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BLOW-UPS IN GENERALIZED COMPLEX GEOMETRY

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Abstract. We study blow-ups in generalized complex geometry. To that end we introduce the concept of holomorphic ideals, which allows one to define a blow-up in the category of smooth manifolds. We then investigate which generalized complex submanifolds are suitable for blowing up. Two classes naturally appear: generalized Poisson submanifolds and generalized Poisson transversals. These are submanifolds for which the geometry normal to the submanifold is complex, respectively symplectic. We show that generalized Poisson submanifolds carry a canonical holomorphic ideal, and we give a necessary and sufficient condition for the corresponding blow-up to be generalized complex. For generalized Poisson transversals we prove a normal form theorem for a neighborhood of the submanifold and use it to define a generalized complex blow-up.

Contents

1. Introduction

The notion of blowing up was invented by algebraic geometers in the study of birational transformations. Although it is unclear to the authors when and by whom precisely the notion of blowing up was invented, Zariski [18] introduced it in modern language and used it to study singularities. This work culminated in results by Abhyankar and Hironaka on resolutions of singularities in all dimensions. Later Hopf [12] introduced the corresponding notion in the context of complex analytic geometry. Blowing up a submanifold preserves the class of Kähler manifolds, and it was pointed out by Gromov in [9] that it can be defined in the symplectic category as well. This was then used by McDuff in [15] to produce examples of simply-connected non-Kählerian symplectic manifolds.

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The fact that blow-ups exist in both complex and symplectic geometry naturally raises the question whether the same is true in generalized complex geometry, a concept introduced by Hitchin and developed by Gualtieri [10] and which unifies complex and symplectic structures into one framework. This question was first investigated in [7], where it was shown that a blow-up exists for a non-degenerate point of complex type in a generically symplectic 4-manifold. This was then used to produce new examples of generalized complex structures on the manifolds $m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ for m odd.

In this paper we study blow-ups in generalized complex geometry. The first step is to understand which submanifolds are suitable for blowing up. In the complex and symplectic categories these are the complex, respectively symplectic, submanifolds. There are a number of possible ways to define a generalized complex submanifold, and the one which we will use has complex and symplectic submanifolds as special examples. However, for blowing up this notion is too general, and we will restrict ourselves to two special subclasses. The first are the *generalized Poisson submanifolds*, where the geometry normal to the submanifold is complex. Using the normal form theorem of the first author [2] we prove that these submanifolds come naturally equipped with a special ideal which gives them a holomorphic flavor, and we use that to construct the blow-up as a differentiable manifold. The question of whether this blow-up has a generalized complex structure for which the blow-down map is holomorphic then boils down to the analogous question in the context of holomorphic Poisson geometry. This has been answered by Polishchuk in [16], and, building on that, we give necessary and sufficient conditions for blowing up a generalized Poisson submanifold.

The second class of submanifolds comprises the *generalized Poisson transversals*, where the geometry normal to the submanifold is symplectic. As in the symplectic category, to blow them up we first need a normal form for the generalized complex structure in a neighborhood of the submanifold. Such a neighborhood theorem was constructed in [8] in the context of Poisson geometry, and it has a direct counterpart in our setting. We then blow up the submanifold globally. An elegant way to perform this last step uses reduction methods, just as the symplectic blow-up can be performed using symplectic cuts, as shown in [13]. In contrast with generalized Poisson submanifolds, the blow-up is not canonical but depends on some additional choices. This is also the case for blow-ups in symplectic geometry, for even if one specifies the symplectic volume of the exceptional divisor, it is not known in general whether the blow-up is unique up to symplectomorphism.

Organization: In Section 2 we briefly review all the necessary ingredients from generalized complex geometry that are needed in the paper. Most of this material is due to [10], and all statements without explicit references are from there. We then proceed in Section 3 to the blow-up procedure. We first define the notion of a *holomorphic ideal* and argue that this is the natural input to define a blow-up procedure in the category of smooth manifolds. Then, in Section 3.1 we introduce generalized Poisson manifolds and explain the extra assumptions that are needed for the blow-up. In Section 3.2 we define generalized Poisson transversals, give a normal form for their neighborhoods, and use it to blow them up. Finally, in Section 3.3 we discuss other types of generalized complex submanifolds and give a concrete example of one that cannot be blown-up.

2. Generalized complex geometry

Let M be a real $2n$-dimensional manifold equipped with a closed real 3-form H. The main idea of generalized geometry is to replace the tangent bundle TM by the bundle $\mathbb{T}M := TM \oplus T^*M$. The latter carries two natural structures, the first being a fiberwise *natural pairing*

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}(\xi(Y)+\eta(X)),$$

which is a non-degenerate metric of signature $(2n, 2n)$. The second is a bracket on its space of sections which replaces the Lie bracket and is called the *Courant bracket*. It is given by

$$[\![X+\xi, Y+\eta]\!] := [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi - \iota_Y\iota_X H.$$

This version of the Courant bracket is not skew-symmetric but does satisfy the Jacobi identity.

Definition 2.1. A *generalized complex structure* on (M,H) is a complex structure $\mathcal{J}$ on $\mathbb{T}M$ which is orthogonal with respect to the natural pairing and whose $(+i)$-eigenbundle $L \subset \mathbb{T}M_{\mathbb{C}}$ is involutive.[1]

A Lagrangian, involutive subbundle $L \subset \mathbb{T}M_{\mathbb{C}}$ is also called a *Dirac structure*, and it follows from the definition that generalized complex structures correspond in a one-to-one fashion with Dirac structures L satisfying the non-degeneracy condition $L \cap \bar{L} = 0$.

Example 2.2. The main examples are provided by complex and symplectic geometry:

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{2.1}$$

The associated Dirac structures are given by $L_I = T^{0,1}M \oplus T^{*1,0}M$, where $T^{0,1}M$ denotes the $(-i)$-eigenbundle of I, and $L_\omega = \{X - i\omega(X) |\ X \in TM_{\mathbb{C}}\}$. Another important example is provided by a holomorphic Poisson structure (I,σ). If $\sigma = Q - iIQ$, then

$$\mathcal{J}_\sigma = \begin{pmatrix} -I & 4IQ \\ 0 & I^* \end{pmatrix} \tag{2.2}$$

is generalized complex, with Dirac structure $L_{I,\sigma} = \{X + \sigma(\xi) + \xi | X \in T^{0,1}M, \xi \in T^{*1,0}M\}$. In these examples the 3-form is taken to be 0.

A useful way to look at generalized complex structures is through spinors. There is a natural action of the Clifford algebra of $(\mathbb{T}M, \langle,\rangle)$ on differential forms given by

$$(X+\xi)\cdot\rho = \iota_X\rho + \xi\wedge\rho,$$

yielding an identification between the space of differential forms and the space of spinors for $\mathrm{Cl}(\mathbb{T}M, \langle,\rangle)$. A line subbundle $K \subset \Lambda^\bullet T^*M_{\mathbb{C}}$ gives rise to an isotropic subbundle $L \subset \mathbb{T}M_{\mathbb{C}}$ by taking its annihilator

$$L = \{X+\xi \in \mathbb{T}M_{\mathbb{C}} | (X+\xi)\cdot K = 0\}.$$

[1]A subbundle of $\mathbb{T}M$ is called involutive if its space of sections is closed with respect to the Courant bracket.

This gives rise to a one-to-one correspondence between Dirac structures $L \subset \mathbb{T}M_{\mathbb{C}}$ and complex line bundles $K \subset \Lambda^\bullet T^*M_{\mathbb{C}}$ which satisfy the following two conditions. Firstly, K has to be generated by *pure spinors*, i.e., forms ρ which at each point x admit a decomposition

$$\rho_x = e^{B+i\omega} \wedge \Omega, \tag{2.3}$$

where $B + i\omega$ is a 2-form and Ω is decomposable. This condition is equivalent to L being of maximal rank. Secondly, if ρ is a local section of K there should exist $X + \xi \in \Gamma(\mathbb{T}M_{\mathbb{C}})$ with

$$d^H\rho = (X + \xi) \cdot \rho.$$

This condition amounts to the involutivity of L. The condition $L \cap \bar{L} = 0$ can then be expressed in spinor language using the *Chevalley pairing*: If $\rho \in \Gamma(K)$ is non-vanishing, then

$$L \cap \bar{L} = 0 \Longleftrightarrow (\rho, \bar{\rho})_{Ch} := (\rho \wedge \bar{\rho}^T)_{top} \neq 0.$$

The superscript T stands for transposition, acting on a degree l-form by $(\beta_1 \wedge \ldots \wedge \beta_l)^T = \beta_l \wedge \ldots \wedge \beta_1$, and the subscript *top* stands for the highest degree component. If ρ is given by (2.3) at a particular point x, then this condition becomes

$$\omega^{n-k} \wedge \Omega \wedge \bar{\Omega} \neq 0, \tag{2.4}$$

where $2n$ is the real dimension of M and $k = \deg(\Omega)$. The line bundle K associated to a generalized complex structure $\mathcal{J}$ is called the *canonical line bundle*, and the integer k appearing in (2.4) is called the *type* of $\mathcal{J}$ at x. Structures of type 0 are called symplectic, and those of maximal type n complex.[2] Another description of the type is as follows. Every generalized complex structure naturally induces a Poisson structure given by the composition

$$\pi_{\mathcal{J}} : T^*M \hookrightarrow \mathbb{T}M \xrightarrow{\mathcal{J}} \mathbb{T}M \twoheadrightarrow TM. \tag{2.5}$$

The conormal bundle to the leaves, i.e., the kernel of $\pi_{\mathcal{J}}$, is given by the complex distribution

$$\nu_{\mathcal{J}} := T^*M \cap \mathcal{J}(T^*M).$$

Note that $\nu_{\mathcal{J}}$ might be singular as its complex dimension can jump in even steps from one point to the next. The type at a point x is then given by

$$\mathrm{type}_x(\mathcal{J}) = \dim_{\mathbb{C}}(\nu_{\mathcal{J}})_x = \frac{1}{2}\mathrm{corank}_{\mathbb{R}}(\pi_{\mathcal{J}})_x.$$

Having laid out the relevant geometric structures we need to define morphisms between them.

Definition 2.3. A *generalized map* between (M_1, H_1) and (M_2, H_2) is a pair $\Phi := (\varphi, B)$, where $\varphi : M_1 \to M_2$ is a smooth map and $B \in \Omega^2(M_1)$ satisfies $\varphi^*H_2 = H_1 + dB$.

[2]The reason being that they are equivalent to symplectic or complex structures, where equivalence is defined in Definition 2.4.

