 + I_4^2 = I_1^2 \right\}$$

performs the reduction to one degree of freedom by identifying points related through (2).

The phase portraits are obtained by intersecting, within \mathbb{R}^3 , the level sets of the Hamiltonian $H = H(I_2, I_3, I_4)$ with S^2 . Where H is a Morse function, this yields finitely many centres and saddles, with no heteroclinic connections between the latter. Under variation of parameters local and global bifurcations may occur.

4. The universal unfolding

In this section we state our main theorem. First we provide a context for the theorem by introducing the notion of stable mappings under left-right-equivalence.

4.1. Equivalence classes for S^1 -invariant Hamiltonian systems

The meaning of ‘universal unfolding’ depends on the universe in which we work and the notion of equivalence. As explained in section 3.2 we consider Hamiltonian systems on \mathbb{R}^4 that are S^1 -invariant and can be reduced to S^2 . If we content ourselves with characterising the reduced dynamics of H by the orbits only we just need to specify values of H and H_2 . That is the orbits of the reduced Hamiltonian systems are the fibres of the mapping (H, H_2) . Note however that $H_2(I) = I_1$ and H_2 is an integral of the Hamiltonian system. So I_1 is constant and therefore not to be considered as a variable but rather a parameter. Furthermore note that the fibres of the mappings (H, H_2) and (H, H_2^2) are identical. Using the relation of the generators of the invariants I , we have $H_2(I)^2 = I_1^2 = I_2^2 + I_3^2 + I_4^2$. This leads us to define $K(I) = H_2(I)^2$ and

consider the mapping $\mathcal{F}(I) = (H(I), K(I))$ on our universe $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$, the S^1 -invariant C^∞ -mappings from \mathbb{R}^4 to \mathbb{R}^2 taking $(0, 0)$ to $(0, 0)$.

A natural notion of equivalence on $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ is provided by so called *left-right-equivalences*, see Definition 4.1 below. For if \mathcal{F} and \mathcal{G} are left-right-equivalent then the fibres of \mathcal{F} and \mathcal{G} are diffeomorphic. This in turn implies that the orbits of the S^1 -invariant Hamiltonian systems in $\mathcal{F} = (H, K)$ and $\mathcal{G} = (H', K')$ can be mapped to each other by a simple diffeomorphism.

Definition 4.1. The mappings $\mathcal{F}, \mathcal{G} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ are called *left-right-equivalent* if $(\psi, \phi) \in \text{Diff}(\mathbb{R}^2)_0 \times \text{Diff}(\mathbb{R}^4)_0^{S^1}$ exists such that $(\psi, \phi) \cdot \mathcal{F} = \mathcal{G}$, where $(\psi, \phi) \cdot \mathcal{F} = \psi \circ \mathcal{F} \circ \phi$.

4.1.1. *Stable S^1 -invariant mappings, co-dimension and unfolding*

The idea of stability of a mapping \mathcal{F} is that every mapping \mathcal{G} nearby \mathcal{F} is equivalent to \mathcal{F} , or put differently, that \mathcal{G} is an element of the orbit of \mathcal{F} under left-right-equivalence. Here we give a short overview in a series of definitions and theorems.

Definition 4.2. The *orbit* of $\mathcal{F} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ under left-right-equivalences is given by

$$\text{Orb}_{\mathcal{F}} = \{(\psi, \phi) \cdot \mathcal{F} \mid (\psi, \phi) \in \text{Diff}(\mathbb{R}^2)_0 \times \text{Diff}(\mathbb{R}^4)_0^{S^1}\}.$$

To define ‘nearby’ we use the definition of a deformation.

Definition 4.3. A *deformation (or unfolding)* of a mapping $\mathcal{F} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ is a C^∞ -mapping $\mathcal{F} : \mathbb{R}^4 \times \mathbb{R}^p \rightarrow \mathbb{R}^2$ defining a family of S^1 -equivariant $\mathcal{F}_\nu, \nu \in \mathbb{R}^p$, such that $\mathcal{F}_0 = \mathcal{F}$.

This allows to formulate a parametric version of \mathcal{F} being an interior point of the orbit of \mathcal{F} .

Definition 4.4. A mapping $\mathcal{F} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ is called *stable* if for every deformation \mathcal{F}_ν there is an open neighbourhood U of $0 \in \mathbb{R}^p$ such that for all $\nu \in U, \mathcal{F}_\nu \in \text{Orb}_{\mathcal{F}}$.

The conditions of stability in this sense are hard to check. The conditions of *infinitesimal stability* are much easier to check and this notion of stability turns out to be equivalent with the previous one.

Definition 4.5. \mathcal{F} is called *infinitesimally stable* if the tangent space of $\text{Orb}_{\mathcal{F}}$ at \mathcal{F} is equal to the tangent space of $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ at \mathcal{F} .

A proof of the next theorem can be found in [28].

