Universiteit Utrecht

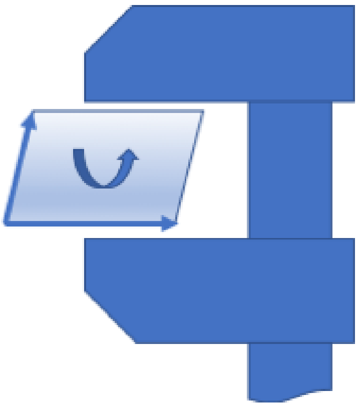
Faculteit Bètawetenschappen

Introduction to Geometric Algebra

A powerful tool for mathematics and physics

Bachelor Thesis (TWIN)

Denis Lamaker



Supervisor:

Dr. J.W. van de Leur

Mathematical Institute

June 14, 2019

Abstract

In this thesis an introduction to geometric or Clifford algebra is given, with an emphasis on the geometric aspects of this algebra. The aim is to show that this algebra is a powerful tool for both mathematics and physics and results in compact, coordinate free expressions. The main focus will be on Euclidean spaces of 2 and 3 dimensions, but it will be shown that it is possible to extend the results to higher dimensions. Finally a start will be made to further extend to more general algebras with non-degenerate bilinear forms with a mixed signature.

Contents

(This page is intentionally left blank.)

1 Introduction

Starting from familiar Euclidean spaces in 2 and 3 dimensions ($\mathbb{R}^2$ and $\mathbb{R}^3$), which are vector spaces with an inner product, we extend them to an associative algebra called Geometric algebra or Clifford algebra by introducing the geometric product, after first having had a look at the exterior algebra. Geometric algebra contains elements which can represent geometric objects such as points, lines, planes and volumes, but at the same time these objects can be used as operators. We will derive elegant coordinate free expressions for a number of geometric operations such as projection, reflection and rotation. We will discover that several other well known algebras like complex numbers, quaternions and the Pauli algebra, are subalgebras of geometric algebra. Geometric algebra will also provide a natural representation for what are called polar and axial vectors. After getting familiar with the properties of the geometric algebra in 2 and 3 dimensions, we will take a more general approach which allows us to extend this type of algebra to arbitrary dimensions and quadratic forms with mixed signature. In this general approach we will take the geometric product as basic and formulate other products in terms of it. We will limit ourselves to vector spaces over the field $\mathbb{R}$.

2 The outer product in 2 and 3 dimensions

From physics we are familiar with products between vectors. Examples are the inner or dot product, which results in a scalar, and the vector product which results in another vector. The inner product is a symmetric bilinear product of two vectors which results in a scalar. The vector product is an anti-symmetric bilinear product which results in a vector which is orthogonal to both vectors in the product and has a direction which is determined by the 'right hand rule'. Although these two products have proved to be very useful in physics, there are a few shortcomings. Both products miss the very desirable algebraic property of associativity and do not support the definition of a multiplicative inverse. The definition of the vector product limits its use to 3 dimensions and does not allow for easy generalization to higher dimensions. The inner product is only defined between 2 vectors and produces a scalar and not a vector again so it is not an operation that closes in the vector space. Considered from a 2D standpoint, the vector product does not close in the 2D space, it needs the third dimension.

We can remove the disadvantages mentioned for the vector product by defining a new product which is called the outer product (due to Grassmann). It will retain some of the properties of the vector product, but most importantly: it will be associative.

First we define what we mean by the orientation of a subspace of some Euclidean space.
A line through the origin is a 1-dimensional subspace of a Euclidean space of dimension ≥ 1. We can turn the line into an oriented line by choosing a basis (in this case 1 vector $\{e_1\}$). For an arbitrary vector $v = \lambda e_1$ in this 1-dimensional subspace we now can define its orientation by taking the sign of its coordinate λ.

A plane through the origin is a 2-dimensional subspace of a Euclidean space of dimension ≥ 2. We can turn the plane into an oriented plane by choosing an ordered basis (in this case 2 vectors $\mathfrak{B} = \{e_1, e_2\}$). For an ordered set of arbitrary independent vectors $\{v_1, v_2\}$ in this 2-dimensional subspace we now can define its orientation by taking the sign of the determinant of the coordinates of the vectors relative to the chosen basis: $\det_{\mathfrak{B}}(v_1, v_2)$. If we would choose $\{v_1, v_2\}$ as our ordered basis, this determinant would be clearly equal to 1 and we would have a positive orientation.

We can generalize this to subspaces of dimension k of a space with dimension $n \geq k$ by choosing an ordered basis $\mathfrak{B} = \{e_1, e_2, \ldots, e_k\}$ for this subspace and defining the orientation of an ordered set of independent vectors $\{v_1, v_2, \ldots, v_k\}$ as the sign of the determinant of coordinates $\det_{\mathfrak{B}}(v_1, v_2, \ldots, v_k)$. From its definition we see that the orientation of a set of vectors is inverted when we exchange any two vectors of the set.

A vector v in a Euclidean space has a magnitude which is given by the length of the vector $\|v\| = \sqrt{v \cdot v}$ and a direction which is given by the line ℓ that is spanned by the vector v. A vector has also an orientation when the line ℓ is oriented.

The outer product $a \wedge b$ of two vectors will be defined as a new kind of object, different from a vector, called a bivector or 2-vector, which also has properties of magnitude, direction and orientation just like a vector. The symbol used for the outer product is '$\wedge$' and therefore it is also referred to as the 'wedge' product. We can interpret it as an amount of oriented area in the plane spanned by the vectors a and b, where the magnitude is equal to the area of the parallelogram that is determined by a and b, the direction is equal to that of the plane spanned by a and b and the orientation is given by the orientation of the ordered set of vectors $\{a, b\}$ relative to a chosen orientation of the plane.

Geometrically it is clear that the magnitude of $b \wedge a$ is equal to the magnitude of $a \wedge b$ because they both determine the same parallelogram. From the definition of orientation we see that the orientation of $b \wedge a$ is the opposite of that of $a \wedge b$, so a property of the outer product of two vectors is: $a \wedge b = -b \wedge a$. We should not think of the bivector as having a shape like that of the parallelogram, but as a certain amount of area in a particular plane, which can be given a sign when we choose an orientation for the plane. We will see that choosing an orientation is equivalent to a choice of some non-zero bivector in that plane and assume its orientation to be positive; the orientation of the bivector $a \wedge b$ can then be calculated.

Like the product of 2 vectors, the outer product of 3 vectors will produce a new type of object called a trivector or 3-vector, different from vectors and bivectors. It can be interpreted as an amount of volume in a 3 dimensional (sub)space spanned by the 3 vectors. For the outer product of 3 vectors, we have two possibilities: $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$. The magnitude of $(a \wedge b) \wedge c$ is defined as the magnitude of $a \wedge b$ times the magnitude of the component of c perpendicular to the plane defined by $a \wedge b$. The magnitude of $a \wedge (b \wedge c)$ is defined as the magnitude of $b \wedge c$ times the magnitude of the component of a perpendicular to the plane defined by $b \wedge c$. Geometrically, both magnitudes are equal to the volume of a parallelepiped formed by the vectors a, b, c and therefore equal to each other. The orientation of $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$ is defined as the orientation of the ordered set of vectors $\mathfrak{B} = \{a, b, c\}$ relative to some chosen orientation for the subspace spanned by $\mathfrak{B}$. Because it only depends on the order of the vectors, the orientation for the products $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$ is the same. The direction of $a \wedge b \wedge c$ is the 3-dimensional space spanned by the vectors a, b and c. Because magnitude, direction and orientation of $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$ are equal we obtain the result that the outer product is associative.

Similarly we can define the outer product of k vectors and call it a k-vector, where the number k is called the grade of the k-vector.

If we want to turn the bivectors into a vector space, we have to define the product of a bivector with a scalar and the sum of two bivectors. The product of a scalar λ with a bivector B is

simply a bivector λB with a magnitude equal to $|\lambda|\|B\|$, the same direction as B and the same orientation as B when $\lambda > 0$ and opposite orientation if $\lambda < 0$. The sum of two bivectors B and C is a bit more complicated. Bivectors B and C are associated to planes $\mathcal{B}$ and $\mathcal{C}$ through the origin through their direction. When the two planes are the same, the sum of B and C will also be in that plane and the magnitude will be defined as the sum of the oriented magnitudes. When the two planes are different, they will intersect in a line through the origin and we can find a non-zero vector a which spans that line and is common to the planes $\mathcal{B}$ and $\mathcal{C}$. Therefore we can find vectors b in $\mathcal{B}$ and c in $\mathcal{C}$ such that $B = a \wedge b$ and $C = a \wedge c$. We now define $B + C = a \wedge b + a \wedge c = a \wedge (b + c)$ and this states (by definition) that the outer product distributes over bivector addition. It is not hard to show that the sum does not depend on the choice of the vector a and therefore is well defined. Therefore the bivectors in 3D form a vector space. Addition of k-vectors in higher dimensions can also be defined by choosing a basis and adding the projections of k-vectors on the basis k-vectors to obtain a new k-vector.
We do not need to define the outer product of two bivectors or higher in 3D, since this requires 4 vectors and one of the vectors would always be a linear combination of the others and, as a consequence of the anti-commutativity of the outer product, the product would always be zero.

