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Bilattice logic properly displayed ☆

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Abstract

We introduce a proper multi-type display calculus for bilattice logic (with conflation) for which we prove soundness, completeness, conservativity, standard subformula property and cut elimination. Our proposal builds on the product representation of bilattices and applies the guidelines of the multi-type methodology in the design of display calculi.

Keywords: Non-classical logics; Bilattice logic; Many-valued logics; Substructural logics; Algebraic proof theory; Sequent calculi; Cut elimination; Display calculi; Multi-type calculi

1. Introduction

Bilattices are algebraic structures introduced in [26] in the context of a multivalued approach to deductive reasoning, and have subsequently found applications in a variety of areas in computer science and artificial intelligence. The basic intuition behind the bilattice formalism, which can be traced back to the work of Dunn and Belnap [15,4,5] and even earlier, to Kleene's proposal of a three-valued logic, is to carry out reasoning within a space of truth-values that results from expanding the classical set $\{\mathtt{f}, \mathtt{t}\}$ with a value $\bot$, representing lack of information, and a value $\top$, representing over-defined or contradictory information.

More generally, Ginsberg [26] argued that one could take as space of truth-values a set equipped with *two* lattice orderings (a *bilattice*), reflecting respectively the *degree of truth* and the *degree of information* associated with propositions. The bilattice framework may thus be viewed as an attempt at combining the many-valued approach to vagueness of fuzzy logic with the Dunn–Belnap–Kleene treatment of partial and inconsistent information. In fact, a number of works has shown how bilattice-like structures naturally arise in the context of fuzzy logic when one tries to account for uncertainty, imprecision and incompleteness of information [16,39,14,13].

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During the last two decades, the theory of bilattices has been investigated in depth from a proof-theoretic and algebraic point of view: complete (Hilbert- and Gentzen-style) presentations of bilattice-based logics were introduced in [1,2], followed by [9] which focuses on the implication-free reduct of the logic. The calculi introduced in these papers share many features with those considered e.g. in [17] for the Belnap–Dunn logic, of which bilattice logics are conservative expansions.

Negation plays a very special role. Indeed, it is because of this connective that bilattice logics are not *self-extensional* [42] (or, as other authors say, *congruential*), i.e. the inter-derivability relation of the logic is not a congruence of the formula algebra. This means that there are formulas φ and ψ such that $\varphi \dashv\vdash \psi$ and yet $\neg\varphi \not\dashv\vdash \neg\psi$ (this is not the case of the Belnap–Dunn logic, which is self-extensional). In the Gentzen-style calculus for bilattice logic *GBL* introduced in [1, Section 3.2], there are four introduction rules for each binary connective, two of which are standard and introduce it as main connective on the left and on the right of the turnstile, and two are non-standard and introduce the same connective under the scope of negation. From a proof-theoretic perspective, this solution has the disadvantage that the resulting calculus is not fully modular, does not enjoy the standard subformula property, and violates some key criteria about introduction rules for connectives adopted in the literature on display calculi, structural proof theory and dynamic logics on the basis of technical considerations, and in the literature on proof-theoretic semantics on more philosophical ground and concerns (see [40,38,21,41]).

In this paper, we introduce a *proper multi-type display calculus* for bilattice logic that circumvents all the above-mentioned disadvantages.[1] As a first approximation to the problem of providing a calculus for the full Arieli–Avron logic [1,10], we shall here focus on its implication-free fragment, which is precisely the logic axiomatized by means of a Hilbert-style calculus in [9]. We consider this to be a reasonable tradeoff: on the one hand because, thanks to the modularity of our calculus, we do not anticipate any major technical difficulties in introducing further rules to account for the implication (this is current work in progress); on the other hand because the characteristic behaviour of the bilattice negation (and the problems that arise in its proof-theoretic treatment) already manifest in the context of the implicationless logic. Another natural future project will be providing a display calculus for modal expansions of bilattice logic such as those introduced in [33] – see the concluding remarks in Section 7.

The design of our display calculus follows the principles of the *multi-type* methodology (cf. Section 2.3), introduced in [27,20,18,19] for displaying dynamic epistemic logic and propositional dynamic logic, and subsequently applied to displaying several other well known logics (e.g. linear logic with exponentials [31], inquisitive logic [22], semi-De Morgan logic [28], lattice logic [30]) which are not properly displayable in their single-type presentation, and also to design families of novel logical frameworks in a modular and principled way [7]. Our multi-type syntactic presentation of bilattice logic is based on the algebraic insight provided by the product representation theorems (see e.g. [8]) and possesses all the desirable properties of *proper* display calculi. In particular, our calculus enjoys the standard subformula property, supports a proof-theoretic semantics and is fully modular. These features make it possible to prove important results about the logics in a principled way and are key for developing interactive and automated reasoning tools [3].

Structure of the paper In Section 2 we recall basic definitions and results about bilattices and bilattice logics and discuss the general motivations and insights underlying (multi-type) display calculi. Section 3 develops an algebraic analysis of bilattices as heterogeneous structures which provides a basis for our multi-type approach to their proof theory. In Section 4, we introduce the multi-type bilattice logic which corresponds to heterogenous bilattices. Our display calculi are introduced in Section 5, and we prove its soundness, completeness, conservativity, subformula property and cut elimination in Section 6. In Section 7 we outline some directions for future work.

2. Preliminaries

2.1. Bilattices

The following definitions and results can be found e.g. in [1,9].

[1] The notion of proper display calculus has been introduced in [40]. Properly displayable logics, i.e. those which can be captured by some proper display calculus, have been characterized in a purely proof-theoretic way in [11]. In [29], an alternative characterization of properly displayable logics was introduced which builds on the algebraic theory of unified correspondence [12].

Definition 2.1. A *bilattice* is a structure $\mathbb{B} = (B, \leq_t, \leq_k, \neg)$ such that B is a non-empty set, $(B, \leq_t)$, $(B, \leq_k)$ are lattices, and $\neg$ is a unary operation on B having the following properties:

- if $a \leq_t b$, then $\neg b \leq_t \neg a$,
- if $a \leq_k b$, then $\neg a \leq_k \neg b$,
- $\neg\neg a = a$.

We use $\wedge, \vee$ for the lattice operations which correspond to $\leq_t$ and $\otimes, \oplus$ for those that correspond to $\leq_k$. If present, the lattice bounds of $\leq_t$ are denoted by $\mathtt{f}$ and $\mathtt{t}$ (minimum and maximum, respectively) and those of $\leq_k$ by $\bot$ and $\top$. The smallest non-trivial bilattice is the four-element one (called **Four**) with universe $\{\mathtt{f}, \mathtt{t}, \bot, \top\}$.

Fact 2.2. *The following equations (*De Morgan laws for negation*) hold in any bilattice:*

$$\neg(x \wedge y) = \neg x \vee \neg y, \qquad \neg(x \vee y) = \neg x \wedge \neg y,$$
$$\neg(x \otimes y) = \neg x \otimes \neg y, \qquad \neg(x \oplus y) = \neg x \oplus \neg y.$$

Moreover, if the bilattice is bounded, then

$$\neg\mathtt{t} = \mathtt{f}, \quad \neg\mathtt{f} = \mathtt{t}, \quad \neg\top = \top, \quad \neg\bot = \bot.$$

Definition 2.3. A bilattice is called *distributive* when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold:

$$x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z) \quad \text{for all } \circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$$

If a distributive bilattice is bounded, then

$$\mathtt{t} \otimes \mathtt{f} = \bot, \quad \mathtt{t} \oplus \mathtt{f} = \top, \quad \top \wedge \bot = \mathtt{f}, \quad \top \vee \bot = \mathtt{t}.$$

In the following, we use B to denote the class of bounded distributive bilattices.

Theorem 2.4 *(Representation of distributive bilattices). Let $\mathbb{L}$ be a bounded distributive lattice with join $\sqcup$ and meet $\sqcap$. Then the algebra $\mathbb{L} \odot \mathbb{L}$ having as universe the direct product $L \times L$ is a distributive bilattice with the following operations:*

$$\begin{aligned}
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \sqcap b_1, a_2 \sqcup b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \sqcup b_1, a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle &:= \langle a_1 \sqcap b_1, a_2 \sqcap b_2\rangle \\
\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle &:= \langle a_1 \sqcup b_1, a_2 \sqcup b_2\rangle \\
\neg\langle a_1, a_2\rangle &:= \langle a_2, a_1\rangle \\
\mathtt{f} &:= \langle 0, 1\rangle \\
\mathtt{t} &:= \langle 1, 0\rangle \\
\bot &:= \langle 0, 0\rangle \\
\top &:= \langle 1, 1\rangle
\end{aligned}$$

Theorem 2.5. *Every distributive bilattice is isomorphic to $\mathbb{L} \odot \mathbb{L}$ for some distributive lattice $\mathbb{L}$.*

Definition 2.6. A structure $\mathbb{B} = (B, \leq_t, \leq_k, \neg, -)$ is a *bilattice with conflation* if $(B, \leq_t, \leq_k, \neg)$ is a bilattice and the *conflation* $- : B \to B$ is an operation satisfying:

- if $a \leq_t b$, then $-a \leq_t -b$;
- if $a \leq_k b$, then $-b \leq_k -a$;
- $- - a = a$.

We say that $\mathbb{B}$ is *commutative* if it also satisfies the equation: $\neg - x = -\neg x$.

Fact 2.7. *The following equations (*De Morgan laws for conflation*) hold in any bilattice with conflation:*

$$-(x \wedge y) = -x \wedge -y \qquad -(x \vee y) = -x \vee -y$$
$$-(x \otimes y) = -x \oplus -y \qquad -(x \oplus y) = -x \otimes -y$$

Moreover, if the bilattice is bounded, then

$$-\mathtt{t} = \mathtt{t}, \quad -\mathtt{f} = \mathtt{f}, \quad -\top = \bot, \quad -\bot = \top.$$

We denote by CB the class of bounded commutative distributive bilattices with conflation.

Theorem 2.8. *Let $\mathbb{D} = (D, \sqcap, \sqcup, \sim, 0, 1)$ be a De Morgan algebra, then $\mathbb{D} \odot \mathbb{D}$ is a bounded commutative distributive bilattice with conflation where:*

- $(D, \sqcap, \sqcup, 0, 1) \odot (D, \sqcap, \sqcup, 0, 1)$ *is a bounded distributive bilattice;*
- $-(a, b) = (\sim b, \sim a)$.