Theorem 4.6. *A mapping is stable if and only if it is infinitesimally stable.*

Stable mappings form an open and dense subset of $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$, see [32]. A mapping that fails to be stable has therefore non-zero co-dimension

Definition 4.7. Two deformations \mathcal{F}_ν and \mathcal{G}_μ are *left-right-equivalent* if there are (ψ_ν, ϕ_ν) and $\mu(\nu)$ with $\psi_\nu \circ \mathcal{F}_\nu \circ \phi_\nu = \mathcal{G}_{\mu(\nu)}$.

This allows to generalise the previous discussion of mappings to deformations.

Definition 4.8. A versal unfolding is a stable deformation.

The minimal number of parameters of a versal unfolding of a mapping \mathcal{F}_0 coincides for $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ with the co-dimension of \mathcal{F}_0 .

4.1.2. *The tangent space of $Orb_{\mathcal{F}}$ at (H, K)*

Let $\mathcal{X}(\mathbb{R}^4)$ be the Lie algebra of $\text{Diff}(\mathbb{R}^4)_0$ and $\mathcal{X}(\mathbb{R}^4)^{S^1}$ be the Lie algebra of $\text{Diff}(\mathbb{R}^4)_0^{S^1}$.

Lemma 4.9. *The tangent space of $Orb_{\mathcal{F}}$, $\mathcal{F} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$, at (H, K) is given by*

$$\{X(\mathcal{F}) + d\mathcal{F}(Y) \mid X \in \mathcal{X}(\mathbb{R}^2), Y \in \mathcal{X}(\mathbb{R}^4)^{S^1}\}.$$

Proof. For every near-identity transformation $(\psi, \phi) \in \text{Diff}(\mathbb{R}^2)_0 \times \text{Diff}(\mathbb{R}^4)_0^{S^1}$ there exist $X \in \mathcal{X}(\mathbb{R}^2)$ and $Y \in \mathcal{X}(\mathbb{R}^4)^{S^1}$ such that for some $t \in \mathbb{R}$ we have $(\psi, \phi) = (e^{tX}, e^{tY})$. Then the tangent vectors are $\frac{d}{dt}(e^{tX} \circ \mathcal{F} \circ e^{tY})|_{t=0} = X(\mathcal{F}) + d\mathcal{F}(Y)$. \square

Taking a closer look at the tangent space of $Orb_{\mathcal{F}}$ at $\mathcal{F} = (H, K)$ in Lemma 4.9; we explicitly have

$$X(\mathcal{F}) + d\mathcal{F}(Y) = (X_1(H, K) + Y(H), X_2(H, K) + Y(K)). \tag{4}$$

In this expression X is any vector field on \mathbb{R}^2 , but Y is an S^1 -equivariant vector field on \mathbb{R}^4 . Using Theorem 4.6 we have to check that every S^1 -equivariant map germ can be written as $(X_1(H, K) + Y(H), X_2(H, K) + Y(K))$ for a suitable choice of X and Y .

4.1.3. *The restricted tangent space of $Orb_{\mathcal{F}}$ at (H, K)*

The S^1 -equivariant vector fields are such that $Y(K)$ can be any function of degree 2 and higher in the set of S^1 -invariant functions on \mathbb{R}^4 . This follows from an explicit calculation of these vector fields in section 5.2. Thus the stability of \mathcal{F} is determined by the first component. More precisely we have the following.

Proposition 4.10. *The co-dimension of (H, K) in $C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ with the full group of left-right-equivalences is equal to the co-dimension of H in $C^\infty(\mathbb{R}^4)_0^{S^1}$ with the group of left-right-equivalences that fix K .*

Therefore we restrict to vector fields in $Y \in \mathcal{X}(\mathbb{R}^4)^{S^1}$ such that $X_2(H, K) + Y(K) = 0$. Or, from a slightly different point of view, we look for a normal form of the mapping $\mathcal{F} = (H, K)$. But the second component can already be regarded as being in normal form. Therefore we may restrict to transformations that preserve K , that is $X_2(H, K) + Y(K) = 0$.

Lemma 4.11. *The set of S^1 -equivariant vector fields Y with $Y(K) \in \mathbb{R}[[H, K]]_0$ can be decomposed as the direct sum of two modules. The first is a module over $\mathbb{R}[[I]]_0/\sim$ and consists of vector fields $Y \in \mathcal{X}(\mathbb{R}^4)^{S^1}$ taking K to zero. The second is a module over $\mathbb{R}[[H, K]]_0$, generated by vector fields $Y \in \mathcal{X}(\mathbb{R}^4)^{S^1}$ taking K to K or to H .*

Proof. S^1 -equivariant vector fields Y such that $Y(K) \in \mathbb{R}[[H, K]]_0$ are generated by S^1 -equivariant vector fields satisfying one of the three equations $Y(K) = 0$, $Y(K) = K$ and $Y(K) = H$. \square

From now on we consider the restricted tangent space of Orb_H under left-right-transformations and we call it T_1 . The restricted tangent space of Orb_H is again the sum of two (function) modules $\mathcal{J} \oplus \mathcal{M}$. Suppose U_1, \dots, U_k generate the solutions of $Y(K) = 0$ and V_1 and V_2 solve $Y(K) = K$ and $Y(K) = H$, respectively. Furthermore let $F_i = U_i(H)$ for $i \in \{1, \dots, k\}$ and $G_j = V_j(H)$ for $j \in \{1, 2\}$. Then we have the following.