In 2D we have the subspaces of scalars, vectors and bivectors and from these we can define a vector space which is given by the direct sum of these subspaces. Although adding scalars, vectors and bivectors might seem strange at first sight, it is not different to the addition between real and imaginary numbers as used in the algebra of the complex numbers or adding monomials to form polynomials. Adding the outer product to this vector space gives an algebra which is called the exterior algebra $\Lambda(\mathbb{R}^2)$.

In 3D we have the subspaces of scalars, vectors, bivectors and trivectors which now gives the exterior algebra $\Lambda(\mathbb{R}^3)$.

The use of the outer product can be generalized to vector spaces of dimension higher than 3. In that case we will define it to be associative, distributive, anti-commutative and bilinear. The outer product of k vectors is called a k-blade and we assume they can also be added and satisfy the properties of a vector space. In dimensions higher than 3, the sum of a number of k-blades cannot not always be written as a product of vectors (so again a k-blade) and we will use the more general term k-vector for this. If the vectors in a blade are taken from a vector space of dimension n, we can at most have n-blades because any number of vectors higher than n will always be linearly dependent and result in an outer product of 0. The dimension of a k-vector subspace based on a vector space with dimension n will be $\binom{n}{k}$. Given that we have a basis $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$ for the vector space, we can form $n!/(n-k)!$ possible k-blades, but because of the anti-commutativity of the outer product, any two k-blades containing the same basis vectors in any order will differ only by a sign and therefore we have to divide by an extra factor of $k!$.

Finally we can create an even bigger vector space by allowing k-vectors of different grade to be added together and define things so that they obey the axioms of a vector space. This vector space contains objects which are called multivectors and which are linear combinations of scalars, vectors, 2-vectors, ..., n-vectors. The dimension of this vector space is $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$. The nice thing is that this vector space can now be turned into an algebra because the outer product now closes, provided we identify the outer product of a scalar with any multivector with the vector space product of a scalar with a multivector and consider it to commute so: $\lambda \wedge M = M \wedge \lambda = \lambda M = M\lambda$. This algebra is called the exterior algebra $\bigwedge(\mathbb{R}^n)$.

The grade of a multivector can be obtained by the grade operator $\langle . \rangle_k$. So $\langle A \rangle_k$ returns a multivector containing all k-vector parts of A. In order to explicitly mention that a multivector A has grade k we write it as A_k. When confusion might arise with k being an index, we will write $A_{<k>}$. A multivector which is a sum of only k-vectors is called a homogeneous multivector. A short way to express this is as a condition: $\langle A \rangle_k = A$.

3 Geometric Algebra in Euclidean spaces

Geometric algebra will be defined as an extension of the exterior algebra, which is built on the same vector space of multivectors as the exterior algebra, but the outer product has been replaced by a new product, called the geometric product, which is also associative and distributive, but not commutative or anti-commutative. The geometric product of two homogeneous multivectors A and B, which we will write simply by AB, can now be a sum of homogeneous multivectors of higher and lower grade than A and B, whereas the outer product only produced a multivector of higher grade (or equal for scalars) than A and B. An additional note on notation is that we will use lower case Greek letters to denote scalars, lower case Latin letters to denote 1-vectors and upper case Latin letters to denote general multivectors. In expressions that use indices, we will use the Einstein summation convention by default and will write '(no summation)' behind the equation when it is not used.

The geometric product of two vectors combines the properties of the inner and outer products and is defined as:

$$ab = a \cdot b + a \wedge b \tag{1}$$

Although the first to mention it was Grassmann, the first to really put it into action was Clifford. This resulted in what mathematicians nowadays call Clifford algebras, although the original name given by Clifford himself was geometric algebra [Doran et al., 2003, p.20].

From this definition it follows that the geometric product of two orthogonal vectors is equal to the outer product and therefore anti-commutes, whereas the geometric product of two collinear vectors is equal to their inner product which in that case is a scalar. For the geometric product of a vector with itself we have $a^2 = aa = a \cdot a = \|a\|^2$. This also means that every non-zero vector has an inverse under the geometric product: $a^{-1} = a/a^2 = a/\|a\|^2$.

It is also possible to define the inner and outer product in terms of the geometric product. Because $ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b$, we can derive the following relations for the inner and outer product of two vectors:

$$\begin{aligned} a \cdot b &= \frac{ab + ba}{2} \\ a \wedge b &= \frac{ab - ba}{2} \end{aligned} \tag{2}$$

In the following we will denote the spaces of multivectors based on the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, as $\mathcal{G}_2$ and $\mathcal{G}_3$. Later, when we will generalize to higher dimensions and other spaces than $\mathcal{G}_2$ and $\mathcal{G}_3$, we will use the geometric product as the basic multiplication operation and define other products, among which the inner and outer products, in terms of it.

3.1 Geometric algebra in 2D

In this section we will limit ourselves to the geometric algebra of $\mathcal{G}_2$. From the definition of the geometric product we saw that collinear vectors commute and give a scalar result whereas orthogonal vectors anti-commute and give a bivector result. These properties are important and will be used over and over in the following. The geometric product of two arbitrary vectors will always be a sum of a scalar part and a bivector part.

Every multivector in $\mathcal{G}_2$ can be written as a linear combination of basis vectors derived from an orthonormal basis $\{e_1, e_2\}$ of the original vector space $\mathbb{R}^2$. Since the basis vectors are orthonormal, their geometric product is equal to the outer product and therefore we get the bivector $e_1 \wedge e_2 = e_1 e_2$, which we will also write as e_{12}. This is the highest grade we can get because in a product with more than 2 factors we will have to repeat basis vectors and therefore the outer product will give the result 0. The dimension of the bivector subspace is equal to 1 because all products of 2 different basis vectors only differ by a sign and are therefore linearly dependent. The highest grade in a multivector space is therefore equal to the dimension of the vector space on which it is based, so for $\mathcal{G}_2$ that is 2. The elements of the subspace with the highest grade are also called pseudoscalars and the element of the orthonormal basis that generates it is called the unit pseudoscalar and is denoted by the symbol I. The properties of I can differ depending on the grade but it is used often so it deserves a special symbol. A possible basis for the vector space $\mathcal{G}_2$ can be: $\{1, e_1, e_2, e_1 e_2\}$. The bivector $e_1 e_2$ represents an oriented area in the plane and because the vectors e_i form an orthonormal basis, the magnitude of $e_1 \wedge e_2$ is equal to 1 so $I = e_1 e_2$. If we multiply a basis vector say e_1 from the left with the pseudoscalar I we get:

$$\begin{aligned} Ie_1 &= e_1 e_2 e_1 = -e_1 e_1 e_2 = -e_2 \quad \text{and} \\ Ie_2 &= e_1 e_2 e_2 = e_1. \end{aligned} \tag{3}$$

This result is the same as when we had rotated the two basis vectors clockwise over an angle of $\pi/2$. Because every vector is a linear combination of the basis vectors and rotation is a linear operation, multiplying any vector by $e_1 e_2$ to the left corresponds to a clockwise rotation over $\pi/2$.

Similarly multiplying a vector to the right by $e_1 e_2$ corresponds to an anti-clockwise rotation over $\pi/2$:

$$\begin{aligned} e_1 I &= e_1 e_1 e_2 = e_2 \qquad\qquad\qquad \text{and} \\ e_2 I &= e_2 e_1 e_2 = -e_2 e_2 e_1 = -e_1. \end{aligned} \tag{4}$$

From this we can also conclude that the pseudoscalar $I = e_1 e_2$ anti-commutes with a vector since rotating a vector a ninety degrees clockwise gives the opposite of rotating that vector anti-clockwise over ninety degrees, which is formulated algebraically as:

$$aI = a_i e_i I = -a_i I e_i = -I a_i e_i = -Ia \tag{5}$$

3.1.1 Projection and Rejection

For vectors a and b we have: $a = a_{\parallel b} + a_{\perp b}$ where $a_{\parallel b}$ is the component of a parallel to b which is called the projection of a onto b and $a_{\perp b}$ is the component of a perpendicular to b which is called the rejection of a onto b.