Theorem 2.9. *Every bounded commutative distributive bilattice with conflation is isomorphic to $\mathbb{D} \odot \mathbb{D}$ for some De Morgan algebra $\mathbb{D}$.*

2.2. Bilattice logic

In the present subsection we introduce Bilattice Logic (BL) and Bilattice Logic with Conflation (CBL). The language of CBL $\mathcal{L}$ over a denumerable set $\mathsf{AtProp} = \{p, q, r, \ldots\}$ of atomic propositions is generated as follows:

$$A ::= p \mid \mathtt{t} \mid \mathtt{f} \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \otimes A \mid A \oplus A \mid -A,$$

the language of BL is the conflation-free reduct of $\mathcal{L}$, where conflation is the name of the connective '$-$'. Bilattice Logic consists of the following axioms:

$$A \vdash A, \quad \neg\neg A \dashv\vdash A,$$

$$\mathtt{f} \vdash A, \;\; A \vdash \mathtt{t}, \;\; \bot \vdash A, \;\; A \vdash \top,$$

$$A \vdash \neg\mathtt{f}, \;\; \neg\mathtt{t} \vdash A, \;\; \neg\bot \vdash A, \;\; A \vdash \neg\top,$$

$$A \wedge B \vdash A, \;\; A \wedge B \vdash B, \;\; A \vdash A \vee B, \;\; B \vdash A \vee B,$$

$$A \otimes B \vdash A, \;\; A \otimes B \vdash B, \;\; A \vdash A \oplus B, \;\; B \vdash A \oplus B,$$

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C),$$

$$A \otimes (B \oplus C) \vdash (A \otimes B) \vee (A \oplus C),$$

$$\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B, \;\; \neg(A \vee B) \dashv\vdash \neg A \wedge \neg B,$$

$$\neg(A \otimes B) \dashv\vdash \neg A \otimes \neg B, \;\; \neg(A \oplus B) \dashv\vdash \neg A \oplus \neg B,$$

and the following rules:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \vee C \vdash B}$$

$$\frac{A \vdash B \qquad A \vdash C}{A \vdash B \otimes C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \oplus C \vdash B}$$

CBL consists of the axioms and rules of BL plus the following axioms:

$$--A \dashv\vdash A, \quad -\neg A \dashv\vdash \neg - A,$$
$$-\mathtt{f} \vdash A, \quad A \vdash -\mathtt{t}, \quad -\top \vdash A, \quad A \vdash -\bot,$$
$$-(A \wedge B) \dashv\vdash -A \wedge -B, \quad -(A \vee B) \dashv\vdash -A \vee -B,$$
$$-(A \otimes B) \dashv\vdash -A \oplus -B, \quad -(A \oplus B) \dashv\vdash -A \otimes -B.$$

The algebraic semantics of BL (resp. CBL) is given by B (resp. CB). We use $A \models_{\mathsf{B}} C$ (resp. $A \models_{\mathsf{CB}} C$) to mean: for any $\mathbb{B} \in \mathsf{B}$ (resp. $\mathbb{B} \in \mathsf{CB}$), if $A^{\mathbb{B}} \in F_{\mathtt{t}}$ then $C^{\mathbb{B}} \in F_{\mathtt{t}}$. Here $A^{\mathbb{B}}$ and $C^{\mathbb{B}}$ mean the interpretations of A and C in $\mathbb{B}$, respectively; and $F_{\mathtt{t}} = \{a \in B : \mathtt{t} \leq_k a\}$ is the set of designated elements of $\mathbb{B}$ (using the terminology of [1, Definition 2.13], $F_{\mathtt{t}}$ is the *least bifilter* of $\mathbb{B}$).

Soundness of BL (resp. CBL) is straightforward. In order to show completeness, we can prove that every axiom and rule of Arieli and Avron's *GBL* (resp. *GBS*, cf. [1]) is derivable in BL (resp. CBL).[2] Then the completeness of BL (resp. CBL) follows from the completeness of *GBL* (resp. *GBS*, [1, Theorem 3.7]).

Theorem 2.10 *(Completeness). $A \vdash_{\mathrm{BL}} C$ iff $A \models_{\mathsf{B}} C$ (resp. $A \vdash_{\mathrm{CBL}} C$ iff $A \models_{\mathsf{CB}} C$).*

2.3. Display calculi and multi-type display calculi

A major issue in structural proof theory is the design of *analytic* calculi, that is, calculi in which formulas are deduced through a process of step-wise decomposition, in which no elements extraneous to the formula to be proved are allowed. The best known analytic calculi are Gentzen's sequent calculi [25], the analiticity of which takes the form of the *cut elimination* theorem, stating that every sequent for which a deduction exists can be proven by a deduction in which a certain rule (the cut rule, the only rule violating analiticity) is not applied. The syntactic proof of cut elimination is very informative, but is also lengthy and intricate, and hence error-prone. Moreover, it is not robust: that is, it does not extend modularly from a given calculus to any of its extensions obtained by adding a given rule. Various extensions and refinements of Gentzen's sequent calculi have been introduced to improve modularity while retaining cut-elimination. One of the most elegant and successful such proposals is Belnap's framework of *display calculi* [6]. Belnap's refinement is based on the introduction of a richer syntax for the constituents of each sequent, which includes structural connectives along with logical connectives. This syntax allows for the definition of an environment in which the essentials of syntactic cut elimination can be precisely described. In this environment, a cut elimination *meta-theorem* can be proved, which gives a set of sufficient conditions for the cut elimination theorem to hold of sequent calculi. Most of these conditions are easily verified by inspecting the shape of the rules. Meta-theorems provide much smoother, robust and modular routes to cut elimination than the original proof devised by Gentzen. In a slogan, cut elimination via meta-theorems is to ordinary cut elimination what canonicity is to completeness. Indeed, canonicity provides a uniform strategy to achieve completeness; likewise, the conditions of Belnap's meta-theorem guarantee that one and the same transformation strategy achieves cut elimination for any calculus satisfying them. Belnap's display calculi account simultaneously for large families of logical systems, including modal logics and substructural logics. However, the scope of display calculi, as proposed by Belnap and later refined by Wansing by means of the notion of *proper display calculi* [40], does not encompass many important logical systems, and in [11,29], a characterization is given of the logics which can be endowed with (single-type) proper display calculi. The theory of *multi-type calculi* is a generalization of Belnap's original framework, capable to encompass those logics which – like linear logic, dynamic epistemic logic, propositional dynamic logic, and inquisitive logic – fall out of the scope of the characterization given in [29], and uniformly endow them with the same excellent properties enjoyed by (single-type) proper display calculi. What sets multi-type calculi apart from other proof-theoretic methodologies is that, in multi-type calculi, entities of different types can coexist and interact on equal ground: each type has its own internal logic (i.e. language and deduction relation), and the interaction between logics of different types is facilitated by special heterogeneous connectives, primitive to the language, and treated on a par with all the others. This enriched environment is specifically designed to address the problem of expressing the interactions between entities of different types by

[2] In order to do this, we view a sequent $\Gamma \Rightarrow \Delta$ of *GBL* (*GBS*) as the equivalent sequent $\bigwedge \Gamma \Rightarrow \bigvee \Delta$.

means of analytic rules. Indeed, although the source of the mathematical difficulties was different for each of the logics mentioned above, a common core to these difficulties was identified precisely in the encoding of key interactions between entities of different types. For instance, for dynamic epistemic logic the difficulties lay in the interactions between (epistemic) actions, agents' beliefs, and facts of the world; for linear logic, in the interaction between general resources and reusable resources; for propositional dynamic logic, between general and iterative actions; for inquisitive logic, between general and flat formulas. In each case, precisely the formal encoding of these interactions gave rise to non-analytic axioms in the original formulations of the logics. In each case, the multi-type approach allowed to redesign the logics, so as to encode the key interactions into analytic multi-type rules, and define a multi-type proper display calculus for each of them. Adding types makes it possible to move to a richer and more expressive environment in which these interactions can be unravelled and reformulated with analytic (multi-type) terms.

A key feature towards the implementation of the multi-type methodology on specific logics, such as bilattice logic, is the use of algebraic information for proof-theoretic purposes. That is, we aim at reformulating the algebraic semantics of the given logic in a way which highlights the existence and interaction of different algebras, which can be taken as potential interpreters of different types, as well as of natural maps connecting these algebras, which can be taken as potential interpreters of heterogeneous connectives spanning between these types. In the case of bilattices, this reformulation pivots on the representation theorem of bilattices as twist-product of lattices.

3. Multi-type algebraic presentation of bilattices

In the present section we introduce the algebraic environment which justifies semantically the multi-type approach to bilattice logic presented in Section 5. The main insight is that (bounded) bilattices (with conflation) can be equivalently presented as heterogeneous structures, i.e. tuples consisting of two (bounded) distributive lattices (De Morgan algebras) together with two maps between them.

3.1. Multi-type semantic environment

For a bilattice $\mathbb{B}$, let $\mathrm{Reg}(\mathbb{B}) = \{a \in B : a = \neg a\}$ be the set of *regular elements* [8]. It is easy to show that $\mathrm{Reg}(\mathbb{B})$ is closed under $\otimes$ and $\oplus$, hence $(\mathrm{Reg}(\mathbb{B}), \otimes, \oplus)$ is a sublattice of $(B, \otimes, \oplus)$. For every $a \in B$, we let

$$\mathrm{reg}(a) := (a \vee (a \otimes \neg a)) \oplus \neg(a \vee (a \otimes \neg a))$$

be the regular element associated with a. It follows from the representation result of [8, Theorem 3.2] that

$$\mathbb{B} \cong (\mathrm{Reg}(\mathbb{B}), \otimes, \oplus) \odot (\mathrm{Reg}(\mathbb{B}), \otimes, \oplus)$$

where the isomorphism $\pi : B \to \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B})$ is defined, for all $a \in B$, as $\pi(a) := \langle \mathrm{reg}(a), \mathrm{reg}(\neg a)\rangle$. The inverse map $f : \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B}) \to B$ is defined, for all $\langle a, b\rangle \in \mathrm{Reg}(\mathbb{B}) \times \mathrm{Reg}(\mathbb{B})$, as

$$f(\langle a, b\rangle) := (a \otimes (a \vee b)) \oplus (b \otimes (a \wedge b)).$$

3.2. Heterogeneous bilattices

Definition 3.1. A *heterogeneous bilattice* (HBL) is a tuple $\mathbb{H} = (\mathbb{L}_1,\ \mathbb{L}_2,\ \mathrm{n},\ \mathrm{p})$ satisfying the following conditions:

(H1) $\mathbb{L}_1, \mathbb{L}_2$ are bounded distributive lattices.
(H2) $\mathrm{n} : \mathbb{L}_1 \to \mathbb{L}_2$ and $\mathrm{p} : \mathbb{L}_2 \to \mathbb{L}_1$ are mutually inverse lattice isomorphisms.

