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The Skolemization of prenex formulas in intermediate logics

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Abstract

The skolem class of a logic consists of the formulas for which the derivability of the formula is equivalent to the derivability of its Skolemization. In contrast to classical logic, the skolem classes of many intermediate logics do not contain all formulas. In this paper it is proven for certain classes of propositional formulas that any instance of them by (independent) predicate sentences in prenex normal form belongs to the skolem class of any intermediate logic complete with respect to a class of well-founded trees. In particular, all prenex sentences belong to the skolem class of these logics, and this result extends to the constant domain versions of these logics.

Keywords: Skolemization; Prenex formulas; Intuitionistic logic; Intermediate logics

1. Introduction

The connection between quantifiers and functions has a rich history and there are several topics within logic where this connection plays a major role, such as the axiom of choice in set theory, the Dialectica interpretation in constructive mathematics and the Skolemization method in proof theory and computer science.

The first two cases are about the quantifier combination $\forall\exists$, in the sense that the derivability of a formula of the form $\forall x \exists y \varphi(x, y)$ implies the derivability of $\forall x \varphi(x, fx)$ for an appropriate function f. In the case of Skolemization, however, one focusses on the opposite quantifier combination $\exists\forall$ and studies in how far the derivability of a formula of the form $\exists x \forall y \varphi(x, y)$ is equivalent to the derivability of $\exists x \varphi(x, gx)$ for a function symbol g not occurring in φ. This can be viewed as a quantifier combination $\forall\exists$ in disguise if one thinks of g as providing a

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counterexample to $\exists x \forall y \varphi(x, y)$: if $\exists x \forall y \varphi(x, y)$ is not derivable, that is, if $\forall x \exists y \neg\varphi(x, y)$ has a model, then there is a function g such that $\forall x \neg\varphi(x, gx)$ has a model. The question then is in how far such counterexamples g exist. This paper studies this question for propositional combinations of prenex formulas in the setting of intermediate logics.

Usually Skolemization is only defined for prenex formulas, but since in some intermediate logics not every formula has a prenex normal form, the notion of Skolemization has to be extended to infix formulas in the context of such logics. This Skolemization method replaces the strong quantifier occurrences in a formula in the following way, where a quantifier occurrence is *strong* if it is a positive occurrence of a universal quantifier or a negative occurrence of an existential quantifier (strong quantifier occurrences are the ones that become universal under prenexification).

Given a formula φ with n occurrences of strong quantifiers, we define formulas $\varphi_0 = \varphi, \dots, \varphi_n$, where φ_{i+1} is the result of replacing the leftmost strong quantifier occurrence $Qx\psi(x, \bar{y})$ in φ_i by $Q(f_i(\bar{y}), \bar{y})$, where $\bar{y}$ are the variables of the weak quantifiers in the scope of which $Qx\psi(x, \bar{y})$ occurs, and the $f_1, \dots, f_n$ are all distinct and do not occur in φ. In this case we say that f_i *corresponds* to that quantifier occurrence Qx. The *skolemization* φ^s of φ is defined to be φ_n. Note that skolemization is unique up to the name of the skolem functions.

For a given logic L, its *skolem class* consists exactly of those formulas φ such that

$$\vdash_L \varphi \Leftrightarrow \vdash_L \varphi^s. \tag{1}$$

A logic *admits* skolemization if its skolem class consists of all formulas. As is well-known, the skolem class of classical logic consists of all formulas. But for many intermediate logics the skolem class is considerably smaller. For intuitionistic logic IQC the following are two formulas that do not belong to the skolem class of IQC.

$$\varphi_1 = \neg\neg\forall x(\psi(x) \vee \neg\psi(x)) \quad \varphi_2 = \neg\neg\exists x\psi(x) \to \exists x\neg\neg\psi(x).$$

Since the direction from left to right of (1) holds in all intermediate logics, it follows that the above two formulas are not derivable, while their skolemizations

$$\varphi_1^s = \neg\neg(\psi(c) \vee \neg\psi(c)) \quad \varphi_2^s = \neg\neg\psi(c) \to \exists x\neg\neg\psi(x)$$

are derivable in IQC.

For many intermediate logics, including IQC, the skolem class is not completely understood. Mints provided a necessary condition on formulas for belonging to the skolem class of IQC and showed that in the presence of equality even certain prenex formulas do no belong to that class [10–12]. In a series of papers, Baaz and the author have described the skolem class of certain conservative extensions of IQC [2–4]. Among other things it is shown in these papers that if intuitionistic logic is extended with an existence predicate, then every formula without strong universal quantifiers belongs to the skolem class of that logic. Also in the setting of substructural and fuzzy logics, the Skolemization method has been investigated [1,5–7].

In this paper we concentrate on prenex formulas and ask which propositional combinations of prenex sentences belong to the skolem class of intermediate logics. In other words, given a propositional formula $A(p_1, \dots, p_k)$ in which no atom occurs more than once (a *rigid* formula), we ask for which prenex sentences $\varphi_1, \dots, \varphi_k$ the formula $A(\varphi_1, \dots, \varphi_k)$ belongs to the skolem class of a given logic. The logics to which our proofs apply are the logics that are complete with respect to a class of frames which are well-founded trees, and therefore include intuitionistic logic. For the latter logic, several of the theorems below can also be established using proof systems, but for many of the other logics that fall in that class no useful proof system is known, and we do not know of a syntactic way to obtain our results.

The first example above shows that already for the simple propositional formula $\neg\neg p$ there is a prenex formula φ, for example $\varphi = \forall x(\psi(x) \vee \neg\psi(x))$, such that $\neg\neg\varphi$ does not belong to the skolem class of IQC. As far as we understand it at the moment, this is related to the negative occurrence of implication in the formula, which is why in this paper we concentrate on formulas without such negative occurrences. We show for all intermediate logics that we consider, including IQC, that all prenex formulas belong to the skolem class. Moreover, the same is proven for the logics obtained by restricting the models to those with constant domains. For propositional formulas A without negative occurrences of implication the two main things that we prove are the following. If no atom occurs positively in A, then any instance with prenex sentences belongs to the skolem class. If A does contain such positive occurrences, the same holds provided the prenex sentences have no predicates in common. Further results can be found in Sections 6 and 7.

Although we do not obtain a full description of the skolem class of our intermediate logics, we provide examples in Section 7.2 that show that the propositional combinations for which we have proved that their instances belong to the skolem classes cannot be extended much further.

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2. Preliminaries

In this paper propositional formulas are indicated by the roman letters from the beginning of the alphabet, A, B, C. The propositional language consists of atoms, which are denoted by p, q, r or indexed versions thereof, the constants $\top$ and $\bot$ and the connectives $\wedge, \vee, \rightarrow$, where $\neg A$ is defined as $A \rightarrow \bot$. When we write $A(p_1, \dots, p_k)$ we assume that the variables that occur in A are among $p_1, \dots, p_k$. If every atom occurs at most once in a formula, the formula is called *rigid*. If no implication occurs negatively in a formula, the formula is an *nni (no negative implications)* formula.

Predicate formulas are defined as usual and indicated by lower case Greek letters. We consider an arbitrary language $\mathcal{L} \cup \mathcal{V} \cup \mathcal{F}$ for predicate logic, where $\mathcal{V}$ is an infinite set of variables, $\mathcal{L}$ a set of function and relation symbols, and $\mathcal{F}$ a set of function symbols that contains infinitely many functions of every arity, and such that $\mathcal{L} \cap \mathcal{F} = \varnothing$. The results in this paper are independent of the number and arity of the relation and function symbols in $\mathcal{L}$. The functions in $\mathcal{F}$ will function as *skolem functions*, and the requirement guarantees that there are sufficiently many available.

A predicate formula without free variables is called a *sentence*. Given a sequence of elements $\bar{a} = a_1, \dots, a_m$ and a set X, $\bar{a} \in X$ denotes that all elements of $\bar{a}$ belong to X, and for $i, j \leq m$, $\bar{a}^i_j$ denotes the sequence $a_i, \dots, a_j$ and $\bar{a}_j$ the sequence $a_1, \dots, a_j$. Given a formula χ, let $\chi[d/x]$ denote the results of replacing the free occurrences of x in χ by d.