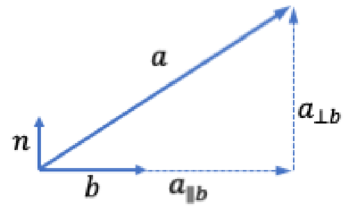


Figure 1: Projection and Rejection of a vector

They can be calculated using geometric algebra as follows:

$$\begin{aligned} a &= a(bb^{-1}) \\ &= (ab)b^{-1} \\ &= (a \cdot b + a \wedge b)b^{-1} \\ &= (a \cdot b)b^{-1} + (a \wedge b)b^{-1} \\ &= a_{\parallel b} + a_{\perp b} \end{aligned} \tag{6}$$

We did not use the dimension of the vector space in this derivation, so the results hold in spaces $\mathcal{G}_n$ with arbitrary n.

3.1.2 Reflection in a vector (or line)

We can calculate the reflection $\mathsf{M}(a)$ of a vector a in a line spanned by a vector b as follows:

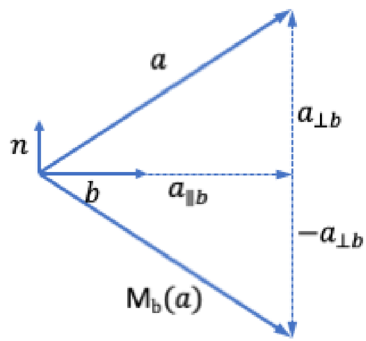


Figure 2: Reflection of a vector

$$\begin{aligned} \mathsf{M}_b(a) &= (a_{\parallel b}) - (a_{\perp b}) \\ &= (a \cdot b)b^{-1} - (a \wedge b)b^{-1} \\ &= (b \cdot a)b^{-1} + (b \wedge a)b^{-1} \\ &= (b \cdot a + b \wedge a)b^{-1} \\ &= bab^{-1} \end{aligned} \tag{7}$$

Again the result holds for arbitrary spaces $\mathcal{G}_n$.

From this expression it follows that the result does not depend on the length of the vector b and therefore it simplifies the expression even further when b is a unit vector because in that case we have $b^{-1} = b$. The expression for the reflection then becomes $\mathsf{M}_b(a) = bab$.
Sometimes it is more convenient to specify the line by a unit normal vector n and in that case we get:

$$\begin{aligned} \mathsf{M}_{\perp n}(a) &= a_{\perp n} - a_{\parallel n} \\ &= -nan \end{aligned} \tag{8}$$

The formula shows in an easy way that if we reflect two vectors a and b in a unit vector n, the inner product between the two is invariant so lengths and angles are preserved by reflection. Let $\mathsf{M}_n(a) = nan$ and $\mathsf{M}_n(b) = nbn$ then we have:

$$\begin{aligned} \mathsf{M}_n(a)\mathsf{M}_n(b) &= (nan)(nbn) \\ &= nannbn \\ &= nabn \end{aligned} \tag{9}$$

and therefore

$$\begin{aligned} \mathsf{M}_n(a) \cdot \mathsf{M}_n(b) &= \tfrac{1}{2}(\mathsf{M}_n(a)\mathsf{M}_n(b) + \mathsf{M}_n(b)\mathsf{M}_n(a)) \\ &= \tfrac{1}{2}(nabn + nban) \\ &= \tfrac{1}{2}n(ab + ba)n \\ &= n\tfrac{1}{2}(ab + ba)n \\ &= n(a \cdot b)n \\ &= nn(a \cdot b) \\ &= a \cdot b \end{aligned} \tag{10}$$

3.1.3 Rotation in 2D

We might remember from geometry that every rotation can be obtained by successive reflection in two lines where the angle of rotation that is obtained is twice the angle between the two lines.

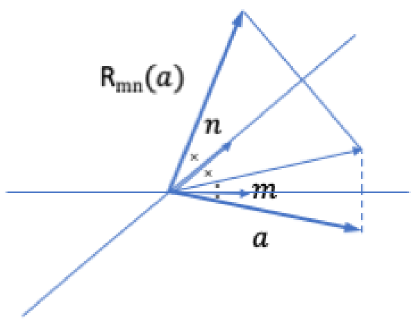


Figure 3: Rotation of a vector in a plane

If we take independent unit vectors m and n to represent the direction of two lines and we first reflect a vector a in the line with direction m, followed by reflection in the line with direction n, we get a rotated vector $\mathsf{R}(a)$ given by:

$$\mathsf{R}(a) = n(mam)n = (nm)a(mn) \tag{11}$$

If we denote the angle between the unit vectors m and n by ϕ and the unit pseudoscalar $e_1 \wedge e_2$ by I, and remember that the magnitude of $m \wedge n$ is equal to $\sin(\phi)$, we can write:

$$mn = m \cdot n + m \wedge n = \cos(\phi) + \sin(\phi)I \tag{12}$$

Because $I^2 = -1$ and comparing this to $i^2 = -1$ for the complex numbers, we can define an exponential by using power series and get an expression similar to Euler's formula:

$$\cos(\phi) + \sin(\phi)I = e^{\phi I} \tag{13}$$

Similarly we get for nm:

$$nm = n \cdot m + n \wedge m = m \cdot n - m \wedge n = \cos(\phi) - \sin(\phi)I = e^{-\phi I} \tag{14}$$

The angle of rotation θ is twice the angle ϕ between m and n therefore in 2D we can write for a rotation of a vector a over an angle θ:

$$\mathsf{R}(a) = (nm)a(mn) = e^{-I\theta/2}\, a\, e^{I\theta/2} \tag{15}$$

Because we have seen that the unit pseudoscalar I in 2D anti-commutes with vectors, we have:

$$e^{-\phi I}a = (\cos(\phi) - \sin(\phi)I)a = a(\cos(\phi) + \sin(\phi)I) = ae^{\phi I} \tag{16}$$

Therefore we can write for rotations over an angle θ in 2D:

$$\mathsf{R}(a) = e^{-I\theta/2}\, a\, e^{I\theta/2} = e^{-I\theta}\, a = a\, e^{I\theta} \tag{17}$$

An important note is that the exponents can only be added because they commute. We will see that we can not simply pull the exponents over to one side for a vector a in 3D which is not in the plane of rotation, but the product with exponentials on both sides will be still valid.

Because the geometric product of scalars and pseudoscalars commutes, it is not hard to show that the (even) subalgebra of the direct sum of the scalar and pseudoscalar subspaces is isomorphic to the algebra of the complex numbers. So the complex numbers are 'contained' in geometric algebra.

3.2 Geometric algebra in 3D

We now have a vector space of 3 dimensions so an orthonormal basis with 3 vectors e_1, e_2 and e_3.

Since these basis vectors are orthonormal, we now have 3 independent anti-commuting bivectors: e_1e_2, e_2e_3 and e_3e_1 and 1 trivector $e_1e_2e_3$.
The basis for $\mathcal{G}_3$ will now consist of $2^3 = 8$ basis vectors:

1 basis vector for the 0-dimensional scalar subspace (the number 1)
3 basis vectors for the 1-dimensional vector subspace, (e_1, e_2, e_3)
3 basis vectors for the 2-dimensional bivector subspace, (e_1e_2, e_2e_3, e_3e_1)
1 basis vector for the 3-dimensional trivector subspace, $(e_1e_2e_3)$, the unit pseudoscalar, which

we will again denote by I.

The basis vectors for the bivector subspace determine planes embedded in the 3D space, and similarly to what we saw in 2D, they square to -1 and right and left multiplication of a vector with the basis bivector in the same plane will rotate the vector 90 degrees anti-clockwise or clockwise respectively.

The unit pseudoscalar I also squares to -1 (it takes only 1 exchange to reverse the order of I):

$$I^2 = (e_1e_2e_3)(e_1e_2e_3) = -(e_1e_2e_3)(e_3e_2e_1) = -1 \tag{18}$$

This will not always be the case. For example in 4 dimensions it takes 2 exchanges to reverse the order of the basis vectors so we get:

$$I^2 = (e_1e_2e_3e_4)(e_1e_2e_3e_4) = (-1)^2(e_1e_2e_3e_4)(e_4e_3e_2e_1) = 1 \tag{19}$$

With the above choice of basis vectors we can write the bivectors in the basis as:

$$E_k = Ie_k. \tag{20}$$

We also can write:

$$e_i \wedge e_j = \varepsilon_{ijk}E_k. \tag{21}$$

From this it follows that:

$$\begin{aligned} e_ie_j &= e_i \cdot e_j + e_i \wedge e_j \\ &= \delta_{ij} + \varepsilon_{ijk}E_k. \end{aligned} \tag{22}$$

For the product of the bivectors E_i and E_j we find:

$$\begin{aligned} E_iE_j &= Ie_iIe_j \\ &= IIe_ie_j \\ &= -e_ie_j \\ &= -\delta_{ij} - \varepsilon_{ijk}E_k. \end{aligned} \tag{23}$$

From this we see that different basis bivectors will anti-commute and their product will again be a bivector and that the product of equal basis bivectors is equal to -1.