We will often abbreviate $(\varphi, 0)$ by φ and drop the prefix "generalized". An important role is played by *B-field transformations*, maps of the form[3] $(Id, -B) =: e^B_*$. They act on $\mathbb{T}M$ via

$$e^B_* : X + \xi \mapsto X + \xi - \iota_X B. \tag{2.6}$$

Given $u \in \Gamma(\mathbb{T}M)$ we denote by $\mathrm{ad}(u) : \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ the adjoint action with respect to the Courant bracket. This infinitesimal symmetry has a flow, i.e., a family of isomorphisms $\psi_t : \mathbb{T}M \to \mathbb{T}M$ characterized by the equation

$$\frac{d}{dt}\psi_t(v) = -[\![u, \psi_t(v)]\!].$$

Concretely, if $u = X + \xi$ and φ_t denotes the flow of X, then

$$\psi_t = (\varphi_t)_* \circ e_*^{-\int_0^t B_s ds}, \tag{2.7}$$

where $B_s := \varphi_s^*(d\xi + \iota_X H)$. A map $\Phi = (\varphi, B)$ gives rise to a correspondence

$$X + \xi \underset{\Phi}{\sim} Y + \eta \overset{\text{def}}{\Longleftrightarrow} \varphi_* X = Y, \quad \xi = \varphi^*\eta - \iota_X B.$$

Definition 2.4. A map $\Phi : (M_1, H_1, \mathcal{J}_1) \to (M_2, H_2, \mathcal{J}_2)$ is called *generalized holomorphic* if

$$X + \xi \underset{\Phi}{\sim} Y + \eta \Longrightarrow \mathcal{J}_1(X + \xi) \underset{\Phi}{\sim} \mathcal{J}_2(Y + \eta).$$

It is called an *isomorphism* if it is in addition invertible.

Remark 2.5. It follows immediately from the definition that φ is a Poisson map, i.e., $\varphi_*\pi_{\mathcal{J}_1} = \pi_{\mathcal{J}_2}$. This is quite restrictive; for example if the target is symplectic, then φ has to be a submersion. In the complex category we recover the usual notion of holomorphic maps.

In case φ is a diffeomorphism a more concrete description in terms of spinors can be given. If K_i is the canonical bundle for $\mathcal{J}_i$, Φ being an isomorphism amounts to

$$K_1 = e^B \wedge \varphi^* K_2.$$

We now state the analogue of the Newlander-Nirenberg and Darboux theorems in generalized complex geometry.

Theorem 2.6 ([2])**.** *Let $(M, H, \mathcal{J})$ be a generalized complex manifold. If $x \in M$ is a point where $\mathcal{J}$ has type k, then a neighborhood of x is isomorphic to a neighborhood of $(0,0)$ in*

$$(\mathbb{R}^{2n-2k}, \omega_{st}) \times (\mathbb{C}^k, \sigma), \tag{2.8}$$

where ω_{st} is the standard symplectic form, σ is a holomorphic Poisson structure which vanishes at 0, and the 3-form is zero.

Finally we come to the notion of a generalized complex submanifold. For this the notion of holomorphic map as defined above is actually too restrictive. Let $\Phi = (\varphi, B)$ be a map and let L_2 be a Dirac structure on (M_2, H_2). We define the *backward image* of L_2 along Φ by

$$\mathfrak{B}\Phi(L_2) := \{X + \varphi^*\xi - \iota_X B \,|\, \varphi_* X + \xi \in L_2\} \subset \mathbb{T}M_1. \tag{2.9}$$

[3] The minus sign is chosen so that $e^B \wedge ((X + \xi) \cdot \rho) = (e^B_*(X + \xi)) \cdot e^B \wedge \rho$.

This is a Dirac structure on (M_1, H_1), provided it is a smooth vector bundle. A sufficient condition for that is that $\ker(d\varphi^*) \cap \varphi^* L$ is of constant rank. Similarly, the *forward image* of a Dirac structure L_1 on (M_1, H_1) is given by

$$\mathfrak{F}\Phi(L_1) := \{\varphi_* X + \xi | X + \varphi^*\xi - \iota_X B \in L_1\} \subset \varphi^* \mathbb{T} M_2.$$

This will be smooth if $\ker(\varphi_*) \cap e_*^{-B} L$ has constant rank and projects down to M_2 if it is constant along the fibers of φ. In case φ is a diffeomorphism we have $\mathfrak{F}\Phi(L) = \varphi_*(e_*^{-B}(L))$. A more detailed description of forward and backward images in Dirac geometry, including proofs of the above statements, can be found in [5].

Definition 2.7. A *generalized complex submanifold* is a submanifold $i : Y \hookrightarrow (M, H, \mathcal{J})$ such that $\mathfrak{B}i(L)$ is generalized complex, i.e., is smooth and satisfies $\mathfrak{B}i(L) \cap \overline{\mathfrak{B}i(L)} = 0$.

Remark 2.8. A sufficient condition for smoothness is that $N^*Y \cap \mathcal{J}(N^*Y)$ is of constant rank. Moreover, the second condition is equivalent to $\mathcal{J}(N^*Y) \cap (N^*Y)^\perp \subset N^*Y$. In complex or symplectic manifolds we recover the usual notion of complex, respectively symplectic, submanifolds. Also, a point is always a generalized complex submanifold. Note that in the symplectic case the inclusion map is only generalized holomorphic if Y is an open subset.

3. Blowing up submanifolds

Before considering blow-ups in generalized complex geometry we discuss the notion of blowing up a submanifold in the general context of smooth manifolds. We emphasize that the blow-ups that will be considered here refer to complex blow-ups and in particular should not be confused with so-called real (oriented) blow-ups.

Definition 3.1. Let M be a smooth manifold and let C^∞_M be the sheaf of complex valued smooth functions on M. Let $Y \subset M$ be a closed[4] submanifold of real codimension $2l$, with $l \geq 1$. A *holomorphic ideal* for Y is an ideal sheaf $I_Y \subset C^\infty_M$ with the following properties:

(i) $I_Y|_{M \setminus Y} = C^\infty_M|_{M\setminus Y}$.
(ii) Each $y \in Y$ has a neighborhood U together with $z^1, \dots, z^l \in I_Y(U)$, such that $z := (z^1, \dots, z^l) : U \to \mathbb{C}^l$ is a submersion with $Y \cap U = z^{-1}(0)$, and $I_Y|_U = \langle z^1, \dots, z^l\rangle$.

A holomorphic ideal turns NY into a complex vector bundle via $N^*Y_{\mathbb{C}} = N^{*1,0}Y \oplus N^{*0,1}Y$, where $N^{*1,0}_y Y := \langle d_y z | \; z \in I_Y\rangle$. Given a complex structure on NY there are many holomorphic ideals inducing it, and one way to obtain them is as follows. The zero section in NY carries a natural holomorphic ideal generated by $\Gamma(N^{*1,0}Y)$, viewed as fiberwise linear functions on NY. We will call this ideal I_Y^{lin}. Using a tubular embedding of NY into M we can then glue this ideal on NY to the trivial ideal on $M\setminus Y$. In fact, all holomorphic ideals for Y arise in this way.

Proposition 3.2. *Let I_Y be a holomorphic ideal for Y, inducing a complex structure on NY. Then there exists a tubular embedding $\iota : NY \to M$ such that $\iota^* I_Y = I_Y^{lin}$.*

[4]Closed in the sense of topological subspace; we do not necessarily assume that Y is compact.

Proof. Let $\kappa : NY \to M$ be a tubular embedding and consider $\kappa^* I_Y$, which is a holomorphic ideal for Y on NY inducing the same complex structure[5] on NY as I_Y^{lin}. It suffices to show that there exists a diffeomorphism φ of NY defined in a neighborhood of Y that fixes Y and satisfies $\varphi^*\kappa^* I_Y = I_Y^{\mathrm{lin}}$. Then $\iota := \kappa \circ \varphi$ will be the desired tubular embedding. We will construct φ on $N^{1,0}Y$, which is isomorphic as a complex vector bundle to NY.

Pick an open cover $\{U_\alpha\}$ of Y together with trivializing frames $e^\alpha = (e_1^\alpha, \dots, e_k^\alpha)$ for $N^{1,0}Y$ over U_α, with $k = \mathrm{codim}_{\mathbb{C}}(Y)$. Such a local frame induces an identification $q^{-1}U_\alpha \cong U_\alpha \times \mathbb{C}^k$ by letting $(x, z_\alpha) \in U_\alpha \times \mathbb{C}^k$ correspond to $\sum_i z_\alpha^i e_i^\alpha(x)$. Here $q : N^{1,0}Y \to Y$ denotes the projection. Let $e_i^\alpha = (g_{\alpha\beta})_i{}^j e_j^\beta$ be the transition on a double overlap $U_\alpha \cap U_\beta$, so that in particular $z_\beta^i = (g_{\alpha\beta})_j{}^i z_\alpha^j$. By taking the U_α sufficiently small we may assume that $\kappa^* I_Y$ is generated, on a neighborhood of U_α in $q^{-1}U_\alpha$, by functions $w_\alpha^1, \dots, w_\alpha^k$. By assumption, we know that

$$dw_\alpha^i|_{U_\alpha} = (h_\alpha)^i{}_j dz_\alpha^j|_{U_\alpha} \tag{3.1}$$

for some family of invertible matrices h_α on U_α. We may absorb h_α in the local frame e_α and assume without loss of generality that $h_\alpha = \mathrm{Id}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. The expression $w_\alpha q^*(\rho_\alpha e^\alpha)$ defines a map from a neighborhood of Y in $N^{1,0}Y$ to $N^{1,0}Y$, given by

$$v \mapsto \sum_i w_\alpha^i(v)\rho_\alpha(q(v))e_i^\alpha(q(v)).$$

It sends fibers to fibers and restricts to the identity on Y, and outside U_α it maps all fibers to zero. Note that the same expression without the ρ_α would only be defined over U_α. Define

$$\psi := \sum_\alpha w_\alpha q^*(\rho_\alpha e^\alpha) : N^{1,0}Y \to N^{1,0}Y, \tag{3.2}$$

which is well-defined in a neighborhood of Y in NY. Again, this map is fiber preserving and restricts to the identity on Y. We claim that its derivative along Y is the identity, so that it induces a diffeomorphism of neighborhoods of Y in NY. Indeed, to check this we look in a particular coordinate chart $q^{-1}U_\alpha \cong U_\alpha \times \mathbb{C}^k$. There, ψ is given by

$$\psi : (x, z_\alpha^i) \mapsto \big(x, \sum_\beta \rho_\beta(x)(g_{\beta\alpha}(x))_j{}^i(x) w_\beta^j(x, z_\alpha)\big). \tag{3.3}$$

Using (3.1) with $h_\alpha = \mathrm{Id}$ we see that

$$\begin{aligned} d\Big(\sum_\beta \rho_\beta(x)(g_{\beta\alpha}(x))_j{}^i w_\beta^j(x, z_\alpha)\Big)\Big|_Y &= \sum_\beta \rho_\beta(x)(g_{\beta\alpha}(x))_j{}^i dw_\beta^j(x, z_\alpha)|_Y \\ &= \sum_\beta \rho_\beta(x) dz_\alpha^i|_Y = dz_\alpha^i|_Y, \end{aligned}$$

which implies that $d\psi|_Y = \mathrm{Id}$; hence ψ is a local diffeomorphism around Y. From the local expression (3.3) it is clear that $\psi^* I_Y^{\mathrm{lin}} = \kappa^* I_Y$, so $\varphi = \psi^{-1}$ is the desired diffeomorphism. $\square$

[5] This is because $d\kappa|_Y : T(NY)|_Y \to TM|_Y$ induces the identity map on NY.