2.1. Models

A *Kripke model* (or *model*) $\mathcal{M}$ consists of a tuple $(W, R, D, I, \Vdash)$, where (W, R) is a rooted frame, meaning that R is a binary relation on the set W of nodes with a root w_0, and D is a collection $D = \{\mathcal{D}_w \mid w \in W\}$ of sets, which are *domains*, and I a collection $I = \{I_w \mid w \in W\}$ of *interpretations*. The domains are nondecreasing: $\mathcal{D}_w \subseteq \mathcal{D}_v$ whenever $w \preccurlyeq v$. $\mathcal{D}$ is the union of all domains in $\mathcal{M}$. The domain $\mathcal{D}_{w_0}$ at the root is also denoted by $\mathcal{D}_0$. Interpretation I_w is similar to an interpretation for $\mathcal{D}_w$ viewed as a classical model, but only for functions: for any n-ary function symbol g, $I_w(g)$ is a function $I_w(g) : \mathcal{D}_w^n \to \mathcal{D}_w$ such that

$$\forall \bar{a} \in \mathcal{D}_w^n : \; w \preccurlyeq v \text{ implies } I_w(g)(\bar{a}) = I_v(g)(\bar{a}).$$

The interpretation is then extended to all terms in $\mathcal{T}(\mathcal{L} \cup \mathcal{F} \cup \mathcal{V})$ in the usual way.

The interpretation of predicates is defined via the forcing relation, which is defined as usual, with the requirement of upwards persistency for atomic formulas $P(\bar{x})$:

$$\forall \bar{a} \in \mathcal{D}_w^n : \; \text{ if } w \preccurlyeq v \text{ and } \mathcal{M}, w \Vdash P(\bar{a}), \text{ then } \mathcal{M}, v \Vdash P(\bar{a}).$$

A *well-founded tree model* is a model whose frame is a well-founded tree. A logic is *Kripke complete* if it is complete with respect to a class of frames, and it is a *well-founded tree* logic if it is complete with respect to a class of frames which are well-founded trees. It is a *constant domain well-founded tree* logic if there exists a class of frames that are well-founded trees such that the logic is complete with respect to the class of all constant domain models on these frames. Given a well-founded tree logic L, let Lcd denote the logic that is complete with respect to the class of all constant domain models on the frames of L. Thus if L is a constant domain well-founded tree logic, then $L = L_{CD}$.

Our results apply to all well-founded tree logics. The following theorem shows that this class includes IQC. The result was first obtained by Takeuti [13] and can also be found in [8,9, Corollary 6.4.19].

Theorem 1. IQC *is sound and complete with respect to the class of well-founded tree models.*

2.2. *Standard prenex formulas*

Throughout the paper, $\Diamond$ is an element of $\{\exists, \forall\}$, and

$$\overline{\Diamond} \equiv_{def} \begin{cases} \forall & \text{if } \Diamond = \exists \\ \exists & \text{if } \Diamond = \forall. \end{cases}$$

When a set of prenex formulas is considered, it is tacitly assumed that no variable occurs in more than one formula. A sequence of formulas $\varphi_1, \ldots, \varphi_k$ is *independent* if no predicate occurs in more than one of the φ_i.

A prenex formula φ is *standard* if it consists of a block of alternating quantifiers, followed by a quantifier free formula: for some variables x_i and y_i,

$$\varphi = \exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \exists x_n \forall y_n \psi(\bar{x}, \bar{y}) \tag{2}$$

or

$$\varphi = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \ldots \forall x_n \exists y_n \psi(\bar{x}, \bar{y}), \tag{3}$$

where ψ is quantifier-free and $n \geq 1$. Thus every standard prenex formula starts with a block of at least two quantifiers, an existential one followed by a universal one, or vice versa. In case (2) φ is also called $\exists$-*standard* and in case (3) it is called $\forall$-*standard*.

In an arbitrary prenex formula $\varphi = \Diamond_1 x_1 \ldots \Diamond_n x_n \psi$, where the $\Diamond_i$ are quantifiers and ψ is quantifier-free, a quantifier $\Diamond_j$ is a *dummy quantifier* if x_j does not occur in the formula ψ. Clearly, by introducing dummy quantifiers, every prenex formula $\varphi = \Diamond_1 x_1 \ldots \Diamond_n x_n \psi (x_1, \ldots, x_n)$, where ψ is quantifier-free, is equivalent in IQC to a $\Diamond$-standard prenex formula $\varphi_{sf}^{\Diamond}$: $\varphi_{sf}^{\Diamond}$ is that $\Diamond$-standard prenex formula such that φ can be obtained from $\varphi_{sf}^{\Diamond}$ by removing some dummy quantifiers from $\varphi_{sf}^{\Diamond}$, and such that no $\Diamond$-standard prenex formula with less quantifiers than $\varphi_{sf}^{\Diamond}$ has that property. For example, if $\varphi = \exists x \exists y P(x, y)$, then $\varphi_{sf}^{\exists} = \exists x \forall z_1 \exists y \forall z_2 P(x, y)$ and $\varphi_{sf}^{\forall} = \forall z_1 \exists x \forall z_2 \exists y P(x, y)$. Clearly, $\varphi_{sf}^{\Diamond}$ is unique up to the renaming of variables, but we assume that we have fixed one particular choice of variables, so that we can think of $\varphi_{sf}^{\Diamond}$ as being uniquely determined by φ.

Given a $\Diamond$-standard prenex formula φ, define:

$$\begin{aligned} \varphi_h^{\Diamond} &\equiv_{def} \Diamond x_h \bar{\Diamond} y_h \Diamond x_{h+1} \bar{\Diamond} y_{h+1} \dots \Diamond x_n \bar{\Diamond} y_n \psi(\bar{x}, \bar{y}) \\ \varphi_{h\Diamond} &\equiv_{def} \Diamond x_h \Diamond x_{h+1} \dots \Diamond x_n \psi(\bar{x}, \bar{y}). \end{aligned}$$

In particular, $\varphi_1^{\Diamond} = \varphi$. Furthermore, φ_{n+1} is by definition ψ.

To indicate its possible free variables, $\varphi_{h\Diamond}$ is denoted by $\varphi_h(x_1, \dots, x_{h-1}, \bar{y})$, meaning that all free variables that actually occur in $\varphi_{h\Diamond}$ are among $x_1, \dots, x_{h-1}, \bar{y}$. Accordingly, we use the convention that for any sequences $\bar{a} = a_1, \dots, a_{h-1}$, $\bar{c} = c_1, \dots, c_{h-1}$ and $\bar{e} = e_1, \dots, e_n$:

$$\begin{aligned} \varphi_h^{\Diamond}(\bar{a}, \bar{c}) &= \Diamond x_h \bar{\Diamond} y_h \dots \Diamond x_n \bar{\Diamond} y_n \psi(\bar{a}, x_h, \dots, x_n, \bar{c}, y_h, \dots, y_n) \\ &= \Diamond x_h \bar{\Diamond} y_h \dots \Diamond x_n \bar{\Diamond} y_n \psi(\bar{a}, \bar{x}_n^h, \bar{c}, \bar{y}_n^h) \\ \varphi_{h\Diamond}(\bar{a}, \bar{e}) &= \Diamond x_h \Diamond x_{h+1} \dots \Diamond x_n \psi(\bar{a}, x_h, \dots, x_n, \bar{e}) \\ &= \Diamond x_h \Diamond x_{h+1} \dots \Diamond x_n \psi(\bar{a}, \bar{x}_n^h, \bar{e}). \end{aligned}$$

2.3. *Skolemization*

The Skolemization method for infix formulas has been defined in the introduction. Below we define for standard prenex formulas φ, the $\exists$-*skolemization* $\varphi_{\exists}^s$ and the $\forall$-*skolemization* $\varphi_{\forall}^s$, which are formulas without universal and existential quantifiers, respectively. As we will see, for a standard prenex formula φ, $\varphi^s = \varphi_{\exists}^s$. The use of these two skolemization variants is explained in Remark 1.

