Universality in enumerative geometry and Vafa-Witten theory

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Universality in enumerative geometry and Vafa-Witten theory

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Chapter 0

Introduction

Hilbert schemes of points on algebraic surfaces and their intersection theory play a prominent role in modern enumerative geometry. In many examples, solutions to enumerative problems can be expressed in terms of intersection numbers of such Hilbert schemes. In this thesis, I will discuss two new instances of this. The first problem is a 'classical' enumerative problem, and concerns counts of nodal curves on (families of) surfaces. It is related to the Göttsche conjecture [Göt98]. The second problem is a 'modern' physics-related enumerative problem and concerns the virtual enumeration of solutions to the Vafa-Witten equations [TT17a, VW94].

0.1 The Göttsche Conjecture

The Göttsche conjecture gives a qualitative solution to the problem of enumerating certain singular curves on a surface. The most basic instance of this problem is the observation that the number of lines in a plane, passing through distinct points P and Q is exactly 1:

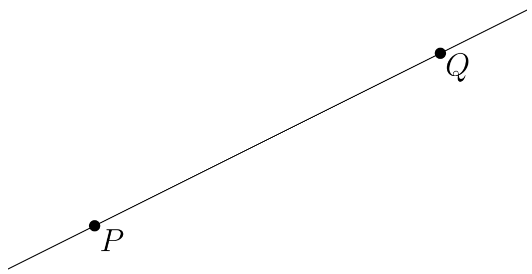

Figure 1: A line through P and Q.

More generally, one can solve a similar counting problems for plane curves of higher degree. For example, a conic in the (complex projective) plane is given by a degree 2 homogeneous equation

$$a_0x^2 + a_1xy + a_2y^2 + a_3yz + a_4z^2 + a_5xz = 0\,,$$

with $a_i \in \mathbb{C}$ for $i = 0, \ldots, 5$, not all equal to zero. Two such equations define the same curve, precisely when they differ by non-zero scalar. It follows that the space of conics form a 5-dimensional projective space, with homogeneous coordinates $a_0, \ldots, a_5$.

For a point P in the projective plane, the condition "P lies on the curve" corresponds to a hyperplane in the projective 5-space of conics. For five points $P_1, \ldots, P_5$ in general position, i.e., no three points are co-linear, these hyperplanes intersect in a unique point, which then corresponds to a conic containing the points $P_1, \ldots, P_5$:

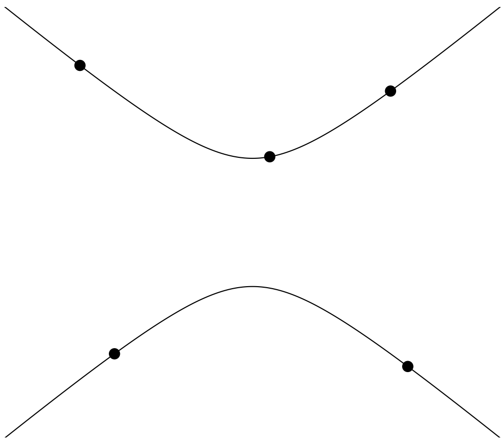

Figure 2: A conic containing five points in general position.

In general, there exists exactly one curve of degree d that passes though any generally chosen points $P_1, \ldots, P_N \in \mathbb{P}^2$, with $N = \frac{d(d+3)}{2}$.

The problem becomes a more interesting when one imposes other conditions on the curves. For example, we can consider curves with certain singularities. The simplest kind of singularity is the *node*, i.e. a point where the curve locally looks like the union of the x- and y-axes in the plane:

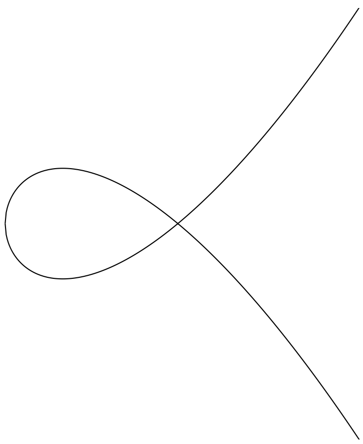

Figure 3: A curve with a node.

For example, nodal plane curves of degree 3 form a 8-dimensional subvariety of the 9-dimensional projective space of cubics. It has been known for a long time [Ste54] that it is a subvariety of degree 12. This means that the intersection of the variety with 8 sufficiently general hyperplanes consists of 12 distinct points. If we consider hyperplanes that correspond to point conditions as above, we find that the number of nodal cubics in the projective plane that intersect 8 general points, is 12.

In general, nodal plane curves of degree $d \geq 1$ form a locally closed codimension-one subvariety of the projective space of all plane curves of degree d. The degree of this subvariety is given by the polynomial $3(d-1)^2$. One of the first mentions of this fact can be found in [Ste54]. Similar statements for curves with two or three nodes have been know since the nineteenth century (see [KP04, Remark 3.7] for a detailed historical overview).

The Göttsche conjecture [Göt98], which is now a theorem due to Tzeng [Tze12], concerns a generalisation of the counting problem. It considers curves on an arbitrary smooth projective surfaces S, having any fixed number δ of nodes, and asserts a certain polynomial property of the curve count.

Definition 0.1.1. *Let $\delta \in \mathbb{Z}_{\geq 0}$. A curve C is called* δ-nodal *if there exist points*

$$p_1, \ldots, p_\delta \in C$$

such that

$$C \backslash \{p_1, \ldots, p_\delta\}$$

is smooth, and C has a nodal singularity in p_i for each $i = 1, \ldots, \delta$.

In general, a curve on a surface S is not given by a homogeneous equation, but rather as the vanishing locus of a section of a line bundle on S. Again, all curves that can be realised by sections of a fixed line bundle L form a projective space, called the complete linear system of L. It will be denoted by $|L|$.

We assume that L has enough sections, in a suitable sense (depending on δ). Let

$$\nu_\delta(S, L)$$

denote the number of δ-nodal curves in a general linear system $\mathbb{P}^\delta \subset |L|$. E.g., if we can realise $\mathbb{P}^\delta$ as the intersection of hyperplanes in $|L|$ corresponding to point conditions as above, the number $\nu_\delta(S, L)$ can be interpreted "the number of δ-nodal curves in the complete linear system $|L|$ intersecting points $P_1, \ldots, P_n \in S$", where

$$n = \dim(|L|) - \delta \,.$$

Now the Göttsche conjecture states that $\nu_\delta(S, L)$ is given by a polynomial (depending only on δ) in certain topological invariants of the surface S and the line bundle L, namely the intersection numbers

$$L^2, LK_S, K_S^2, c_2(S) \,,$$

where K_S is the canonical divisor, $c_2(S)$ the second Chern class of the tangent bundle of S, and we write $L^2 = c_1(L)^2$ etc.

Theorem 0.1.2 (Göttsche Conjecture)**.** *For each $\delta \geq 0$, there exists a universal polynomial $P_\delta = P_\delta(x, y, z, t) \in \mathbb{Q}[x, y, z, t]$ of degree δ, such that for any surface S and any sufficiently ample line bundle L on S we have*

$$\nu_\delta(S, L) = P_\delta(L^2, LK_S, K_S^2, c_2(S)) \,.$$

The numbers $\nu_\delta(S,L)$ satisfy the following multiplicative property:

$$\nu_\delta(S \sqcup S', L \sqcup L') = \sum_{i+j=\delta} \nu_i(S,L)\nu_j(S',L') \qquad (0.1.3)$$

for surfaces S and S' and sufficiently ample line bundles L and L' on S and S' respectively. This follows essentially from the fact that a nodal curve on the disjoint union $S \sqcup S'$ is a union of nodal curves on S and S'. Because of this multiplicative property, the universal polynomials of Theorem 0.1.2 have a special form:

Corollary 0.1.4. *[Göt98, Proposition 2.3] There are universal power series*

$$A_1, A_2, A_3, A_4 \in \mathbb{Q}[[q]]$$

with

$$\sum_{\delta \geq 0} P_\delta\, q^\delta = A_1^x\, A_2^y\, A_3^z\, A_4^t\,.$$

As an important consequence, the polynomials P_δ are completely determined by their values on a set of pairs (S,L) for which the vectors

$$(L^2, LK_S, K_S^2, c_2(S))$$

are linearly independent. This means in particular that if we know the numbers $\nu_\delta(S,L)$ for $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, and all line bundles L on S, we know the full solution to the counting problem. This type of reasoning is used several times in this thesis. It allows us to do many computations (see especially Chapter 2), and serves as a motivation for all our universality results.

A short proof of the Göttsche Conjecture was given in [KST11] by Kool, Shende and Thomas. We will briefly discuss Hilbert scheme of points, and their role in the proof of loc. cit.

0.2 Hilbert schemes of points

For a surface S and an integer $n \geq 0$, let

$$S^{[n]} = \Big\{\text{finite subschemes } Z \subset S \text{ of length } n\Big\}$$

be the Hilbert scheme of n points on S. A general point of the Hilbert scheme corresponds to a subscheme of S consisting of n distinct points. When the points (on S) collide, the corresponding point of the Hilbert scheme remembers some information about the direction of the collision. For example, a subscheme of S of length 2, that is supported at a single point, can be represented by an infinitesimal arrow (see Figure 4). By a well-known theorem of Fogarty [Fog68], $S^{[n]}$ is a smooth variety of dimension $2n$.

In [KST11], the numbers

$$\nu_\delta(S,L)$$

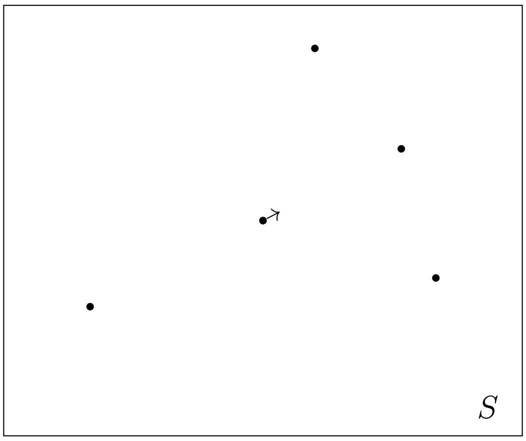

Figure 4: A point of $S^{[6]}$.

are given by certain tautological integrals over Hilbert schemes of points on S. More precisely, let $L^{[n]}$ denote the vector bundle on $S^{[n]}$ characterised by

$$(L^{[n]})_{[Z]} = \Gamma(Z, L|_Z)\,, \quad [Z] \in S^{[n]}\,,$$

where $(L^{[n]})_{[Z]}$ denotes the fibre of $L^{[n]}$ at $[Z]$, and $\Gamma(Z, L|_Z)$ the vector space of global sections of $L|_Z$. Then, by the results of [KST11, Theorem 3.4, Equation (4.2)], which are based on the BPS-calculus of [PT10], for $n = 1, \ldots, \delta$, there exist polynomials $P_n^\delta(T_{S^{[n]}}, L^{[n]})$ in the Chern classes of $L^{[n]}$ and the tangent bundle $T_{S^{[n]}}$ of $S^{[n]}$, such that $\nu_\delta(S, L)$ is a linear combination of the integrals

$$\int_{S^{[n]}} P_n^\delta(T_{S^{[n]}}, L^{[n]})\,. \tag{0.2.1}$$

By an algorithm of Ellingsrud, Göttsche and Lehn [EGL01], any tautologcal integral can be expressed by a *universal* polynomial $Q_n^\delta(S, L)$, i.e., depending only on (δ, n), in the intersection numbers

$$L^2, LK_S, K_S^2, c_2(S)\,.$$

This leads to a proof of the Göttsche conjecture.

0.3 A conjecture of Kleiman and Piene

The first result of this thesis is a generalisation of the Göttsche conjecture, in which we consider the counting problem relative to a base scheme B. This means that we allow the surface S to vary in a smooth family over B, and moreover that the nodal curves are 'enumerated' by a homology class on B, rather than by a number.

As a motivation, we will discuss an example. Consider planar curves of degree 3 in $\mathbb{P}^3$, and say that we are interested in counting such curves that have a node. For example, we could impose that the curves intersect 11 general lines in $\mathbb{P}^3$, so that this number is finite. In fact, it equals 12960, which has been known for at least 140 years [Sch79].

A planar curve of degree 3 that admits a node lies in a (unique) plane $\mathbb{P}^2 \subset \mathbb{P}^3$. Hence, we could say that rather than counting nodal curves on a surface (as in the Göttsche conjecture), we are counting curves on a family of surfaces, namely on the universal surface over the Grassmannian of planes in $\mathbb{P}^2$.

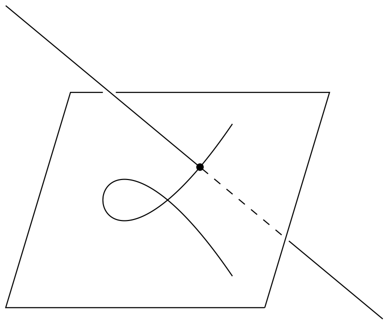

Figure 5: A nodal plane curve intersecting a line in $\mathbb{P}^3$.

We can generalize the problem by considering plane curves of any degree d in $\mathbb{P}^3$. Such curves form the underlying set of a scheme B. For $d > 1$, a plane curve of degree d in $\mathbb{P}^3$ lies on a (necessarily unique) plane $V \subset \mathbb{P}^3$, and the scheme B has the structure of a projective bundle over the Grassmannian $\mathbb{G}(2,4)$ of planes in $\mathbb{P}^3$ with fibre $|\mathcal{O}_{\mathbb{P}^2}(d)|$. Consider the universal curve $\mathcal{C}$ over B, which maps to the universal plane $\mathbb{V}$ in $\mathbb{P}^3$:

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathbb{V} & \longrightarrow & \mathbb{P}^3 \\ \downarrow & & \downarrow & & \\ B & \longrightarrow & \mathbb{G}(2,4)\,. & & \end{array}$$

In the language of Chapter 1, $\mathcal{C}$ is a relative effective divisor on the smooth projective family of surfaces

$$\mathbb{V}_B = \mathbb{V} \times_{\mathbb{G}(2,4)} B \to B\,.$$

We are interested in enumerating the nodal fibres of $\mathcal{C} \to B$. As an application of the general Theorem 0.3.3, which we will discuss below, the following result is proved in Section 1.6.

Theorem 0.3.1 (Theorem 1.B)**.** *The number of planar δ-nodal curves of degree $d \geq \max\{1, \delta\}$ in $\mathbb{P}^3$ intersecting $\frac{d(d+3)}{2} + 3 - \delta$ general lines is given by a polynomial $N_\delta(d)$ in d.*

In Section 1.7 we discuss how one can use Atiyah-Bott localisation to calculate the polynomials $N_\delta(d)$, which I did for $\delta \leq 12$. Finally, we will verify some of the results using techniques from 19th century geometry by Schubert.

In general, let $q\colon \mathcal{S} \to B$ be a smooth projective family of surfaces. Let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor, i.e. a subscheme of pure codimension 1 for which the morphism $\mathcal{C} \to B$ is flat. Let $\delta \geq 0$ be an integer. We are interested in the locus in B over which the fibres of the family $\mathcal{C} \to B$ are nodal curves. Kleiman and Piene conjecture (and prove for $\delta \leq 8$) that, as in the Göttsche conjecture, the homology class of this locus (with the correct multiplicities) can be expressed in the Chern data of $\mathcal{S}$ and $\mathcal{C}$ [KP04].

Consider the locus

$$B(\delta) = \{b \in B : \mathcal{C}_b \text{ is a } \delta\text{-nodal curve}\} \subset B$$

of nodal fibres of $\mathcal{C}$. Write

$$\epsilon(a,b,c) = q_* \left(c_1(\mathcal{O}_{\mathcal{S}}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c \right) \in A^{a+b+2c-2}(B)$$

for $a, b, c \geq 0$, where $A^*(B)$ denotes the (cohomological) Chow ring of B, and $T_{\mathcal{S}/B}$ the relative tangent bundle.

Conjecture 0.3.2 (Kleiman-Piene)**.** *Under certain genericity conditions, here exists a natural cycle on B, with support equal to the closure $\overline{B(\delta)}$ of $B(\delta)$. Moreover, its Chow homology class can be expressed by a universal polynomial in the classes $\epsilon(a,b,c)$.*

In Chapter 1, we will define a class $\gamma(\mathcal{C}) \cap [B]$ by integrals over the relative Hilbert schemes of points on the fibres of the family $\mathcal{S} \to B$

$$\mathrm{Hilb}^n(\mathcal{S}/B) \to B\,, \quad n = 0, \ldots, \delta,$$

that are closely modelled on the integrals (0.2.1). We will show that, under a mild ampleness condition on $\mathcal{C}$, the class $\gamma(\mathcal{C}) \cap [B]$ the the homology class of a cycle that satisfies the description of Conjecture 0.3.2. In particular, cf. the proof of [KST11] of the Göttsche conjecture, the polynomiality follows by a relative version of the algorithm of [EGL01].

Theorem 0.3.3 (Theorem 1.A)**.** *Conjecture 0.3.2 holds for sufficiently ample relative effective divisors $\mathcal{C} \subset S/B$.*

0.4 Vafa-Witten invariants

Chapters 2 and 3 of this thesis are devoted to Vafa-Witten invariants, which (virtually) count solutions of the Vafa-Witten equations [VW94]. The invariants have recently been defined in algebraic geometry by Tanaka and Thomas in [TT17a, TT17b], where the counting problem is interpreted as the enumeration of certain sheaves on a local surface, i.e., the total space of the canonical bundle of a smooth projective surface S. Moreover, a refined invariant has been defined in [Tho18a].

Vafa-Witten invariants can be written as the sum of an *instanton* part, and a *monopole* part [TT17a]. Instanton contributions are given by *virtual* Euler characteristics (or virtual χ_y-genenera for the refined invariants) of moduli spaces of sheaves on the surface S. Such moduli spaces have been studied extensively. For example, when such a moduli space is smooth of the expected dimension, its virtual Euler characteristic agrees with the topological Euler characteristic, which has been computed in many cases (see [GK18] for an overview).

In [GK18], Göttsche and Kool compute instanton contributions to rank 2 and 3 Vafa-Witten invariants for surfaces satisfying $p_g(S) > 0$ and $H_1(S, \mathbb{Z}) = 0$, making use of the work of Mochizuki [Moc09]. Inspired by their their computations, they conjecture explicit formulas for generating series of such contributions. Moreover, based on [VW94], they conjecture modularity (*S-duality*) of the generating series of the full invariant. The predictions extend formulas for the unrefined rank 2 invariants from the original paper of Vafa and Witten.

One of the proposed modular transformations exchanges the monopole contributions with (part of) the instanton contributions. Using this, and the formulas for the instanton contributions, Göttsche and Kool also find conjectural formulas for generating series of monopole contributions. In Chapters 2 and 3 we will discuss monopole contributions to Vafa-Witten invariants. We will prove part of the structure of the formulas of Göttsche and Kool, and provide computational evidence for their conjectures.

More precisely, we will study *vertical* contributions to Vafa-Witten invariants, which form a part of the monopole contributions. However, for prime rank these for the entire monopole contribution, by a theorem of Thomas [Tho18a]. In Chapter 2, we will work under the condition that stability and semi-stability of sheaves on the local surface agree (the *stable* case). In Chapter 3, we also treat sheaves that possibly admit non-trivial automorphisms (the *semistable case*). In fact, we show that vertical contributions to Vafa-Witten invariants are well defined in the semistable case, proving part of the conjectures of [TT17b] and [Tho18a]. Moreover, we show that they are computed by the same tautological integrals as in the stable case, treated in Chapter 2.

0.4.1 Vertical Vafa-Witten invariants

Let (S, H) be a polarised algebraic surface, and consider the local surface $X \to S$, i.e. the total space of the canonical bundle. A sheaf on X can be encoded by a *Higgs pair* (E, ϕ), where E is a sheaf on S and $\phi\colon E \to E \otimes \omega_S$ a homomorphism of $\mathcal{O}_S$-modules. The sheaves we are interested in correspond to pairs (E, ϕ), with E a torsion free sheaf of finite rank r, with fixed determinant $\det(E) = L$ and $\operatorname{tr}(\phi) = 0$.

Assume that S and L and a second Chern class $c_2 \in H^4(S, \mathbb{Z}) = \mathbb{Z}$ are chosen such that semistability and stability of such Higgs pairs agree (see Section 2.3). The natural $\mathbb{C}^*$ action on X, which scales the fibres over S, lifts to an action on the moduli space $\mathcal{N}^{\perp}_{r,L,c_2}$ of stable Higgs pairs as above (and satisfying $c_2(E) = c_2$). In [Tho18a] the latter is equipped with a symmetric perfect obstruction theory [BF08], and the Vafa-Witten invariant is defined by an integral over the $\mathbb{C}^*$-fixed locus, given by the virtual localisation formula [GP99].

The $\mathbb{C}^*$-fixed locus of $\mathcal{N}^{\perp}_{r,L,c_2}$ contains an open and closed locus that consists of Higgs pairs (E, ϕ) of the form

$$E = E_0 \oplus \ldots \oplus E_{1-r}\,, \quad \phi = (\phi_1, \ldots, \phi_{r-1})\,, \tag{0.4.1}$$

where the E_i are torsion free sheaves of rank one, and ϕ is given by non-zero maps

$$\phi_i\colon E_{-(i-1)} \to E_{-i} \otimes \omega_S\,, \quad i = 1, \ldots, r-1\,.$$

This locus can be described in terms of *nested Hilbert schemes* of points on the surface S.

In fact, for a Higgs pair (E, ϕ) as above, and for each $i = 0, \ldots, r-1$, the sheaf E_{-i} can be uniquely written as

$$E_{-i} = L_i \otimes I_i \,,$$

where L_i is a line bundle on S, and $I_i \subseteq \mathcal{O}_S$ is an ideal sheaf corresponding to a zero-dimensional subscheme of S. The maps ϕ_i can then be viewed as *nestings* of ideal sheaves. Nested Hilbert schemes of points are of independent interest, and have recently been studied in a number of papers [GSY18, GSY17, GT17, GT19].

In the terminology of Chapter 3, we will refer to the contribution of the above locus to the Vafa-Witten invariant, as the *vertical* contribution, and denote it by

$$\mathrm{VW}^{\mathrm{vert}}_{r,L,c_2} \,.$$

In [GT19] it was noted that for surfaces with $p_g(S) = h^{0,2}(S) > 0$, the vertical contribution to Vafa-Witten invariants can be expressed by tautological integrals over products

$$S^{[n_0]} \times \cdots \times S^{[n_{r-1}]}$$

of Hilbert schemes, and hence is given by a universal expression in certain intersection numbers on S, again by the algorithm of [EGL01].

In Chapter 3 we will consider Higgs pairs that have been rigidified by a *Joyce-Song section*, i.e. a section

$$s \in H^0(S, E \otimes \mathcal{O}_S(mH))$$

for some integer $m \gg 0$. In [TT17b], a partially conjectural definition for the Vafa-Witten invariant is given as an invariant of the moduli space of rigidified Higgs pairs. The new definition extends the one of [TT17a], but also includes choices of L for which there exist strictly semistable Higgs sheaves on S. In Chapter 3, we will show that the vertical contributions in the semistable case are computed by the same tautological integrals as in the stable case, which agrees with the physics predictions [VW94].

Consider the generating series

$$\mathsf{Z}_{S,r,L}(q) = \frac{q^{\frac{1-r}{2r}c_1^2}}{\#\,\mathrm{Pic}(S)[r]} \sum_{c_2 \in \mathbb{Z}} \mathrm{VW}^{\mathrm{vert}}_{r,L,c_2}\, q^{c_2} \,,$$

where $\mathrm{Pic}(S)[r] \coloneqq \{[L] \in \mathrm{Pic}(S) : L^{\otimes r} \cong \mathcal{O}_S\}$. The observation that the tautologicals integrals satisfy a multiplicative property similar to (0.1.3), will lead to the following result.

Theorem 0.4.2 (Theorem 2.A, Theorem 3.B). *Fix a rank $r \geq 1$. There exist universal Laurent series*

$$A, B, C_{ij} \in \mathbb{Q}(\!(q^{\frac{1}{2r}})\!) \,, \quad 1 \leq i \leq j < r \,,$$

depending only on r, such that for any surface S with $p_g(S) > 0$, and any line bundle L on S, we have

$$\mathsf{Z}_{S,r,L}(q) = A^{\chi(\mathcal{O}_S)} B^{K_S^2} \sum_{\beta} \mathrm{SW}(\beta^1) \cdots \mathrm{SW}(\beta^{r-1}) \prod_{i \leq j} C_{ij}^{\beta^i \beta^j}$$

where the sum is taken over classes $\beta^1, \dots, \beta^{r-1} \in H^2(S, \mathbb{Z})$ *with*

$$c_1(L) \equiv \sum_i i\beta^i \mod rH^2(S, \mathbb{Z}),$$

and $\mathrm{SW}(\beta^i)$ *denotes the Seiberg-Witten invariant of* β^i.

In Chapter 2, we compute part of the Laurent series A, B, C_{ij} by various methods. In Section 2.9 we discuss how, for low ranks r, we can compute the first few coefficients of the series by evaluating the tautological integrals discussed above on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where we have access to toric methods.

The series A (for any rank) is determined the generating series $\mathsf{Z}_{S,r,0}(q)$ evaluated on a K3 surface, which has been computed in [TT17b] (unrefined) and [Tho18a] (refined). As an alternative, I show how the series A can be determined by evaluating the tautological integrals on a K3 surface. In Section 2.10, we use the second method to give an alternative proof of the Göttsche-Kool conjectures of the power series A. For another application of our method, see [GKW19], where it is used in the context of K-theoretic Verlinde numbers on surfaces.

In Section 2.11 we consider a surface of general type, for which the Picard group $\mathrm{Pic}(S)$ is generated by a smooth canonical curve. In some cases, we can compute the vertical contributions to the Vafa-Witten invariant by a more direct method. In fact, for $r = 2$, $L = K_S$, and $c_2 \leq 2$, the locus of Higgs pairs of the form (0.4.1) is smooth, and we have explicit expressions for the virtual class. We proceed by intersection theoretic computations in the Chow rings of C, C^2 and $C^{[2]}$.

The results of the computations above are given in Theorems 2.B, 2.B′ and 2.C.

0.5 Cross-references

Chapter 1 has appeared in Selecta Mathematica [Laa18a]. Chapters 2 and 3 have been published as a preprints on `https://arxiv.org/` [Laa18b, Laa19]. Although Chapter 3 is a sequel to Chapter 2, it is written as an independent paper, and non-standard notation will be reintroduced.

Chapter 1

The Kleiman-Piene conjecture and plane curves in $\mathbb{P}^3$

1.1 Introduction

1.1.1 The Kleiman-Piene Conjecture

All schemes we consider are separated and of finite type over $\mathbb{C}$. Let B be a base scheme, and let $q\colon \mathcal{S} \to B$ be a smooth family of surfaces, i.e. a smooth projective morphism of relative dimension 2. By a *curve*, we mean a proper 1-dimensional scheme, not necessary irreducible or reduced. Let $\mathcal{C}$ be a *relative effective (Cartier) divisor* on the family $\mathcal{S} \to B$, i.e. an effective Cartier divisor on $\mathcal{S}$, such that the morphism $\mathcal{C} \to B$ is flat. Fix a non-negative integer δ. We call a curve δ-*nodal* if it is reduced, has δ nodes and no other singularities. Consider the following counting problem:

Problem 1. What is, if finite, the number of δ-nodal curves in the family $\mathcal{C} \to B$?

More generally, consider the locus

$$B(\delta) \coloneqq \left\{ b \in B \mid \mathcal{C}_b \text{ is a } \delta\text{-nodal curve} \right\}$$

and write $\overline{B(\delta)}$ for its closure in B.

Problem 2. What is the class $\left[\overline{B(\delta)}\right] \in A_*(B)$?

Assume that B is Cohen-Macaulay and of pure dimension n. In [KP04], Kleiman and Piene construct a natural effective cycle $U(\delta)$ with support equal to the closure of the locus of δ-nodal curves. For $\delta \leq 8$ they prove that the class $[U(\delta)]$ is given in a rather specific form as a polynomial in the classes

$$\epsilon(a, b, c) \coloneqq q_*(c_1(\mathcal{O}_{\mathcal{S}}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)\,.$$

Kleiman and Piene work with certain assumptions on the dimensions of equisingular strata in the family. For a curve C, let $\mathcal{D}$ be its equisingularity type. It can be represented by an Enriques diagram, which encodes the numerical invariants of the singularities of C [KP99]. Conversely, for an equisingularity type (or Enriques diagram) $\mathcal{D}$, we write $B(\mathcal{D}) \subset B$ for the locus of curves in the family $\mathcal{C} \to B$, with equisingularity type $\mathcal{D}$.

One of the invariants of an equisingularity type is the codimension $cod(\mathcal{D})$. It is the 'expected codimension' in which curves with equisingularity type $\mathcal{D}$ appear in a family. More precisely, in [KP99], it is characterized as the codimension of the locus of curves with equisingularity type $\mathcal{D}$ in the universal family $\mathcal{C} \to |L|$ of any sufficiently ample complete linear system. The hypotheses on the family $\mathcal{C} \to B$ under which the class $U(\delta)$ is constructed and which we will denote by $\mathbf{DIM_{KP}}$ are the following:

- The locus of non-reduced curves $B(\infty)$ has codimension $> \delta$;
- For each equisingularity type $\mathcal{D}$, the locus $B(\mathcal{D})$ has at least the expected codimension $cod(\mathcal{D})$, or codimension $> \delta$.

Here we use the convention $\operatorname{codim}(\emptyset) = \infty$. In [KP04] and [KP], the authors prove the following theorem:

Theorem 1.1.1 (Kleiman-Piene)**.** *Under the above hypotheses* $\mathbf{DIM_{KP}}$*, the locus* $B(\delta)$ *of* δ*-nodal curves is either empty, or has pure codimension* δ*. There is a natural non-negative cycle* $U(\delta)$ *with support* $\overline{B(\delta)}$*. For* $\delta \leq 8$*, the rational equivalence class* $[U(\delta)]$ *is given by a universal polynomial*[1] *in the classes* $\epsilon(a, b, c)$*.*

Moreover, in [KP04] the following conjecture is made.

Conjecture 1.1.2 (Kleiman-Piene)**.** *Theorem 1.1.1 holds for all* $\delta \geq 0$*.*

In this paper we propose a class $\gamma(\mathcal{C}) \in A^\delta(B)$, enumerating the δ-nodal curves, inspired by the BPS calculus of Pandharipande and Thomas [PT10]. We will show that if B is complete, but not necessarily Cohen-Macaulay, the class $\gamma(\mathcal{C}) \cap [B]$ is the rational equivalence class of a natural cycle with support $\overline{B(\delta)}$. For this we work with hypotheses **DIM**, similar to but slightly weaker than $\mathbf{DIM_{KP}}$, and an additional ampleness assumption **AMP**. By means of a family version of an algorithm by Ellingsrud, Göttsche and Lehn [EGL01], we show that without assumptions, the class $\gamma(\mathcal{C})$ is a universal polynomial in the classes $\epsilon(a, b, c)$. This will be the content of Theorem 1.A below.

1.1.2 BPS numbers

Let C be a locally planar, reduced curve of arithmetic genus g. In [PT10] the authors consider the following transformation of the generating series of topological Euler characteristics of Hilbert schemes $C^{[i]}$ of i points on C, which defines the *BPS numbers* $n_{r,C}$ of C.

$$\sum_{i=0}^{\infty} e(C^{[i]})\, q^i = \sum_{r=-\infty}^{g} n_{r,C}\, q^{g-r}(1-q)^{2r-2}\,.$$

They prove the following:

Theorem 1.1.3 (Pandharipande-Thomas)**.** *The numbers* $n_{r,C}$ *are zero, unless* $g - \delta \leq r \leq g$*, where* δ *is the* δ*-invariant of* C*, i.e.,* $g - \delta$ *is the geometric genus of* C*.*

[1] In fact, their statement is more precise: The polynomials are of the form $P_\delta(a_1, \dots, a_\delta)/\delta!$, in which P_δ is the δ-th Bell polynomial, and a_i is a linear combination of classes $\epsilon(a, b, c)$ with $a + b + 2c = i + 2$, so that $a_i \in A^i(B)$. Moreover, an algorithm is given that produces these classes.

Shende proves in [She12] that the number $n_{g-i,C}$ equals the degree of the subvariety of i-nodal curves in the versal deformation space of C. In particular it is positive for $0 \le i \le \delta$.

Let B be a scheme and let $p\colon \mathcal{C} \to B$ be a family of (not necessarily reduced) curves, i.e., a projective flat morphism of relative dimension 1. Assume that the fibres are locally planar curves of genus g. Let

$$p^{[i]}\colon \mathcal{C}_B^{[i]} = \mathrm{Hilb}^i(\mathcal{C}/B) \to B$$

be the relative Hilbert scheme of i points on the fibres of $\mathcal{C} \to B$. We define constructible functions $n_r = n_r(\mathcal{C})$ on B by

$$\sum_{i=0}^{\infty} p_*^{[i]}(1_{\mathcal{C}_B^{[i]}})\, q^i = \sum_{r=-\infty}^{g} n_r\, q^{g-r}(1-q)^{2r-2}\,. \tag{1.1.4}$$

In other words, n_r is the function that assigns the number $n_{r,\mathcal{C}_b^{red}}$ to a point $b \in B$. By Theorem 1.1.3, the function $n_{g-\delta}$ is supported on the locus of curves that have δ-invariant $\ge \delta$ or are non-reduced. In the same paper it is shown that $n_{g-\delta,C} = 1$ for a δ-nodal curve C.

Let $q\colon \mathcal{S} \to B$ be a smooth family of surfaces and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor. We will use the embedding $\mathcal{C} \hookrightarrow \mathcal{S}$ to analogously define classes $n_r^{cl} \in A^*B$. In fact, $\mathcal{C}_B^{[i]}$ is a subscheme of $\mathcal{S}_B^{[i]} = \mathrm{Hilb}^i(\mathcal{S}/B)$, cut out regularly (in particular, in the expected codimension i) by the tautological bundle $\mathcal{O}(\mathcal{C})_B^{[i]}$ (see Lemma 1.2.7). The scheme $\mathcal{S}_B^{[i]}$ is smooth over B [AIK77] and the *virtual tangent bundle* $T_{\mathcal{C}_B^{[i]}/B}$, as defined in [Ful98, B.7.6], is given by the class

$$\left[T_{\mathcal{S}_B^{[i]}/B}\Big|_{\mathcal{C}_B^{[i]}} - \mathcal{O}(\mathcal{C})_B^{[i]}\Big|_{\mathcal{C}_B^{[i]}} \right]$$

in the Grothendieck group $K(\mathcal{C}_B^{[i]})$ of vector bundles on $\mathcal{C}_B^{[i]}$. Let

$$c\colon K \Rightarrow (A^*)^\times$$

be the total Chern class. Then the classes $n_r^{cl} = n_r^{cl}(\mathcal{C}) \in A^*(B)$ are defined by the equation

$$\sum_{i=0}^{\infty} p_*^{[i]} c(T_{\mathcal{C}_B^{[i]}/B})\, q^i = \sum_{r=-\infty}^{g} n_r^{cl}\, q^{g-r}(1-q)^{2r-2}\,. \tag{1.1.5}$$

Here the homomorphism

$$p_*^{[i]}\colon A^*(\mathcal{C}_B^{[i]}) \to A^*(B)$$

denotes the Gysin push-forward as defined in [Ful98, Chapter 17]. We define

$$\gamma(\mathcal{C}) = \left\{ n_{g-\delta}^{cl}(\mathcal{C}) \right\}_\delta \in A^\delta(B)$$

to be equal to the degree-δ part of $n_{g-\delta}^{cl}(\mathcal{C})$. We will show that it reflects some of the properties of $n_{g-\delta}(\mathcal{C})$. In fact, in Proposition 1.3.1, we will compare n_r and n_r^{cl} by means of the Chern-Schwartz-MacPherson class.

Remark 1.1.6. Göttsche and Shende [GS14] also mention the CSM-class of the constructible function n_r as an invariant counting nodal curves. Moreover they consider an analogous class, using the virtual tangent bundle of Hilbert schemes of points of the curve. However, the use of the *relative* tangent bundle, which is natural from the point of view of [KP04], is essential for our results.

1.1.3 Results

Recall that a line bundle L on a smooth projective surface S is called *δ-very ample* if for any finite subscheme $Z \subset S$ of length $\delta + 1$, the map $H^0(S, L) \to H^0(Z, L|_Z)$ is surjective [BS91]. For a line bundle $\mathcal{L}$ on a smooth family of surfaces $\mathcal{S} \to B$, consider the following ampleness hypotheses, which we denote by **AMP**:

- For every $b \in B$, the line bundle $\mathcal{L}_b = \mathcal{L}|_{\mathcal{S}_b}$ on $\mathcal{S}_b$ is δ-very ample.
- The dimension of the vector spaces $H^0(\mathcal{S}_b, \mathcal{L}_b)$ is locally constant on B.

Now let $\mathcal{C}$ and $q\colon \mathcal{S} \to B$ be given as above, and let $\mathcal{L} = \mathcal{O}(\mathcal{C})$. Without making any assumptions on the dimensions of the equisingular strata, **AMP** guarantees that the class $\gamma(\mathcal{C}) \cap [B]$ is supported on the locus of curves with δ-invariant $\geq \delta$. If B is equidimensional, it will follow (Proposition 1.4.5) that $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural effective cycle with support $\overline{B(\delta)}$ if we assume the following hypotheses, which we denote by **DIM**:

- The locus of δ-nodal curves, if non-empty, has codimension δ.
- The loci of curves of the following type have codimension $> \delta$:
  - Curves with δ-invariant $> \delta$;
  - Curves with δ-invariant $= \delta$, but with singularities other than nodes;
  - Non-reduced curves.

As explained in [KP99], for an equisingularity type $\mathcal{D}$ we have *cod*$(\mathcal{D}) \geq \delta(\mathcal{D})$, with equality only for δ-nodal curves. It follows that **DIM** is slightly weaker than $\mathbf{DIM_{KP}}$. To summarize, we will prove the following theorem:

Theorem 1.A. *Let B be a scheme and fix an integer δ. Let $\mathcal{C}$ be a relative effective divisor on a flat family of smooth surfaces $\mathcal{S} \to B$. Then the class $\gamma(\mathcal{C})$ can be expressed universally as a polynomial of degree δ in classes of the form*

$$\epsilon(a, b, c) = q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)\,.$$

Now assume that B is complete of pure dimension n, that the line bundle $\mathcal{O}_{\mathcal{S}}(\mathcal{C})$ satisfies **AMP**, *and moreover assume that $\mathcal{C} \to B$ satisfies* **DIM**. *Then the class $\gamma(\mathcal{C}) \cap [B] \in A_{n-\delta}(B)$ is the class of a natural cycle on B with support $\overline{B(\delta)}$.*

Remark 1.1.7. The natural cycle in the theorem is constructed in Section 1.4. See Proposition 1.4.5.

The conjecture of Kleiman and Piene is a family version of the Göttsche conjecture [Göt98]. For a sufficiently ample line bundle L on a smooth surface S, the latter asserts that the degree of the Severi locus of δ-nodal curves in the complete linear system $|L|$, is given by a universal polynomial in the numbers L^2, $(L.K)$, K^2 and $c_2(S)$, for K the canonical divisor on S. Equivalently, the number of δ-nodal curves in a general linear system $\mathbb{P}^\delta \subset |L|$ is given by such a polynomial.

The Göttsche conjecture was first proved using algebraic methods by Tzeng in [Tze12]. Other proofs were given in [Liu00], [Kaz03] and [KST11]. In [LT14] and [Ren17] the result is generalized to other singularity types.

Our theorem implies the Göttsche conjecture for δ-very ample L, but it is not independent from existing results. In fact, our method can be seen as a family version of the proof in [KST11]. Moreover, sharper results are known in terms of the required ampleness [KS13].

1.1.4 Application to plane curves in $\mathbb{P}^3$

The motivation for the project was the following counting problem. For fixed integers $\delta \geq 0$ and $d > 1$ write

$$n = \frac{d(d+3)}{2} + 3 - \delta\,.$$

and consider lines $\ell_1, \ldots, \ell_n \subset \mathbb{P}^3$. The space of curves of degree d that lie on a plane in $\mathbb{P}^3$ and that intersect the lines $\ell_1, \ldots, \ell_n$ has expected dimension δ. We will show that, if we choose the lines $\ell_1, \ldots, \ell_n$ sufficiently general, the subspace of δ-nodal curves is finite (and reduced, as a scheme). For $d \geq \delta$, we can use our method to calculate the number $N_{\delta,d}$ of δ-nodal plane curves of degree d intersecting the lines $\ell_1, \ldots, \ell_n$.

Let $\mathcal{C} \to B$ be the universal plane curve of degree d in $\mathbb{P}^3$. We will show in Proposition 1.6.3 that for $d \geq \delta$, we have $N_{\delta,d} = \gamma(\mathcal{C}) \cap [B_{\ell_1,\ldots,\ell_n}]$, in which $B_{\ell_1,\ldots,\ell_n} \subset B$ is the closed subvariety of curves intersecting the general lines $\ell_1, \ldots, \ell_n$. We use this to prove our second main result:

Theorem 1.B. *Let $\delta \geq 0$. The number of planar δ-nodal curves of degree $d \geq \max\{1, \delta\}$ in $\mathbb{P}^3$ intersecting $n = \frac{d(d+3)}{2} + 3 - \delta$ general lines is given by a polynomial $N_\delta(d)$ in d of degree $\leq 9 + 2\delta$. Moreover, for $\delta \leq 12$ these polynomials are the ones given in Section 1.9.*

Remark 1.1.8. Our computations suggest that $N_\delta(d)$ has degree *exactly* $9 + 2\delta$, with leading coefficient $\frac{3^\delta}{162\,\delta!}$, but we do not prove this.

1.2 Preliminaries

1.2.1 Chern-Schwartz-MacPherson classes

To any constructible function f on a complete scheme X, one can assign a class $c_{SM}(f)$ in the Chow group of X, called the *Chern-Schwartz-MacPherson class* of f. The existence of well-behaved Chern classes for constructible functions was conjectured by Deligne and Grothendieck and proved by MacPherson in [Mac74]. Several other constructions are known. See [Alu06] for an overview and a new construction in a more general set-up.

For a subset V of a scheme X, write $1_V\colon X \to \mathbb{Z}$ for the function with constant value 1 on its support V. Recall that a constructible function is a map $f\colon X \to \mathbb{Z}$ that can be written as a finite sum

$$f = \sum_{i\in I} \alpha_i 1_{V_i}$$

with $\alpha_i \in \mathbb{Z}$ and $V_i \subset X$ closed. Let $F(X)$ be the group of constructible functions on X. For a proper morphism $g : X \to Y$, there is a homomorphism $g_*\colon F(X) \to F(Y)$ given by

$$g_*(1_V)(y) = e(V \cap g^{-1}(y)), \quad y \in Y\,,$$

in which $V \subset X$ is closed and e is the topological Euler characteristic. For a scheme X let $A_*(X)$ denote the Chow group of X. In the following theorem, we will view A_* as a (covariant) functor on the category of complete schemes with proper morphisms to the category of abelian groups. Let c denote the total Chern class.

Theorem 1.2.1 (MacPherson)**.** *There is a unique natural transformation*

$$c_{SM}\colon F \Rightarrow A_*$$

satisfying $c_{SM}(1_X) = c(T_X)$ for X smooth projective.

Remark 1.2.2. The uniqueness of such a natural transformation follows from resolution of singularities. MacPherson proved the naturality of the homology class, but in fact his argument gives this stronger result (see [Ful98, 19.1.7]).

Definition 1.2.3. *For a complete scheme X and a constructible function $f \in F(X)$, we call $c_{SM}(f) = c_{SM}(X)(f)$ the* Chern-Schwartz-MacPherson class *of f. We also write*

$$c_{SM}(X) = c_{SM}(1_X)$$

and call this class the Chern-Schwartz-MacPherson class of X.

It follows directly from the definitions that for a complete scheme X, we have

$$\int c_{SM}(X) = \pi_*(1_X) = e(X)$$

with $\pi : X \to \{*\}$ the morphims to a point. At the other extreme, we have the following lemma.

Lemma 1.2.4. *Let X be a scheme and let $V \subset X$ be a locally closed subset of dimension n and let $\overline{V}$ be its closure (with the reduced scheme structure) . Then*

$$c_{SM}(1_V) = [\overline{V}] + \textit{cycles of dimension} < n\,.$$

Proof. Let $g\colon \widetilde{V} \to \overline{V}$ be a birational morphism from a nonsingular projective variety $\widetilde{V}$. Let $U \subset \overline{V}$ be a dense open over which g is an isomorphism and let $Z = \overline{V}\backslash U$ be its complement. Write

$$\partial V = \overline{V}\backslash V$$

for the boundary of V. Then the we have

$$\begin{aligned} g_* 1_{\widetilde{V}} &= 1_{\overline{V}} + f \\ &= 1_V + 1_{\partial V} + f \end{aligned}$$

with f a constructible function on Z. On the other hand, we have

$$c_{SM}(1_{\widetilde{V}}) = c(T_{\widetilde{V}}) = [\widetilde{V}] + \text{cycles of dimension} < n\,.$$

Since the functions f and $1_{\partial V}$ are supported on closed subsets of dimension $< n$, by naturality the same holds for their Chern-Schwartz-Macpherson classes. It follows that we have

$$\begin{aligned} c_{SM}(1_V) &= g_* c_{SM}(\widetilde{V}) - c_{SM}(f) - c_{SM}(1_{\partial V}) \\ &= [\overline{V}] + \text{cycles of dimension} < n\,. \end{aligned}$$

□

1.2.2 Hilbert schemes of points

Let S be a smooth projective surface, and let $S^{[n]}$ be the Hilbert scheme of n points on S. Let $\mathcal{Z}$ be the universal subscheme of length n, with natural morphisms

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\ i\ } & S \\ \big\downarrow{\scriptstyle \pi} & & \\ S^{[n]} & & \end{array}$$

For a vector bundle F on S, we write $F^{[n]} := \pi_* i^* F$ for the tautological vector bundle on $S^{[n]}$ with fibre

$$F^{[n]}\big|_{[Z]} = H^0(Z, F|_Z)$$

over a point $[Z] \in S^{[n]}$.