The main point in the above proof is the construction of the map ψ in (3.2) which relates any holomorphic ideal for Y on NY to its linearization. Since the functions $w_\alpha(t) := (1-t)z_\alpha + tw_\alpha$ all satisfy $dw_\alpha(t)|_Y = dz_\alpha|_Y$, the family ψ_t, defined by the same equation as ψ but with w_α replaced by $w_\alpha(t)$, defines an isotopy from the identity to $\psi = \psi_1$ on a small enough neighborhood of Y in NY. Consequently, if Y is compact we may use the isotopy extension theorem to obtain the following corollary.

Corollary 3.3. *If Y is compact and if I_Y and I'_Y are two holomorphic ideals for Y that induce the same complex structure on NY, then there is a diffeomorphism φ of M with $\varphi^* I_Y = I'_Y$.*

We will mainly be interested in holomorphic ideals for smooth submanifolds, but in order to state the definition of the blow-up we also consider singular submanifolds of codimension 1.

Definition 3.4. A *divisor* on M is an ideal sheaf $I_Y \subset C^\infty_M$ which locally can be generated by a single function and whose zero set Y is nowhere dense in M.

Remark 3.5. To define the blow-up it is enough to work only with divisors whose zero set Y is smooth. This is not necessary though, and working with general divisors gives a slightly more general universal property in Definition 3.6 below. The condition on Y being nowhere dense is necessary because we are working with functions that are not necessarily analytic.

Equipped with these definitions we can define the notion of blowing up in the same way as is usually done in algebraic geometry.

Definition 3.6. Let $Y \subset M$ be a closed submanifold and let I_Y be a holomorphic ideal for Y. The *blow-up* of I_Y in M is defined as a smooth manifold $\widetilde{M}$ together with a smooth *blow-down map* $p : \widetilde{M} \to M$ such that $I_{\widetilde{Y}} := p^* I_Y$ is a divisor and is universal in the following sense: For any smooth map $f : X \to M$ such that $f^* I_Y$ is a divisor, there is a unique $\widetilde{f} : X \to \widetilde{M}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}"] \arrow[dr, "f"'] & \widetilde{M} \arrow[d, "p"] \\ & M \end{tikzcd}$$

Theorem 3.7. *The blow-up $(\widetilde{M}, p)$ exists and is unique up to unique isomorphism. Moreover, $p : \widetilde{M}\backslash\widetilde{Y} \to M\backslash Y$ is a diffeomorphism, $I_{\widetilde{Y}}$ is smooth, and $p : \widetilde{Y} \to Y$ is isomorphic to*[6] $\mathbb{P}(NY) \to Y$.

Proof. By definition we can cover M by charts which are either disjoint from Y or are of the form $\mathbb{C}^l \times \mathbb{R}^m$ with coordinates $(z^1, \ldots, z^l, x^1, \ldots, x^m)$, where the z^i are as in Definition 3.1 (ii) and x^i are coordinates on Y. If we can construct the blow-up on each individual chart, then the universal property implies that all the local constructions can be glued into the desired manifold $\widetilde{M}$. On a chart not intersecting Y we do nothing as I_Y is already (trivially) a divisor there. On a chart

[6] Here $\mathbb{P}(NY)$ denotes the complex projectivization of NY with respect to the complex structure on NY induced by the holomorphic ideal.

$U = \mathbb{C}^l \times \mathbb{R}^m$ as above with $Y \cap U = \{0\} \times \mathbb{R}^m$ we define $\widetilde{U} := \widetilde{\mathbb{C}^l} \times \mathbb{R}^m$ and $p = (\pi, \mathrm{Id}) : \widetilde{U} \to U$ where $\pi : \widetilde{\mathbb{C}^l} \to \mathbb{C}^l$ is the blow-up of the origin. Recall that

$$\widetilde{\mathbb{C}^l} = \{(z, [x]) | z \in [x]\} \subset \mathbb{C}^l \times \mathbb{P}^{l-1}$$

has a cover by l charts on which π has the form

$$(v^1, \ldots, v^{i-1}, z^i, v^{i+1}, \ldots, v^l) \mapsto (z^i v^1, \ldots, z^i v^{i-1}, z^i, z^i v^{i+1}, \ldots, z^i v^l) \tag{3.4}$$

for $i \leq l$. Now suppose that $f : X \to U$ is a map such that $f^*(I_Y|_U)$ is a divisor with nowhere dense zero set D. The desired lift $\widetilde{f} : X \to U$ is already uniquely defined on $X \backslash D$ because π is an isomorphism over $\mathbb{C}^l \backslash \{0\}$, so we only have to show that $\widetilde{f}$ extends smoothly over D. To that end write $f = (f^1, \ldots, f^l, f'^1, \ldots, f'^m)$, so that $f^*(I_Y|_U) = \langle f^1, \ldots, f^l \rangle$. By definition of being a divisor there exists, on a neighborhood V of any $x_0 \in D$, a function g with $\langle g \rangle = \langle f^1, \ldots, f^l \rangle$. Therefore there exist $a^i, b_i \in C^\infty(V)$ with $f^i = a^i g$ and $g = \sum_i b_i f^i$, and so, since $g \neq 0$ on a dense set, we obtain $\sum_i a^i b_i = 1$. In particular there is an index i_0 such that, after possibly shrinking V, a^{i_0} is nowhere zero. The map $\widetilde{f} : V \backslash D \to \widetilde{U}$ maps into the chart (3.4) for $i = i_0$, where it is necessarily of the form

$$\widetilde{f} : x \mapsto \Big(\frac{f^1(x)}{f^{i_0}(x)}, \ldots, f^{i_0}(x), \ldots, \frac{f^l(x)}{f^{i_0}(x)}, f'^1(x), \ldots, f'^m(x) \Big).$$

Since $f^i / f^{i_0} = a^i / a^{i_0}$ we see that $\widetilde{f}$ indeed extends smoothly over the whole of V and therefore over the whole of D. So, from the above discussion the blow-up $p : \widetilde{M} \to M$ indeed exists and is unique. Its further mentioned properties are easily verified from the construction. □

Remark 3.8. It follows from the universal property that the blow-up construction is functorial; i.e., for any map $f : (M_1, I_{Y_1}) \to (M_2, I_{Y_2})$ with $f^* I_{Y_2} = I_{Y_1}$, there is a unique map $\widetilde{f} : \widetilde{M_1} \to \widetilde{M_2}$ making the obvious diagram commute. Note that $f^* I_{Y_2} = I_{Y_1}$ implies that the induced map $df : NY_1 \to NY_2$ is complex linear and injective. One case where this occurs is when $f : M_1 \to (M_2, I_{Y_2})$ is transverse to Y_2. Then $f^* I_{Y_2}$ is a holomorphic ideal for $Y_1 := f^{-1} Y_2$.

If $Y \subset M$ is a compact submanifold whose normal bundle NY is equipped with a complex structure, then we obtain a holomorphic ideal for Y on M which, by Corollary 3.3, is unique up to (non-canonical) diffeomorphism. In particular, we can blow up Y in M to obtain another smooth manifold $\widetilde{M}$, which is unique up to (non-canonical) diffeomorphism.

Definition 3.9. Let $Y \subset M$ be a compact submanifold with complex normal bundle. The *blow-up* $\widetilde{M}$ of Y in M is defined as the blow-up with respect to any holomorphic ideal inducing the given complex structure on NY.

Remark 3.10. We call $\widetilde{M}$ the blow-up of Y in M, even though $\widetilde{M}$ is only unique up to non-unique diffeomorphism.

3.1. **Generalized Poisson submanifolds.** In this section we will look at generalized complex submanifolds for which the geometry in normal directions is complex. The precise definition is as follows.

Definition 3.11. Let $\mathcal{J}$ be a generalized complex structure on M. A *generalized Poisson submanifold* is a submanifold $Y \subset M$ such that $\mathcal{J}(N^*Y) = N^*Y$.

This condition is equivalent to $\mathcal{J}(N^*Y) \cap (N^*Y)^\perp = N^*Y$; hence generalized Poisson submanifolds are automatically generalized complex[7] in the sense of Definition 2.7. Since $\mathcal{J}$ is orthogonal it also preserves $(N^*Y)^\perp$, and this gives an explicit description of the generalized complex structure induced on Y via $\mathbb{T}Y \cong (N^*Y)^\perp/N^*Y$. In this description it is clear that the inclusion map is generalized holomorphic, and so Y is a Poisson submanifold for $\pi_{\mathcal{J}}$, justifying the terminology. The key fact in the blow-up theory of generalized Poisson submanifolds is the following.