2.3.1. *Skolemization of standard prenex formulas*

Given a standard prenex formula φ with $2n$ quantifiers, choose skolem functions $f_1, \dots, f_n \in \mathcal{F}$, where the arity of f_h is h, and let

$$\mathcal{F}_{\varphi} \equiv_{def} \{f_1, \dots, f_n\}.$$

When the skolemization of more than one formula is considered it is always assumed that no skolem function is used twice.

Given a sequence $\bar{a}$ of length n and a sequence $\bar{c}$ of length h, we define the following sequences of skolem terms:

$$f_h^{\uparrow}(\bar{a}) \equiv_{def} f_h(\bar{a}_h), f_{h+1}(\bar{a}_{h+1}), \dots, f_n(\bar{a}) \quad f_h^{\downarrow}(\bar{c}) \equiv_{def} f_1(\bar{c}_1), f_2(\bar{c}_2), \dots, f_h(\bar{c}).$$

The skolem function $\mathrm{sk}(\varphi, h, \Diamond)$ replaces the variables $y_h, \dots, y_n$ in formula $\varphi_h^{\Diamond}$ by the skolem terms $f_h(\bar{x}_h), f_{h+1}(\bar{x}_{h+1}) \dots, f_n(\bar{x})$ and deletes the corresponding quantifiers:

$$\begin{aligned} \varphi_{h\Diamond}^s \equiv_{def} & \Diamond x_h \Diamond x_{h+1} \dots \Diamond x_n \psi\big(\bar{x}, \bar{y}_{h-1}, f_h^{\uparrow}(\bar{x})\big) \\ & \Diamond x_h \Diamond x_{h+1} \dots \Diamond x_n \psi\big(\bar{x}, \bar{y}_{h-1}, f_h(\bar{x}_h), f_{h+1}(\bar{x}_{h+1}), \dots, f_n(\bar{x})\big). \end{aligned}$$

The $\exists$-*skolemization* $\varphi_{\exists}^s$ and $\forall$-*skolemization* $\varphi_{\forall}^s$ of φ are defined as follows.

$$\varphi_{\Diamond}^s \equiv_{def} \begin{cases} \exists x_1 \dots \exists x_n \psi\big(\bar{x}, f_1(x_1), \dots, f_n(\bar{x})\big) & \text{if } \Diamond = \exists \text{ and } \varphi \text{ is } \exists\text{-standard} \\ \forall x_1 \dots \forall x_n \psi\big(\bar{x}, f_1(x_1), \dots, f_n(\bar{x})\big) & \text{if } \Diamond = \forall \text{ and } \varphi \text{ is } \forall\text{-standard.} \end{cases}$$

Observe that the free variables in $\varphi^s_{h\Diamond}$ are $\bar{x}_{h-1}, \bar{y}_{h-1}$. Therefore, using the same convention as in the previous section, we define for $\bar{a}, \bar{c}$ of length $h-1$:

$$\varphi^s_{h\Diamond}(\bar{a}, \bar{c}) \;\equiv_{def}\; \Diamond x_h \dots \Diamond x_n \psi\big(\bar{a}, \bar{x}^h_n, \bar{c}, f_h(\bar{a}, x_h), f_{h+1}(\bar{a}, \bar{x}^h_{h+1}), \dots, f_n(\bar{a}, \bar{x}^h_n)\big).$$

Remark 1. From the definitions above it follows that for standard prenex formulas φ, $\varphi^s = \varphi^s_\exists$. It also follows that for any propositional formula $A(p_1, \dots, p_k)$ and prenex predicate formulas $\varphi_1, \dots, \varphi_k$, the formula $(A(\varphi_1, \dots, \varphi_k))^s$ is equivalent to the result of substituting $\varphi^s_\exists$ for p_i in case p_i occurs positively in A and substituting $\varphi^s_\forall$ for p_i otherwise. This is exactly the way in which ∃-skolemization and ∀-skolemization are used below.

2.3.2. Prenex formulas and skolem classes

The following lemma implies that for any prenex sentence φ, φ belongs to the skolem class of a logic if and only if $\varphi^\exists_{sf}$ belongs to that skolem class, which is again equivalent to $\varphi^\forall_{sf}$ belonging to that skolem class. For this reason, the main theorems below are about standard prenex sentences.

Lemma 1. *Suppose $\varphi = \Diamond_1 x_1 \dots \Diamond_n x_n \psi(\bar{x})$ is an arbitrary prenex sentence. Let φ_y be the result of inserting a dummy quantifier of the form $\exists y$ or $\forall y$ and denoted by Qy*[2] *between quantifiers $\Diamond_h$ and $\Diamond_{h+1}$, where y is a fresh variable that does not occur in φ. Then in all (constant domain) Kripke complete logics for $\Diamond \in \{\exists, \forall\}$:*

$$\vdash (\varphi_y)^s_\Diamond \;\Leftrightarrow \vdash \varphi^s_\Diamond.$$

In particular, $\vdash (\varphi_y)^s$ if and only if $\vdash \varphi^s$.

Proof. With φ, ψ as in the lemma we have

$$\varphi_y = \Diamond_1 x_1 \dots \Diamond_h x_h Q y \Diamond_{h+1} x_{h+1} \dots \Diamond_n x_n \psi(\bar{x}).$$

Suppose φ is a formula in $\mathcal{L}$ and $\varphi^s_\Diamond$ in $\mathcal{L}_f = \mathcal{L} \cup \mathcal{F}_{\varphi^s_\Diamond} = \mathcal{L} \cup \{f_1, \dots, f_m\}$. If $Q \neq \Diamond$, then $(\varphi_y)^s_\Diamond$ is the result of replacing the symbol f_i in $\varphi^s_\Diamond$ by a skolem function g_i, and the lemma clearly holds.

If $Q = \Diamond$, the number of skolem functions in $\varphi^s_\Diamond$ and $(\varphi_y)^s_\Diamond$ is equal. Let $\mathcal{L}_g = \mathcal{L} \cup \mathcal{F}_{(\varphi_y)^s_\Diamond} = \mathcal{L} \cup \{g_1, \dots, g_m\}$, and we can without loss of generality assume that g_i is a skolem function for the same quantifier as f_i. Then $(\varphi_y)^s_\Diamond$ is the result of replacing terms $f_i(\bar{s})$ in $\varphi^s_\Diamond$ by

$$\begin{cases} g_i(\bar{s}) & \text{if } i \le h \\ g_i(\bar{s}, y) & \text{if } i > h. \end{cases}$$

Thus f_i and g_i may have different arity. To prove the direction from right to left in the lemma, assume $\vdash \varphi^s_\Diamond$. Let χ be the result of removing quantifier Qy from the quantifiers in $(\varphi_y)^s_\Diamond$ and keeping all else the same. Then $\vdash \chi$ follows. Thus so does $\vdash Qy\chi$. And since the quantifiers in $Qy\chi$ are either all existential or all universal, $\vdash (\varphi_y)^s_\Diamond$ follows.

For the other direction, let $\mathcal{M} = (W, R, D, I, \Vdash)$ be a model such that $\mathcal{M} \nVdash \varphi^s_\Diamond$. It suffices to show that there exists a model $\mathcal{M}_g$ such that $\mathcal{M}_g \nVdash (\varphi_y)^s_\Diamond$. As we will see, if $\mathcal{M}$ is a constant domain model, so is $\mathcal{M}_g$. Let $\mathcal{M}_g = (W, R, D, J, \Vdash)$ be the model for $\mathcal{L}_g$ that is defined as $\mathcal{M}$, except for the interpretation of the skolem functions $g_1, \dots, g_m$. For any such g_i of arity k and for every node w define

$$J_w(g_i)(a_1, \dots, a_k) \;\equiv_{def}\; \begin{cases} I_w(f_i)(a_1, \dots, a_k) & \text{if } i \le h \\ I_w(f_i)(a_1, \dots, a_{k-1}) & \text{if } i > h. \end{cases}$$

[2] The symbol Q instead of $\Diamond$ is used here to distinguish it from the other uses of $\Diamond$ in the proof.