In [EGL01], Ellingsrud, Göttsche and Lehn describe a method to calculate certain tautological integrals on $S^{[n]}$. In fact, they give a constructive proof of the following theorem:

Theorem 1.2.5 (Ellingsrud, Göttsche, Lehn)**.** *Let $F_1, \dots, F_l$ be vector bundles on S of respective ranks $r_1, \dots, r_l$. Let P be a polynomial in the Chern classes of $T_{S^{[n]}}$ and the Chern classes of the bundles $F_i^{[n]}$. Then there is a universal polynomial Q, depending only on P, in numbers*

$$\int_S p(T_S, F_1, \dots, F_l)\,,$$

in which p is a polynomial in the Chern classes of T_S, the ranks r_i and the Chern classes of the bundles F_i, such that

$$\int_{S^{[n]}} P = Q\,.$$

Now let B be a base scheme and let $q\colon \mathcal{S} \to B$ be proper and smooth of relative dimension 2. Let $\mathcal{S}_B^{[i]} = \mathrm{Hilb}^i(\mathcal{S}/B)$ denote the relative Hilbert scheme of i points on the fibres of $\mathcal{S} \to B$, with structure morphism $q^{[i]}\colon \mathcal{S}_B^{[i]} \to B$. For a vector bundle $\mathcal{F}$ on $\mathcal{S}$ we can define as above the tautological bundle $\mathcal{F}_B^{[i]} = \pi_* i^* \mathcal{F}$ on $\mathcal{S}_B^{[i]}$, in which π and i are the natural morphisms in

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\;i\;} & \mathcal{S} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle q} \\ \mathcal{S}_B^{[i]} & \xrightarrow{\;q^{[i]}\;} & B \end{array}$$

from the universal length n subscheme $\mathcal{Z}$ on the fibres of $\mathcal{S} \to B$. It restricts to

$$\mathcal{F}_B^{[i]}\Big|_{\mathcal{S}_b^{[i]}} = (\mathcal{F}|_{\mathcal{S}_b})^{[i]}$$

on the fibre $\mathcal{S}_b^{[i]}$ over a point $b \in B$.

The following is proved in [AIK77] and generalizes a well known result by Fogarty:

Lemma 1.2.6. *The morphism $q^{[i]}\colon \mathcal{S}_B^{[i]} \to B$ is smooth of relative dimension $2i$.*

Now let $\mathcal{S} \to B$ be given as above, and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor.

Lemma 1.2.7. *The Hilbert scheme $\mathcal{C}_B^{[i]}$ is cut out regularly (in particular, in the expected codimension i) by the canonical section of the vector bundle $\mathcal{O}(\mathcal{C})_B^{[i]}$ on $\mathcal{S}_B^{[i]}$. Moreover, the morphism $\mathcal{C}_B^{[i]} \to B$ is flat.*

Proof. We follow [AIK77]. Let $b \in B$ be an arbitrary point and consider the fibres $C = \mathcal{C}_b$ and $S = \mathcal{S}_b$ over b. By [Gro66, §11.3.8], it suffices to check that the Hilbert scheme $C^{[i]}$ is cut out regularly by the canonical section of the bundle $\mathcal{O}(C)^{[i]}$ on $S^{[i]}$. Since $S^{[i]}$ is smooth, we only need to check that for a divisor $C \subset S$ on a smooth surface S, the Hilbert scheme $C^{[i]}$ has the expected dimension i. But this can be seen by inspection of the fibres of the Hilbert-Chow morphism $C^{[i]} \to C^{(i)}$. In fact, by [Iar72], the locus in $S^{[i]}$ of subschemes of length i supported at a point has dimension $i-1$. From this it follows directly that the locus in $C^{(i)}$ over which the fibres of the morphism $C^{[i]} \to C^{(i)}$ have dimension r, has codimension $\geq r$. □

1.3 The smooth case

Let B be a scheme. Let $\mathcal{S} \to B$ be smooth projective of relative dimension 2 and let $\mathcal{C} \subset \mathcal{S}$ be a relative effective divisor. For a fixed δ, let

$$n_{g-\delta}^{cl} = n_{g-\delta}^{cl}(\mathcal{C}) \in A^*(B)$$

be the class defined by equation (1.1.5) in the introduction. The following situation is the model for our results.

Proposition 1.3.1. *Assume B is* projective *and that the relative Hilbert schemes of points $\mathcal{C}_B^{[i]}$ for $i = 0, \ldots, \delta$ are non-singular. (So in particular B is non-singular). Then we have the following identity in $A_*(B)$:*

$$c(T_B)\, n^{cl}_{g-\delta} \cap [B] = c_{\mathrm{SM}}(n_{g-\delta})\,. \tag{1.3.2}$$

In particular, the class $n^{cl}_{g-\delta} \cap [B]$ is supported on the locus of curves that have δ-invariant $\geq \delta$ or are non-reduced, i.e. it is the push forward of a class on this locus. If B is of pure dimension n and the family of curves $\mathcal{C} \to B$ satisfies **DIM***, we find*

$$n^{cl}_{g-\delta} \cap [B] = \left[\overline{B(\delta)}\right] + \beta$$

with β a sum of cycles of dimension $< n - \delta$.

Proof. By the defining equation (1.1.4), the constructible function $n_{g-\delta}$ is a linear combination of the terms $p^{[i]}_*(1_{\mathcal{C}_B^{[i]}})$. It is easy to see that only the terms with $i = 0, \ldots, \delta$ are involved. Similarly, $n^{cl}_{g-\delta}$ is a linear combination (with the same coefficients) of the classes $p^{[i]}_*(c(T_{\mathcal{C}_B^{[i]}/B}))$, with $i = 0 \ldots \delta$. Therefore it suffices to verify that relation (1.3.2) holds for the first $\delta + 1$ terms in the left hand sides of (1.1.4) and (1.1.5).

Recall that we have defined the class $T_{\mathcal{C}_B^{[i]}/B}$ in the Grothendieck group $K(\mathcal{C}_B^{[i]})$ by

$$T_{\mathcal{C}_B^{[i]}/B} = T_{\mathcal{S}_B^{[i]}/B}\Big|_{\mathcal{C}_B^{[i]}} - \mathcal{O}(\mathcal{C})^{[i]}\big|_{\mathcal{C}_B^{[i]}}\,.$$

For $i = 0, \ldots, \delta$, we have by Lemma 1.2.6 and by Lemma 1.2.7 the following relations in $K(\mathcal{S}_B^{[i]})$ and $K(\mathcal{C}_B^{[i]})$ respectively:

$$\begin{aligned} T_{\mathcal{S}_B^{[i]}} &= T_{\mathcal{S}_B^{[i]}/B} + T_B\,; \\ T_{\mathcal{S}_B^{[i]}}\Big|_{\mathcal{C}_B^{[i]}} &= T_{\mathcal{C}_B^{[i]}} + \mathcal{O}(\mathcal{C})_B^{[i]}\Big|_{\mathcal{C}_B^{[i]}}\,. \end{aligned}$$

It follows that

$$T_{\mathcal{C}_B^{[i]}/B} = T_{\mathcal{C}_B^{[i]}} - T_B$$

and hence

$$c(T_{\mathcal{C}_B^{[i]}/B}) = c(T_{\mathcal{C}_B^{[i]}})\, c(T_B)^{-1} \in A^*(\mathcal{C}_B^{[i]})\,.$$

By the defining properties of Chern-Schwartz-MacPherson classes (Theorem 1.2.1) we obtain

$$\begin{aligned} c_{SM}(p^{[i]}_*(1_{\mathcal{C}_B^{[i]}})) &= p^{[i]}_* c_{SM}(1_{\mathcal{C}_B^{[i]}}) \\ &= p^{[i]}_*\Big(c(T_{\mathcal{C}_B^{[i]}}) \cap [\mathcal{C}_B^{[i]}]\Big) \\ &= p^{[i]}_*\Big(c(T_B)\, c(T_{\mathcal{C}_B^{[i]}/B}) \cap [\mathcal{C}_B^{[i]}]\Big) \\ &= c(T_B)\, p^{[i]}_*\Big(c(T_{\mathcal{C}_B^{[i]}/B})\Big) \cap [B]\,. \end{aligned}$$

By Theorem 1.1.3 the support of the constructible function $n_{g-\delta}$ is lies in locus in B over which the curves in the family $\mathcal{C} \to B$ have δ-invariant $\geq \delta$. By the functoriality of the Chern-Schwartz-MacPherson class, the cycle class

$$c(T_B)\, n^{cl}_{g-\delta} \cap [B]$$

is the push-forward of a cycle class on this locus. Since $c(T_B)$ is invertible, the same holds for $n^{cl}_{g-\delta} \cap [B]$. By [PT10, Prop. 3.23], the function $n_{g-\delta}$ has constant value 1 on the locus of δ-nodal curves. If $\mathcal{C}$ satisfies **DIM**, it follows that the support of the constructible function $n_{g-\delta} - 1_{B(\delta)}$ lies in a closed subset of codimension $> \delta$. Hence the last assertion follows from Lemma 1.2.4. □

Example 1.3.3. In [KST11] it is shown that both conditions of the proposition are satisfied by the universal curve $\mathcal{C} \to |L|$ over the linear system of a δ-very ample line bundle L on a smooth surface S, i.e., $\mathcal{C}$ satisfies **DIM**, and the relative Hilbert schemes $\mathcal{C}^{[i]}_{|L|}$ are non-singular for $i \leq \delta$. By Bertini's theorem it then follows that the same holds for the restriction $\mathcal{C}_{\mathbb{P}^\delta}$ of the universal curve to a general linear system $\mathbb{P}^\delta \subset |L|$. In particular the set $\mathbb{P}^\delta(\delta)$ is finite, and it follows that the degree

$$\int_{\mathbb{P}^\delta} n^{cl}_{g-\delta}(\mathcal{C}_{\mathbb{P}^\delta}) \cap [\mathbb{P}^\delta] = \int_{\mathbb{P}^\delta} \left[\mathbb{P}^\delta(\delta)\right]$$

equals the number of δ-nodal curves in the linear system $\mathbb{P}^\delta$.

Remark 1.3.4. It should be noted that the integrals differ slightly from the ones in [KST11]. In fact, in loc. cit. the authors consider a linear combination of the integrals

$$\int_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}} c\left(T_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}}\right) \tag{1.3.5}$$

whereas we use the relative (virtual) tangent bundles

$$T_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}/\mathbb{P}^\delta} = T_{\mathcal{C}^{[i]}_{|L|}/|L|}\Big|_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}}$$

and consider the integrals

$$\int_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}} c\left(T_{\mathcal{C}^{[i]}_{\mathbb{P}^\delta}/\mathbb{P}^\delta}\right) \tag{1.3.6}$$

Interestingly, (1.3.5) does not equal (1.3.6) in general, but after taking the BPS linear combination of the integrals for $i = 0, \ldots, \delta$, they both calculate the number of δ nodal curves in the linear system $\mathbb{P}^\delta$.

More generally, let B be a scheme and let $p\colon \mathcal{S} \to B$ be smooth projective of relative dimension 2. Let $\mathcal{L}$ be a line bundle on $\mathcal{S}$ and assume $\mathcal{L}$ satisfies **AMP**. Then $p_*\mathcal{L}$ is a vector bundle and the fiber of the projective bundle

$$|\mathcal{L}/B| := \mathbb{P}(p_*\mathcal{L}) = \operatorname{Proj}(\operatorname{Sym}((p_*\mathcal{L})^*)) \to B$$

over a point $b \in B$ is the δ-very ample complete linear system $|\mathcal{L}_b|$. Consider the family

$$\mathcal{S}_{|\mathcal{L}/B|} = \mathcal{S} \times_B |\mathcal{L}/B| \to |\mathcal{L}/B|$$

and let $\mathcal{C} \subset \mathcal{S}_{|\mathcal{L}/B|}$ be the universal divisor.

Lemma 1.3.7. *The family of curves $\mathcal{C} \to |\mathcal{L}/B|$ satisfies* **DIM**. *If B is smooth, the Hilbert schemes $\mathcal{C}^{[i]}_{|\mathcal{L}/B|}$ are non-singular for $i \leq \delta$.*

Proof. The first statement can be checked fibrewise over B. By Lemma 1.2.7, the Hilbert schemes of points $\mathcal{C}^{[i]}_{|\mathcal{L}/B|}$ are flat over $|\mathcal{L}/B|$, and hence over B. Hence, if B is smooth, also the second statement can be checked fibrewise over B. Hence, the lemma follows from Example 1.3.3. □

1.4 Functoriality and support

Let B be a scheme. Let $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $q\colon \mathcal{S} \to B$. Let $f\colon B' \to B$ be a morphism and consider the relative effective divisor $\mathcal{C}_{B'} = \mathcal{C} \times_B B'$ on the smooth family of surfaces

$$\mathcal{S}_{B'} = \mathcal{S} \times_B B' \to B' .$$

Then we have the following:

Lemma 1.4.1. *i) $f^*(n^{cl}_{g-\delta}(\mathcal{C})) = n^{cl}_{g-\delta}(\mathcal{C}_{B'})$ in $A^*(B')$;*

ii) $f_*(n^{cl}_{g-\delta}(\mathcal{C}_{B'}) \cap [B']) = n^{cl}_{g-\delta}(\mathcal{C}) \cap f_*[B']$ in $A_*(B)$.

Proof. It suffices to show the relations for the LHS of equation (1.1.5). Note that we have Cartesian squares

$$\begin{array}{ccc} \mathcal{C}^{[i]}_{B'} & \xrightarrow{f_i} & \mathcal{C}^{[i]}_{B} \\ \downarrow{\scriptstyle p^{[i]}_{B'}} & & \downarrow{\scriptstyle p^{[i]}_{B}} \\ B' & \xrightarrow{f} & B . \end{array}$$

By flatness of the vertical maps, we have

$$\begin{aligned} f^* p^{[i]}_{B*} c(T_{\mathcal{C}^{[i]}_B/B}) &= p^{[i]}_{B'*} f_i^* c(T_{\mathcal{C}^{[i]}_B/B}) \\ &= p^{[i]}_{B'*} c(f_i^* T_{\mathcal{C}^{[i]}_B/B}) \\ &= p^{[i]}_{B'*} c(T_{\mathcal{C}^{[i]}_{B'}/B'}) . \end{aligned}$$

The second part follows from the projection formula. □

Let $\mathcal{C}$ be given as above. Write $\mathcal{L} = \mathcal{O}(\mathcal{C})$ and assume $\mathcal{L}$ satisfies **AMP**. As in the previous section, we can form the projective bundle $|\mathcal{L}/B| := \mathbb{P}(q_*\mathcal{L})$ over B, with fibre over b the δ-very ample complete linear system $|\mathcal{L}_b|$. Let $\mathcal{C}'' \subset \mathcal{S}_{|\mathcal{L}/B|}$ denote the universal relative effective divisor on the family $\mathcal{S}_{|\mathcal{L}/B|} \to |\mathcal{L}/B|$. The bundle $\pi\colon |\mathcal{L}/B| \to B$ has a canonical section

$$\begin{array}{c} |\mathcal{L}/B| \\ {\scriptstyle \pi}\downarrow \uparrow{\scriptstyle s} \\ B \end{array} \tag{1.4.2}$$

induced by the inclusion $\mathcal{O}(-\mathcal{C}) \hookrightarrow \mathcal{O}$, and the divisor $\mathcal{C}$ is obtained by restricting the divisor $\mathcal{C}''$.

Lemma 1.4.3. *Assume that B is equidimensional, and let $W \subset \overline{|\mathcal{L}/B|(\delta)}$ be an irreducible component. Then $\pi(W)$ is an irreducible component of B.*

Proof. Clearly $\pi(W)$ is closed and irreducible, so it suffices to show that we have $\dim(\pi(W)) = \dim(B)$. Write W° for the open $W \cap |\mathcal{L}/B|(\delta)$ of W. By Lemma 1.3.7 and the first statement of Theorem 1.1.1[2], $|\mathcal{L}/B|(\delta)$ has pure codimension δ. Hence $W^\circ \subset |\mathcal{L}/B|$ has codimension δ. By Example 1.3.3, a fibre W_b° over $b \in B$ is empty, or has codimension δ in $|\mathcal{L}/B|_b = |\mathcal{L}_b|$. It follows that

$$\begin{aligned}\dim(\pi(W)) &= \dim(\pi(W^\circ)) \\ &= \dim(|\mathcal{L}/B|) - \delta - (\dim(|\mathcal{L}_b|) - \delta) \\ &= \dim(B)\,.\end{aligned}$$

□

Definition 1.4.4. *Define a cycle*

$$U(\delta) := \sum_W l(\mathcal{O}_{\pi(W),B})\,[W] \in A_*(|\mathcal{L}/B|)$$

in which the sum is taken over the irreducible components of $\overline{|\mathcal{L}/B|(\delta)}$. Here the multiplicity $l(\mathcal{O}_{\pi(W),B})$ denotes the length of the local ring along the subvariety $\pi(W)$ of B.

As in the introduction, we will use the notation

$$\gamma(\mathcal{C}) = \{n^{cl}_{g-\delta}(\mathcal{C})\}_\delta \in A^\delta(B)\,.$$

Proposition 1.4.5. *Let B be a* complete *base scheme, and $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $\mathcal{S} \to B$. Assume the line bundle $\mathcal{L} = \mathcal{O}(\mathcal{C})$ satisfies* **AMP**. *Then we have*

$$n^{cl}_{g-\delta}(\mathcal{C}) \in A^{\geq\delta}(B)\,.$$

Now assume that B is equidimensional, and let s be given as in (1.4.2). *Then we have*

$$\gamma(\mathcal{C}) \cap [B] = s^!(U(\delta))\,.$$

In particular, $\gamma(\mathcal{C}) \cap [B]$ is supported on the locus of curves with δ-invariant $\geq \delta$. Finally, if moreover $\mathcal{C}$ satisfies **DIM**, *then $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural cycle with support $\overline{B(\delta)}$.*

Proof. Assume that B is equidimensional. Without loss of generality, we may assume that B is connected so that $H^0(\mathcal{S}_b, \mathcal{L}_b)$ is constant and $|\mathcal{L}/B|$ equidimensional. By resolution of singularities, there is a proper surjective morphism $f\colon B' \to B$ from a smooth projective scheme B' with $f_*[B'] = [B]$. In fact, we consider the union of $l(\mathcal{O}_{Z,B})$ copies of Z for each irreducible component $Z \subset B$. Then we choose for each component a birational morphism from a smooth projective variety.

[2]Our assumptions are slightly weaker, but the argument given in [KP04] still holds if we replace $\mathbf{DIM_{KP}}$by **DIM**.

Let $\mathcal{C}'$, $\mathcal{S}'$ and $\mathcal{L}'$ be the base changes of $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{L}$ along f. Then

$$\mathcal{L}' = f_{\mathcal{S}}^* \mathcal{O}_{\mathcal{S}}(\mathcal{C}) = \mathcal{O}_{\mathcal{S}'}(\mathcal{C}')$$

satisfies **AMP** and we have the following diagram with Cartesian squares:

$$\begin{array}{ccc} \mathcal{C}''' & \longrightarrow & \mathcal{C}'' \\ \downarrow & & \downarrow \\ |\mathcal{L}'/B'| & \xrightarrow{f_\pi} & |\mathcal{L}/B| \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \uparrow{\scriptstyle s} \\ B' & \xrightarrow{f} & B\,, \end{array}$$

in which $\mathcal{C}'''$ and $\mathcal{C}''$ are the universal divisors on $\mathcal{S}'_{|\mathcal{L}'/B'|}$ and $\mathcal{S}_{|\mathcal{L}/B|}$ in the linear systems $|\mathcal{L}'/B'|$ and $|\mathcal{L}/B|$ respectively.

Note that $(f_\pi)_*[|\mathcal{L}'/B'|] = [|\mathcal{L}/B|]$. By Lemma 1.3.7 and Proposition 1.3.1 we have

$$n_{g-\delta}^{cl}(\mathcal{C}''') \cap |\mathcal{L}'/B'| = \left[\overline{|\mathcal{L}'/B'|(\delta)}\right] + \alpha \in A_*(|\mathcal{L}'/B'|)\,,$$

with α a sum of classes with codimension $> \delta$. We have (set-theoretically)

$$f_\pi(\overline{|\mathcal{L}'/B'|(\delta)}) = \overline{|\mathcal{L}/B|(\delta)}\,.$$

Now let W be an irreducible component of $\overline{|\mathcal{L}/B|(\delta)}$. We will prove that the multiplicity of W in $(f_\pi)_*([\overline{|\mathcal{L}'/B'|(\delta)}])$ is given by $l(\mathcal{O}_{\pi(W),B})$ ($\star$). By Lemma 1.4.3, $\pi(W)$ (with the reduced scheme structure) is an irreducible component of B. Any irreducible component W' of $\overline{|\mathcal{L}'/B'|(\delta)}$ that maps onto W lies over one of the $l(\mathcal{O}_{\pi(W),B})$ copies of a resolution of singularities

$$\rho \colon \widetilde{\pi(W)} \to \pi(W)\,,$$

so W' lies in $W \times_{\pi(W)} \widetilde{\pi(W)}$. Let $U \subset \pi(W)$ be an open over which ρ is an isomorphism. Then

$$W \times_{\pi(W)} \rho^{-1}(U) \to W \cap \pi^{-1}(U)$$

is an isomorphism. Since W' is irreducible and maps onto W, we have

$$W' = \overline{W \times_{\pi(W)} \rho^{-1}(U)}$$

and the morphism $W' \to W$ is generically of degree 1. We have now proved $\star$. It follows that

$$(f_\pi)_*\left(\left[\overline{|\mathcal{L}'/B'|(\delta)}\right]\right) = U(\delta)\,.$$

Hence, by Lemma 1.4.1 we have

$$\begin{aligned} n_{g-\delta}^{cl}(\mathcal{C}) \cap [B] &= s^*(n_{g-\delta}^{cl}(\mathcal{C}'')) \cap [B] \\ &= s^!(n_{g-\delta}^{cl}(\mathcal{C}'') \cap [|\mathcal{L}/B|]) \\ &= s^!(n_{g-\delta}^{cl}(\mathcal{C}'') \cap (f_\pi)_*[|\mathcal{L}'/B'|]) \\ &= s^!(f_\pi)_*(n_{g-\delta}^{cl}(\mathcal{C}''') \cap [|\mathcal{L}'/B'|]) \\ &= s^!(f_\pi)_*\left(\left[\overline{|\mathcal{L}'/B'|(\delta)}\right]\right) + \beta \\ &= s^!\,(U(\delta)) + \beta \end{aligned}$$

with $\beta \in A_*(B)$ a sum of classes with codimension $> \delta$. In particular we find

$$\gamma(\mathcal{C}) \cap [B] = s^! \left(U(\delta) \right) .$$

If $\mathcal{C}$ satisfies **DIM**, it follows that $s(B)$ and $\overline{|\mathcal{L}/B|(\delta)}$ intersect properly in $|\mathcal{L}/B|$, i.e. in dimension $\dim B - \delta$, and we have set theoretically

$$s^{-1}\left(\overline{|\mathcal{L}/B|(\delta)}\right) = \overline{B(\delta)} . \tag{1.4.6}$$

To see this, note that since $|\mathcal{L}/B|(\delta)$ has codimension δ in $|\mathcal{L}/B|$, an irreducible component of $s^{-1}\left(\overline{|\mathcal{L}/B|(\delta)}\right)$ has codimension $\leq \delta$. On the other hand, it consists of curves with δ-invariant $\geq \delta$. So by **DIM**, it has pure codimension δ in B. Moreover, the set

$$s^{-1}\left(\overline{|\mathcal{L}/B|(\delta)}\right) \Big\backslash \, B(\delta)$$

consists of curves with δ-invariant $\geq \delta$ that are not δ-nodal. By **DIM**, it has codimension $> \delta$. Hence $B(\delta)$ lies dense in $s^{-1}\left(\overline{|\mathcal{L}/B|(\delta)}\right)$, proving equation (1.4.6).

It follows that $\gamma(\mathcal{C}) \cap [B]$ is a the class of a natural effective cycle with support equal to $\overline{B(\delta)}$.

Now let B be any complete scheme and let $V \to B$ be a morphism from an n-dimensional variety V. By the above we have

$$n^{cl}_{g-\delta}(\mathcal{C}) \cap [V] = n^{cl}_{g-\delta}(\mathcal{C}_V) \cap [V] \in A_{\leq n-\delta}(V)$$

and hence we have an equality of bivariant classes

$$n^{cl}_{g-\delta}(\mathcal{C}) = \gamma(\mathcal{C}) + \alpha \in A^*(B)$$

with $\alpha \in A^*(B)$ a sum of classes of degree $> \delta$. □

1.5 Universality: relative EGL

Let B be a scheme, and let $\mathcal{C}$ be a relative effective divisor on a smooth family of surfaces $q \colon \mathcal{S} \to B$. The arithmetic genus of a curve in the family $p \colon \mathcal{C} \to B$ is denoted by g, which we view as a locally constant function on B. We consider the transformation of power series (1.1.5) and rewrite it as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} p_*^{[n]}(c(T_{\mathcal{C}^{[n]}/B})\, q^n &= \sum_{r=-\infty}^{g} n^{cl}_r \, q^{g-r}(1-q)^{2r-2} \\
&= \sum_{i=0}^{\infty} n^{cl}_{g-i} \, q^i (1-q)^{2(g-i)-2} \\
&= \sum_{i=0}^{\infty} n^{cl}_{g-i} \, q^i \sum_{j=0}^{\infty} (-q)^j \binom{2(g-i)-2}{j} \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} n^{cl}_{g-i} \, (-1)^{n-i} \binom{2(g-i)-2}{n-i} q^n .
\end{aligned}$$

It follows that we can write

$$n^{cl}_{g-\delta} = \sum_{i=0}^{\delta} a_i\, p^{[i]}_*(c(T_{\mathcal{C}^{[i]}_B/B}))\,, \tag{1.5.1}$$

in which the a_i are polynomials of degree $\delta - i$ in g, depending only on δ and i. In fact, $a_i = a_{i\delta}$ can be found by inverting the upper triangular matrix with 1's on the diagonal

$$\left[(-1)^{j-i}\binom{2(g-i)-2}{j-i}\right]_{0\le i,j\le\delta}\,.$$

Proposition 1.5.2. *The class $\gamma(\mathcal{C})$, can be expressed universally as a polynomial of degree δ in the classes*

$$\epsilon(a,b,c) := q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)\,,$$

in which q_* denotes the Gysin push-forward.

We prove the existence of a universal polynomial in Lemma 1.5.3. In Lemma 1.5.6 we compute its degree.

Lemma 1.5.3. *There exists a polynomial as in the proposition of degree $\le \delta$ in the classes $\epsilon(a,b,c)$.*

Proof. We can view g as an element of $A^0(B)$. In fact we have

$$\begin{aligned} 2g-2 &= p_*(c_1(T^\vee_{\mathcal{C}/B})) \\ &= p_*(c_1(\mathcal{O}(\mathcal{C})|_{\mathcal{C}}) - c_1(T_{\mathcal{S}/B}|_{\mathcal{C}})) \\ &= q_*((c_1(\mathcal{O}(\mathcal{C})))^2 - c_1(\mathcal{O}(\mathcal{C}))c_1(T_{\mathcal{S}/B}))\,. \end{aligned}$$

By the equation (1.5.1), it suffices therefore to prove that the degree-δ parts of the classes $p^{[i]}_*(c(T_{\mathcal{C}^{[i]}_B/B}))$ can be expressed universally as polynomials of degree i in the classes $\epsilon(a,b,c)$. By Lemma 1.2.7, we have the equality

$$p^{[i]}_*(c(T_{\mathcal{C}^{[i]}_B/B})) = q^{[i]}_*\left(\frac{c(T_{\mathcal{S}^{[i]}_B/B})}{c(\mathcal{O}(\mathcal{C})^{[i]})} c_i(\mathcal{O}(\mathcal{C})^{[i]})\right)\,, \tag{1.5.4}$$

and hence the lemma follows by the following generalisation of Theorem 1.2.5. □

Let $q\colon \mathcal{S}\to B$ proper and and smooth relative dimension 2. Write

$$q^{[n]}\colon \mathcal{S}^{[n]}_B = \mathrm{Hilb}^n(\mathcal{S}/B)\to B \quad\text{and}\quad q^n\colon \mathcal{S}^n_B = \mathcal{S}\times_B\cdots\times_B\mathcal{S}\to B$$

for the structure morphisms.

Theorem 1.5.5. *Let $\mathcal{F}_1,\ldots,\mathcal{F}_l$ be vector bundles on $\mathcal{S}$ of respective ranks $r_1,\ldots,r_l$. Let P be a polynomial in the Chern classes of $T_{\mathcal{S}^{[n]}_B/B}$ and the Chern classes of the bundles $\mathcal{F}_i^{[n]}$. Then there is a universal polynomial Q, depending only on P, of degree $\le n$ in the classes in $A^*(B)$ of the form $q_*p(T_{\mathcal{S}/B},\mathcal{F}_1,\ldots,\mathcal{F}_l)$, with p a polynomial in the Chern classes of the bundles in the brackets and the ranks $r_1,\ldots,r_l$, such that we have*

$$q^{[n]}_* P = Q\,.$$

Proof. The argument given in [EGL01] directly generalises to the relative case. For the case $\mathcal{S} = S \times B$, see also [KT14], Section 4. In fact, Proposition 3.1 in [EGL01] still holds if we replace $S^{[n+1]} \times S^m$ by $\mathcal{S}_B^{[n+1]} \times_B \mathcal{S}_B^m$, the bundle $T_{S^{[i]}}$ by the relative tangent bundle $T_{\mathcal{S}_B^{[n]}/B}$ and the integrals by push-forward to B. It follows that we can find a universal polynomial $\widetilde{P}$ in the Chern classes of the sheaves $T_{\mathcal{S}/B}$, $\mathcal{O}_\Delta$ and the $\mathcal{F}_i$, pulled back along the several projections

$$pr_i \colon \mathcal{S}_B^n \to \mathcal{S} \quad \text{and} \quad pr_{ij} \colon \mathcal{S}_B^n \to \mathcal{S} \times_B \mathcal{S}\,,$$

such that we have an equation

$$q_*^{[n]} P = q_*^n \widetilde{P}\,.$$

By Grothendieck-Riemann-Roch we have

$$ch(\mathcal{O}_\Delta) = \Delta_*(td(-T_{\mathcal{S}/B}))$$

for the projections $pr_i \colon \mathcal{S} \times_B \mathcal{S} \to \mathcal{S}$. It follows that $\widetilde{P}$ is a polynomial in the Chern classes the bundles $pr_i^* T_{\mathcal{S}/B}$ and $pr_i^* \mathcal{F}_i$ and the classes $p_{ij}^*[\Delta]$. By the excess intersection formula, a product of classes of the latter form is a polynomial in Chern classes of the bundles $pr_i^* T_{\mathcal{S}/B}$, intersected with a product

$$\Delta^{k_1} \times \cdots \times \Delta^{k_m} \colon \mathcal{S}_B^m \hookrightarrow \mathcal{S}_B^n$$

of diagonals

$$\Delta^{k_i} \colon \mathcal{S} \hookrightarrow \mathcal{S}_B^{k_i}$$

for integers

$$k_1, \ldots, k_m \geq 1\,, \quad k_1 + \ldots + k_m = n\,.$$

It follows that $q_*^n \widetilde{P}$ is a sum of classes $q_*^m \widetilde{P}_m$ for $m = 1, \ldots, n$ and polynomials $\widetilde{P}_m$ in the Chern classes of the bundles $pr_i^* T_{\mathcal{S}/B}$ and $pr_i^* \mathcal{F}_i$, pulled back along the several projections $\mathcal{S}_B^m \to \mathcal{S}$. Now use the fact that for classes $\alpha_1, \ldots, \alpha_m \in A^*(\mathcal{S})$ we have

$$q_*\alpha_1 \cdots q_*\alpha_m = (q^m)_*(pr_1^*\alpha_1 \cdots pr_m^*\alpha_m)$$

in $A^*(B)$. □

Lemma 1.5.6. *The degree of the polynomial of Lemma 1.5.3 is* δ*.*

Proof. By Lemma 1.5.3, the class $\gamma(\mathcal{C})$ can be expressed universally as polynomial γ in classes $\epsilon(a, b, c)$ of degree $\leq \delta$. Now let $\mathcal{C}$ be the universal curve in a complete linear system $|L|$ on a surface S, and let $\mathbb{P}^\delta \subset |L|$ be a general linear system. Let $\omega \in A^1(\mathbb{P}^\delta)$ be the class of a hyperplane. As explained in the proof of [KST11, Thm. 4.1], the algorithm of [EGL01] applied to the right hand side of (1.5.4) for $i = \delta$ produces a term $c_2(S)^\delta/\delta!$ coming from the term

$$c_{2\delta}(T_{S^{[\delta]}})\omega^\delta = c_{2\delta}(T_{S^{[\delta]}_{\mathbb{P}^\delta}/\mathbb{P}^\delta})\omega^\delta\,.$$

As noted in Remark 1.3.4, the integrals in [KST11] differ by a factor $c(T_{\mathbb{P}^\delta})$. However, this does not affect the term $c_{2\delta}(T_{S^{[\delta]}})\omega^\delta$. It follows that γ is a polynomial of degree δ in classes the classes $\epsilon(a, b, c)$. □

Proof of Proposition 1.5.2. Combine Lemmas 1.5.3 and 1.5.6. □

This completes the proof of our first main result:

Proof of Theorem 1.A. Combine Proposition 1.4.5 and Proposition 1.5.2. □

1.5.1 Multiplicativity

We will check that the class γ has the expected multiplicative behaviour, cf. [KP04] and [Göt98]. Let B be a base scheme. For $k = 1, 2$, let $\mathcal{S}_k \to B$ be proper and smooth of relative dimension 2, and let $\mathcal{C}_k$ be a relative effective divisor on $\mathcal{S}_k$, and write $p_k \colon \mathcal{C}_k \to B$ for the morphism to B. Let $\mathcal{C}$ be the union

$$\mathcal{C} \coloneqq \mathcal{C}_1 \amalg \mathcal{C}_2 \subset \mathcal{S}_1 \amalg \mathcal{S}_2 \to B\,.$$

We have the following relations:

$$\begin{aligned} \mathcal{C}_B^{[n]} &= \coprod_{i+j=n} (\mathcal{C}_1)_B^{[i]} \times_B (\mathcal{C}_2)_B^{[j]}\,; \\ p_*^{[i]} c(T_{\mathcal{C}_B^{[n]}/B}) &= \sum_{i+j=n} p_{1*}^{[i]} c(T_{(\mathcal{C}_1)_B^{[i]}/B}) \cdot p_{2*}^{[i]} c(T_{(\mathcal{C}_2)_B^{[j]}/B})\,. \end{aligned} \tag{1.5.7}$$

For $k = 1, 2$, let g_k be the arithmetic genus of a curve in the family $\mathcal{C}_k \to B$, so we have

$$g - 1 = g_1 - 1 + g_2 - 1\,,$$

with g the genus of a curve in the family $\mathcal{C} \to B$. It follows easily from (1.5.7) that we have the identity

$$n_{g-\delta}^{cl}(\mathcal{C}) = \sum_{i+j=\delta} n_{g_1-i}^{cl}(\mathcal{C}_1)\, n_{g_2-j}^{cl}(\mathcal{C}_2)\,. \tag{1.5.8}$$

For any $i \geq 0$, let γ_i be the degree-i part of n_{g-i}^{cl}. We record the following lemma.

Lemma 1.5.9. *Let B be complete and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be given as above. Assume that for $k = 1, 2$, the line bundle $\mathcal{O}(\mathcal{C}_k)$ on $\mathcal{S}_k$ satisfies* **AMP**. *Then we have the relation*

$$\gamma_\delta(\mathcal{C}) = \sum_{i+j=\delta} \gamma_i(\mathcal{C}_1)\, \gamma_j(\mathcal{C}_2)$$

Proof. By Proposition 1.4.5, it follows directly from (1.5.8), by taking degree-δ parts on both sides of the equation. □

Remark 1.5.10. As was shown in [Göt98], this gives, in the case of the Göttsche conjecture, the generating series $\gamma = \sum \gamma_\delta\, q^\delta$ in terms of the Bell polynomials cf. [KP04]. In this case, there are four power series, corresponding to L^2, KL, K^2 and $c_2(S)$, that determine the generating series. In the general case, one needs to determine the coefficients of the classes $\epsilon(a, b, c)$ with $a + b + 2c = \delta + 2$ for each δ, rather than just the ones with $b + 2c \leq 2$. Expectedly, it suffices to evaluate γ on a sufficiently large class of examples to obtain the qualitative result.

1.6 Application: Plane Curves in $\mathbb{P}^3$

We will apply the results to the problem of counting δ-nodal plane curves of degree $d > 1$ in $\mathbb{P}^3$. As we will see below, the space of such curves has dimension

$$n \coloneqq \frac{d(d+3)}{2} + 3 - \delta\,. \tag{1.6.1}$$

Let $N_{\delta,d}$ denote the number of δ-nodal plane curves of degree d that intersect general lines $\ell_1, \dots, \ell_n \subset \mathbb{P}^3$. The main result of this section is that for each δ, and $d \geq \delta$, the numbers $N_{\delta,d}$ are given by polynomial of degree $\leq 2\delta + 9$ in d.

Let $\mathrm{Gr} := \mathrm{Gr}(2, \mathbb{P}^3)$ be the Grassmannian of planes in $\mathbb{P}^3$ and let $\mathcal{U}$ be the tautological vector bundle on Gr. Let $\mathcal{O}_{\mathrm{Gr}}(1)$ be the bundle corresponding to the hyperplane class via the identification $\mathrm{Gr} = \check{\mathbb{P}}^3$. These two bundles are related via the tautological short exact sequence

$$0 \to \mathcal{U} \to \mathbb{C}^4 \otimes \mathcal{O}_{\mathrm{Gr}} \to \mathcal{O}_{\mathrm{Gr}}(1) \to 0 \,. \tag{1.6.2}$$

Let $q \colon \mathcal{S} = \mathbb{P}(\mathcal{U}) \to \mathrm{Gr}$ be the universal plane. As a family of subvarieties of $\mathbb{P}^3$, it comes with a relatively very ample line bundle $\mathcal{O}_{\mathcal{S}}(1)$. Choose an integer $d > 1$ and consider the line bundle $\mathcal{L} := \mathcal{O}_{\mathcal{S}}(d)$. As in Section 1.3, we can form the projective bundle

$$B := |\mathcal{L}/\mathrm{Gr}| = \mathbb{P}(q_*\mathcal{L}) \xrightarrow{\ \ \pi\ \ } \mathrm{Gr}\,,$$

which comes with a canonical bundle $\mathcal{O}_B(1)$. The fibre over a point $[V] \in \mathrm{Gr}$ corresponding to a plane $V \subset \mathbb{P}^3$ is the complete linear system $|\mathcal{O}_V(d)|$. In particular π is of relative dimension $r - 1$, with

$$r = \mathrm{rank}(q_*\mathcal{L}) = \frac{(d+1)(d+2)}{2}\,.$$

Moreover, it follows that B parametrizes planes in $\mathbb{P}^3$, together with a degree d curve on that plane. As a planar curve in $\mathbb{P}^3$ of degree > 1 lies in a unique plane, the variety in fact parametrizes planar curves in $\mathbb{P}^3$.

Let $p \colon \mathcal{C} \to B$ be the universal curve. Then $\mathcal{C}$ is a relative effective divisor on the family $\mathcal{S}_B = \mathcal{S} \times_{\mathrm{Gr}} B$ and we have

$$\mathcal{O}_{\mathcal{S}_B}(\mathcal{C}) = \mathcal{L}(1) = \mathcal{L} \otimes \mathcal{O}_B(1)\,.$$

Now let $\delta \leq d$. Note that for $b \in B$, we have

$$\mathcal{S}_b \cong \mathbb{P}^2 \quad \text{and} \quad \mathcal{L}(1)|_{\mathcal{S}_b} \cong \mathcal{O}_{\mathbb{P}^2}(d)\,,$$

so $\mathcal{L}(1)$ satisfies **AMP**, as $\mathcal{O}(d)$ is δ-very ample [BS91]. By Lemma 1.3.7, the locus $B(\delta) \subset B$ has codimension δ and we have $\gamma(\mathcal{C}) = \left[\overline{B(\delta)}\right] \subset B$. For lines $\ell_1, \dots, \ell_n$ in $\mathbb{P}^3$, denote the (reduced) locus of curves intersecting the line ℓ_i by $B_{\ell_i} \subset B$, and let

$$B_{\ell_1,\dots,\ell_n} = B_{\ell_1} \cap \ldots \cap B_{\ell_n} \subset B$$

be the (scheme theoretic) intersection. We will use the notation

$$\partial B(\delta) = \overline{B(\delta)} \Big\backslash B(\delta)\ .$$

Proposition 1.6.3. *For*

$$n = \frac{d(d+3)}{2} + 3 - \delta$$

as above and general lines $\ell_1, \dots, \ell_n$, we have

$$B_{\ell_1,\dots,\ell_n} \cap \partial B(\delta) = \emptyset\,.$$

Moreover, the intersection $B_{\ell_1,\dots,\ell_n} \cap B(\delta)$ *is finite and reduced, and its degree is given by*

$$N_{\delta,d} = \int_{B_{\ell_1,\dots,\ell_n}} \gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_n}}) \, .$$

Remark 1.6.4. More precisely, in the proof we will construct a non-empty Zariski open subset

$$U \subset \mathrm{Gr}(1, \mathbb{P}^3)^n$$

of the n-fold product of the Grassmannian of lines in $\mathbb{P}^3$, such that the proposition holds for any n-tuple of lines $(\ell_1, \dots, \ell_n) \in U$.

Remark 1.6.5. It should be noted that for $\delta > 0$, the scheme $B_{\ell_1,\dots,\ell_n}$ is singular. In fact, for every i, a local computation shows that the singular locus of the variety B_{ℓ_i} is the divisor of curves $C \in B_{\ell_i}$ such that ℓ_i lies in the plane spanned by C.

Proof. We will prove the first two statements by an argument in the spirit of Lemma 4.7 in [KP99]. We have $n = r + 2 - \delta = \dim B - \delta$. It follows that, cf. loc. cit., the expected dimension of $B_{\ell_1,\dots,\ell_n} \cap B(\delta)$ is

$$\dim B - n - \delta = 0 \, . \tag{1.6.6}$$

Let $\mathrm{Gr}(1, \mathbb{P}^3)$ be the Grassmannian of lines in $\mathbb{P}^3$, and let $\mathbb{L} \to \mathrm{Gr}(1, \mathbb{P}^3)$ be the universal line. Let P be the limit of the following diagram:

$$\begin{array}{ccccc} & & & & \overline{B(\delta)} \\ & & & & \downarrow \\ & & \mathcal{C}_B^n & \longrightarrow & B \\ & & \downarrow & & \\ \mathbb{L}^n & \longrightarrow & (\mathbb{P}^3)^n & & \\ \downarrow & & & & \\ \mathrm{Gr}(1, \mathbb{P}^3)^n \, , & & & & \end{array} \tag{1.6.7}$$

in which we use the notation

$$\mathcal{C}_B^n = \mathcal{C} \times_B \cdots \times_B \mathcal{C} \, .$$

Then P parametrises the following data:

- lines $\ell_1, \dots \ell_n \subset \mathbb{P}^3$;
- points $p_1, \dots, p_n \in \mathbb{P}^3$;
- a plane $V \subset \mathbb{P}^3$;
- a curve $C \in \overline{B(\delta)}$;

subject to the following conditions:

- $C \subset V$;
- $p_i \in \ell_i$ for $i = 1, \ldots, n$;
- $p_i \in C$ for $i = 1, \ldots, n$.

The horizontal maps in the diagram are flat with relative dimensions $2n$ and n respectively. Since $\overline{B(\delta)}$ has dimension n, it follows that

$$\dim(P) = 4n = \dim(\mathrm{Gr}(1, \mathbb{P}^3)^n) .$$

A curve $C \in B(\delta)$ has a dense smooth open subset C° (as it is nodal). Therefore also the universal curve $\mathcal{C} \to B$ restricted to $\overline{B(\delta)}$ has this property (the latter being reduced). As $P \to \mathcal{C}^n_B|_{\overline{B(\delta)}}$ is smooth (with fibre $\cong (\mathbb{P}^2)^n$), we see that P is generically smooth. In particular, the singular locus P^{sing} of P has dimension $< 4n$. Consider the morphism

$$\phi\colon P \to \mathrm{Gr}(1, \mathbb{P}^3)^n .$$

Let $U_1 = \mathrm{Gr}(1, \mathbb{P}^3)^n \backslash \phi(P^{\mathrm{sing}})$ be the complement of the image of the singular locus in $\mathrm{Gr}(1, \mathbb{P}^3)^n$. As $\mathrm{Gr}(1, \mathbb{P}^3)^n$ has dimension $4n$, the open $U_1 \subset \mathrm{Gr}(1, \mathbb{P}^3)^n$ is non-empty. Since $\phi^{-1}(U_1)$ is smooth, there is a non-empty open $U_2 \subset U_1$ such that the morphism ϕ is finite and reduced over U_2. Moreover, the closed subsets

$$Z_i = \{(\vec{\ell}, \vec{p}, V, C) \in P \mid \ell_i \subset V\} \subset P , \quad i = 0, \ldots, n$$

and

$$P_{\partial B(\delta)} = \{(\vec{\ell}, \vec{p}, V, C) \in P \mid C \in \partial B(\delta)\} \subset P$$

have positive codimension. Therefore the open

$$U = U_2 \cap \left(\mathrm{Gr}(1, \mathbb{P}^3)^n \Bigg\backslash \phi \left(\bigcup_{i=1}^{n} Z_i \cup P_{\partial B(\delta)} \right) \right)$$

is non-empty.