Lemma 4.12. *The restricted tangent space of Orb_H is the sum of two modules $\mathcal{J} \oplus \mathcal{M}$, the first is a module over $\mathbb{R}[[I]]_0/\sim$ and the second is a module over $\mathbb{R}[[H, K]]_0$. That is, every function f in the tangent space of Orb_H is of the form $f = \xi_1 F_1 + \dots + \xi_k F_k + \eta_0 + \eta_1 G_1 + \eta_2 G_2$, with $\xi_i \in \mathbb{R}[[I]]_0/\sim$ and $\eta_i \in \mathbb{R}[[H, K]]_0$.*

Thus the question about the co-dimension and universal unfolding of the mapping $\mathcal{F} \in C^\infty(\mathbb{R}^4, \mathbb{R}^2)_0^{S^1}$ reduces to finding the co-dimension and a complement of the first component of the tangent space of \mathcal{F} with respect to restricted left-right transformations. This in turn can be reformulated as follows. Let G be the mapping

$$G : \left(C^\infty(\mathbb{R}^4, \mathbb{R})_0^{S^1} \right)^k \times \left(C^\infty(\mathbb{R}^2, \mathbb{R}) \right)_0^3 \longrightarrow C^\infty(\mathbb{R}^4, \mathbb{R})_0^{S^1} .$$

$$(\xi_1, \dots, \xi_k, \eta_0, \eta_1, \eta_2) \mapsto X_1(H, K) + Y(H)$$

Then the questions we want to answer are:

1. What is the co-dimension of the image of G in $\mathbb{R}[[I]]_0/\sim$?
2. If the latter is nonzero, then what is a complement?

4.2. Statement of main theorem

Our main theorem is about the universal unfolding of the mapping $(H, K) : C^\infty(\mathbb{R}^4)_0^{S^1} \longrightarrow \mathbb{R}^2$ with respect to restricted left-right-equivalence from the previous section. That is we consider all left-right transformations that preserve $K = H^2_2$. As explained in section 1.2 we are interested in the fibres of the mapping (H, K) . For this mapping we have the following result.

Theorem 4.13. *The universal unfolding of the mapping (H, K) with respect to restricted left-right-equivalence is given by*

$$H(I; \mu) = I_1 + a_1 I_2^2 + a_2 I_3^2 + a_3 I_4^2 + b_1 I_2^3 + b_2 I_3^3 + b_3 I_4^3$$

$$+ \mu_1 I_2 + \mu_2 I_3 + \mu_3 I_4 + \mu_4 I_2^3 + \mu_5 I_3^3$$

$$K(I) = I_2^2 + I_3^2 + I_4^2$$

provided that the real coefficients a_1, a_2, a_3 and b_1, b_2 and b_3 satisfy the non-degeneracy condition

$$(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) b_1 b_2 b_3 \neq 0.$$

The parameters μ_4 and μ_5 are moduli.

5. Proof of main theorem

We now prove our main theorem. Our starting point is the mapping $\mathcal{F} = (H, K)$. We split the higher order terms of H into two parts, H_4 is of degree 2 in I , H_6 is of degree 3.

The proof consists of several steps which we now list.

- 1) Apply preliminary transformations to \mathcal{F} to get rid of as many coefficients as possible.
- 2) Determine the tangent space of $Orb_{\mathcal{F}}$ at \mathcal{F} .
- 3) Find the S^1 -equivariant vector fields on \mathbb{R}^4 .
- 4) Observe that we can restrict to the first component of \mathcal{F} using restricted vector fields.
- 5) Observe that we can proceed by degree when we split $C^\infty(\mathbb{R}^4, \mathbb{R})_0^{S^1}$ as a direct sum of spaces of homogeneous polynomials. The cases of relative large degree turn out to be the easiest. Then we are left with a finite number of low degree cases that have to be treated separately.

5.1. Preliminary transformations

We start with the mapping $\mathcal{F} = (H, K)$, where H is a polynomial of degree 3 in I , that is $H = H_2 + H_4 + H_6$. We assume that symplectic transformations already have been used exhaustively. But since we consider \mathcal{F} in a more general context, more transformations are allowed.

The first observation is that we can always subtract H_2 from H because H_2 is a conserved function in the sense of Hamiltonian systems. Thus we have $H = H_4 + H_6$. Furthermore, since $H_2(I) = I_1$ we consider I_1 as a parameter. Therefore I_1 appears at most in the coefficients of H . So in fact H and K only depend on I_2, I_3 and I_4 , without further restrictions or relations.

$$K(I_2, I_3, I_4) = I_2^2 + I_3^2 + I_4^2$$

$$H_4(I_2, I_3, I_4) = a_1 I_2^2 + a_2 I_3^2 + a_3 I_4^2 + a_{23} I_2 I_3 + a_{24} I_2 I_4 + a_{34} I_3 I_4$$

$$H_6(I_2, I_3, I_4) = b_1 I_2^3 + b_2 I_3^3 + b_3 I_4^3$$

The second observation is that by a transformation from $id \times SO(3)$ we can always achieve $a_{23} = 0$, $a_{24} = 0$ and $a_{34} = 0$. Note that such a transformation preserves both K and the relation $I_1^2 = I_2^2 + I_3^2 + I_4^2$.