Define for a quantifier-free subformula χ of $\varphi^s_\Diamond$, χ_g to be the result of replacing, starting with the outermost terms, any term $f_i(s_1,\dots,s_k)$ in χ by

$$\begin{cases} g_i(s_1,\dots,s_k) & \text{if } i \leq h \\ g_i(s_1,\dots,s_k,y) & \text{if } i > h. \end{cases}$$

We show that for any such subformula $\chi(\bar{z})$ of $\varphi^s_\Diamond$, for all nodes w:

$$\forall \bar{a}, b \in \mathcal{D}_w \colon \mathcal{M}, w \Vdash \chi(\bar{a}) \Leftrightarrow \mathcal{M}_g, w \Vdash \chi_g(\bar{a})[b/y]. \tag{4}$$

This is a simple induction on χ, we treat the case that χ is an atomic formula and leave the connectives to the reader. As $\chi(\bar{z})$ is an atomic subformula of $\varphi^s_\Diamond$, it does not contain y and y can only occur in $\chi_g(\bar{z})$ in terms of the form $g_i(t_1,\dots,t_l,y)$ for some terms $t_1,\dots,t_l$ such that $i > h$ and l is the arity of f_i. Hence (4) holds by the definition of $J_w(g_i)$. This proves (4).

Assume that $\varphi^s_\Diamond = \Diamond_1 z_1 \dots \Diamond_m z_m \psi(\bar{z})$. Then for some $1 \leq j \leq m$ we have $(\varphi_y)^s_\Diamond = \Diamond_1 z_1 \dots \Diamond_j z_j \Diamond y \Diamond_{z+1} z_{j+1} \dots \Diamond_m z_m \psi_g(\bar{z}, y)$. By assumption we have $\mathcal{M}_g \nVdash (\varphi_y)^s_\Diamond$. It remains to be shown that $\mathcal{M} \nVdash \varphi^s_\Diamond$. For $\Diamond = \forall$, there are $v \succcurlyeq w$ and $\bar{a}, b \in \mathcal{D}_v$ such that $\mathcal{M}_g, v \nVdash \psi_g(\bar{a}, b)$. Hence $\mathcal{M}, v \nVdash \psi(\bar{a})$ by (4). And thus $\mathcal{M} \nVdash \varphi^s_\Diamond$. For $\Diamond = \exists$, for all $\bar{a}, b \in \mathcal{D}_{w_0}$ we have $\mathcal{M}_g, w_0 \nVdash \psi_g(\bar{a}, b)$. Hence $\mathcal{M}, w_0 \nVdash \psi(\bar{a})$ by (4). Therefore $\mathcal{M} \nVdash \varphi^s_\Diamond$ also in this case. □

Corollary 1. *In all (constant domain) Kripke complete logics L, for all prenex sentences φ:*

$$\vdash \varphi^s \Leftrightarrow \vdash (\varphi_{sf})^s.$$

Therefore, φ belongs to the skolem class of L if and only if $(\varphi_{sf})^s$ belongs to the skolem class of L.

3. Proof idea

The proof that standard prenex sentences all belong to the skolem class of well-founded tree logics uses the semantics by showing how a countermodel $\mathcal{M} \nVdash \varphi$ in the language $\mathcal{L}$ can be transformed into a countermodel $\mathcal{M}^\exists_\varphi \nVdash \varphi^s_\exists$ in the language $\mathcal{L} \cup \mathcal{F}_\varphi$. For this construction we use its counterpart as well: any model $\mathcal{M} \Vdash \varphi$ in the language $\mathcal{L}$ can be transformed into a model $\mathcal{M}^\forall_\varphi \Vdash \varphi^s_\forall$ in the language $\mathcal{L} \cup \mathcal{F}_\varphi$. We treat the two quantifiers in parallel. Recall from Section 2.2 that $\Diamond$ denotes an arbitrary element in $\{\exists, \forall\}$.

For simplicity we assume in this proof sketch that φ contains only two quantifiers and no connectives: $\varphi = \exists x \forall y \psi(x,y)$ in case $\Diamond = \exists$, and $\varphi = \forall x \exists y \psi(x,y)$ in case $\Diamond = \forall$, and in both cases ψ is a predicate. Thus $\varphi^s_\exists = \exists x \psi(x, f(x))$ in the first case and $\varphi^s_\forall = \forall x \psi(x, f(x))$ in the second case. In both cases $\mathcal{F}_\varphi = \{f\}$.

Let $\mathcal{M} = (W, R, I, D, \Vdash)$ be a well-founded tree model and recall that $\mathcal{D}$ denotes the union of all domains in D. The term model $\mathcal{M}^\Diamond_\varphi = (W, R, I^\Diamond, D^\Diamond, \Vdash^\Diamond)$ has the same frame as the original model but its domains are

$$\mathcal{D}^\Diamond_w \equiv_{def} \begin{cases} \mathcal{T}(\mathcal{L} \cup \mathcal{F}_\varphi \cup \mathcal{D}) & \text{if } \Diamond = \exists \\ \mathcal{T}(\mathcal{L} \cup \mathcal{F}_\varphi \cup \mathcal{D}_w) & \text{if } \Diamond = \forall. \end{cases}$$

The interpretations are defined by simply putting $I^\Diamond_w(t) = t$ for all terms $t \in \mathcal{D}^\Diamond_w$. Forcing $\Vdash^\Diamond$ is defined as follows, where we write $\Vdash$ for $\Vdash^\Diamond$ as long as no confusion is possible.

Case $\Diamond = \exists$: Suppose $\mathcal{M} \nVdash \varphi$. To show that $\mathcal{M}^\exists_\varphi \nVdash \varphi^s_\exists$ one has to show that $\mathcal{M}^\exists_\varphi \nVdash \psi(a, f(a))$ for all $a \in \mathcal{D}^\exists_{w_0}$, where w_0 is the root. To this end we choose for every $a \in \mathcal{D}_{w_0}$ a node w and an element $e \in \mathcal{D}_w$ such that $\mathcal{M}, w \nVdash \psi(a, e)$. Such w and e exist because $\mathcal{M} \nVdash \varphi$. The tuple

$(a\colon e\colon w)$ is called an $\exists$-*witness*. The set of all tuples $(a\colon e\colon w)$ chosen in this way is denoted by $\mathcal{W}_1^{\exists}(\mathcal{M},\varphi)$.

Case $\Diamond = \forall$: Suppose $\mathcal{M}, w \Vdash \varphi$. To show that $\mathcal{M}_\varphi^{\forall} \Vdash \varphi_\forall^s$ one has to show that $\mathcal{M}_\varphi^{\forall}, w \Vdash \psi(a, f(a))$ for all nodes w and all $a \in \mathcal{D}_w^{\forall}$. Choose for every w and $a \in \mathcal{D}_w$ an $e \in \mathcal{D}_w$ such that $\mathcal{M}, w \Vdash \psi(a, e)$. Such an e exists because $\mathcal{M}, w \Vdash \varphi$. The tuple $(a\colon e\colon w)$ is called a $\forall$-*witness*. The set of all witnesses chosen in this way is denoted by $\mathcal{W}_1^{\forall}(\mathcal{M},\varphi)$.

The idea is to let $f(a)$ behave at w in $\mathcal{M}_\varphi^{\Diamond}$ as e does at w in $\mathcal{M}$. In order to do so we define a reduction $\iota_w^{\Diamond}$, which is a partial function from $\mathcal{D}_w^{\Diamond}$ to $\mathcal{D}$. Roughly, $\iota_w^{\Diamond}$ equals I_w on terms that do not contain f, is the identity on $\mathcal{D}$, and for nodes w and $a \in \mathcal{D}_w$ such that $(a\colon e\colon w) \in \mathcal{W}_1^{\Diamond}(\mathcal{M},\varphi)$ for some e, $\iota_w^{\Diamond}(f(a))$ is equal to e. For all nodes v and $a \in \mathcal{D}_v$ for which the latter is not the case, define

$$\iota_v^{\Diamond}(a) \equiv_{def} \begin{cases} e & \text{if } w \preccurlyeq v \text{ and } (a\colon e\colon w) \in \mathcal{W}_1^{\Diamond}(\mathcal{M},\varphi) \text{ for some } w, e \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For the first case of the reduction to be well-defined requires that every w and $a \in \mathcal{D}_w$ occur together in at most one tuple in $\mathcal{W}_1^{\Diamond}(\mathcal{M},\varphi)$. If $\Diamond = \exists$, this is the case. If $\Diamond = \forall$, this is achieved by allowing only those $(a\colon e\colon w)$ in $\mathcal{W}_1^{\forall}(\mathcal{M},\varphi)$ such that no node u below w forces $\exists y\psi(a, y)$ if a belongs to $\mathcal{D}_u$. Note that such a node exists because of the well-foundedness of the model. Under this modification there are, for every node v and element a, at most one $w \preccurlyeq v$ and e such that $(a\colon e\colon w) \in \mathcal{W}_1^{\forall}(\mathcal{M},\varphi)$. Further details are provided in the full proof in Section 4.2.