Now let $\vec{\ell} = (\ell_1, \ldots, \ell_n) \in U$ be an n-tuple of lines and let

$$P_{\vec{\ell}} = \phi^{-1}(\vec{\ell})$$

be the fibre over $\vec{\ell}$. Consider the morphism

$$\psi\colon P \to \overline{B(\delta)} \subset B .$$

For a point $[C] \in \overline{B(\delta)}$, the fibre of $P_{\vec{\ell}}$ over $[C]$ is the scheme

$$P_{\vec{\ell}} \cap \psi^{-1}([C]) = \prod_{i=1}^{n} (\ell_i \cap C) \subset (\mathbb{P}^3)^n . \tag{1.6.8}$$

Let $V \subset \mathbb{P}^3$ be the plane spanned by C. By definition of U, we have $\ell_i \not\subset V$ for the lines $\ell_1, \ldots, \ell_n$ in $\vec{\ell}$. It follows that the intersection

$$C \cap \ell_i \subset \ell_i \cap V$$

is a reduced point, if it is non-empty. Hence $P_{\vec{\ell}}$ maps isomorphically to its image in B.

On the other hand, the scheme (1.6.8) is non-empty if and only if C intersects the lines ℓ_i. By definition of U, it follows that we have

$$\begin{aligned}\psi(P_{\vec{\ell}}) &= (B_{\ell_1,\dots,\ell_n} \cap \overline{B(\delta)})^{red} \\ &= (B_{\ell_1,\dots,\ell_n} \cap B(\delta))^{red} .\end{aligned}$$

Finally, we will show that in fact

$$\psi(P_{\vec{\ell}}) = B_{\ell_1,\dots,\ell_n} \cap B(\delta) .$$

To see this, first note that $\psi(P_{\vec{\ell}})$ lies in the open subset $W \subset B$ consisting of curves C with $\ell_i \not\subset V$ for $i = 1, \dots, n$ and V the plane spanned by C. As noted before, for a curve $C \in W$, the scheme $C \cap \ell_i$ is empty or consists of a single reduced point. It follows that over W, the scheme

$$\ell_i \times_{\mathbb{P}^3} \mathcal{C}$$

is mapped isomorphically to its image B_{ℓ_i} in B, since a *scheme theoretic* fibre of

$$\ell_i \times_{\mathbb{P}^3} \mathcal{C}_W \to B_{\ell_i} \cap W$$

is a *reduced* point. We conclude that

$$\begin{aligned}\psi(P_{\vec{\ell}}) &= \psi\left((\ell_1 \times_{\mathbb{P}^3} \mathcal{C}) \times_B \cdots \times_B (\ell_n \times_{\mathbb{P}^3} \mathcal{C}) \times_B \overline{B(\delta)}\right) \\ &= B_{\ell_1} \cap \cdots \cap B_{\ell_n} \cap \overline{B(\delta)} \\ &= B_{\ell_1,\dots,\ell_n} \cap B(\delta) .\end{aligned}$$

As the intersection $B(\delta) \cap B_{\ell_1,\dots,\ell_n}$ is finite and reduced, it is *transverse*, by (1.6.6). It follows by Lemma 1.4.1, Proposition 1.4.5 and Lemma 1.3.7 that we have

$$\begin{aligned}\#(B(\delta) \cap B_{\ell_1,\dots,\ell_n}) &= \int_B ([\overline{B(\delta)}].[B_{\ell_1,\dots,\ell_n}]) \\ &= \int_B \gamma(\mathcal{C}) \cap [B_{\ell_1,\dots,\ell_n}] \\ &= \int_{B_{\ell_1,\dots,\ell_n}} \gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_n}}) .\end{aligned}$$ □

The following lemma is essentially [Zin13], Exercise 3.4. See also [Ful98], Example 3.2.22. For completeness, we will include the proof.

Lemma 1.6.9. *For a line $\ell \subset \mathbb{P}^3$, the closed subvariety $B_\ell \subset B$ is a divisor, cut out by a section of the line bundle*

$$\pi^* \mathcal{O}_{\mathrm{Gr}}(d) \otimes \mathcal{O}_B(1) .$$

Proof. It suffices to construct the section outside the codimension two subvariety $Z \subset \mathrm{Gr}$ of planes $[V] \in \mathrm{Gr}$ containing the line ℓ. Let $U := \mathrm{Gr} - Z$ be the complement. Consider the fibre product $\mathcal{S}_\ell$ of the following diagram:

$$\begin{array}{ccc} & & \mathcal{S} \\ & & \downarrow \\ \ell & \longrightarrow & \mathbb{P}^3 . \end{array}$$

Then the morphism $\mathcal{S}_\ell \to \mathrm{Gr}$ has fibre $V \cap \ell \subset \mathbb{P}^3$ over a point $[V] \in \mathrm{Gr}$. In particular, it restricts to an isomorphism over U. Now consider the fibre product $\mathcal{C}_\ell$ of the diagram

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow \\ \ell & \longrightarrow & \mathbb{P}^3 . \end{array}$$

The morphism $\mathcal{C}_\ell \to B$ has *scheme theoretic* fibre $C \cap \ell$ over a point $[C] \in B$. It follows that $\mathcal{C}_\ell$ is mapped onto B_ℓ and the morphism restricts to an isomorphism over $B_\ell \times_{\mathrm{Gr}} U$.

As noted before, $\mathcal{C}$ is the zero locus of a canonical section of the line bundle $\mathcal{O}_{\mathcal{S}}(d) \otimes \mathcal{O}_B(1)$ on $\mathcal{S} \times_{\mathrm{Gr}} B$, in which $\mathcal{O}_{\mathcal{S}}(d)$ denotes the pull-back of the bundle $\mathcal{O}_{\mathbb{P}^3}(d)$ on $\mathbb{P}^3$ to the universal plane $\mathcal{S}$. It follows that $\mathcal{C}_\ell \subset \mathcal{S}_\ell \times_{\mathrm{Gr}} B$ is cut out by a section of the bundle $\mathcal{O}_\ell(d) \otimes \mathcal{O}_B(1)$, in which $\mathcal{O}_\ell(d)$ is the restriction of $\mathcal{O}_{\mathbb{P}^3}(d)$ to $\mathbb{P}^1 \cong \ell \subset \mathbb{P}^3$. Hence, it suffices to show that $\mathcal{O}_{\mathbb{P}^3}(d)$ equals $\mathcal{O}_{\mathrm{Gr}}(d)$ when pulled back to $\mathcal{S}_\ell \times_{\mathrm{Gr}} B$. But this can be seen easily by noting that in the diagram (with Cartesian square)

$$\begin{array}{ccccc} \mathcal{S}_\ell & \longrightarrow & \mathcal{S} & \longrightarrow & \mathrm{Gr} \\ \downarrow & & \downarrow & & \\ \ell & \longrightarrow & \mathbb{P}^3 & & \end{array}$$

the fibre $(\mathcal{S}_\ell)_x$ over a point $x \in \ell$ is the inverse image of the divisor on Gr of planes in $\mathbb{P}^3$ containing the point x. □

The following corollary of Proposition 1.6.3 is the first part of Theorem 1.B, our second main result.

Corollary 1.6.10. *For every $\delta \geq 0$, there is polynomial N_δ of degree $\leq 9 + 2\delta$ such that $N_{\delta,d} = N_\delta(d)$ for $d \geq \delta$.*

Proof. By Proposition 1.6.3, we need to compute the integral

$$\int_{B_{\ell_1,\ldots,\ell_n}} \gamma(\mathcal{C}|_{B_{\ell_1,\ldots,\ell_n}}) = \int_B \gamma(\mathcal{C}) \cap [B_{\ell_1,\ldots,\ell_n}] .$$

Let $H = c_1(\mathcal{O}_{\mathrm{Gr}}(1))$ and $\xi = c_1(\mathcal{O}_B(1))$. Then, by the Lemma 1.6.9, we have for general lines $\ell_1, \ldots, \ell_n \subset \mathbb{P}^3$ the equation

$$\begin{aligned} [B_{\ell_1,\ldots,\ell_n}] &= (dH + \xi)^n \\ &= \sum_{i=0}^{3} \binom{n}{i} (dH)^i \xi^{n-i} \end{aligned}$$

in $A^*(B)$. On the other hand, we know by Theorem 1.A that $\gamma(\mathcal{C})$ is a polynomial of degree δ in classes

$$\epsilon(a,b,c) = (q_B)_*(c_1(\mathcal{O}(\mathcal{C})^a c_1(T_{\mathcal{S}/\mathrm{Gr}})^b c_2(T_{\mathcal{S}/\mathrm{Gr}})^c)\,.$$

It will follow that $\gamma(\mathcal{C})$ is a polynomial in H, ξ and d. To see this, if suffices to show that the classes $\epsilon(a,b,c)$ are polynomials in H, ξ and d. Let $\eta = c_1(\mathcal{O}_{\mathcal{S}}(1))$. Then we have

$$\begin{aligned} c_1(\mathcal{O}(\mathcal{C})^a c_1(T_{\mathcal{S}/\mathrm{Gr}})^b c_2(T_{\mathcal{S}/\mathrm{Gr}})^c &= (d\eta+\xi)^a(3\eta+c_1(\mathcal{U}))^b(3\eta^2+2\eta c_1(\mathcal{U})+c_2(\mathcal{U}))^c \\ &= (d\eta+\xi)^a(3\eta-H)^b(3\eta^2-2\eta H+H^2)^c\,, \end{aligned}$$

in which we use the short exact sequences

$$0 \to \mathcal{O} \to \mathcal{U} \otimes \mathcal{O}_{\mathcal{S}}(1) \to T_{\mathcal{S}/\mathrm{Gr}} \to 0$$

and (1.6.2). The structure of the Chow ring of $\mathcal{S}_B$ is given by

$$A^*(\mathcal{S}) = A^*(B)[\eta]/(\eta^3 + c_1(\mathcal{U})\eta^2 + c_2(\mathcal{U})\eta + c_3(\mathcal{U}))$$

and the push-forward

$$\epsilon(a,b,c) = (q_B)_*(c_1(\mathcal{O}(\mathcal{C})^a c_1(T_{\mathcal{S}/\mathrm{Gr}})^b c_2(T_{\mathcal{S}/\mathrm{Gr}})^c)$$

is computed by repeatedly substituting the equation

$$\begin{aligned} \eta^3 &= -c_1(\mathcal{U})\eta^2 - c_2(\mathcal{U})\eta - c_3(\mathcal{U}) \\ &= H\eta^2 - H^2\eta + H^3\,. \end{aligned}$$

and taking the coefficient of η^2. Since $H^4 = 0$, the substitution procedure terminates after 3 steps. For fixed a, b an c, we obtain a polynomial in H, ξ and d. It follows that $\gamma(\mathcal{C})$ can be written as a polynomial in H, ξ and d.

Note that the coefficient of H^i in $\epsilon(a,b,c)$ has degree at most $2+i$ as a polynomial in d. As $\gamma(\mathcal{C})$ is a polynomial of degree δ in classes $\epsilon(a,b,c)$, the coefficient of H^i in this $\gamma(\mathcal{C})$ has degree at most $2\delta + i$, as a polynomial in d.

We consider the class

$$\gamma(\mathcal{C}|_{B_{\ell_1,\ldots,\ell_n}}) = \gamma(\mathcal{C}) \cap [B_{\ell_1,\ldots,\ell_n}]$$

in $A_0(B)$. We will show that its degree is a polynomial in d. The Chow ring of $B = \mathbb{P}(q_*\mathcal{L})$ is given by

$$\begin{aligned} A_*(B) &= A^*(\mathrm{Gr})[\xi]/(\xi^r + c_1(q_*\mathcal{L})\xi^{r-1} + c_2(q_*\mathcal{L})\xi^{r-2} + c_3(q_*\mathcal{L})\xi^{r-3}) \\ &= \mathbb{Z}[H,\xi]/(H^4, \xi^r + c_1(q_*\mathcal{L})\xi^{r-1} + c_2(q_*\mathcal{L})\xi^{r-2} + c_3(q_*\mathcal{L})\xi^{r-3})\,. \end{aligned}$$

We have

$$q_*\mathcal{L} = \mathrm{Sym}^d(\mathcal{U}^*)\,,$$

and hence its Chern class is a polynomial in d and H. In fact, we have

$$c(q_*\mathcal{L}) = 1 + \frac{d\,(d+1)\,(d+2)}{6}\,H + \frac{d\,(d+1)\,(d+2)\,(d+3)\,(d^2+2)}{72}\,H^2 + \frac{d\,(d+1)\,(d+2)\,(d+3)\,(d^2+2)\,(d^3+3\,d^2+2\,d+12)}{1296}\,H^3\,.$$

Note that coefficient of H^i is a polynomial in d of degree $3i$. We can compute the degree

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_n}}) = \int_B \gamma(\mathcal{C}) \cap \sum_{i=0}^{3} \binom{n}{i} (dH)^i \xi^{n-i}$$

as follows. Note that the class $\gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_n}})$ is homogeneous of degree $r+2$ in H and ξ. Using the relations

$$\xi^r = -c_1(q_*\mathcal{L})\xi^{r-1} - c_2(q_*\mathcal{L})\xi^{r-2} - c_3(q_*\mathcal{L})\xi^{r-3} \quad \text{and} \quad H^4 = 0$$

we can rewrite it as

$$uH^3\xi^{r-1}$$

for a polynomial u in d depending only on δ. Now we have

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_n}}) = u\,.$$

Recall that we write $\pi\colon B \to \mathrm{Gr}$ for the projection. Then the class

$$uH^3 = \pi_*(uH^3\xi^{r-1})$$

is a product of classes of the following types:

- The coefficients of $\gamma(\mathcal{C})$ as a polynomial in ξ;
- Classes of the form $\binom{n}{i}(dH)^i$;
- Polynomials in the Chern classes of $q_*\mathcal{L}$.

As remarked before, the coefficient of H^i in $\gamma(\mathcal{C})$ has degree $2\delta + i$ in d. In other words, every factor $d^{2\delta+i}$ appearing in the terms of $\gamma(\mathcal{C})$ is accompanied by a factor H^i. Similarly, the coefficients of H^i of classes of the second and third type, are polynomials of degree $3i$ in d, so every factor d^j appearing in the terms of these classes is accompanied by a factor $H^{\lceil j/3 \rceil}$. It follows that uH^3 has degree at most $2\delta + 9$ in d. □

1.7 Torus localization

As in the previous section, let $\mathrm{Gr} = \mathrm{Gr}(2, \mathbb{P}^3)$ be the Grassmannian of planes in $\mathbb{P}^3$, with universal plane $q\colon \mathcal{S} = \mathbb{P}(\mathcal{U}) \to \mathrm{Gr}$, in which $\mathcal{U}$ is the tautological vector bundle on Gr. On $\mathcal{S}$ we have defined the line bundle $\mathcal{L} = \mathcal{O}_{\mathcal{S}}(d)$, which we use to construct the projective bundle $B = \mathbb{P}(q_*\mathcal{L})$ over Gr parametrizing planar curves of degree d in $\mathbb{P}^3$, with universal curve $\mathcal{C} \to B$. The variety B has dimension $r+2$, with

$$r = \frac{(d+1)(d+2)}{2}$$

the rank of $q_*\mathcal{L}$. Finally, we define

$$n \coloneqq r + 2 - \delta = \frac{d(d+3)}{2} + 3 - \delta$$

and write $B_{\ell_1,\ldots,\ell_n}$ for the locus of curves intersecting general lines $\ell_1, \ldots, \ell_n$.

Rather than using the algorithm of [EGL01], we can use the Bott residue formula to evaluate the integral

$$\int_B \gamma(\mathcal{C}|_{B_{\ell_1,\ldots,\ell_n}}) = \int_B \gamma(\mathcal{C}) \cap [B_{\ell_1,\ldots,\ell_n}]\,.$$

of Proposition 1.6.3. By the Lemmas 1.6.9 and 1.2.7, we need to compute

$$\begin{aligned}&\int_{\mathcal{C}_B^{[i]}} c(T_{\mathcal{C}_B^{[i]}/B}) \cap \left[\mathcal{C}^{[i]}_{B_{\ell_1,\ldots,\ell_n}}\right] = \\ &\int_{\mathcal{S}_B^{[i]}} \frac{c(T_{\mathcal{S}_B^{[i]}/B})}{c(\mathcal{O}(\mathcal{C})_B^{[i]})}\, c_i(\mathcal{O}(\mathcal{C})_B^{[i]})\, c_1(\mathcal{O}_{\mathrm{Gr}}(d) \otimes \mathcal{O}_B(1))^n\end{aligned} \tag{1.7.1}$$

for $i = 1, \ldots, \delta$. We have equations

$$\mathcal{O}_{\mathcal{S}_B}(\mathcal{C})_B^{[i]} = (\mathcal{L} \otimes \mathcal{O}_B(1))_B^{[i]} = \mathcal{L}_{\mathrm{Gr}}^{[i]} \otimes \mathcal{O}_B(1)$$

and

$$T_{\mathcal{S}_B^{[i]}/B} = \pi^*_{\mathcal{S}^{[i]}} T_{\mathcal{S}_{\mathrm{Gr}}^{[i]}/\mathrm{Gr}}\,.$$

It follows that the class on the right hand side of (1.7.1) is a polynomial in classes pulled back from $\mathcal{S}_{\mathrm{Gr}}^{[i]}$, and the first Chern class of the line bundle $\mathcal{O}_B(1)$. We will continue to use the notation $\xi = c_1(\mathcal{O}_B(1))$ and $H = c_1(\mathcal{O}_{\mathrm{Gr}}(1))$. We can rewrite the factors involving the bundle $\mathcal{O}_B(1)$ in the integral as follows:

$$\begin{aligned}
c_1(\mathcal{O}_{\mathrm{Gr}}(d)\otimes\mathcal{O}_B(1))^n &= (dH+\xi)^n\\
&= \left(\xi^3 + n\,dH\,\xi^2 + \binom{n}{2}(dH)^2\,\xi + \binom{n}{3}(dH)^3\right)\xi^{n-3}\\
c_i(\mathcal{L}^{[i]}_{\mathrm{Gr}}\otimes\mathcal{O}_B(1)) &= \sum_{k=0}^{i} c_k(\mathcal{L}^{[i]}_{\mathrm{Gr}})\,\xi^{i-k}\\
\frac{1}{c(\mathcal{L}^{[i]}_{\mathrm{Gr}}\otimes\mathcal{O}_B(1))} &= \sum_{j=0}^{\infty}\Big(1-c(\mathcal{L}^{[i]}_{\mathrm{Gr}}\otimes\mathcal{O}_B(1))\Big)^j\\
&= \sum_{j=0}^{\infty}\left(1-\sum_{k=0}^{i} c_k(\mathcal{L}^{[i]}_{\mathrm{Gr}})\,(1+\xi)^{i-k}\right)^j\\
&= \sum_{j=0}^{3+2i+\delta}\left(1-\sum_{k=0}^{i} c_k(\mathcal{L}^{[i]}_{\mathrm{Gr}})\,(1+\xi)^{i-k}\right)^j + \alpha\,.
\end{aligned}$$

In the last expression, $\alpha \in A^*(\mathcal{S}^{[i]}_B)$ is a sum of classes of degree $> 3+2i+\delta$. It follows that

$$\alpha\,\xi^{n-3} = 0\,,$$

as we have

$$(n-3)+(3+2i+\delta) = r+2+2i = \dim(\mathcal{S}^{[i]}_B)\,.$$

Hence we can compute (1.7.1) by integrating the class

$$q^{[i]}_*\Bigg(c(T_{\mathcal{S}^{[i]}_{\mathrm{Gr}}/\mathrm{Gr}}) \quad\times\quad \left(\xi^3 + n\,dH\,\xi^2 + \binom{n}{2}(dH)^2\,\xi + \binom{n}{3}(dH)^3\right)\xi^{n-3} \quad\times$$
$$\sum_{k=0}^{i} c_k(\mathcal{L}^{[i]}_{\mathrm{Gr}})\,\xi^{i-k} \quad\times\quad \sum_{j=0}^{3+2i+\delta}\left(1-\sum_{k=0}^{i} c_k(\mathcal{L}^{[i]}_{\mathrm{Gr}})\,(1+\xi)^{i-k}\right)^j\Bigg)\,.$$

As in the proof of Corollary 1.6.10, the push-forward along the projection

$$\pi\colon \mathcal{S}^{[i]}_B = \mathbb{P}(q_*\mathcal{L})\times_{\mathrm{Gr}}\mathcal{S}^{[i]}_{\mathrm{Gr}} \to \mathcal{S}^{[i]}_{\mathrm{Gr}}$$

can be computed by substituting the equation

$$\xi^r = -c_1(q_*\mathcal{L})\xi^{r-1} - c_2(q_*\mathcal{L})\xi^{r-2} - c_3(q_*\mathcal{L})\xi^{r-3}$$

and taking the coefficient of ξ^{r-1}. Hence we can rewrite (1.7.1) as an integral

$$\int_{\mathcal{S}^{[i]}_{\mathrm{Gr}}} P(T_{\mathcal{S}^{[i]}_{\mathrm{Gr}}/\mathrm{Gr}}, \mathcal{L}^{[i]}_{\mathrm{Gr}}, \mathcal{O}_{\mathrm{Gr}}(1), d)\,, \tag{1.7.2}$$

in which P is a polynomial[3] in d and the Chern classes of the bundles in the brackets.

[3] The fact that the expression is polynomial in d is not important for the computation. However, it does give another proof of the polynomiality of $N_{\delta,d}$ for $d \geq \delta$, that does not depend on the algorithm of [EGL01].

We can evaluate this integral using the Bott residue formula. For notation and definitions, see [EG98]. Recall that for a torus T acting on a smooth variety X, the fixed locus X^T is smooth [Ive72], so a connected component $F \subset X^T$ has normal bundle $N_F X$ of rank equal to the codimension d_F of F in X.)

Theorem 1.7.3 (Bott residue formula ([EG98])). *Let $E_1, \dots, E_r$ be T-equivariant vector bundles on a complete, smooth n-dimensional variety X with a torus action by T. Let $p(E)$ be a polynomial in the Chern classes of the bundles E_i. Then*

$$\int_X p(E) \cap [X] = \sum_{F \subset X^T} \int_F \left(\frac{p^T(E|_F) \cap [F]_T}{c^T_{d_F}(N_F X)} \right),$$

in which we sum over the connected components F of the fixed locus X^T of the torus action.

Consider the natural action of the torus $T = (\mathbb{G}_m)^4$ on $\mathbb{P}^3$ given by

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \cdot (a_0 : a_1 : a_2 : a_3) = (\lambda_0 a_0 : \lambda_1 a_1 : \lambda_2 a_2 : \lambda_3 a_3).$$

It induces a dual action of T on the Grassmanian $\mathrm{Gr} = \check{\mathbb{P}}^3$ which lifts to an equivariant structure on $\mathcal{O}_{\mathrm{Gr}}(1)$. The action on $\check{\mathbb{P}}^3 \times \mathbb{P}^3$ restricts to an action on $\mathcal{S}$ (which is simply the incidence variety). This action, in turn, lifts to an equivariant structure on the line bundle $\mathcal{O}_{\mathcal{S}}(1)$. Moreover, we obtain actions on the Hilbert schemes $\mathcal{S}^{[i]}$ and induced equivariant structures on the the bundles $T_{\mathcal{S}^{[i]}/\mathrm{Gr}}$ and $\mathcal{L}^{[i]} = \mathcal{O}_{\mathcal{S}}(d)^{[i]}$.

Lemma 1.7.4. *The fixed locus of the action of T on $\mathcal{S}^{[i]}_{\mathrm{Gr}}$ is finite and reduced.*

Proof. As the variety $\mathcal{S}^{[i]}_{\mathrm{Gr}}$ is smooth, so is the fixed locus, as remarked above. Hence it suffices to show that the underlying set is finite. The fixed points of Gr are the four planes

$$V_k = \mathcal{Z}(x_i) \subset \mathbb{P}^3 = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2, x_3]$$

for $k = 0, \dots, 3$, given by the vanishing of a coordinate. Note that the morphism $\mathcal{S}^{[i]}_{\mathrm{Gr}} \to \mathrm{Gr}$ is equivariant for the T-action. It follows that T acts on the fibres $V_k^{[i]}$. Now we follow [ES96], Section 4. Let $Z \subset V_0$ be a subscheme of length i, fixed under the action of T. Then Z is supported on the T-invariant locus $\{P_1, P_2, P_3\} \subset V_0$, with

$$P_1 = (0:1:0:0), \quad P_2 = (0:0:1:0), \quad P_3 = (0:0:0:1).$$

For $k = 1, 2, 3$, let Z_k the component of Z, supported on $\{P_k\}$, and let i_k the length of Z_k. On an open neighbourhood of P_1, we have coordinates $u = x_2/x_1$ and $v = x_3/x_1$ on the plane V_0. Now T acts on the coordinate ring $\mathbb{C}[u, v]$ by

$$\lambda \cdot u = \frac{\lambda_1}{\lambda_2} u \quad \text{and} \quad \lambda \cdot v = \frac{\lambda_1}{\lambda_3} v.$$

As Z is invariant, so is the ideal $\mathcal{I}(Z_1)$ of Z_1 in $\mathbb{C}[u, v]$. It follows that $\mathcal{I}(Z_1)$ is generated by monomials in u and v. The coordinate ring $\mathbb{C}[u, v]/\mathcal{I}(Z_1)$ is spanned by the i_1 monomials $u^k v^l$ not contained in $\mathcal{I}(Z_1)$. For every $k \geq 0$, define

$$d_k = \max\{l \mid u^k v^l \notin \mathcal{I}(Z_1)\}.$$

It is easy to see that the d_k define a partition

$$P_{Z_1} = (d_0 \geq \ldots \geq d_n)$$

of length l_1. Similarly, we get partitions P_{Z_2} and P_{Z_3}. Conversely, any *tripartition* (P_1, P_2, P_3) of length i, consisting of three partitions P_1, P_2, P_3 with $|P_1|+|P_2|+|P_3| = i$, corresponds to a T-invariant length-i subscheme of V_0. It is clear that there are finitely many such tripartitions. By repeating this argument for the other planes V_k, the result follows. □

We will apply Theorem 1.7.3 to the integral (1.7.2). By Lemma 1.7.4, the fixed locus consists of isolated points. Hence, for a fixed point $[Z] \in \mathcal{S}_{\mathrm{Gr}}^{[i]}$, the normal bundle $N_{[Z]}\mathcal{S}_{\mathrm{Gr}}^{[i]}$ is just the restriction of the tangent bundle of $\mathcal{S}_{\mathrm{Gr}}^{[i]}$. The restrictions of the bundles T_{Gr}, $T_{\mathcal{S}_{\mathrm{Gr}}^{[i]}/\mathrm{Gr}}$, $\mathcal{L}^{[i]}$ and $\mathcal{O}_{\mathrm{Gr}}(1)$ to $[Z]$ are T-representations. Lemma 3 of [EG98] gives the equivariant Chern classes explicitly as polynomials in the characters of the torus. The details are similar to the computation in [ES96].

Proof Theorem 1.B, second part. We have performed the calculation using Maple. We have computed the polynomials N_δ up to $\delta = 12$. The results are printed in Section 1.9. □

Remark 1.7.5. Up to $\delta = 7$, our answers are in agreement with polynomials communicated by Ritwik Mukherjee, which he calculated using the methods from [BM16] [BM15] and [Zin17] and verified by means of the algorithm of [KP04].

Remark 1.7.6. This method of calculating node polynomials seems to be quite efficient. For example, consider the integral

$$\int_B \gamma(\mathcal{C})\, c_1(\mathcal{O}_{\mathrm{Gr}}(1))^3 \cap [B_{\ell_1,\ldots,\ell_{n-3}}]\,.$$

It is easy to see that it computes the number of δ-nodal curves of degree d intersecting $n-3$ general points in a *fixed* plane $\mathbb{P}^2$. By an minor adaptation of our code, we were able to compute the node polynomials up to $\delta = 15$, finding agreement with the polynomials up to $\delta = 14$, published by Block in [Blo11]. However, Göttsche has computed the polynomials up to $\delta \leq 28$ [Göt98]. The polynomial for $\delta = 15$ is given in Section 1.10.

1.8 Low degree checks

Let $\delta, d \geq 1$. We want to determine the contribution of reducible curves to the number $N_{\delta,d}$ of planar δ-nodal curves of degree d in $\mathbb{P}^3$ intersecting general lines $\ell_1, \ldots, \ell_n \subset \mathbb{P}^3$. For certain δ and d, *all* curves contributing to $N_{\delta,d}$ are reducible, thereby giving consistency checks of our formulae. If in addition the irreducible components of these reducible curves are smooth or 1-nodal, these can be calculated by classical methods.

Let C be a curve in our counting problem, and assume we can write C as the union of irreducible curves $C = C_1 \cup \ldots \cup C_r$. The curves C_i are necessarily nodal (if singular), and intersect transversely. For $i = 1, \ldots, r$, let δ_i be the number of nodes of C_i, and d_i its degree. As C lies in a plane, two curves C_i and C_j with $1 \le i < j \le r$ intersect in $d_i d_j$ points. We have

$$\begin{aligned} d &= \sum_{i=1}^{r} d_i\,, \\ \delta &= \sum_{1\le i<j\le r} d_i d_j + \sum_{i=1}^{r} \delta_i\,. \end{aligned} \tag{1.8.1}$$

Moreover, there is a partition

$$\{\ell_1, \ldots, \ell_n\} = \bigsqcup_{i=1}^{r} \Sigma_i \tag{1.8.2}$$

such that C_i intersects the lines in $\Sigma_i \subset \{\ell_1, \ldots, \ell_n\}$.

Conversely, choose a partition as in (1.8.2) and integers d_i and δ_i for $i = 1, \ldots, r$ such that the equations (1.8.1) hold. We will determine the number of curves contributing to $N_{\delta,d}$ that decompose as described above, with these fixed data.

For $i \in \{1, \ldots, r\}$, let $B_i := \mathbb{P}(\mathrm{Sym}^{d_i}(\mathcal{U}^*)) \xrightarrow{\pi_i} \mathrm{Gr} = \mathrm{Gr}(2, \mathbb{P}^3)$ be the projective bundle parametrizing planar curves of degree d_i. Consider the locus

$$W_i := W(d_i, \delta_i, \Sigma_i) \subset B_i$$

of *irreducible* δ_i-nodal curves that intersect the lines in Σ_i.

We have the following lemma.

Lemma 1.8.3. *For general $\ell_1, \ldots, \ell_n$ the number of curves that contribute to $N_{\delta,d}$ that decompose with data fixed above, is given by*

$$\int_{\mathrm{Gr}} \prod_{i=1}^{r} \pi_{i*}[\overline{W}_i]\,,$$

in which $[\overline{W}_i]$ denotes the class in $A_* B_i$ of the closure of W_i.

Remark 1.8.4. More precisely, in the proof we will construct a non-empty Zariski open

$$U \subset \mathrm{Gr}(1, \mathbb{P}^3)^n$$

such that for an n-tuple $(\ell_1, \ldots, \ell_n) \in U$, the statement of the lemma holds.

Proof. The argument is similar to the proof of Proposition 1.6.3, so we will not give all the details. For $i = 1, \ldots, r$, let $n_i = \#\Sigma_i$ and form the limit P_i as in the diagram (1.6.7). We have natural morphisms $\phi_i \colon P_i \to \mathrm{Gr}(1, \mathbb{P}^3)^{n_i}$. Consider the morphism

$$\phi = (\phi_1, \ldots, \phi_r) \colon P_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} P_r \to \prod_{i=1}^{r} \mathrm{Gr}(1, \mathbb{P}^3)^{n_i} = \mathrm{Gr}(1, \mathbb{P}^3)^n\,.$$

As in Proposition 1.6.3, ϕ is finite and smooth over a non-empty open

$$U_0 \subset \mathrm{Gr}(1, \mathbb{P}^3)^n\,.$$

Here we use the fact due to Severi, that for the line bundles $\mathcal{O}(d)$ on $\mathbb{P}^2$, the locus of irreducible δ-nodal curves in $|\mathcal{O}(d)|$, if non-empty, has codimension δ [Sev68]. There is a non-empty open $U_1 \subset U_0$, such that for a point $\Sigma = (\Sigma_1, \ldots, \Sigma_r) \in U_1$, the fibre over Σ is

$$\begin{aligned}\phi^{-1}(\Sigma) &= \overline{W}_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} \overline{W}_r \\ &= W_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} W_r \,.\end{aligned}$$

Finally, there is an non-empty open $U_2 \subset U_1$, such that for $\Sigma \in U_2$, and any point $(C_1, \ldots, C_r) \in \phi^{-1}(\Sigma)$, the curves $C_1, \ldots, C_r$ intersect transversely, i.e.

$$C = C_1 \cup \ldots \cup C_r \subset V$$

is a nodal curve, in which the union is taken in the plane $V \subset \mathbb{P}^3$ corresponding to the image of $(C_1, \ldots, C_r)$ in Gr. By a count of dimensions, the sets $\pi_i(W_i)$ intersect properly in Gr. It follows that the contribution to $N_{\delta,d}$ by curves of this type is given by

$$\begin{aligned}\#\phi^{-1}(\Sigma) &= \int_{P_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} P_r} [\overline{W}_1 \times_{\mathrm{Gr}} \cdots \times_{\mathrm{Gr}} \overline{W}_r] \\ &= \int_{\mathrm{Gr}} \prod_{i=1}^{r} \pi_{i*}[\overline{W}_i] \,.\end{aligned}$$ □

Notation 1.8.5. Let $\ell_1, \ldots, \ell_n$ general lines in $\mathbb{P}^3$, and let $\Sigma_i \subset \{\ell_1, \ldots, \ell_n\}$ be a subset with $\#\Sigma_i = n_i$. Let $W_i = W(d_i, \delta_i, \Sigma_i)$ be the locus in $B_i = \mathbb{P}(\mathrm{Sym}^{d_i}(\mathcal{U}^*)) \xrightarrow{\pi} \mathrm{Gr} = \mathrm{Gr}(2, \mathbb{P}^3)$ of irreducible δ_i-nodal degree d_i plane curves in $\mathbb{P}^3$ intersecting the lines in Σ_i. We will write ν_{d_i,δ_i,n_i} for the class $\pi_*[\overline{W}_i] \in A_*(\mathrm{Gr})$.

We formulate the conclusion of the discussion above in the following proposition.

Proposition 1.8.6. *The number of δ-nodal plane curves in $\mathbb{P}^3$ of degree d, intersecting $n = \frac{d(d+3)}{2} + 3 - \delta$ general lines $\Sigma = \{\ell_1, \ldots, \ell_n\}$, is given by*

$$N_{\delta,d} = \sum_{r=1}^{\infty} \sum_{\Theta} \mu(\Theta)\, \nu_{d_1,\delta_1,n_1} \cdots \nu_{d_r,\delta_r,n_r}$$

in which the second sum is taken over unordered *r-tuples (or multisets) of triples*

$$\Theta = \{(d_i, \delta_i, n_i)\}_{i=1,\ldots,r}$$

of integers $d_i \geq 1$, $\delta_i \geq 0$, $n_i \geq 0$ satisfying (1.8.1) *and*

$$n = n_1 + \ldots + n_r \,,$$

and in which the multiplicity $\mu(\Theta)$ is the number of unordered partitions of the set Σ in sets of lengths $n_1, \ldots, n_r$, labeled by d_i and δ_i. In fact, this number is given by

$$\mu(\Theta) = \frac{1}{\#\mathrm{Stab}_{S_r}\overline{(d, \delta, n)}} \binom{n}{n_1, \ldots, n_r}$$

in which the denominator is the order of the stabilizer subgroup of

$$\overline{(d, \delta, n)} = ((d_i, \delta_i, n_i))_{i=1,\ldots,r} \in (\mathbb{Z}^3)^r$$

for the action of the symmetric group S_r on $(\mathbb{Z}^3)^r$.

Remark 1.8.7. The generality condition in the proposition means that the lines have to be general in the sense of Lemma 1.8.3, for every triple $(\bar{d}, \bar{\delta}, \bar{n})$ appearing in the second sum.

Using the proposition, we will compute the numbers $N_{\delta,d}$, for $0 \leq \delta \leq 6$ and $\delta = 8$ and certain low d. We will compare results with with the numbers $N_\delta(d)$, with N_δ the node polynomial as computed in the previous section, and given in the appendix for $\delta \leq 12$. In these cases, we can choose d in such a way that the irreducible components of the curves are smooth or 1-nodal.

Lemma 1.8.8. *For the following δ and d, the irreducible components of δ-nodal plane curves of degree d of are lines.*

Table 1.1: Nodal plane curves consisting of lines in $\mathbb{P}^3$

δ	1	3	6
d	2	3	4
$N_\delta(d)$	140	7280	261800*

In the following cases, a δ-nodal plane curve of degree d has only linear components besides one smooth conic component.

Table 1.2: Nodal plane curves with a smooth conic component

δ	0	2	5
d	2	3	4
$N_\delta(d)$	92	15660	1303500*

Finally, δ-nodal plane curves of degree d of the following types have only linear components besides two smooth conic components, or a nodal cubic component.

Table 1.3: Nodal plane curves with two conics or a nodal cubic.

δ	4	8
d	4	5
$N_\delta(d)$	3071796	385022820*

Remark 1.8.9. Since we can apply Theorem 1.A only under the assumption $d \geq \delta$, we have not proved that the value of the polynomial $N_\delta(d)$ equals the curve count $N_{\delta,d}$ in the cases indicated with *. However, as we will prove below, in these cases the polynomials give the right numbers. In general, the *Göttsche threshold* $d \geq \lceil \delta/2 \rceil + 1$ for nodal curves in $\mathbb{P}^2$, determined in [KS13], seems to hold also in our case, i.e. that the node polynomials N_δ have value $N_{\delta,d}$ in these d.

Proof. For an integral curve C, and its normalisation $\widetilde{C}$, we have

$$g(C) - \delta(C) = g(\widetilde{C}) \geq 0 .$$

It follows that the number of nodes of an irreducible plane curve of degree d is bounded by its arithmetic genus $\frac{(d-1)(d-2)}{2}$. Now use the equations (1.8.1). □

We have the following elementary lemma.

Lemma 1.8.10. *Let H be the hyperplane class in* $\mathrm{Gr} \cong \check{\mathbb{P}}^3$. *Then we have*

$$\begin{array}{llllll} \nu_{1,0,2} & = 1 & \nu_{2,0,5} & = 1 & \nu_{3,1,8} & = 12 \\ \nu_{1,0,3} & = 2H & \nu_{2,0,6} & = 8H & \nu_{3,1,9} & = 216H \\ \nu_{1,0,4} & = 2H^2 & \nu_{2,0,7} & = 34H^2 & \nu_{3,1,10} & = 2040H^2 \\ \nu_{1,0,5} & = 0 & \nu_{2,0,8} & = 92H^3 & \nu_{3,1,11} & = 12960H^3 \end{array}$$

Proof. Let $n := \frac{d(d+3)}{2} + 3 - \delta$. First note that for $d \geq \delta + 2$, all δ-nodal curves of degree d are irreducible, so we have by Lemma 1.3.7, Proposition 1.4.5 and a slightly adapted version of Proposition 1.6.3 the identity

$$\nu(d, \delta, n-i) = \pi_*(\gamma(\mathcal{C}|_{B_{\ell_1,\dots,\ell_{n-i}}})) \in A_i(\mathrm{Gr}) .$$

Hence we can compute the classes by the methods of the previous sections. In the case that $\delta = 0, 1$, however, the classes can be computed by elementary means. Let $\delta = 0$. The locus of curves in $B = \mathbb{P}(\mathrm{Sym}^d(\mathcal{U}^*))$ intersecting a line, is cut out by a section of $\mathcal{O}_{\mathrm{Gr}}(d) \otimes \mathcal{O}_B(1)$. Note that a general such curve is smooth. It follows that we have the equation

$$\nu_{d,0,n-i} = \pi_*(c_1(\mathcal{O}_{\mathrm{Gr}}(d) \otimes \mathcal{O}_B(1))^{n-i}) ,$$

the right hand side of which can easily be calculated.

Now let $\delta = 1$. For a curve $C \subset \mathbb{P}^2$, given by a degree d polynomial f, the singular locus is given by the equations

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 0 , \quad f = 0 .$$

The rules $f \mapsto df$ and $f \mapsto f$ define homomorphisms

$$\mathcal{O}_B(-1) \to \Omega_{\mathcal{S}/\mathrm{Gr}} \otimes \mathcal{O}_{\mathcal{S}}(d) \quad \text{and} \quad \mathcal{O}_B(-1) \to \mathcal{O}_{\mathcal{S}}(d)$$

of bundles on $\mathcal{S} \times_{\mathrm{Gr}} B$, in which $\mathcal{S} = \mathbb{P}(\mathcal{U}) \to \mathrm{Gr}$ is the universal $\mathbb{P}^2$-bundle over the Grassmannian. The homomorphisms simultaneously vanish on the singular locus of the fibres of the universal curve $\mathcal{C} \to B$. As curves with one node lie dense in this locus, it follows that the class of the closure of the locus of 1-nodal curves in B is given by

$$\alpha = (pr_B)_*(c_3((\Omega_{\mathcal{S}/\mathrm{Gr}} \oplus \mathcal{O}_{\mathcal{S}\times_{\mathrm{Gr}} B}) \otimes \mathcal{O}_{\mathcal{S}}(d) \otimes \mathcal{O}_B(1)) \in A^1(B) .$$

We conclude that

$$\nu_{d,1,n-i} = \pi_*(c_1(\mathcal{O}_{\mathrm{Gr}}(d) \otimes \mathcal{O}_B(1))^{n-i} \cap \alpha) .$$

Again, by a straight-forward computation, we obtain the numbers in the third column. □

For $n = \frac{d(d+3)}{2} + 3 - \delta$, the number

$$\int \nu_{d,\delta,n-i}\, H^i$$

has the following interpretation: it is the number of planar curves C in $\mathbb{P}^3$ of degree d, with δ nodes, intersecting general lines $\ell_1, \ldots, \ell_{n-i} \subset \mathbb{P}^3$, such that the plane of the curve contains general points $P_1, \ldots, P_i \in \mathbb{P}^3$. For certain cases, this enumerative problem has already been studied by Schubert using his calculus introduced in [Sch79]. E.g. he treats conics, planar and twisted cubics and planar quartic curves in $\mathbb{P}^N$ that intersect points, lines and planes. The curves are allowed to have nodal singularities, or a cusp in the case of the planar cubic. The degrees of the classes in the second and third column of Lemma 1.8.10 can be found in §20 and §24 of loc. cit. respectively. By Proposition 1.8.6 it follows that the numbers in Tables 1.1 - 1.3 can computed by 19th century geometry and some elementary combinatorics.