Remark 5.1. We may include more third degree terms in H_6 , like $I_2 I_4^2$. However, they turn out to be unimportant.

5.2. S^1 -equivariant vector fields

Considering the mapping (H, K) instead of (H, H_2) where I_1 is a parameter, we take I_2, I_3 and I_4 as co-ordinates on \mathbb{R}^3 without any restrictions. Now (H, K) is a mapping in $C^\infty(\mathbb{R}^3, \mathbb{R}^2)_0$. Origin preserving transformations on \mathbb{R}^3 are generated by the vector fields

$$\begin{aligned}
 X_1 &= I_2 \frac{\partial}{\partial I_2}, X_2 = I_3 \frac{\partial}{\partial I_2}, X_3 = I_4 \frac{\partial}{\partial I_2}, \\
 X_4 &= I_2 \frac{\partial}{\partial I_3}, X_5 = I_3 \frac{\partial}{\partial I_3}, X_6 = I_4 \frac{\partial}{\partial I_3}, \\
 X_7 &= I_2 \frac{\partial}{\partial I_4}, X_8 = I_3 \frac{\partial}{\partial I_4}, X_9 = I_4 \frac{\partial}{\partial I_4}.
 \end{aligned}
 \tag{5}$$

To define the restricted tangent space of the mapping \mathcal{F} we have to find the vector fields solving $X(K) = 0$, $X(K) = K$ and $X(K) = H$.

Lemma 5.2. *The vector fields solving $X(K) = 0$ are generated by*

$$U_1 = X_2 - X_4, \quad U_2 = X_3 - X_7, \quad U_3 = X_6 - X_8.$$

The vector fields solving $X(K) = K$ and $X(K) = H$ respectively are generated by

$$\begin{aligned}
 V_1 &= \frac{1}{2}(X_1 + X_5 + X_9) \\
 V_2 &= \frac{1}{2}((a_1 + b_1 I_2)X_1 + (a_2 + b_2 I_3)X_5 + (a_3 + b_3 I_4)X_9).
 \end{aligned}$$

Proof. Let $X = \frac{1}{2} \sum_{i=1}^9 \xi_i X_i$ then

$$X(K) = \xi_1 I_2^2 + \xi_5 I_3^2 + \xi_9 I_4^2 + (\xi_2 + \xi_4) I_2 I_3 + (\xi_3 + \xi_7) I_2 I_4 + (\xi_6 + \xi_8) I_3 I_4$$

and after some straightforward calculations the results follow. \square

5.3. The structure of the restricted tangent space

The restricted tangent space T_1 , see section 4.1.3, is the sum of a module \mathcal{M} and an ideal \mathcal{J} both subsets of $\mathbb{R}[[I_2, I_3, I_4]]_0$. \mathcal{M} is a module over $\mathbb{R}[[H, K]]_0$ and generated by the functions $1, G_1$ and G_2 . \mathcal{J} is the ideal generated by F_1, F_2, F_3 . So if $f \in T_1$ then $f = \xi_1 F_1 + \xi_2 F_2 + \xi_3 F_3 + \eta_0 + \eta_1 G_1 + \eta_2 G_2$, with $\xi_i \in \mathbb{R}[[I_2, I_3, I_4]]_0$ and $\eta_i \in \mathbb{R}[[H, K]]_0$.

In Lemma 5.2 we defined the vector fields U_1, U_2, U_3, V_1 and V_2 . Thus we know the generators of \mathcal{J} and \mathcal{M}

$$\begin{aligned}
 F_1 &:= U_1(H) = (a_1 - a_2) I_2 I_3 + \text{h.o.t.} \\
 F_2 &:= U_2(H) = (a_1 - a_3) I_2 I_4 + \text{h.o.t.} \\
 F_3 &:= U_3(H) = (a_2 - a_3) I_3 I_4 + \text{h.o.t.} \\
 G_1 &:= V_1(H) - H = H_6(I_2, I_3, I_4) \\
 G_2 &:= V_2(H) = a_1^2 I_2^2 + a_2^2 I_3^2 + a_3^2 I_4^2 + \text{h.o.t.}
 \end{aligned}$$

Defining G_1 as $V_1(H) - H$ instead of $V_1(H)$ is just convenient but not essential. In the definition above we only show the *leading terms* of F_1, \dots, G_2 .

In principle each term in $f \in \mathbb{R}[[I_2, I_3, I_4]]_0$ is an infinite series, but with a term of lowest degree. For our purposes it makes sense to call this the *degree* of f and the term with lowest degree the *leading term*. Recall that the degree is at least 1 as we only consider formal series without constant term. Before using this to define a filtration on T_1 we formally define the degree of f and the leading term.