We let HBL denote the class of HBLs. An HBL is *perfect* if:

(H3) both $\mathbb{L}_1$ and $\mathbb{L}_2$ are perfect lattices[3];
(H4) p, n are complete lattice isomorphisms.

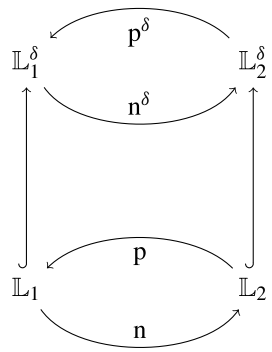
[3] A distributive lattice $\mathbb{A}$ is *perfect* (cf. [23]) if it is complete, completely distributive and completely join-generated by the set $J^{\infty}(\mathbb{A})$ of its completely join-irreducible elements (as well as completely meet-generated by the set $M^{\infty}(\mathbb{A})$ of its completely meet-irreducible elements).

By (H2) we have that $\mathrm{np} = \mathrm{Id}_{\mathbb{L}_1}$ and $\mathrm{pn} = \mathrm{Id}_{\mathbb{L}_2}$, from which it straightforwardly follows that n and p are both right and left adjoints of each other. The definition of *the heterogeneous bilattice with conflation* (HCBL) is analogous, except that we replace (H1) with the following condition:

(H1′) $\mathbb{L}_1$ and $\mathbb{L}_2$ are De Morgan algebras, with De Morgan negations denoted $\sim_1$ and $\sim_2$ respectively.

We let HCBL denote the class of HCBLs. In what follows, we let $\mathbb{L}^\delta$ denote the canonical extension of the lattice $\mathbb{L}$. The following lemma is an easy consequence of the results in [24, Theorems 2.3 and 3.2].

Lemma 3.2. *If* $(\mathbb{L}_1, \mathbb{L}_2, \mathrm{n}, \mathrm{p})$ *is an HBL (HCBL), then* $(\mathbb{L}_1^\delta, \mathbb{L}_2^\delta, \mathrm{n}^\delta, \mathrm{p}^\delta)$ *is a perfect HBL (resp. HCBL).*



3.3. Equivalence of the two presentations

The following result is a straightforward verification of Definition 3.1.

Proposition 3.3. *For any bounded distributive bilattice* $\mathbb{B}$*, the tuple*

$$\mathbb{B}^+ = (\mathrm{Reg}(\mathbb{B}),\ \mathrm{Reg}(\mathbb{B}),\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})},\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})})$$

is an HBL., where $\sqcap_1 = \sqcap_2 = \otimes, \sqcup_1 = \sqcup_2 = \oplus, 1_1 = 1_2 = \top$ *and* $0_1 = 0_2 = \bot$.

For any CB $\mathbb{B}$*, the tuple*

$$\mathbb{B}^+ = ((\mathrm{Reg}(\mathbb{B}), \sim_1),\ (\mathrm{Reg}(\mathbb{B}), \sim_2),\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})},\ \mathrm{Id}_{\mathrm{Reg}(\mathbb{B})})$$

is an HCBL, where $\sim_2 = \sim_1 = -$.

Proposition 3.4. *If* $(\mathbb{L}_1,\ \mathbb{L}_2,\ n,\ p)$ *is an HBL (resp. HCBL), then* $L_1 \times L_2$ *is a bilattice (resp. a bilattice with conflation) when endowed with the following structure:*

$$\begin{array}{rcl}
\langle a_1, a_2\rangle \otimes \langle b_1, b_2\rangle & := & \langle a_1 \sqcap_1 b_1,\ a_2 \sqcap_2 b_2\rangle \\
\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle & := & \langle a_1 \sqcup_1 b_1,\ a_2 \sqcup_2 b_2\rangle \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle & := & \langle a_1 \sqcap_1 b_1,\ a_2 \sqcup_2 b_2\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle & := & \langle a_1 \sqcup_1 b_1,\ a_2 \sqcap_2 b_2\rangle \\
\neg\langle a_1, a_2\rangle & := & \langle p(a_2), n(a_1)\rangle \\
-\langle a_1, a_2\rangle & := & \langle \mathrm{p}(\sim_2 a_2), \mathrm{n}(\sim_1 a_1)\rangle \\
f & := & \langle 0, 1\rangle \\
t & := & \langle 1, 0\rangle \\
\bot & := & \langle 0, 0\rangle \\
\top & := & \langle 1, 1\rangle
\end{array}$$

A lattice isomorphism $h : \mathbb{L} \to \mathbb{L}'$ is *complete* if it satisfies the following properties for each $X \subseteq \mathbb{L}$:

$$h(\bigvee X) = \bigvee h(X) \qquad h(\bigwedge X) = \bigwedge h(X).$$

Proof. Firstly, we show that $\langle L_1 \times L_2, \otimes, \oplus\rangle$ and $\langle L_1 \times L_2, \wedge, \vee\rangle$ are bounded distributive lattices. It is obvious that they are both bounded lattices. We only need to show that the distributivity law holds. We have:

$$
\begin{array}{lll}
 & \langle a_1,\ a_2\rangle \otimes (\langle b_1,\ b_2\rangle \oplus (\langle c_1,\ c_2\rangle) & \\
= & \langle a_1,\ a_2\rangle \otimes (\langle b_1 \sqcup_1 c_1,\ b_2 \sqcup_2 c_2\rangle) & \text{Def. of } \oplus \\
= & \langle a_1 \sqcap_1 (b_1 \sqcup_1 c_1),\ a_2 \sqcap_2 (b_2 \sqcup_2 c_2)\rangle & \text{Def. of } \otimes \\
= & \langle (a_1 \sqcap_1 b_1) \sqcup_1 (a_1 \sqcap_1 c_1),\ (a_2 \sqcap_2 b_2) \sqcup_2 (a_2 \sqcap_2 c_2)\rangle & \text{Distributivity of } \mathbb{L}_1 \text{ and } \mathbb{L}_2 \\
= & \langle (a_1 \sqcap_1 b_1),\ (a_2 \sqcap_2 b_2)\rangle \oplus \langle (a_1 \sqcap_1 c_1),\ (a_2 \sqcap_2 c_2)\rangle & \text{Def. of } \oplus \\
= & (\langle a_1,\ a_2\rangle \otimes \langle b_1,\ b_2\rangle) \oplus (\langle a_1,\ a_2\rangle \otimes \langle c_1,\ c_2\rangle) & \text{Def. of } \otimes
\end{array}
$$

As to $\langle L_1 \times L_2, \wedge, \vee\rangle$, the argument is analogous.

Now we show that the properties of $\neg$ are also met. Assume that $\langle a_1,\ a_2\rangle \leq_t \langle b_1,\ b_2\rangle$, equivalently, $a_1 \leq_1 b_1$ and $b_2 \leq_2 a_2$. By the definition of $\neg$, we have $\neg\langle a_1,\ a_2\rangle = \langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle$ and $\neg\langle b_1,\ b_2\rangle = \langle \mathrm{p}(b_2),\ \mathrm{n}(b_1)\rangle$. Hence $\mathrm{p}(b_2) \leq_1 \mathrm{p}(a_2)$ and $\mathrm{n}(a_1) \leq_2 \mathrm{n}(b_1)$ by (H2). Thus $\neg\langle b_1,\ b_2\rangle \leq_t \neg\langle a_1,\ a_2\rangle$. A similar reasoning shows that the corresponding property involving $\neg$ and $\leq_k$ also holds. The following argument shows that $\neg$ is involutive.

$$
\begin{array}{lll}
 & \neg\neg\langle a_1,\ a_2\rangle & \\
= & \neg\langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= & \langle \mathrm{pn}(a_1),\ \mathrm{np}(a_2)\rangle & \text{Def. of } \neg \\
= & \langle a_1,\ a_2\rangle & \mathrm{np} = \mathrm{Id}_{\mathbb{L}_1} \text{ and } \mathrm{pn} = \mathrm{Id}_{\mathbb{L}_2}
\end{array}
$$

As to conflation, assume $\langle a_1,\ a_2\rangle \leq_t \langle b_1,\ b_2\rangle$, equivalently, $a_1 \leq_1 b_1$ and $b_2 \leq_2 a_2$. By the definition of $-$ we have $-\langle a_1,\ a_2\rangle = \langle \mathrm{p}(\sim_2 a_2),\ \mathrm{n}(\sim_1 a_1)\rangle$ and $-\langle b_1,\ b_2\rangle = \langle \mathrm{p}(\sim_2 b_2),\ \mathrm{n}(\sim_1 b_1)\rangle$. Hence $\mathrm{p}(\sim_2 a_2) \leq_1 \mathrm{p}(\sim_2 b_2)$ and $\mathrm{n}(\sim b_1) \leq_2 \mathrm{n}(\sim b_2)$ by (H2). Thus $-\langle a_1,\ a_2\rangle \leq_t -\langle b_1,\ b_2\rangle$. A similar reasoning shows that the corresponding property involving $-$ and $\leq_k$ also holds. The following arguments show that $-$ is involutive and $-$ and $\neg$ are commutative.

$$
\begin{array}{ll}
\ \ - - \langle a_1,\ a_2\rangle & \\
= -\langle \mathrm{p}(\sim_2 a_2),\ \mathrm{n}(\sim_1 a_1)\rangle & \text{Def. of } - \\
= \langle \mathrm{p}(\sim_2 \mathrm{n}(\sim_1 a_1)),\ \mathrm{n}(\sim_1 \mathrm{p}(\sim_2 a_2))\rangle & \text{Def. of } - \\
= \langle \mathrm{p}(\sim_2\sim_2 \mathrm{n}(a_1)),\ \mathrm{n}(\sim_1\sim_1 \mathrm{p}(a_2))\rangle & \text{H2} \\
= \langle \mathrm{pn}(a_1),\ \mathrm{np}(a_2)\rangle & \text{H1} \\
= \langle a_1,\ a_2\rangle & \mathrm{np} = \mathrm{Id}_{\mathbb{L}_1} \text{ and } \mathrm{pn} = \mathrm{Id}_{\mathbb{L}_2}
\end{array}
$$

$$
\begin{array}{ll}
\ \ -\neg\langle a_1,\ a_2\rangle & \\
= -\langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= \langle \mathrm{p}(\sim_2 \mathrm{n}(a_1)),\ \mathrm{n}(\sim_1 \mathrm{p}(a_2))\rangle & \text{Def. of } - \\
= \neg\langle \sim_1 \mathrm{p}(a_2),\ \sim_2 \mathrm{n}(a_1)\rangle & \text{Def. of } \neg \\
= \neg\langle \mathrm{p}(\sim_2 a_2),\ \mathrm{n}(\sim_1 a_2)\rangle & \text{H2} \\
= \neg - \langle a_1,\ a_2\rangle & \text{Def. of } - \quad \square
\end{array}
$$

Definition 3.5. For any HBL $\mathbb{H} = (\mathbb{L}_1,\ \mathbb{L}_2,\ n,\ p)$, we let $\mathbb{H}_+ = (L_1 \times L_2,\ \wedge,\ \vee,\ \otimes,\ \oplus,\ \neg)$ denote the product algebra where the four lattice operations are defined as in $\mathbb{L}_1 \odot \mathbb{L}_2$ (Theorem 2.4) and the negation is given by $\neg\langle a_1, a_2\rangle := \langle \mathrm{p}(a_2),\ \mathrm{n}(a_1)\rangle$ for all $\langle a_1, a_2\rangle \in L_1 \times L_2$. If $\mathbb{L}_1$ and $\mathbb{L}_2$ are isomorphic De Morgan algebras, then we define $\mathbb{H}_+ = (L_1 \times L_2,\ \wedge,\ \vee,\ \otimes,\ \oplus,\ \neg,\ -)$ as before, with the conflation given by $-\langle a_1, a_2\rangle := \langle \mathrm{p}(\sim_2 a_2),\ \mathrm{n}(\sim_1 a_1)\rangle$ for all $\langle a_1, a_2\rangle \in L_1 \times L_2$.