Curves with only linear components

- $\boldsymbol{\delta = 1, d = 2}$

$$N_{1,2} = \binom{7}{3,4} \times 2^2 = 140 = N_1(2)\,.$$

- $\boldsymbol{\delta = 3, d = 3}$

$$N_{3,3} = \binom{9}{4,3,2} \times 2^2 + \frac{1}{3!} \times \binom{9}{3,3,3} \times 2^3 = 7280 = N_3(3)\,.$$

- $\boldsymbol{\delta = 6, d = 4}$

$$N_{6,4} = \frac{1}{2!} \times \binom{11}{4,3,2,2} \times 2^2 + \frac{1}{3!} \times \binom{11}{3,3,3,2} \times 2^3 = 261800 = N_6(4)\,.$$

Curves with one conic component

- $\boldsymbol{\delta = 0, d = 2}$

$$N_{0,2} = \nu_{2,0,8} = 92 = N_2(0)\,.$$

- $\boldsymbol{\delta = 2, d = 3}$

$$N_{2,3} = \binom{10}{8} \times 92 + \binom{10}{7} \times 34 \times 2 + \binom{10}{6} \times 8 \times 2 = 15660 = N_2(3)\,.$$

- $\boldsymbol{\delta = 5, d = 4}$

$$N_{5,4} = \frac{1}{2!} \times \binom{12}{8,2,2} \times 92 + \binom{12}{7,3,2} \times 34 \times 2 + \binom{12}{6,4,2} \times 8 \times 2 + \frac{1}{2!} \times \binom{12}{6,3,3} \times 8 \times 2^2 + \binom{12}{5,4,3} \times 2^2 = 1303500 = N_5(4)\,.$$

Curves with a nodal cubic or two conic components

- $\boldsymbol{\delta = 4, d = 4}$ The contribution of curves with two conic components is

$$\binom{13}{8} \times 92 + \binom{13}{7} \times 34 \times 8 = 585156\,.$$

The contribution of curves consisting of a nodal cubic and a line is

$$\binom{13}{11} \times 12960 + \binom{13}{10} \times 2040 \times 2 + \binom{13}{9} \times 216 \times 2 = 2486640\,.$$

It total we have:

$$N_{4,4} = 585156 + 2486640 = 3071796 = N_4(4)\,.$$

- $\boldsymbol{\delta = 8, d = 5}$ Two conics and a line:

$$\begin{pmatrix} 15 \\ 8,5,2 \end{pmatrix} \times 92 + \begin{pmatrix} 15 \\ 7,6,2 \end{pmatrix} \times 34 \times 8 + \begin{pmatrix} 15 \\ 7,5,3 \end{pmatrix} \times 34 \times 2 + \\ \frac{1}{2!} \times \begin{pmatrix} 15 \\ 6,6,3 \end{pmatrix} \times 8 \times 8 \times 2 + \begin{pmatrix} 15 \\ 6,5,4 \end{pmatrix} \times 8 \times 2 = 122942820\,.$$

A nodal cubic, and two lines:

$$\frac{1}{2!} \times \begin{pmatrix} 15 \\ 11,2,2 \end{pmatrix} \times 12960 + \begin{pmatrix} 15 \\ 10,3,2 \end{pmatrix} \times 2040 \times 2 + \begin{pmatrix} 15 \\ 9,4,2 \end{pmatrix} \times 216 \times 2 + \\ \frac{1}{2!} \times \begin{pmatrix} 15 \\ 9,3,3 \end{pmatrix} \times 216 \times 2 \times 2 + \begin{pmatrix} 15 \\ 8,4,3 \end{pmatrix} \times 12 \times 2 \times 2 = 262080000\,.$$

Total:

$$N_{8,5} = 122942820 + 262080000 = 385022820 = N_8(5)\,.$$

1.9 Appendix: Node polynomials for $\delta = 0, \ldots, 12$.

In order to keep the denominators under control, we will print the node polynomials for curves with *ordered* nodes, i.e. $N_\delta^o = \delta!\, N_\delta$.

$$N_0^o = \frac{1}{324}\, d\,(d-1)\,(d+2)\,(d+1)\,(d^2+4\,d+6)\,(2\,d^3+6\,d^2+13\,d+3)$$

$$N_1^o = \frac{1}{108}\, d\,(d+3)\,(d+2)\,(2\,d^4+4\,d^3+d^2-10\,d-6)\,(d-1)^2\,(d+1)^2$$

$$N_2^o = \frac{1}{108}\, d\,(d-1)\,(d-2)\,(d+2)\,(d+1)\,(6\,d^8+30\,d^7-25\,d^6-255\,d^5-142\,d^4 \\ +333\,d^3+629\,d^2+18\,d+198)$$

$$N_3^o = \frac{1}{108}\, d\,(d-1)\,(d-2)\,(18\,d^{12}+108\,d^{11}-315\,d^{10}-2664\,d^9+470\,d^8 \\ +21919\,d^7+19103\,d^6-58136\,d^5-106948\,d^4+7039\,d^3+129360\,d^2 \\ -165798\,d+110700)$$

$$N_4^o = \frac{1}{36}\,(d-1)\,(d-3)\,(18\,d^{15}+90\,d^{14}-747\,d^{13}-3843\,d^{12}+11660\,d^{11} \\ +63140\,d^{10}-75352\,d^9-486678\,d^8+73143\,d^7+1773729\,d^6+1150606\,d^5 \\ -4123550\,d^4-3282032\,d^3+12893256\,d^2-11795040\,d+3404160)$$

$$N_5^o = \frac{1}{36}\,(d-1)\,(54\,d^{18}-4545\,d^{16}+1152\,d^{15}+159342\,d^{14}-67218\,d^{13} \\ -2985967\,d^{12}+1450512\,d^{11}+32041927\,d^{10}-12936036\,d^9-198254910\,d^8 \\ +9946932\,d^7+752976733\,d^6+563804514\,d^5-2869526338\,d^4 \\ -1811459616\,d^3+11267964504\,d^2-12007211040\,d+4224182400)$$

$$N_6^o = \frac{1}{36}\,(162\,d^{21}-486\,d^{20}-17901\,d^{19}+56781\,d^{18}+836361\,d^{17}-2772558\,d^{16} \\ -21438711\,d^{15}+73412631\,d^{14}+327808568\,d^{13}-1138677007\,d^{12} \\ -3072121759\,d^{11}+10302259428\,d^{10}+18632510223\,d^9-50159288793\,d^8 \\ -99732049025\,d^7+130703793592\,d^6+629801216266\,d^5 \\ -777706339956\,d^4-2089991213304\,d^3+5446674186768\,d^2 \\ -4582360442880\,d+1359752313600)$$

$$N_7^o = \frac{1}{12}\,(162\,d^{23}-810\,d^{22}-22275\,d^{21}+117045\,d^{20}+1315044\,d^{19}-7305633\,d^{18} \\ -43435062\,d^{17}+257593851\,d^{16}+875704283\,d^{15}-5620623440\,d^{14} \\ -11055698265\,d^{13}+77840061643\,d^{12}+89179790228\,d^{11} \\ -672462975543\,d^{10}-563743329044\,d^9+3506892852821\,d^8 \\ +4693983485919\,d^7-13574568995962\,d^6-37376320692374\,d^5 \\ +84863008074540\,d^4+101290677876264\,d^3-419002213496112\,d^2 \\ +415086981865920\,d-136551736742400)$$

$$
\begin{aligned}
N_8^o = \frac{1}{12}\,(&486\,d^{25} - 3402\,d^{24} - 80271\,d^{23} + 603369\,d^{22} + 5736609\,d^{21}\\
&- 47210985\,d^{20} - 230681484\,d^{19} + 2141947278\,d^{18} + 5649412578\,d^{17}\\
&- 62197110162\,d^{16} - 84069436618\,d^{15} + 1201119124190\,d^{14}\\
&+ 695539180710\,d^{13} - 15500834280650\,d^{12} - 2727660315107\,d^{11}\\
&+ 131722402261845\,d^{10} + 25466213716945\,d^{9} - 750756824927669\,d^{8}\\
&- 664023356945796\,d^{7} + 3782983980383618\,d^{6} + 6489582893159132\,d^{5}\\
&- 24182782626411432\,d^{4} - 11635999979827824\,d^{3} + 98923354020446400\,d^{2}\\
&- 113846941521653760\,d + 40910206904985600)\\
N_9^o = \frac{1}{12}\,(&1458\,d^{27} - 13122\,d^{26} - 282123\,d^{25} + 2810295\,d^{24} + 23620086\,d^{23}\\
&- 269654670\,d^{22} - 1102117023\,d^{21} + 15284004291\,d^{20} + 30114876816\,d^{19}\\
&- 567444295476\,d^{18} - 422910483264\,d^{17} + 14442428462976\,d^{16}\\
&- 173655449080\,d^{15} - 256024966449048\,d^{14} + 117007607498013\,d^{13}\\
&+ 3154205513887891\,d^{12} - 1917888357827630\,d^{11}\\
&- 26939284058835262\,d^{10} + 9381328237342969\,d^{9}\\
&+ 170575538835999315\,d^{8} + 81091482477623574\,d^{7}\\
&- 1043320220595663742\,d^{6} - 1048469186302651972\,d^{5}\\
&+ 6715930311282223672\,d^{4} + 3733173716347953 6\,d^{3}\\
&- 24266279644066088640\,d^{2} + 32202247356878376960\,d\\
&- 12516744443551488000)\\
N_{10}^o = \frac{1}{4}\,(&1458\,d^{29} - 16038\,d^{28} - 324405\,d^{27} + 4083129\,d^{26} + 30991005\,d^{25}\\
&- 471257676\,d^{24} - 1603277307\,d^{23} + 32578597143\,d^{22} + 43377665589\,d^{21}\\
&- 1500353595792\,d^{20} - 174393237924\,d^{19} + 48387112634196\,d^{18}\\
&- 31729963605856\,d^{17} - 1117426679453368\,d^{16} + 1278946167008861\,d^{15}\\
&+ 18584197566356041\,d^{14} - 25929838100941987\,d^{13}\\
&- 222292527680013236\,d^{12} + 301840036425933217\,d^{11}\\
&+ 1945082392976187963\,d^{10} - 1702396861961183657\,d^{9}\\
&- 13840054897129551538\,d^{8} - 869477353690603586\,d^{7}\\
&+ 97969270160191718168\,d^{6} + 43400439640826602848\,d^{5}\\
&- 629779784557096225952\,d^{4} + 234491922967527070944\,d^{3}\\
&+ 2074544035031110584960\,d^{2} - 3163830549287964595200\,d\\
&+ 1320649590395681280000)
\end{aligned}
$$

$$
\begin{aligned}
N_{11}^{o} = \frac{1}{4}\,(&4374\,d^{31} - 56862\,d^{30} - 1102977\,d^{29} + 16973307\,d^{28} + 117608112\,d^{27} \\
&- 2318176989\,d^{26} - 6425985798\,d^{25} + 191682452511\,d^{24} \\
&+ 133119473328\,d^{23} - 10693744028253\,d^{22} + 5846987132217\,d^{21} \\
&+ 424349459112114\,d^{20} - 577264977532776\,d^{19} - 12296681379527388\,d^{18} \\
&+ 23489692414637363\,d^{17} + 263030758384442289\,d^{16} \\
&- 593247075529299782\,d^{15} - 4162564652290610993\,d^{14} \\
&+ 9903092880496934734\,d^{13} + 49018479445283034499\,d^{12} \\
&- 106385448246367215702\,d^{11} - 447930561908076256091\,d^{10} \\
&+ 645540608889693443477\,d^{9} + 3606790056461753863832\,d^{8} \\
&- 1341780384161439521626\,d^{7} - 28540272186090313415704\,d^{6} \\
&+ 12057954350574986515 84\,d^{5} + 182406168443172371488448\,d^{4} \\
&- 128952276571759318016928\,d^{3} - 557203918390573072878720\,d^{2} \\
&+ 976485597554969367782400\,d - 435294406202292274176000)
\end{aligned}
$$

$$
\begin{aligned}
N_{12}^{o} = \frac{1}{4}\,(&13122\,d^{33} - 196830\,d^{32} - 3706965\,d^{31} + 68070375\,d^{30} + 432815319\,d^{29} \\
&- 10850751657\,d^{28} - 23535697932\,d^{27} + 1055997538326\,d^{26} \\
&+ 50357440881\,d^{25} - 70021319228739\,d^{24} + 87374064448161\,d^{23} \\
&+ 3341431959709527\,d^{22} - 7111742962317408\,d^{21} \\
&- 118120713345379188\,d^{20} + 326384557105326777\,d^{19} \\
&+ 3137002724874226941\,d^{18} - 10106444734270420903\,d^{17} \\
&- 62918353809936417707\,d^{16} + 220730277300344083152\,d^{15} \\
&+ 956623940326083332050\,d^{14} - 3393890378620526954445\,d^{13} \\
&- 11241179417671261959041\,d^{12} + 35363022169384877426927\,d^{11} \\
&+ 109549515125017919639429\,d^{10} - 229128080595756761453742\,d^{9} \\
&- 1002037783274877109543198\,d^{8} + 840677011541967726001824\,d^{7} \\
&+ 8641146394045844077954112\,d^{6} - 4223694280033586640137824\,d^{5} \\
&- 54779602892548858064166240\,d^{4} + 56217660837944500164819456\,d^{3} \\
&+ 156589791424366871478896640\,d^{2} - 316225057234071161731737600\,d \\
&+ 149867365795069610096640000)
\end{aligned}
$$

1.10 Appendix: A node polynomial for curves in $\mathbb{P}^2$

The number of 15-nodal curves degree $d \geq 9$ (by [KS13]) in $\mathbb{P}^2$ containing $\frac{d(d+1)}{2} - 15$ points in general position is given by the following polynomial.

$$
\begin{aligned}
&\frac{1}{15!}(14348907\,d^{30} - 430467210\,d^{29} - 789189885\,d^{28} + 144134770815\,d^{27}\\
&- 800302316310\,d^{26} - 21505566260997\,d^{25} + 206046709321635\,d^{24}\\
&+ 1830389081571180\,d^{23} - 25973085837797631\,d^{22} - 90805122781323093\,d^{21}\\
&+ 2106764580151475244\,d^{20} + 1842311595032520885\,d^{19}\\
&- 120731061785804511795\,d^{18} + 83105496803044790514\,d^{17}\\
&+ 5106565375968131056197\,d^{16} - 8800802481614659877511\,d^{15}\\
&- 162890506083253675564674\,d^{14} + 397425775424906515333221\,d^{13}\\
&+ 3952008654242554161166365\,d^{12} - 11546375323786656779457252\,d^{11}\\
&- 72858625897371563437077825\,d^{10} + 232182939704411137229570133\,d^{9}\\
&+ 1010825449711157998476650988\,d^{8} - 3241105115881805786551102893\,d^{7}\\
&- 10336040203392280930456480032\,d^{6} + 30163840992557581783875044832\,d^{5}\\
&+ 74721661229894928962601063456\,d^{4} - 168817217722446315040796818224\,d^{3}\\
&- 347671495806428829919633280640\,d^{2} + 429634898369604339129576633600\,d\\
&+ 794015010296634348660582144000)
\end{aligned}
$$

Chapter 2

Monopole contributions to refined Vafa-Witten invariants

2.1 Introduction

2.1.1 Vafa-Witten invariants

In [TT17a], Yuuji Tanaka and Richard Thomas proposed a definition of an SU(r) Vafa-Witten invariant [VW94]. Let (S,H) be a polarized smooth complex surface with canonical bundle ω_S. A *Higgs pair* is a pair

$$(E,\phi) \quad \text{with} \quad E \in \mathrm{Coh}(S),\, \phi\colon E \to E \otimes \omega_S\,.$$

Choose a rank r, Chern classes c_1, c_2 on S, and a line bundle M on S with $c_1(M) = c_1$. Assume that r, c_1 and c_2 are chosen in such a way that stability and semistability of Higgs pairs coincide (see Section 2.3). Let

$$\mathcal{N}^{\perp}_{r,M,c_2} = \{(E,\phi) \mid \operatorname{tr}\phi = 0, \operatorname{rk}(E) = r, \det E \cong M, c_2(E) = c_2\}$$

be the moduli space of Gieseker stable trace free Higgs pairs with fixed determinant. In [TT17a] a symmetric perfect obstruction theory on $\mathcal{N}^{\perp}_{r,M,c_2}$ is constructed. Its dual complex is given by the cone

$$R\mathscr{H}om_\pi(\mathbb{E},\mathbb{E})_0 \xrightarrow{[\ ,\phi]} R\mathscr{H}om_\pi(\mathbb{E},\mathbb{E}\otimes\omega_S)_0 \to T\,, \tag{2.1.1}$$

where $(\mathbb{E},\phi)$ is a universal Higgs pair on $\mathcal{N}^{\perp}_{r,M,c_2} \times S$ and

$$\pi\colon \mathcal{N}^{\perp}_{r,M,c_2} \times S \to \mathcal{N}^{\perp}_{r,M,c_2}$$

denotes the projection. The $\mathbb{C}^*$-action on $\mathcal{N}^{\perp}_{r,M,c_2}$, which is given by scaling the Higgs field, can be lifted to an equivariant structure on $\mathbb{E}$. It gives rise to a localized virtual class, which is used to define the Vafa-Witten invariant by

$$\mathrm{VW}_{r,c_1,c_2}(S) = \int_{[(\mathcal{N}^{\perp}_{r,M,c_2})^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}\,, \tag{2.1.2}$$

where N^{vir} is the virtual normal bundle to $(\mathcal{N}^{\perp}_{r,M,c_2})^{\mathbb{C}^*}$ in $\mathcal{N}^{\perp}_{r,M,c_2}$, and $e(N^{\mathrm{vir}})$ denotes its equivariant Euler class.

A Higgs pair (E,ϕ) in the fixed locus $(\mathcal{N}^{\perp}_{r,M,c_2})^{\mathbb{C}^*}$ can be equipped with a $\mathbb{C}^*$-action, and hence decomposes into weight spaces. We may assume the 0 is the highest weight appearing in the decomposition. As explained in [TT17a], the Higgs field acts with weight -1. Hence we can write

$$E = \bigoplus_{i=0}^{k} E_i \otimes \mathfrak{t}^{-i}$$
$$\phi = (\phi_1, \ldots, \phi_k) \colon E \to E \otimes \omega_S \otimes \mathfrak{t}\,,$$

where the E_i are torsion free sheaves of rank r_i and ϕ decomposes into maps

$$\phi_i \colon E_{i-1} \to E_i \otimes \omega_S \otimes \mathfrak{t} \quad \text{for} \quad i = 1, \ldots, k\,.$$

We will write

$$\mathcal{M}_{(r_0,\ldots,r_k)} = \mathcal{M}_{(r_0,\ldots,r_k),c_1,c_2} \subset (\mathcal{N}^{\perp}_{r,M,c_2})^{\mathbb{C}^*}$$

for the open and closed locus of Higgs pairs with weight spaces of dimensions $r_0, \ldots, r_k$. The locus

$$\mathcal{M}_{(r)} = \left\{(E,\phi) \in (\mathcal{N}^{\perp}_{r,M,c_2})^{\mathbb{C}^*} \mid \phi = 0\right\}$$

is called the *instanton branch* [GK18]. It is isomorphic to the moduli space of torsion free rank r sheaves, and its contribution to the Vafa-Witten invariant is the (localized) virtual Euler characteristic (up to a sign). Its complement in the $\mathbb{C}^*$-fixed locus is called the *monopole branch*. In this paper, we will discuss the contribution of the locus $\mathcal{M}_{1^r} = \mathcal{M}_{(1\cdots 1)}$ of Higgs pairs with 1-dimensional weight spaces to the monopole branch. As an application of [GT19], we will describe the structure of the generating series of the contributions of $\mathcal{M}_{1^r}$ to the Vafa-Witten invariant, and compute them in some cases.

In [MT] (see also [Tho18a]), Maulik and Thomas define a refined version of the Vafa-Witten invariant. It is a rational function in $\sqrt{y}$, rather than a rational number. It specializes to the unrefined invariant at $y = 1$. The instanton contribution to the refined Vafa-Witten invariant is given, up to a sign and a power of y, by the χ_y-genus [FG10] of the component $\mathcal{M}_{(r)}$, which refines the virtual Euler characteristic [GK17]. We will discuss the contribution of $\mathcal{M}_{1^r}$ to the refined invariant.

2.1.2 Nested Hilbert schemes

Fix a rank r. For an r-tuple of non-negative integers $n = (n_0, \ldots, n_{r-1})$, and an $(r-1)$-tuple $\beta = (\beta_1, \ldots, \beta_{r-1})$ of classes in $H^2(S, \mathbb{Z})$, let

$$S^{[n_i]} := \mathrm{Hilb}^{n_i}(S)$$

denote the Hilbert schemes of n_i points on S, and let $\mathrm{Hilb}_{\beta_i}(S)$ be the Hilbert schemes of curves on S with class β_i. We will also write

$$\mathrm{Hilb}^n_\beta(S) := S^{[n_0]} \times \cdots \times S^{[n_{r-1}]} \times \mathrm{Hilb}_{\beta_1}(S) \times \cdots \times \mathrm{Hilb}_{\beta_s}(S)\,.$$

The *nested Hilbert scheme*

$$i : S^{[n]}_\beta \hookrightarrow \mathrm{Hilb}^n_\beta(S) \tag{2.1.3}$$

is defined as the incidence locus

$$\{I_0, \ldots, I_{r-1}, C_1, \ldots, C_{r-1} \mid I_{i-1}(-C_i) \subset I_i\}\,.$$

The nested Hilbert schemes are studied in [GSY18], in which a perfect obstruction theory is constructed. Write $\mathcal{I}^{[n_i]}$ for the universal ideal sheaf on $S^{[n_i]} \times S$, and let

$$\mathcal{D}_i \to \mathrm{Hilb}_{\beta_i}(S) \times S$$

be the universal curve with class β_i. Finally, write

$$\pi\colon S^{[n]}_\beta \times S \to S^{[n]}_\beta$$

for the projection.

Theorem 2.1.4. [GSY18] *The nested Hilbert scheme $S^{[n]}_\beta$ admits a perfect obstruction theory, the dual of which is given by a cone on*

$$\left(\bigoplus_{i=0}^{r-1} R\mathscr{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]})\right)_0 \to \bigoplus_{i=1}^{r-1} R\mathscr{H}om_\pi\left(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\mathcal{D}_i)\right)\,,$$

in which the LHS is the cocone of the trace map

$$\bigoplus_{i=0}^{r-1} R\mathscr{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}) \to R\pi_*\mathcal{O}_S\,.$$

In [GT19], Gholampour and Thomas give another construction of the perfect obstruction theory, using virtual resolutions of degeneracy loci of complexes. Moreover, they give a formula for the induced virtual class in the ambient space (2.1.3). We will give the statement in the following restricted setting.

Let S be a surface satisfying

$$H^1(\mathcal{O}_S) = 0 \quad \text{and} \quad p_g(S) > 0\,.$$

For $i = 0, \ldots, r-1$, let $\mathcal{O}_S(\beta_i)$ be the line bundle with $c_1(\mathcal{O}_S(\beta_i)) = \beta_i$, so we have

$$\mathrm{Hilb}_{\beta_i}(S) = |\mathcal{O}_S(\beta_i)| \coloneqq \mathbb{P}(H^0(\mathcal{O}_S(\beta_i)))\,.$$

We will write

$$\mathcal{F}(\beta_i) = \mathcal{F} \otimes \mathcal{O}_S(\beta_i)$$

for any sheaf $\mathcal{F}$ on S.

Theorem 2.1.5. [GT19, Theorem 5.6] *After push-forward by i, the virtual class of $S^{[n]}_\beta$ is given by*

$$\begin{aligned} i_*[S^{[n]}_\beta]^{\mathrm{vir}} = &\prod_{i=1}^{r-1} e\left(R\pi_*\mathcal{O}_S(\beta_i) - R\mathscr{H}om_\pi\left(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)\right)\right) \\ &\cap \left[S^{[n_0]} \times \cdots \times S^{[n_{r-1}]}\right] \times \mathrm{SW}(\beta_1) \times \cdots \times \mathrm{SW}(\beta_{r-1}) \\ \in\ & A_{n_0+n_s}\left(\mathrm{Hilb}^n_\beta(S)\right) \end{aligned}$$

in which

$$\mathrm{SW}(\beta_i) \in A_0(|\mathcal{O}_S(\beta_i)|) \cong \mathbb{Z}$$

is the Seiberg-Witten invariant of β_i, considered as a 0-cycle.

Remark 2.1.6. We write

$$e\left(R\pi_*\mathcal{O}_S(\beta_i) - R\mathscr{H}om_\pi\left(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)\right)\right)$$

for $(n_{i-1}+n_i)$th Chern class of the K-theory class in the brackets, which behaves in some sense like a rank $n_{n-1}+n_i$ vector bundle. E.g., by the generalized Carlsson-Okounkov vanishing [GT17], its Chern classes vanish beyond its rank.

Remark 2.1.7. For the definition and some basic properties of Seiberg-Witten classes of algebraic surfaces with $H^1(S)=0$ and $p_g>0$, we refer to [Moc09, Section 6.3.1] or [GSY17, Section 4].

Remark 2.1.8. It is Theorem 2.1.5 that allows us to compute the $\mathcal{M}_{1^r}$ contributions to the Vafa-Witten invariant. A large part of our paper should be seen as an application of the result by Gholampour and Thomas.

2.1.3 Results

The moduli space $\mathcal{M}_{1^r}$ is a union of nested Hilbert schemes $S_\beta^{[n]}$ [GSY17, TT17a]. Moreover, the $\mathbb{C}^*$-localized virtual class from [TT17a] agrees with the virtual class from Theorem 2.1.4. It follows that the contribution of each component $\mathcal{M}_{1^r}$ to the Vafa-Witten invariant is *topological* [GT19]. The observation that the generating series of these contributions is multiplicative cf. [Göt98], leads to the following result.

Notation 2.1.9. We will write

$$\mathrm{VW}_{1^r,c_1,c_2}(S,y)$$

for the contribution of $\mathcal{M}_{1^r}=\mathcal{M}_{1^r,c_1,c_2}$ to the refined Vafa-Witten invariant of [MT], and

$$\mathsf{Z}_{S,r,c_1}(q,y) = \frac{q^{\frac{1-r}{2r}c_1^2}}{\#H^2(S,\mathbb{Z})[r]}\sum_{c_2\in\mathbb{Z}}\mathrm{VW}_{1^r,c_1,c_2}(S,y)\,q^{c_2}$$

for the generating series of such contributions. Here

$$H^2(S,\mathbb{Z})[r] := \ker\left(H^2(S,\mathbb{Z}) \xrightarrow{\cdot r} H^2(S,\mathbb{Z})\right)$$

denotes the r-torsion subgroup of $H^2(S,\mathbb{Z})$.

Theorem 2.A. *Fix a rank $r\geq 1$. There are universal Laurent series*

$$A,\, B,\, C_{ij} \in \mathbb{Q}(\sqrt{y})(\!(q^{\frac{1}{2r}})\!) \quad 1\leq i\leq j\leq r-1\,,$$

depending only *on r, such that for any surface S with $H^1(\mathcal{O}_S)=0$ and $p_g(S)>0$, and any class $c_1\in H^2(S,\mathbb{Z})$ such that semistability of Higgs pairs implies stability for all c_2, we have*

$$\mathsf{Z}_{S,r,c_1}(q,y) = A^{\chi(\mathcal{O}_S)}B^{K_S^2}\sum_{\beta}\mathrm{SW}(\beta^1)\cdots\mathrm{SW}(\beta^{r-1})\prod_{i\leq j}C_{ij}^{\beta^i\beta^j} \tag{2.1.10}$$

where the sum is taken over tuples $\beta=(\beta^1,\dots,\beta^{r-1})\in(H^2(S,\mathbb{Z}))^{r-1}$ with

$$c_1 \equiv \sum_i i\beta^i \mod rH^2(S,\mathbb{Z})\,.$$

Remark 2.1.11. The condition $H^1(\mathcal{O}_S) = 0$ in Theorem 2.A is superfluous. However, for expository reasons, we will work with this condition throughout the paper. Moreover, (2.1.10) can be shown to hold for *all* c_1 when we extend the definition of the LHS to the semistable case. This will be subject of future work.

Remark 2.1.12. For odd rank r, the Laurent series have coefficients in $\mathbb{Q}(y)$, rather than in $\mathbb{Q}(\sqrt{y})$ (see Proposition 2.8.4).

The following corollary is implicitly in the statement of Theorem 2.A.

Corollary 2.1.13. *Fix a rank r, and let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. Let c_1 be a Chern classes, for which semistability implies stability for all c_2. Then $\mathsf{Z}_{S,r,c_1}(q, y)$ is independent of the choice of a polarization of the surface S.*

We will define the Laurent series in the theorem explicitly in terms of tautological integrals over products of Hilbert schemes of points on the surface S (see Sections 2.5, 2.7 and 2.8). Although for surfaces with $\deg(K_S) < 0$, the locus $\mathcal{M}_{1^r}$ is empty by stability, the Hilbert schemes and the integrals are still defined. We will prove universality of these integrals for *all* surfaces (Proposition 2.7.6). As usual [Göt98], the coefficients of the power series can be determined by evaluating these integrals on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where we have access to toric methods, as we explain in Section 2.9.

2.1.4 Göttsche-Kool conjectures

In [GK18], Lothar Göttsche and Martijn Kool conjecture a formula for the generating series of the χ_y-genus of the instanton branch for rank 2 and 3. Moreover they conjecture, motivated by S-duality [VW94], that the generating series of refined Vafa-Witten invariants has modular properties that relate the contributions of the instanton branch to those of the monopole branch. Using this, they give a conjectural formula for the contribution of the monopole branch to refined Vafa-Witten invariants of rank 2 and 3. For rank 2, their conjectures refine the predictions in the physics literature [VW94].

The formulas of [GK18] that predict the monopole contributions to the Vafa-Witten invariants in rank 2 and 3, have precisely the structure of the generating series (2.1.10) of the $\mathcal{M}_{1^r}$ contributions. This suggests that $\mathcal{M}_{1^r}$ accounts for the entire monopole contribution.

Conjecture 2.1.14. *For S and c_1 as in Theorem 2.A, and r prime, we have*

$$\mathrm{VW}^{\mathrm{monopole}}_{r,c_1,c_2}(S, y) = \mathrm{VW}_{1^r,c_1,c_2}(S, y)$$

for all $c_2 \in \mathbb{Z}$.

The conjecture has now been proved by Thomas in [Tho18a].

Theorem 2.1.15 (Thomas)**.** *Conjecture 2.1.14 holds.*

It follows that Theorem 2.A and Theorem 2.1.15 prove the *structure* of [GK18, Conjecture 1.5], generalized to arbitrary prime rank. The rank 2 and 3 conjectures of [GK18] give the universal series appearing in the formula explicitly in terms of functions

$$\phi_{-2,1}(x,y),\ \Delta(x),\ \Theta_{A_2,(1,0)}(x,y),\ \eta(x),\ \theta_2(x,y),\ \theta_3(x,y),\ \text{and } W_{\pm}(x,y),$$

which we give in Section 2.13. The following conjectures imply [GK18, Remark 1.7 and Conjecture 1.5].

Notation 2.1.16. In order to emphasize the dependency on r, we will write $A^{(r)}$, $B^{(r)}$ and $C_{ij}^{(r)}$ for the series appearing in Theorem 2.A.

Conjecture 2.1.17. *For rank* 2*, the universal series appearing in Theorem 2.A, and defined in Section 2.8, are given by*

$$\begin{aligned} A^{(2)}(y) &= \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^2,y^2)^{\frac{1}{2}}\,\tilde{\Delta}(q^2)^{\frac{1}{2}}}\,, \\ B^{(2)}(y) &= \frac{\tilde{\eta}(q)^2}{\theta_3(q,y)}\,, \\ C_{11}^{(2)}(y) &= \frac{-\theta_3(q,y)}{\theta_2(q,y)}\,. \end{aligned}$$

Conjecture 2.1.18. *For rank* 3*, we have*

$$\begin{aligned} A^{(3)}(y) &= \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^3,y^3)^{\frac{1}{2}}\,\tilde{\Delta}(q^3)^{\frac{1}{2}}}\,, \\ B^{(3)}(y) &= \frac{\tilde{\eta}(q)^3\,W_-(q^{\frac{1}{2}},y)}{\Theta_{A_2,(1,0)}(q^{\frac{1}{2}},y)}\,, \\ C_{12}^{(3)}(y) &= W_+(q^{\frac{1}{2}},y)\,W_-(q^{\frac{1}{2}},y)\,, \\ C_{11}^{(3)}(y) &= C_{22}^{(3)}(y) \\ &= \frac{1}{W_-(q^{\frac{1}{2}},y)}\,. \end{aligned}$$

2.1.5 Toric computations

As remarked before, the universality result of Proposition 2.7.6 allows us to determine the first few terms of the power series of Theorem 2.A by toric computations. I implemented the Atiyah-Bott localization formula for the surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in Sage [Sag15] and found agreement with Conjectures 2.1.17 and 2.1.18.

Define multiplicative subgroups

$$U_N^{(r)} := 1 + q^N\,\mathbb{Q}(y^{\frac{1}{2}})[[q]] \subset \mathbb{Q}(y^{\frac{1}{2}})(\!(q^{\frac{1}{2r}})\!)^*\,.$$

for all $r, N \geq 1$, and consider series

$$\begin{aligned} P &= cq^{\frac{z}{2r}}(1 + p_1 q + p_2 q^2 + \ldots) \quad \text{and} \\ P' &= c' q^{\frac{z'}{2r}}(1 + p_1' q + p_2' q^2 + \ldots) \end{aligned}$$

with $c, c' \in \mathbb{Q}(\sqrt{y})^*$, $p_i, p'_i \in \mathbb{Q}(\sqrt{y})$ and $z, z' \in \mathbb{Z}$. The Laurent series appearing in Theorem 2.A and Conjectures 2.1.17 and 2.1.18 are all of this form. Then we have

$$P \equiv P' \mod U^{(r)}_{N+1} \quad (\text{i.e. } P'P^{-1} \in U^{(r)}_{N+1})$$

if and only if

$$c = c', \quad z = z', \quad \text{and} \quad p_1 = p'_1, \ldots, p_N = p'_N .$$

Theorem 2.B. *Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. The rank 2 conjectures of [GK18] correctly predict the first 15 terms of the universal series of Theorem 2.A. The rank 3 conjectures correctly predict the first 11 terms. In other words, the equations of Conjecture 2.1.17 hold modulo $U^{(2)}_{15}$, and the equations of Conjecture 2.1.18 hold modulo $U^{(3)}_{11}$.*

2.1.6 K3 surfaces

Let S be a K3 surface. Then $0 \in H^2(S, \mathbb{Z})$ is the only Seiberg-Witten basic class of S, and for $c_1 = 0$, equation (2.1.10) becomes

$$\text{“}\mathsf{Z}_{S,r,0}(q, y)\text{”} = \left(A^{(r)}\right)^2 .$$

Note that in our setting, the left-hand side has not been defined for $r > 1$, due to the existence of strictly semistable sheaves. Hence we cannot apply Theorem 2.A directly to determine the power series $A^{(r)}$. We can, however, evaluate on S the tautological integrals that are used to define the universal series $A^{(r)}$. This leads to the following result, which we prove in Section 2.10.

Theorem 2.C. *We have*

$$A^{(r)}(y) = \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^r, y^r)^{\frac{1}{2}}\, \tilde{\Delta}(q^r)^{\frac{1}{2}}} \tag{2.1.19}$$

for any $r \geq 1$.

Remark 2.1.20. Refined Vafa-Witten invariants of K3 surfaces have been computed in [Tho18a]. In Theorem 3.B we extend Theorem 2.A to the semistable case, so Theorem 2.C will follow also from the results of loc. cit..

2.1.7 A special case

Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and assume the Picard group

$$\mathrm{Pic}(S) = \mathbb{Z} \cdot [C]$$

of S is generated by a smooth very ample canonical curve $C \in |K_S|$. Let $c_1 = K_S$. For rank 3, only $\beta = (K_S, 0)$ contributes to the right-hand side of (2.1.10) (see Lemma 2.11.1). In rank 2, and in a slightly more general setting [TT17a], the only contribution is given by $\beta = (K_S)$. By Theorem 2.A, we have

$$\mathsf{Z}_{S,r,K_S}(q, y) = \left(-A^{(r)}(y)\right)^{\chi(\mathcal{O}_S)} \left(B^{(r)}(y) C^{(r)}_{11}(y)\right)^{K_S^2} \tag{2.1.21}$$

for $r = 2, 3$. Here we have used the equation

$$\mathrm{SW}(K_S) = (-1)^{\chi(\mathcal{O}_S)} \tag{2.1.22}$$

by e.g. [Moc09, Proposition 6.3.4]. In this setting, our computations are slightly faster, and we find the following result.

Theorem 2.B′. *Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and assume that the Picard group of S is generated by a smooth very ample canonical curve. Then we have*

$$\mathsf{Z}_{S,2,K_S}(q,y) \equiv \left(\frac{-(y^{\frac{1}{2}} - y^{-\frac{1}{2}})}{\phi_{-2,1}(q^2,y^2)^{\frac{1}{2}}\tilde{\Delta}(q^2)^{\frac{1}{2}}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{-\tilde{\eta}(q)^2}{\theta_2(q,y)}\right)^{K_S^2} \mod U_{18}^{(2)}\,,$$

$$\mathsf{Z}_{S,3,K_S}(q,y) \equiv \left(\frac{-(y^{\frac{1}{2}} - y^{-\frac{1}{2}})}{\phi_{-2,1}(q^3,y^3)^{\frac{1}{2}}\tilde{\Delta}(q^3)^{\frac{1}{2}}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{\tilde{\eta}(q)^3}{\Theta_{A_2,(1,0)}(q^{\frac{1}{2}},y)}\right)^{K_S^2} \mod U_{14}^{(3)}\,.$$

For S a surface as in Theorem 2.B′, and rank $r = 2$, the moduli space $\mathcal{M}_{1^2,K_S,c_2}$ is smooth for $c_2 \leq 3$. In [TT17a] and [Tho18a], this is used to compute the Vafa-Witten invariant by direct intersection-theoretic calculations. The rank 2 equation of Theorem 2.B′ is proved modulo $U_3^{(2)}$ in [Tho18a]. In [TT17a], it is proved modulo $U_4^{(2)}$ in the unrefined case.

For rank 3, the moduli space $\mathcal{M}_{1^3,K_s,c_2}$ is smooth if and only if $c_2 \leq 2$ (see Proposition 2.11.2). This allows us to compute the Vafa-Witten invariants by the methods of [TT17a, Tho18a]. As a result, we obtain an *alternative proof* by direct calculations, discussed in Section 2.11, for the rank 3 equation of Theorem 2.B′, modulo $U_3^{(3)}$.

2.2 The moduli space

Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$, and fix a rank r. As mentioned in the introduction, the locus $\mathcal{M}_{1^r}$ of Higgs pairs with 1-dimensional weight spaces is a union of nested Hilbert schemes. In this section, we will introduce some notation and describe universal Higgs pairs over the components.

Write $s \coloneqq r - 1$ and let $L = (L_0, \ldots, L_s)$ be an r-tuple of line bundles on S.

Notation 2.2.1. Define classes

$$\beta_i = c_1(L_i \otimes L_{i-1}^* \otimes \omega_S) \\ \in H^2(S, \mathbb{Z})$$

for $i = 1, \ldots, s$, and write

$$\beta = \beta(L) = (\beta_1, \ldots, \beta_s)\,.$$

We will also write

$$\beta^i = K_S - \beta_i$$

for $i = 1, \dots, s$ and an s-tuple $\beta = (\beta_1, \dots, \beta_s) \in (H^2(S, \mathbb{Z}))^s$. In particular, when $\beta = \beta(L)$, we have

$$\beta^i = c_1(L_{i-1} \otimes L_i^*)$$

for $i = 1, \dots, s$.

Remark 2.2.2. We will use the convention

$$s \coloneqq r - 1$$

throughout the paper. Furthermore, L will always denote an $s+1$-tuple of line bundles on S, and β an s-tuple of classes in $H^2(S, \mathbb{Z})$.

Consider the product of complete linear systems

$$\begin{aligned}\mathrm{Hilb}_\beta(S) &= \mathrm{Hilb}_{\beta_1}(S) \times \cdots \times \mathrm{Hilb}_{\beta_s}(S) \\ &= |\mathcal{O}_S(\beta_1)| \times \cdots \times |\mathcal{O}_S(\beta_s)| \,.\end{aligned}$$

and write

$$\begin{array}{ccccc} |\mathcal{O}_S(\beta_i)| \times S & \xleftarrow{\mathrm{pr}_i} & \mathrm{Hilb}_\beta(S) \times S & \xrightarrow{q} & S \\ & & \downarrow{\scriptstyle \pi} & & \\ & & \mathrm{Hilb}_\beta(S) & & \end{array}$$

for the projections, where $i = 1, \dots, s$. We will write $\mathcal{O}_{\beta_i}(1)$ for the canonical line bundle on $|\mathcal{O}_S(\beta_i)|$.

Define the following line bundles on $\mathrm{Hilb}_\beta(S) \times S$:

$$\begin{aligned}\mathcal{L}_0 &\coloneqq L_0 \\ \mathcal{L}_1 &\coloneqq L_1 \otimes \mathrm{pr}_1^* \mathcal{O}_{\beta_1}(1) \\ &\cdots \\ \mathcal{L}_s &\coloneqq L_s \otimes \mathrm{pr}_1^* \mathcal{O}_{\beta_1}(1) \otimes \cdots \otimes \mathrm{pr}_s^* \mathcal{O}_{\beta_s}(1) \,.\end{aligned}$$

The tautological sections

$$\mathcal{O}_{|\mathcal{O}_S(\beta_i)| \times S} \to \mathcal{O}_{\beta_i}(1) \boxtimes \mathcal{O}_S(\beta_i)$$

induce maps

$$\phi_{\mathcal{L},i} \colon \mathcal{L}_{i-1} \to \mathcal{L}_i \otimes q^* \omega_S$$

for $i = 1, \dots, s$.

Let $\mathfrak{t}$ be an equivariant parameter for the trivial $\mathbb{C}^*$-action on a point. Define the locally free sheaf

$$E_{\mathcal{L}} \coloneqq (\mathcal{L}_0 \otimes \mathfrak{t}^0) \oplus \ldots \oplus (\mathcal{L}_s \otimes \mathfrak{t}^{-s}) \,.$$

The maps $\phi_{\mathcal{L},i}$ define a $\mathbb{C}^*$-equivariant Higgs field

$$\phi_{\mathcal{L}} = (\phi_{\mathcal{L},1}, \dots, \phi_{\mathcal{L},s}) : E_{\mathcal{L}} \to E_{\mathcal{L}} \otimes q^* \omega_S \otimes \mathfrak{t} \,.$$

Now choose non-negative integers $n = (n_0, \dots, n_s)$ and write

$$\mathrm{Hilb}^n_\beta(S) = S^{[n_0]} \times \cdots \times S^{[n_s]} \times \mathrm{Hilb}_\beta(S)$$

as in the introduction. Let $\mathcal{I}^{[n_i]}$ denote the universal ideal sheaf on $S^{[n_i]} \times S$. Define the following sheaf on $\mathrm{Hilb}^n_\beta(S) \times S$, suppressing obvious pull-backs:

$$E^{[n]}_{\mathcal{L}} := (\mathcal{L}_0 \otimes \mathcal{I}^{[n_0]} \otimes \mathfrak{t}^0) \oplus \ldots \oplus (\mathcal{L}_s \otimes \mathcal{I}^{[n_s]} \otimes \mathfrak{t}^{-s}) .$$

The nested Hilbert scheme is by definition the maximal subscheme

$$i\colon S^{[n]}_\beta \hookrightarrow \mathrm{Hilb}^n_\beta(S)$$

over which $\phi_{\mathcal{L}}$ restricts to a Higgs field

$$\phi^{[n]}_{\mathcal{L}} : E^{[n]}_{\mathcal{L}} \to E^{[n]}_{\mathcal{L}} \otimes \omega_S \otimes \mathfrak{t} ,$$

i.e., there exist (necessarily unique) dotted arrows completing in the diagram

$$\begin{array}{ccc} \mathcal{L}_{i-1} & \xrightarrow{\phi_{\mathcal{L},i}} & \mathcal{L}_i \otimes q^*\omega_S \\ \downarrow & & \downarrow \\ \mathcal{L}_{i-1} \otimes \mathcal{I}^{[n_{i-1}]} & \overset{\phi^{[n]}_{\mathcal{L},i}}{\dashrightarrow} & \mathcal{L}_i \otimes \mathcal{I}^{[n_i]} \otimes q^*\omega_S \end{array}$$

of sheaves on $S^{[n]}_\beta \times S$ (where we suppress obvious pull-backs), defining the Higgs field $\phi^{[n]}_{\mathcal{L}}$, and the subscheme $S^{[n]}_\beta \subseteq \mathrm{Hilb}^n_\beta(S)$ is maximal with this property.

Remark 2.2.3. Throughout the paper, the letter n is reserved for $s+1$-tuples of non-negative integers, and $\mathrm{Hilb}^n_\beta(S)$ will always denote a product of Hilbert schemes as above.

Proposition 2.2.4 ([GSY17], [TT17a])**.** *The scheme $\mathcal{M}_{1^r}$ is a disjoint union of components that are uniquely represented by a triple*

$$(S^{[n]}_\beta, E^{[n]}_{\mathcal{L}}, \phi^{[n]}_{\mathcal{L}}) ,$$

as constructed above.

By Proposition 2.2.4, $\mathcal{M}_{1^r}$ consists of components that are naturally indexed by tuples $L = (L_0, \ldots, L_s)$ and $n = (n_0, \ldots, n_s)$.

Definition 2.2.5. *We denote a component of $\mathcal{M}_{1^r}$ represented by a triple*

$$(S^{[n]}_\beta, E^{[n]}_{\mathcal{L}}, \phi^{[n]}_{\mathcal{L}})$$

by $\mathcal{M}^{[n]}_L$.

Obviously, not every pair (L, n) corresponds to a component of $\mathcal{M}_{1^r}$. The nested Hilbert scheme might be empty, or the Higgs pairs in the family $(E^{[n]}_{\mathcal{L}}, \phi^{[n]}_{\mathcal{L}})$ might be unstable. The other restriction is the Chern data of the Higgs pairs. We will address stability in Section 2.3. We finish this section with a lemma regarding the second issue.

Lemma 2.2.6. *The total Chern class (in cohomology) of any fibre E of $E_{\mathcal{L}}^{[n]}$ over $S_\beta^{[n]}$ is given by*

$$\begin{aligned} c(E) &= 1 + c_1 + |n|\cdot pt + \sum_{0\le i<j\le s} c_1(L_i)c_1(L_j) \\ &= 1 + (s+1)c_1(L_s) + \sum_{i=1}^{s} i\beta^i \\ &\quad + |n|\cdot pt + \frac{s}{2(s+1)}c_1^2 - \sum_{1\le i<j\le s}\frac{i(s+1-j)}{s+1}\beta^i\beta^j - \sum_{1\le i\le s}\frac{i(s+1-i)}{2(s+1)}(\beta^i)^2 , \end{aligned}$$

where

$$\begin{aligned} c_1 &= c_1(L_0) + \ldots + c_1(L_s) , \\ |n| &= n_0 + \ldots + n_s , \end{aligned}$$

and pt denotes the Poincaré dual of the homology class of a point.