Definition 5.3. For $0 \neq f \in \mathbb{R}[[I_2, I_3, I_4]]_0$ we define the *degree* of f as $k \in \mathbb{N}$ for which $0 < \lim_{t \rightarrow 0} t^{-k} f(tI) < \infty$. Suppose $k = \text{degree}(f)$ then we call $L(f) = \lim_{t \rightarrow 0} t^{-k} f(tI)$ the *leading term* of f .

The following properties of degree and leading term are almost obvious.

Lemma 5.4. Let f and g be functions (germs) in $\mathbb{R}[[I_2, I_3, I_4]]_0$ and let m and n be monomials in $\mathbb{R}[[I_2, I_3, I_4]]_0$, then

- i) if $m(I) = I^l$ then $\text{degree}(m) = |l| := l_1 + l_2 + l_3$
- ii) if $\text{degree}(m) < \text{degree}(n)$ then $\text{degree}(m + n) = \text{degree}(m)$ and $L(m + n) = m$
- iii) $\text{degree}(f + g) = \min(\text{degree}(f), \text{degree}(g))$ and if $\text{degree}(f) < \text{degree}(g)$ then $L(f + g) = L(f)$
- iv) $\text{degree}(f \cdot g) = \text{degree}(f) + \text{degree}(g)$ and $L(f \cdot g) = L(f)L(g)$

With this notion of degree we define a filtration on $\mathbb{R}[[I_2, I_3, I_4]]_0$. Since \mathcal{J} and \mathcal{M} are subsets of $\mathbb{R}[[I_2, I_3, I_4]]_0$ they immediately inherit the filtration.

Definition 5.5. For $k \in \mathbb{N}_{>0}$ let \mathcal{R}_k be the set $\{f \in \mathbb{R}[[I_2, I_3, I_4]]_0 \mid \text{degree}(f) = k\}$. Then we have $\mathcal{R}_{k+1} \subset \mathcal{R}_k$ and $\mathcal{R}_1 = \mathbb{R}[[I_2, I_3, I_4]]_0$, therefore \mathcal{R}_k is a filtration of $\mathbb{R}[[I_2, I_3, I_4]]_0$. Similarly $\{\mathcal{J}_k\}$ and $\{\mathcal{M}_k\}$ are filtrations.

Remark 5.6. As an analogy of a Gröbner basis for polynomial ideals, see [4], we could hope that T_1 is generated by $L(F_1), L(F_2), L(F_3), 1, L(G_1), L(G_2)$ in the following sense: every $f \in T_1$ can be written as $\xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3) + \eta_0 + \eta_1 L(G_1) + \eta_2 L(G_2)$, with $\xi_i \in \mathbb{R}[[I_2, I_3, I_4]]_0$ and $\eta_i \in \mathbb{R}[[H, K]]_0$.

5.4. Splitting into homogeneous parts

Since the co-dimension of \mathcal{F} as a smooth mapping is the same as the co-dimension of the mapping as a formal power series, we can simplify the problem by looking at homogeneous functions and add the co-dimensions found for each degree starting at degree one. This is carried out in the following chain of assertions.

Let $\mathcal{H}_k(I_2, I_3, I_4)$ be the set of all homogeneous functions of degree k in I_2, I_3 and I_4 . In fact we have $\mathcal{H}_k(I_2, I_3, I_4) = \mathcal{R}_k / \mathcal{R}_{k+1}$. Furthermore let $\mathcal{H}_k(H, K)$ be the set of all homogeneous functions of degree k in K and H , then $\mathcal{H}_m(H, K) \subset \mathcal{R}_{2m}$. Since $\mathcal{H}_m(H, K)$ is not homogeneous in I we use a projection $\Pi_k : \mathcal{R}_k \rightarrow \mathcal{H}_k(I_2, I_3, I_4)$ selecting the homogeneous part of a function $f \in \mathcal{R}_k$. The following general result leaves us with a small number of cases.

Proposition 5.7. The co-dimension of $\Pi_k(T_1)$ in $\mathcal{H}_k(I_2, I_3, I_4)$ is zero for $k = 4$ and $k \geq 6$. Or, put differently, the mapping (odd degree)

$$\begin{aligned} &\mathcal{H}_{2m-1}(I_2, I_3, I_4)^3 \times \mathcal{H}_{m-1}(H, K) \rightarrow \mathcal{H}_{2m+1}(I_2, I_3, I_4) : \\ &(\xi_1, \xi_2, \xi_3, \eta_1) \mapsto \Pi_{2m+1}(\xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3) + \eta_1 L(G_1)) \end{aligned}$$

is onto for $m \geq 3$ and also the mapping (even degree)

$$\mathcal{H}_{2m-2}(I_2, I_3, I_4)^3 \times \mathcal{H}_m(H, K) \times \mathcal{H}_{m-1}(H, K) \rightarrow \mathcal{H}_{2m}(I_2, I_3, I_4) : \\ (\xi_1, \xi_2, \xi_3, \eta_0, \eta_2) \mapsto \Pi_{2m}(\xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3) + \eta_0 + \eta_2 L(G_2))$$

is onto for $m \geq 2$.

Thus we have to investigate degrees 1, 2, 3 and 5 separately. First we prove Proposition 5.7 in three lemmas. In order to do so it is useful to introduce some notation, which is motivated by the fact that the projection of $\xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3)$ on $\langle I_2^k, I_3^k, I_4^k \rangle$ is always zero.