Forcing is defined for the predicate ψ and any $a, c \in \mathcal{D}_w^{\Diamond}$ as

$$\mathcal{M}_\varphi^{\Diamond}, w \Vdash \psi(a, c) \equiv_{def} \iota_w^{\Diamond}a, \iota_w^{\Diamond}c \in \mathcal{D}_w \text{ and } \mathcal{M}, w \Vdash \psi(\iota_w^{\Diamond}a, \iota_w^{\Diamond}c).$$

In particular, $\mathcal{M}_\varphi^{\Diamond}, w \nVdash \psi(a, c)$, for any w for which $\iota_w^{\Diamond}a \notin \mathcal{D}_w$ or $\iota_w^{\Diamond}c \notin \mathcal{D}_w$, which guarantees the upwards persistency of the forcing relation.

Since for every $(a\colon e\colon w) \in \mathcal{W}_1^{\Diamond}(\mathcal{M},\varphi)$: $\iota_w^{\Diamond}a = a$, $\iota_w^{\Diamond}f(a) = e \in \mathcal{D}_w$, this implies that $\mathcal{M}_\varphi^{\Diamond}, w \Vdash \psi(a, f(a))$ if and only if $\mathcal{M}, w \Vdash \psi(a, e)$. Hence $\mathcal{M}_\varphi^{\exists}, w_0 \nVdash \exists x\forall y\psi(x, y)$ and $\mathcal{M}_\varphi^{\forall}, w_0 \Vdash \forall x\exists y\psi(x, y)$, as desired.

Some subtleties are left out here that are addressed in the full proof. In case there are more skolem functions, the proof clearly becomes more complicated, but the main idea remains the same.

4. Witnesses

In Section 3 the interpretation of the skolem functions is explained in case they are unary. In this section this idea is extended to the general case, which in the next section is used to define the term models.

Throughout this section, φ is a fixed $\Diamond$-standard prenex sentence and $\mathcal{M}$ is an arbitrary well-founded tree model for the language $\mathcal{L}$ with root w_0, such that φ is of form (2) and $\mathcal{M} \nVdash \varphi$ if $\Diamond = \exists$, and φ is of form (3) and $\mathcal{M} \Vdash \varphi$ if $\Diamond = \forall$. Thus φ has $2n$ quantifiers and $\mathcal{F}_\varphi = \{f_1, \dots, f_n\}$. In the terminology of Section 3, for any $\bar{a} = a_1, \dots, a_n$ a $\Diamond$-witness $(a_1, \dots, a_n\colon e_1, \dots, e_n\colon w_1, \dots, w_n)$ has to be chosen such that skolem function $f_j(a_1, \dots, a_j)$ behaves like e_j for all $j \leq n$.

4.1. Witness construction

The set $\mathcal{W}^{\Diamond}(\mathcal{M},\varphi)$ of $\Diamond$-*witnesses for* $\mathcal{M}, \varphi$ (simply $\Diamond$-*witnesses* if $\mathcal{M}$ and φ are clear from the context) is the union $\mathcal{W}^{\Diamond}(\mathcal{M},\varphi) = \bigcup_{h=0}^{n} \mathcal{W}_h^{\Diamond}(\mathcal{M},\varphi)$, where the inductively defined $\mathcal{W}_h^{\Diamond}(\mathcal{M},\varphi)$

consist of expressions of the form $(\bar{a}\colon \bar{e}\colon \bar{w})$ such that $\bar{a}, \bar{e}, \bar{w}$ are three sequences of h elements; $\bar{w}$ consists of nodes and $\bar{a}, \bar{e}$ both of elements in domains in $\mathcal{M}$. The *length* of such a tuple is by definition h. The set $\mathcal{W}_0^{\Diamond}(\mathcal{M},\varphi)$, which by definition is empty, is included for notational convenience. For $i > 0$, the sets $\mathcal{W}_i^{\forall}(\mathcal{M},\varphi)$ and $\mathcal{W}_i^{\exists}(\mathcal{M},\varphi)$ are defined separately:

Construction $\mathcal{W}_1^{\exists}(\mathcal{M},\varphi)$*:* Let $\mathcal{W}_1^{\exists}(\mathcal{M},\varphi)$ be the smallest set consisting of tuples of length 1, so that for every $a_1 \in \mathcal{D}_{w_0}$ it contains exactly one tuple $(a_1\colon e_1\colon w_1)$ such that $e_1 \in \mathcal{D}_{w_1}$ and $w_1 \nVdash \varphi_2^{\exists}(a_1, e_1)$. Note that such w_1 and e_1 can be found because $\mathcal{M} \nVdash \varphi$.

Construction $\mathcal{W}_{h+1}^{\exists}(\mathcal{M},\varphi)$*:* Suppose $\mathcal{W}_h^{\exists}(\mathcal{M},\varphi)$ has been constructed and $h < n$. We let $\mathcal{W}_{h+1}^{\exists}(\mathcal{M},\varphi)$ be the smallest set consisting of tuples of length of $h+1$, such that for every $(\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^{\exists}(\mathcal{M},\varphi)$ and all $a_{h+1} \in \mathcal{D}_{w_h}$, there is exactly one tuple $(\bar{a}, a_{h+1}\colon \bar{e}, e_{h+1}\colon \bar{w}, w_{h+1}) \in \mathcal{W}_{h+1}^{\exists}(\mathcal{M},\varphi)$ such that $w_{h+1} \succcurlyeq w_h$, $e_{h+1} \in \mathcal{D}_{w_{h+1}}$, and $w_{h+1} \nVdash \varphi_{h+2}^{\exists}(\bar{a}, a_{h+1}, \bar{e}, e_{h+1})$. Note that such w_{h+1} and e_{h+1} can be found because $w_h \nVdash \varphi_{h+1}^{\exists}(\bar{a}, \bar{e})$.

Construction $\mathcal{W}_1^{\forall}(\mathcal{M},\varphi)$*:* Let $\mathcal{W}_1^{\forall}(\mathcal{M},\varphi)$ be the smallest set consisting of tuples of length 1, so that for every node w_1 and every element $a_1 \in \mathcal{D}_{w_1}$ such that for no node v below w_1, $a_1 \in \mathcal{D}_v$ and v forces $\exists y_1 \varphi_2^{\forall}(a_1, y_1)$, it contains exactly one tuple $(a_1\colon e_1\colon w_1)$ such that $e_1 \in \mathcal{D}_{w_1}$ and $w_1 \Vdash \varphi_2^{\forall}(a_1, e_1)$. Note that such an e_1 can be found because $\mathcal{M} \Vdash \varphi$ and $\mathcal{M}$ is a well-founded tree.