Proof. This is a straight forward computation. For the second equation, note that we have

$$(s+1)c_1(L_i) = \sum_{k=1}^{i} -k\beta^k + \sum_{k=i+1}^{s}(s+1-k)\beta^k + c_1$$

for $i = 0, \ldots, s$. Substituting this into

$$\sum_{0\le i<j\le s} c_1(L_i)c_1(L_j)$$

and interchanging sums gives the result. □

2.3 Stability

By Proposition 2.2.4, $\mathcal{M}_{1^r}$ consists of components that are isomorphic to nested Hilbert schemes $S_\beta^{[n]}$, with

$$\beta = (\beta_1, \ldots, \beta_s) \quad \text{and} \quad n = (n_0, \ldots, n_s)$$

tuples of divisor classes on S and integers respectively. The Hilbert scheme is empty if and only if one of the β_i's is not effective, or $\beta_i = 0$ and $n_{i-1} < n_i$ for some i. Obviously, the virtual class of the nested Hilbert scheme vanishes in this case.

We will give dual conditions on β and n, which hold whenever the Higgs pairs parametrized by $S_\beta^{[n]}$ are Gieseker unstable, and which in turn imply the vanishing of the virtual class. We recall the definition of stability of Higgs pairs.

Definition 2.3.1. *Let H be a polarization of the surface S. A Higgs pair (E, ϕ) is called* slope stable *(resp.* slope semistable*) if*

$$\frac{\deg(F)}{\mathrm{rk}(F)} < \frac{\deg(E)}{\mathrm{rk}(E)} \quad (\textit{resp.} \quad \frac{\deg(F)}{\mathrm{rk}(F)} \le \frac{\deg(E)}{\mathrm{rk}(E)})$$

for every ϕ-invariant subsheaf $0 \neq F \subsetneq E$ *with* $\mathrm{rk}(F) < \mathrm{rk}(E)$*. It is called* Gieseker stable *(resp.* Gieseker semistable*) if we have inequalities of polynomials in* m

$$\frac{\chi(F(mH))}{\mathrm{rk}(F)} < \frac{\chi(E(mH))}{\mathrm{rk}(E)} \quad (\textit{resp.} \quad \frac{\chi(F(mH))}{\mathrm{rk}(F)} \leq \frac{\chi(E(mH))}{\mathrm{rk}(E)})$$

for every proper ϕ-invariant subsheaf $0 \neq F \subsetneq E$*. By "(semi)stable", we will always mean Gieseker (semi)stable.*

Let $E = E_0 \oplus \ldots \oplus E_s$ be a sum of torsion free rank 1 sheaves, equipped with a Higgs field

$$\phi = (\phi_1, \ldots, \phi_s) \colon E \to E \otimes \omega_S$$

given by homomorphisms

$$\phi_i \colon E_{i-1} \to E_i \otimes \omega_S \quad \text{for } i = 1, \ldots, s\,.$$

Note that all Higgs pairs in $\mathcal{M}_{1^r}$ are of this form.

Lemma 2.3.2. *Assume that (E, ϕ) is indecomposable, i.e. $\phi_i \neq 0$ for $i = 1, \ldots, s$ and assume that*

$$\deg(E_{i-1}) \geq \deg(E_i) \quad \textit{for} \quad i = 1, \ldots, s\,.$$

Then the pair (E, ϕ) is slope semistable. It is slope stable unless

$$\deg(E_0) = \ldots = \deg(E_s)\,.$$

Proof. Let $F \subset E$ be a ϕ-invariant Higgs field. Let j be maximal, such that

$$F \subset E_j \oplus \ldots \oplus E_s\,. \tag{2.3.3}$$

I claim that F has rank $s + 1 - j$. It follows that if F is a destabilizing subsheaf, so is $E_j \oplus \ldots \oplus E_s$.

In order to prove the claim, consider the filtration

$$F = F^0 \supset \ldots \supset F^{s-j} \supset F^{s+1-j} = 0$$

of F, given by

$$F^i = K_S^{-i} \otimes \phi^{\circ i}(F)\,,$$

so we have

$$F^i \subset E_{j+i} \oplus \ldots \oplus E_s$$

for $i = 0, \ldots, s + 1 - j$. Note that for $i = 0, \ldots, s - j$, by injectivity of $\phi_{j+i} \cdots \phi_{j+1}$, and by the choice of j, the composition

$$F^i \subset E_{j+i} \oplus \ldots \oplus E_s \to E_{j+i}$$

is non-zero, and hence its image has rank 1, since E_{j+i} is torsion-free. On the other hand, its kernel contains F^{i+1}. It follows that we have

$$\mathrm{rk}\, F > \mathrm{rk}\, F^1 > \ldots > \mathrm{rk}\, F^{s+1-j} = 0$$

and hence, $\mathrm{rk}(F) = s+1-j$ by (2.3.3), proving the claim.

It follows that (E,ϕ) is slope semistable if and only if

$$\frac{\sum_{i=j}^{s}\deg(E_i)}{s+1-j} \leq \frac{\sum_{i=0}^{s}\deg(E_i)}{s+1} = \frac{\deg(E)}{\mathrm{rk}(E)}$$

for $j = 0,\ldots,s$. This clearly holds when $\deg(E_i) \leq \deg(E_{i-1})$ for all i. Finally note that (E,ϕ) is slope stable if one of the inequalities is strict. □

The hypothesis of Lemma 2.3.2 certainly holds when $c_1(E_{i-1})-c_1(E_i)$ is effective for each i. In this case, the condition

$$\deg(E_0) = \ldots = \deg(E_s)$$

implies that

$$c_1(E_0) = \ldots = c_1(E_s)\,.$$

Although such a Higgs pair is not slope stable, it might still be Gieseker (semi)stable.

Lemma 2.3.4. *Assume that (E,ϕ) is indecomposable, and assume that*

$$c_1(E_0) = \ldots = c_1(E_s) \quad \textit{and} \quad c_2(E_0) \leq \ldots \leq c_2(E_s)\,.$$

Then the pair (E,ϕ) is Gieseker semistable. It is Gieseker stable unless

$$c_2(E_0) = \ldots = c_2(E_s)\,.$$

Proof. The proof is similar to the proof of Lemma 2.3.2. Simply note that by Grothendieck-Riemann-Roch the hypothesis implies

$$\chi(E_{i-1}(m)) \geq \chi(E_i(m))$$

for $i = 1,\ldots,s$, with equality whenever $n_{i-1} = n_i$. □

Now let S be a surface with $p_g(S) > 0$ and $H^1(\mathcal{O}_S) = 0$. Let $L_0,\ldots,L_s$ be line bundles on S and let $n = (n_0,\ldots,n_s)$ be non-negative integers. Let $\beta = \beta(L)$ (and β_i and β^i for $i = 1,\ldots,s$) be given as in Notation 2.2.1, and consider the flat family of Higgs pairs $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ over the base $S_\beta^{[n]}$, as defined in Section 2.2.

In terms of β and n, Lemma 2.3.2 and Lemma 2.3.4 tell us that whenever the family $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ is not Gieseker semistable, there is an $i \in \{1,\ldots,s\}$ such that the divisor class β^i is not effective, or such that $\beta^i = 0$ and $n_{i-1} > n_i$ (compare to the introduction of this section!). As we will see in the following proposition, this suffices to show that we have $i_*[S_\beta^{[n]}]^{\mathrm{vir}} = 0$ in this case (recall that we write $i\colon S_\beta^{[n]} \hookrightarrow \mathrm{Hilb}_\beta^n(S)$).

Proposition 2.3.5. *Assume that*

$$i_*[S_\beta^{[n]}]^{\mathrm{vir}} \neq 0\,.$$

Then the family $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ of Higgs pairs is (Gieseker) semistable for any polarization of S. It is stable unless $L_0 = \ldots = L_s$ and $n_0 = \ldots = n_s$.

Proof. By Theorem 2.1.5, equation (2.1.22) and the hypothesis, we have

$$\begin{aligned}\mathrm{SW}(\beta^1)\cdots\mathrm{SW}(\beta^s) &= (-1)^{s\cdot\chi(\mathcal{O}_S)}\mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_s)\\ &\neq 0\,.\end{aligned}$$

It follows that $\beta^i \geq 0$ for $i = 0, \ldots, s$, by definition of the Seiberg-Witten class. By Lemma 2.3.2, the fibres of $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ are slope-stable, and hence Gieseker stable, unless $L_0 = \ldots = L_s$. Assume the latter. By Lemma 2.3.4, we need to show that $n_{i-1} \leq n_i$ for all i. Assume that $n_{i-1} > n_i$ for some i. Then the nested Hilbert scheme

$$i\colon S^{[n_i,n_{i-1}]} \hookrightarrow S^{[n_i]} \times S^{[n_{i-1}]}$$

is empty, and we have by Serre duality and Theorem 2.1.5

$$\begin{aligned}&e\Big(R\pi_*\omega_S - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]} \otimes \omega_S)\Big)\\ &= (-1)^{n_{i-1}+n_i} e\left(R\pi_*(\mathcal{O}_S) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_{i-1}]})\right)\\ &= (-1)^{n_{i-1}+n_i} i_*[S^{[n_i,n_{i-1}]}]^{\mathrm{vir}}\\ &= 0\,.\end{aligned} \tag{2.3.6}$$

By the assumption $L_0 = \ldots = L_s$, we have in particular $\beta_i = K_S$. It follows that the expression of Theorem 2.1.5 has a factor (2.3.6). We find $i_*[S_\beta^{[n]}]^{\mathrm{vir}} = 0$, which contradicts the hypothesis. □

Recall from Proposition 2.2.4, that $\mathcal{M}_{1^r,c_1,c_2}$ is a union of Hilbert schemes $S_\beta^{[n]}$. We will also write i for the morphism

$$i\colon \mathcal{M}_{1^r,c_1,c_2} \to \bigsqcup_{\beta,n} \mathrm{Hilb}_\beta^n(S)\,,$$

which is given on each component of $\mathcal{M}_{1^r,c_1,c_2}$ by the inclusion

$$i\colon S_\beta^{[n]} \to \mathrm{Hilb}_\beta^n(S)\,.$$

By the vanishing of Proposition 2.3.5, we can sum in the following proposition over all pairs (L, n) or (β, n), rather than the ones that correspond to components of *stable* Higgs pairs. In particular, the push-forward by i of the virtual class does not depend on the polarization of the surface S.

Proposition 2.3.7. *Let S be a surface with $p_g(S) > 0$ and $H^1(\mathcal{O}_S) = 0$. Fix r, c_1 and c_2 such that Gieseker semistability of Higgs pairs implies Gieseker stability. Then we have*

$$\begin{aligned}i_*[\mathcal{M}_{1^r,c_1,c_2}]^{\mathrm{vir}} &= \sum_{L,n} i_*[S_{\beta(L)}^{[n]}]^{\mathrm{vir}}\\ &= \sum_{\beta,n} i_*[S_\beta^{[n]}]^{\mathrm{vir}} \cdot \#\ker\left(H^2(S,\mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S,\mathbb{Z})\right)\end{aligned} \tag{2.3.8}$$

where the sums are taken over

$$L, n \quad \textit{with} \quad \begin{aligned} c_1 &= \sum_{i=1}^{s} c_1(L_i) \\ c_2 &= |n| + \sum_{0 \le i < j \le s} c_1(L_i) c_1(L_j)\,; \end{aligned}$$

and, respectively,

$$\beta, n \quad \textit{with} \quad \begin{aligned} c_1 &\equiv \sum_{i=1}^{s} i\beta^i \mod (s+1)H^2(S,\mathbb{Z}) \\ c_2 &= |n| + \frac{s}{2(s+1)} c_1^2 - \sum_{1 \le i < j \le s} \frac{i(s+1-j)}{s+1} \beta^i \beta^j \\ &\quad - \sum_{1 \le i \le s} \frac{i(s+1-i)}{2(s+1)} (\beta^i)^2\,. \end{aligned}$$

Proof. The c_2-conditions on the pairs (L, n) and (β, n) appearing in the sums are given by Lemma 2.2.6. Moreover, it is easy to see that for an s-tuple of curve classes $\beta = (\beta_1, \ldots, \beta_s) \in (H^2(S,\mathbb{Z}))^s$, there is a tuple of vector bundles $L = (L_0, \ldots, L_s)$ with

$$c_1 = \sum_{i=1}^{s} c_1(L_i) \quad \text{and} \quad \beta = \beta(L)$$

if and only if

$$c_1 \equiv \sum_{i=1}^{s} i\beta^i \mod (s+1)H^2(S,\mathbb{Z})\,.$$

Now assume that $S_\beta^{[n]} \cong \mathcal{M}_L^{[n]} \subset \mathcal{M}_{1^r, c_1, c_2}$ is a component (see Definition 2.2.5) . For a curve class

$$\gamma \in \ker\Big(H^2(S,\mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S,\mathbb{Z})\Big) =: K$$

there is a component

$$S_\beta^{[n]} \cong \mathcal{M}_{L(\gamma)}^{[n]} \subset \mathcal{M}_{1^r, c_1, c_2}\,,$$

where

$$L(\gamma) = (L_0 \otimes \mathcal{O}_S(\gamma), \ldots, L_s \otimes \mathcal{O}_S(\gamma))\,.$$

In fact, there is a K-torsor of components of $\mathcal{M}_{1^r, c_1, c_2}$ that are isomorphic to $S_\beta^{[n]}$. This expains the second equation of (2.3.8).

Finally, note that pairs (β, n) for which the scheme $S_\beta^{[n]}$ is empty obviously do not contribute to the right-hand side of (2.3.8). By Proposition 2.3.5, we have for pairs (β, n) for which $S_\beta^{[n]}$ parametrizes unstable Higgs pairs, that the push-forward of the virtual class $i_*[S_\beta^{[n]}]$ vanishes. Hence these pairs do also not contribute to the right-hand side of (2.3.8). For this reason, the sum can be taken over *all* pairs (β, n) rather than only over the ones corresponding to components of $\mathcal{M}_{1^r, c_1, c_2}$. □

2.4 Tautological integrals

Choose line bundles $L = (L_0, \dots, L_s)$ on S and let $\beta = \beta(L) = (\beta_1, \dots, \beta_s)$ and $\mathcal{L} = (\mathcal{L}_0, \dots, \mathcal{L}_s)$ be defined as in Section 2.2. Let $n = (n_0, \dots, n_s)$ be non-negative integers. Recall that we write

$$E_{\mathcal{L}}^{[n]} = \mathcal{L}_0 \otimes \mathcal{I}^{[n_0]} \otimes \mathfrak{t}^0 \oplus \ldots \oplus \mathcal{L}_s \otimes \mathcal{I}^{[n_s]} \otimes \mathfrak{t}^{-s}$$

for the sheaf on

$$\mathrm{Hilb}_\beta^n(S) \times S = S^{[n_0]} \times \cdots \times S^{[n_s]} \times |\mathcal{O}_S(\beta_1)| \times \cdots \times |\mathcal{O}_S(\beta_s)| \times S$$

and for its restriction to the nested Hilbert scheme

$$i : S_\beta^{[n]} \hookrightarrow \mathrm{Hilb}_\beta^n(S)\,,$$

over which we have a canonically defined Higgs field $\phi_{\mathcal{L}} \colon E_{\mathcal{L}}^{[n]} \to E_{\mathcal{L}}^{[n]} \otimes \omega_S \otimes \mathfrak{t}$.

Define a class

$$\begin{aligned} T_{\mathcal{L}}^{[n]} :=\ & R\mathscr{H}om_\pi(E_{\mathcal{L}}^{[n]}, E_{\mathcal{L}}^{[n]} \otimes \omega_S \otimes \mathfrak{t})_0 - R\mathscr{H}om_\pi(E_{\mathcal{L}}^{[n]}, E_{\mathcal{L}}^{[n]})_0 \\ & \in K_0^{\mathbb{C}^*}(\mathrm{Hilb}_\beta^n(S))\,, \end{aligned}$$

and denote its pull-back to $S_\beta^{[n]}$ by the same symbol. Note that $T_{\mathcal{L}}^{[n]}$ depends only on β, rather than on L (or on $\mathcal{L}$). We will write

$$N_{\mathcal{L}}^{[n]} := T_{\mathcal{L}}^{[n]} - \left(T_{\mathcal{L}}^{[n]}\right)^{\mathbb{C}^*}$$

for its moving part. Let e denote the $\mathbb{C}^*$-equivariant Euler class, and define the rational number

$$\mathrm{VW}_\beta^{[n]} := \int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{1}{e\left(N_{\mathcal{L}}^{[n]}\right)}\,. \tag{2.4.1}$$

In the case that $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ represents a component

$$\mathcal{M}_L^{[n]} = (S_\beta^{[n]}, E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]}) \subset \mathcal{M}_{1^r, c_1, c_2}\,,$$

$T_{\mathcal{L}}^{[n]}$ is the class in K-theory of the cone (2.1.1) in the introduction, and hence equals the $\mathbb{C}^*$-localized perfect obstruction theory of [TT17a] on $\mathcal{M}_L^{[n]}$. Over $\mathcal{M}_L^{[n]}$, the class $N_{\mathcal{L}}^{[n]}$ is the virtual normal bundle to the $\mathbb{C}^*$ fixed locus $(\mathcal{N}_{r,M,c_2}^\perp)^{\mathbb{C}^*}$ in $\mathcal{N}_{r,M,c_2}^\perp$. By definition of the Vafa-Witten invariant (2.1.2), the contribution of the component $\mathcal{M}_L^{[n]}$ is given by $\mathrm{VW}_\beta^{[n]}$.

If the Higgs pair $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ contains fibres that are not Gieseker semistable, it does not represent a component of any $\mathcal{M}_{1^r, c_1, c_2}$, and hence does not contribute to the Vafa-Witten invariant. On the other hand, by Proposition 2.3.5, we have $\mathrm{VW}_\beta^{[n]} = 0$ in this case. It follows that, using the notation from Proposition 2.3.7, we have:

$$\begin{aligned} \mathrm{VW}_{1^r, c_1, c_2} &= \sum_{L,n} \mathrm{VW}_{\beta(L)}^{[n]} \\ &= \sum_{\beta,n} \mathrm{VW}_\beta^{[n]} \cdot \#\ker\left(H^2(S, \mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S, \mathbb{Z})\right)\,. \end{aligned}$$

Now define a line bundle

$$K_{\mathcal{L}}^{[n]} := \det\left(T_{\mathcal{L}}^{[n]\vee}\right)$$

on $\mathrm{Hilb}_\beta^{[n]}(S)$. Note that $T_{\mathcal{L}}^{[n]}$ is defined as the difference between a complex and its dual, up to a factor $\mathfrak{t}$. Hence its determinant is by construction a square, up to a factor $\mathfrak{t}$. Hence, after choosing once and for all a square root of $\mathfrak{t}$, the line bundle $K_{\mathcal{L}}^{[n]}$ has a canonical square root, denoted by $(K_{\mathcal{L}}^{[n]})^{\frac{1}{2}}$. Over $S_\beta^{[n]}$, the bundle $K_{\mathcal{L}}^{[n]}$ restricts to the virtual canonical bundle [Tho18a], and its square root restricts to the canonical square root of [Tho18a, Proposition 2.6].

By [Tho18a], the contribution to the refined invariant can be computed by

$$\mathrm{VW}_\beta^{[n]}(y) := \left[\int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{\mathrm{ch}\left((K_{\mathcal{L}}^{[n]})^{\frac{1}{2}}\right)}{\mathrm{ch}\left(\Lambda^\bullet(N_{\mathcal{L}}^{[n]\vee})\right)} \mathrm{Td}\left((T_{\mathcal{L}}^{[n]})^{\mathbb{C}^*}\right)\right]_{\mathrm{ch}(\mathfrak{t})=y}, \tag{2.4.2}$$

where ch and Td denote the $\mathbb{C}^*$-equivariant Chern character and Todd class respectively. Again, in the language of Proposition 2.3.7, we have

$$\begin{aligned}\mathrm{VW}_{1^r,c_1,c_2}(y) &= \sum_{L,n} \mathrm{VW}_{\beta(L)}^{[n]}(y)\\ &= \sum_{\beta,n} \mathrm{VW}_\beta^{[n]}(y)\cdot \#\ker\left(H^2(S,\mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S,\mathbb{Z})\right).\end{aligned} \tag{2.4.3}$$

By Theorem 2.1.5, we have

$$\begin{aligned}i_*[S_\beta^{[n]}]^{\mathrm{vir}} = \prod_{i=1}^s e\left(R\pi_*(\mathcal{O}_S(\beta_i) - R\mathscr{H}om_\pi\left(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)\right)\right)\\ \cap \left[S^{[n_0]}\times\cdots\times S^{[n_s]}\right]\times \mathrm{SW}(\beta_1)\times\cdots\times\mathrm{SW}(\beta_s).\end{aligned}$$

The factor

$$\begin{aligned}\mathrm{SW}(\beta) &:= \mathrm{SW}(\beta_1)\times\cdots\times\mathrm{SW}(\beta_s)\\ &\in A_0\big(|\mathcal{O}_S(\beta_1)|\times\cdots\times|\mathcal{O}_S(\beta_s)|\big)\end{aligned}$$

annihilates all Chern classes in the integrants of (2.4.1) and (2.4.2) that are pulled back from

$$|\mathcal{O}_S(\beta_1)|\times\cdots\times|\mathcal{O}_S(\beta_s)|.$$

It follows that we can rewrite (2.4.1) and (2.4.2) as integrals over

$$\mathrm{Hilb}^n(S) = S^{[n_0]}\times\cdots\times S^{[n_s]}.$$

Define the sheaf

$$E_L^{[n]} := L_0\otimes\mathcal{I}^{[n_0]}\otimes\mathfrak{t}^0 \oplus \ldots, \oplus L_s\otimes\mathcal{I}^{[n_s]}\otimes\mathfrak{t}^{-s} \quad \text{on} \quad \mathrm{Hilb}^n(S)\times S$$

and classes

$$\begin{aligned} T_L^{[n]} &:= R\mathscr{H}om_\pi(E_L^{[n]}, E_L^{[n]} \otimes \omega_S \otimes \mathfrak{t})_0 - R\mathscr{H}om_\pi(E_L^{[n]}, E_L^{[n]})_0\,, \\ N_L^{[n]} &:= T_L^{[n]} - \left(T_L^{[n]}\right)^{\mathbb{C}^*}\,, \quad \text{and} \\ K_L^{[n]} &:= \det\left(T_L^{[n]\vee}\right) \end{aligned}$$

in $K_0(\mathrm{Hilb}^n(S))$. Again, note that since $H^1(\mathcal{O}_S) = 0$, the classes $T_L^{[n]}$, $N_L^{[n]}$, and $K_L^{[n]}$ depend on $\beta = \beta(L)$, rather than on L. We have, now considering $\mathrm{SW}(\beta)$ as an integer,

$$\begin{aligned} \mathrm{VW}_\beta^{[n]} = \mathrm{SW}(\beta) \int_{[\mathrm{Hilb}^n(S)]} & \frac{1}{e\left(N_L^{[n]}\right)} \\ & \times \prod_{i=1}^{s} e\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right) \end{aligned} \tag{2.4.4}$$

and

$$\begin{aligned} \mathrm{VW}_\beta^{[n]}(y) = \mathrm{SW}(\beta) \Bigg[\int_{[\mathrm{Hilb}^n(S)]} & \frac{\mathrm{ch}\left((K_L^{[n]})^{\frac{1}{2}}\right)}{\mathrm{ch}(\Lambda^\bullet(N_L^{[n]\vee}))} \mathrm{Td}\left((T_L^{[n]})^{\mathbb{C}^*}\right) \\ \times \prod_{i=1}^{s} & e\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right) \Bigg]_{\mathrm{ch}(\mathfrak{t}) = y} . \end{aligned} \tag{2.4.5}$$

2.5 Removing trace

The generating series

$$\sum_n \mathrm{VW}_\beta^{[n]} q^n$$

has leading term

$$\mathrm{VW}_\beta^{[0]} = \mathrm{SW}(\beta) \frac{1}{e(N_L^{[0]})}\,.$$

We can renormalize the series by dividing through the factor $\frac{1}{e(N_L^{[0]})}$ ($= F(S,\beta)$ in the notation below). In terms of the integrals of (2.4.4), this comes down to considering 'traceless' integrants. By this we mean the following. Note that the integrant of (2.4.4) can be written as product of (equivariant) Euler classes of terms of the form

$$R\mathscr{H}om_\pi(\mathcal{E}, \mathcal{F})$$

with $\mathcal{E}$ and $\mathcal{F}$ torsion free rank 1 sheaves. Then a traceless version of the integral of (2.4.4) (denoted by $Q_n(S,\beta)$ below) is given by replacing each such term by

$$R\mathscr{H}om_\pi(\mathcal{E}, \mathcal{F})_0 = R\mathscr{H}om_\pi(\mathcal{E}, \mathcal{F}) - R\mathscr{H}om_\pi(\det \mathcal{E}, \det \mathcal{F})\,.$$

It is easy to see that the resulting expression computes the renormalized generating series (see (2.5.1) below). In Section 2.6, we will deal with the normalizing factor separately.

We keep the notation from the previous section. Moreover, we will write

$$E_L := E_L^{[0]} = L_0 \otimes \mathfrak{t}^0 \oplus \ldots \oplus L_s \otimes \mathfrak{t}^{-s}$$

for the vector bundle on S, and furthermore

$$\begin{aligned} T_L &:= T_L^{[0]} = R\operatorname{Hom}(E_L, E_L \otimes \omega_S \otimes \mathfrak{t})_0 - R\operatorname{Hom}(E_L, E_L)_0\,, \\ N_L &:= N_L^{[0]} = T_L - (T_L)^{\mathbb{C}^*}\,, \\ K_L &:= K_L^{[0]} = \det T_L^\vee\,. \end{aligned}$$

for the classes in the equivariant K-group of a point. Finally, we will also use the notation

$$T_{L,0}^{[n]} = T_L^{[n]} - T_L\,, \quad N_{L,0}^{[n]} = N_L^{[n]} - N_L\,, \quad \text{and} \quad K_{L,0}^{[n]} = K_L^{[n]} \otimes K_L^*\,,$$

for the classes in $K_0(\mathrm{Hilb}^n(S))$, where we suppress pull-backs from the point. Define

$$\begin{aligned} F(S,\beta) &:= \frac{1}{e(N_L)} \quad \text{and} \\ Q_n(S,\beta) &:= \int_{[\mathrm{Hilb}^n(S)]} \frac{1}{e\left(N_{L,0}^{[n]}\right)} \prod_{i=1}^{s} e\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\,, \end{aligned}$$

so we have

$$\mathrm{VW}_\beta^{[n]} = \mathrm{SW}(\beta)\, F(S,\beta)\, Q_n(S,\beta)\,. \tag{2.5.1}$$

In the refined case, define

$$\begin{aligned} F(S,\beta,y) &:= \left[\frac{\mathrm{ch}\left(K_L^{\frac{1}{2}}\right)}{\mathrm{ch}(\Lambda^\bullet(N_L^\vee))} \mathrm{Td}\left(T_L^{\mathbb{C}^*}\right)\right]_{\mathrm{ch}(\mathfrak{t})=y} \quad \text{and} \\ Q_n(S,\beta,y) &:= \left[\int_{[\mathrm{Hilb}^n(S)]} \frac{\mathrm{ch}\left((K_{L,0}^{[n]})^{\frac{1}{2}}\right)}{\mathrm{ch}(\Lambda^\bullet((N_{L,0}^{[n]})^\vee))} \mathrm{Td}\left((T_{L,0}^{[n]})^{\mathbb{C}^*}\right) \right. \\ &\qquad \left. \times \prod_{i=1}^{s} e\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\right]_{\mathrm{ch}(\mathfrak{t})=y}, \end{aligned}$$

so that

$$\mathrm{VW}_\beta^{[n]}(y) = \mathrm{SW}(\beta)\, F(S,\beta,y)\, Q_n(S,\beta,y)\,. \tag{2.5.2}$$

Remark 2.5.3. A priori, $Q_n(S,\beta,y)$ is a rational function in $\sqrt{y}$, due to the fractional exponent of the virtual canonical bundle. However, an easy computation shows that the equivariant parameter $\mathfrak{t}$ appears in $K_{\beta,0}^{[n]}$, with *even* exponent, and hence, $Q_n(S,\beta,y)$ is in fact a rational function in y.

In the Section 2.6 we will compute $F(S,\beta)$ under the assumption $\mathrm{SW}(\beta) \neq 0$. In Section 2.7 we will show that the numbers $Q_n(S,\beta)$ are given by universal polynomials $P_n(S,\beta)$ in the Chern numbers of S and $\beta = (\beta_1,\ldots,\beta_s)$. We will deal with the refined version at the same time.

2.6 The leading term

In this section we compute the factor $F(S,\beta,y)$. Let $L=(L_0,\dots,L_s)$ be an $(s+1)$-tuple of line bundles on S, and let $\beta=\beta(L)$ be given as in Notation 2.2.1. Also recall that we write

$$E_L = L_0\otimes \mathfrak{t}^0 \oplus \ldots \oplus L_s \otimes \mathfrak{t}^{-s}\,.$$

Assume that

$$\mathrm{SW}(\beta)=\mathrm{SW}(\beta_1)\cdots\mathrm{SW}(\beta_s)\neq 0\,.$$

Then, by [Moc09, Proposition 6.29], we have $(\beta^i)^2=(\beta^i K_S)$, or equivalently $\chi(\beta^i)=\chi(\mathcal{O}_S)$ for $i=1,\dots,s$. Using Serre duality, we can write

$$\begin{aligned}
T_L &= R\,\mathrm{Hom}(E_L, E_L\otimes\omega_S\otimes\mathfrak{t})_0 - R\,\mathrm{Hom}(E_L,E_L)_0\\
&= \sum_{i=0}^{s}(\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})\left(\sum_{j=0}^{s-i}\chi(L_{i+j}^*\otimes L_j\otimes\omega_S)-\sum_{j=1}^{s-i}\chi(L_{i+j}^*\otimes L_{j-1})\right)\\
&\quad -(\mathfrak{t}-1)\cdot\chi(\mathcal{O}_S)\\
&= \sum_{i=1}^{s}(\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})\left(\sum_{j=0}^{s-i}\chi(\beta^{1+j}+\ldots+\beta^{i+j}+K_S)-\sum_{j=1}^{s-i}\chi(\beta^{j}+\ldots+\beta^{i+j})\right)\,,
\end{aligned}$$

where the second sum starts with $i=1$, since the coefficient of $(\mathfrak{t}-1)$ equals

$$\left(\sum_{j=0}^{s}\chi(K_S)-\sum_{j=1}^{s}\chi(\beta^j)\right)-\chi(\mathcal{O}_S)=0$$

by the assumption $\mathrm{SW}(\beta)\neq 0$. Note that in particular, we have

$$T_L = N_L\,. \tag{2.6.1}$$

Moreover, note that

$$\begin{aligned}
\chi(\beta^{1+j}+\ldots+\beta^{i+j}+K_S) &= \frac{(\beta^{1+j}+\ldots+\beta^{i+j}+K_S)\cdot(\beta^{1+j}+\ldots+\beta^{i+j})}{2}\\
&\quad+\chi(\mathcal{O}_S)\\
&= \sum_{j<k\le l\le i+j}\beta^k\beta^l+\chi(\mathcal{O}_S)\,.
\end{aligned}$$

and similarly

$$\begin{aligned}
\chi(\beta^{j}+\ldots+\beta^{i+j}) &= \frac{(\beta^{j}+\ldots+\beta^{i+j})\cdot(\beta^{j}+\ldots+\beta^{i+j}-K_S)}{2}+\chi(\mathcal{O}_S)\\
&= \sum_{j\le k<l\le i+j}\beta^k\beta^l+\chi(\mathcal{O}_S)\,.
\end{aligned}$$

It follows that for $k\le l$, the multiplicity with which the term

$$(\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})\cdot\beta^k\beta^l$$

appears in T_L is given by

$$\begin{aligned}\mu(i,k,l) &:= \#\{j \mid 0 \le j < k \le l \le i+j \le s\} - \#\{j \mid 0 < j \le k < l \le i+j \le s\} \\ &= \#\{j \mid 0, l-i \le j \le k-1, s-i\} - \#\{j \mid 1, l-i \le j \le k, s-i; k<l\} \\ &= \begin{cases} \left.\begin{cases} -1 & \text{if } l-k \le i < \min(l, s-k+1) \\ 1 & \text{if } \max(l, s-k+1) \le i \le s \\ 0 & \text{else} \end{cases}\right\} & \text{if } k<l \\ \min\{i, s-k+1, k, s-i+1\} & \text{if } k=l\,, \end{cases}\end{aligned}$$

so we have

$$N_L = T_L = \sum_{i=1}^{s} \left(\chi(\mathcal{O}_S) + \sum_{k \le l} \mu(i,k,l) \cdot \beta^k \beta^l \right) \cdot (\mathsf{t}^{i+1} - t^{-i})\,. \tag{2.6.2}$$

We define the following rational numbers:

$$\begin{aligned} F_0^{(s+1)} &:= \frac{(-1)^s}{s+1}\,; \\ F_{kk}^{(s+1)} &:= \frac{(-1)^{sk}}{\binom{s+1}{k}} && \text{for} \quad 1 \le k \le s\,; \\ F_{kl}^{(s+1)} &:= \frac{l(s+1-k)}{(l-k)(s+1)} && \text{for} \quad 1 \le k < l \le s\,. \end{aligned}$$

Proposition 2.6.3. *Recall that we assume* $\mathrm{SW}(\beta) \neq 0$. *We have*

$$F(S,\beta) = \left(F_0^{(s+1)}\right)^{\chi(\mathcal{O}_S)} \prod_{k \le l} \left(F_{kl}^{(s+1)}\right)^{\beta^k \beta^l}\,.$$

Proof. Applying $\frac{1}{e()}$ to equation (2.6.2), we obtain

$$\begin{aligned} F(S,\beta) &= \frac{1}{e(N_L)} \\ &= e\left(\sum_{i=1}^{s} \mathsf{t}^{i+1} - t^{-i}\right)^{-\chi(\mathcal{O}_S)} \prod_{k \le l} e\left(\sum_{i=1}^{s} \mu(i,k,l)(\mathsf{t}^{i+1} - t^{-i})\right)^{-\beta^k \beta^l} \end{aligned}$$

Note that we have

$$\frac{1}{e(\mathsf{t}^{i+1} - \mathsf{t}^{-i})} = \frac{-i}{i+1}\,,$$

and hence

$$e\left(\sum_{i=1}^{s} (\mathsf{t}^{i+1} - \mathsf{t}^{i})\right)^{-1} = \frac{-1 \cdots - i}{2 \cdots (i+1)} = \frac{(-1)^s}{s+1}$$

For $k < l$ we find

$$\begin{aligned} e\left(\sum_{i=1}^{s}\mu(i,k,l)\cdot(\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})\right)^{-1} &= \prod_{i=l-k}^{\min(l-1,s-k)}\left(\frac{-i}{i+1}\right)^{-1}\cdot\prod_{i=\max(l,s-k+1)}^{s}\frac{-i}{i+1} \\ &= \frac{l(s+1-k)}{(l-k)(s+1)}\,. \end{aligned}$$

Finally, write

$$a \coloneqq \min(k, s+1-k) \quad \text{and} \quad b \coloneqq \max(k, s+1-k)\,,$$

so we have

$$\begin{aligned} e\left(\sum_{i=1}^{s}\mu(i,k,k)\cdot(\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})\right)^{-1} &= \prod_{i=1}^{a}\left(\frac{-i}{i+1}\right)^{i}\prod_{i=a+1}^{b-1}\left(\frac{-i}{i+1}\right)^{a}\prod_{i=b}^{s}\left(\frac{-i}{i+1}\right)^{s+1-i} \\ &= (-1)^{sk}\frac{1\cdots a}{(b+1)\cdots(s+1)} \\ &= \frac{(-1)^{sk}}{\binom{s+1}{k}} \end{aligned}$$

□

Notation 2.6.4. We will use *quantum integers*, which are given by

$$[i]_y \coloneqq \frac{y^{i/2}-y^{-i/2}}{y^{1/2}-y^{-1/2}}\,.$$

We will also use the notation

$$\binom{i}{j}_y \coloneqq \frac{[i]_y\cdots[i-j+1]_y}{[1]_y\cdots[j]_y}$$

for non-negative integers $i \leq j$.

Define the following rational functions in $y^{1/2}$:

$$\begin{aligned} F_0^{(s+1)}(y) &\coloneqq \frac{(-1)^s}{[s+1]_y}\,; \\ F_{kk}^{(s+1)}(y) &\coloneqq \frac{(-1)^{sk}}{\binom{s+1}{k}_y} && \text{for} \quad 1\leq k\leq s\,; \\ F_{kl}^{(s+1)}(y) &\coloneqq \frac{[l]_y[s+1-k]_y}{[l-k]_y[s+1]_y} && \text{for} \quad 1\leq k<l\leq s\,. \end{aligned}$$

Proposition 2.6.5. *Assume that* $\mathrm{SW}(\beta)\neq 0$*. Then we have*

$$F(S,\beta,y) = \left(F_0^{(s+1)}(y)\right)^{\chi(\mathcal{O}_S)}\prod_{k\leq l}\left(F_{kl}^{(s+1)}(y)\right)^{\beta^k\beta^l}\,.$$

Proof. Recall (2.6.1) that T_L has no fixed part, so we have

$$F(S,\beta,y) = \left[\frac{\mathrm{ch}(K_L^{\frac{1}{2}})}{\mathrm{ch}(\Lambda^\bullet(N_L^\vee))}\,\mathrm{Td}((T_L)^{\mathbb{C}^*})\right]_{\mathrm{ch}(\mathfrak{t})=y}$$
$$= \left[\frac{\mathrm{ch}(\det(N_L^\vee))^{\frac{1}{2}}}{\mathrm{ch}(\Lambda^\bullet(N_L^\vee))}\right]_{\mathrm{ch}(\mathfrak{t})=y}.$$

Note that we have

$$\left[\frac{\mathrm{ch}(\det((\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})^\vee)^{\frac{1}{2}})}{\mathrm{ch}(\Lambda^\bullet((\mathfrak{t}^{i+1}-\mathfrak{t}^{-i})^\vee))}\right]_{\mathrm{ch}(\mathfrak{t})=y} = -y^{1/2}\frac{y^i-1}{y^{i+1}-1}$$
$$= -\frac{[i]_y}{[i+1]_y}.$$

Now follow the proof of Proposition 2.6.3.

□

Remark 2.6.6. If $r=s+1$ is odd, note that $F(S,\beta,y)$ is a function in y, rather than in $\sqrt{y}$, for any $\beta=(\beta_0,\ldots,\beta_s)$ with $\mathrm{SW}(\beta)\neq 0$.

Example 2.6.7. For rank 2 we have

$$F(S,\beta,y) = \left(\frac{-y^{1/2}}{1+y}\right)^{\chi(\mathcal{O}_S)+\beta^1\beta^1},$$

and for rank 3

$$F(S,\beta,y) = \left(\frac{y}{1+y+y^2}\right)^{\chi(\mathcal{O}_S)+\beta^1\beta^1+\beta^2\beta^2}\left(\frac{(y+1)^2}{1+y+y^2}\right)^{\beta^1\beta^2}.$$

2.7 Universality

Let S be a smooth projective surface, not necessarily with $H^1(\mathcal{O}_S)=0$ or $p_q>0$. For non-negative integers $n=(n_0,\ldots,n_s)$ and classes $\beta=(\beta_1,\ldots,\beta_s)$, consider the rational number $Q_n(S,\beta)$ defined in Section 2.5 as an integral over

$$\mathrm{Hilb}^n(S) = S^{[n_0]}\times\cdots\times S^{[n_s]}.$$

Using the notation

$$q^n := q_0^{n_0}\cdots q_s^{n_s}$$

we form the generating series

$$\sum_n Q_n(S,\beta)\,q^n.$$

The following universality result Proposition 2.7.3, or rather its refinement Proposition 2.7.6, is the main ingredient for the proof of Theorem 2.A.

Remark 2.7.1. In Section 2.5, the integrals $Q_n(S,\beta)$ were defined in terms of a lift of β to a vector of line bundles L, such that $\beta = \beta(L)$ (see Notation 2.2.1). Since we do not assume $H^1(\mathcal{O}_S) = 0$ in this section, this lift involves a a lift of the β_i to divisor classes of S. We assume we have made such choice, and we consider β as a vector of classes in $A^1(S)$. By Proposition 2.7.3, $Q_n(S,\beta)$ does not depend on this choice.

Notation 2.7.2. In the following we will use the formal symbols

$$\underline{\chi(\mathcal{O}_S)},\ \underline{K_S^2},\ \underline{K_S\beta^i},\ \underline{\beta^i\beta^j} \quad \text{for} \quad 1 \le i \le j \le s\,.$$

Formally, they form a dual basis of the vector space of cobordism classes (with $\mathbb{Q}$-coefficients) of surfaces together with an s-tuple of divisors. In particular they can be evaluated on (classes of) such pairs. For example, we have

$$\underline{K_S\beta^2}(\mathbb{P}^2,(H,H,0)) = K_{\mathbb{P}^2}H = -3\,,$$

where H is the class of a hyperplane. In general we write, a bit informally,

$$\underline{\mathfrak{N}}(S,\beta) = \mathfrak{N}\,.$$

Proposition 2.7.3. *For each symbol*

$$\underline{\mathfrak{N}} \in \left\{\underline{\chi(\mathcal{O}_S)}, \underline{K_S^2}, \underline{K_S\beta^i}, \underline{\beta^i\beta^j}\right\}_{1\le i\le j\le s}$$

there is a power series $A_{\underline{\mathfrak{N}}}^{(s+1)} \in \mathbb{Q}[[q_0,\dots,q_s]]$*, starting with* 1*, and depending only on* s*, such that*

$$\sum_n Q_n(S,\beta)\,q^n = \prod_{\underline{\mathfrak{N}}} (A_{\underline{\mathfrak{N}}}^{(s+1)})^{\mathfrak{N}}$$

for any *smooth projective surface* S *and classes* $\beta_1,\dots,\beta_s \in A^1(S)$*.*

Proof. By the techniques of [EGL01] (see also [GNY08]), the integral $Q_n(S,\beta)$ can be universally expressed as a polynomial $P_n(S,\beta)$ in the Chern numbers of S and the classes $\beta^1,\dots,\beta^s$. Following [Göt98, Proposition 2.3], it suffices to show that the generating series is multiplicative, i.e., that we have

$$\sum Q_n(S\sqcup S',\beta+\beta')\,q^n = \sum Q_n(S,\beta)\,q^n \cdot \sum Q_n(S',\beta')\,q^n$$

for surfaces S and S' and s-tuples β and β' of classes in $A^1(S)$ and $A^1(S')$ respectively.

Note that

$$\begin{aligned}
\mathrm{Hilb}^n(S\sqcup S') &= (S\sqcup S')^{[n_0]} \times \cdots \times (S\sqcup S')^{[n_s]}\\
&= \bigsqcup_{i_0+j_0=n_0} (S^{[i_0]}\times S'^{[j_0]}) \times \cdots \times \bigsqcup_{i_s+j_s=n_s} (S^{[i_s]}\times S'^{[j_s]})\\
&= \bigsqcup_{\substack{i_0+j_0=n_0,\\ \cdots\\ i_s+j_s=n_s}} S^{[i_0]}\times S'^{[j_0]} \times\cdots\times S^{[i_s]}\times S'^{[j_s]}\\
&= \bigsqcup_{i+j=n} \mathrm{Hilb}^i(S)\times \mathrm{Hilb}^j(S')\,,
\end{aligned} \tag{2.7.4}$$

in which the last sum is taken over $s+1$-tuples $i = (i_0, \ldots, i_s)$ and $j = (j_0, \ldots, j_s)$ of non-negative integers with $n = i + j$. Consider the universal ideal sheaves

$$\mathcal{I}_{S \sqcup S'}^{[n_k]} \quad \text{for} \quad k = 0, \ldots, s$$

on

$$\mathrm{Hilb}^n(S \sqcup S') \times (S \sqcup S') .$$

For fixed i and j with $i + j = n$ and for $k = 0, \ldots, s$, we will write

$$p_k \colon \mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S') \times S \to S^{[i_k]} \times S$$
$$q_k \colon \mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S') \times S' \to S'^{[j_k]} \times S'$$

for the projections. Over the components in the decomposition (2.7.4), the universal sheaves are given by

$$\mathcal{I}_{S \sqcup S'}^{[n_k]}\Big|_{\mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S') \times (S \sqcup S')} = p_k^* \mathcal{I}_S^{[i_k]} \oplus q_k^* \mathcal{I}_{S'}^{[j_k]} .$$

Write

$$\pi \colon S \to * , \quad \pi' \colon S' \to * \quad \text{and} \quad \pi \sqcup \pi' \colon S \sqcup S' \to *$$

for the projections. Let M and M' be a line bundles on S and S' respectively. It follows that for $0 \le k, l \le s$ we have

$$R\mathscr{H}om_{\pi \sqcup \pi'}(\mathcal{I}_{S \sqcup S'}^{[n_k]}, \mathcal{I}_{S \sqcup S'}^{[n_l]} \otimes (M \oplus M')) = \sum_{i+j=n} R\mathscr{H}om_{\pi}(p_k^* \mathcal{I}_S^{[i_k]}, p_l^* \mathcal{I}_S^{[i_l]} \otimes M) \oplus R\mathscr{H}om_{\pi'}(q_k^* \mathcal{I}_{S'}^{[j_k]}, q_l^* \mathcal{I}_{S'}^{[j_l]} \otimes M') \quad (2.7.5)$$

in the ring

$$K_0(\mathrm{Hilb}^n(S \sqcup S')) = \bigoplus_{i+j=n} K_0(\mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S')) .$$

For any pair i, j of $(s+1)$-tuples of non-negative integers, write

$$p \colon \mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S') \to \mathrm{Hilb}^i(S)$$
$$q \colon \mathrm{Hilb}^i(S) \times \mathrm{Hilb}^j(S') \to \mathrm{Hilb}^j(S') .$$

Let L and L' be $s+1$-tuples of line bundles on S and S' respectively, such that $\beta = \beta(L)$ and $\beta' = \beta(L')$ (see notation 2.2.1). Consider K-theory classes

$$N_{L \oplus L',0}^{[n]} , \quad N_{L,0}^{[i]} \quad \text{and} \quad N_{L',0}^{[j]}$$

as defined in Section 2.5. (Note that these classes may depend on the choice of L and L', but will only depend on β and β' after passing to numerical K-theory. See also Remark 2.7.1.) By definition, $N_{L+L',0}^{[n]}$ is linear combination of classes of the form (2.7.5), and we find

$$N_{L+L',0}^{[n]} = \sum_{i+j=n} p^* N_{L,0}^{[i]} + q^* N_{L',0}^{[j]} .$$

It follows that

$$\frac{1}{e(N^{[n]}_{L+L',0})} = \sum_{i+j=n} \frac{1}{e(N^{[i]}_{L,0})} \cdot \frac{1}{e(N^{[j]}_{L',0})} \,.$$

Finally, the corresponding multiplicative property of the factor

$$\prod_{i=k}^{s} e\left(R\pi_* \mathcal{O}(\beta_k) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{k-1}]}, \mathcal{I}^{[n_k]}(\beta_k))\right)$$

in the integrant of $Q_n(S,\beta)$ follows from the generalized Carlsson-Okounkov vanishing of [GT17] (see Remark 2.1.6). Integrating gives the result. □

The proof of Proposition 2.7.3 also gives the following refined result.