Proposition 3.12. *Let $Y \subset (M, \mathcal{J})$ be a closed generalized Poisson submanifold. There is a canonical holomorphic ideal I_Y whose associated complex structure on N^*Y is given by $\mathcal{J}$.*

Proof. Consider a generalized complex chart $U = (\mathbb{R}^{2n-2k}, \omega_{st}) \times (\mathbb{C}^k, \sigma)$ around a point in Y as provided by Theorem 2.6. Since Y is a union of symplectic leaves we have $Y \cap U = W \times Z$ where $W \subset \mathbb{R}^{2n-2k}$ is open and $Z \subset \mathbb{C}^k$ is a complex submanifold which is Poisson for σ. By choosing appropriate holomorphic coordinates z^i on $\mathbb{C}^k$ we may assume that $Z = \{z^1, \ldots, z^l = 0\}$, and a natural choice of holomorphic ideal for Y in U is then given by $\langle z^1, \ldots, z^l \rangle$. To patch these local ideals into a global one we need to show that on the overlap of two charts the corresponding ideals match. So suppose $(\mathbb{R}^{2n-2k}, \omega_i) \times (\mathbb{C}^k, \sigma_i)$, $i = 1, 2$, are two local models[8] and suppose that (φ, B) is a generalized complex isomorphism between them which maps Y to itself. Let (x, z) and (y, w) be coordinates on the two charts, where x, y and z, w denote the symplectic, respectively complex, directions, and such that I_Y is given by $\langle z^1, \ldots, z^l \rangle$, respectively $\langle w^1, \ldots, w^l \rangle$. By symmetry it suffices to show that $\varphi^* w^i \in \langle z^1, \ldots, z^l \rangle$ for all $i \leq l$. As is shown in [14, Ch. VI], this condition may be verified on the level of Taylor series, and since $\varphi^* w^i \in \langle z^1, \ldots, z^l, \bar{z}^1, \ldots, \bar{z}^l \rangle$ because $\varphi(Y) = Y$, we only need to verify that

$$\left. \frac{\partial^r w^i}{\partial \bar{z}^{i_1} \ldots \partial \bar{z}^{i_r}} \right|_Y = 0, \quad \forall r \geq 0,\ \forall i, i_1, \ldots, i_r \in \{1, \ldots, l\}. \tag{3.5}$$

Here we are abbreviating $w^i := \varphi^* w^i$. The case $r = 0$ reads $w^i|_Y = 0$, which is satisfied since $\varphi(Y) = Y$. To verify (3.5) we first write what it means for (φ, B) to be an isomorphism:

$$e^{i\omega_1} \wedge e^{\sigma_1}(dz^1 \ldots dz^k) = e^{f+B+i\omega_2} \wedge e^{\sigma_2}(dw^1 \ldots dw^k). \tag{3.6}$$

The factor e^f is there because we are taking representatives of the spinor line. At Y, using that Y is Poisson, (3.6) becomes

$$e^{i\omega_1} \wedge dz^1 \ldots dz^l \wedge e^{\sigma_1}(dz^{l+1} \ldots dz^k) = e^{f+B+i\omega_2} \wedge dw^1 \ldots dw^l \wedge e^{\sigma_2}(dw^{l+1} \ldots dw^k).$$

Now apply $dw^i \wedge \iota_{\partial_{\bar{z}^{i_1}}}$, with $i, i_1 \leq l$, to both sides. The left hand side vanishes, while the only survivor on the right is given by

$$\frac{\partial w^i}{\partial \bar{z}^{i_1}} e^{f+B+i\omega_2} \wedge dw^1 \ldots dw^l \wedge e^{\sigma_2}(dw^{l+1} \ldots dw^k),$$

[7] Note that $N^*Y_{\mathbb{C}} \cap i^*L = N^*Y_{\mathbb{C}}$ so $\mathfrak{Bi}(L)$ is automatically smooth, where $i : Y \hookrightarrow M$ denotes the inclusion.

[8] Strictly speaking we should look at open neighborhoods of 0, but for the sake of notation we suppress this. Also note that we can assume that the "k" in both charts is the same, as the type can only jump in even steps and $(\mathbb{R}^{4s}, \omega_{st})$ is isomorphic to $(\mathbb{C}^{2s}, \sigma_0)$ for σ_0 an invertible holomorphic Poisson structure.

so (3.5) holds for $r \leq 1$. This implies in particular that the forms $dz^1 \dots dz^l$ and $dw^1 \dots dw^l$ are proportional along Y, where again we think of w^i as a function of (x, z).

Suppose inductively that for some $m \geq 1$ equation (3.5) is satisfied for all $r \leq m$. Apply $dw^i \wedge \mathcal{L}_{\partial_{\bar{z}^{i_1}}} \dots \mathcal{L}_{\partial_{\bar{z}^{i_m}}}$, for any $i, i_1, \dots, i_m \leq l$, to both sides of (3.6) and evaluate the resulting expression at Y. The left hand side will vanish again because ω_1 is independent of z and σ_1 is holomorphic. Using multi-index notation, the Leibniz rule gives

(3.7)
$$0 = dw^i \wedge \sum_{\substack{I \sqcup J \sqcup K \sqcup L \\ = \{i_1, \dots, i_m\}}} \mathcal{L}_{\partial_{\bar{z}^I}}(e^{f+B+i\omega_2}) \mathcal{L}_{\partial_{\bar{z}^J}}(e^{\sigma_2}) \mathcal{L}_{\partial_{\bar{z}^K}}(dw^1 \dots dw^l) \mathcal{L}_{\partial_{\bar{z}^L}}(dw^{l+1} \dots dw^k).$$

Claim. We have $\mathcal{L}_{\partial_{\bar{z}^J}} \sigma_2(dw^j)|_Y = 0$ for all $J \subset \{i_1, \dots, i_m\}$ and $j \leq l$.

Let us accept this claim for the moment and continue with the proof. We compute

(3.8)
$$\mathcal{L}_{\partial_{\bar{z}^K}} dw^j = \sum_{1 \leq a \leq k} \frac{\partial^{|K|+1} w^j}{\partial z^a \partial \bar{z}^K} dz^a + \sum_{1 \leq a \leq k} \frac{\partial^{|K|+1} w^j}{\partial \bar{z}^a \partial \bar{z}^K} d\bar{z}^a + \sum_{1 \leq b \leq 2n-2k} \frac{\partial^{|K|+1} w^j}{\partial x^b \partial \bar{z}^K} dx^b.$$

If $j \leq l$, the function $\partial^{|K|} w^j / \partial \bar{z}^{|K|}$ vanishes along Y by the induction hypothesis. Hence, at Y the first and second terms above with $a > l$ together with the entire third term vanish, because we differentiate in directions tangent to Y. If in addition $|K| < m$, the second term vanishes by the induction hypothesis. It follows that for $K \subsetneq \{i_1, \dots, i_m\}$, $\mathcal{L}_{\partial_{\bar{z}^K}}(dw^1 \dots dw^l)|_Y$ is proportional to $(dw^1 \dots dw^l)|_Y$. Using the Claim, these terms all disappear from (3.7) because we wedge everything with dw^i. It is then readily verified that (3.7) reduces to

$$0 = e^{f+B+i\omega_1} e^{\sigma_2} \sum_{1 \leq i_{m+1} \leq l} \frac{\partial^{m+1} w^i}{\partial \bar{z}^{i_1} \dots \partial \bar{z}^{i_{m+1}}} d\bar{z}^{i_{m+1}} dw^1 \dots dw^k$$

at Y. So (3.5) holds for $r = m + 1$ as well and therefore for all r by induction. $\square$

Proof of Claim. If we write $\sigma_2 = \sigma_2^{ab} \partial_{w^a} \partial_{w^b}$, the Poisson condition implies that σ_2^{ab} vanishes at Y for $a \leq l$ or $b \leq l$. A repeated Lie derivative on σ_2 will be a sum of terms of the form

$$\frac{\partial^r \sigma_2^{ab}}{\partial \bar{z}^{i_1} \dots \partial \bar{z}^{i_r}} (\mathcal{L}_{\partial_{\bar{z}^{j_1}}} \dots \mathcal{L}_{\partial_{\bar{z}^{j_s}}} \partial_{w^a})(\mathcal{L}_{\partial_{\bar{z}^{k_1}}} \dots \mathcal{L}_{\partial_{\bar{z}^{k_t}}} \partial_{w^b}). \tag{3.9}$$

Using the chain rule and the fact that σ_2 is holomorphic we can rewrite the first term in terms of w-derivatives. By the induction hypothesis there are no derivatives in the w^i-directions for $i \leq l$, because these come together with a term of the form $\partial w^i / \partial \bar{z}^{i_j}$ or a further derivative thereof. Moreover, if either $a \leq l$ or $b \leq l$ there are also no w^i-derivatives for $i > l$ because these are tangent to Y along which σ^{ab} is constantly equal to zero. Hence (3.9) will only be non-zero at Y for $a, b > l$, and so to prove the Claim it suffices to show that $(\mathcal{L}_{\partial_{\bar{z}^{j_1}}} \dots \mathcal{L}_{\partial_{\bar{z}^{j_s}}} \partial_{w^a})(dw^j)|_Y = 0$ for $a > l, j \leq l$. Abbreviating $J = \{j_1, \dots, j_s\}$ we have

$$0 = \mathcal{L}_{\bar{z}^J}(dw^j(\partial_{w^a})) = \sum_{J_1 \sqcup J_2 = J} (\mathcal{L}_{\bar{z}^{J_1}} dw^j)(\mathcal{L}_{\bar{z}^{J_2}} \partial_{w^a}).$$

From equation (3.8) and the comments below it we see that $\mathcal{L}_{\bar{z}^{J_1}} dw^j$ is a linear combination of $dw^{j'}$ with $j' \leq l$. The result then follows by induction over s. $\square$

Having a canonical holomorphic ideal for Y we obtain a canonical blow-up $\widetilde{M}$. We now investigate whether $\widetilde{M}$ carries a generalized complex structure for which the blow-down map p is holomorphic. Clearly this structure exists and is unique on $\widetilde{M}\backslash\widetilde{Y}$, and we only need to verify whether it extends over $\widetilde{Y}$. From the definition of the ideal I_Y and the blow-up construction, p is locally given by

$$\mathbb{R}^{2(n-k)} \times \mathrm{Bl}_Z \mathbb{C}^k \to \mathbb{R}^{2(n-k)} \times \mathbb{C}^k,$$

where $\mathrm{Bl}_Z \mathbb{C}^k$ is the complex blow-up of $Z \subset \mathbb{C}^k$. The target is equipped with the generalized complex structure determined by the standard symplectic form on $\mathbb{R}^{2(n-k)}$ and a holomorphic Poisson structure σ on $\mathbb{C}^k$. Clearly this structure lifts if and only if σ lifts. So we are led to the following question: When does a holomorphic Poisson structure lift to a blow-up? This was addressed by Polishchuk in [16], and for completeness we review the results here in more differential geometric language. Recall that $Z \subset (X, \sigma)$ is a holomorphic Poisson submanifold if and only if its holomorphic ideal sheaf I_Z of functions vanishing on Z is a Poisson ideal. In that case $N^{*1,0}Z$ inherits a fiberwise Lie algebra structure, given by the Poisson bracket under the natural isomorphism $N^{*1,0}Z \cong I_Z/I_Z^2$. To state the blow-up conditions on Z we need the following terminology.

Definition 3.13. A Lie algebra $\mathfrak{g}$ is *degenerate* if the map $\Lambda^3\mathfrak{g} \to Sym^2(\mathfrak{g})$ given by

$$x \wedge y \wedge z \mapsto [x,y]z + [y,z]x + [z,x]y$$

vanishes.