Proposition 3.6. *For any $\mathbb{B} \in \mathsf{B}$ (resp. $\mathbb{B} \in \mathsf{CB}$) and any HBL (resp. HCBL) $\mathbb{H}$, we have*

$$\mathbb{B} \cong (\mathbb{B}^+)_+ \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

Proof. Immediately follows from Propositions 3.3 and 3.4. $\square$

4. Multi-type bilattice logic

The results of Section 3.3 show that HBL (resp. HCBL) is an equivalent presentation of B (resp. CB), and motivate from a semantic perspective the syntactic shift we take in the present section, from a single-type language to a multi-type language.[4] Indeed, heterogeneous algebras provide a natural interpretation for the following multi-type language $\mathcal{L}_{MT}$ consisting of terms of types L_1 and L_2.

$$\mathsf{L}_1 \ni A_1 ::= p_1 \mid 1_1 \mid 0_1 \mid \mathrm{p}A_2 \mid \sim_1 A_1 \mid A_1 \sqcap_1 A_1 \mid A_1 \sqcup_1 A_1$$
$$\mathsf{L}_2 \ni A_2 ::= p_2 \mid 1_2 \mid 0_2 \mid \mathrm{n}A_1 \mid \sim_2 A_2 \mid A_2 \sqcap_2 A_2 \mid A_2 \sqcup_2 A_2$$

The interpretation of $\mathcal{L}_{MT}$-terms into HCBLs is defined as the easy generalization of the interpretation of propositional languages in universal algebra; namely, L_1-terms (resp. L_2-terms) are interpreted in the first and second De Morgan algebras of any HCBL, respectively.

The toggle between CB and HCBL (cf. Sections 3.3) is reflected syntactically by the translations $t_1(\cdot), t_2(\cdot) : \mathcal{L} \to \mathcal{L}_{MT}$ defined as follows:

$t_1(p)$	$:=$	p_1	$t_2(p)$	$:=$	p_2
$t_1(\mathtt{t})$	$:=$	1_1	$t_2(\mathtt{t})$	$:=$	0_2
$t_1(\mathtt{f})$	$:=$	0_1	$t_2(\mathtt{f})$	$:=$	1_2
$t_1(\top)$	$:=$	1_1	$t_2(\top)$	$:=$	1_2
$t_1(\bot)$	$:=$	0_1	$t_2(\bot)$	$:=$	0_2
$t_1(A \wedge B)$	$:=$	$t_1(A) \sqcap_1 t_1(B)$	$t_2(A \wedge B)$	$:=$	$t_2(A) \sqcup_2 t_2(B)$
$t_1(A \vee B)$	$:=$	$t_1(A) \sqcup_1 t_1(B)$	$t_2(A \vee B)$	$:=$	$t_2(A) \sqcap_2 t_2(B)$
$t_1(A \otimes B)$	$:=$	$t_1(A) \sqcap_1 t_1(B)$	$t_2(A \otimes B)$	$:=$	$t_2(A) \sqcap_2 t_2(B)$
$t_1(A \oplus B)$	$:=$	$t_1(A) \sqcup_1 t_1(B)$	$t_2(A \oplus B)$	$:=$	$t_2(A) \sqcup_2 t_2(B)$
$t_1(\neg A)$	$:=$	$\mathrm{p}t_2(A)$	$t_2(\neg A)$	$:=$	$\mathrm{n}t_1(A)$
$t_1(-A)$	$:=$	$\mathrm{p}{\sim_2}t_2(A)$	$t_2(-A)$	$:=$	$\mathrm{n}{\sim_1}t_1(A)$

The translations above are compatible with the toggle between B (resp. CB) and HBL (resp. HCBL). Indeed, recall that $\mathbb{B}^+$ denotes the heterogeneous algebra associated with a given $\mathbb{B} \in \mathsf{B}$ (cf. Definition 3.5). The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 4.1. *For all $\mathcal{L}$-formulas A and B and every $\mathbb{B} \in \mathsf{B}$ (resp. $\mathbb{B} \in \mathsf{CB}$),*

$$\mathbb{B} \models A \leq B \quad \textit{iff} \quad \mathbb{B}^+ \models t_1(A) \leq t_1(B).$$

5. Multi-type proper display calculus

In this section we introduce the proper display calculus D.BL (D.CBL) for bilattice logic (with conflation).

5.1. Language

The language $\mathcal{L}_{MT}$ of D.CBL is given by the union of the sets $\mathcal{L}_1$ and $\mathcal{L}_2$ defined as follows. $\mathcal{L}_1$ is given by simultaneous induction over the set $\mathsf{AtProp}_1 = \{p_1, q_1, r_1, \ldots\}$ of L_1-type atomic propositions as follows:

$$A_1 ::= p_1 \mid 1_1 \mid 0_1 \mid \mathrm{p}\, A_2 \mid \sim_1 A_1 \mid A_1 \sqcap_1 A_1 \mid A_1 \sqcup_1 A_1$$
$$X_1 ::= A_1 \mid \hat{1}_1 \mid \check{0}_1 \mid \mathrm{P}\, X_2 \mid *_1 X_1 \mid X_1 \hat{\sqcap}_1 X_1 \mid X_1 \check{\sqcup}_1 X_1 \mid X_1 \check{\sqsupset}_1 X_1 \mid X_1 \hat{\sqsubset}_1 X_1$$

$\mathcal{L}_2$ is given by simultaneous induction over the set $\mathsf{AtProp}_2 = \{p_2, q_2, r_2, \ldots\}$ of L_2-type atomic propositions as follows:

$$A_2 ::= p_2 \mid 1_2 \mid 0_2 \mid \mathrm{n}\, A_1 \mid \sim_2 A_2 \mid A_2 \sqcap_2 A_2 \mid A_2 \sqcup_2 A_2$$
$$X_2 ::= A_2 \mid \hat{1}_2 \mid \check{0}_2 \mid \mathrm{N}\, X_1 \mid *_2 X_2 \mid X_2 \hat{\sqcap}_2 X_2 \mid X_2 \check{\sqcup}_2 X_2 \mid X_2 \check{\sqsupset}_2 X_2 \mid X_2 \hat{\sqsubset}_2 X_2$$

The language of D.BL is the $\{*_1, *_2, \sim_1, \sim_2\}$-free fragment of $\mathcal{L}_{MT}$.

[4] In what follows, we only introduce the multi-type language associated with HCBL. The language associated with HBL can be obtained by removing the unary operators $\sim_1$ and $\sim_2$.

5.2. Rules

For $i \in \{1, 2\}$,

- Pure L_i-type display rules

$$\text{res}\ \genfrac{}{}{3pt}{}{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i}{X_i \vdash Y_i \mathbin{\check{\sqsupset}} Z_i} \qquad \genfrac{}{}{3pt}{}{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}{X_i \mathbin{\hat{\sqsubset}_i} Y_i \vdash Z_i}\ \text{res}$$

- Multi-type display rules

$$\text{adj}\ \genfrac{}{}{3pt}{}{\mathrm{P}\, X_2 \vdash Y_1}{X_2 \vdash \mathrm{N}\, Y_1} \qquad \genfrac{}{}{3pt}{}{\mathrm{N}\, X_1 \vdash Y_2}{X_1 \vdash \mathrm{P}\, Y_2}\ \text{adj}$$

- Pure L_i-type identity and cut rules

$$\text{Id}_\text{i}\ \frac{}{p_i \vdash p_i} \qquad \frac{X_i \vdash A_i \qquad A_i \vdash Y_i}{X_i \vdash Y_i}\ \text{Cut}$$

- Pure L_i-type structural rules

$$\hat{1}_i\ \frac{X_i \mathbin{\hat{\sqcap}_i} \hat{1}_i \vdash Y_i}{X_i \vdash Y_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} \check{0}_i}{X_i \vdash Y_i}\ \check{0}_i$$

$$\text{E}\ \frac{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i}{Y_i \mathbin{\hat{\sqcap}_i} X_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}{X_i \vdash Z_i \mathbin{\check{\sqcup}_i} Y_i}\ \text{E}$$

$$\text{A}\ \frac{(X_i \mathbin{\hat{\sqcap}_i} Y_i) \mathbin{\hat{\sqcap}_i} Z_i \vdash W_i}{X_i \mathbin{\hat{\sqcap}_i} (Y_i \mathbin{\hat{\sqcap}_i} Z_i) \vdash W_i} \qquad \frac{X_i \vdash (Y_i \mathbin{\check{\sqcup}_i} Z_i) \mathbin{\check{\sqcup}_i} W_i}{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} (Z_i \mathbin{\check{\sqcup}_i} W_i)}\ \text{A}$$

$$\text{W}\ \frac{X_i \vdash Z_i}{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i}{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Z_i}\ \text{W}$$

$$\text{C}\ \frac{X_i \mathbin{\hat{\sqcap}_i} X_i \vdash Z_i}{X_i \vdash Z_i} \qquad \frac{X_i \vdash Y_i \mathbin{\check{\sqcup}_i} Y_i}{X_i \vdash Y_i}\ \text{C}$$