Proposition 2.7.6. *For each symbol*

$$\underline{\mathfrak{N}} \in \left\{ \underline{\chi(\mathcal{O}_S)}, \underline{K_S^2}, \underline{K_S\beta^i}, \underline{\beta^i\beta^j} \right\}_{1\le i\le j\le s}$$

there is a power series $A^{(s+1)}_{\underline{\mathfrak{N}}}(y) \in \mathbb{Q}(y)[[q_0,\ldots,q_s]]$, *starting with* 1, *such that*

$$\sum_n Q_n(S,\beta,y)\, q^n = \prod_{\underline{\mathfrak{N}}} (A^{(s+1)}_{\underline{\mathfrak{N}}}(y))^{\mathfrak{N}}$$

for any *smooth projective surface* S *and classes* $\beta_1,\ldots,\beta_s \in A^1(S)$.

Proof. A similar proof holds, using the multiplicative properties of ch, $\Lambda^\bullet$, det, and Td. Note that by Remark 2.5.3, the universal series take coefficients in $\mathbb{Q}(y)$, rather than in $\mathbb{Q}(\sqrt{y})$. □

2.8 Proof of Theorem A

As usual, we fix a rank $r = s+1$. In this section, we will identify

$$q \coloneqq q_0 = \ldots = q_s \,,$$

so the equation in Proposition 2.7.6 becomes

$$\sum_n Q_n(S,\beta,y)\, q^{|n|} = \prod_{\underline{\mathfrak{N}}} (A^{(r)}_{\underline{\mathfrak{N}}}(y))^{\mathfrak{N}} \tag{2.8.1}$$

in the ring $\mathbb{Q}(y)[[q]]$, where we use the notation $|n| = n_0 + \ldots + n_s$.

Let $L = (L_0,\ldots,L_s)$ be line bundles on a surface S, and let

$$\begin{aligned} \beta = \beta(L) &= (\beta_1,\ldots,\beta_s) \\ &= (K_S - \beta^1, \ldots, K_S - \beta^s) \end{aligned}$$

be given as in Notation 2.2.1. For non-negative integers $n = (n_0,\ldots,n_s)$ and ideal sheaves $I_i \in S^{[n_i]}$, consider the sheaf

$$E = L_0 \otimes I_0 \oplus \ldots \oplus L_s \otimes I_s \,.$$

In the notation of Section 2.2, E is a fibre of the family $E_{\mathcal{L}}^{[n]}$ of sheaves on S over $\mathrm{Hilb}^n_\beta(S)$. By Lemma 2.2.6 we have

$$c_2(E) = |n| + \frac{r-1}{2r} c_1(E)^2 - \sum_{i<j} \frac{i(r-j)}{r} \beta^i \beta^j - \sum_i \frac{i(r-i)}{2r} (\beta^i)^2 .$$

We will write

$$d(\beta) := -\sum_{i<j} \frac{i(r-j)}{r} \beta^i \beta^j - \sum_i \frac{i(r-i)}{2r} (\beta^i)^2 ,$$

so that we have

$$q^{\frac{1-r}{2r} c_1(E)^2} q^{c_2(E)} = q^{|n|+d(\beta)} . \tag{2.8.2}$$

Finally, recall that for any surface S with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and for β with

$$\mathrm{SW}(\beta) = \mathrm{SW}(\beta_1) \cdots \mathrm{SW}(\beta_s) \neq 0$$

we have, by Proposition 2.6.5,

$$F(S,\beta,y) = \left(F_0^{(r)}(y)\right)^{\chi(\mathcal{O}_S)} \prod_{k\leq l} \left(F_{kl}^{(r)}(y)\right)^{\beta^k \beta^l} , \tag{2.8.3}$$

where $F(S,\beta,y)$ is defined as in Section 2.5.

Define the following Laurent series in $q^{\frac{1}{2r}}$ with coefficients in $\mathbb{Q}(\sqrt{y})$:

$$\begin{aligned}
A &:= F_0^{(r)}(y)\, A^{(r)}_{\underline{\chi(\mathcal{O}_S)}}(y)\, (-1)^{(r-1)} \\
B &:= A^{(r)}_{\underline{K_S^2}}(y) \\
C_{ij} &:= q^{\frac{i(j-r)}{r}}\, F_{ij}^{(r)}(y)\, A^{(r)}_{\underline{\beta^i\beta^j}}(y) \quad \text{for} \quad 1 \leq i < j \leq r-1 ; \\
C_{ii} &:= q^{\frac{i(i-r)}{2r}}\, F_{ii}^{(r)}(y)\, A^{(r)}_{\underline{\beta^i\beta^i}}(y)\, A^{(r)}_{\underline{\beta^i K_S}}(y) \quad \text{for} \quad 1 \leq j \leq r-1 .
\end{aligned}$$

Proof of Theorem 2.A. First note that, by definition, the Laurent series are universal in the sense that they *only* depend on r. Now let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. We have

$$\begin{aligned}
\mathsf{Z}_{S,r,c_1}(q,y) &= \frac{q^{\frac{1-r}{2r}c_1^2}}{\#H^2(S,\mathbb{Z})[r]} \sum_{c_2\in\mathbb{Z}} \mathrm{VW}_{1^r,c_1,c_2}(S,y)\, q^{c_2} \\
&= q^{\frac{1-r}{2r}c_1^2} \sum_{c_2\in\mathbb{Z}} \hat{\sum_{\beta,n}} \mathrm{VW}^{[n]}_\beta(y)\, q^{c_2} && \text{by (2.4.3)} \\
&= \sum_\beta \sum_{n\in(\mathbb{Z}_{\geq 0})^r} \mathrm{VW}^{[n]}_\beta(y)\, q^{|n|+d(\beta)} && \text{by (2.8.2)} \\
&= \sum_\beta \mathrm{SW}(\beta) F(S,\beta,y) \sum_{n\in(\mathbb{Z}_{\geq 0})^r} Q_n(S,\beta,y)\, q^{|n|+d(\beta)} && \text{by (2.5.2)} \\
&= \sum_\beta \mathrm{SW}(\beta^\vee) A^{\chi(\mathcal{O}_S)} B^{K_S^2} \prod_{i\leq j} C_{ij}^{\beta^i\beta^j} . && \text{by (2.8.1) and (2.8.3)}
\end{aligned}$$

Here the symbol $\sum_{\beta}$ denotes a sum over $(r-1)$-tuples $\beta = (\beta_1, \ldots, \beta_{r-1})$ satisfying

$$c_1 \equiv \sum_{i=1}^{r-1} i\beta^i \mod rH^2(S, \mathbb{Z})\,.$$

The sum $\hat{\sum}_{\beta,n}$ is taken over β as above, and $n \in (\mathbb{Z}_{\geq 0})^r$ with

$$c_2 = |n| + \frac{r-1}{2r}c_1^2 + d(\beta)$$

cf. Proposition 2.3.7. Finally, we have used the notation $\beta^\vee = (\beta^1, \ldots, \beta^{r-1})$, and the equation

$$\begin{aligned}\mathrm{SW}(\beta^\vee) &= \mathrm{SW}(\beta^1)\cdots\mathrm{SW}(\beta^{r-1})\\ &= (-1)^{(r-1)\chi(\mathcal{O}_S)}\,\mathrm{SW}(\beta)\,,\end{aligned}$$

which follows from [Moc09, Proposition 6.3.4]. □

Proposition 2.8.4. *Let S be surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and let Chern r, c_1 and c_2 be Chern classes such that semistability implies stability. If r is odd, we have*

$$\mathrm{VW}_{1^r,c_1,c_2}(S, y) \in \mathbb{Q}(y) \subset \mathbb{Q}(\sqrt{y})\,.$$

Proof. By Proposition 2.7.6 and Remark 2.6.6, the Laurent series A, B and C_{ij} have coefficients in $\mathbb{Q}(y)$. □

2.9 Toric computations

We will see how to compute the coefficients of the series

$$A^{(s+1)}_{\underline{\mathfrak{N}}} \quad \text{for} \quad \underline{\mathfrak{N}} \in \mathcal{N} := \left\{\underline{\chi(\mathcal{O}_S)}, \underline{K_S^2}, \underline{K_S\beta^i}, \underline{\beta^i\beta^j}\right\}_{1\leq i\leq j\leq s}\,,$$

up to some degree N. In fact, cf. [Göt98], and as we will explain in this section, it suffices to evaluate the integrals

$$Q_n(S,\beta) = \int_{[\mathrm{Hilb}^n(S)]} \frac{\prod_{i=1}^s e\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)}{e\left(N^{[n]}_{L,0}\right)} \tag{2.9.1}$$

for $|n| \leq N$ on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ and sufficiently many different β (see Section 2.5 for notation and definitions).

Let $\omega_{2,1^s}$ denote the $\mathbb{Q}$-vector space of cobordism classes of surfaces with s-tuples of line bundles, as defined in [LP12], and let B be a basis. For rank $s+1 = 2$, we could take

$$B = \left([\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \omega_S^{\pm 1}]\right)\,,$$

and for rank $s+1=3$

$$B = \big([\mathbb{P}^1\times\mathbb{P}^1,(\mathcal{O},\mathcal{O})],[\mathbb{P}^2,(\mathcal{O},\mathcal{O})],[\mathbb{P}^2,(\omega_S^{\pm 1},\mathcal{O})],[\mathbb{P}^2,(\mathcal{O},\omega_S^{\pm 1})],[\mathbb{P}^2,(\omega_S,\omega_S)]\big)\ .$$

We can view the symbols $\underline{\mathfrak{N}}$ as coordinate functions on $\omega_{2,1^s}$. For a surface S, with an s-tuple $\beta\in(H^2(S,\mathbb{Z}))^s$, we write

$$\underline{\mathfrak{N}}(S,\beta)=\mathfrak{N}\,,$$

so we have

$$\underline{\chi(\mathcal{O}_S)}(S,\beta)=\chi(\mathcal{O}_S)\,,\quad \underline{K_S^2}(S,\beta)=K_S^2\,,\quad \underline{K_S\beta^i}(S,\beta)=K_S\beta^i\,,\quad \ldots\,.$$

Then the fact that B is a basis can be expressed by the fact that the matrix

$$M^{(s+1)}:=\Big[\underline{\mathfrak{N}}(S,\beta)\Big]_{[S,\beta]\in B,\ \underline{\mathfrak{N}}\in\mathcal{N}}$$

is invertible. For the bases for rank 2 and 3 given above, we have

$$M^{(2)}=\begin{pmatrix}1&8&0&0\\1&9&0&0\\1&9&9&9\\1&9&-9&9\end{pmatrix}$$

and

$$M^{(3)}=\begin{pmatrix}1&8&0&0&0&0&0\\1&9&0&0&0&0&0\\1&9&9&9&0&0&0\\1&9&-9&9&0&0&0\\1&9&0&0&9&9&0\\1&9&0&0&-9&9&0\\1&9&9&9&9&9&9\end{pmatrix}$$

respectively.

Recall that by Proposition 2.7.3, we have

$$\sum_n Q_n(S,\beta)\,q^n=\prod_{\underline{\mathfrak{N}}}(A_{\underline{\mathfrak{N}}}^{(s+1)})^{\mathfrak{N}}$$

for any surface S, and curve classes $\beta=(\beta_1,\ldots,\beta_s)$. Taking the natural logarithm, we obtain

$$\log\sum_n Q_n(S,\beta)\,q^n=\sum_{\underline{\mathfrak{N}}}\mathfrak{N}\log A_{\underline{\mathfrak{N}}}^{(s+1)}\,.$$

By definition of M, we have

$$\left[\log\sum_n Q_n(S,\beta)q^n\right]_{[S,\beta]\in B}=M\cdot\left[\log A_{\underline{\mathfrak{N}}}^{(s+1)}\right]_{\underline{\mathfrak{N}}\in\mathcal{N}}\,.$$

Now assume we want to compute the power series $A_{\underline{\mathfrak{N}}}^{(s+1)}$ up to order N. Since M is invertible, it suffices evaluate the integrals $Q_n(S,\beta)$ for all $n\in(\mathbb{Z}_{\geq 0})^{s+1}$ with

$|n| \leq N$. Note that by Proposition 2.7.6, the discussion above also applies to the refined case.

Let S be any toric surface with a torus T, and assume that we have equipped all line bundles appearing in the integral (2.9.1) with an equivariant structure. Then, by applying the Atiyah-Bott localization formula, we obtain

$$\begin{aligned} Q_n(S,\beta) &= \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int \frac{\prod_{i=1}^s e\left(\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\big|_F\right)}{e\left(N_{L,0}^{[n]}\big|_F\right) e(T_{\mathrm{Hilb}^n(S),F})} \\ &= \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int e\left(-T_{L,0}^{[n]}\big|_F\right) \end{aligned} \tag{2.9.2}$$

in which $e()$ denotes the equivariant Euler class for the torus $T \times \mathbb{C}^*$.

Remark 2.9.3. In the factor

$$e\left(\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\big|_F\right)$$

in the formula above, the Euler class $e(\,)$ should a priori be the T-equivariant Chern class $c^T_{n_{i-1}+n_i}(\,)$, but by [CO12, Lemma 6] and Lemma 2.9.5 below, the K-theory class

$$\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\big|_F \in K_0^T(F)$$

can be represented by an honest $n_{i-1} + n_i$-dimensional representation of the torus T. It follows that the T-equivariant top Chern class agrees with T-equivariant Euler class.

Remark 2.9.4. The compact form of the expression 2.9.2 is due to the fact that it is obtained by applying the Atiyah-Bott localization formula twice. The virtual version of the formula, due to Graber and Pandharipande [GP99], expresses by definition the contributions (2.4.1) of nested Hilbert schemes $S_\beta^{[n]}$ to the monopole branch of the Vafa-Witten invariant for a surface S with $p_g(S) > 0$. The second time, however, we applied the formula to Hilbert schemes of points on a toric surface.

Similarly, we have

$$\begin{aligned} Q_n(S,\beta,y) &= \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int \frac{\mathrm{ch}\left((K_{L,0}^{[n]})^{\frac{1}{2}}|_F\right)}{\mathrm{ch}\left(\Lambda^\bullet(N_{L,0}^{[n]\vee}|_F)\right)} \mathrm{Td}\left((T_{L,0}^{[n]})^{\mathbb{C}^*}|_F\right) \\ &\quad \times \frac{\prod_{i=1}^s e\left(\left(R\pi_*\mathcal{O}(\beta_i) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))\right)\big|_F\right)}{e(T_{\mathrm{Hilb}^n(S),F})} \\ &= \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int \frac{\mathrm{ch}\left((K_{L,0}^{[n]})^{\frac{1}{2}}|_F\right)}{\mathrm{ch}\left(\Lambda^\bullet(T_{L,0}^{[n]\vee}|_F)\right)}, \end{aligned}$$

where ch and Td denote the $T \times \mathbb{C}^*$-equivariant Chern character and Todd class respectively. We have suppressed that by convention, we denote the Chern character of $\mathfrak{t}$ by $y = \mathrm{ch}(\mathfrak{t})$. Finally, note that the second equation follows from the identity

$$\mathrm{ch}(\Lambda^\bullet(L^*)) = 1 - \exp(-\alpha) = \frac{e(L)}{\mathrm{Td}(L)}$$

for any 1-dimensional T-representation L with $c_1(L) = \alpha$.

Let $F \in \mathrm{Hilb}^n(S)$ be a T-fixed point. Let $0 \leq i, j \leq s$, and write

$$I = \mathcal{I}_F^{[n_i]} \quad \text{and} \quad J = \mathcal{I}_F^{[n_j]} .$$

The class $T_{L,0}^{[n]}\Big|_F$ is a linear combination of classes of the form

$$\Big(R\pi_* M - R\mathscr{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_j]} \otimes M)\Big)\Big|_F = \chi(M) - R\,\mathrm{Hom}_S(I, J \otimes M) ,$$

where M is a $T \times \mathbb{C}^*$-equivariant line bundle on S.

Lemma 2.9.5. *Let $\{U_\sigma\}_{\sigma=1,\dots,e(S)}$ be the maximal open cover of S by affine T-fixed subsets, cf. [GK17, Section 4]. Then we have*

$$\chi(M) - R\,\mathrm{Hom}_S(I, J \otimes M) = \sum_{\sigma=1}^{e(S)} \Gamma(U_\sigma, M) - R\,\mathrm{Hom}_{U_\sigma}(I|_{U_\sigma}, J|_{U_\sigma} \otimes M|_{U_\sigma}) . \quad (2.9.6)$$

Proof. Write $U_{\sigma\tau} = U_\sigma \cap U_\tau$ for $\sigma < \tau$. Since I and J are ideal sheaves of $\mathbb{C}^*$-fixed 0-dimensional subschemes of S, and $U_{\sigma\tau}$ does not contain any fixed points, we have

$$\begin{aligned} \Gamma(U_{\sigma\tau}, \mathcal{E}xt^i(I, J \otimes M)) &= \Gamma(U_\sigma \cap U_\tau, \mathcal{E}xt^i(I|_{U_{\sigma\tau}}, (J \otimes M)|_{U_{\sigma\tau}})) \\ &= \Gamma(U_{\sigma\tau}, \mathcal{E}xt^i(\mathcal{O}, M)) \end{aligned}$$

for any i , and a similarly for intersections $U_\sigma \cap U_\tau \cap U_\upsilon$. Now use the local-to-global spectral sequence and the Čech complex for the covering $\{U_\alpha\}$ (cf. [MNOP06, Section 4.6]), to compare the classes $\chi(M)$ and $R\,\mathrm{Hom}_S(I, J \otimes M)$. □

Now [CO12, Lemma 6], and also the proof of [GK17, Proposion 4.1], give an explicit expression for the right-hand side of (2.9.6). This allows us to compute the integrals $Q_n(S, \beta)$ and $Q_n(S, \beta, y)$. We have implemented the computation in Sage [Sag15] for $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $S = \mathbb{P}^2$ and for any β, n and rank. Part of the results for rank 3 are listed in Section 2.14.

2.10 The universal series A

Fix K3 surface S and non-negative integers $n = (n_0, \dots, n_s)$. Consider the inclusion

$$i \colon S^{[n]} \hookrightarrow \mathrm{Hilb}^n(S) = S^{[n_0]} \times \cdots \times S^{[n_s]}$$

of the nested Hilbert scheme. In the case that $n = (k, \dots, k)$ for some non-negative integer k, we write

$$\Delta_{S^{[k]} \times \cdots \times S^{[k]}} \cong S^{[k]} \subset S^{[k]} \times \cdots \times S^{[k]}$$

for the diagonal.

Lemma 2.10.1.

$$i_*[S^{[n]}]^{\mathrm{vir}} = \begin{cases} [\Delta_{S^{[k]} \times \cdots \times S^{[k]}}] & \textit{if } n = (k, \dots, k) \\ 0 & \textit{else} \end{cases}$$

Proof. For the first case, see [GSY18, Theorem 2] (note that in this case we have

$$S^{[n]} = S^{[k,\dots,k]} \cong \Delta_{S^{[k]}\times\dots\times S^{[k]}} \cong S^{[k]}\,,$$

and the perfect obstruction theory is just the cotangent bundle). For the second case, note that we have by Theorem 2.1.5 and Serre duality:

$$\begin{aligned} i_*[S^{[n_0,\dots,n_s]}]^{\mathrm{vir}} &= \prod_i e(\chi(\mathcal{O}_S) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]})) \cap [\mathrm{Hilb}^n(S)] \\ &= (-1)^{n_0+n_s} \prod_i e(\chi(\mathcal{O}_S) - R\mathscr{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_{i-1}]})) \cap [\mathrm{Hilb}^n(S)] \\ &= (-1)^{n_0+n_s} j_*[S^{[n_s,\dots,n_0]}]^{\mathrm{vir}}\,, \end{aligned}$$

where j is the inclusion

$$j\colon S^{[n_s,\dots,n_0]} \hookrightarrow \mathrm{Hilb}^n(S)\,.$$

Now note that $S^{[n_0,\dots,n_s]}$ or $S^{[n_s,\dots,n_0]}$ is empty, unless $n_0 = \ldots = n_s$. □

Set $\beta = (0,\dots,0)$, so we have $\beta^i = K_S - \beta_i = 0$ for $i = 1,\dots,s$. By Proposition 2.7.3 we have

$$\sum_n Q_n(S,\beta)\, q^n = \left(A^{(s+1)}_{\underline{\chi(\mathcal{O}_S)}}\right)^2\,.$$

Recall that $Q_n(S,\beta)$ is by definition integral over the virtual class of $S^{[n]}_\beta = S^{[n]}$. Hence, by Lemma 2.10.1, we have $Q_n(S,\beta) = 0$ or $n_0 = \ldots = n_s$. Assume that we have $n = (k,\dots,k)$ for a non-negative integer k. We will compute $Q_n(S,\beta)$.

We let $L = (\mathcal{O}_S,\dots,\mathcal{O}_S)$ the $(s+1)$-tuple of copies of the trivial line on S, so we have $\beta = \beta(L)$. We write

$$E^{[n]} := E^{[n]}_L = \mathrm{pr}_0^*\mathcal{I}^{[k]}\otimes \mathfrak{t}^0 \oplus \ldots \oplus \mathrm{pr}_s^*\mathcal{I}^{[k]} \otimes \mathfrak{t}^{-s}$$

for the sheaf on $\mathrm{Hilb}^n(S)\times S$, where pr_i denotes the ith projection

$$\mathrm{pr}_i\colon \mathrm{Hilb}^n(S) = S^{[k]}\times\cdots\times S^{[k]} \to S^{[k]}\,,$$

and its base change to S. We will write $\Delta\colon S^{[k]} \to S^{[k]}\times\cdots\times S^{[k]}$ for the diagonal embedding (which of course can be identified with the embedding i), and denote its base change to S by the same symbol. We have an isomorphism

$$\Delta^* E^{[n]} \cong \mathcal{I}^{[k]}\otimes\mathfrak{t}^0 \oplus\ldots\oplus \mathcal{I}^{[k]}\otimes\mathfrak{t}^{-s}$$

of sheaves on $S^{[k]}\times S$. Write

$$E = (\mathfrak{t}^0 \oplus\ldots\oplus \mathfrak{t}^{-s})\otimes\mathcal{O}_S$$

for the vector bundle on S. Using the notation Section 2.5, we have the following equality in the $\mathbb{C}^*$-equivariant K-group of $S^{[k]}$:

$$\begin{aligned} \Delta^* T^{[n]}_{L,0} &= R\mathscr{H}om_\pi(\Delta^*E^{[n]}, \Delta^*E^{[n]}\otimes\mathfrak{t})_0 - R\mathscr{H}om_\pi(\Delta^*E^{[n]}, \Delta^*E^{[n]})_0 \\ &\quad - (R\,\mathrm{Hom}(E, E\otimes\mathfrak{t})_0 - R\,\mathrm{Hom}(E,E)_0) \\ &= \sum_{i=1}^{s+1} R\mathscr{H}om_\pi(\mathcal{I}^{[k]},\mathcal{I}^{[k]})_0\otimes\mathfrak{t}^i + \sum_{i=0}^{s} - R\mathscr{H}om_\pi(\mathcal{I}^{[k]},\mathcal{I}^{[k]})_0\otimes\mathfrak{t}^{-i} \\ &= T_{S^{[k]}} + \sum_{i=1}^{s}\left(T_{S^{[k]}}\otimes\mathfrak{t}^{-i} - T_{S^{[k]}}\otimes\mathfrak{t}^i\right) - T_{S^{[k]}}\otimes\mathfrak{t}^{s+1}\,. \end{aligned} \tag{2.10.2}$$

Note that $T_{S^{[k]}}$ has even rank, so we have

$$e(T_{S^{[k]}} \otimes \mathsf{t}^{-i}) = e(T_{S^{[k]}} \otimes \mathsf{t}^{i}),$$

and hence

$$\begin{aligned} e\left(\sum_{i=1}^{s} \left(T_{S^{[k]}} \otimes \mathsf{t}^{-i} - T_{S^{[k]}} \otimes \mathsf{t}^{i}\right)\right) &= \prod_{i=1}^{s} \frac{e\left(T_{S^{[k]}} \otimes \mathsf{t}^{-i}\right)}{e\left(T_{S^{[k]}} \otimes \mathsf{t}^{i}\right)} \\ &= 1 \end{aligned}$$

It follows that we have

$$\begin{aligned} Q_n(S,\beta) &= \int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{1}{e(N_{L,0}^{[n]})} \\ &= \int_{S^{[k]}} \frac{1}{e\left(\Delta^*\left(T_{L,0}^{[n]}\right)^{\mathrm{mov}}\right)} \\ &= \int_{S^{[k]}} e(T_{S^{[k]}} \otimes \mathsf{t}^{s+1}) \\ &= \int_{S^{[k]}} e(T_{S^{[k]}}) \\ &= e(S^{[k]}). \end{aligned}$$

By Göttsche's formula [Göt90], we have

$$\begin{aligned} \left(A^{(s+1)}_{\underline{\chi(\mathcal{O}_S)}}\right)^2 &= \sum_{n \in (\mathbb{Z}_{\geq 0})^{s+1}} Q_n(S,\beta)\, q^{|n|} \\ &= \sum_{k \in \mathbb{Z}_{\geq 0}} e(S^{[k]})\, q^{k(s+1)} \\ &= \left(\prod_{k \geq 1} \frac{1}{1 - q^{k(s+1)}}\right)^{24}. \end{aligned}$$

It follows that

$$F_0^{(s+1)} A^{(s+1)}_{\underline{\chi(\mathcal{O}_S)}} (-1)^{s+1} = \frac{1}{s+1} \left(\prod_{k \geq 1} \frac{1}{1 - q^{k(s+1)}}\right)^{12},$$

which is the specialization of the right-hand side of (2.1.19) at $y = 1$, as expected.

Now consider the integral

$$Q_n(S,\beta,y) = \left[\int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{\mathrm{ch}\left((K_{L,0}^{[n]})^{\frac{1}{2}}\right)}{\mathrm{ch}(\Lambda^\bullet((N_{L,0}^{[n]})^\vee))} \mathrm{Td}\left((T_{L,0}^{[n]})^{\mathbb{C}^*}\right)\right]_{\mathrm{ch}(\mathsf{t}) = y}.$$

Since $T_{S^{[k]}}$ is self-dual and has even rank, we have by [Tho18a, equation 2.28] the following equation in the $\mathbb{C}^*$-equivariant K-group on $S^{[k]}$:

$$\begin{aligned} \frac{(\det(T_{S^{[k]}} \otimes \mathsf{t}^{-i} - T_{S^{[k]}} \otimes \mathsf{t}^{i})^\vee)^{\frac{1}{2}}}{\Lambda^\bullet((T_{S^{[k]}} \otimes \mathsf{t}^{-i} - T_{S^{[k]}} \otimes \mathsf{t}^{i})^\vee)} &= \frac{\det(T^*_{S^{[k]}} \otimes \mathsf{t}^{i}) \otimes \Lambda^\bullet(T^*_{S^{[k]}} \otimes \mathsf{t}^{-i})}{\Lambda^\bullet(T^*_{S^{[k]}} \otimes \mathsf{t}^{i})} \\ &= 1. \end{aligned}$$

It follows that, as above, the middle terms of (2.10.2) do not contribute to $Q_n(S,\beta,y)$.

Remark 2.10.3. Note that taking the square root involves a choice. First of all, our choice here is consistent with the one we made before. More importantly, after choosing a root $\sqrt{\mathfrak{t}}$, the choice is unique up to 2-torsion in $\mathrm{Pic}(S_\beta^{[n]})$, which is killed by ch.

Since $T_{S^{[k]}}$ is self-dual, so is $K_{S^{[k]}}$, and hence $\mathrm{ch}(K_{S^{[k]}}) = 1$. Writing $r = s+1$, we find

$$
\begin{aligned}
Q_n(S,\beta,y) &= \left[\int_{S^{[k]}} \frac{\mathrm{ch}\left((\det(T_{S^{[k]}} - T_{S^{[k]}} \otimes \mathfrak{t}^r)^\vee)^{\frac{1}{2}}\right)}{\mathrm{ch}(\Lambda^\bullet(-T_{S^{[k]}} \otimes \mathfrak{t}^r)^\vee)} \mathrm{Td}(T_{S^{[k]}})\right]_{\mathrm{ch}(\mathfrak{t})=y} \\
&= \left[\int_{S^{[k]}} \mathrm{ch}\left(K_{S^{[k]}} \otimes \mathfrak{t}^{rk}\right) \mathrm{ch}(\Lambda^\bullet(T_{S^{[k]}} \otimes \mathfrak{t}^{-r}))\,\mathrm{Td}(T_{S^{[k]}})\right]_{\mathrm{ch}(\mathfrak{t})=y} \\
&= y^{rk} \sum_{i=0}^{2k} (-1)^i y^{-ri} \int_{S^{[k]}} \mathrm{ch}(\Lambda^i T_{S^{[k]}})\,\mathrm{Td}(T_{S^{[k]}}) \\
&= y^{rk} \sum_{i=0}^{2k} (-1)^i y^{-ri} \chi(\Lambda^i T_{S^{[k]}}) \\
&=: \chi_{-y^r}(S^{[k]})\,.
\end{aligned}
$$

The generating series of χ_y-genera of the Hilbert schemes $S^{[k]}$ has been computed in [GS93]. It follows that we have

$$
\begin{aligned}
\left(A^{(r)}_{\underline{\chi(\mathcal{O}_S)}}(y)\right)^2 &= \sum_{n\in(\mathbb{Z}_{\geq 0})^r} Q_n(S,\beta,y)\, q^{|n|} \\
&= \sum_{k\in\mathbb{Z}_{\geq 0}} \chi_{-y^r}(S^{[k]}) q^{rk} \\
&= \prod_{k\geq 1} \frac{1}{(1-q^{rk})^{20}(1-y^{-r}q^{rk})^2(1-y^r q^{rk})^2}\,.
\end{aligned}
$$

We conclude

$$
\begin{aligned}
A^{(r)}(y) &= F_0^{(r)}(y) A^{(r)}_{\underline{\chi(\mathcal{O}_S)}}(y)(-1)^r \\
&= \frac{1}{[r]_y} \prod_{k\geq 1} \frac{1}{(1-q^{rk})^{10}(1-y^{-r}q^{rk})(1-y^r q^{rk})} \\
&= \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^r,y^r)^{\frac{1}{2}}\,\tilde{\Delta}(q^r)^{\frac{1}{2}}}\,,
\end{aligned}
$$

proving Theorem 2.C.

2.11 Smooth components

In the case that the monopole branch of the moduli space of $\mathbb{C}^*$-fixed Higgs pairs is smooth, there is a direct method to compute the Vafa-Witten invariants. Let S

be a surface with $H^1(\mathcal{O}_S) = 0$, $p_g > 0$, and assume that $\mathrm{Pic}(S)$ is generated by a smooth very ample canonical curve C. In this case, the only Seiberg-Witten basic classes of S are 0 and K_S. For rank 2, the monopole branch

$$\mathcal{M}_{1^2} = \mathcal{M}_{1^2,K_S,c_2} \subset (\mathcal{N}^{\perp}_{2,\omega_S,c_2})^{\mathbb{C}^*}$$

is smooth precisely when $c_2 = 0, 1, 2, 3$ [TT17a]. In particular, the virtual class is given by the Euler class of the obstruction bundle and the Vafa-Witten invariants can be computed using the intersection theory of (smooth) nested Hilbert schemes of points on the surface and the smooth canonical curve. This method, which is carried out in [TT17a] (unrefined) and [Tho18a] (refined), can be generalized to rank 3, but only for $c_2 = 0, 1, 2$. We have done the computation in this setting, and have found that they confirm our results (see the discussion after Theorem 2.B′).

Let $(E, \phi) \in \mathcal{M}_{1^3,K_S,c_2}$ be a Higgs pair, so E can be written as

$$E = I_0 \otimes \omega_S^a \oplus I_1 \otimes \omega_S^b \oplus I_2 \otimes \omega_S^c$$

where $I_i \in S^{[n_i]}$ for $i = 0, 1, 2$ and $a, b, c \in \mathbb{Z}$ such that $a + b + c = 1$. Moreover, we have $\phi = (\phi_1, \phi_2)$ for non-zero homomorphisms

$$\phi_1 \colon I_0 \otimes \omega_S^a \to I_1 \otimes \omega_S^{b+1}\,,$$
$$\phi_2 \colon I_1 \otimes \omega_S^b \to I_2 \otimes \omega_S^{c+1}\,.$$

Lemma 2.11.1. *We have* $(a, b, c) = (1, 0, 0)$.

Proof. Slope semistability of E implies that

$$c \leq \frac{1}{3}\,, \quad \frac{b+c}{2} \leq \frac{1}{3}\,.$$

On the other hand, by the existence of the maps ϕ_1 and ϕ_2 we have

$$a \leq b + 1\,, \quad b \leq c + 1\,.$$

It is easy to see that the only integral solution to these inequalities together with $a + b + c = 1$ is $(a, b, c) = (1, 0, 0)$. □

Proposition 2.11.2. *Let S be given as above. Then $\mathcal{M}_{1^3,K_S,c_2}$ is smooth if and only if $c_2 \leq 2$. In particular, we have*

$$\mathcal{M}_{1^3,K_S,1} \cong \big(S^{[1]} \times |K_S|\big) \sqcup \mathcal{C}\,;$$
$$\mathcal{M}_{1^3,K_S,2} \cong \big(S^{[2]} \times |K_S|\big) \sqcup \big(S^{[1]} \times |K_S|\big) \sqcup \big(S^{[1]} \times \mathcal{C}\big) \sqcup \mathcal{C}^{[2]}_{|K_S|}$$

in which

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & S \\ \big\downarrow & & \\ |K_S| & & \end{array}$$

is the universal canonical curve, and $\mathcal{C}^{[2]}_{|K_S|} \to |K_S|$ the relative Hilbert scheme of pairs of points.

Proof. Note that for $I_i \in S^{[n_i]}$, $i = 0,1,2$, we have

$$c_2(I_0 \otimes \omega_S \oplus I_1 \oplus I_2) = n_0 + n_1 + n_2 \,.$$

By Lemma 2.11.1, we find

$$\mathcal{M}_{1^3,K_S,c_2} \cong \bigsqcup_{|n|=c_2, n_0 \geq n_1} S^{[n_0,n_1,n_2]}_{(0,K_S)} \,,$$

where we have used that $S^{[n_0,n_1,n_2]}_{(0,K_S)}$ is empty whenever $n_0 < n_1$. In particular, we have

$$\begin{aligned}
\mathcal{M}_{1^3,K_S,1} &= S^{[1,0,0]}_{(0,K_S)} \sqcup S^{[0,0,1]}_{(0,K_S)} \\
&\cong S^{[1]} \times |K_S| \sqcup \mathcal{C} \,; \\
\mathcal{M}_{1^3,K_S,2} &= S^{[2,0,0]}_{(0,K_S)} \sqcup S^{[1,1,0]}_{(0,K_S)} \sqcup S^{[1,0,1]}_{(0,K_S)} \sqcup S^{[0,0,2]}_{(0,K_S)} \\
&\cong \left(S^{[2]} \times |K_S|\right) \sqcup \left(S^{[1]} \times |K_S|\right) \sqcup \left(S^{[1]} \times \mathcal{C}\right) \sqcup \mathcal{C}^{[2]}_{|K_S|} \,.
\end{aligned}$$

The total spaces of the universal canonical curve $\mathcal{C}$, and of relative Hilbert scheme of points $\mathcal{C}^{[2]}_{|K_S|}$ are smooth by the the assumption that K_S is very ample.

The component

$$S^{[1,1,1]}_{(0,K_S)} \cong (S \times \mathcal{C}) \cup (\Delta_S \times |K_S|) \overset{i}{\hookrightarrow} S \times S \times |K_S|$$

of $\mathcal{M}_{1^3,K_S,3}$ has two irreducible components with non-empty intersection. More generally, let $c_2 \geq 3$. For an ideal sheaf I on S, let Z_I denote the corresponding subscheme. Then the component

$$\begin{aligned}
S^{[1,1,c_2-2]}_{0,K_S} &\cong S^{[1,c_2-2]}_{K_S} \\
&= \{p \in S, I \in S^{[c_2-2]}, C \in |K_S| : Z_I \subset C \cup p\}
\end{aligned}$$

has two components given by the conditions $p \in Z_I$, and $Z_I \subset C$ respectively. Hence, it is singular at points in the intersection given by $p \in Z_I \subset C$. It follows that $\mathcal{M}_{1^r,K_S,c_2}$ is singular. □

The connected components of $\mathcal{M}_{1^3,K_S,1}$, together with the restrictions of the universal sheaf on $\mathcal{M}_{1^3,K_S,1} \times S$ are given as follows:

$$\begin{aligned}
S^{[1]} \times |K_S| \,, &\quad \mathcal{I}^{[1]} \otimes \omega_S \oplus \mathfrak{t}^{-1} \oplus \mathcal{O}_{|K_S|}(1) \otimes \mathfrak{t}^{-2} \,; \\
\mathcal{C} \,, &\quad \omega_S \oplus \mathfrak{t}^{-1} \oplus j^*(\mathcal{I}^{[1]} \otimes \mathcal{O}_{|K_S|}(1)) \otimes \mathfrak{t}^{-2} \,.
\end{aligned}$$

in which $j \colon \mathcal{C} \times S \to S^{[1]} \times |K_S| \times S$ is the inclusion. We have suppressed pull-backs along the several projections. For $c_2 = 2$, we have

$$\begin{aligned}
S^{[2]} \times |K_S| \,, &\quad \left(\mathcal{I}^{[2]} \otimes \omega_S\right) \oplus \mathfrak{t}^{-1} \oplus \left(\mathcal{O}_{|K_S|}(1) \otimes \mathfrak{t}^{-2}\right) ; \\
S^{[1]} \times |K_S| \,, &\quad \left(\mathcal{I}^{[1]} \otimes \omega_S\right) \oplus \left(\mathcal{I}^{[1]} \otimes \mathfrak{t}^{-1}\right) \oplus \left(\mathcal{O}_{|K_S|}(1) \otimes \mathfrak{t}^{-2}\right) ; \\
S^{[1]} \times \mathcal{C} \,, &\quad \left(\mathcal{I}^{[1]} \otimes \omega_S\right) \oplus \mathfrak{t}^{-1} \oplus \left(j^* \left(\mathcal{I}^{[1]} \otimes \mathcal{O}_{|K_S|}(1)\right) \otimes \mathfrak{t}^{-2}\right) ; \\
\mathcal{C}^{[2]}_{|K_S|} \,, &\quad \omega_S \oplus \mathfrak{t}^{-1} \oplus \left(j_2^* \left(\mathcal{I}^{[2]} \otimes \mathcal{O}_{|K_S|}(1)\right) \otimes \mathfrak{t}^{-2}\right) .
\end{aligned}$$

in which we have written

$$j_2 \colon \mathcal{C}^{[2]}_{|K_S|} \times S \hookrightarrow S^{[2]} \times |K_S| \times S$$

for the inclusion. Again, we have suppressed pull-backs along projections. Now define Higgs fields $\phi = (\phi_1, \phi_2)$ by the several natural inclusions of ideal sheaves.

As the moduli spaces are smooth, we can compute the virtual class of each component by taking the Euler class of the obstruction bundle. Write $H \coloneqq c_1(\mathcal{O}_{|K_S|}(1))$. Using Theorem 2.1.4, we find

$$\begin{aligned}
[S^{[1]} \times |K_S|]^{\mathrm{vir}} &= e(K_S^* + \Omega_{|K_S|}(1)) \\
&= (-1)^{p_g} \cdot [C]\,, \\
[\mathcal{C}]^{\mathrm{vir}} &= e(\Omega_{|K_S|}(1)) \\
&= (-1)^{p_g - 1} \cdot [C]\,, \\
[S^{[2]} \times |K_S|]^{\mathrm{vir}} &= e(\omega_S^{[2]*} + \Omega_{|K_S|}(1)) \\
&= [\mathcal{C}^{[2]}_{|K_S|}] \cap (-H)^{p_g - 1} \\
&= (-1)^{p_g - 1} \cdot [C^{[2]}]\,, \\
[S^{[1]} \times |K_S|]^{\mathrm{vir}} &= e(H^0(\omega_S)) \\
&= 0\,, \\
[S^{[1]} \times \mathcal{C}]^{\mathrm{vir}} &= e(\omega_S^* + \Omega_{|K_S|}(1)) \\
&= (-1)^{p_g} \cdot [C \times C]\,, \\
[\mathcal{C}^{[2]}_{|K_S|}]^{\mathrm{vir}} &= e(\Omega_{|K_S|}(1)) \\
&= (-1)^{p_g - 1} \cdot [C^{[2]}]\,.
\end{aligned}$$

It follows that the computation of the contribution of the Vafa-Witten invariant reduces to computations in the intersection rings of C and $C^{[2]}$. Using Grothendieck-Riemann-Roch to compute the Chern classes of the relative Hom complexes, this is a straight forward computation. The details are similar to the computations in [TT17a] and [Tho18a].

2.12 Comparison to the Göttsche-Kool conjectures

For rank 2, the Laurent series that appear in Theorem 2.A, and are defined in Section 2.8, are given by

$$\begin{aligned}
A^{(2)} &= \frac{1}{y^{-\frac{1}{2}} + y^{\frac{1}{2}}} A^{(2)}_{\underline{\chi(\mathcal{O}_S)}}(y)\,, \\
B^{(2)} &= A^{(2)}_{\underline{K_S^2}}(y)\,, \\
C^{(2)}_{11} &= -q^{-\frac{1}{4}} \frac{1}{y^{-\frac{1}{2}} + y^{\frac{1}{2}}} A^{(2)}_{\underline{\beta^1 \beta^1}}(y) A^{(2)}_{\underline{\beta^1 K_S}}(y)\,,
\end{aligned}$$

and for rank 3 by

$$\begin{aligned}
A^{(3)} &= \frac{1}{y^{-1}+1+y} A^{(3)}_{\underline{\chi(\mathcal{O}_S)}}(y)\,, \\
B^{(3)} &= A^{(3)}_{\underline{K_S^2}}(y)\,, \\
C^{(3)}_{11} &= q^{-\frac{1}{3}} \frac{1}{y^{-1}+1+y} A^{(3)}_{\underline{\beta^1\beta^1}}(y) A^{(3)}_{\underline{\beta^1 K_S}}(y)\,, \\
C^{(3)}_{22} &= q^{-\frac{1}{3}} \frac{1}{y^{-1}+1+y} A^{(3)}_{\underline{\beta^2\beta^2}}(y) A^{(3)}_{\underline{\beta^2 K_S}}(y)\,, \\
C^{(3)}_{12} &= q^{-\frac{1}{3}} \frac{(1+y)^2}{1+y+y^2} A^{(3)}_{\underline{\beta^1\beta^2}}(y)\,.
\end{aligned}$$

In Section 2.9, we have discussed a method for computing the terms of the generating series $A^{(r)}_{\underline{\mathfrak{N}}}$ appearing above. In Section 2.14 we have listed the first few terms of the rank 3 power series. The computations allow us to check the equations of Conjectures 2.1.17 and 2.1.18 term by term, leading to Theorem 2.B. As an example, let us just verify one term of $C^{(3)}_{12}$. We have

$$\begin{aligned}
C^{(3)}_{12} &= q^{-\frac{1}{3}} \frac{(1+y)^2}{1+y+y^2} A^{(3)}_{\underline{\beta^1\beta^2}}(y) \\
&= q^{-\frac{1}{3}} \frac{(1+y)^2}{1+y+y^2} \left(1 + \frac{y^4+6y^3+6y^2+6y+1}{(y+1)^2 y} q + \dots \right).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
W(q^{\frac{1}{2}}, y) &= \frac{\Theta_{A_2,(0,0)}(q^{\frac{1}{2}}, y)}{\Theta_{A_2,(1,0)}(q^{\frac{1}{2}}, y)} \\
&= \frac{1 + (y^2+2y+2y^{-1}+y^{-2})\, q + \dots}{(y+1+y^{-1}) q^{\frac{1}{3}} + (y^2+1+y^{-2})\, q^{\frac{4}{3}} + \dots} \\
&= q^{-\frac{1}{3}} \frac{1}{y+1+y^{-1}} \frac{1 + (y^2+2y+2y^{-1}+y^{-2})\, q + \dots}{1 + (y-1+y^{-1})\, q + \dots} \\
&= q^{-\frac{1}{3}} \left(\frac{1}{y+1+y^{-1}} + \frac{y^2+y+1+y^{-1}+y^{-2}}{y+1+y^{-1}} q + \dots \right),
\end{aligned}$$

and hence

$$W(q^{\frac{1}{2}}, 1) = q^{-\frac{1}{3}} \frac{1}{3} (1 + 5\, q + \dots)\,.$$

It follows that

$$\begin{aligned}
&W_+(q^{\frac{1}{2}}, y)W_-(q^{\frac{1}{2}}, y) \\
&= W(q^{\frac{1}{2}}, y) + 3W(q^{\frac{1}{2}}, 1) \\
&= q^{-\frac{1}{3}} \left(1 + \frac{1}{y+1+y^{-1}} + \left(\frac{y^2+y+1+y^{-1}+y^{-2}}{y+1+y^{-1}} + 5\right) q + \dots\right) \\
&= q^{-\frac{1}{3}} \left(\frac{y+2+y^{-1}}{y+1+y^{-1}} + \left(\frac{y^2+y+1+y^{-1}+y^{-2}+5(y+1+y^{-1})}{y+1+y^{-1}}\right) q + \dots\right) \\
&= q^{-\frac{1}{3}} \frac{(1+y)^2}{1+y+y^2} \left(1 + \frac{y^4+6y^3+6y^2+6y+1}{(y+1)^2 y} q + \dots\right) \\
&\equiv C^{(3)}_{12}(y) \mod U^{(3)}_2 ,
\end{aligned}$$

where have used the notation

$$U^{(3)}_2 = 1 + q^2\, \mathbb{Q}(y^{\frac{1}{2}})[[q]] \subset \mathbb{Q}(y^{\frac{1}{2}})(\!(q^{\frac{1}{6}})\!)^* .$$

from the introduction.