Remark 3.14. Geometrically, this is equivalent to the condition that the Lie bracket of any two elements lies in the plane spanned by them. In particular, it depends on the base field over which $\mathfrak{g}$ is defined. For instance, the complexification of a degenerate Lie algebra over $\mathbb{R}$ is degenerate over $\mathbb{C}$, but a degenerate Lie algebra over $\mathbb{C}$ need not be degenerate over $\mathbb{R}$ when we restrict scalars. It is shown in [16] that degeneracy is equivalent to being either Abelian or isomorphic to the algebra generated by $e_1, \ldots, e_{n-1}, f$, with relations $[e_i, e_j] = 0$ and $[f, e_i] = e_i$. Note that 2-dimensional Lie algebras are always degenerate.

If Z is Poisson we call $N^{*1,0}Z$ degenerate if its fiberwise Lie algebra structure is degenerate over $\mathbb{C}$. This is equivalent to the condition

$$\{f,g\}h + \{g,h\}f + \{h,f\}g \in I_Z^3 \quad \forall f,g,h \in I_Z. \tag{3.10}$$

Now let $p : \widetilde{X} \to X$ denote the complex blow-up along a complex submanifold Z, and let $\widetilde{Z}$ be the exceptional divisor. We say that σ can be lifted if there exists a holomorphic Poisson structure $\widetilde{\sigma}$ on $\widetilde{X}$ for which p is a Poisson map. Note that a lift is necessarily unique, because p is an isomorphism almost everywhere. The proof of the following proposition is due to Polishchuk [16]; for the convenience of the reader we reproduce it here.

Proposition 3.15 ([16])**.** *There exists a lift $\widetilde{\sigma}$ on $\widetilde{X}$ if and only if Z is a Poisson submanifold and $N^{*1,0}Z$ is degenerate. The exceptional divisor $\widetilde{Z}$ is a Poisson submanifold if and only if $N^{*1,0}Z$ is Abelian.*

Proof. Let $z^1, \dots, z^k$ be local coordinates on X with $Z = \{z^1, \dots, z^l = 0\}$ for some $l \leq k$. This is covered by l charts on $\widetilde{X}$ on which the projection has the form (cf. (3.4))

$$p : (v^1, \dots, z^a, \dots, v^l, z^{l+1}, \dots, z^k) \mapsto (z^a v^1, \dots, z^a, \dots, z^a v^l, z^{l+1}, \dots, z^k) \tag{3.11}$$

for $a \leq l$. Then p is an isomorphism on the open dense set $\{z^a \neq 0\}$, where we have $v^j = z^j/z^a$. We have to verify when the brackets extend smoothly over the exceptional divisor $\{z^a = 0\}$. There are two types of brackets that cause trouble. Firstly,

$$\{z^i, v^j\} = \{z^i, \frac{z^j}{z^a}\} = \frac{1}{z^a}\{z^i, z^j\} - \frac{z^j}{(z^a)^2}\{z^i, z^a\}, \tag{3.12}$$

for $i = a$ or $i > l$, and $j \leq l$ with $j \neq a$. Secondly,

$$\{v^i, v^j\} = \{\frac{z^i}{z^a}, \frac{z^j}{z^a}\} = \frac{1}{(z^a)^3}\big(z^a\{z^i, z^j\} + z^i\{z^j, z^a\} + z^j\{z^a, z^i\}\big), \tag{3.13}$$

for $1 \leq i, j \leq l$, $i \neq a \neq j$. Now (3.12) extends smoothly over $z^a = 0$ for all a if and only if I_Z is Poisson, while (3.13) extends over $z^a = 0$ for all a if and only if I_Z is degenerate in the sense of (3.10). Finally, $I_{\widetilde{Z}}$ is generated by z^a, and this is a Poisson ideal if and only if the right hand side of (3.12) for $i = a$ is divisible by z^a, which is equivalent to $\{I_Z, I_Z\} \subset I_Z^2$. □

If $Y \subset (M, \mathcal{J})$ is a generalized Poisson submanifold, then Y is in particular a Poisson submanifold for $\pi_{\mathcal{J}}$, and so N^*Y inherits a fiberwise Lie algebra structure in the same manner as discussed above in the holomorphic Poisson context. As e.g. shown in the proof below, this Lie bracket is complex linear with respect to the complex structure on N^*Y induced by $\mathcal{J}$. We call N^*Y degenerate if the Lie algebra structure is degenerate over $\mathbb{C}$.

Theorem 3.16. *Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson submanifold and let $p : \widetilde{M} \to M$ denote the blow-up with respect to the canonical holomorphic ideal I_Y. Then $\widetilde{M}$ has a generalized complex structure for which p is holomorphic if and only if N^*Y is degenerate.*

Proof. Pick a local chart where $Y = W \times Z \subset (\mathbb{R}^{2(n-k)}, \omega_0) \times (\mathbb{C}^k, \sigma)$ with W open and Z a holomorphic Poisson submanifold (cf. the proof of Proposition 3.12). As explained in the discussion above, the generalized complex structure lifts to the blow-up if and only if σ lifts to the blow-up of Z in $\mathbb{C}^k$, which we now know to be equivalent to $N^{*1,0}Z$ being degenerate. Denote by N^*Z the normal bundle of Z considered as a real submanifold, which carries a complex structure because Z is a complex submanifold. If $Q = \mathrm{Re}(\sigma)$ we have

$$\begin{aligned}
[\alpha, \beta]_Q = dQ(\alpha, \beta) &= d(\frac{1}{2}\sigma(\alpha^{1,0}, \beta^{1,0}) + \frac{1}{2}\bar{\sigma}(\alpha^{0,1}, \beta^{0,1})) \\
&= \frac{1}{2}[\alpha^{1,0}, \beta^{1,0}]_\sigma + \frac{1}{2}[\alpha^{0,1}, \beta^{0,1}]_{\bar{\sigma}},
\end{aligned}$$

for $\alpha, \beta \in N^*Z$. Consequently the complex isomorphism $N^*Z \to N^{*1,0}Z$ given by $\alpha \mapsto \alpha^{1,0}$ carries $[,]_Q$ over to $\frac{1}{2}[,]_\sigma$. In particular, $N^{*1,0}Z$ is degenerate if and only if $(N^*Z, [,]_Q)$ is degenerate as a complex Lie algebra. Now in the local chart $N^*Y = N^*Z$ and $\pi_{\mathcal{J}} = -\omega_0^{-1} \oplus 4IQ$. Hence $[,]_Q$ and $[,]_{\pi_{\mathcal{J}}}$ agree up to a complex multiple, and so one is degenerate over $\mathbb{C}$ if and only if the other one is. □

Since degeneracy is automatic in codimension 2, we obtain Corollary 3.17.

Corollary 3.17. *If Y is a generalized Poisson submanifold of complex codimension 2, then it has a generalized complex blow-up.*

Example 3.18. Let $(M, \mathcal{J})$ be a generalized complex manifold. In [3] it is shown that the complex locus, i.e., the points of type 0, carries canonically the structure of a complex analytic space. Any complex submanifold of the complex locus is then a generalized Poisson submanifold and can be blown up as soon as its conormal bundle is degenerate. The easiest applications are in complex codimension 2 where degeneracy is automatic. For example, any point in the complex locus on a generalized complex 4-manifold can be blown up. This generalizes the corresponding result from [7], where it was assumed that the point lies in the smooth part of the complex locus.

Example 3.19. We mention here an explicit example of a generalized Poisson submanifold of positive dimension that can be blown up. In [11] it is shown that any even-dimensional compact Lie group admits a *generalized Kähler structure*, i.e., a commuting pair of generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ for which $(u, v) \mapsto \langle \mathcal{J}_1 u, \mathcal{J}_2 v\rangle$ is positive definite on $\mathbb{T}M$. As shown in [17], one can construct such a generalized Kähler structure out of left- and right-invariant complex structures on the group in such a way that a maximal torus is a generalized Poisson submanifold for $\mathcal{J}_1$. Applying this to the Lie group $S^3 \times S^3$ yields a generalized Poisson submanifold $S^1 \times S^1 \subset S^3 \times S^3$, which can be blown-up because it has complex codimension 2.

The following result shows how blow-ups can be used to desingularize the type change locus of a generalized complex 4-manifold.

Theorem 3.20. *Let $(M, \mathcal{J})$ be a 4-dimensional generalized complex manifold which is of symplectic type on an open dense set and with complex locus $Z \subset M$. Then there exists a generalized complex manifold $(\widetilde{M}, \widetilde{\mathcal{J}})$ whose complex locus $\widetilde{Z}$ has at most normal crossing singularities, together with a generalized holomorphic map $p : \widetilde{M} \to M$ that induces an isomorphism between $\widetilde{M} \backslash \widetilde{Z}$ and $M \backslash Z$.*

Proof. Since Z is locally described by the vanishing of a holomorphic Poisson tensor in two complex dimensions, it locally looks like a complex curve. By Example 3.18 we can blow up any point on Z to obtain another generalized complex manifold. In general, if $C \subset X$ is a complex curve on a complex smooth surface X, one can perform a locally finite number of blow-ups on X so that the underlying analytic set of the total transform of the curve C has only ordinary double points. A proof of this fact can be found e.g. in [4]. In particular, the total transform[9] itself will be a normal crossing divisor with possible multiplicities (so in local coordinates z_1, z_2 it will be given by $z_1^a z_2^b = 0$ for some $a, b \in \mathbb{Z}_{>0}$). Now we do not have a global complex structure available, but this desingularization procedure is purely local, so we conclude that after a finite number of blow-ups we get a generalized complex manifold whose complex locus, as a complex analytic space, has at most normal crossings as singularities. □

[9]The *total transform* of a subset C under a blow-up equals $p^{-1}(C)$ where p is the blow-down map, while the *proper transform* equals $\overline{p^{-1}(C) \backslash E}$.

3.2. **Generalized Poisson transversals.** We now focus on submanifolds where the geometry normal to the submanifold is symplectic.

Definition 3.21. Let $(M, \mathcal{J})$ be a generalized complex manifold. A *generalized Poisson transversal* is a submanifold $Y \subset M$ satisfying

$$\mathcal{J}(N^*Y) \cap (N^*Y)^{\perp} = 0. \tag{3.14}$$

Remark 3.22. The above condition automatically implies $N^*Y \cap \mathcal{J}(N^*Y) = 0$; hence generalized Poisson transversals are in particular generalized complex submanifolds.[10] Note that (3.14) is equivalent to $\pi_{\mathcal{J}}(N^*Y) + TY = TM|_Y$; i.e., Y is a Poisson transversal for the underlying Poisson structure $\pi_{\mathcal{J}}$. Geometrically, Y intersects the symplectic leaves of $\pi_{\mathcal{J}}$ transversally and symplectically. Note that if $\mathcal{J}$ is complex, then Y has to be an open subset, while if $\mathcal{J}$ is symplectic, then Y has to be a symplectic submanifold.