Construction $\mathcal{W}_{h+1}^{\forall}(\mathcal{M},\varphi)$*:* Suppose $\mathcal{W}_h^{\forall}(\mathcal{M},\varphi)$ has been constructed and $h < n$. We let $\mathcal{W}_{h+1}^{\forall}(\varphi)$ be the smallest set consisting of tuples of length of $h+1$, such that for every $(\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^{\forall}(\mathcal{M},\varphi)$, every $w_{h+1} \succcurlyeq w_h$ and every $a_{h+1} \in \mathcal{D}_{w_{h+1}}$ such that no domain at a node below w_{h+1} both contains a_{h+1} and forces $\exists y_{h+1} \varphi_{h+2}^{\forall}(a_{h+1}, y_{h+1})$, there is exactly one tuple $(\bar{a}, a_{h+1}\colon \bar{e}, e_{h+1}\colon \bar{w}, w_{h+1}) \in \mathcal{W}_{h+1}^{\forall}(\mathcal{M},\varphi)$ such that $e_{h+1} \in \mathcal{D}_{w_{h+1}}$ and $w_{h+1} \Vdash \varphi_{h+2}^{\forall}(\bar{a}, a_{h+1}, \bar{e}, e_{h+1})$. Note that such an e_{h+1} can be found because $w_h \Vdash \varphi_{h+1}^{\forall}(\bar{a}, \bar{e})$ and the model is a well-founded tree.

The construction is such that the following two lemmas hold, where the proof of the first is straightforward and therefore omitted.

Lemma 2. *For all* $1 \le h \le n$ *and* $(\bar{a}\colon \bar{e}\colon \bar{w}) = (a_1, \dots, a_h\colon e_1, \dots, e_h\colon w_1, \dots, w_h) \in \mathcal{W}_h^{\Diamond}(\mathcal{M},\varphi)$*:*

- $w_1 \preccurlyeq w_1 \preccurlyeq \dots \preccurlyeq w_h$ *and* $e_h \in \mathcal{D}_{w_h}$,
- $a_h \in \mathcal{D}_{w_{h-1}}$ *if* $\Diamond = \exists$, *and* $a_h \in \mathcal{D}_{w_h}$ *if* $\Diamond = \forall$,
- $w_h \nVdash \varphi_{h+1}(\bar{a}, \bar{e})$ *if* $\Diamond = \exists$, *and* $w_h \Vdash \varphi_{h+1}(\bar{a}, \bar{e})$ *if* $\Diamond = \forall$,
- *if* $h < n$, *for all* $b \in \mathcal{D}_{w_h}$ *there exist* $w \succcurlyeq w_h$ *and* $d \in \mathcal{D}_w$ *such that* $(a_1, \dots, a_h, b\colon e_1, \dots, e_h, d\colon w_1, \dots, w_h, w) \in \mathcal{W}_{h+1}^{\exists}(\mathcal{M},\varphi)$,
- *if* $h < n$, *for all* $w \succcurlyeq w_h$ *and all* $b \in \mathcal{D}_w$ *there is a* $d \in \mathcal{D}_w$ *such that* $(a_1, \dots, a_h, b\colon e_1, \dots, e_h, d\colon w_1, \dots, w_h, w) \in \mathcal{W}_{h+1}^{\forall}(\mathcal{M},\varphi)$.

Lemma 3.

1. *If* $(\bar{a}\colon \bar{c}\colon \bar{v}), (\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^{\exists}(\mathcal{M},\varphi)$, *then* $\bar{w} = \bar{v}$ *and* $\bar{c} = \bar{e}$.
2. *If* $(\bar{a}\colon \bar{c}\colon \bar{v}), (\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^{\forall}$, *then* $c_h = e_h$, *or* $v_h \not\preccurlyeq w_h$ *and* $w_h \not\preccurlyeq v_h$.

Proof. The proof of 1 is a straightforward induction on h. We prove 2 with induction on h. Case $h = 0$ is trivial.

Case $h = 1$: Suppose $(a\colon c_1\colon v), (a\colon e_1\colon w) \in \mathcal{W}_1^{\forall}$. If $w \preccurlyeq v$, then $w = v$ and $c_1 = e_1$ by the definition of $\mathcal{W}_1^{\forall}$, and similarly if $v \preccurlyeq w$.

Case $h+1$: Suppose $(\bar{a}, a_{h+1} : \bar{c}, c_{h+1} : \bar{v}, v_{h+1})$ and $(\bar{a}, a_{h+1} : \bar{e}, e_{h+1} : \bar{w}, w_{h+1})$ belong to $\mathcal{W}^{\forall}_{h+1}$. If $w_{h+1} \preccurlyeq v_{h+1}$, then $w_{h+1} = v_{h+1}$ and $c_{h+1} = e_{h+1}$ by the definition of $\mathcal{W}^{\forall}_{h+1}$, and similarly if $v_{h+1} \preccurlyeq w_{h+1}$. That is what had to be shown. □

4.2. *Witness functions*

We define the following partial *witness functions*. For $\bar{a} = a_1, \ldots, a_h \in \mathcal{D}$ of length at most n:

$$ws^{\Diamond}_v(\bar{a}) \equiv_{def} e_h \text{ if } (\bar{a} : \bar{e} : \bar{w}) \in \mathcal{W}^{\Diamond}_h(\mathcal{M}, \varphi) \text{ and } w_h \preccurlyeq v \text{for some} \bar{e}, \bar{w} \text{ and } h.$$

$ws^{\exists}_v(\cdot)$ is a partial function, as for certain v and $\bar{a}$ there may exist no $w_h \preccurlyeq v$ that fulfil the requirements. But the above lemma implies that when there does exist such a tuple, it is unique. On the other hand, $ws^{\forall}_v(\cdot)$ is a total function by Lemma 4.

Lemma 4.

- *If* $\bar{a} \in \mathcal{D}_v$, *then* $ws^{\forall}_v(\bar{a})$ *is defined.*
- *If* $w \preccurlyeq v$ *and* $\bar{a} \in \mathcal{D}_w$, *then* $ws^{\forall}_w(\bar{a}) = ws^{\forall}_v(\bar{a})$.
- *If* $w \preccurlyeq v$ *and* $ws^{\exists}_w(\bar{a})$ *is defined, then* $ws^{\exists}_w(\bar{a}) = ws^{\exists}_v(\bar{a})$.
- *For* $\Diamond \in \{\exists, \forall\}$: $\bar{a}, \bar{e}, ws^{\Diamond}_{w_h}(\bar{a}) \in \mathcal{D}_{w_h}$ *for any* $(\bar{a} : \bar{e} : \bar{w}) \in \mathcal{W}^{\Diamond}_h(\mathcal{M}, \varphi)$.
- *For* $\Diamond \in \{\exists, \forall\}$: *if* $\bar{a} \in \mathcal{D}_v$ *and* $ws^{\Diamond}_v(\bar{a})$ *is defined, then* $ws^{\Diamond}_v(\bar{a}) \in \mathcal{D}_v$.

Proof. We only prove the first statement, the others follow easily from that and the construction of $\mathcal{W}^{\exists}_h$ and $\mathcal{W}^{\forall}_h$. We use induction on the length of $\bar{a}$. If the length is one, then for any node v and $a_1 \in \mathcal{D}_v$, there exists a tuple $(a_1 : e_1 : w_1)$ in $\mathcal{W}^{\forall}_1(\mathcal{M}, \varphi)$ such that $w_1 \preccurlyeq v$ and $a_1 \in \mathcal{D}_{w_1}$. Hence $ws^{\forall}_v(a_1)$ is defined.

Assume that $\bar{a}, a_{h+1} \in \mathcal{D}_v$. By the induction hypothesis, $ws^{\forall}_v(\bar{a})$ exists. Thus there exists a tuple $(\bar{a} : \bar{e} : \bar{w}) \in \mathcal{W}^{\forall}_h(\mathcal{M}, \varphi)$ such that $w_h \preccurlyeq v$. From the construction of $\mathcal{W}^{\forall}_{h+1}(\mathcal{M}, \varphi)$ and the assumption that $a_{h+1} \in \mathcal{D}_v$ it follows that for some tuple $(\bar{a}, a_{h+1} : \bar{e}, e_{h+1} : \bar{w}, w_{h+1})$ in $\mathcal{W}^{\forall}_{h+1}(\mathcal{M}, \varphi)$ we have $w_{h+1} \preccurlyeq v$ and $a_{h+1} \in \mathcal{D}_{w_{h+1}}$. This implies that $ws^{\forall}_v(\bar{a}, a_{h+1})$ is defined, which is what had to be shown. □

Remark 2. The first statement in Lemma 4 does not hold for $ws^{\exists}_w$: it may well be the case that in a certain model $\mathcal{M}$ there exists a node v and an element $a_1 \in \mathcal{D}_{w_0} \subseteq \mathcal{D}_v$ such that for no $(a_1 : e_1 : w_1) \in \mathcal{W}^{\exists}_1(\mathcal{M}, \varphi)$ it holds that $w_1 \preccurlyeq v$, in which case $ws^{\exists}_v(a_1)$ is undefined.