Definition 5.8. The space $\mathcal{H}_k(I_2, I_3, I_4)$ has a monomial basis denoted by $b_k = \{\dots, I_2^k, I_3^k, I_4^k\}$. Let b_k^\sharp be the set of monomials I_2^k, I_3^k and I_4^k . Furthermore let b_k^b be the set of monomials in b_k with the monomials in b_k^\sharp excluded. Finally let B_k^\sharp be the subspace of $\mathcal{H}_k(I_2, I_3, I_4)$ spanned by b_k^\sharp , similarly B_k^b is spanned by b_k^b .

The next three lemmas treat different parts of Proposition 5.7. The following lemma shows that the mapping from $\mathcal{H}_{k-2}(I_2, I_3, I_4)^3$ to B_k^b is onto for each $k \geq 2$. Thus we get rid of the first factor of the mapping in Proposition 5.7. Later on we use this lemma again for the remaining low degree cases.

Lemma 5.9. The mapping $\mathcal{H}_{k-2}(I_2, I_3, I_4)^3 \rightarrow B_k^b : (\xi_1, \xi_2, \xi_3) \mapsto \xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3)$ is onto provided that $a_1 - a_2 \neq 0, a_2 - a_3 \neq 0$ and $a_3 - a_1 \neq 0$ and $k \geq 2$.

Proof. Every monomial in B_k^b can be written as either $I^l I_2 I_3, I^l I_3 I_4$ or $I^l I_2 I_4$ for some multi-index l with $|l| = k - 2$. Therefore every $f \in B_k^b$ can be expressed as $\xi_1 L(F_1) + \xi_2 L(F_2) + \xi_3 L(F_3)$ for some $\xi_i \in \mathcal{H}_{k-2}(I_2, I_3, I_4)$, but only if $a_1 - a_2 \neq 0, a_2 - a_3 \neq 0$ and $a_3 - a_1 \neq 0$. If for example $a_1 - a_2 = 0$, then $I_2 I_3 \notin T_1$. \square

The following two lemmas show that the second factor of the mapping in Proposition 5.7 maps onto B_k^\sharp , but we have to distinguish the odd and even degree cases.

Lemma 5.10 (Odd degree). The mapping $\mathcal{H}_{m-1}(H, K) \rightarrow \mathcal{H}_{2m+1}(I_2, I_3, I_4) : \eta_1 \mapsto \Pi_{2m+1}(\eta_1 L(G_1))$ followed by projection on B_{2m+1}^\sharp is onto provided that $a_1 - a_2 \neq 0, a_2 - a_3 \neq 0, a_3 - a_1 \neq 0$ and $b_1 b_2 b_3 \neq 0$ and $m \geq 3$.

Proof. The projection of the functions $K^{m-1} L(G_1), K^{m-2} H L(G_1), \dots, H^{m-1} L(G_1)$ on B_{2m+1}^\sharp is given by the vectors in the matrix

$$\begin{pmatrix} b_1 & a_1 b_1 & a_1^2 b_1 & \dots & a_1^{m-1} b_1 \\ b_2 & a_2 b_2 & a_2^2 b_2 & \dots & a_2^{m-1} b_2 \\ b_3 & a_3 b_3 & a_3^2 b_3 & \dots & a_3^{m-1} b_3 \end{pmatrix}$$

which has rank three as soon as the conditions are met. \square

Finally we state and prove a lemma for the even degree case.

Lemma 5.11 (Even degree). *The mapping $\mathcal{H}_m(H, K) \times \mathcal{H}_{m-1}(H, K) \rightarrow \mathcal{H}_{2m}(I_2, I_3, I_4) : (\eta_0, \eta_2) \mapsto \Pi_{2m}(\eta_0 + \eta_2 L(G_2))$ followed by projection on B_{2m}^\sharp is onto provided that $a_1 - a_2 \neq 0$, $a_2 - a_3 \neq 0$ and $a_3 - a_1 \neq 0$ and $m \geq 2$.*

Proof. The projection of the functions

$$K^m, K^{m-1}H, \dots, H^m, K^{m-1}L(G_2), K^{m-2}HL(G_2), \dots, H^{m-1}L(G_2)$$

on B_{2m}^\sharp is given by the vectors in the matrix

$$\begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^m \\ 1 & a_2 & a_2^2 & \dots & a_2^m \\ 1 & a_3 & a_3^2 & \dots & a_3^m \end{pmatrix} \tag{6}$$

which has rank three as soon as the conditions are met. \square

With these three lemmas we prove Proposition 5.7.