3.2.1. *A normal form theorem.* Let $Y \hookrightarrow (M, J)$ be a generalized Poisson transversal. To blow up Y we need a description of a neighborhood of Y in M. Since Y is a generalized complex submanifold it has its own generalized complex structure $\mathcal{J}_Y$. Moreover, the splitting $TM|_Y = TY \oplus NY$, with $NY := \pi_{\mathcal{J}}(N^*Y)$, induces a decomposition $(\pi_{\mathcal{J}})|_Y = \pi_{\mathcal{J}_Y} + \omega_Y$, where $\pi_{\mathcal{J}_Y}$ equals the Poisson structure on Y induced by $\mathcal{J}_Y$ and $\omega_Y \in \Gamma(\Lambda^2 NY)$ is non-degenerate. The suggestive notation for the latter indicates that we will consider ω_Y as a symplectic structure on the bundle N^*Y. In what follows we will implicitly identify Y with the zero section in N^*Y and use the decomposition

$$T(N^*Y)|_Y = N^*Y \oplus TY. \tag{3.15}$$

We will first show that associated to $(\mathcal{J}_Y, \omega_Y)$, there is a family of generalized complex structures on a neighborhood of Y in N^*Y. For that we need the following lemma.

Lemma 3.23. *There exists a closed 2-form σ on the total space of N^*Y, which along Y is given by $\omega_Y \oplus 0$.*

Proof. Choose a Hermitian structure (g, I) on N^*Y compatible with ω_Y. Let e_j be a local unitary frame with dual frame e^j, such that $\omega_Y = \frac{i}{2}\sum_j e^j \wedge \bar{e}^j$. We obtain local coordinates (x, z) on N^*Y by identifying (x, z) with the point $\sum_j z^j e_j(x)$. Note that the z-coordinates are complex. If ρ_α is a partition of unity and e_j^α are local frames as above, define

$$\lambda := \sum_{\alpha, j} p^*(\rho_\alpha) \frac{i}{2} z^{\alpha j} d\bar{z}^{\alpha j}. \tag{3.16}$$

Then $\sigma := d\lambda$ restricts to ω_Y on Y, and its restriction to each fiber of N^*Y is the translation invariant extension of ω_Y. Note that this particular choice of λ is also $U(1)$-invariant. $\square$

Theorem 3.24. *Associated to the data $(\mathcal{J}_Y, \omega_Y)$ there is a family of mutually isotopic generalized complex structures on a neighborhood of Y in N^*Y.*

[10] As $i^*L \cap N^*Y_{\mathbb{C}} = 0$, $\mathfrak{B}i(L)$ is smooth. Here $i : Y \hookrightarrow M$ denotes the inclusion.

Proof. If σ is a closed extension of ω_Y as in Lemma 3.23 we define a Dirac structure L_σ on the total space of N^*Y by

$$L_\sigma := e_*^{i\sigma}(\mathfrak{B}p(L_Y)), \tag{3.17}$$

where $p : N^*Y \to Y$ is the projection. It is integrable with respect to the 3-form $\widetilde{H} := p^*H_Y$ where $H_Y := i^*H$, and along the zero section we have

$$L_\sigma|_Y = \{X + \xi + e - i\omega_Y(e) | X + \xi \in L_Y, e \in N^*Y\},$$

where we used the decomposition (3.15). In particular $L_\sigma \cap \overline{L_\sigma} = 0$ at Y, hence also in a neighborhood of Y in N^*Y. We will denote the resulting generalized complex structure by $\mathcal{J}_\sigma$. The family of the theorem is by definition the set of $\mathcal{J}_\sigma$, where σ ranges over the closed extensions of ω_Y. If σ and σ' are two closed extensions, we obtain a family of closed 2-forms $\sigma_t := (1-t)\sigma + t\sigma'$ all extending ω_Y. We can now apply Lemma 3.25 below, which is a generalization of the well-known Moser trick, to conclude that $\mathcal{J}_\sigma$ and $\mathcal{J}_{\sigma'}$ are isotopic. □

Lemma 3.25. *Let σ_t be a family of closed 2-forms extending ω_Y and let L_{σ_t} be the corresponding family of Dirac structures. Then there is a neighborhood U of Y in N^*Y and a family of embeddings*[11] *$\Phi_t = (\varphi_t, B_t) : U \to N^*Y$, satisfying $\Phi_0 = (\mathrm{Id}, 0)$ and $\mathfrak{F}\Phi_t(L_{\sigma_0}) = L_{\sigma_t}$. Furthermore, $\Phi_t = (\varphi_t, B_t)$ fixes Y up to first order, in the sense that $\varphi_t|_Y = \mathrm{Id}$, $d\varphi_t|_Y = \mathrm{Id}$, and $B_t|_Y = 0$ for all t.*

Proof. Since $\sigma_t - \sigma_0$ vanishes on Y, Lemma 3.26 below provides a family $\eta_t \in \Omega^1(N^*Y)$ with $\sigma_t - \sigma_0 = d\eta_t$ and such that the 1-jet of η_t vanishes[12] along Y. By definition,

$$L_t := L_{\sigma_t} = e_*^{i\sigma_t}(\mathfrak{B}p(L_Y)) = e_*^{id\eta_t}(L_{\sigma_0}).$$

Since $d\eta_t$ vanishes at Y, L_t defines a family of generalized complex structures $\mathcal{J}_t$ in a neighborhood U' of Y, integrable with respect to the 3-form $\widetilde{H}$. Consider the time-dependent generalized vector field $\mathcal{J}_t\dot{\eta}_t =: X_t + \xi_t$ on U' and let $\psi_{t,s}$ be its flow, given by

$$\psi_{t,s} = (\varphi_{t,s})_* \circ e_*^{-\int_s^t \varphi_{r,s}^*(d\xi_r + \iota_{X_r}\widetilde{H})dr}, \tag{3.18}$$

where $\varphi_{t,s}$ is the flow of the time-dependent vector field X_t. Since the 1-jet of η_t vanishes on Y, we can find a smaller neighborhood $U \subset U'$ with the property that $\varphi_{t,s} : U \to U'$ is a well-defined embedding, fixing Y to first order. We claim that

$$L_t = \psi_{t,0}L_0. \tag{3.19}$$

From the formula for L_t this amounts to showing that $e_*^{-id\eta_t}\psi_{t,0}L_0 = L_0$. We have

$$\begin{aligned} \frac{d}{dt}e_*^{-id\eta_t}\psi_{t,0}(u) &= -i[\![\dot{\eta}_t, e_*^{-id\eta_t}\psi_{t,0}(u)]\!] - e_*^{-id\eta_t}[\![\mathcal{J}_t\dot{\eta}_t, \psi_{t,0}(u)]\!] \\ &= [\![-i\dot{\eta}_t - \mathcal{J}_0\dot{\eta}_t, e_*^{-id\eta_t}\psi_{t,0}(u)]\!]. \end{aligned} \tag{3.20}$$

This shows that $e_*^{-id\eta_t}\psi_{t,0}$ integrates the adjoint action of $-i\dot{\eta}_t - \mathcal{J}_0\dot{\eta}_t \in \Gamma(L_0)$. Since $\Gamma(L_0)$ is involutive, (3.19) indeed holds. The desired family is then given by

$$\Phi_t = (\varphi_t, B_t) := (\varphi_{t,0}, \int_0^t \varphi_{r,0}^*(d\xi_r + \iota_{X_r}H)dr).$$ □

[11] We call a generalized map (φ, B) an embedding if the underlying smooth map φ is.

[12] That is, if we write $\eta_t = \sum_i \eta_{t,i}(x)dx^i$ in a local coordinate system, then the functions $\eta_{t,i}$ and $\partial_j\eta_{t,i}$ all vanish on Y for each t.

Lemma 3.26. *Let $\alpha_t \in \Omega^k_{\mathrm{cl}}(E)$ be a family of closed forms on the total space of a vector bundle E over Y, which all vanish at Y. Then there exists a family $\eta_t \in \Omega^{k-1}(E)$ with $d\eta_t = \alpha_t$, such that for each t the 1-jet of the form η_t vanishes along Y.*

Proof. Let V denote the Euler vector field on E, i.e., $V_\xi = \xi$ for $\xi \in E$. Its flow is given by $\varphi_s(\xi) = e^s\xi$, and we have

$$\alpha_t = \lim_{s\to-\infty}\left(\varphi_0^*\alpha_t - \varphi_s^*\alpha_t\right) = \int_{-\infty}^0 \frac{d}{ds}\varphi_s^*\alpha_t ds = d\left(\iota_V \int_{-\infty}^0 \varphi_s^*\alpha_t ds\right) =: d\eta_t.$$

Another formula for η_t is given by $\eta_t = \iota_V \int_0^1 \frac{1}{s}L_s^*\alpha_t ds$, where L_s denotes left-multiplication by s on E. The forms η_t then satisfy all the properties of the lemma. □

Theorem 3.24 shows that for any symplectic vector bundle over a generalized complex manifold, a neighborhood of the zero section carries a generalized complex structure for which the zero section is a generalized Poisson transversal. Our next aim is to show that all compact generalized Poisson transversals locally arise from this construction. To establish that, we will construct an embedding of a neighborhood of Y in N^*Y into M that pulls back $\mathcal{J}$ to one of the structures of Theorem 3.24. This embedding will only depend on the choice of a connection on TM, and all such embeddings will turn out to be isotopic to each other.