- Pure L_i type operational rules

$$1_i\ \frac{\hat{1}_i \vdash X_i}{1_i \vdash X_i} \qquad \frac{}{\hat{1}_i \vdash 1_i}\ 1_i$$

$$0_i\ \frac{}{0_i \vdash \check{0}_i} \qquad \frac{X_i \vdash \check{0}_i}{X_i \vdash 0_i}\ 0_i$$

$$\sqcap_i\ \frac{A_i \mathbin{\hat{\sqcap}_i} B_i \vdash X_i}{A_i \sqcap_i B_i \vdash X_i} \qquad \frac{X_i \vdash A_i \qquad Y_i \vdash B_i}{X_i \mathbin{\hat{\sqcap}_i} Y_i \vdash A_i \sqcap_i B_i}\ \sqcap_i$$

$$\sqcup_i\ \frac{A_i \vdash X_i \qquad B_i \vdash Y_i}{A_i \sqcup_i B_i \vdash X_i \mathbin{\check{\sqcup}_i} Y_i} \qquad \frac{X_i \vdash A_i \mathbin{\check{\sqcup}_i} B_i}{X_i \vdash A_i \sqcup_i B_i}\ \sqcup_i$$

- Multi-type structural rules

$$\text{N}\ \genfrac{}{}{3pt}{}{X_1 \vdash Y_1}{\mathrm{N}\, X_1 \vdash \mathrm{N}\, Y_1} \qquad \genfrac{}{}{3pt}{}{X_2 \vdash Y_2}{\mathrm{P}\, X_2 \vdash \mathrm{P}\, Y_2}\ \text{P}$$

$$\mathrm{P}\hat{1}_2\ \frac{\hat{1}_1 \vdash X_1}{\mathrm{P}\, \hat{1}_2 \vdash X_1} \qquad \frac{X_1 \vdash \check{0}_1}{X_1 \vdash \mathrm{P}\, \check{0}_2}\ \mathrm{P}\check{0}_2$$

- Multi-type operational rules

$$\text{n}\,\frac{\mathrm{N}\,A_1 \vdash X_2}{\mathrm{n}\,A_1 \vdash X_2} \qquad \frac{X_2 \vdash \mathrm{N}\,A_1}{X_2 \vdash \mathrm{n}\,A_1}\,\text{n}$$

$$\text{p}\,\frac{\mathrm{P}\,A_2 \vdash X_1}{\mathrm{p}\,A_2 \vdash X_1} \qquad \frac{X_1 \vdash \mathrm{P}\,A_2}{X_1 \vdash \mathrm{p}\,A_2}\,\text{p}$$

The multi-type display calculus D.CBL also includes the following rules:

- Pure L_i display structural rules:

$$\text{adj}*\,\frac{*_i\, X_i \vdash Y_i}{*_i\, Y_i \vdash X_i} \qquad \frac{X_i \vdash *_i\, Y_i}{Y_i \vdash *_i\, X_i}\,\text{adj}*$$

- Pure L_i structural rules:

$$\text{cont}\,\frac{X_i \vdash Y_i}{*_i\, Y_i \vdash *_i\, X_i}$$

- Multi-type structural rules:

$$*_2\mathrm{N}\,\frac{\mathrm{N} *_1 X_1 \vdash Y_2}{*_2\,\mathrm{N}\,X_1 \vdash Y_2} \qquad \frac{X_2 \vdash \mathrm{N} *_1 Y_1}{X_2 \vdash *_2\,\mathrm{N}\,Y_1}\,*_2\mathrm{N}$$

- Pure L_i operational rules:

$$\sim_i\,\frac{*_i\; X_i \vdash Y_i}{\sim_i\; X_i \vdash Y_i} \qquad \frac{X_i \vdash *_i\; Y_i}{X_i \vdash \sim_i Y_i}\,\sim_i$$

An essential feature of our calculus is that the logical rules are standard introduction rules of display calculi. This is key for achieving a canonical proof of cut elimination. The special behaviour of negation is captured by a suitable translation in a multi-type environment, which makes it possible to circumvent the technical difficulties created by the non-standard introduction rules of [1].

6. Properties

In this section, we sketch the proofs of the main properties of the calculi D.BL and D.CBL. We only sketch them since these proofs are instances of general facts of the theory of multi-type calculi.

6.1. Soundness

We outline the verification of soundness of the rules of D.BL (resp. D.CBL) w.r.t. the semantics of *perfect* HBL (resp. HCBL). The first step consists in interpreting structural symbols as their corresponding logical symbols. This induces a natural interpretation of structural terms as logical/algebraic terms, which we omit. Then we interpret sequents as inequalities, and rules as quasi-inequalities. The verification of soundness of the rules of D.BL (resp. D.CBL) then consists in checking the validity of their corresponding quasi-inequalities in perfect HBL (resp. HCBL). For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{\mathrm{P}\,X_2 \vdash Y_1}{X_2 \vdash \mathrm{N}\,Y_1} \rightsquigarrow \forall a_2 \forall b_1\,[\mathrm{p}(a_2) \leq_1 b_1 \;\Leftrightarrow\; a_2 \leq_2 \mathrm{n}(b_1)]$$

$$\frac{X_i \vdash Y_i}{*_i\, Y_i \vdash *_i\, X_i} \rightsquigarrow \forall a_i \forall b_i\,[a_i \leq_i b_i \;\Leftrightarrow\; \sim_i b_i \leq_i \sim_i a_i]$$

The verification of soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to multi-type structural rules follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm

ALBA [29, Section 3.4] on one of the defining inequalities of HBL (resp. HCBL).[5] For instance, the soundness of the first rule above is due to p and n being inverse to each other (see discussion after Definition 3.1).

6.2. Completeness

Proposition 6.1. *For every formula A of* BL *(resp. CBL), the sequents $t_1(A) \vdash t_1(A)$ and $t_2(A) \vdash t_2(A)$ are derivable in D.BL (resp. D.CBL).*

Proof. By induction on the complexity of the formula A. If A is an atomic formula, the translation of $t_i(A) \vdash t_i(A)$ with $i \in \{1, 2\}$ is $A_i \vdash A_i$, which is derivable using (Id) in L_1 and L_2, respectively. If $A = B \otimes C$, then $t_i(B \otimes C) = t_i(B) \sqcap_i t_i(C)$ and if $A = B \oplus C$, then $t_i(B \oplus C) = t_i(B) \sqcup_i t_i(C)$. By induction hypothesis, $t_i(A_i) \vdash t_i(A_i)$. The following derivations complete the proof:

$$\dfrac{\dfrac{\dfrac{\text{W}\,\dfrac{t_i(B) \vdash t_i(B)}{t_i(B)\,\hat{\sqcap}_i\, t_i(C) \vdash t_i(B)} \qquad \text{E}\,\dfrac{\text{W}\,\dfrac{t_i(C) \vdash t_i(C)}{t_i(C)\,\hat{\sqcap}_i\, t_i(B) \vdash t_i(C)}}{t_i(B)\,\hat{\sqcap}_i\, t_i(C) \vdash t_i(C)}}{(t_i(B)\,\hat{\sqcap}_i\, t_i(C))\,\hat{\sqcap}_i\,(t_i(B)\,\hat{\sqcap}_i\, t_i(C)) \vdash t_i(B) \sqcap_i t_i(C)}\,\text{C}}{t_i(B)\,\hat{\sqcap}_i\, t_i(C) \vdash t_i(B) \sqcap_i t_i(C)}}{t_i(B) \sqcap_i t_i(C) \vdash t_i(B) \sqcap_i t_i(C)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{t_i(B) \vdash t_i(B)}{t_i(B) \vdash t_i(B)\,\check{\sqcup}_i\, t_i(C)}\,\text{W} \qquad \dfrac{\dfrac{t_i(C) \vdash t_i(C)}{t_i(C) \vdash t_i(C)\,\check{\sqcup}_i\, t_i(B)}\,\text{W}}{t_i(C) \vdash t_i(B)\,\check{\sqcup}_i\, t_i(C)}\,\text{E}}{t_i(B) \sqcup_i t_i(C) \vdash (t_i(B)\,\check{\sqcup}_i\, t_i(C))\,\check{\sqcup}_i\,(t_i(B)\,\check{\sqcup}_i\, t_i(C))}\,\text{C}}{t_i(B) \sqcup_i t_i(C) \vdash t_i(B)\,\check{\sqcup}_i\, t_i(C)}}{t_i(B) \sqcup_i t_i(C) \vdash t_i(B) \sqcup_i t_i(C)}$$

The arguments for $A = B \wedge C$ and $A = B \vee C$ are similar and they are omitted.

If $A = \neg B$, then $t_1(\neg B) = \mathrm{p}t_2(B)$ and $t_2(\neg B) = \mathrm{n}t_1(B)$. By induction hypothesis $t_i(A) \vdash t_i(A)$. Hence, the following derivations complete the proof:

$$\dfrac{\dfrac{\dfrac{t_2(B) \vdash t_2(B)}{\mathrm{P}\,t_2(B) \vdash \mathrm{P}\,t_2(B)}\,\text{P}}{\mathrm{P}\,t_2(B) \vdash \mathrm{p}\,t_2(B)}}{\mathrm{p}\,t_2(B) \vdash \mathrm{p}\,t_2(B)} \qquad \dfrac{\dfrac{\text{N}\,\dfrac{t_1(B) \vdash t_1(B)}{\mathrm{N}\,t_1(B) \vdash \mathrm{N}\,t_1(B)}}{\mathrm{N}\,t_1(B) \vdash \mathrm{n}\,t_1(B)}}{\mathrm{n}\,t_1(B) \vdash \mathrm{n}\,t_1(B)}$$

If $A = -B$, then $t_1(-B) = \mathrm{p} \sim_2 t_2(B)$ and $t_2(-B) = \mathrm{n} \sim_1 t_1(B)$. By induction hypothesis $t_i(B) \vdash t_i(B)$. Hence, the following derivations complete the proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t_2(B) \vdash t_2(B)}{*_2\, t_2(B) \vdash *_2\, t_2(B)}\,\text{cont}}{*_2\, t_2(B) \vdash \sim_2 t_2(B)}}{\sim_2 t_2(B) \vdash \sim_2 t_2(B)}}{\mathrm{P} \sim_2 t_2(B) \vdash \mathrm{P} \sim_2 t_2(B)}\,\text{P}}{\mathrm{p} \sim_2 t_2(B) \vdash \mathrm{P} \sim_2 t_2(B)}}{\mathrm{p} \sim_2 t_2(B) \vdash \mathrm{p} \sim_2 t_2(B)} \qquad \dfrac{\dfrac{\text{N}\,\dfrac{\dfrac{\dfrac{\dfrac{t_1(B) \vdash t_1(B)}{*_1\, t_1(B) \vdash *_1\, t_1(B)}\,\text{cont}}{*_1\, t_1(B) \vdash \sim_1 t_1(B)}}{\sim_1 t_2(B) \vdash \sim_1 t_1(B)}}{\mathrm{N} \sim_1 t_1(B) \vdash \mathrm{N} \sim_1 t_1(B)}}{\mathrm{N} \sim_1 t_1(B) \vdash \mathrm{n} \sim_1 t_1(B)}}{\mathrm{n} \sim_1 t_1(B) \vdash \mathrm{n} \sim_1 t_1(B)} \qquad \square$$

Proposition 6.2. *For all formulas A, B of* BL *(resp. CBL), if $A \vdash B$ is derivable in BL (resp. CBL), then $t_1(A) \vdash t_1(B)$ is derivable in D.BL (resp. D.CBL).*

[5] As discussed in [29], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

Proof. In what follows we show that the translations of the axioms and rules of BL (resp C.BL) are derivable in D.BL (resp. D.CBL). Since BL (resp C.BL) is complete w.r.t. the class of bilattice algebras (by Theorem 2.10), and hence w.r.t. their associated heterogeneous algebras (by Propositions 3.3 and 3.4), this is enough to show the completeness of D.BL (resp. D.CBL). For the sake of readability, in what follows, for each BL-formula A and $i \in \{1, 2\}$, we let $t_i(A) := A_i$.