If we look at the direct sum of the scalars and the bivectors with the geometric product, we get a subalgebra which looks very similar to that of the quaternions except for the fact that the product of the three basis bivectors E_1, E_2, E_3 results in 1 instead of -1.

$$E_1E_2E_3 = (e_2e_3)(e_3e_1)(e_1e_2) = e_2e_2 = 1 \tag{24}$$

In order to solve this last discrepancy, we invert all basis vectors to $E_i' = -E_i$ to get:

$$E_1'E_2'E_3' = (e_3e_2)(e_1e_3)(e_2e_1) = (e_3e_2e_1)(e_3e_2e_1) = I^{-1}I^{-1} = (-I)(-I) = -1 \tag{25}$$

So with the correspondence: $1 \leftrightarrow \mathbf{1}, e_3e_2 \leftrightarrow \mathbf{i}, e_1e_3 \leftrightarrow \mathbf{j}, e_2e_1 \leftrightarrow \mathbf{k}$, we see that the even subalgebra of scalars and bivectors in $\mathcal{G}_3$ is isomorphic to the quaternions.

3.2.1 The pseudoscalar in $\mathcal{G}_3$

The basis vector with the highest grade in $\mathcal{G}_3$ is the vector $e_1e_2e_3$ which is an oriented volume with unit magnitude and it spans a 1-dimensional subspace of $\mathcal{G}_3$. This vector is again called the pseudoscalar with symbol I.

The product of a vector with the pseudoscalar commutes because the pseudoscalar commutes with every basis vector. It takes 3 interchanges to move the basis vector through the pseudoscalar from the leftmost to the rightmost position. Although it would seem that this would product a minus sign, every basis vector gets exchanged with itself in the proces and the product of a vector with itself commutes of course. Therefore the effective number of exchanges is 2 and therefore there is no sign change overall. Because we can repeat this process for a product of several basis 1-vectors the pseudoscalar commutes with basis vectors of all grades in $\mathcal{G}_3$ and therefore with all multivectors.

The product of the pseudoscalar I with a vector maps a vector to a bivector and we get the inverse mapping by multiplying a bivector by $I^{-1} = -I$. Therefore the spaces of vectors and bivectors are called dual spaces. We also can consider the spaces of scalars and pseudoscalars as dual and this forms the basis for the term pseudoscalar. We already saw that dual spaces have the same number of basis vectors and now we can also define a bijection between dual spaces by using the pseudoscalar. Similar to the terms scalar and pseudoscalar, we also have the terms vector and pseudovector, where a pseudovector is an element of the dual space of the vectors. The term pseudovector is also used in the context of polar and axial vectors and we will say more about this when we have had a look at reflections and rotations in 3D.

3.2.2 The product of a vector with a bivector

In 2D we saw that multiplying a vector with a bivector resulted in a rotation over 90 degrees together with a scaling equal to the magnitude of the bivector. The direction of rotation depended on the orientation of the bivector and whether we multiplied from the left or from the right. Therefore multiplication of a vector with a bivector was anti-commutative. In 3D we can have a component of the vector a perpendicular to the plane determined by the bivector B and therefore we can decompose a as: $a = a_{\parallel B} + a_{\perp B}$.

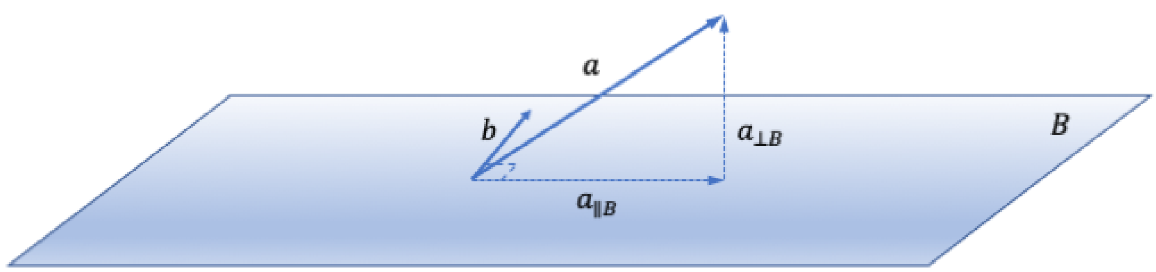


Figure 4: Multiplication of a vector a with a bivector B

The parallel component $a_{\parallel B}$ will be rotated and scaled by the bivector B in the same way as it did in 2D. In the plane determined by the bivector B we can always find a unique vector b perpendicular to $a_{\parallel B}$ such that $B = a_{\parallel B} \wedge b = a_{\parallel B}b$. We have:

$$a_{\parallel B}B = a_{\parallel B}a_{\parallel B}b = a_{\parallel B}^2 b \tag{26}$$

and since $a_{\parallel B}^2$ is a scalar, $a_{\parallel B}B$ therefore is a vector.

To see what happens to the the perpendicular component $a_{\perp B}$ we observe that the vectors $a_{\perp B}, a_{\parallel B}$ and b are mutually perpendicular, they all anti-commute and therefore their product is a 3-vector. Also we have:

$$a_{\perp B}B = a_{\perp B}a_{\parallel B}b = -a_{\parallel B}a_{\perp B}b = a_{\parallel B}ba_{\perp B} = Ba_{\perp B} \tag{27}$$

which shows that $a_{\perp B}$ commutes with B.

So in 3D the product of a vector with a bivector will in general be the sum of a vector (grade 1) and a pseudoscalar (grade 3).

Up till now we have defined the geometric product in terms of the inner and outer product for two vectors. Now we want to extend this definition to products of three vectors in such a way that the geometric product is associative.

Similar to the definition for two vectors, we would like to define the geometric product of a vector a with a bivector B as a combination of the inner and outer product as follows:

$$\begin{aligned} aB &= a \cdot B + a \wedge B \\ Ba &= B \cdot a + B \wedge a \end{aligned} \tag{28}$$

Now looking at the geometric product of a vector a with a bivector $b \wedge c$ we have:

$$a(b \wedge c) = a \cdot (b \wedge c) + a \wedge (b \wedge c) \tag{29}$$

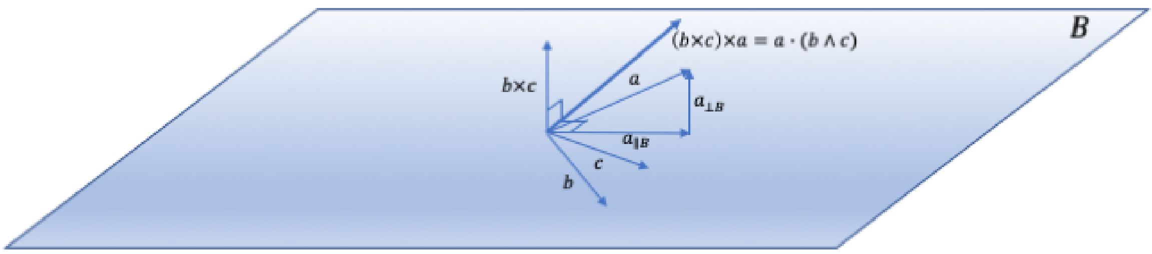


Figure 5: How to calculate $a \cdot (b \wedge c)$?