Let $p : T^*M \to M$ be the cotangent bundle and choose a connection ∇ on TM, whose dual connection on T^*M we also denote by ∇. Using the Poisson structure $\pi_{\mathcal{J}}$ we obtain a vector field V on the total space of the bundle T^*M, whose value at $\xi \in T^*M$ is given by $V_\xi := \pi_{\mathcal{J}}(\xi)^h_\xi$, the horizontal lift of the vector $\pi_{\mathcal{J}}(\xi)$ at the point $\xi \in T^*M$ with respect to the connection ∇. We denote by $\varphi_t : T^*M \to T^*M$ the flow of V and define

$$\exp := p \circ \varphi_1|_{N^*Y} : N^*Y \to M.$$

Lemma 3.27. *The map* exp *restricts to a diffeomorphism between a neighborhood of Y in N^*Y and a neighborhood of Y in M. If ∇' is a different connection, then* $\exp'$ *is isotopic to* exp *via maps which all agree along Y up to first order.*[13]

Proof. By definition of V we have $L_s^*V = sV$ for $s \in \mathbb{R}$, where L_s denotes multiplication by s on the fibers of T^*M. It follows that[14] $\varphi_t(L_s\xi) = L_s(\varphi_{st}(\xi))$ for $\xi \in T^*M$. Hence,

$$d_y\varphi_t(\xi) = \left.\frac{d}{ds}\right|_{s=0} \varphi_t(L_s\xi) = \xi + t\pi_{\mathcal{J}}(\xi),$$

for $y \in Y \subset N^*Y$. Since V vanishes at Y we have $\exp|_Y = \mathrm{Id}$, and so

$$d_y\varphi_t(\xi, v) = (\xi, v + t\pi_{\mathcal{J}}(\xi)) \tag{3.21}$$

in terms of the decomposition (3.15). Composing with p gives $d_y \exp(\xi, v) = v + \pi_{\mathcal{J}}(\xi)$; hence by transversality of Y we see that exp is a local diffeomorphism. Since $\exp|_Y = \mathrm{Id}$ and Y is properly embedded, exp gives a diffeomorphism between a neighborhood of Y in N^*Y and a neighborhood of Y in M. As the space of

[13] That is, they all restrict to the identity on Y, and their derivatives $T_y(N^*Y) \to T_yM$ all coincide for $y \in Y$.

[14] This equality is similar to the more familiar equality $\gamma_{sX}(t) = \gamma_X(st)$ for geodesics.

connections is affine, we can connect any other connection ∇' to ∇ by a path of connections $\{\nabla^t\}_{t\in[0,1]}$ (e.g. the linear path), whose exponentials $\exp^t$ give the desired isotopy. Since (3.21) is independent of ∇^t, the maps $\exp^t$ all agree up to first order along Y. □

We will now explicitly construct one of the generalized complex structures from Theorem 3.24 which agrees with $\exp^*(\mathcal{J})$ (up to B-field transform). Denote by ω_{can} the canonical symplectic form on T^*M, and recall that for $y \in M$ and $X, Y \in T_yM$, $\alpha, \beta \in T_y^*M$,

$$(\omega_{\mathrm{can}})_y(\alpha + X, \beta + Y) = \alpha(Y) - \beta(X) \tag{3.22}$$

in terms of $T(T^*M)|_M = T^*M \oplus TM$.

Lemma 3.28. *Define*

$$\widetilde{\sigma}_t := -\int_0^t (\varphi_s)^*\omega_{\mathrm{can}} ds \ \in \ \Omega^2_{\mathrm{cl}}(T^*M), \tag{3.23}$$

where φ_s is the flow of the vector field V. Then $\sigma := i^*\widetilde{\sigma}_1$ is a closed extension of ω_Y, where $i : N^*Y \hookrightarrow T^*M$ denotes the inclusion.

Proof. Recall that $T(N^*Y)|_Y = N^*Y \oplus TY$, so that elements of $T_y(N^*Y)$ can be written as $\alpha + X$, for $\alpha \in N_y^*Y$ and $X \in T_yY$. Using (3.21), (3.22), and the definition of $\widetilde{\sigma}_t$, we obtain

$$\begin{aligned}\sigma_y(\alpha + X, \beta + Y) &= -\int_0^1 (\omega_{\mathrm{can}})_y\big(\alpha + (X + s\pi_{\mathcal{J}}(\alpha)), \beta + (Y + s\pi_{\mathcal{J}}(\beta))\big)ds \\ &= -\int_0^1 2s\alpha(\pi_{\mathcal{J}}(\beta))ds = \omega_Y(\alpha, \beta)\end{aligned}$$

for all $\alpha, \beta \in N_y^*Y$ and $X, Y \in T_yY$. This shows that σ is indeed an extension of ω_Y. □

We are now ready to state the normal form theorem.

Theorem 3.29. *Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson transversal.*[15] *Then a neighborhood of Y in $(M, \mathcal{J})$ is isomorphic to a neighborhood of Y in $(N^*Y, \widetilde{J})$, where $\widetilde{J}$ is one of the generalized complex structures of Theorem* 3.24.

Remark 3.30. In particular, this result tells us that on a neighborhood of Y, $\mathcal{J}$ is completely determined by the induced generalized complex structure $\mathcal{J}_Y$ on Y and the induced symplectic structure on the vector bundle N^*Y.

Proof. The vector field V defined above is part of the generalized vector field $\mathcal{V}$ on T^*M defined by $\mathcal{V}_\xi := (\mathcal{J}\xi)^h_\xi$, the horizontal lift of $\mathcal{J}\xi \in \mathbb{T}_{p(\xi)}M$ to $\xi \in T^*M$ with respect to the connection ∇. If ψ_t denotes the flow of $\mathcal{V}$, then a computation similar to (3.20) shows that

$$\frac{d}{dt}\psi_t e_*^{i\widetilde{\sigma}_t}(u) = [\![-i\lambda_{\mathrm{can}} - \mathcal{V}, \psi_t e_*^{i\widetilde{\sigma}_t}(u)]\!].$$

Since $(-i\lambda_{\mathrm{can}} - \mathcal{V})_\xi = (-i\xi - \mathcal{J}\xi)^h_\xi \in \mathfrak{B}p(L)$ and $\mathfrak{B}p(L)$ is involutive, $\psi_t e_*^{i\widetilde{\sigma}_t}$ preserves $\mathfrak{B}p(L)$ and so

$$e_*^{i\widetilde{\sigma}_t}\mathfrak{B}p(L) = \psi_{-t}\mathfrak{B}p(L), \tag{3.24}$$

[15] Note that Y is not assumed to be compact, but we do require it to be closed in M.

as Dirac structures on T^*M. Here is an overview of all the maps involved:

$$\begin{tikzcd} N^*Y \arrow[r, "i"] \arrow[d, "p"] & T^*M \arrow[r, "\varphi_t"] \arrow[d, "p"] & T^*M \arrow[dl, "p"] \\ Y \arrow[r] & M & \end{tikzcd}$$

The left square is commutative, but the right triangle is not. Now if we apply $\mathfrak{B}i$ to (3.24) at $t = 1$, the left hand side becomes $e_*^{i\sigma}\mathfrak{B}i\mathfrak{B}p(L) = e_*^{i\sigma}\mathfrak{B}p(L_Y)$ where $\sigma = i^*\widetilde{\sigma}_1$. This is precisely one of the structures from Theorem 3.24. If we write $\psi_t = (\varphi_t)_* e_*^{-B_t}$ (see (2.7)), the right hand side becomes

$$\mathfrak{B}i(\psi_{-1}\mathfrak{B}p(L)) = \mathfrak{B}i\mathfrak{B}\Phi_1\mathfrak{B}p(L) = \mathfrak{B}(p \circ \Phi_1 \circ i)(L),$$

where $\Phi_t := (\varphi_t, B_t)$. Now $p \circ \Phi_1 \circ i = (\exp, i^*B_1)$, so if we define $B := i^*B_1$, then $(\exp, B)$ is indeed holomorphic with respect to $\mathcal{J}$ and $\widetilde{\mathcal{J}}$. ☐

3.2.2. *Blowing up.* In this section we will use the normal form theorem for Y to construct the symplectic version of the blow-up. To motivate the upcoming discussion let us recall how to blow up a point using symplectic cuts (cf. [13]). Let $\omega_{\rm st} = \frac{i}{2}(dw \wedge d\bar{w} + \sum_j dz^j \wedge d\bar{z}^j)$ be the standard symplectic structure on $\mathbb{C} \times \mathbb{C}^n$ and consider the Hamiltonian S^1-action given by $e^{i\theta} \cdot (w, z) = (e^{i\theta}w, e^{-i\theta}z)$, with moment map

$$\mu(w, z) = \frac{1}{2}(|z|^2 - |w|^2). \tag{3.25}$$

Now S^1 acts freely on $\mu^{-1}(\frac{1}{2}\varepsilon^2)$ for $\varepsilon > 0$, and the map $\kappa : \mu^{-1}(\frac{1}{2}\varepsilon^2) \to \mathbb{C}^n \times \mathbb{P}^{n-1}$ given by

$$\kappa : (w, z) \mapsto (\frac{wz}{|z|}, [z])$$

induces a diffeomorphism from $\mu^{-1}(\frac{1}{2}\varepsilon^2)/S^1$ onto $\widetilde{\mathbb{C}}^n = \{(x, l) | x \in l\}$, the blow-up of $\mathbb{C}^n$ at the origin. It is a well-known fact that $\kappa^*(\mathrm{pr}_1^*\omega_{\rm st} + \varepsilon^2 \mathrm{pr}_2^*\omega_{FS}) = \omega_{\rm st}$, giving an explicit description of the symplectic form on the reduced space. Finally, consider the following slice for the S^1-action:

$$\varphi : \mathbb{C}^n \backslash \overline{B_\varepsilon} \to \mu^{-1}(\frac{1}{2}\varepsilon^2), \qquad u \mapsto (\sqrt{|u|^2 - \varepsilon^2}, u).$$

Here B_ε is the ball of radius ε. Clearly $\varphi^*\omega_{\rm st} = \frac{i}{2}\sum_j du^j \wedge d\bar{u}^j$, which shows that the symplectic quotient $\mu^{-1}(\frac{1}{2}\varepsilon^2)/S^1$ is symplectomorphic, away from the exceptional divisor, to $(\mathbb{C}^n \backslash \overline{B_\varepsilon}, \omega_{\rm st})$.

To use this in our setting we need a reduction procedure for generalized complex structures. A general reduction theory has been introduced in [6], but we only need a very special case, which we will present here. In what follows, an S^1-action on $(Z, H, \mathcal{J})$ is understood to be an S^1-action on the manifold Z which preserves $\mathcal{J}$ and for which $\iota_X H = 0$, where X is the associated action vector field. In analogy with symplectic geometry we call $\mu : Z \to \mathbb{R}$ a *moment map* if $\mathcal{J}X = d\mu$.