The identity axiom $A \vdash A$ is proved in Proposition 6.1.

The derivations of the binary rules are standard and we omit them.

As to $\mathtt{f} \vdash A$, by the translation, $t_1(\mathtt{f}) = 0_1$, hence, the sequent $0_1 \vdash A_1$ is derivable in D.BL as follows

$$\begin{array}{cl}
0_1 \vdash \check{0}_1 & \\ \hline
0_1 \vdash \check{0}_1 \check{\sqcup}_1 A_1 & \mathrm{W} \\ \hline
0_1 \vdash A_1 & \check{0}_1
\end{array}$$

The proofs of the translations of $A \vdash \mathtt{t}$, $\bot \vdash A$ and $A \vdash \top$ are analogous.

As to $A \vdash \neg\mathtt{f}$, by the translation, $t_1(\neg\mathtt{f}) = \mathrm{p}t_2(\mathtt{f}) = \mathrm{p}1_2$, hence we can prove $A_1 \vdash \mathrm{p}1_2$ by the introduction rule of (1_2) on the left side, (W), $(\hat{1}_2)$, (adj), (p) and the introduction rule of (1_2) on the right side. The proofs of $\neg\mathtt{t} \vdash A$, $\neg\bot \vdash A$, $A \vdash \neg\top$ are analogous.

In what follows, we let the sequent on the right side of $\rightsquigarrow$ denote the result of the translation. The translations of the remaining axioms in BL are derivable in D.BL as follows:

$\neg\neg A \dashv\vdash A \quad \rightsquigarrow \quad \mathrm{p}\,\mathrm{n}\,A_1 \dashv\vdash A_1$

$$\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\mathrm{N}\,A_1 \vdash \mathrm{N}\,A_1 & \mathrm{N} \\ \hline
\mathrm{n}\,A_1 \vdash \mathrm{N}\,A_1 & \\ \hline
\mathrm{P}\,\mathrm{n}\,A_1 \vdash A_1 & \mathrm{adj} \\ \hline
\mathrm{p}\,\mathrm{n}\,A_1 \vdash A_1 &
\end{array}
\qquad
\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\mathrm{N}\,A_1 \vdash \mathrm{N}\,A_1 & \mathrm{N} \\ \hline
\mathrm{N}\,A_1 \vdash \mathrm{n}\,A_1 & \\ \hline
A_1 \vdash \mathrm{P}\,\mathrm{n}\,A_1 & \mathrm{adj} \\ \hline
A_1 \vdash \mathrm{p}\,\mathrm{n}\,A_1 &
\end{array}$$

$- - A \dashv\vdash A \quad \rightsquigarrow \quad \mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1 \dashv\vdash A_1$

$$\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\ast_1 A_1 \vdash \ast_1 A_1 & \mathrm{cont} \\ \hline
\ast_1 A_1 \vdash \sim_1 A_1 & \\ \hline
\mathrm{N} \ast_1 A_1 \vdash \mathrm{N} \sim_1 A_1 & \mathrm{N} \\ \hline
\mathrm{N} \ast_1 A_1 \vdash \mathrm{n} \sim_1 A_1 & \\ \hline
\ast_2 \mathrm{N} A_1 \vdash \mathrm{n} \sim_1 A_1 & \ast_2\mathrm{N} \\ \hline
\ast_2 \mathrm{n} \sim_1 A_1 \vdash \mathrm{N} A_1 & \mathrm{adj}\ast \\ \hline
\sim_2 \mathrm{n} \sim_1 A_1 \vdash \mathrm{N} A_1 & \\ \hline
\mathrm{P} \sim_2 \mathrm{n} \sim_1 A_1 \vdash A_1 & \mathrm{adj} \\ \hline
\mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1 \vdash A_1 &
\end{array}
\qquad
\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\ast_1 A_1 \vdash \ast_1 A_1 & \mathrm{cont} \\ \hline
\sim_1 A_1 \vdash \ast_1 A_1 & \\ \hline
\mathrm{N} \sim_1 A_1 \vdash \mathrm{N} \ast_1 A_1 & \mathrm{N} \\ \hline
\mathrm{n} \sim_1 A_1 \vdash \mathrm{N} \ast_1 A_1 & \\ \hline
\mathrm{n} \sim_1 A_1 \vdash \ast_2 \mathrm{N} A_1 & \ast_2\mathrm{N} \\ \hline
\mathrm{N} A_1 \vdash \ast_2 \mathrm{n} \sim_1 A_1 & \mathrm{adj}\ast \\ \hline
\mathrm{N} A_1 \vdash \sim_2 \mathrm{n} \sim_1 A_1 & \\ \hline
A_1 \vdash \mathrm{P} \sim_2 \mathrm{n} \sim_1 A_1 & \mathrm{adj} \\ \hline
A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} \sim_1 A_1 &
\end{array}$$

$-\neg A \dashv\vdash \neg - A \quad \rightsquigarrow \quad \mathrm{p} \sim_2 \mathrm{n}\,A_1 \dashv\vdash \mathrm{p}\,\mathrm{n} \sim_1 A_1$

$$\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\ast_1 A_1 \vdash \ast_1 A_1 & \mathrm{cont} \\ \hline
\ast_1 A_1 \vdash \sim_1 A_1 & \\ \hline
\ast_1 \sim_1 A_1 \vdash A_1 & \ast \\ \hline
\mathrm{N} \ast_1 \sim_1 A_1 \vdash \mathrm{N} A_1 & \mathrm{N} \\ \hline
\mathrm{N} \ast_1 \sim_1 A_1 \vdash \mathrm{n} A_1 & \\ \hline
\ast_2 \mathrm{N} \sim_1 A_1 \vdash \mathrm{n} A_1 & \ast_2\mathrm{N} \\ \hline
\ast_2 \mathrm{n} A_1 \vdash \mathrm{N} \sim_1 A_1 & \mathrm{adj}\ast \\ \hline
\sim_2 \mathrm{n} A_1 \vdash \mathrm{N} \sim_1 A_1 & \\ \hline
\sim_2 \mathrm{n} A_1 \vdash \mathrm{n} \sim_1 A_1 & \\ \hline
\mathrm{P} \sim_2 \mathrm{n} A_1 \vdash \mathrm{P}\mathrm{n} \sim_1 A_1 & \mathrm{P} \\ \hline
\mathrm{P} \sim_2 \mathrm{n} A_1 \vdash \mathrm{p}\mathrm{n} \sim_1 A_1 & \\ \hline
\mathrm{p} \sim_2 \mathrm{n} A_1 \vdash \mathrm{p}\mathrm{n} \sim_1 A_1 &
\end{array}
\qquad
\begin{array}{cl}
A_1 \vdash A_1 & \\ \hline
\ast_1 A_1 \vdash \ast_1 A_1 & \mathrm{cont} \\ \hline
\sim_1 A_1 \vdash \ast_1 A_1 & \\ \hline
A_1 \vdash \ast_1 \sim_1 A_1 & \mathrm{adj}\ast \\ \hline
\mathrm{N} A_1 \vdash \mathrm{N} \ast_1 \sim_1 A_1 & \mathrm{N} \\ \hline
\mathrm{n} A_1 \vdash \mathrm{N} \ast_1 \sim_1 A_1 & \\ \hline
\mathrm{n} A_1 \vdash \ast_2 \mathrm{N} \sim_1 A_1 & \ast_2\mathrm{N} \\ \hline
\mathrm{N} \sim_1 A_1 \vdash \ast_2 \mathrm{n} A_1 & \mathrm{adj}\ast \\ \hline
\mathrm{n} \sim_1 A_1 \vdash \ast_2 \mathrm{n} A_1 & \\ \hline
\mathrm{n} \sim_1 A_1 \vdash \sim_2 \mathrm{n} A_1 & \\ \hline
\mathrm{P}\mathrm{n} \sim_1 A_1 \vdash \mathrm{P} \sim_2 \mathrm{n} A_1 & \mathrm{P} \\ \hline
\mathrm{P}\mathrm{n} \sim_1 A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} A_1 & \\ \hline
\mathrm{p}\mathrm{n} \sim_1 A_1 \vdash \mathrm{p} \sim_2 \mathrm{n} A_1 &
\end{array}$$