The only term that is not yet clear is $a \cdot (b \wedge c)$, but we can use the vector product to help us find it. We define the inner product to be distributive and we make the (reasonable) assumption that the inner product of a vector that is perpendicular to some plane B will vanish for every bivector with the same direction as that plane.

$$
\begin{aligned}
a \cdot (b \wedge c) &= (a_{\|B} + a_{\perp B}) \cdot (b \wedge c) \\
&= a_{\|B} \cdot (b \wedge c) + a_{\perp B} \cdot (b \wedge c) \\
&= a_{\|B} \cdot (b \wedge c) \\
&= a_{\|B} \cdot (b \wedge c) + a_{\|B} \wedge (b \wedge c) \\
&= a_{\|B}(b \wedge c)
\end{aligned} \tag{30}
$$

Right multiplying $a_{\|B}$ by $b \wedge c$ will rotate $a_{\|B}$ over 90 degrees with the same orientation as $b \wedge c$ and will scale by the magnitude of $b \wedge c$ which is equal to $|b||c| \sin(b \angle c)$. But we get the same result if we calculate the vector product $(b \times c) \times a_{\|B} = (b \times c) \times a$ and therefore:

$$
\begin{aligned}
a \cdot (b \wedge c) &= a_{\|B}(b \wedge c) \\
&= (b \times c) \times a_{\|B} \\
&= (b \times c) \times a \\
&= (a \cdot b)c - (a \cdot c)b,
\end{aligned} \tag{31}
$$

where the last step uses an identity for vector products. So the final expression for $a(b \wedge c)$ becomes:

$$
a(b \wedge c) = (a \cdot b)c - (a \cdot c)b + a \wedge (b \wedge c) \tag{32}
$$

Similarly we have:

$$
\begin{aligned}
(b \wedge c)a &= (b \wedge c) \cdot a + (b \wedge c) \wedge a \\
&= (b \wedge c) \cdot a + a \wedge b \wedge c
\end{aligned} \tag{33}
$$

With a shortened derivation similar to that of $a \cdot (b \wedge c)$ we get:

$$
\begin{aligned}
(b \wedge c) \cdot a &= (b \wedge c) \cdot a_{\|b \wedge c} \\
&= (b \wedge c) a_{\|b \wedge c} \\
&= -a_{\|b \wedge c}(b \wedge c) \\
&= -a \cdot (b \wedge c) \\
&= (a \cdot c)b - (a \cdot b)c.
\end{aligned} \tag{34}
$$

Substituting this result gives:

$$
\begin{aligned}
(b \wedge c)a &= (b \wedge c) \cdot a + (b \wedge c) \wedge a \\
&= (a \cdot c)b - (a \cdot b)c + a \wedge b \wedge c
\end{aligned} \tag{35}
$$

Now looking at the geometric product of three vectors we have:

$$
\begin{aligned}
a(bc) &= a(b \cdot c + b \wedge c) \\
&= (b \cdot c)a + a \cdot (b \wedge c) + a \wedge (b \wedge c) \\
&= (a \cdot b)c - (a \cdot c)b + (b \cdot c)a + a \wedge b \wedge c,
\end{aligned} \tag{36}
$$

Similarly we have:

$$
\begin{aligned}
(ab)c &= (a \cdot b + a \wedge b)c \\
&= (a \cdot b)c + (a \wedge b) \cdot c + (a \wedge b) \wedge c \\
&= (a \cdot b)c - (a \cdot c)b + (b \cdot c)a + a \wedge b \wedge c,
\end{aligned} \tag{37}
$$

which shows that with these definitions the geometric product is indeed associative.

We have been defining the geometric product in terms of the inner and outer product, but now we can also define the inner and outer product in terms of the geometric product. This will be the strategy used when we want to generalize further to higher dimensions, where we can rely less on our geometric intuition and cannot make use of the vector product. From our previous results we can deduce:

$$\begin{aligned} a \cdot B &= a \cdot (b \wedge c) \\ &= (a \cdot b)c - (a \cdot c)b \\ &= -(b(a \cdot c) - c(a \cdot b)) \\ &= -(b \wedge c) \cdot a \\ &= -B \cdot a \end{aligned} \tag{38}$$

and

$$\begin{aligned} a \wedge B &= a \wedge (b \wedge c) \\ &= (b \wedge c) \wedge a \\ &= B \wedge a \end{aligned} \tag{39}$$

Using this and: $aB = a \cdot B + a \wedge B$ and $Ba = B \cdot a + B \wedge a = -a \cdot B + a \wedge B$ we find:

$$\begin{aligned} a \cdot B &= \frac{aB - Ba}{2} \quad \text{and} \\ a \wedge B &= \frac{aB + Ba}{2} \end{aligned} \tag{40}$$

Notice that with respect to the definition for two vectors, the plus and minus signs have changed places.

Finally we can also define the inner product of two bivectors: $(a \wedge b) \cdot (c \wedge d)$.

We can write the dot product of two vectors a and b as $a \cdot b = (a \cdot b)b^{-1}b$, which is the geometric product of the projection of a onto the line spanned by b, with b. Similarly the dot product of two bivectors $a \wedge b$ and $c \wedge d$ can be defined as the projection of the bivector $a \wedge b$ on the plane spanned by $c \wedge d$, multiplied by $c \wedge d$. The projection of $a \wedge b$ on $c \wedge d$ is equal to the outer product of the projections of a and b on $c \wedge d$. If we write $B = c \wedge d$) then the projection of a on B is given by $(a \cdot B)B^{-1}$ and because it is in the same plane, it anti-commutes with B. Similar relationships hold for the projection of b on B. Therefore we can write:

$$\begin{aligned} (a \wedge b) \cdot (c \wedge d) &= ((a \cdot B)B^{-1} \wedge (b \cdot B)B^{-1})B \\ &= \tfrac{1}{2}((a \cdot B)B^{-1}(b \cdot B)B^{-1} - (b \cdot B)B^{-1}(a \cdot B)B^{-1})B \\ &= \tfrac{1}{2}((a \cdot B)B^{-1}(b \cdot B) - (b \cdot B)B^{-1}(a \cdot B)) \\ &= -\tfrac{1}{2}((a \cdot B)(b \cdot B)B^{-1} - (b \cdot B)(a \cdot B)B^{-1}) \\ &= -\tfrac{1}{2}((a \cdot B)(b \cdot B) - (b \cdot B)(a \cdot B))B^{-1} \\ &= -((a \cdot B) \wedge (b \cdot B))B^{-1} \\ &= -(a \cdot (c \wedge d)) \wedge (b \cdot (c \wedge d))(c \wedge d)^{-1} \\ &= -(((a \cdot c)d - (a \cdot d)c) \wedge ((b \cdot c)d - (b \cdot d)c))(c \wedge d)^{-1} \\ &= -((a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c))(c \wedge d)(c \wedge d)^{-1} \\ &= -((a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)) \\ &= -\begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix} \end{aligned} \tag{41}$$

The inner product of a bivector $c \wedge d$ with itself gives:

$$\begin{aligned}(c \wedge d) \cdot (c \wedge d) &= (c \cdot d)(d \cdot c) - (c \cdot c)(d \cdot d) \\ &= -\begin{vmatrix} c \cdot c & c \cdot d \\ d \cdot c & d \cdot d \end{vmatrix} \\ &= -(c^2 d^2 - (c \cdot d)^2) \\ &= -c^2 d^2 \sin^2(c \angle d),\end{aligned} \tag{42}$$

which is a negative number, which could be expected because the square of a bivector is a negative scalar.

3.2.3 Trivectors

The highest grade objects in 3D that we can have are 3-vectors and they form a 1-dimensional subspace of $\mathcal{G}_3$. We can envisage them as volumes in space with a certain magnitude and orientation and because their subspace is 1 dimensional they all have the same direction. If we have an orthonormal basis $\{e_1, e_2, e_3\}$ for the underlying vector space, a basis vector for the subspace of trivectors could be $e_1 e_2 e_3$. Because the subspace of scalars also has a dimension of 1, the elements of the trivector subspace are also called pseudoscalars. The pseudoscalar which has a magnitude of 1 and the same orientation as a chosen orientation for the 3D space is called the unit pseudoscalar and is denoted by the symbol I. As soon as we have chosen a basis vector $I = e_1 e_2 e_3$ for the trivector subspace, we can find the orientation of any trivector by multiplying it by I. If the product is a negative scalar then it has positive orientation and negative orientation if the product is a positive scalar. The product of the unit pseudoscalar with itself is $II = (e_1 e_2 e_3)(e_1 e_2 e_3) = -(e_1 e_2 e_3)(e_3 e_2 e_1) = -1$ and therefore I has an inverse equal to $-I$. Suppose we multiply a basis vector e_1 by $I = e_1 e_2 e_3$ we get $e_1(e_1 e_2 e_3) = e_2 e_3$ which is a bivector. It easy to see that this is true for all basis vectors and therefore also for all vectors. Because I has an inverse, there exists a bijective mapping between vectors and bivectors. In a similar ways as for scalars and trivectors we call the bivectors pseudovectors. This relationship is denoted by the term duality so scalars are the dual of trivectors and vectors the dual of bivectors. The notation used is that of a star: $a^* = aI$.

The duality relation also provides us with a way to relate the familiar vector product to the outer product. If we define the bivectors E_k as $E_k = I e_k$ then we have:

$$\begin{aligned} I E_k &= I I e_k \\ &= -e_k \end{aligned} \tag{43}$$

And therefore we have:

$$\begin{aligned} -I a \wedge b &= -I \varepsilon_{ijk} a_i b_j E_k \\ &= \varepsilon_{ijk} a_i b_j (-I E_k) \\ &= \varepsilon_{ijk} a_i b_j e_k \\ &= a \times b \end{aligned} \tag{44}$$

and therefore the inverse relationship $a \wedge b = I a \times b$.