Proof of Proposition 5.7. The odd degree part of the proposition is covered by combining Lemmas 5.9 and 5.10 showing that the product mapping is onto $\mathcal{H}_{2m+1}(I_2, I_3, I_4)$. Similarly combining Lemmas 5.9 and 5.11 shows that in case of even degree the product mapping is onto $\mathcal{H}_{2m}(I_2, I_3, I_4)$. \square

Finally we consider the remaining cases: degrees 1, 2, 3 and 5. In all cases we follow the same pattern, we determine the co-dimension of $\Pi_k(\xi_1 F_1 + \xi_2 F_2 + \xi_3 F_3 + \eta_0 + \eta_1 G_1 + \eta_2 G_2)$ in $\mathcal{H}_k(I_2, I_3, I_4)$ for $k \in \{1, 2, 3, 5\}$. But in view of Lemma 5.9 we only have to consider the projection on B_k^\sharp . The main result of this part is the next proposition.

Proposition 5.12. *A complement of T_1 in \mathcal{R} is spanned by the functions $\langle I_2, I_3, I_4, I_2^3, I_3^3 \rangle$ or $\langle I_2, I_3, I_4, I_2^3, I_4^3 \rangle$ or $\langle I_2, I_3, I_4, I_3^3, I_4^3 \rangle$ as a linear space.*

We prove this proposition in several lemmas. The following lemma is immediately clear.

Lemma 5.13 (Degree one). *A monomial basis of functions of degree one is $\{I_2, I_3, I_4\}$. Since T_1 does not contain functions of degree one, the co-dimension in this space is three and a complement is B_1^\sharp itself.*

Thus we get unfolding terms: $\mu_1 I_2, \mu_2 I_3$ and $\mu_3 I_4$.

Lemma 5.14 (Degree two). *Functions of degree two with a nonzero projection on B_2^\sharp are K, H and G_2 . These three functions are independent as soon as $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) \neq 0$.*

Proof. The projection of K , H and G_2 onto B_2^\sharp is given by the matrix

$$A_2^\sharp = \begin{pmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{pmatrix},$$

cf. (6). The determinant of A_2^\sharp is $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$. \square

Lemma 5.15 (Degree three). *There is only one function in T_1 with a nonzero projection on B_3^\sharp , namely G_1 . Thus the co-dimension of T_1 in the space of homogeneous functions of degree three is two. As a complement any pair of I_2^3 , I_3^3 and I_4^3 will do. We take for example $\mu_4 I_2^3$ and $\mu_5 I_3^3$ as unfolding terms, then we must impose the condition $b_3 \neq 0$.*

Proof. The projection of G_1 , $\mu_4 I_2^3$ and $\mu_5 I_3^3$ on B_3^\sharp is given by the matrix

$$A_3^\sharp = \begin{pmatrix} b_1 & \mu_4 & 0 \\ b_2 & 0 & \mu_5 \\ b_3 & 0 & 0 \end{pmatrix} \square$$

Lemma 5.16 (Degree five). *There are only two functions of degree five in T_1 , namely KG_1 and HG_1 , with a nonzero projection on B_5^\sharp . However, a function $F_5 \in T_1$ exists such that $\Pi_k(F_5) = 0$ for $k \leq 4$ and $\Pi_5(F_5) \neq 0$. With F_5 the co-dimension of T_1 in the space of homogeneous functions of degree five is zero, provided that $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)b_1b_2b_3 \neq 0$.*

Proof. Let

$$F_5 = \xi_1 I_2 I_3 F_1 + \xi_2 I_2 I_4 F_2 + \xi_3 I_3 I_4 F_3 + \eta_{01} K^2 + \eta_{02} KH + \eta_{03} H^2 + \eta_{21} KG_2 + \eta_{22} HG_2$$

be a function of degree 4, with $\xi_1, \dots, \eta_{22} \in \mathbb{R}$. Then a non-trivial solution of $\Pi_4(F_5) = 0$ exists while $\Pi_5(F_5) \neq 0$. The projection of the functions $\Pi_5(KG_1)$, $\Pi_5(HG_1)$ and $\Pi_5(F_5)$ onto B_5^\sharp has the matrix

$$A_5^\sharp = \begin{pmatrix} b_1 & a_1 b_1 & -(a_1^2 - a_2 a_3 + a_1(a_2 + a_3))b_1 \\ b_2 & a_2 b_2 & -(a_2^2 - a_1 a_3 + a_2(a_1 + a_3))b_2 \\ b_3 & a_3 b_3 & -(a_3^2 - a_1 a_2 + a_3(a_1 + a_2))b_3 \end{pmatrix}$$

and $\det(A_5^\sharp) = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1)b_1b_2b_3$. \square

The last lemma is about the modal parameters.

Lemma 5.17. *Parameters μ_4 and μ_5 are moduli.*

Proof. Let $H(I; \mu)$ be as in the main Theorem 4.13. From the previous proofs it follows almost immediately that the unfoldings of $(H(I; 0, 0, 0, 0, 0), K)$ and $(H(I; 0, 0, 0, \mu_4, \mu_5), K)$ are equal for small values of μ_4 and μ_5 . Therefore μ_4 and μ_5 are moduli. \square

The proof of Theorem 4.13 follows from Proposition 5.7, Proposition 5.12 and Lemma 5.17.

Remark 5.18. As a by product we find that T_1 is not generated by

$$L(F_1), L(F_2), L(F_3), 1, L(G_1), L(G_2).$$

See Remark 5.6.