$\neg A \wedge \neg B \dashv\vdash \neg(A \vee B) \quad \rightsquigarrow \quad \mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \dashv\vdash \mathrm{p}\,(A_2 \sqcap_2 B_2)$ and
$\neg A \otimes \neg B \dashv\vdash \neg(A \otimes B) \quad \rightsquigarrow \quad \mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \dashv\vdash \mathrm{p}\,(A_2 \sqcap_2 B_2)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{\mathrm{P}\,A_2 \vdash \mathrm{P}\,A_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{p}\,A_2 \vdash \mathrm{P}\,A_2}}{\mathrm{p}\,A_2 \,\hat{\sqcap}_1\, \mathrm{p}\,B_2 \vdash \mathrm{P}\,A_2}\,{\scriptstyle \mathrm{W}}}{\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \vdash \mathrm{P}\,A_2}}{\mathrm{N}\,(\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2) \vdash A_2}\,{\scriptstyle \mathrm{adj}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{\mathrm{P}\,B_2 \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{p}\,B_2 \vdash \mathrm{P}B_2}}{\mathrm{p}\,B_2 \,\hat{\sqcap}_1\, \mathrm{p}\,A_2 \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{W}}}{\mathrm{p}\,A_2 \,\hat{\sqcap}_1\, \mathrm{p}\,B_2 \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{E}}}{\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \vdash \mathrm{P}\,B_2}}{\mathrm{N}\,(\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2) \vdash B_2}\,{\scriptstyle \mathrm{adj}}}{\mathrm{N}\,(\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2) \,\hat{\sqcap}_2\, \mathrm{N}\,(\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2) \vdash A_2 \sqcap_2 B_2}}{\mathrm{N}\,(\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2) \vdash A_2 \sqcap_2 B_2}\,{\scriptstyle \mathrm{C}}}{\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \vdash \mathrm{P}\,(A_2 \sqcap_2 B_2)}\,{\scriptstyle \mathrm{adj}}}{\mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2 \vdash \mathrm{p}\,(A_2 \sqcap_2 B_2)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{A_2 \,\hat{\sqcap}_2\, B_2 \vdash A_2}\,{\scriptstyle \mathrm{W}}}{A_2 \sqcap_2 B_2 \vdash A_2}}{\mathrm{P}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{P}\,A_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{P}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,A_2}}{\mathrm{p}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,A_2} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{B_2 \,\hat{\sqcap}_2\, A_2 \vdash B_2}\,{\scriptstyle \mathrm{W}}}{A_2 \,\hat{\sqcap}_2\, B_2 \vdash B_2}\,{\scriptstyle \mathrm{E}}}{A_2 \sqcap_2 B_2 \vdash B_2}}{\mathrm{P}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{P}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,B_2}}{\mathrm{p}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,B_2}}{\mathrm{p}\,(A_2 \sqcap_2 B_2) \,\hat{\sqcap}_1\, \mathrm{p}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2}}{\mathrm{p}\,(A_2 \sqcap_2 B_2) \vdash \mathrm{p}\,A_2 \sqcap_1 \mathrm{p}\,B_2}\,{\scriptstyle \mathrm{C}}$$

$\neg(A \wedge B) \dashv\vdash \neg A \vee \neg B \quad \rightsquigarrow \quad \mathrm{p}\,(A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p}\,A_2 \sqcup_1 \mathrm{p}\,B_2$ and
$\neg(A \oplus B) \dashv\vdash \neg A \oplus \neg B \quad \rightsquigarrow \quad \mathrm{p}\,(A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p}\,A_2 \sqcup_1 \mathrm{p}\,B_2$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{\mathrm{P}\,A_2 \vdash \mathrm{P}\,A_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{P}\,A_2 \vdash \mathrm{p}\,A_2}}{\mathrm{P}\,A_2 \vdash \mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2}\,{\scriptstyle \mathrm{W}}}{A_2 \vdash \mathrm{N}\,(\mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2)}\,{\scriptstyle \mathrm{adj}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{\mathrm{P}\,B_2 \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{P}\,B_2 \vdash \mathrm{p}\,B_2}}{\mathrm{P}\,B_2 \vdash \mathrm{p}\,B_2 \,\check{\sqcup}_1\, \mathrm{p}\,A_2}\,{\scriptstyle \mathrm{W}}}{\mathrm{P}\,B_2 \vdash \mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2}\,{\scriptstyle \mathrm{E}}}{B_2 \vdash \mathrm{N}\,(\mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2)}\,{\scriptstyle \mathrm{adj}}}{A_2 \sqcup_2 B_2 \vdash \mathrm{N}\,(\mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2) \,\check{\sqcup}_1\, \mathrm{N}\,(\mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2)}}{A_2 \sqcup_2 B_2 \vdash \mathrm{N}\,(\mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2)}\,{\scriptstyle \mathrm{C}}}{\mathrm{P}\,(A_2 \sqcup_2 B_2) \vdash \mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2}\,{\scriptstyle \mathrm{adj}}}{\mathrm{p}\,(A_2 \sqcup_2 B_2) \vdash \mathrm{p}\,A_2 \,\check{\sqcup}_1\, \mathrm{p}\,B_2}}{\mathrm{p}\,(A_2 \sqcup_2 B_2) \vdash \mathrm{p}\,A_2 \sqcup_1 \mathrm{p}\,B_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_2 \vdash A_2}{\mathrm{P}\,A_2 \vdash \mathrm{P}\,A_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{p}\,A_2 \vdash \mathrm{P}\,A_2}}{\mathrm{N}\,\mathrm{p}\,A_2 \vdash A_2}\,{\scriptstyle \mathrm{adj}}}{\mathrm{N}\,\mathrm{p}\,A_2 \vdash A_2 \,\check{\sqcup}_2\, B_2}\,{\scriptstyle \mathrm{W}}}{\mathrm{N}\,\mathrm{p}\,A_2 \vdash A_2 \sqcup_2 B_2}}{\mathrm{p}\,A_2 \vdash \mathrm{P}\,(A_2 \sqcup_2 B_2)}\,{\scriptstyle \mathrm{adj}}}{\mathrm{p}\,A_2 \vdash \mathrm{p}\,(A_2 \sqcup_2 B_2)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B_2 \vdash B_2}{\mathrm{P}\,B_2 \vdash \mathrm{P}\,B_2}\,{\scriptstyle \mathrm{P}}}{\mathrm{p}\,B_2 \vdash \mathrm{P}\,B_2}}{\mathrm{N}\,\mathrm{p}B_2 \vdash B_2}\,{\scriptstyle \mathrm{adj}}}{\mathrm{N}\,\mathrm{p}\,B_2 \vdash B_2 \,\check{\sqcup}_2\, A_2}\,{\scriptstyle \mathrm{W}}}{\mathrm{N}\,\mathrm{p}\,B_2 \vdash A_2 \,\check{\sqcup}_2\, B_2}\,{\scriptstyle \mathrm{E}}}{\mathrm{N}\,\mathrm{p}\,B_2 \vdash A_2 \sqcup_2 B_2}}{\mathrm{p}\,B_2 \vdash \mathrm{P}\,(A_2 \sqcup_2 B_2)}\,{\scriptstyle \mathrm{adj}}}{\mathrm{p}\,B_2 \vdash \mathrm{p}\,(A_2 \sqcup_2 B_2)}}{\mathrm{p}\,A_2 \sqcup_1 \mathrm{p}\,B_2 \vdash \mathrm{p}\,(A_2 \sqcup_2 B_2) \,\check{\sqcup}_1\, \mathrm{p}\,(A_2 \sqcup_2 B_2)}}{\mathrm{p}\,A_2 \sqcup_1 \mathrm{p}\,B_2 \vdash \mathrm{p}\,(A_2 \sqcup_2 B_2)}\,{\scriptstyle \mathrm{C}}$$

$-(A \wedge B) \dashv\vdash -A \wedge -B \ \rightsquigarrow\ \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \dashv\vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2$

$$\begin{array}{cl}
A_2 \vdash A_2 & \\ \hline
A_2 \vdash A_2 \check{\sqcup}_2 B_2 & \mathrm{W} \\ \hline
A_2 \vdash A_2 \sqcup_2 B_2 & \\ \hline
\ast_2 (A_2 \sqcup_2 B_2) \vdash \ast_2 A_2 & \mathrm{cont} \\ \hline
\ast_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 A_2 & \\ \hline
\sim_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 A_2 & \\ \hline
\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{P} \sim_2 A_2 & \mathrm{P} \\ \hline
\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 & \\ \hline
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 &
\end{array}
\qquad
\begin{array}{cl}
B_2 \vdash B_2 & \\ \hline
B_2 \vdash B_2 \check{\sqcup}_2 A_2 & \mathrm{W} \\ \hline
B_2 \vdash A_2 \check{\sqcup}_2 B_2 & \mathrm{E} \\ \hline
B_2 \vdash A_2 \sqcup_2 B_2 & \\ \hline
\ast_2 (A_2 \sqcup_2 B_2) \vdash \ast_2 B_2 & \mathrm{cont} \\ \hline
\ast_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 B_2 & \\ \hline
\sim_2 (A_2 \sqcup_2 B_2) \vdash \sim_2 B_2 & \\ \hline
\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{P} \sim_2 B_2 & \mathrm{P} \\ \hline
\mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 B_2 & \\ \hline
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 B_2 &
\end{array}$$

$$\begin{array}{cl}
\hline
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \hat{\sqcap}_1 \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2 & \\ \hline
\mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2 & \mathrm{C}
\end{array}$$

$$\begin{array}{cl}
A_2 \vdash A_2 & \\ \hline
\ast_2 A_2 \vdash \ast_2 A_2 & \mathrm{cont} \\ \hline
\sim_2 A_2 \vdash \ast_2 A_2 & \\ \hline
\mathrm{P} \sim_2 A_2 \vdash \mathrm{P} \ast_2 A_2 & \mathrm{P} \\ \hline
\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} \ast_2 A_2 & \\ \hline
\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \ast_2 A_2 & \mathrm{W} \\ \hline
\mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) \vdash \ast_2 A_2 & \mathrm{adj} \\ \hline
B_2 \vdash \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 A_2) & \mathrm{adj}^*
\end{array}
\qquad
\begin{array}{cl}
B_2 \vdash B_2 & \\ \hline
\ast_2 B_2 \vdash \ast_2 B_2 & \mathrm{cont} \\ \hline
\sim_2 B_2 \vdash \ast_2 B_2 & \\ \hline
\mathrm{P} \sim_2 B_2 \vdash \mathrm{P} \ast_2 B_2 & \mathrm{P} \\ \hline
\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \ast_2 B_2 & \\ \hline
\mathrm{p} \sim_2 B_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 A_2 \vdash \mathrm{P} \ast_2 B_2 & \mathrm{W} \\ \hline
\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \ast_2 B_2 & \mathrm{E} \\ \hline
\mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) \vdash \ast_2 B_2 & \mathrm{adj} \\ \hline
B_2 \vdash \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) & \mathrm{adj}^*
\end{array}$$

$$\begin{array}{cl}
\hline
A_2 \sqcup_2 B_2 \vdash \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) \check{\sqcup}_2 \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) & \\ \hline
A_2 \sqcup_2 B_2 \vdash \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) & \mathrm{C} \\ \hline
\mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) \vdash \ast_2 (A_2 \sqcup_2 B_2) & \mathrm{adj}^* \\ \hline
\mathrm{N}(\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2) \vdash \sim_2 (A_2 \sqcup_2 B_2) & \\ \hline
\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcup_2 B_2) & \mathrm{adj} \\ \hline
\mathrm{p} \sim_2 A_2 \hat{\sqcap}_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) & \\ \hline
\mathrm{p} \sim_2 A_2 \sqcap_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcup_2 B_2) & \mathrm{C}
\end{array}$$