We could have used this to find the expression for $a \cdot (b \wedge c)$ from our experience with the vector product:

$$
\begin{aligned}
a \cdot (b \wedge c) &= \frac{a(b \wedge c) - (b \wedge c)a}{2} \\
&= I\frac{a(-I)(b \wedge c) - (-I)(b \wedge c)a}{2} \\
&= I\frac{a(b \times c) - (b \times c)a}{2} \\
&= I(a \wedge (b \times c)) \\
&= -a \times (b \times c) \\
&= -((a \cdot c)b - (a \cdot b)c) \\
&= (a \cdot b)c - (a \cdot c)b
\end{aligned} \tag{45}
$$

Earlier we found for the geometric product of two basis vectors e_i and e_j:

$$
e_i e_j = \delta_{ij} + I\epsilon_{ijk}e_k \tag{46}
$$

This looks very similar to the expression for the Pauli matrices which is given by:

$$
\sigma_i \sigma_j = \delta_{ij}\mathsf{I} + i\,\epsilon_{ijk}\sigma_k \tag{47}
$$

where I is the identity matrix and the Pauli matrices σ_i are given by:

$$
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{48}
$$

In quantum mechanics, the Pauli matrices are used to model the spin of particles, but we now see that they also form a matrix representation for the algebra of 3-space.

3.2.4 Projection and Rejection in 3D

The formulas for projection, rejection and reflection of a vector in a vector (line) are identical to the ones we found in 2D. With one dimension extra, we now also have the possibility to reflect a vector in a plane. We first recall the projection $a_{\|b}$ and rejection $a_{\perp b}$ of a vector a on a vector b, where $a = a_{\|b} + a_{\perp b}$.

$$
\begin{aligned}
a_{\|b} &= (a \cdot b)b^{-1} \\
a_{\perp b} &= (a \wedge b)b^{-1}
\end{aligned} \tag{49}
$$

With one dimension extra, we now also have the possibility to reflect a vector in a plane determined by a bivector B.
Similar to the procedure in 2D, we can determine the projection $a_{\|B}$ of a vector a on a bivector B, which represents the component of a in the plane and the rejection $a_{\perp B}$ which is the component of a perpendicular to the plane. Because we assume that the bivector B is not equal to 0, it has an inverse B^{-1} so we can write:

$$
\begin{aligned}
a &= a(BB^{-1}) \\
&= (aB)B^{-1} \\
&= (a \cdot B + a \wedge B)B^{-1} \\
&= (a \cdot B)B^{-1} + (a \wedge B)B^{-1} \\
&= a_{\|B} + a_{\perp B}
\end{aligned} \tag{50}
$$

If we write $B = c \wedge d$ and $B^{-1} = (c \wedge d)^{-1} = (c \wedge d)/(c \wedge d)^2$ we get:

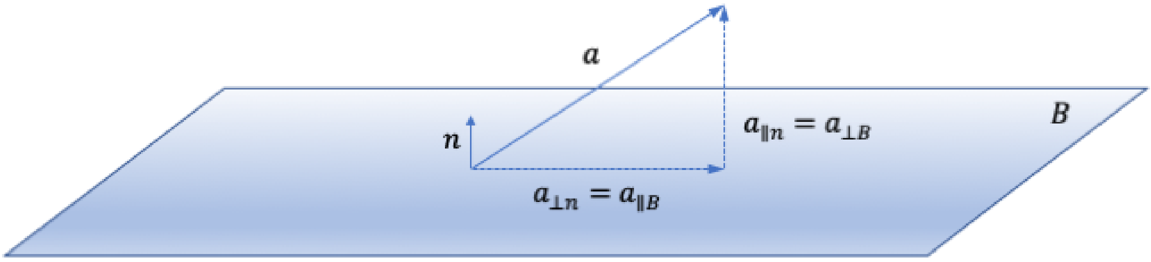


Figure 6: Projection of a vector on a plane B

$$
\begin{aligned}
(a \cdot B)B^{-1} &= (a \cdot (c \wedge d))(c \wedge d)/(c \wedge d)^2 \\
&= ((a \cdot c)d - (a \cdot d)c)(c \wedge d)/(c \wedge d)^2
\end{aligned}
\tag{51}
$$

Substituting $c(c \wedge d) = (c \cdot c)d - (c \cdot d)c$, $d(c \wedge d) = (d \cdot c)d - (d \cdot d)c$ and $(c \wedge d)^2 = (c \wedge d) \cdot (c \wedge d) = (c \cdot d)^2 - (c \cdot c)(d \cdot d)$, we can write this as:

$$
\frac{\begin{vmatrix} a \cdot c & a \cdot d \\ c \cdot d & d \cdot d \end{vmatrix} c - \begin{vmatrix} a \cdot c & a \cdot d \\ c \cdot c & d \cdot c \end{vmatrix} d}{\begin{vmatrix} c \cdot c & c \cdot d \\ c \cdot d & d \cdot d \end{vmatrix}}
\tag{52}
$$

We can also project a bivector $A = a \wedge b$ on a plane determined by a bivector $B = c \wedge d$. We define this as the outer product of the projections $a_{\|B}$ of the vector a and $b_{\|B}$ of the vector b on the plane determined by B. The result is a bivector again:

$$
\begin{aligned}
a_{\|B} \wedge b_{\|B} &= (a \cdot (c \wedge d)) \wedge (b \cdot (c \wedge d)) \\
&= ((a \cdot c)d - (a \cdot d)c) \wedge ((b \cdot c)d - (b \cdot d)c) \\
&= ((a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c))(c \wedge d)
\end{aligned}
\tag{53}
$$

which can also be written as:

$$
\frac{\begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix}}{\begin{vmatrix} c \cdot c & c \cdot d \\ d \cdot c & d \cdot d \end{vmatrix}}(c \wedge d)
\tag{54}
$$

3.2.5 Reflections in 3D

In 3D we will be looking at reflection in a plane and therefore it is natural to use the unit normal vector n to the plane or the bivector B that determines the plane to describe the reflection. First we decompose the vector a into a component $a_{\|n}$ parallel to n and $a_{\perp n}$ perpendicular to n.

For reflection of a vector a in a plane we can use the unit normal vector n to the plane. The reflected vector $\mathsf{M}(a)$ is given by:

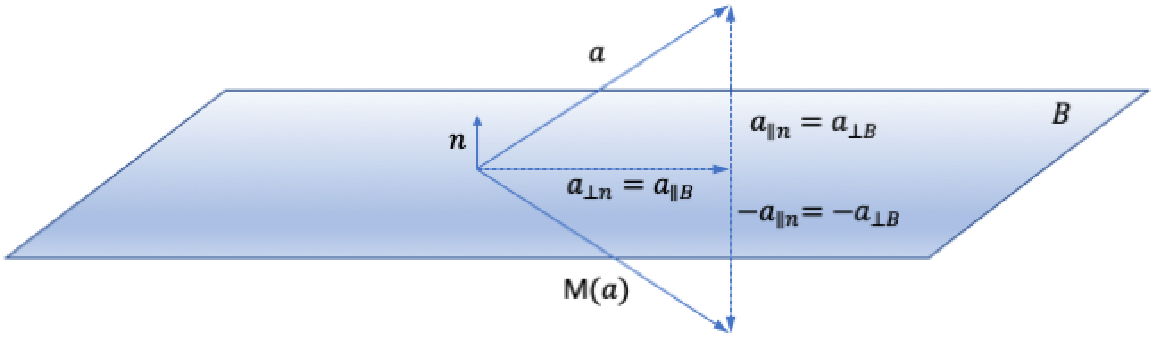


Figure 7: Reflection of a vector in 3D

$$
\begin{aligned}
\mathsf{M}_{\perp \mathsf{n}}(a) &= a_{\perp n} - a_{\| n} \\
&= (a \wedge n)n - (a \cdot n)n \\
&= -((a \cdot n) - (a \wedge n))n \\
&= ((n \cdot a) + (n \wedge a))n \\
&= -nan
\end{aligned}
\tag{55}
$$

If we use the bivector B that determines the plane we get:

$$
\begin{aligned}
\mathsf{M}_{\mathsf{B}}(a) &= a_{\| B} - a_{\perp B} \\
&= (a \cdot B)B^{-1} - (a \wedge B)B^{-1} \\
&= (a \cdot B - a \wedge B)B^{-1} \\
&= (\tfrac{1}{2}(aB - Ba) - \tfrac{1}{2}(aB + Ba))B^{-1} \\
&= -BaB^{-1}
\end{aligned}
\tag{56}
$$