6. Discussion

The dynamics of an n -degree-of-freedom Hamiltonian system locally around an elliptic equilibrium at the origin is characterised by an n -tuple $\omega \in \mathbb{R}^n$ of frequencies. When the frequencies satisfy an integer relation $\langle m | \omega \rangle \neq 0$ with $m \in \mathbb{Z}^n$ we say that the frequencies are *resonant*. For most equilibria the frequencies are non-resonant. However, when the system depends on parameters there are resonances at a dense subset of parameter values. Since low order resonances are accompanied by bifurcations the corresponding points in parameter space are of special interest.

Here we consider two-degree-of-freedom systems. In that case $\omega = (\omega_1, \omega_2)$, so ω is resonant if ω_1/ω_2 is an element of \mathbb{Q} . We may assume without loss of generality that ω_1 and ω_2 are relative prime integers k and l at resonance. The linear part of the vector field is determined by $\omega = (k, l)$ if $k \neq \pm l$. In linear Hamiltonian systems imaginary eigenvalues, in case the frequencies k, l have a sign. The sign is related to the Morse index of the Hamiltonian. Therefore a $k:l$ resonance is not equivalent to a $k:-l$ resonance; in particular the 1:1 and 1:-1 resonances are not equivalent. Moreover, eigenvalues with equal sign are always semi-simple, whereas the 1:-1 resonance can also be nilpotent. Thus there are three resonances with equal frequencies, namely the semi-simple 1:-1, the nilpotent 1:-1 and the 1:1 resonance. The latter is always semi-simple. The nilpotent 1:-1 resonance is what triggers the Hamiltonian Hopf bifurcation.

As indicated in the introduction the $k:l$ resonances, with $k, l \in \mathbb{N}$, are very similar. In particular, in the sense of section 4.1.1 the co-dimension is 2, provided that $k:l$ is not equal to 1:1, 1:2 or 1:3. The last two exceptional cases have co-dimension 1 and 3, respectively. Thus all definite resonances except 1:1 have in common that they occur persistently in 1-parameter families and if more parameters are present these are moduli, see [10]. In this respect our case of the 1:1 resonance is very exceptional: its co-dimension is 5, it occurs persistently in 3-parameter families and two of the unfolding parameters are moduli. When we restrict to the linear unfolding, there is a transformation group acting on the unfolding. This can be used to reduce the number of parameters. Using invariants of this transformation group we find that one of the generators is $\mu_1^2 + \mu_2^2 + \mu_3^2$. Then in a *reduced linear unfolding* the 1:1 resonance occurs persistently in a 1-parameter family, see [17] for more details.

Before applying singularity theory we reduce the S^1 -symmetric system using invariants. Another approach is that in [4] where the system is first reduced to a planar system. Then singularity theory using right equivalence is applied to obtain an unfolding. With a different notion of equivalence one may expect different co-dimensions. In [4], by nature of the method, one finds lower bounds for the co-dimensions. For the resonances 1:2, 1:3 and 1:4 these lower bounds are computed and they coincide with the co-dimensions found in [10], namely 1, 3 and 2, respectively. However, the non-degeneracy conditions of [4] and [10] differ. It would be interesting to compare both methods for the 1:1 resonance.

The results obtained so far are a starting point for extensions and applications. Let us list a few. In general, when a system passes a resonance upon varying one or more parameters, one expects a

bifurcation to occur. We see this phenomenon in the resonances mentioned earlier. Therefore we would like to explore the bifurcation scenario of the 1:1 resonance, or more general explore the geometry of level sets of the momentum mapping depending on parameters near 1:1 resonance. A similar program can be carried out for Hamiltonian systems in 1:1 resonance which are also reversible, see [25], or symmetric (other than the S^1 symmetry induced by the 1:1 resonance). For a reversible unfolding with an additional \mathbb{Z}_2 -symmetry see [26,27] and references therein.

For unfoldings with a single parameter $\lambda \in \mathbb{R}$ Theorem 4.13 yields a left–right equivalence with

$$H(I; \mu(\lambda)) = H(I) + \mu_1(\lambda)I_2 + \mu_2(\lambda)I_3 + \mu_3(\lambda)I_4 + \mu_4(\lambda)I_2^3 + \mu_5(\lambda)I_3^3 .$$

In [30] 1-parameter unfoldings of H_2 are studied that are invariant under the linear symplectic action of a compact symmetry group Γ . In case the 1-parameter unfolding is universal with respect to Γ -symmetric unfoldings it is shown that the bifurcation diagram is determined by the linear part — the moduli $\mu_4(\lambda)$ and $\mu_5(\lambda)$ can be removed, conforming to Lemma 5.17.

The unfolding of the semisimple 1:–1 resonance is similar to the unfolding of the 1:1 resonance, but the bifurcation scenario is most likely very different. A well-known system in 1:1 resonance is the Hénon–Heiles system. Our original plan, to apply the unfolding and bifurcation results, now comes within reach. Furthermore we wish to relate our results to the results in a series of articles by Elipe, Lanchares et al. and Frauendiener [11,12,21–25] for families of S^1 -symmetric Hamiltonian systems. These are the subjects of future publications.

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