$-(A \otimes B) \dashv\vdash -A \oplus -B \ \rightsquigarrow\ \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \dashv\vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2$

$$\begin{array}{cl}
A_2 \vdash A_2 & \\ \hline
\ast_2 A_2 \vdash \ast_2 A_2 & \mathrm{cont} \\ \hline
\ast_2 A_2 \vdash \sim_2 A_2 & \\ \hline
\mathrm{P} \ast_2 A_2 \vdash \mathrm{P} \sim_2 A_2 & \mathrm{P} \\ \hline
\mathrm{P} \ast_2 A_2 \vdash \mathrm{p} \sim_2 A_2 & \\ \hline
\mathrm{P} \ast_2 A_2 \vdash \mathrm{p} \sim_2 A_2 \check{\sqcup}_1 \mathrm{p} \sim_2 B_2 & \mathrm{W} \\ \hline
\mathrm{P} \ast_2 A_2 \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 & \\ \hline
\ast_2 A_2 \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) & \mathrm{adj} \\ \hline
\ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2 & \mathrm{adj}\ast
\end{array}
\qquad
\begin{array}{cl}
B_2 \vdash B_2 & \\ \hline
\ast_2 B_2 \vdash \ast_2 B_2 & \mathrm{cont} \\ \hline
\ast_2 B_2 \vdash \sim_2 B_2 & \\ \hline
\mathrm{P} \ast_2 B_2 \vdash \mathrm{P} \sim_2 B_2 & \mathrm{P} \\ \hline
\mathrm{P} \ast_2 B_2 \vdash \mathrm{p} \sim_2 B_2 & \\ \hline
\mathrm{P} \ast_2 B_2 \vdash \mathrm{p} \sim_2 B_2 \check{\sqcup}_1 \mathrm{p} \sim_2 A_2 & \mathrm{W} \\ \hline
\mathrm{P} \ast_2 B_2 \vdash \mathrm{p} \sim_2 A_2 \check{\sqcup}_1 \mathrm{p} \sim_2 B_2 & \mathrm{E} \\ \hline
\mathrm{P} \ast_2 B_2 \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 & \\ \hline
\ast_2 B_2 \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) & \mathrm{adj} \\ \hline
\ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash B_2 & \mathrm{adj}\ast
\end{array}$$

$$\begin{array}{cl}
\hline
\ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \hat{\sqcap}_2 \ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2 \sqcap_2 B_2 & \\ \hline
\ast_2 \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) \vdash A_2 \sqcap_2 B_2 & \mathrm{C} \\ \hline
\ast_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) & \mathrm{adj}\ast \\ \hline
\sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{N}(\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2) & \\ \hline
\mathrm{P} \sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 & \mathrm{adj} \\ \hline
\mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \vdash \mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 &
\end{array}$$

$$
\begin{array}{c}
A_2 \vdash A_2 \\ \hline
*_2 A_2 \vdash *_2 A_2 \;\;(\text{cont}) \\ \hline
\sim_2 A_2 \vdash *_2 A_2 \\ \hline
\mathrm{P} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2 \;\;(\mathrm{P}) \\ \hline
\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} *_2 A_2 \\ \hline
\mathrm{N}\mathrm{p} \sim_2 A_2 \vdash *_2 A_2 \;\;(\text{adj}) \\ \hline
A_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 A_2 \;\;(\text{adj}*) \\ \hline
A_2 \hat{\sqcap}_2 B_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 A_2 \;\;(\text{W}) \\ \hline
A_2 \sqcap_2 B_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 A_2 \\ \hline
\mathrm{N}\mathrm{p} \sim_2 A_2 \vdash *_2 (A_2 \sqcap_2 B_2) \;\;(\text{adj}*) \\ \hline
\mathrm{N}\mathrm{p} \sim_2 A_2 \vdash \sim_2 (A_2 \sqcap_2 B_2) \\ \hline
\mathrm{p} \sim_2 A_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcap_2 B_2) \;\;(\text{adj}) \\ \hline
\mathrm{p} \sim_2 A_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)
\end{array}
\qquad
\begin{array}{c}
B_2 \vdash B_2 \\ \hline
*_2 B_2 \vdash *_2 B_2 \;\;(\text{cont}) \\ \hline
\sim_2 B_2 \vdash *_2 B_2 \\ \hline
\mathrm{P} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2 \;\;(\mathrm{P}) \\ \hline
\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} *_2 B_2 \\ \hline
\mathrm{N}\mathrm{p} \sim_2 B_2 \vdash *_2 B_2 \;\;(\text{adj}) \\ \hline
B_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 B_2 \;\;(\text{adj}*) \\ \hline
B_2 \hat{\sqcap}_2 A_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 B_2 \;\;(\text{W}) \\ \hline
A_2 \hat{\sqcap}_2 B_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 B_2 \;\;(\text{E}) \\ \hline
A_2 \sqcap_2 B_2 \vdash *_2 \mathrm{N}\mathrm{p} \sim_2 B_2 \\ \hline
\mathrm{N}\mathrm{p} \sim_2 B_2 \vdash *_2 (A_2 \sqcap_2 B_2) \;\;(\text{adj}*) \\ \hline
\mathrm{N}\mathrm{p} \sim_2 B_2 \vdash \sim_2 (A_2 \sqcap_2 B_2) \\ \hline
\mathrm{p} \sim_2 B_2 \vdash \mathrm{P} \sim_2 (A_2 \sqcap_2 B_2) \;\;(\text{adj}) \\ \hline
\mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2)
\end{array}
$$

$$
\begin{array}{c}
\hline
\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \check{\sqcup}_1 \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \\ \hline
\mathrm{p} \sim_2 A_2 \sqcup_1 \mathrm{p} \sim_2 B_2 \vdash \mathrm{p} \sim_2 (A_2 \sqcap_2 B_2) \;\;(\text{C})
\end{array}
\qquad \square
$$

6.3. Conservativity

To argue that the calculus introduced in Section 5 is conservative w.r.t. BL (resp. CBL), we follow the standard proof strategy discussed in [29,27]. Denote by $\vdash_{\mathrm{BL}}$ (resp. $\vdash_{\mathrm{CBL}}$) the consequence relation defined by the calculus for BL (resp. CBL) introduced in Section 2, and by $\models_{\mathsf{HBL}}$ (resp. $\models_{\mathsf{HCBL}}$) the semantic consequence relation arising from the class of (perfect) HBLs (resp. HCBLs). We need to show that, for all formulas A and B of the original language of BL (resp. CBL), if $t_1(A) \vdash t_1(B)$ is a D.BL-derivable (resp. D.CBL-derivable) sequent, then $A \vdash_{\mathrm{BL}} B$ (resp. $A \vdash_{\mathrm{CBL}} B$). This can be proved using the following facts: (a) the rules of D.BL (resp. D.CBL) are sound w.r.t. perfect HBLs (resp. HCBLs); (b) BL (resp. CBL) is complete w.r.t. B (resp. CB); and (c) B (resp. CB) are equivalently presented as HBL (resp. HCBL, cf. Section 3.3), so that the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Propositions 6.1 and 6.2). Let then A, B be formulas of the original language of BL (resp. CBL). If $t_1(A) \vdash t_1(B)$ is a D.BL (resp. D.CBL)-derivable sequent, then, by (a), $t_1(A) \models_{\mathsf{HBL}} t_1(B)$ (resp. $t_1(A) \models_{\mathsf{HCBL}} t_1(B)$). By (c) and Proposition 4.1, this implies that $A \models_{\mathsf{B}} B$ (resp. $A \models_{\mathsf{CB}} B$). By (b), this implies that $A \vdash_{\mathrm{BL}} B$ (resp. $A \vdash_{\mathrm{CBL}} B$), as required.

6.4. Subformula property and cut elimination

Let us briefly sketch the proof of cut elimination and subformula property for D.BL (resp. D.CBL). As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi [6]. The meta-theorem to which we will appeal for D.BL (resp. D.CBL) was proved in [19].

All conditions in [19, Theorem 4.1] except C'_8 are readily seen to be satisfied by inspection of the rules. Condition C'_8 requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we only show C'_8 for the unary connectives $\sim$ and n (the proof for p is analogous). The cases of lattice connectives are standard and hence omitted.

L_i-type connectives

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X_i \vdash *_i A_i\end{array}}{X_i \vdash \sim_i A_i} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ *_i A_i \vdash Y_i\end{array}}{\sim_i A_i \vdash Y_i}}{X_i \vdash Y_i}
\quad \rightsquigarrow \quad
\text{cont}\;\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ *_i A_i \vdash Y_i\end{array}}{*_i Y_i \vdash A_i} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ X_i \vdash *_i A_i\end{array}}{A_i \vdash *_i X_i}}{*_i Y_i \vdash *_i X_i}}{X_i \vdash Y_i}
$$

Multi-type connectives

$$
\dfrac{\dfrac{\begin{matrix}\vdots\, \pi_1 \\ X_2 \vdash \mathrm{N}A_1\end{matrix}}{X_2 \vdash \mathrm{n}A_1} \qquad \dfrac{\begin{matrix}\vdots\, \pi_2 \\ \mathrm{N}A_1 \vdash Y_2\end{matrix}}{\mathrm{n}A_1 \vdash Y_2}}{X_2 \vdash Y_2}
\qquad \rightsquigarrow \qquad
\mathrm{P}\,\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, \pi_1 \\ X_2 \vdash \mathrm{N}A_1\end{matrix}}{\mathrm{P}X_2 \vdash A_1} \qquad \dfrac{\begin{matrix}\vdots\, \pi_2 \\ \mathrm{N}A_1 \vdash Y_2\end{matrix}}{A_1 \vdash \mathrm{P}Y_2}}{\mathrm{P}X_2 \vdash \mathrm{P}Y_2}}{X_2 \vdash Y_2}
$$

7. Conclusions and future work

The modular character of proper multi-type display calculi makes it possible to easily extend our formalism so as to capture axiomatic extensions (e.g. the logic of *classical bilattices with conflation* [1, Definition 2.11]) as well as language expansions of the basic bilattice logics treated in the present paper. Expansions of bilattice logic have been extensively studied in the literature as early as in [1], which introduces an implication enjoying the deduction-detachment theorem (see also [10]). More recently, modal operators have been added to bilattice logics, motivated by potential applications to computer science and in particular verification of programs [33,36]; as well as dynamic modalities, motivated by applications in the area of dynamic epistemic logic [34,35].

Yet more recently, bilattices with a negation not necessarily satisfying the involution law ($\neg\neg a = a$) have been introduced with motivations of domain theory and topological duality (see [32]), and the study of the corresponding logics has been started [37]. These logics are weaker than the one considered in the present paper, and so adapting our display calculus formalism to them might prove a more challenging task (in particular, the translations introduced in Section 6 may need to be redefined, as they rely on the maps p and n being lattice isomorphisms, which is no longer true in the non-involutive case).

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