2.13 Appendix: Functions appearing in the conjectures by Göttsche and Kool

$$
\begin{aligned}
\phi_{-2,1}(x,y) &\coloneqq (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \prod_{n=1}^{\infty} \frac{(1-x^n y)^2 (1-x^n y^{-1})^2}{(1-x^n)^4} \\
\tilde{\eta}(x) &\coloneqq \prod_{n\in\mathbb{Z}_{>0}} (1-x^n) \\
\tilde{\Delta}(x) &= \prod_{n\in\mathbb{Z}_{>0}} (1-x^n)^{24} \\
\theta_2(x,y) &\coloneqq \sum_{n\in\mathbb{Z}+\frac{1}{2}} x^{n^2} y^n \\
\theta_3(x,y) &\coloneqq \sum_{n\in\mathbb{Z}} x^{n^2} y^n \\
\Theta_{A_2,(0,0)}(x,y) &\coloneqq \sum_{(m,n)\in\mathbb{Z}^2} x^{2(m^2-mn+n^2)} y^{m+n} \\
\Theta_{A_2,(1,0)}(x,y) &\coloneqq \sum_{(m,n)\in\mathbb{Z}^2} x^{2(m^2-mn+n^2+m-n+\frac{1}{3})} y^{m+n} \\
W(x,y) &\coloneqq \frac{\Theta_{A_2,(0,0)}(x,y)}{\Theta_{A_2,(1,0)}(x,y)} \,.
\end{aligned}
$$

The functions $W_{\pm}(x,y)$ are defined by following polynomial equation in ω:

$$
\omega^2 - \left(W(x,y)^2 + 3W(x,y)W(x,1)\right)\omega + W(x,y) + 3W(x,1) = 0\,.
$$

We will use the convention that $W_-(x,y)$ is the one with leading term

$$
x^{\frac{2}{3}}(y^{-1}+1+y)\,,
$$

so we have

$$
\begin{aligned}
W_-(q^{\frac{1}{2}},y) &= \frac{y^2+y+1}{y}\, q^{\frac{1}{3}} \left(1 + \frac{2y^2+3y+2}{(y+1)^2} q + \dots\right) \\
W_+(q^{\frac{1}{2}},y) &= \frac{(y+1)^2\, y}{y^2+y+1} q^{-\frac{2}{3}} \left(1 + \frac{y^4+4y^3+3y^2+4y+1}{(y+1)^2} q + \dots\right) \,.
\end{aligned}
$$

2.14 Appendix: Rank 3 results

We set $q := q_0 = q_1 = q_2$ and print the first few terms of $A^{(3)}_{\underline{\mathfrak{N}}}(y)$ for

$$\underline{\mathfrak{N}} \in \left\{ \underline{\chi(\mathcal{O}_S)}, \underline{K_S^2}, \underline{\beta^1 K_S}, \underline{\beta^2 K_S}, \underline{\beta^1\beta^1}, \underline{\beta^2\beta^2}, \underline{\beta^1\beta^2} \right\}.$$

$$\begin{aligned}
A^{(3)}_{\underline{\chi(\mathcal{O}_S)}}(y) &\equiv 1 + \frac{y^6+10y^3+1}{y^3}\, q^3 \mod q^4 \\
A^{(3)}_{\underline{K_S^2}}(y) &\equiv 1 - \frac{(y^2+y+1)^2}{(y+1)^2\, y}\, q \\
&- \frac{(2\,y^4+7\,y^3+12\,y^2+7\,y+2)\,(y^2+y+1)^2}{y^2\,(y+1)^4}\, q^2 \\
&+ \frac{1}{y^3(y+1)^6}\Big(5y^{12}+39y^{11}+150y^{10}+382y^9+705y^8+1002y^7 \\
&+ 1121y^6+1002y^5+705y^4+382y^3+150y^2+39y+5\Big)\, q^3 \mod q^4 \\
A^{(3)}_{\underline{\beta^1 K_S}}(y) &\equiv A^{(3)}_{\underline{\beta^2 K_S}}(y) \\
&\equiv 1 + \frac{1}{2}\frac{(y^2+y+1)(y-1)^2}{(y+1)^2 y}\, q \\
&+ \frac{1}{8}\frac{(23y^4+68y^3+142y^2+68y+23)(y^2+y+1)^2}{(y+1)^4 y^2}\, q^2 \\
&- \frac{y^2+y+1}{16y^3(y+1)^6}\Big(15y^{10}+244y^9+1006y^8+2790y^7+4719y^6+5780y^5 \\
&+ 4719y^4+2790y^3+1006y^2+244y+15\Big)\, q^3 \mod q^4 \\
A^{(3)}_{\underline{\beta^1\beta^1}}(y) &\equiv A^{(3)}_{\underline{\beta^2\beta^2}}(y) \\
&\equiv 1 - \frac{1}{2}\frac{y^4+3y^3+6y^2+3y+1}{(y+1)^2 y}\, q \\
&- \frac{1}{8}\frac{1}{(y+1)^4 y^2}\Big(5y^8+30y^7+109y^6+218y^5+280y^4 \\
&+ 218y^3+109y^2+30y+5\Big)\, q^2 \\
&+ \frac{1}{16y^3(y+1)^6}\Big(11y^{12}+115y^{11}+571y^{10}+1868y^9+4205y^8+6845y^7 \\
&+ 8026y^6+6845y^5+4205y^4+1868y^3+571y^2+115y+11\Big)\, q^3 \mod q^4 \\
A^{(3)}_{\underline{\beta^1\beta^2}}(y) &\equiv 1 + \frac{y^4+6y^3+6y^2+6y+1}{(y+1)^2 y}\, q \\
&- \frac{y^6+y^5+8y^4+8y^3+8y^2+y+1}{(y+1)^2 y^2}\, q^2 \\
&+ \frac{y^6+3y^4+4y^3+3y^2+1}{y^2(y+1)^2}\, q^3 \mod q^4
\end{aligned}$$

Chapter 3

Vertical Vafa-Witten invariants via Joyce-Song pairs

3.1 Introduction

In Chapter 2 we have studied monopole contributions to Vafa-Witten invariants, which virtually count certain sheaves on a local surface. We have worked under the assumption that, for the relevant sheaves, semi-stability implies stability (we call this the *stable* case). In this chapter, we will extend our results to the *semi-stable* case, where we don't make this assumption.

3.1.1 Joyce-Song pairs

Let S be an smooth algebraic surface over $\mathbb{C}$ with a polarisation H, and let

$$q\colon X \to S$$

be the total space of the canonical bundle of S. We will consider moduli spaces of certain compactly supported coherent sheaves on X. We are particularly interested in strictly semistable sheaves. Since semistable sheaves may have non-trivial automorphisms, we will work with sheaves that have been rigidified by a Joyce-Song section. The resulting *Joyce-Song pairs* and their moduli spaces have been studied in [JS12] and (in our context) in [TT17b].

Definition 3.1.1. *Let $m \gg 0$ be an integer. A* Joyce-Song pair *is a pair $(\mathcal{E}, s)$ consisting of a coherent compactly supported sheaf $\mathcal{E}$ on X, and a non-zero section*

$$s\colon \mathcal{O}_X(-m) \to \mathcal{E}\,.$$

A Joyce-Song pair $(\mathcal{E}, s)$ is called stable *if $\mathcal{E}$ is Gieseker semistable, and s does not factor through any strict subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ for which we have an equality*

$$p(\mathcal{F}) = p(\mathcal{E})$$

of reduced Hilbert polynomials.

We fix a *charge* $\gamma = (r, c_1, c_2) \in H^{\text{even}}(S)$ with $r \geq 1$, and a line bundle L on S with $c_1(L) = c_1$. We will write $\gamma^\perp = (r, L, c_2)$. Let $\mathcal{E}$ be a compactly supported coherent sheaf on X. We will say that $\mathcal{E}$ is of *type* $\gamma^\perp$ if

- $\mathrm{rk}(q_*\mathcal{E}) = r$

- $\det(q_*\mathcal{E}) \cong L$
- $c_2(q_*\mathcal{E}) = c_2$
- $\mathcal{E}$ has zero *centre of mass* (see [TT17a]).

We will write

$$\mathcal{P}^\perp = \mathcal{P}^\perp_\gamma(m) = \Big\{\text{stable Joyce-Song pairs } (\mathcal{E}, s) \text{ with } \mathcal{E} \text{ of type } \gamma^\perp\Big\}$$

for the moduli space of Joyce-Song pairs.

3.1.2 Vafa-Witten invariants

A point $(\mathcal{E}, s) \in \mathcal{P}^\perp$ can be viewed as an object

$$I^\bullet = [\mathcal{O}_X(-m) \xrightarrow{s} \mathcal{E}] \in \mathcal{D}^b(X)$$

in the derived category of X, where $\mathcal{O}_X(-m)$ is placed in degree 0. By [TT17b, JS12], the moduli space $\mathcal{P}^\perp$ carries a perfect obstruction theory governed by

$$R\operatorname{Hom}_X(I^\bullet, I^\bullet)_\perp\,,$$

which is given by

$$R\operatorname{Hom}_X(I^\bullet, I^\bullet) \cong R\operatorname{Hom}_X(I^\bullet, I^\bullet)_\perp \oplus H^*(\mathcal{O}_X) \oplus H^{\geq 1}(\mathcal{O}_S) \oplus H^{\leq 1}(K_S)[-1]\,.$$

Vafa-Witten invariants have been defined *conjecturally* in [TT17b, Tho18a]. We will give their definition for an algebraic surface S with $p_g(S) > 0$.

Consider the natural $\mathbb{C}^*$ action on $X \to S$, given by scaling the fibres. It induces a $\mathbb{C}^*$ action on $\mathcal{P}^\perp$. The unrefined invariant will be defined by the virtual localization formula [GP99].

Conjecture-Definition 3.1.2. [TT17b] *There exists a rational number* VW_γ*, called the* Vafa-Witten invariant *of* $(S, H, \gamma^\perp)$*, such that for all* $m \gg 0$*, we have*

$$\begin{aligned}\int_{[\mathcal{P}^\perp_\gamma(m)]^{\mathrm{vir}}} 1 &:= \int_{[(\mathcal{P}^\perp_\gamma(m))^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} \\ &= (-1)^{\chi(\gamma(m))-1}\chi(\gamma(m))\mathrm{VW}_\gamma\,.\end{aligned}$$

Refined invariants have been defined in [Tho18a] by the K-theoretic virtual localisation formula [Qu18, CFK09]. For notation and definitions, see [Tho18a].

Conjecture-Definition 3.1.3. [Tho18a] *There exists a rational function* $\mathrm{VW}_\gamma(t)$ *in* $\sqrt{t}$*, called the* refined Vafa-Witten invariant *of* $(S, H, \gamma^\perp)$*, such that for all* $m \gg 0$*, we have*

$$\begin{aligned}\chi_t\left(\mathcal{P}^\perp_\gamma(m), \hat{\mathcal{O}}^{\mathrm{vir}}_{\mathcal{P}^\perp_\gamma(m)}\right) &:= \chi_t\left(\left(\mathcal{P}^\perp_\gamma(m)\right)^{\mathbb{C}^*}, \frac{\mathcal{O}^{\mathrm{vir}}_{\left(\mathcal{P}^\perp_\gamma(m)\right)^{\mathbb{C}^*}}}{\Lambda^\bullet(N^{\mathrm{vir}})^\vee} \otimes K^{\frac{1}{2}}_{\mathcal{P}^\perp_\gamma(m),\mathrm{vir}}\Big|_{\left(\mathcal{P}^\perp_\gamma(m)\right)^{\mathbb{C}^*}}\right) \\ &= (-1)^{\chi(\gamma(m))-1}[\chi(\gamma(m))]_t\mathrm{VW}_\gamma(t)\,,\end{aligned}$$

where

$$[\chi(\gamma(m))]_t := \frac{t^{\chi(\gamma(m))-1} + \ldots + t + 1}{t^{\frac{\chi(\gamma(m))-1}{2}}}$$

denotes the quantum integer.

3.1.3 The fixed locus

Following [GSY17], the fixed locus $(\mathcal{P}^\perp)^{\mathbb{C}^*}$ can be written as a disjoint union of open and closed subschemes as follows. Let $(\mathcal{E}, s) \in (\mathcal{P}^\perp)^{\mathbb{C}^*}$ be a $\mathbb{C}^*$ fixed Joyce-Song pair. By Lemma 3.6.1 below, we have a canonical weight space decomposition

$$E = q_*\mathcal{E} = E^0 \oplus \ldots \oplus E^{-k}\,.$$

We will write

$$\lambda_{(\mathcal{E},s)} \coloneqq (\operatorname{rk} E^0, \ldots, \operatorname{rk} E^{-k})$$

for the vector of ranks. We now have, cf. loc. cit., a decomposition

$$(\mathcal{P}^\perp)^{\mathbb{C}^*} = \coprod_{|\lambda|=r} \mathcal{P}^\perp_\lambda\,,$$

where $\mathcal{P}^\perp_\lambda$ is the open and closed subscheme defined by

$$\lambda_{(\mathcal{E},s)} = \lambda$$

for an ordered partition $\lambda = (\lambda_0, \ldots, \lambda_k)$ of r. We will study the contribution of the locus $\mathcal{P}^\perp_{1^r} = \mathcal{P}^\perp_{(1,\ldots,1)}$ of *vertical* Joyce-Song pairs to the Vafa-Witten invariants.

Using [GSY18] and [GT19], we will construct the schemes $\mathcal{P}^\perp_{1^r}$ and their virtual classes directly in Sections 3.2-3.4.

3.1.4 Results

Similar to Conjecture-Definition 3.1.2, we can define *vertical* contributions to the Vafa-Witten invariant by

$$\int_{[\mathcal{P}^\perp_{1^r}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} = (-1)^{\chi(\gamma(m))-1}\chi(\gamma(m))\mathrm{VW}^{\mathrm{vert}}_\gamma$$

for $m \gg 0$ (note that the left hand side depends implicitly on m). The vertical contribution $\mathrm{VW}^{\mathrm{vert}}_\gamma(t)$ to the refined invariant can be defined by

$$\chi_t\left(\mathcal{P}^\perp_{1^r}, \frac{\mathcal{O}^{\mathrm{vir}}_{\mathcal{P}^\perp_{1^r}}}{\Lambda^\bullet(N^{\mathrm{vir}})^\vee} \otimes K^{\frac{1}{2}}_{\mathcal{P}^\perp_\gamma(m),\mathrm{vir}}\Big|_{\mathcal{P}^\perp_{1^r}}\right) = (-1)^{\chi(\gamma(m))-1}[\chi(\gamma(m))]_t\mathrm{VW}^{\mathrm{vert}}_\gamma(t)$$

for $m \gg 0$.

Theorem 3.A. *Let S be a surface with $p_g(S) > 0$, with a polarisation H and*

$$\gamma^\perp = (r, L, c_2)$$

given as above. The vertical contributions

$$\mathrm{VW}^{\mathrm{vert}}_\gamma \quad \textit{and} \quad \mathrm{VW}^{\mathrm{vert}}_\gamma(t)$$

to the (refined) Vafa-Witten invariant of $(S, H, \gamma^\perp)$ are well-defined.

Let (S, H) be a polarised surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. Let γ be a charge for which any semistable sheaf on X of type γ is stable. In Chapter 2 (see Theorem 2.A) we have seen that the vertical contributions to the Vafa-Witten invariant of (S, H, γ) can be expressed in terms of the coefficients of universal Laurent series $A, B, C_{ij} \in \mathbb{Q}(\!(q^{\frac{1}{2r}})\!)$ for $1 \leq i \leq j < r$ and Seiberg-Witten invariants $\mathrm{SW}(\beta^i)$ of classes $\beta^i \in H^2(S, \mathbb{Z})$. The Laurent series are in turn defined by certain tautological integrals over products of Hilbert schemes of points on S.

We will show that the tautological integrals compute the Vafa-Witten invariants for any surface S with $p_g(S) > 0$ and any $\gamma^\perp$. More precisely, consider the following generating series

$$\mathsf{Z}_{S,r,L}(q) = \frac{q^{\frac{1-r}{2r}c_1(L)^2}}{\#\operatorname{Pic}(S)[r]} \sum_{c_2 \in \mathbb{Z}} \mathrm{VW}^{\mathrm{vert}}_{(r,L,c_2)}\, q^{c_2}\,,$$

where $\#\operatorname{Pic}(S)[r]$ is the r-torsion of the Picard group of S. Then the series A, B, C_{ij} in the following theorem, or rather their refined counterparts, are precisely the ones of Theorem 2.A.

Theorem 3.B. *Fix a rank $r \geq 1$. There exist universal Laurent series*

$$A, B, C_{ij} \in \mathbb{Q}(\!(q^{\frac{1}{2r}})\!)\,, \quad 1 \leq i \leq j < r\,,$$

depending only on r, such that for any surface S with $p_g(S) > 0$, and any line bundle L on S, we have

$$\mathsf{Z}_{S,r,L}(q) = A^{\chi(\mathcal{O}_S)} B^{K_S^2} \sum_{\beta} \mathrm{SW}(\beta^1) \cdots \mathrm{SW}(\beta^{r-1}) \prod_{i \leq j} C_{ij}^{\beta^i \beta^j}$$

where the sum is taken over classes $\beta^1, \ldots, \beta^{r-1} \in H^2(S, \mathbb{Z})$ with

$$c_1(L) \equiv \sum_i i\beta^i \mod rH^2(S, \mathbb{Z})\,.$$

The same statement holds for generating series of vertical contributions to refined Vafa-Witten invariants, when one allows the Laurent series to have coefficients in $\mathbb{Q}(\sqrt{t})$.

Remark 3.1.4. In particular $\mathrm{VW}^{\mathrm{vert}}_\gamma$ and $\mathrm{VW}^{\mathrm{vert}}_\gamma(t)$ do not depend on the polarisation H, or on the lift $\gamma^\perp$ of γ.

3.2 A tautological family of Joyce-Song pairs

Recall that via the spectral construction (see e.g. [TT17a]), a sheaf $\mathcal{E}$ on X can be viewed as Higgs pair (E, ϕ) on S, where $E = q_*\mathcal{E}$ is a sheaf on S, and $\phi\colon E \to E \otimes \omega_S$ is a map that encodes the $\mathcal{O}_X$-module structure of $\mathcal{E}$. In Chapter 2, we have studied families of Higgs pairs that are flags of sheaves of rank one. Such families form (étale covers of) nested Hilbert schemes. We will equip the Higgs pairs, or their corresponding sheaves on X, with a Joyce-Song section. The resulting Joyce-Song pairs will form a projective bundle over the space of Higgs pairs.

Choose classes $\alpha_0, \ldots, \alpha_s \in H^2(S, \mathbb{Z})$ and a line bundle L on S with

$$c_1(L) = \alpha_0 + \ldots + \alpha_s\,.$$

Define classes

$$\begin{aligned}\beta_1 &= \alpha_1 - \alpha_0 + c_1(\omega_S)\\ &\vdots\\ \beta_s &= \alpha_s - \alpha_{s-1} + c_1(\omega_S)\,,\end{aligned}$$

and write S_{β_i} for the Hilbert scheme of curves on S with class β_i. Let M_α be the limit of the following (solid) diagram:

$$\begin{array}{ccccc} M_\alpha & \dashrightarrow & & & S_{\beta_1} \times \cdots \times S_{\beta_s} \\ \vdots & \searrow & & & \downarrow \\ \vdots & & \mathrm{Pic}_{\alpha_0}(S) \times \cdots \times \mathrm{Pic}_{\alpha_s}(S) & \xrightarrow{\partial} & \mathrm{Pic}_{\beta_1}(S) \times \cdots \times \mathrm{Pic}_{\beta_s}(S) \\ \downarrow & & \downarrow \mathrm{det} & & \\ [L] & \longrightarrow & \mathrm{Pic}_{\alpha_0 + \ldots + \alpha_s}(S) & & \end{array}$$

where the map det is given by the rule

$$(L_0, \ldots, L_s) \mapsto L_0 \otimes \cdots \otimes L_s\,,$$

and the map ∂ by

$$(L_0, \ldots, L_s) \mapsto (L_0^* \otimes L_1 \otimes \omega_S, \ldots, L_{s-1}^* \otimes L_s \otimes \omega_S)\,.$$

Then M_α parametrizes sums of line bundles

$$L_0 \oplus \ldots \oplus L_s\,,$$

with constant determinant L, together with non-zero maps

$$\phi_i \colon L_{i-1} \to L_i \otimes \omega_S \quad \text{for } i = 1, \ldots, s\,. \tag{3.2.1}$$

For non-negative integers $n_0, \ldots, n_s$, let $S^{[n_i]}$ be the Hilbert scheme of n_i points on S. We will write

$$S_\beta^{[n]} = S_{\beta_1, \ldots, \beta_s}^{[n_0, \ldots, n_s]} \hookrightarrow S^{[n_0]} \times \cdots \times S^{[n_s]} \times S_{\beta_1} \times \cdots \times S_{\beta_s}$$

for the nested Hilbert scheme, i.e. the subscheme defined by the rule

$$I_{i-1}(-C_i) \subset I_i\,, \quad i = 0, \ldots, s$$

for ideal sheaves $I_i \in S^{[n_i]}$ and curves $C_i \in S_{\beta_i}$. Let M_α^n be the fibre product

$$\begin{array}{ccc} M_\alpha^n & \dashrightarrow & S_\beta^{[n]} \\ \vdots & & \downarrow \\ M_\alpha & \longrightarrow & S_{\beta_1} \times \cdots \times S_{\beta_s}\,. \end{array}$$

Then M^n_α parametrizes Higgs pairs (E, ϕ) on S, given by a sheaf E with $\det(E) \cong L$ of the form

$$E = (L_0 \otimes I_0) \oplus \ldots \oplus (L_s \otimes I_s)\,,$$

with line bundles $L_i \in \mathrm{Pic}_{\alpha_i}(S)$ and ideal sheaves $I_i \in S^{[n_i]}$ for $i = 0, \ldots, s$, together with maps as in (3.2.1), that factor through the ideal sheaves:

$$\begin{array}{ccccccc} L_0 & \xrightarrow{\phi_1} & L_1 \otimes \omega_S & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_s} & L_s \otimes \omega_S^{\otimes s} \\ \uparrow & & \uparrow & & & & \uparrow \\ L_0 \otimes I_0 & \overset{\phi_1}{\dashrightarrow} & L_1 \otimes I_1 \otimes \omega_S & \overset{\phi_2}{\dashrightarrow} & \cdots & \overset{\phi_s}{\dashrightarrow} & L_s \otimes I_s \otimes \omega_S^{\otimes s}\,. \end{array}$$

Remark 3.2.2. A torsion-free sheaf of rank one on S can be uniquely written as $I \otimes L$, with I an ideal sheaf of a finite subscheme of S, and L a line bundle. Writing

$$E^{-i} = L_i \otimes I_i\,, \quad i = 0, \ldots, s\,,$$

it follows that we can view M^n_α is a moduli space of graded sums of rank one torsion-free sheaves

$$E = E^0 \oplus \ldots \oplus E^{-s}$$

on S, with a homogeneous Higgs field $\phi \colon E \to E \otimes \omega_S$ of weight -1 and rank s.

We choose *a* universal Higgs pair (E, ϕ) on $M^n_\alpha \times S$, which is only unique up to twists by elements of $\mathrm{Pic}(M^n_\alpha)$. Let $\mathcal{E}$ be the sheaf on $M^n_\alpha \times X$ corresponding to (E, ϕ).

Fix an integer $m \gg 0$ and write

$$E_m = E \otimes \mathcal{O}_S(m \cdot H) \quad \text{and} \quad \mathcal{E}_m = \mathcal{E} \otimes \mathcal{O}_X(m \cdot q^* H)\,.$$

Let

$$\pi_S \colon M^n_\alpha \times S \to M^n_\alpha\,, \quad \pi_X \colon M^n_\alpha \times X \to M^n_\alpha$$

denote the projections. Define a projective bundle

$$\begin{aligned} \mathbb{P} &:= \mathbb{P}(\pi_{X*}\mathcal{E}_m) \\ &= \mathbb{P}(\pi_{S*}E_m)\,. \end{aligned}$$

with projection map

$$p \colon \mathbb{P} \to M^n_\alpha$$

and canonical line bundle $\mathcal{O}_{\mathbb{P}}(1)$. The tautological section

$$s \colon \mathcal{O}_{\mathbb{P}\times X} \to (p \times \mathrm{id}_X)^* \mathcal{E}_m \otimes \mathcal{O}_{\mathbb{P}}(1)\,,$$

defines a universal family of Joyce-Song pairs on $\mathbb{P} \times X$, which we will denote by

$$(\mathcal{E}(1), s)\,. \tag{3.2.3}$$

3.3 The virtual class

Note that $M_\alpha \to S_{\beta_1} \times \cdots \times S_{\beta_s}$ is an surjective étale morphism of degree $(s+1)^{2\cdot q(S)}$, with

$$q(S) = h^{0,1}(S) = \dim H^1(S, \mathcal{O}_S)$$

the irregularity of S. In fact, it is a torsor under the torsion subgroup

$$(\mathbb{Z}/(s+1)\mathbb{Z})^{2\cdot q(S)} \cong \mathrm{Pic}_0(S)[s+1] \subset \mathrm{Pic}_0(S)\,,$$

which acts on M_α by the rule

$$N \cdot (E, \phi) = (E \otimes N, \phi)$$

for $N \in \mathrm{Pic}_0(S)$ with with $N^{\otimes(s+1)} \cong \mathcal{O}_S$ and $(E, \phi) \in M_\alpha$. The same holds for

$$\eta \colon M_\alpha^n \to S_\beta^{[n]}\,.$$

In particular M_α^n carries a perfect obstruction theory, which is simply the pull-back of the perfect obstruction theory on $S_\beta^{[n]}$ considered in [GSY18] and [GT19]. Note that we have

$$\begin{aligned} \eta^*[S_\beta^{[n]}]^{\mathrm{vir}} &= [M_\alpha^n]^{\mathrm{vir}} \quad \text{and} \\ \eta_*[M_\alpha^n]^{\mathrm{vir}} &= (s+1)^{2\cdot q(S)} \cdot [S_\beta^{[n]}]^{\mathrm{vir}}\,. \end{aligned}$$

Let $T_{M_\alpha^n}$ denote the virtual tangent bundle of M_α^n, i.e. the class in $K^0(M_\alpha^n)$ of the dual of its perfect obstruction theory.

Proposition 3.3.1. *The scheme $\mathbb{P}$ carries a perfect obstruction theory with virtual tangent bundle*

$$T_{\mathbb{P}} = p^* T_{M_\alpha^n} + T_{\mathbb{P}/M_\alpha^n} - T_{\mathbb{P}/M_\alpha^n}^\vee\,.$$

Proof. We will give the proof for rank two. The general case follows directly from the techniques of [GT19, Section 5]. In Section 4.3 of loc.cit., a vector bundle

$$B \to \mathrm{Pic}_\beta(S) \times S^{[n_0]} \times S^{[n_1]}$$

is constructed, together with an open subscheme $U \subset \mathbb{P}(B)$ and a vector bundle F on U, such that $S_\beta^{[n_0,n_1]} \subset U$ is the zero locus of a section

$$s \colon \mathcal{O}_U \to F(1)\,,$$

where we have twisted F by (the restriction of) the canonical line bundle on $\mathbb{P}(B)$. The perfect obstruction theory of $S_\beta^{[n_0,n_1]}$ is now given by

$$\Big[(F(1))^\vee|_{S_\beta^{[n_0,n_1]}} \to \Omega_U|_{S_\beta^{[n_0,n_1]}}\Big] \in \mathcal{D}^b(S_\beta^{[n_0,n_1]})\,.$$

I claim that there exist a cartesian square

$$\begin{array}{ccc} \mathbb{P} & \longrightarrow & \overline{\mathbb{P}} \\ \big\downarrow & & \big\downarrow \\ S_\beta^{[n_0,n_1]} & \longrightarrow & \mathbb{P}(B)\,, \end{array}$$

where the vertical maps are smooth morphims. It follows that $\mathbb{P}$ is the zero locus in $U_{\overline{\mathbb{P}}} = \overline{\mathbb{P}} \times_{\mathbb{P}(B)} U \subset \overline{\mathbb{P}}$ of the section

$$(\mathrm{pr}^* s, 0) \colon \mathcal{O}_{U_{\overline{\mathbb{P}}}} \to \mathrm{pr}^* F(1) \oplus T^{\vee}_{\overline{\mathbb{P}}/\mathbb{P}(B)}\,,$$

where $\mathrm{pr} \colon U_{\overline{\mathbb{P}}} \to U$ denotes the projection. The complex

$$\Big[((\mathrm{pr}^* F(1))^{\vee} \oplus T_{\overline{\mathbb{P}}/\mathbb{P}(B)})|_{\mathbb{P}} \to \Omega_{U_{\overline{\mathbb{P}}}}|_{\mathbb{P}} \Big] \in \mathcal{D}^b(\mathbb{P})$$

defines a perfect obstruction theory on $\mathbb{P}$, which satisfies the description of the proposition. In fact, we have

$$\begin{aligned} T_{\mathbb{P}} &= \Big(T_{U_{\overline{\mathbb{P}}}} - F(1) - T^{\vee}_{\overline{\mathbb{P}}/\mathbb{P}(B)} \Big)\Big|_{\mathbb{P}} \\ &= \Big(\mathrm{pr}^* T_U + T_{\overline{\mathbb{P}}/\mathbb{P}(B)} - F(1) - T^{\vee}_{\overline{\mathbb{P}}/\mathbb{P}(B)} \Big)\Big|_{\mathbb{P}} \\ &= p^* \eta^* (T_U - F(1))\big|_{S_\beta^{[n_0,n_1]}} + T_{\mathbb{P}/S_\beta^{[n_0,n_1]}} - T^{\vee}_{\mathbb{P}/S_\beta^{[n_0,n_1]}} \\ &= p^* T_{M^n_\alpha} + T_{\mathbb{P}/M^n_\alpha} - T^{\vee}_{\mathbb{P}/M^n_\alpha}\,. \end{aligned}$$

We will now prove the claim.

Let ι, σ and τ be the morphisms in the diagram

$$\begin{array}{ccc} S_\beta^{[n_0,n_1]} & \xrightarrow{\ \iota\ } & \mathbb{P}(B) \\ & & \downarrow \\ \sigma \downarrow & \searrow^{\tau} & \mathrm{Pic}_\beta(S) \times S^{[n_0]} \times S^{[n_1]} \\ & & \downarrow \\ S_\beta & \longrightarrow & \mathrm{Pic}_\beta(S)\,, \end{array}$$

and let $\mathcal{D}_\beta \subset S_\beta \times S$ be the universal divisor. The bundle B depends on the choice of a Poincaré bundle $\mathcal{L}_{\beta_i}$ on $\mathrm{Pic}_\beta(S) \times S$, and by [GT19, Section 4.2 - Equation (4.21)] we have an isomorphism

$$(\tau \times \mathrm{id}_S)^* \mathcal{L}_{\beta_i} \otimes (\iota \times \mathrm{id}_S)^* \mathcal{O}_{\mathbb{P}(B)}(1) \cong (\sigma \times \mathrm{id}_S)^* \mathcal{O}(\mathcal{D}_\beta) \tag{3.3.2}$$

of line bundles on $S_\beta^{[n_0,n_1]} \times S$. Let Y be scheme completing the following cartesian diagram with étale vertical morphisms

$$\begin{array}{ccccc} M^n_\alpha & \longrightarrow & Y & \longrightarrow & (\mathrm{Pic}_{\alpha_0}(S) \times \mathrm{Pic}_{\alpha_1}(S))_{[L]} \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\ S_\beta^{[n_0,n_1]} & \longrightarrow & \mathbb{P}(B) & \longrightarrow & \mathrm{Pic}_\beta(S)\,. \end{array}$$

Let $\mathcal{L}_{\alpha_0}$ be the Poincaré bundle on $\mathrm{Pic}_{\alpha_0}(S) \times S$, and for $i = 0, 1$, let $\mathcal{I}^{[n_i]}$ denote the universal ideal sheaf on $S^{[n_i]} \times S$. Define a sheaf

$$\overline{E} = \Big(\mathcal{L}_{\alpha_0} \otimes \mathcal{I}^{[n_0]} \Big) \oplus \Big(\mathcal{L}_{\alpha_0} \otimes \omega_S^* \otimes \mathcal{L}_\beta \otimes \mathcal{O}_{\mathbb{P}(B)}(1) \otimes \mathcal{I}^{[n_1]} \Big)$$

on $Y \times S$, where we have suppressed the various pull-backs. By (3.3.2), its restriction to $M_\alpha^n \times S$ equals the sheaf E defined in Section 3.2 (up to a twist by an element of $\mathrm{Pic}(M_\alpha^n)$) . Hence, possibly after increasing m, we can define

$$\overline{\mathbb{P}} := \mathbb{P}\Big(\pi_{S*}\Big(\overline{E} \otimes \mathcal{O}_S(m \cdot H)\Big)\Big) \to Y\,,$$

proving the claim. □

Proposition 3.3.1, or more precisely the claim in its proof, has the following consequences.

Corollary 3.3.3. *The scheme* $\mathbb{P}$ *carries a virtual virtual given by*

$$[\mathbb{P}]^{\mathrm{vir}} = p^*[M_\alpha^n]^{\mathrm{vir}} \cap e(T^\vee_{\mathbb{P}/M_\alpha^n})\,.$$

Corollary 3.3.4. *The scheme* $\mathbb{P}$ *carries a virtual structure sheaf given by*

$$\mathcal{O}_{\mathbb{P}}^{\mathrm{vir}} = p^*\mathcal{O}_{M_\alpha^n}^{\mathrm{vir}} \otimes \Lambda^\bullet\left(T_{\mathbb{P}/M_\alpha^n}\right)\,.$$

3.4 Localising the virtual class

The family of sheaves on $\mathbb{P} \times X$

$$\mathcal{E}(1) = (p \times \mathrm{id}_X)^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1)$$

constructed in Section 3.2 is invariant under the natural $\mathbb{C}^*$ action on the fibres of $X \to S$, and can be equipped with an equivariant structure. In terms of the Higgs pair (E, ϕ) on $M_\alpha^n \times S$, it is given by a $\mathbb{C}^*$ action on E, which is chosen in such a way that the Higgs field ϕ acts with weight -1 on E. We may assume that the highest weight occurring in the weight space decomposition of E is zero. Following Remark 3.2.2, we write

$$E = E^0 \oplus \ldots \oplus E^{-s}\,,$$

and let $\mathbb{C}^*$ act on E^{-i} with weight $-i$.

The action induces an action on the family of Joyce-Song pairs $\mathbb{P}$. In fact, we have

$$\mathbb{P} = \mathbb{P}\left(\pi_{S*}E_m^0 \oplus \ldots \oplus \pi_{S*}E_m^{-s}\right)$$

with

$$E_m^{-i} = E^{-i} \otimes \mathcal{O}_S(mH)\,, \quad i = 0, \ldots, s\,,$$

so the $\mathbb{C}^*$ action on $\mathbb{P}$ is given by

$$\lambda \cdot (x_0 : \ldots : x_s) \to (x_0\lambda^0 : \ldots : x_s\lambda^{-s})$$

for $\lambda \in \mathbb{C}^*$ and $(x_0 : \ldots : x_s) \in \mathbb{P}$. For $i = 0, \ldots, s$, we will write

$$\mathbb{P}_i := \mathbb{P}\left(\pi_{S*}E_m^{-i}\right) \xrightarrow{p_i} M_\alpha^n$$

for the projective bundle, so the $\mathbb{C}^*$ fixed locus of $\mathbb{P}$ is given by

$$\mathbb{P}_0 \sqcup \ldots \sqcup \mathbb{P}_s = \mathbb{P}^{\mathbb{C}^*} \subset \mathbb{P}\,.$$

Let $\mathfrak{t}$ be an equivariant parameter for the $\mathbb{C}^*$ action on a point. We can equip the perfect obstruction theory of Proposition 3.3.1 with an equivariant structure, so that its virtual tangent bundle is given by

$$T_{\mathbb{P}} = p^* T_{M^n_\alpha} + T_{\mathbb{P}/M^n_\alpha} - T^\vee_{\mathbb{P}/M^n_\alpha} \otimes \mathfrak{t} \tag{3.4.1}$$

in $K^0_{\mathbb{C}^*}(\mathbb{P})$. On each $\mathbb{P}_i$ with $i > 0$, we have an induced $\mathbb{C}^*$ localized perfect obstruction theory with virtual tangent bundle

$$\begin{aligned}(T_{\mathbb{P}}|_{\mathbb{P}_i})^{\mathrm{fix}} &= p_i^* T_{M^n_\alpha} + \Big(\big(T_{\mathbb{P}/M^n_\alpha} - T^\vee_{\mathbb{P}/M^n_\alpha} \otimes \mathfrak{t}\big)\big|_{\mathbb{P}_i}\Big)^{\mathrm{fix}} \\ &= p_i^* T_{M^n_\alpha} + T_{\mathbb{P}_i/M^n_\alpha} - \big(p_i^* \pi_{S*} E_m^{-(i-1)}(1)\big)^\vee\end{aligned} \tag{3.4.2}$$

and virtual class

$$[\mathbb{P}_i]^{\mathrm{vir}} = p_i^* [M^n_\alpha]^{\mathrm{vir}} \cap e\left(\big(p_i^* \pi_{S*} E_m^{-(i-1)}(1)\big)^\vee \right). \tag{3.4.3}$$

The $\mathbb{C}^*$ localized virtual tangent bundle on $\mathbb{P}_0$ is given by

$$(T_{\mathbb{P}}|_{\mathbb{P}_0})^{\mathrm{fix}} = p_0^* T_{M^n_\alpha} + T_{\mathbb{P}_0/M^n_\alpha}\,,$$

and we have

$$[\mathbb{P}_0]^{\mathrm{vir}} = p_0^* [M^n_\alpha]^{\mathrm{vir}}\,. \tag{3.4.4}$$

Similarly we have

$$\mathcal{O}^{\mathrm{vir}}_{\mathbb{P}_i} = \begin{cases} p_0^* \mathcal{O}^{\mathrm{vir}}_{M^n_\alpha} & \text{if } i = 0 \\ p_i^* \mathcal{O}^{\mathrm{vir}}_{M^n_\alpha} \otimes \Lambda^\bullet \left(p_i^* \pi_{S*} E_m^{i-1}(1)\right) & \text{if } i > 0\,. \end{cases}$$

3.5 Stability

Let (E, ϕ) be the family of Higgs pairs on

$$\pi_S \colon M^n_\alpha \times S \to M^n_\alpha$$

constructed in Section 3.2. Recall that ϕ is given by maps

$$\phi_i \colon E^{-(i-1)} \to E^{-i} \otimes \omega_S\,, \quad i = 1, \ldots, s\,,$$

where the torsion free sheaves of rank one E^{-i} are the summands of

$$E = E^0 \oplus \ldots \oplus E^{-s}\,.$$

We have defined the perfect obstruction theory of M^n_α as the pull-back along the étale map $\eta\colon M^n_\alpha \to S^{[n]}_\beta$ of the perfect obstruction theory on the nested Hilbert scheme S_β^n. It follows directly by the description of [GSY18] that it is given by given by the dual of a cone on

$$\left(\bigoplus_{i=0}^{s} R\mathscr{H}om_{\pi_S}(E^{-i},E^{-i})\right)_0 \xrightarrow{\circ\phi-\phi\circ} \bigoplus_{i=1}^{s} R\mathscr{H}om_{\pi_S}(E^{-(i-1)},E^{-i}\otimes\omega_S)\,, \qquad (3.5.1)$$

where the left hands side is given by the kernel of the trace map

$$\bigoplus_{i=0}^{s} R\mathscr{H}om_{\pi_S}(E^{-i},E^{-i}) \hookrightarrow R\mathscr{H}om_{\pi_S}(E,E) \xrightarrow{\mathrm{tr}} R\pi_{S*}\mathcal{O}_S.$$

Proposition 3.5.2. *For each $i=1,\ldots,s$, we have an exact sequence*

$$ob_{M_\alpha^n} \xrightarrow{\sigma_i} R^2\pi_{S*}\mathcal{O}_S \xrightarrow{\phi_i} \mathscr{E}xt^2_{\pi_S}(E^{-(i-1)},E^{-i}\otimes\omega_S) \longrightarrow 0\,. \qquad (3.5.3)$$

Proof. By (3.5.1) have an exact sequence

$$\begin{array}{ccccccc} ob_{M_\alpha^n} & \longrightarrow & \left(\bigoplus_{i=0}^{s}\mathscr{E}xt^2_{\pi_S}(E^{-i},E^{-i})\right)_0 & \longrightarrow & \bigoplus_{i=1}^{s}\mathscr{E}xt^2_{\pi_S}(E^{-(i-1)},E^{-i}\otimes\omega_S) & \longrightarrow & 0\,. \\ & & \downarrow\cong & \overset{\phi}{\nearrow} & & & \\ & & \bigoplus_{i=1}^{s} R^2\pi_{S*}\mathcal{O}_S & & & & \end{array}$$

Let the vertical isomorphism is given by

$$(a_0,\ldots,a_s)\mapsto(\mathrm{tr}(a_1)-\mathrm{tr}(a_0),\ldots,\mathrm{tr}(a_s)-\mathrm{tr}(a_{s-1}))\,,$$

so the dotted arrow ϕ is the map given by the rule

$$(b_1,\ldots,b_s)\mapsto(b_1\cdot\phi_1,\ldots,b_s\cdot\phi_s)\,.$$

It follows that for each summand ϕ_i of ϕ, we have an exact sequence (3.5.3). □

Remark 3.5.4. In particular, we see that if $p_g(S)>0$, and if $\alpha_{i-1}-\alpha_i$ is not the class of an effective divisor (so $\mathscr{E}xt^2_{\pi_S}(E^{-(i-1)},E^{-i}\otimes\omega_S)=0$ by Serre duality), the obstruction sheaf $ob_{M_\alpha^n}$ has a trivial factor $H^2(\mathcal{O}_S)$. In rank $s+1=2$, this fact is used in [GSY18, Section 2.2] to define a reduced virtual class on $S_\beta^{[n_0,n_1]}$.

Recall that we write $\mathcal{E}$ for the sheaf on $M_\alpha^n\times S$, corresponding to (E,ϕ) via the spectral construction.

Proposition 3.5.5. *Assume $p_g(S)>0$. Then (at least) one of the following statements holds:*

i) The family $\mathcal{E}$ of sheaves on $M_\alpha^n\times X\to M_\alpha^n$ is fibrewise Gieseker stable;

ii) We have $\alpha_0=\ldots=\alpha_s$ and $n_0=\ldots=n_s$ (so in particular $\mathcal{E}$ is fibrewise strictly Gieseker semistable);

iii) The virtual class $[M_\alpha^n]^{\mathrm{vir}}$ vanishes.

Remark 3.5.6. In [GSY18, Section 2.2] (see the discussion above) and in [GT19, Section 4.1], it is shown that $[S_\beta^{[n]}]^{\mathrm{vir}}$ vanishes, unless the classes

$$\alpha_{i-1} - \alpha_i\,, \quad i = 1, \dots, s$$

are classes of effective divisors. By Lemma 2.3.2, $\mathcal{E}$ is slope semistable in the latter case.