3.2.6 Rotations in 3D

Similar to the way it was done in 2D, we get a rotation in 3D by two consecutive reflections, but in stead of reflecting in two lines, we will now use two planes. For this case, it is convenient to describe the planes by their unit normal vectors m and n. The rotation will then take place in the plane represented by $m \wedge n$ over twice the angle between m and n. Vectors that are orthogonal to both m and n will not be changed by the two reflections since they lie in both the planes determined by m and n. By using the formulas for reflecting in a plane by using its normal vector we get:

$$
\begin{aligned}
\mathsf{R}_{\mathsf{mn}}(a) &= \mathsf{M}_{\perp \mathsf{n}}(\mathsf{M}_{\perp \mathsf{m}}(a)) \\
&= -n(-mam)n \\
&= nmamn
\end{aligned}
\tag{57}
$$

The formula shows that this result is the same as if we had done consecutive reflections in the normal vectors m and n, because in stead of $-mam$ for reflection in a plane, we use mam for reflection in a vector, but this does not change the result because the minus signs cancel.
We can now define an object which we will call a rotor R by: $R = nm$ and rewrite our formula for rotation as:

$$
\mathsf{R}_{R}(a) = RaR^{\dagger} \tag{58}
$$

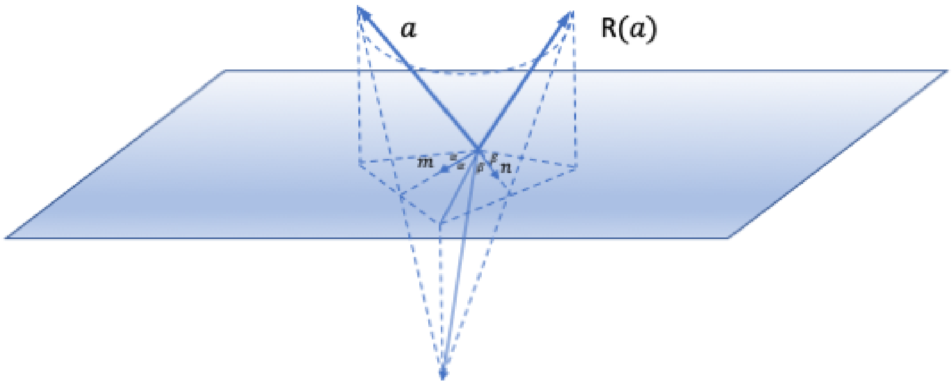


Figure 8: Rotation of a vector in 3D

In this formula we have used an operation called reversion, which is indicated by a dagger (†). As its name already suggests, the reversion operator reverse the order of a product of vectors so in this case we have: $(nm)^\dagger = mn$.

Because the formula was derived using only the formula for reflection in a vector, which works in vector spaces of all dimensions, the rotation formula also works in all dimensions. It is easy to see that:

$$RR^\dagger = R^\dagger R = 1, \tag{59}$$

and therefore the rotation of the product of vectors is the product of the rotated vectors:

$$R(ab)R^\dagger = (RaR^\dagger)(RbR^\dagger). \tag{60}$$

It is clear that this can be extended to sums of products of arbitrary length. For example, we can use it to show that the inner product of two vectors is invariant under rotation:

$$\begin{aligned}
\mathsf{R}(a)\cdot\mathsf{R}(b) &= (RaR^\dagger)\cdot(RbR^\dagger) \\
&= \tfrac{1}{2}(RaR^\dagger RbR^\dagger + RbR^\dagger RaR^\dagger) \\
&= \tfrac{1}{2}(RabR^\dagger + RbaR^\dagger) \\
&= R\tfrac{1}{2}(ab+ba)R^\dagger \\
&= R(a\cdot b)R^\dagger \\
&= (a\cdot b)RR^\dagger \\
&= a\cdot b
\end{aligned} \tag{61}$$

It is also easy to see that the inverse of a rotation mapping is given by $\mathsf{R}^{-1}(a) = R^\dagger a R$ because:

$$\begin{aligned}
\mathsf{R}(\mathsf{R}^{-1}(a)) &= R(R^\dagger aR)R^\dagger = a \quad \text{and} \\
\mathsf{R}^{-1}(\mathsf{R}(a)) &= R^\dagger(RaR^\dagger)R = a
\end{aligned} \tag{62}$$

When we considered rotations in 2D, we already saw that we can write $R = nm$ in exponential form:

$$\begin{aligned}
R &= mn = e^{-\varphi I_B} \quad \text{and} \\
R^\dagger &= nm = e^{\varphi I_B},
\end{aligned} \tag{63}$$

where I_B is the unit bivector $\frac{m\wedge n}{|m\wedge n|}$ which is the unit pseudoscalar for the plane B represented by $m\wedge n$ and φ is the angle between m and n. With this notation our rotation formula becomes:

$$\mathsf{R}(a) = e^{-\varphi I_B}\, a\, e^{\varphi I_B}. \tag{64}$$

We can decompose a into components $a_{\perp B}$ and $a_{\parallel B}$ and because $a_{\perp B}$ is perpendicular to both m and n it will commute with I_B and as we saw in 2D, the parallel component $a_{\parallel B}$ will anticommute with $I_m n$. Therefore we get:

$$\begin{aligned} e^{\varphi I_B} a_{\parallel B} &= (\cos\varphi + I_B \sin\varphi) a_{\parallel B} \\ &= a_{\parallel B}(\cos\varphi - I_B \sin\varphi) \\ &= a_{\parallel B} e^{-\varphi I_B} \end{aligned} \tag{65}$$

and

$$\begin{aligned} e^{\varphi I_B} a_{\perp B} &= (\cos\varphi + I_B \sin\varphi) a_{\perp B} \\ &= a_{\perp B}(\cos\varphi + I_B \sin\varphi) \\ &= a_{\perp B} e^{\varphi I_B} \end{aligned} \tag{66}$$

If we now rewrite our rotation formula:

$$\begin{aligned} \mathsf{R}(a) &= e^{-\varphi I_B}\, a\, e^{\varphi I_B} \\ &= e^{-\varphi I_B}\, (a_{\perp B} + a_{\parallel B})\, e^{\varphi I_B} \\ &= e^{-\varphi I_B}\, a_{\perp B}\, e^{\varphi I_B} + e^{-\varphi I_B}\, a_{\parallel B}\, e^{\varphi I_B} \\ &= a_{\perp B} e^{-\varphi I_B}\, e^{\varphi I_B} + a_{\parallel B}\, e^{\varphi I_B} e^{\varphi I_B} \\ &= a_{\perp B} + a_{\parallel B}\, e^{2\varphi I_B} \end{aligned} \tag{67}$$

We see that the perpendicular component is not affected by the rotation and the parallel component is rotated over an angle 2φ with orientation given by I_B. Therefore if we want to specify a rotor that rotates over an angle θ in an oriented plane B, we should write $R = e^{-(\theta/2) I_B}$.

3.2.7 Polar and axial vectors

In physics there are quantities that can be represented by vectors, but not all vector quantities behave the same way under some symmetry transformations like reflections and rotations. The transformation that is often used to show the difference is called an inversion and it can be viewed as the composition of a reflection in some plane followed by a rotation over an angle π in the same plane. For a vector quantity like velocity, this transformation will map a velocity v to $-v$ and this type of vector is called a polar vector. But for a vector quantity like angular momentum $L = r \times p$, r will be mapped to $-r$ and p to $-p$ but the quantity $L = r \times p = -r \times -p$ will be left unchanged. This type of vector quantity is called an axial vector or sometimes also a pseudovector. As vectors of $\mathbb{R}^3$ we can not tell the difference, but in $\mathcal{G}_3$ we represent polar vectors by 1-vectors and axial vectors by 2-vectors which have exactly the required symmetry properties, but are different objects. If by n we denote the normal vector to our plane of reflection and rotation, our formulas for reflection and rotation of a vector a become:

$$\begin{aligned} \mathsf{M}_{\perp \mathsf{n}}(a) &= -nan, \\ R(\pi) &= \cos(\pi/2) - \sin(\pi/2) In = -In, \\ \mathsf{R}_{\perp n} &= R(\pi)\, a\, R(\pi)^{\dagger} = -InaIn = nan, \\ \mathsf{R}_{\perp n}(\mathsf{M}_{\perp \mathsf{n}}(a)) &= n(-nan)n = -a. \end{aligned} \tag{68}$$

Geometrically we can see that:

$$\begin{aligned} \mathsf{M}_{\perp\mathsf{n}}(a \wedge b) &= \mathsf{M}_{\perp\mathsf{n}}(a) \wedge \mathsf{M}_{\perp\mathsf{n}}(b) \quad \text{and} \\ \mathsf{R}_{\perp\mathsf{n}}(a \wedge b) &= \mathsf{R}_{\perp\mathsf{n}}(a) \wedge \mathsf{R}_{\perp\mathsf{n}}(b). \end{aligned} \tag{69}$$

Mappings which preserve the outer product are called outermorphisms and it is easy to see that compositions of outermorphisms are again outermorphisms.
Therefore we get:

$$\mathsf{R}_{\perp n}(\mathsf{M}_{\perp\mathsf{n}}(a \wedge b)) = (-a) \wedge (-b) = a \wedge b, \tag{70}$$

which shows that vectors and bivectors have the same properties under inversion as polar vectors and axial vectors respectively.