5. Skolemization for standard prenex formulas

Throughout this section $\mathcal{M} = (W, R, D, I, \Vdash)$ is an arbitrary well-founded tree model in a language $\mathcal{L}$ and φ is a fixed $\Diamond$-standard prenex sentence for which we use the same notation as given at the beginning of Section 4: φ is of the form (2) if $\Diamond = \exists$ and of the form (3) if $\Diamond = \forall$. Thus φ has $2n$ quantifiers and its set of skolem functions is $\mathcal{F}_\varphi = \{f_1, \ldots, f_n\}$. For the general idea behind the definitions in this section, the reader may consult Section 3, in which a sketch of the main proof is given.

5.1. Term models

For $\Diamond \in \{\exists, \forall\}$, the *term model* $\mathcal{M}_\varphi^\Diamond = (W, R, I^\Diamond, D^\Diamond, \Vdash^\Diamond)$ is defined as follows. We write $\Vdash$ for $\Vdash^\Diamond$ as the context always determines which model and hence which forcing is meant. Model $\mathcal{M}_\varphi^\Diamond$ has the same frame as $\mathcal{M}$ and the domain at node w is denoted by $\mathcal{D}_w^\Diamond$. Recall that $\mathcal{L}$ contains no variables. Define:

$$\mathcal{D}_w^\Diamond \equiv_{def} \begin{cases} \mathcal{T}(\mathcal{L} \cup \mathcal{F}_\varphi \cup \mathcal{D}) & \text{if } \Diamond = \exists \\ \mathcal{T}(\mathcal{L} \cup \mathcal{F}_\varphi \cup \mathcal{D}_w) & \text{if } \Diamond = \forall. \end{cases}$$

Note that $\mathcal{M}_\varphi^\exists$ is a constant domain model.

Interpretations $I_w^\Diamond$ are the identity: for any $g \in \mathcal{L} \cup \mathcal{F}_\varphi$, $I_w^\Diamond(g)(\bar{s}) \equiv_{def} g(\bar{s})$ for any $\bar{s} \in \mathcal{D}_w^\Diamond$.

To define forcing in the term models we need the inductively defined (partial) functions $\iota_w^\Diamond : \mathcal{D}_w^\Diamond \to \mathcal{D}_w$ given by

$$\iota_w^\Diamond(t) \equiv_{def} \begin{cases} t & \text{if } t \in \mathcal{D}_w \\ I_w(t) & \text{if } t \in \mathcal{L} \text{ and } t \text{ is a constant} \\ I_w(g)(\iota_w^\Diamond s_1, \ldots, \iota_w^\Diamond s_h) & \text{if } g \in \mathcal{L} \text{ and } t = g(s_1, \ldots, s_h) \\ ws_w^\Diamond(\iota_w^\Diamond s_1, \ldots, \iota_w^\Diamond s_h) & \text{if } g \in \mathcal{F}_\varphi \text{ and } t = g(s_1, \ldots, s_h). \end{cases}$$

$\iota_w^\forall$ is a total function by Lemma 4, and $\iota_w^\exists$ is in general not total by Remark 2. When in a term t there occurs a subterm $ws_w^\exists(s)$ that is undefined, then t and $\iota_w^\exists(t)$ are called *undefined*. When we write $\iota_w^\exists(t) \in \mathcal{D}_w$, we mean that $\iota_w^\exists(t)$ is defined and belongs to $\mathcal{D}_w$.

Given a node w and $\bar{a} \in \mathcal{D}_w$, it follows by definition that

$$\iota_w^\Diamond f_1(a_1) = ws_w^\Diamond(a_1) \quad \iota_w^\Diamond f_2(a_1, a_2) = ws_w^\Diamond(a_1, a_2) \quad \ldots$$

Given a sequence of terms $\bar{s} = s_1, \ldots, s_n$ we write $\iota_w^\Diamond \bar{s}$ for $\iota_w^\Diamond(s_1), \ldots, \iota_w^\Diamond(s_n)$.

When χ is a sentence such that all terms that occur in it belong to $\mathcal{D}_w^\Diamond$, then $\iota_w^\Diamond \chi$ denotes the result of replacing every occurrence of term s in χ that is not a subterm of another term, by $\iota_w^\Diamond s$. For example, for a unary predicate P, $a \in \mathcal{D}_w$, and functions $g \in \mathcal{L}$ and $f \in \mathcal{F}_\varphi$:

$$\iota_w^\Diamond P(g(f(a))) = P(\iota_w^\Diamond g(f(a))) = P(I_w(g)(\iota_w^\Diamond f(a))) = P(I_w(g)(ws_w^\Diamond(a))).$$

To define forcing for atomic formulas it suffices to define it for formulas of the form $P(\bar{s})$, where $P(x_1, \ldots, x_m)$ is a predicate and $\bar{s} = s_1, \ldots, s_m$ are m elements in $\mathcal{D}_w^\Diamond$:

$$\mathcal{M}_\varphi^\Diamond, w \Vdash P(\bar{s}) \equiv_{def} \iota_w^\Diamond \bar{s} \in \mathcal{D}_w \text{ and } \mathcal{M}, w \Vdash P(\iota_w^\Diamond \bar{s}).$$

Thus whenever $\iota_w^\Diamond \bar{s} \notin \mathcal{D}_w$, in particular when $\iota_w^\Diamond \bar{s}$ is undefined, $\mathcal{M}_\varphi^\Diamond, w \nVdash P(\bar{s})$.

5.2. Properties of term models

Lemma 5. *For $\Diamond \in \{\exists, \forall\}$: if $t \in \mathcal{D}_w^\Diamond$ and $\iota_w^\Diamond t$ is defined, then $\iota_w^\Diamond t \in \mathcal{D}_w$.*

Proof. With induction to t. If t is an element of $\mathcal{D}_w$, the statement is clear. If t is a constant, it belongs to $\mathcal{L}$ and thus $\iota_w^\Diamond t = I_w(t) \in \mathcal{D}_w$. If $t = g(\bar{s})$, then $\iota_w^\Diamond \bar{s} \in \mathcal{D}_w$ by the induction hypothesis. If $g \in \mathcal{L}$, then $\iota_w^\Diamond t = I_w(g)(\iota_w^\Diamond \bar{s}) \in \mathcal{D}_w$. If $g \in \mathcal{F}_\varphi$, then $\iota_w^\Diamond t = ws_w^\Diamond(\iota_w^\Diamond \bar{s}) \in \mathcal{D}_w$ by Lemma 4. □

Lemma 6. *For $\Diamond \in \{\exists, \forall\}$ and any term $t \in \mathcal{D}_w^\Diamond$: if $\iota_w^\Diamond t$ is defined and $w \preccurlyeq v$, then $\iota_w^\Diamond t = \iota_v^\Diamond t$.*

Proof. Assume $t \in \mathcal{D}_w^{\Diamond}$, $\iota_w^{\Diamond} t$ is defined, and $w \preccurlyeq v$. Hence $\iota_w^{\Diamond} t \in \mathcal{D}_w$ by Lemma 5. We show $\iota_w^{\Diamond} t = \iota_v^{\Diamond} t$ with induction to the structure of t. The case that t is an element of $\mathcal{D}_w$ is trivial. If t is a constant, then it belongs to $\mathcal{L}$ and $\iota_w^{\Diamond} t = I_w(t)$ and $\iota_v^{\Diamond} t = I_v(t)$. Hence $\iota_w^{\Diamond} t = \iota_v^{\Diamond} t$. Finally, suppose $t = g(\bar{s})$. Then $\iota_w^{\Diamond} \bar{s} = \iota_v^{\Diamond} \bar{s}$ by the induction hypothesis. If $g \in \mathcal{L}$, then $\iota_w^{\Diamond} t = I_w(g)(\iota_w^{\Diamond} \bar{s}) = I_v(g)(\iota_v^{\Diamond} \bar{s}) = \iota_v^{\Diamond} t$. In case $g \in \mathcal{F}_\varphi$, note that $\iota_w^{\Diamond} \bar{s}$ is defined since $\iota_w^{\Diamond} t$ is. Therefore $\iota_w^{\Diamond} t = ws_w^{\Diamond}(\iota_w^{\Diamond} \bar{s}) = ws_v^{\Diamond}(\iota_v^{\Diamond} \bar{s}) = \iota_v^{\Diamond} t$ by Lemma 4. □

The next lemma proves that term models are upwards persistent.