Proposition 3.31. *Suppose we have an S^1-action on $(Z, H, \mathcal{J})$ with moment map μ. If $i : \mu^{-1}(c) \hookrightarrow Z$ is a regular level set with quotient $q : \mu^{-1}(c) \to \mu^{-1}(c)/S^1$, then $\mathfrak{F}q(\mathfrak{B}i(L))$ gives a generalized complex structure $\mathcal{J}'$ on $\mu^{-1}(c)/S^1$. If ρ is a local spinor for $\mathcal{J}$ which is S^1-invariant, then $i^*\rho = q^*\rho'$ for a unique form ρ' on the quotient which is a spinor for $\mathcal{J}'$.*

Proof. The inclusion of a regular level set $i : \mu^{-1}(c) \hookrightarrow Z$ has real codimension 1 so that $\mathfrak{B}i(L)$ is automatically smooth, and we have

$$\mathfrak{B}i(L) \cap \mathfrak{B}i(\bar{L}) = \mathbb{C} \cdot X. \tag{3.26}$$

By the assumption $\iota_X H = 0$ we can write $H = q^* H'$ for a (unique) 3-form H' on the quotient, so q is a generalized map. It satisfies $\ker(dq) \cap \mathfrak{B}i(L) = \mathbb{C} \cdot X$, which is of constant rank 1, so the forward image $\mathfrak{F}q(\mathfrak{B}i(L))$ is smooth and projects down to $\mu^{-1}(c)/S^1$ because $\mathfrak{B}i(L)$ is S^1-invariant. It is generalized complex because of (3.26) and the fact that X spans the kernel of q_*. Let ρ be a local spinor for L which is S^1-invariant. Then $i^*\rho$ is non-zero on $\mu^{-1}(c)$ and is an S^1-invariant spinor for $\mathfrak{B}i(L)$. Moreover,

$$0 = (X - i\mathcal{J}X) \cdot \rho = (X - id\mu) \cdot \rho$$

implies that $\iota_X i^*\rho = 0$; hence $i^*\rho$ comes from a unique differential form on $\mu^{-1}(c)/S^1$. This will be a spinor for the induced generalized complex structure on the quotient. □

Consider now a generalized Poisson transversal $Y \subset (M, \mathcal{J})$, with ω_Y the induced symplectic structure on N^*Y. As in the proof of Lemma 3.23 we choose a compatible Hermitian structure (g, I) on the bundle N^*Y and use it to construct an S^1-invariant 1-form λ on the manifold N^*Y of the form (3.16). In particular its differential $\sigma = d\lambda$ is a closed extension of ω_Y which is S^1-invariant and whose restriction to the fibers is translation invariant. Consider the S^1-action on $Z := \mathbb{C} \times N^*Y$ given by

$$e^{i\theta} \cdot (w, z) = (e^{i\theta} w, e^{-i\theta} z),$$

and denote by $X \in \Gamma(TZ)$ the induced action vector field. We equip Z with the 3-form $p^* H_Y$ and the generalized complex structure which is the product of the standard symplectic structure on $\mathbb{C}$ and $\mathcal{J}_\sigma$ on N^*Y as defined by equation (3.17).

Lemma 3.32. *The map $\mu : Z \to \mathbb{R}$ given by $\mu(w, z) := \frac{1}{2} g(z, z) - \frac{1}{2}|w|^2$ is a moment map.*

Proof. We can write $X = (X_1, X_2)$ on $\mathbb{C} \times N^*Y$ with X_i the corresponding action vector field on the separate factors. In particular X_2 is vertical, and by definition of $\mathcal{J}_\sigma$ we have $\mathcal{J}(X_1, X_2) = (\omega_{st}(X_1), \sigma(X_2))$. Since $\omega_{st} + \sigma = d(\lambda_{st} + \lambda)$ where both λ_{st} and λ are S^1-invariant, we get $\mathcal{J}X = -d\iota_X(\lambda_{\text{st}} + \lambda)$. Hence it suffices to show that $-\iota_X(\lambda_{\text{st}} + \lambda) = \mu$. This is a fiberwise equality and can be verified on $\mathbb{C} \times \mathbb{C}^n$. □

Remark 3.33. If one starts with an arbitrary extension $\sigma = d\lambda$ of ω_Y one can average it over S^1 to render it invariant, and the map $-\iota_X(\lambda_{\text{st}} + \lambda)$ is again a moment map. The advantage of our choice above is that the moment map has an explicit description in terms of a metric.

For $\varepsilon > 0$, Proposition 3.31 implies that $\widetilde{N^*Y}_\varepsilon := \mu^{-1}(\frac{1}{2}\varepsilon^2)/S^1$ is generalized complex, which is diffeomorphic to the blow-up of Y in N^*Y with respect to the linear holomorphic ideal. We would like to show that this blow-up can be glued back into the original manifold M to produce the blow-up of Y in M. For that we consider the slice

$$\widetilde{\varphi} : N^*Y \backslash \overline{B_\varepsilon} \hookrightarrow \mu^{-1}(\frac{1}{2}\varepsilon^2) \subset Z, \qquad z \mapsto (\sqrt{|z|^2 - \varepsilon^2}, z). \tag{3.27}$$

Here $\overline{B_\varepsilon}$ is the disc bundle of radius ε. If q denotes the quotient map of the S^1-action, we obtain a diffeomorphism

$$\varphi := q \circ \widetilde{\varphi} : N^*Y \backslash \overline{B_\varepsilon} \longrightarrow \widetilde{N^*Y}_\varepsilon \backslash E,$$

where E denotes the exceptional divisor. To show that φ is holomorphic it suffices, by definition of the generalized complex structure on the quotient, to show that $\widetilde{\varphi}$ pulls back a local spinor on Z to a local spinor for $\mathcal{J}_\sigma$. If $\rho = e^{i\omega_{\rm st}+i\sigma} \wedge p^*\rho_Y$ is such a spinor on Z, then from the definition of $\widetilde{\varphi}$ we see that indeed $\widetilde{\varphi}^*\rho = e^{i\sigma} \wedge p^*\rho_Y$ is a spinor for $\mathcal{J}_\sigma$.

Theorem 3.34. *Let $Y \subset (M, \mathcal{J})$ be a compact generalized Poisson transversal. Then the blow-up*[16] *of Y in M carries a generalized complex structure which, outside a neighborhood of the exceptional divisor, is isomorphic to the complement of a neighborhood of Y in M.*

Remark 3.35. We call $(\widetilde{M}, \widetilde{\mathcal{J}})$ the blow-up of Y in $(M, \mathcal{J})$, even though that terminology is slightly ambiguous. Indeed, the construction depends on several choices and it is not clear, even in the symplectic category, whether different choices lead to isomorphic blow-ups.

Proof. Equip a neighborhood U of Y in N^*Y with the generalized complex structure J_σ where σ is as above. By Theorems 3.24 and 3.29, if U is small enough there is a holomorphic embedding $\iota : (U, \mathcal{J}_\sigma) \to (\iota(U), \mathcal{J})$ with $\iota(U)$ a neighborhood of Y in M. Since Y is compact there is an $\varepsilon > 0$ such that $\overline{B_\varepsilon} \subset U$. Set $\widetilde{U} := \varphi(U) \cup E$ and define the blow-up of Y in M by

$$\widetilde{M} := M \backslash \iota(\overline{B_\varepsilon}) \underset{\iota\circ\varphi^{-1}}{\cup} \widetilde{U}. \tag{3.28}$$

Here the glueing takes place between $\widetilde{U} \backslash E$ and $\iota(U \backslash \overline{B_\varepsilon})$. □

Remark 3.36. The drawback of defining $\widetilde{M}$ by (3.28) is that there is no canonical blow-down map. It is possible to define blow-down maps, but they are not particularly useful because they will not be holomorphic around the exceptional divisor.

Example 3.37. Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold and let $Y \hookrightarrow M$ be a generalized Poisson submanifold for $\mathcal{J}_1$, i.e., $\mathcal{J}_1 N^*Y = N^*Y$. Since $\langle \mathcal{J}_1\alpha, \mathcal{J}_2\alpha \rangle > 0$ for all $\alpha \in N^*Y$, we see that $\mathcal{J}_2 N^*Y \cap (N^*Y)^\perp = 0$; i.e., Y is a generalized Poisson transversal[17] for J_2. In Example 3.19 we discussed how the maximal torus in a compact-even dimensional Lie group is a generalized Poisson submanifold for $\mathcal{J}_1$ which, because of the degeneracy condition, can almost never be blown up. With respect to $\mathcal{J}_2$ however there are no restrictions, so all maximal tori can be blown up for $\mathcal{J}_2$. In [17] a more thorough investigation of these examples is given and it is shown that if the maximal torus can be blown up for $\mathcal{J}_1$ and $\mathcal{J}_2$, then the result is again generalized Kähler.

[16] Since Y is a generalized Poisson transversal, its normal bundle has a complex structure, and we can blow up Y in M as in Definition 3.9.

[17] In Kähler geometry this amounts to the well-known fact that a complex submanifold is automatically symplectic.

3.3. **A remark on other types of submanifolds.** Our definition of a generalized complex submanifold is, besides a smoothness criterion, characterized by

$$\mathcal{J}(N^*Y) \cap (N^*Y)^{\perp} \subset N^*Y. \tag{3.29}$$

In the previous sections we investigated the blow-up theory of the two extreme cases, namely those for which the above inclusion is either an equality (the generalized Poisson case) or the intersection is zero (the generalized Poisson transversals). An obvious question at this point is whether the "intermediate" cases admit a blow-up theory as well. The techniques we used for generalized Poisson submanifolds and generalized Poisson transversals are so different from each other that it does not seem we can use either of them when the type in the normal direction is mixed. We will now give an example where we can explicitly prove that there does not exist a blow-up. The following proposition is a consequence of a more general result by Atiyah [1]; we give a direct proof for the convenience of the reader.

Proposition 3.38. *Let M be a compact 4-dimensional generalized complex manifold of type 1. Then the Euler characteristic $\chi(M)$ is even.*

Proof. A type 1 structure gives rise to a decomposition $TM = L_1 \oplus L_2$, where L_1 is the distribution tangent to the symplectic foliation and L_2 is a choice of normal bundle. In particular L_1 and L_2 are orientable and we can think of them as complex line bundles,[18] giving an almost complex structure on TM. By Wu's formula, using that $c_1(TM) \equiv w_2(M)$ mod 2 and $c_1(TM) = c_1(L_1) + c_1(L_2)$, we obtain

$$\alpha^2 \equiv \alpha \cup c_1(L_1) + \alpha \cup c_1(L_2) \bmod 2 \quad \forall \alpha \in H^2(M, \mathbb{Z}).$$

Applying this to $\alpha = c_1(L_1)$ we see indeed that $\chi(M) = c_1(L_1)c_1(L_2)$ is even. □

Now let M be a compact 4-dimensional generalized complex manifold of type 1. The blow-up of a point in M is differentiably given by $M\#\overline{\mathbb{CP}}^2$, which has Euler characteristic $\chi(M)+1$. If the blow-up would have a generalized complex structure that agrees with the one on M outside a neighborhood of the exceptional divisor, it would have type 1 everywhere since the type can only change in even amounts. By the proposition we conclude that the blow-up cannot be generalized complex, at least not in a way that is reasonably related to the original structure on M.

In the example above, equation (3.29) is neither zero nor an equality. There are, however, generalized complex submanifolds Y for which (3.29) is zero at some points and an equality at others. Further study is needed to see what can be said about these types of submanifolds.

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