3.3 Axiomatic approach to Geometric Algebra

A more general approach to geometric algebra is to provide axioms which formulate the essential properties it should have. An advantage is that it turns out to be possible to find more than one model that satisfies these axioms and therefore reveals the common structure between different applications. In this thesis we can only shortly show how these axioms can be formulated but to further exploration of their consequences must be left for further study and the interested reader is referred to the references mention in the bibliography.

$\mathcal{G}$ is a set with operations addition and multiplication. The elements of $\mathcal{G}$ are called multivectors.

The addition has properties:

$$\begin{aligned} (\mathbf{A1}):&\quad A+(B+C)=(A+B)+C \\ (\mathbf{A2}):&\quad A+0=A \\ (\mathbf{A3}):&\quad A+(-A)=0 \\ (\mathbf{A4}):&\quad A+B=B+A \end{aligned} \tag{71}$$

Multiplication has properties:

$$\begin{aligned} (\mathbf{A5}):&\quad A(BC)=(AB)C \\ (\mathbf{A6}):&\quad 1\,A=A \end{aligned} \tag{72}$$

Multiplication is left and right distributive with respect to addition:

$$\begin{aligned} (\mathbf{A7}):&\quad A(B+C)=AB+AC \\ (\mathbf{A8}):&\quad (A+B)C=AC+BC \end{aligned} \tag{73}$$

We define functions $\langle . \rangle_k : \mathcal{G} \to \mathcal{G}^k$, where $k \in \mathbb{N}$ and $\mathcal{G}^k = \langle \mathcal{G} \rangle_k \subset \mathcal{G}$, with the following properties:

$$\begin{aligned} (\mathbf{A9}):&\quad \langle A+B\rangle_k = \langle A\rangle_k + \langle B\rangle_k \\ (\mathbf{A10}):&\quad \langle \lambda A\rangle_k = \lambda\langle A\rangle_k = \langle A_k\rangle\lambda, \quad (\lambda \in \mathcal{G}^0) \\ (\mathbf{A11}):&\quad \langle B_l\rangle_k = \delta_{kl}B_l \qquad\qquad (B_l \in \mathcal{G}^l) \\ (\mathbf{A12}):&\quad A = \sum_k \langle A\rangle_k \end{aligned} \tag{74}$$

These properties make $\mathcal{G}$ and every $\mathcal{G}^k$ into a vector space over $\mathcal{G}^0$ and $\mathcal{G}^0$ into a field. The elements of $\mathcal{G}^k$ are called k-vectors and the elements of $\mathcal{G}^0$ in particular are also called scalars because of their special relation to the $\mathcal{G}^k$ as vector spaces.

It follows that every multivector in $\mathcal{G}$ can be uniquely written as the sum of elements of the vector spaces $\mathcal{G}^k$:

$$\mathcal{G} = \bigoplus_{k\in\mathbb{N}} \mathcal{G}^k \tag{75}$$

All these properties make $\mathcal{G}$ a unital associative algebra over $\mathcal{G}^0$.

Associated with $\mathcal{G}$ is quadratic form $q : \mathcal{G}^1 \to \mathcal{G}^0$ which is used in the definition of the geometric product of a 1-vector with itself:

$$(\mathbf{A13}): \quad aa = q(a) \tag{76}$$

An important relation which follows from this axiom is:

$$q(a+b) - q(a) - q(b) = (a+b)^2 - a^2 - b^2 = ab + ba, \tag{77}$$

which is a scalar and is closely related to the definition of the inner product.

For the last axiom we need a definition first:

A k-blade is a k-vector B_k that can be written as a product of k mutually anti-commuting vectors: $B_k = a_1 a_2 \dots a_k \quad (a_i a_j = -a_j a_i, i \neq j)$.

$$\textbf{(A14)} \text{ Every } k\text{-vector can be written as a sum of } k\text{-blades.} \tag{78}$$

If we assume that the vector space $\mathcal{G}^1$ has a finite dimension n, then the highest k for which $\mathcal{G}^k$ differs from $\{0\}$ will be also n. This is the case because we cannot have blades which consist of more than n anti-commuting vectors, since the vectors in such a product must be independent and we can not have more than n independent vectors in $\mathcal{G}^1$.

With the help of the above axioms, we can define the inner and outer product using the geometric product:

First the inner product for homogeneous multivectors $\langle A \rangle_r$ and $\langle B \rangle_s$:

$$\begin{aligned} \langle A \rangle_r \cdot \langle B \rangle_s &= \langle A_r B_s \rangle_{|r-s|} && (\text{if } r, s > 0 \\ \langle A \rangle_r \cdot \langle B \rangle_s &= 0, && (\text{if } r = 0, \text{or } s = 0) \end{aligned} \tag{79}$$

and finally for general multivectors:

$$A \cdot B = \sum_r \sum_s \langle A \rangle_r \cdot \langle B \rangle_s. \tag{80}$$

For the outer product of homogeneous multivectors $\langle A \rangle_r$ and $\langle B \rangle_s$:

$$\langle A \rangle_r \wedge \langle B \rangle_s = \langle A_r B_s \rangle_{r+s} \tag{81}$$

and finally for general multivectors:

$$A \wedge B = \sum_r \sum_s \langle A \rangle_r \wedge \langle B \rangle_s. \tag{82}$$

From these basic definitions a lot of useful results can be derived which are important for the effective use of geometric algebra in practice. Baecause of reasons of time and space, the interested reader is referred to the literature. A good reference is for further study is [Hestenes and Sobczyk, 2012].

3.4 Notes and further references

The first chapters of [Doran et al., 2003] were used as the main source for this thesis. I followed roughly the same order of presenting the material, but in a number of cases a different strategy was taken in the proofs. Hopefully it will make the reader interested in the subject because there is still a lot more to study such as the subject of geometric calculus, which brings geometric algebra and analysis together. This combination again results in compact and elegant coordinate free expressions for mathematics and physics.

The source that I used for the axiomatic approach to geometric algebra is [Hestenes and Sobczyk, 2012]. It contains a lot of additional material and can be considered as the defining reference of geometric algebra and geometric calculus.

A book that contains an introduction to geometric algebra and shows the application of geometric algebra to mechanics is [Hestenes, 2012].

A more mathematical approach to Clifford algebras can be found in [Vaz Jr and da Rocha Jr, 2016].

A nice reference to read more about the interesting history of the subject is [Crowe, 1994].

A very readable introductions to geometric algebra and geometric calculus is given in [Macdonald, 2010] and [Macdonald, 2012].

Another area of application of geometric algebra is computer science, especially for calculations involved in visualization of images from different perspectives. A good source which also treats different possible models for the axioms of geometric algebra is [Dorst et al., 2010].

References

[Crowe, 1994] Crowe, M. J. (1994). *A history of vector analysis: The evolution of the idea of a vectorial system*. Courier Corporation.

[Doran et al., 2003] Doran, C., Lasenby, A., and Lasenby, J. (2003). *Geometric algebra for physicists*. Cambridge University Press.

[Dorst et al., 2010] Dorst, L., Fontijne, D., and Mann, S. (2010). *Geometric algebra for computer science: an object-oriented approach to geometry*. Elsevier.

[Hestenes, 2012] Hestenes, D. (2012). *New foundations for classical mechanics*, volume 15. Springer Science & Business Media.

[Hestenes and Sobczyk, 2012] Hestenes, D. and Sobczyk, G. (2012). *Clifford algebra to geometric calculus: a unified language for mathematics and physics*, volume 5. Springer Science & Business Media.

[Macdonald, 2010] Macdonald, A. (2010). *Linear and geometric algebra*. Alan Macdonald.

[Macdonald, 2012] Macdonald, A. (2012). *Vector and geometric calculus*, volume 1. CreateSpace Independent Publishing Platform.

[Vaz Jr and da Rocha Jr, 2016] Vaz Jr, J. and da Rocha Jr, R. (2016). *An introduction to Clifford algebras and spinors*. Oxford University Press.