Remark 3.5.7. In Proposition 2.3.5 we have proven a similar statement for the class

$$\iota_*[S_\beta^{[n]}]^{\mathrm{vir}} \in A^*(S^{[n_0]} \times \dots \times S^{[n_s]} \times S_{\alpha_1} \times \dots \times S_{\alpha_s})$$

under the condition $H^1(S, \mathcal{O}_S) = 0$, using the formula of [GT19, Theorem 5.6].

Proof. Assume that we are not in case i) or ii). By Lemmas 2.3.2 and 2.3.4 there is an $i \in \{1, \dots, s\}$ such that

- $\alpha_{i-1} - \alpha_i$ is not the class of an effective divisor, or
- $\alpha_{i-1} = \alpha_i$ and $n_{i-1} > n_i$.

In either case, we have

$$\mathrm{Ext}^2_{\mathcal{O}_S}(E_x^{-(i-1)}, E_x^{-i} \otimes \omega_S) = \mathrm{Hom}_{\mathcal{O}_S}(E_x^{-i}, E_x^{-(i-1)})^* = 0\,, \quad \forall x \in M_\alpha^n(\mathbb{C})\,.$$

By the exact sequence (3.5.3), the map

$$ob_{M_\alpha^n} \xrightarrow{\sigma_i} R^2\pi_{S*}\mathcal{O}_S \neq 0$$

is surjective. It follows that $[M_\alpha^n]^{\mathrm{vir}} = 0$ by e.g. [KL13]. □

For the universal family of Joyce-Song pairs $(\mathcal{E}(1), s)$ on

$$\pi_X \colon \mathbb{P} \times X \to \mathbb{P}$$

defined in (3.2.3), let $(\mathcal{E}(1), s)|_{\mathbb{P}_i}$ denote its restriction to $\mathbb{P}_i \subset \mathbb{P}$.

Theorem 3.5.8. *Assume $p_g(S) > 0$. For each $i = 0, \dots, s$, (at least) one of the following statements holds:*

i) The family of Joyce-Song pairs $(\mathcal{E}(1), s)|_{\mathbb{P}_i}$ on $\mathbb{P}_i \times X \to \mathbb{P}_i$ is fibrewise stable;

ii) The virtual class $[\mathbb{P}_i]^{\mathrm{vir}}$ vanishes.

Proof. By the equations (3.4.3), and (3.4.4), we may assume that we are in case ii) of Proposition 3.5.5. Hence for $x \in \mathbb{P}$, we have

$$E_x = E_x^0 \oplus \dots \oplus E_x^{-s}$$

with

$$\mathrm{ch}(E_x^0) = \dots = \mathrm{ch}(E_x^{-s})\,.$$

It is easy to see that the Joyce-Song pair $(\mathcal{E}_x, s_x)$ on X is stable, precisely when the first term of

$$s_x = \sum_i s_x^i \in H^0(X, \mathcal{E}_x) = H^0(S, E_x^0) \oplus \ldots \oplus H^0(S, E_x^{-s})$$

is non-zero. It follows that the fibres of $(\mathcal{E}(1), s)|_{\mathbb{P}_i}$ are stable if and only if $i = 0$.

Now let $i > 0$. I claim that we have

$$[M_\alpha^n]^{\mathrm{vir}} = \iota_*[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}}$$

for a cycle class $[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}}$ on the closed subscheme

$$\iota\colon M_\alpha^n(\sigma_i) \coloneqq \left\{ b \in M_\alpha^n \middle| E^{-(i-1)}|_b \cong E^{-i}|_b \right\} \hookrightarrow M_\alpha^n \,. \tag{3.5.9}$$

Consider the Cartesian square

$$\begin{array}{ccc} \mathbb{P}_i(\sigma_i) & \xrightarrow{\iota'} & \mathbb{P}_i \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_i} \\ M_\alpha^n(\sigma_i) & \xrightarrow{\iota} & M_\alpha^n \,. \end{array}$$

Then we have by the claim

$$\begin{aligned} p_i^*[M_\alpha^n] &= p_i^*\iota_*[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}} \\ &= \iota'_* p_i^*[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}} \,. \end{aligned}$$

On the other hand, recall that we write $E_m^{-(i)} = E^{-(i)} \otimes \mathcal{O}_S(mH)$ for fixed $m \gg 0$, so we have by cohomology-and-basechange we have

$$\begin{aligned} \iota'^* p_i^* \pi_{S*} E_m^{-(i-1)}(1) &= p_i^* \pi_{S*}(\iota \times id_S)^* E_m^{-(i-1)}(1) \\ &= p_i^* \pi_{S*}(\iota \times id_S)^* E_m^{-i}(1) \\ &= \iota'^* p_i^* \pi_{S*} E_m^{-i}(1) \\ &= \iota'^* T_{\mathbb{P}_i/M_\alpha^n} + \mathcal{O} \,, \end{aligned}$$

and hence by (3.4.3) and the projection formula for ι' we find

$$\begin{aligned} [\mathbb{P}_i]^{\mathrm{vir}} &= p_i^*[M_\alpha^n]^{\mathrm{vir}} \cap e\left(\left(p_i^* \pi_{S*} E_m^{-(i-1)}(1)\right)^\vee\right) \\ &= \iota'_*\left(p_i^*[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}} \cap \iota'^* e\left(\left(p_i^* \pi_{S*} E_m^{-(i-1)}(1)\right)^\vee\right)\right) \\ &= \iota'_*\left(p_i^*[M_\alpha^n]^{\mathrm{vir}}_{\mathrm{loc}} \cap e\left(\iota'^* T^\vee_{\mathbb{P}_i/M_\alpha^n} + \mathcal{O}\right)\right) \\ &= 0 \,. \end{aligned}$$

We will prove the claim using the cosection (3.5.3). For $b \in M_\alpha^n$ we have $\mathrm{ch}(E^{-i}|_b) = \mathrm{ch}(E^{-(i-1)}|_b)$, and hence

$$\mathrm{Hom}_{\mathcal{O}_S}(E^{-i}|_b, E^{-(i-1)}|_b) = 0 \quad \text{or} \quad E^{-i}|_b \cong E^{-(i-1)}|_b \,,$$

since $E^{-(i-1)}|_b$ and $E^{-i}|_b$ are torsion free and of rank one. By the exact sequence (3.5.3) and Serre duality, it follows that the cosection

$$ob_{M_\alpha^n} \xrightarrow{\sigma_i} R\pi_{S*}\mathcal{O}_S$$

is surjective on the complement of the locus (3.5.9). The claim follows by the work of Kiem and Li [KL13]. □

3.6 Comparison to the VW moduli space

We need the following lemma (compare to [TT17b, Proposition 5.1]).

Lemma 3.6.1. *Let $(\mathcal{E}, s) \in (\mathcal{P}^\perp)^{\mathbb{C}^*}$ be a $\mathbb{C}^*$ fixed Joyce-Song pair. Then $\mathcal{E}$ has a canonical $\mathbb{C}^*$ equivariant structure.*

Proof. We will write

$$\mathbb{C}^* \times X \underset{\mathrm{pr}_2}{\overset{\rho}{\rightrightarrows}} X$$

for the action morphism and the projection respectively. Since $(\mathcal{E}, s)$ is $\mathbb{C}^*$ fixed, there exists an isomorphism $\varphi\colon \rho^*\mathcal{E} \to \mathrm{pr}_2^*\mathcal{E}$ that fits in the diagram

$$\begin{array}{ccc} \mathcal{O}(-m) & \xrightarrow{\rho^* s} & \rho^*\mathcal{E} \\ & \searrow^{\mathrm{pr}_2^* s} & \downarrow \varphi \\ & & pr_2^*\mathcal{E}\,. \end{array}$$

By the stability of $(\mathcal{E}, s)$, we see that $\varphi|_{\{1\}\times X}\colon \mathcal{E} \to \mathcal{E}$ is the identity. The argument of e.g. [Koo11, Proposition 4.4] shows that φ defines an equivariant structure on $\mathcal{E}$, which is unique up to a twist by a character $\mathfrak{t}^z \in \mathrm{Aut}(\mathbb{C}^*) \cong \mathbb{Z}$. Now φ induces a $\mathbb{C}^*$-action on $E = q_*\mathcal{E}$. After multiplying φ by a power of $\mathfrak{t}$, we may assume that the highest weight occurring in the weight space decomposition of E is zero. We call φ the canonical equivariant structure on $\mathcal{E}$. □

Fix a charge $\gamma = (r, c_1, c_2) \in H^{\mathrm{even}}(S)$, and classes $\alpha_0, \ldots, \alpha_s \in H^2(S, \mathbb{Z})$ and $n_0, \ldots, n_s \in H^4(S, \mathbb{Z}) = \mathbb{Z}$ with

$$r = s + 1\,, \quad c_1 = \alpha_0 + \ldots + \alpha_s\,, \quad c_2 = n_0 + \ldots + n_s + \sum_{i<j} \alpha_i\alpha_j\,. \tag{3.6.2}$$

Let L be a vector bundle on S with $c_1(L) = c_1$, as in the introduction, write

$$\gamma^\perp = (r, L, c_2)\,.$$

Recall that we have defined

$$\mathcal{P}^\perp_{1^r} \subset (\mathcal{P}^\perp)^{\mathbb{C}^*}$$

as the (open and closed) locus of Joyce-Song pairs $(\mathcal{E}, s)$ with

$$\mathrm{rk}(E^{-i}) = 1\,, \quad i = 0, \ldots, s$$

in the canonical weight space decomposition

$$E = q_*\mathcal{E} = E^0 \oplus \ldots \oplus E^{-s}\,.$$

induced by Lemma 3.6.1. Now define an open and closed subscheme

$$\mathcal{P}^n_\alpha \subset \mathcal{P}^\perp_{1^r}$$

by the rule

$$c(E^{-i}) = 1 + \alpha_i + n_i\,, \quad i = 0, \dots, s\,.$$

In the terminology of the introduction, the family $\mathcal{E}$ on $M_\alpha^n \times X \to M_\alpha^n$ constructed in Section 3.2 is a family of sheaves of type $\gamma^\perp$ (the centre of mass property is given by the fact that the corresponding Higgs field $\phi\colon E \to E \otimes \omega_S$ is trace-free). It follows that the family of Joyce-Song pairs $(\mathcal{E}(1), s)$ on $\mathbb{P} \times X \to \mathbb{P}$ defines a morphism

$$\mathbb{P} \supset \mathbb{P}^{\mathrm{stab}} \to \mathcal{P}^\perp = \mathcal{P}_\gamma^\perp(m) \tag{3.6.3}$$

on its stable locus.

Proposition 3.6.4. *The map* (3.6.3) *restricts to an isomorphism*

$$(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*} \cong \mathcal{P}_\alpha^n\,.$$

Proof. C.f. [GSY17, Proposition 3.9], this follows directly by comparing functors of points. □

Lemma 3.6.5. *For each $i = 0, \dots, s$ we have $\mathbb{P}_i \subset \mathbb{P}^{\mathrm{stab}}$ or $\mathbb{P}_i \cap \mathbb{P}^{\mathrm{stab}} = \emptyset$.*

Proof. Following the proof of Lemma 2.3.2 the stability of $(\mathcal{E}_x, s_x)$ for $x \in \mathbb{P}_i$ is given by a numerical condition on $\alpha_0, \dots, \alpha_s$ and $n_0, \dots, n_s$, which is constant on $\mathbb{P}_i$. □

Corollary 3.6.6. *$\mathcal{P}_{1^r}^\perp$ is a disjoint union of schemes of the form $\mathbb{P}_i$.*

We will show that the isomorphism of Proposition 3.6.4 respects the virtual structure. Let

$$(\mathscr{E}, s)$$

denote the universal Joyce-Song pair on

$$\pi_X\colon \mathcal{P}^\perp \times X \to \mathcal{P}^\perp\,,$$

and consider the class

$$\mathscr{I}^\bullet = [\mathcal{O} \xrightarrow{s} \mathscr{E}_m] \in \mathcal{D}^b(\mathcal{P}^\perp \times X)\,.$$

There exists a splitting [TT17b]

$$\begin{aligned} R\mathscr{H}om_{\pi_X}(\mathscr{I}^\bullet, \mathscr{I}^\bullet) = {} & R\mathscr{H}om_{\pi_X}(\mathscr{I}^\bullet, \mathscr{I}^\bullet)_\perp \\ & \oplus R\pi_{X*}\mathcal{O}_X \oplus R^{\geq 1}\pi_{S*}\mathcal{O}_S \oplus R^{\leq 1}\pi_{S*}\omega_S \otimes \mathfrak{t}[-1] \end{aligned} \tag{3.6.7}$$

of objects in $\mathcal{D}^b(\mathcal{P}^\perp)$, and by definition the perfect obstruction theory on $\mathcal{P}^\perp$ is given by the dual of

$$R\mathscr{H}om_{\pi_X}(\mathscr{I}^\bullet, \mathscr{I}^\bullet)_\perp[1]\,.$$

In particular, the virtual tangent bundle of $\mathcal{P}_\alpha^n$ is given by

$$\left(- R\mathscr{H}om_{\pi_X}(\mathscr{I}^\bullet, \mathscr{I}^\bullet)_\perp \Big|_{\mathcal{P}_\alpha^n} \right)^{\mathrm{fix}} \in K^0(\mathcal{P}_\alpha^n)\,.$$

Similarly, write

$$I^\bullet = [\mathcal{O}_{X\times\mathbb{P}} \xrightarrow{s} \mathcal{E}_m(1)] \in \mathcal{D}^b(\mathbb{P}\times X)\,,$$

and let the object

$$R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)_\perp \in \mathcal{D}^b(\mathbb{P})$$

be given by the rule (3.6.7) above. Finally, let $R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp \in \mathcal{D}^b(M^n_\alpha)$ be the object considered in [TT17a] satisfying

$$R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E}) = R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp \oplus R\pi_{S*}\mathcal{O}_S \oplus R\pi_{S*}\omega_S \otimes \mathfrak{t}[-1]\,.$$

By definition, pull-back along the morphism (3.6.3) identifies the restrictions to $\mathbb{P}^{\mathrm{stab}}$ of the Joyce-Song pairs $(\mathscr{E}, s)$ and $(\mathcal{E}(1), s)$, and hence we have

$$\mathscr{I}^\bullet|_{\mathbb{P}^{\mathrm{stab}}} \cong I^\bullet|_{\mathbb{P}^{\mathrm{stab}}}$$

and

$$R\mathscr{H}om_{\pi_X}(\mathscr{I}^\bullet,\mathscr{I}^\bullet)_\perp|_{\mathbb{P}^{\mathrm{stab}}} \cong R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)_\perp|_{\mathbb{P}^{\mathrm{stab}}}\,.$$

Following [TT17b, Proposition 6.8], we find

$$\begin{aligned} R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)_\perp &= R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)_0 - R^{\geq 1}\pi_{S*}\mathcal{O}_S + R^{\leq 1}\pi_{S*}\omega_S \otimes \mathfrak{t} \\ &= p^*\, R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E}) - R\pi_{S*}\mathcal{O}_S + R\pi_{S*}\omega_S \otimes \mathfrak{t} \\ &\quad - \pi_{X*}\mathcal{E}_m(1) + (\pi_{X*}\mathcal{E}_m(1))^\vee \otimes \mathfrak{t} + \mathcal{O}_\mathbb{P} - \mathcal{O}_\mathbb{P}\otimes \mathfrak{t} \\ &= p^*\, R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp - T_{\mathbb{P}/M^n_\alpha} + T^\vee_{\mathbb{P}/M^n_\alpha}\otimes \mathfrak{t} \end{aligned} \tag{3.6.8}$$

in $K^0_{\mathbb{C}^*}(\mathbb{P})$. The virtual tangent bundle of M^n_α is precisely the $\mathbb{C}^*$ fixed part of $-R\mathscr{H}om_{\pi_S}(\mathcal{E},\mathcal{E})_\perp$ (use [TT17a, Corollary 2.26] to compare $-R\mathscr{H}om_{\pi_S}(\mathcal{E},\mathcal{E})_\perp$ with (3.5.1)). If follows by (3.4.2) that the isomorphism of Proposition 3.6.4 identifies the virtual tangent bundles of $\mathcal{P}^n_\alpha$ and $(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}$. By Siebert's formula [Sie04] and its K-theoretic analogue [Tho18b, Theorem 4.2], we find that virtual classes and virtual structure sheaves agree. Moreover, by Theorem 3.5.8, we have the following proposition

Proposition 3.6.9. *We have equalities*

$$\begin{aligned} [\mathcal{P}^n_\alpha]^{\mathrm{vir}} &= [(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}]^{\mathrm{vir}} \\ &= [\mathbb{P}^{\mathbb{C}^*}]^{\mathrm{vir}} \end{aligned}$$

where we have identified

$$\begin{aligned} A_*(\mathcal{P}^n_\alpha) &= A_*((\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}) \\ &\subset A_*(\mathbb{P}^{\mathbb{C}^*})\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathcal{O}^{\mathrm{vir}}_{\mathcal{P}^n_\alpha} &= \mathcal{O}^{\mathrm{vir}}_{(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}} \\ &= \mathcal{O}^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}} \end{aligned}$$

in

$$\begin{aligned} K_0(\mathcal{P}^n_\alpha) &= K_0((\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}) \\ &\subset K_0(\mathbb{P}^{\mathbb{C}^*})\,. \end{aligned}$$

3.7 The unrefined case

Theorem 3.5.8 (or its consequence Proposition 3.6.9) allows us compute the contribution of $\mathcal{P}_\alpha^n$ to the Vafa-Witten invariant as a integral over the virtual class of $\mathbb{P}$, which might contain unstable Joyce-Song pairs. We will use this to prove the unrefined part of Theorem 3.A.

Let

$$\begin{aligned} N^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}/\mathbb{P}} &= (T_{\mathbb{P}}|_{\mathbb{P}^{\mathbb{C}^*}})^{\mathrm{mov}} \\ &= \left(\left(T_{\mathbb{P}/M_\alpha^n} - T^\vee_{\mathbb{P}/M_\alpha^n} \otimes \mathfrak{t}\right)|_{\mathbb{P}^{\mathbb{C}^*}}\right)^{\mathrm{mov}} \end{aligned}$$

be the virtual normal bundle to $\mathbb{P}^{\mathbb{C}^*}$ in $\mathbb{P}$, and let $N^{\mathrm{vir}}_{\mathcal{P}_\alpha^n/\mathcal{P}^\perp}$ be the virtual normal bundle to $\mathcal{P}_\alpha^n$ in $\mathcal{P}^\perp$. We will also write

$$N_\alpha^n = (-R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\mathrm{mov}}$$

for the class in $K^0_{\mathbb{C}^*}(M_\alpha^n)$. Note that by (3.6.8) we have

$$\begin{aligned} N^{\mathrm{vir}}_{\mathcal{P}_\alpha^n/\mathcal{P}^\perp} &= \left(-R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)_\perp|_{(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}}\right)^{\mathrm{mov}} \\ &= \left(p^* N_\alpha^n + N^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}/\mathbb{P}}\right)\Big|_{(\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}} \end{aligned}$$

using the identification

$$\mathcal{P}_\alpha^n \cong (\mathbb{P}^{\mathrm{stab}})^{\mathbb{C}^*}$$

of Proposition 3.6.4.

Remark 3.7.1. If the family of sheaves $\mathcal{E}$ on $M_\alpha^n \times X \to M_\alpha^n$ is Gieseker *stable*, M_α^n is an open and closed subscheme of the fixed locus $(\mathcal{N}^\perp)^{\mathbb{C}^*}$ of the moduli space $\mathcal{N}^\perp$ (defined in [TT17a]) of stable compactly supported sheaves on X of type $\gamma^\perp$. In this case, the class N_α^n is the virtual normal bundle to M_α^n in $\mathcal{N}^\perp$.

By virtual $\mathbb{C}^*$ localisation [GP99] and Proposition 3.6.9 we have

$$\begin{aligned} \int_{[\mathbb{P}]^{\mathrm{vir}}} \frac{1}{p^* e(N_\alpha^n)} &= \int_{[\mathbb{P}^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{p^* e(N_\alpha^n) e(N^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}/\mathbb{P}})} \\ &= \int_{[\mathcal{P}_\alpha^n]^{\mathrm{vir}}} \frac{1}{e\left(N^{\mathrm{vir}}_{\mathcal{P}_\alpha^n/\mathcal{P}^\perp}\right)}. \end{aligned}$$

Note that the last line is exactly the contribution of the open and closed locus

$$\mathcal{P}_\alpha^n \subset (\mathcal{P}^\perp)^{\mathbb{C}^*} = (\mathcal{P}_\gamma^\perp(m))^{\mathbb{C}^*}$$

to the integral of Conjecture-Definition 3.1.2. On the other hand, by Corollary 3.3.3 and the projection formula, we have

$$\begin{aligned} \int_{[\mathbb{P}]^{\mathrm{vir}}} \frac{1}{p^* e(N_\alpha^n)} &= \int_{p^*[M_\alpha^n]^{\mathrm{vir}}} \frac{e\left(T^\vee_{\mathbb{P}/M_\alpha^n}\right)}{p^* e(N_\alpha^n)} \\ &= \int_{[M_\alpha^n]^{\mathrm{vir}}} \frac{p_* e\left(T^\vee_{\mathbb{P}/M_\alpha^n}\right)}{e(N_\alpha^n)} \\ &= (-1)^{\chi(\gamma(m))-1} \chi(\gamma(m)) \int_{[M_\alpha^n]^{\mathrm{vir}}} \frac{1}{e(N_\alpha^n)}. \end{aligned}$$

It follows that the vertical contribution $\mathrm{VW}_\gamma^{\mathrm{vert}}$ to the (unrefined) Vafa-Witten invariant (defined in Section 3.1.4), is given by

$$\mathrm{VW}_\gamma^{\mathrm{vert}} = \sum_{\alpha,n} \int_{[M_\alpha^n]^{\mathrm{vir}}} \frac{1}{e(N_\alpha^n)}\,, \tag{3.7.2}$$

where the sum is taken over classes

$$\alpha_0, \dots, \alpha_s \in H^2(S,\mathbb{Z})\,, \quad n_0, \dots, n_s \in H^4(S,\mathbb{Z})$$

satisfying

$$c_1 = \alpha_0 + \ldots + \alpha_s\,, \quad c_2 = n_0 + \ldots + n_s + \sum_{i<j} \alpha_i \alpha_j\,.$$

This proves the unrefined part of Theorem 3.A.

Remark 3.7.3. The computation in this section is used in [TT17b, Proposition 6.8] to show that the different definitions of the Vafa-Witten invariant given in [TT17a] and [TT17b] agree in the case that E is stable.

3.8 The integrals

Recall from Sections 3.2 and 3.3 that we write

$$\begin{aligned} \beta_1 &= \alpha_1 - \alpha_0 + c_1(\omega_S) \\ &\vdots \\ \beta_s &= \alpha_s - \alpha_{s-1} + c_1(\omega_S)\,, \end{aligned}$$

for the classes in $H^2(S,\mathbb{Z})$, and

$$\eta\colon M_\alpha^n \to S_\beta^n$$

for the surjective étale morphism to the nested Hilbert scheme. Note that since

$$R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp = R\mathscr{H}om_{\pi_X}(\mathcal{E}\otimes M,\mathcal{E}\otimes M)_\perp$$

for any line bundle M on S, the class N_α^n is invariant under the action of $\mathrm{Pic}_0(S)[s+1]$ on M_α^n (see Section 3.3). It follows that it descents along η to a class $N_\beta^{[n]}$ in $K^0_{\mathbb{C}^*}(S_\beta^{[n]})$, so we have

$$\begin{aligned} \int_{[M_\alpha^n]^{\mathrm{vir}}} \frac{1}{e(N_\alpha^n)} &= \int_{\eta_*[M_\alpha^n]^{\mathrm{vir}}} \frac{1}{e(N_\beta^{[n]})} \\ &= \#\,\mathrm{Pic}_0(S)[s+1] \int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{1}{e(N_\beta^{[n]})}\,. \end{aligned} \tag{3.8.1}$$

Recall from [TT17a], that we have

$$R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp = R\mathscr{H}om_{\pi_S}(E,E)_0 - R\mathscr{H}om_{\pi_S}(E,E\otimes\omega_S\otimes\mathfrak{t})_0 \tag{3.8.2}$$

in $K^0_{\mathbb{C}^*}(M^n_\alpha)$. Choose line bundles

$$L_i \in \operatorname{Pic}_{\alpha_i}(S)\,, \quad i = 0, \ldots, s$$

and write

$$E_L^{[n]} = \bigoplus_{i=0}^{s} \mathcal{I}^{[n_i]} \otimes L_i \otimes \mathfrak{t}^{-i}$$

for the family of sheaves on

$$\pi_S \colon S^{[n_0]} \times \cdots \times S^{[n_s]} \times S \to S^{[n_0]} \times \cdots \times S^{[n_s]}\,.$$

Finally, define a class

$$N_L^{[n]} := \Big(R\mathscr{H}om_{\pi_S}\Big(E_L^{[n]}, E_L^{[n]} \otimes \omega_S \otimes \mathfrak{t}\Big)_0 - R\mathscr{H}om_{\pi_S}\Big(E_L^{[n]}, E_L^{[n]}\Big)_0 \Big)^{\mathrm{mov}}$$

in $K^0_{\mathbb{C}^*}(S^{[n_0]} \times \cdots \times S^{[n_s]})$.

Write

$$\rho \colon S_\beta^{[n]} \to S^{[n_0]} \times \cdots \times S^{[n_s]}$$

for the natural morphism. By [GT19, Theorem 5.6 and (the discussion before) Theorem 6], we have

$$\begin{aligned} \mathrm{VW}_\beta^{[n]} &:= \int_{[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{1}{e(N_\beta^{[n]})} \\ &= \int_{\rho_*[S_\beta^{[n]}]^{\mathrm{vir}}} \frac{1}{e(N_L^{[n]})} \\ &= \mathrm{SW}(\beta) \int_{S^{[n_0]} \times \cdots \times S^{[n_s]}} \frac{\mathsf{co}_L^{[n]}}{e(N_L^{[n]})} \end{aligned} \tag{3.8.3}$$

where

$$\mathrm{SW}(\beta) = \mathrm{SW}(\beta_1) \cdots \mathrm{SW}(\beta_s)$$

is the product of Seiberg-Witten invariants of the classes $\beta_i \in H^2(S, \mathbb{Z})$, and

$$\begin{aligned} \mathsf{co}_L^{[n]} &:= c_{\mathrm{top}} \left(\bigoplus_{i=1}^{s} R\mathscr{H}om_{\pi_S}\left(\mathcal{I}^{[n_{i-1}]} \otimes L_{i-1}, \mathcal{I}^{[n_i]} \otimes L_i \otimes \omega_S\right)_0 \right) \\ &\in A^{2|n| - n_0 - n_s}(S^{[n_0]} \times \cdots \times S^{[n_s]}) \end{aligned}$$

is a Carlsson-Okounkov type class [CO12, GT19]. The integral (3.8.3) is precisely the one of Equation 2.4.4 in Chapter 2, which computes vertical contributions to the Vafa-Witten invariant in the case that stable = semistable.

Proof of Theorem 3.B (unrefined case). Fix a rank $r = s + 1$. We will use the following convention. For $\beta = (\beta_1, \ldots, \beta_s) \in H^2(S, \mathbb{Z})^s$, we will write

$$\begin{aligned} \beta^\vee &= (\beta^1, \ldots, \beta^s) \\ &= (c_1(\omega_S) - \beta_1, \ldots, c_1(\omega_S) - \beta_s)\,. \end{aligned}$$

In particular we have

$$\begin{aligned}\mathrm{SW}(\beta^\vee) &= \mathrm{SW}(\beta^1)\cdots\mathrm{SW}(\beta^s)\\ &= (-1)^{s\cdot\chi(\mathcal{O}_S)}\mathrm{SW}(\beta)\,.\end{aligned}$$

By Propositions 2.6.3 and 2.7.3 and the proof of Theorem 2.A (see Section 2.8), there exist universal Laurent series (i.e. depending only on r)

$$A, B, C_{ij} \in \mathbb{Q}(\!(q^{\frac{1}{2r}})\!)\,, \quad 1 \le i \le j < r$$

such that

$$q^{d(\beta)} \sum_n \mathrm{VW}_\beta^{[n]} q^{|n|} = \mathrm{SW}(\beta^\vee) A^{\chi(\mathcal{O}_S)} B^{K_S^2} \prod_{i\le j} C_{ij}^{\beta^i\beta^j} \tag{3.8.4}$$

for any surface S and classes $\beta = (\beta_1, \ldots, \beta_s) \in H^2(S,\mathbb{Z})$. We have use a normalizing factor $q^{d(\beta)}$ given by

$$d(\beta) = -\sum_{1\le i,j<r} \frac{i(r-j)}{2r}\beta^i\beta^j\,.$$

Now fix a surface S with $p_g(S) > 0$, a class $c_1 \in H^2(S,\mathbb{Z})$ and a line bundle L on S with $c_1(L) = c_1$. For an r-tuple

$$\alpha = (\alpha_0, \ldots, \alpha_s) \in (H^2(S,\mathbb{Z}))^r\,,$$

write

$$\beta(\alpha) = (\alpha_1 - \alpha_0 + c_1(\omega_S), \ldots, \alpha_s - \alpha_{s-1} + c_1(\omega_S))\,.$$

Also write

$$c_2(\alpha, n) = n_0 + \ldots + n_s + \sum_{i<j} \alpha_i\alpha_j$$

for $n \in (\mathbb{Z}_{\ge 0})^r$ and α as above.

By Lemma 2.2.6 we have have

$$c_2(\alpha, n) = n_0 + \ldots + n_s + d(\beta(\alpha)) + \frac{r-1}{2r}c_1^2.$$

Following the proof of Theorem 2.A in Section 2.8, we can use this to rewrite the sum

$$\sum_{\alpha,n} \mathrm{VW}_{\beta(\alpha)}^{[n]} q^{c_2(\alpha,n)}$$

over $\alpha \in H^2(S,\mathbb{Z})^r$ and $n \in (\mathbb{Z}_{\ge 0})^r$ satisfying

$$\sum_{i=0}^{r-1} \alpha_i = c_1\,,$$

as a sum

$$\#H^2(S,\mathbb{Z})[r] \sum_{\beta,n} \mathrm{VW}_\beta^{[n]} q^{|n| + d(\beta) + \frac{r-1}{2r}c_1^2}$$

over $\beta \in H^2(S,\mathbb{Z})^{r-1}$ and $n \in (\mathbb{Z}_{\geq 0})^r$ satisfying

$$\sum_{i=1}^{r-1} i\beta^i \equiv c_1 \mod rH^2(S,\mathbb{Z})\,. \tag{3.8.5}$$

(To see this, note that the equations

$$\beta(\alpha) = \beta\,, \quad \sum_{i=0}^{s} \alpha_i = c_1$$

have precisely $\# H^2(S,\mathbb{Z})[r]$ solutions α for any β and c_1 satisfying (3.8.5).) By (3.7.2), (3.8.1) and (3.8.4) it follows that

$$\begin{aligned}\sum_{c_2} \mathrm{VW}^{\mathrm{vert}}_{(r,L,c_2)} q^{c_2} &= \sum_{\alpha,n} \int_{[M^n_\alpha]^{\mathrm{vir}}} \frac{1}{e(N^n_\alpha)} q^{c_2(\alpha,n)} \\ &= \#\,\mathrm{Pic}_0(S)[r] \sum_{\alpha,n} \mathrm{VW}^{[n]}_{\beta(\alpha)} q^{c_2(\alpha,n)} \\ &= \#\,\mathrm{Pic}_0(S)[r] \cdot \# H^2(S,\mathbb{Z})[r] \sum_{\beta,n} \mathrm{VW}^{[n]}_{\beta} q^{|n|+d(\beta)+\frac{r-1}{2r}c_1^2} \\ &= \#\,\mathrm{Pic}(S)[r]\, q^{\frac{r-1}{2r}c_1^2} A^{\chi(\mathcal{O}_S)} B^{K_S^2} \sum_{\beta} \mathrm{SW}(\beta^\vee) \prod_{i\leq j} C_{ij}^{\beta^i\beta^j}\,. \qquad \square\end{aligned}$$

3.9 Refined invariants

We will briefly discuss refined invariants. The proofs of Theorems 3.A and 3.B are perfectly analogous to the discussion of Sections 3.7 and 3.8. Using the equivariant structure (3.4.1), we have by Corollary 3.3.4,

$$\mathcal{O}^{\mathrm{vir}}_{\mathbb{P}} = p^*\mathcal{O}^{\mathrm{vir}}_{M^n_\alpha} \otimes \Lambda^\bullet(T_{\mathbb{P}/M^n_\alpha} \otimes \mathfrak{t}^{-1}) \in K_0^{\mathbb{C}^*}(\mathbb{P})\,.$$

Choose a square root

$$K^{\frac{1}{2}} := \det\left(-R\mathscr{H}om_{\pi_X}(I^\bullet, I^\bullet)^\vee_\perp\right)^{\frac{1}{2}}\,.$$

Note that by equations (3.6.8) and (3.8.2), we can take

$$K^{\frac{1}{2}} = p^* \det\left(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp\right)^{\frac{1}{2}} \otimes \omega_{\mathbb{P}/M^n_\alpha} \otimes \mathfrak{t}^{\frac{\chi(\gamma(m))-1}{2}}$$

with

$$\det(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\frac{1}{2}} := \det\left(R\mathscr{H}om_{\pi_S}(E,E)_0 \otimes \mathfrak{t}^{-\frac{1}{2}}\right)\,.$$

By the K-theoretic virtual localisation formula [Qu18], we have, again using 3.6.9:

$$\begin{aligned}\chi_t\left(\mathbb{P}, \frac{\mathcal{O}^{\mathrm{vir}}_{\mathbb{P}} \otimes K^{\frac{1}{2}}}{p^*\Lambda^\bullet(N^n_\alpha)^\vee}\right) &= \chi_t\left(\mathbb{P}^{\mathbb{C}^*}, \frac{\mathcal{O}^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}} \otimes K^{\frac{1}{2}}}{p^*\Lambda^\bullet(N^n_\alpha)^\vee \otimes \Lambda^\bullet(N^{\mathrm{vir}}_{\mathbb{P}^{\mathbb{C}^*}/\mathbb{P}})^\vee}\right) \\ &= \chi_t\left(\mathcal{P}^n_\alpha, \frac{\mathcal{O}^{\mathrm{vir}}_{\mathcal{P}^n_\alpha} \otimes K^{\frac{1}{2}}}{\Lambda^\bullet(N^{\mathrm{vir}}_{\mathcal{P}^n_\alpha/\mathcal{P}^\perp})^\vee}\right)\,.\end{aligned}$$

On the other hand, he have

$$
\begin{aligned}
\chi_t\left(\mathbb{P}, \frac{\mathcal{O}_{\mathbb{P}}^{\mathrm{vir}} \otimes K^{\frac{1}{2}}}{p^*\Lambda^\bullet(N_\alpha^n)^\vee}\right) &= \chi_t\left(M_\alpha^n, p_*\left(\frac{\mathcal{O}_{\mathbb{P}}^{\mathrm{vir}} \otimes K^{\frac{1}{2}}}{p^*\Lambda^\bullet(N_\alpha^n)^\vee}\right)\right) \\
&= \chi_t\Bigg(M_\alpha^n, \frac{\mathcal{O}_{M_\alpha^n}^{\mathrm{vir}} \otimes \det(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\frac{1}{2}}}{\Lambda^\bullet(N_\alpha^n)^\vee} \\
&\qquad \otimes p_*\left(\Lambda^\bullet(T_{\mathbb{P}/M_\alpha^n} \otimes \mathfrak{t}^{-1}) \otimes \omega_{\mathbb{P}/M_\alpha^n}\right) \otimes \mathfrak{t}^{\frac{\chi(\gamma(m))-1}{2}}\Bigg) \\
&= \chi_t\left(M_\alpha^n, \frac{\mathcal{O}_{M_\alpha^n}^{\mathrm{vir}} \otimes \det(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\frac{1}{2}}}{\Lambda^\bullet(N_\alpha^n)^\vee}\right) \\
&\qquad \times (-1)^{\chi(\gamma(m))}[\chi(\gamma(m))]_t\,.
\end{aligned}
$$

Here we have used

$$
\begin{aligned}
p_*\left(\Lambda^\bullet(T_{\mathbb{P}/M_\alpha^n} \otimes \mathfrak{t}^{-1}) \otimes \omega_{\mathbb{P}/M_\alpha^n}\right) \otimes \mathfrak{t}^{\frac{\chi(\gamma(m))-1}{2}} &= (-1)^{\chi(\gamma(m))}\frac{p_*\Lambda^\bullet(\Omega_{\mathbb{P}/M_\alpha^n} \otimes \mathfrak{t})}{\mathfrak{t}^{\frac{\chi(\gamma(m))-1}{2}}} \\
&= (-1)^{\chi(\gamma(m))}\frac{\mathfrak{t}^{\chi(\gamma(m))-1} + \ldots + 1}{\mathfrak{t}^{\frac{\chi(\gamma(m))-1}{2}}} \\
&= (-1)^{\chi(\gamma(m))}[\chi(\gamma(m))]_{\mathfrak{t}}\,.
\end{aligned}
$$

As in the unrefined case, we conclude that the vertical contribution to the refined Vafa-Witten invariant is well defined, and c.f. (3.7.2) given by a sum

$$
\mathrm{VW}_\gamma^{\mathrm{vert}}(t) = \sum_{\alpha,n} \chi_t\left(M_\alpha^n, \frac{\mathcal{O}_{M_\alpha^n}^{\mathrm{vir}} \otimes \det(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\frac{1}{2}}}{\Lambda^\bullet(N_\alpha^n)^\vee}\right).
$$

This finishes the proof of Theorem 3.A.

By the virtual Riemann-Roch formula [CFK09, FG10], we can compute the contribution of M_α^n to the refined Vafa-Witten invariant by the integral

$$
\left[\int_{[M_\alpha^n]^{\mathrm{vir}}} \frac{\mathrm{ch}\left(\det(R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\frac{1}{2}}\right)}{\mathrm{ch}\left(\Lambda^\bullet(N_\alpha^n)^\vee\right)} \mathrm{Td}\left(T_{M_\alpha^n}\right)\right]_{\mathrm{ch}\,\mathfrak{t}=t},
$$

where

$$
T_{M_\alpha^n} = (-R\mathscr{H}om_{\pi_X}(\mathcal{E},\mathcal{E})_\perp)^{\mathrm{fix}}
$$

denotes the virtual tangent bundle of M_α^n. The discussion of Section 3.8 now literally applies to this integral: the integrant descents to a class in $A^*_{\mathbb{C}^*}(S_\beta^{[n]})$, which we can integrate over

$$
\eta_*[M_\alpha^n]^{\mathrm{vir}} = \#\,\mathrm{Pic}_0(S)[r] \cdot [S_\beta^{[n]}]^{\mathrm{vir}}\,.
$$

Again, we can use the results of [GT19] to rewrite the resulting integral as an integral over the product of Hilbert schemes of points

$$
S^{[n_0]} \times \cdots \times S^{[n_s]}\,.
$$

In fact, after taking out the factor $\#\,\mathrm{Pic}_0(S)[r]$, it is precisely the one given in Equation 2.4.5 Copying the proof of the unrefined case given in Section 3.8, Theorem 3.B now follows from Propositions 2.6.5 and 2.7.6.

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Abstract

Chapter 1 of this thesis is devoted to a generalisation of the famous Göttsche Conjecture. For a relative effective divisor $\mathcal{C}$ on a smooth projective family of surfaces $q : \mathcal{S} \to B$, we consider the locus in B over which the fibres of $\mathcal{C}$ are δ-nodal curves. We prove a conjecture by Kleiman and Piene [KP04] on the universality of an enumerating cycle on this locus. We propose a bivariant class $\gamma(\mathcal{C}) \in A^*(B)$ motivated by the BPS calculus of Pandharipande and Thomas, and show that it can be expressed universally as a polynomial in classes of the form

$$q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)\,.$$

Under an ampleness assumption, we show that $\gamma(\mathcal{C}) \cap [B]$ is the class of a natural effective cycle with support equal to the closure of the locus of δ-nodal curves. Finally, we apply our method to calculate node polynomials for plane curves intersecting general lines in $\mathbb{P}^3$. We verify our results using 19th century geometry of Schubert.

In Chapter 2 we study the monopole contribution to (refined) Vafa-Witten invariants, recently defined in [TT17a] and [MT] as invariants of moduli spaces of compactly supported sheaves on a local surfaces. Here we work on under the assumption that stability and semistability of such sheaves agree (the *stable* case). We apply the results of [GT19] to prove a universality result for the generating series of *vertical* contributions, i.e. contributions of $\mathbb{C}^*$-equivariant Higgs pairs with 1-dimensional weight spaces. For prime rank, these account for the entire monopole contribution by a theorem of Thomas. We use toric computations to determine part of the generating series, and find agreement with the conjectures of [GK18] for rank 2 and 3.

In Chapter 3 we show that vertical contributions to Vafa-Witten invariants in the *semistable* case are well defined for surfaces with $p_g(S) > 0$, partially proving conjectures of [TT17b] and [Tho18a]. Moreover, we show that such contributions are computed by the same tautological integrals as in the stable case. Using the work of Kiem and Li, we show that stability of universal families of vertical Joyce-Song pairs is controlled by a cosections of the obstruction sheaves of such families. Combining these results, we extend the universality theorem of Chapter 2 to the semistable case.

Samenvatting

In Hoofdstuk 1 van dit proefschift behandelen we een generalisatie van het beroemde Göttsche-vermoeden. Zij $\mathcal{C}$ een relatieve effectieve divisor op een familie van gladde projectieve oppervlakken $q \colon \mathcal{S} \to B$. Beschouw de locus in B waarover de vezels van $\mathcal{C}$ δ-nodale krommen zijn. We bewijzen een vermoeden van Kleiman en Piene [KP04] over de universaliteit van een algebraïsche cykel op deze locus die de δ-nodale vezels 'telt', in zekere zin. We doen een voorstel voor een klasse $\gamma(\mathcal{C})$ in de Chowring $A^*(B)$, gemotiveerd door de BPS-calculus van Pandharipande en Thomas, en laten zien dat deze op een universele manier kan worden uitgedrukt als een polynoom in klassen van de vorm

$$q_*(c_1(\mathcal{O}(\mathcal{C}))^a c_1(T_{\mathcal{S}/B})^b c_2(T_{\mathcal{S}/B})^c)\,.$$

Onder bepaalde voorwaarden laten we zien dat $\gamma(\mathcal{C}) \cap [B]$ gelijk is aan de rationale-equivalentieklasse van een natuurlijke effectieve cykel, wiens drager gelijk is aan de afsluiting van de locus van δ-nodale vezels. We passen onze methode toe op het tellen van δ-nodale krommen op vlakken in $\mathbb{P}^3$. Voor $\delta \leq 12$ berekenen we het aantal van zulke krommen die bovendien algemeen gekozen lijnen in $\mathbb{P}^3$ doorsnijden. We verifiëren onze resultaten met de negentiende-eeuwse meetkunde van Schubert.

In Hoofdstuk 2 bestuderen we bijdragen van de monopooltak aan (verfijnde) Vafa-Witten invarianten. Deze invarianten zijn recent gedefinieerd (in algebraïsche meetkunde) als invarianten van moduli ruimtes van bepaalde compact gedragen schoven op een lokaal oppervlak [TT17a, MT]. We werken onder de aanname dat stabiliteit en semistabiliteit van zulke schoven overeenkomen (het *stabiele* geval). We gebruiken de resultaten van [GT19] om universaliteit te bewijzen van de voortbrengende reeks van *verticale* bijdragen, d.w.z. bijdragen van $\mathbb{C}^*$-equivariante Higgs-paren met ééndimensionale gewichtsruimten. Volgens een stelling van Thomas geeft deze voortbrengende reeks voor priem rang de bijdragen van de hele monopooltak. We gebruiken torische berekeningen om een deel van de voortbrengende reeks uit te rekenen, en vinden overeenkomst met vermoedens van Göttsche en Kool [GK18] voor rang 2 en rang 3.

In Hoofdstuk 3 laten we zien dat verticale bijdragen aan Vafa-Witten invarianten in het *semistabiele* geval welgedefinieerd zijn voor oppervlakken met $p_g(S) > 0$. Hiermee bewijzen we een deel van vermoedens in [TT17b] en [Tho18a]. Bovendien laten we zien dat de bijdragen gegeven worden door dezelfde tautologische integralen als in het stabiele geval. Gebruikmakend van het werk van Kiem en Li laten we zien dat stabiliteit van universele families van verticale Joyce-Song paren gecontroleerd wordt door cosecties van obstructieschoven van zulke families. Met behulp van deze resultaten breiden we het universaliteitsresultaat van Hoofdstuk 2 uit naar het semistabiele geval.

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Curriculum vitae

Ties Laarakker was born on 20 October, 1989 in Groningen, the Netherlands. In 2006 he graduated from the Gymnasium Celeanum in Zwolle. In September of the same year he started his Bachelor's studies in mathematics at the University of Amsterdam, which he completed in August 2011. Between September 2013 and Februari 2016 he studied for his Master's degree at the same university. Between September 2008 and June 2012, he studied double bass at the Conservatory of Amsterdam, from which he received a Bachelor's degree.

After obtaining his Master's degree with a thesis entitled "Zero-cycles on K3 surfaces", Ties started his PhD at Utrecht University in April 2016, under supervision of Martijn Kool. In April and May 2018, he visited the MSRI in Berkeley, as a member of the spring program "Enumerative geometry beyond numbers". In October of the same year, he visited Imperial College, London at the invitation of Richard Thomas.

Since September 2019, Ties is working at Imperial College as a Research Associate.