Lemma 7. *For $\Diamond \in \{\exists, \forall\}$, for all atomic formulas $\chi(\bar{x})$, terms $t, \bar{s} \in \mathcal{D}_w^{\Diamond}$ and nodes $w \preccurlyeq v$: $I_w^{\Diamond}(t) = I_v^{\Diamond}(t)$, and if $\mathcal{M}_\varphi^{\Diamond}, w \Vdash \chi(\bar{s})$, then $\mathcal{M}_\varphi^{\Diamond}, v \Vdash \chi(\bar{s})$.*

Proof. For interpretations, the lemma is clear. For the other part, note that $\chi(\bar{x})$ is of the form $P(\bar{x}, \bar{t})$ for some predicate symbol P and terms $\bar{t} \in \mathcal{D}_w^{\Diamond}$. Therefore we only have to consider the case that χ is a predicate symbol, say P. Suppose $w \preccurlyeq v$ and $\mathcal{M}_\varphi^{\Diamond}, w \Vdash P(\bar{s})$. Thus $\iota_w^{\Diamond} \bar{s} \in \mathcal{D}_w$ and $\mathcal{M}, w \Vdash P(\iota_w^{\Diamond} \bar{s})$. Hence $\mathcal{M}, v \Vdash P(\iota_w^{\Diamond} \bar{s})$. Since $\iota_w^{\Diamond} \bar{s} = \iota_v^{\Diamond} \bar{s}$ by Lemma 6, $\mathcal{M}_\varphi^{\Diamond}, v \Vdash P(\bar{s})$ follows. □

Lemma 8. *For any quantifier-free sentence χ that only contains terms t such that $t \in \mathcal{D}_w^{\Diamond}$ and $\iota_w^{\Diamond} t \in \mathcal{D}_w$:*

$$\mathcal{M}_\varphi^{\Diamond}, w \Vdash \chi \text{ if and only if } \mathcal{M}, w \Vdash \iota_w^{\Diamond} \chi. \tag{5}$$

Proof. A simple induction on the complexity of χ. □

The next lemma shows that the previous lemma can be strengthened in the case that $\Diamond = \forall$.

Lemma 9. *For any sentence χ that only contains terms in $\mathcal{D}_w^{\forall}$:*

$$\mathcal{M}_\varphi^{\forall}, w \Vdash \chi \text{ if and only if } \mathcal{M}, w \Vdash \iota_w^{\forall} \chi.$$

Proof. With formula induction. As remarked below the definition of $\iota_w^{\Diamond}$ in Section 5.1, $\iota_w^{\forall}$ is total. From Lemma 5 it follows that $\iota_w^{\forall} t \in \mathcal{D}_w$ for any term t in χ. Therefore for quantifier-free χ the lemma follows from Lemma 8. In case χ contains quantifiers, we treat the case that the outermost quantifier is existential and leave the other case to the reader.

Therefore consider a sentence $\exists x \chi(x)$. If $\mathcal{M}, w \Vdash \iota_w^{\forall}(\exists x \chi(x))$, then because $\iota_w^{\forall} a = a$ for all elements a in $\mathcal{D}_w$, there exists an element $a \in \mathcal{D}_w$ such that $\mathcal{M}, w \Vdash \iota_w^{\forall}(\chi(a))$. Therefore $\mathcal{M}_\varphi^{\forall}, w \Vdash \chi(a)$ by the induction hypothesis. For the other direction, assume $\mathcal{M}_\varphi^{\forall}, w \Vdash \exists x \chi(x)$, thus $\mathcal{M}_\varphi^{\forall}, w \Vdash \chi(t)$ for some $t \in \mathcal{D}_w^{\forall}$. Since $\iota_w^{\forall} t \in \mathcal{D}_w$ by Lemma 5, $\mathcal{M}, w \Vdash \iota_w^{\forall}(\exists x \chi(x))$ follows by the induction hypothesis. □

This completes the description of the term models. In the next section we show that they have the desired properties: $\mathcal{M}_\varphi^{\exists} \nVdash \varphi_\exists^s$ if $\mathcal{M} \nVdash \varphi$, and $\mathcal{M}_\varphi^{\forall} \Vdash \varphi_\forall^s$ if $\mathcal{M} \Vdash \varphi$.

5.3. *Witness functions and term models*

Lemma 10. *For all $0 \leq h \leq n$,*

1. *If $\mathcal{M} \nVdash \varphi$, then for all $(\bar{a} : \bar{e} : \bar{w}) \in \mathcal{W}_h^{\exists}(\mathcal{M}, \varphi)$:*
   $\mathcal{M}_\varphi^{\exists}, w_h \nVdash \varphi_{(h+1)\exists}^s(\bar{a}, \bar{e}, f_{h+1}^{\uparrow}(\bar{a}, \bar{x}_n^{h+1}))$.

2. *If* $\mathcal{M} \Vdash \varphi$*, then for all* $(\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^\forall(\mathcal{M}, \varphi)$*:*
$\mathcal{M}_\varphi^\forall, w_h \Vdash \varphi^s_{(h+1)\forall}(\bar{a}, \bar{e}, f^\uparrow_{h+1}(\bar{a}, \bar{x}_n^{h+1}))$.

Proof. Let $\Diamond \in \{\exists, \forall\}$. With induction to $(n-h)$. Consider $(\bar{a}\colon \bar{e}\colon \bar{w}) \in \mathcal{W}_h^\Diamond(\mathcal{M}, \varphi)$. First observe that $\iota^\Diamond_{w_h}(c) = c$ for any $c \in \mathcal{D}_{w_h}$ and $\bar{a}, \bar{e} \in \mathcal{D}_{w_h}$ by Lemma 4. Therefore $\iota^\Diamond_{w_h}(a_i) = a_i$, $\iota^\Diamond_{w_h}(e_j) = e_j$, and

$$\iota^\Diamond_{w_h}(f_h(\bar{a})) = ws^\Diamond_{w_h}(\iota^\Diamond_{w_h}\bar{a}) = ws^\Diamond_{w_h}(\bar{a}) = e_h. \tag{6}$$

For $h = n$, we have to prove that

$$\mathcal{M}_\varphi^\exists, w_n \nVdash \psi(\bar{a}, \bar{e}) \text{ if } q = \exists \tag{7}$$

$$\mathcal{M}_\varphi^\forall, w_n \Vdash \psi(\bar{a}, \bar{e}) \text{ if } q = \forall. \tag{8}$$

This follows from the construction of $\mathcal{W}_i^\exists(\mathcal{M}, \varphi)$ and $\mathcal{W}_i^\forall(\mathcal{M}, \varphi)$ and Lemma 2 in Section 4.

For $h < n$, $\varphi^s_{(h+1)\Diamond} = \Diamond x_{h+1}\varphi^s_{(h+2)\Diamond}$. Thus in order to show 1 and 2, we have to show the following:

$$\forall a_{h+1} \in \mathcal{D}_{w_h} : \mathcal{M}_\varphi^\exists, w_h \nVdash \varphi^s_{(h+2)\exists}(\bar{a}, a_{h+1}, \bar{e}, f^\uparrow_{h+1}(\bar{a}, a_{h+1}, \bar{x}_n^{h+2}))$$

$$\forall w_{h+1} \succcurlyeq w_h \forall a_{h+1} \in \mathcal{D}_{w_{h+1}} : \mathcal{M}_\varphi^\forall, w_{h+1} \Vdash \varphi^s_{(h+2)\forall}(\bar{a}, a_{h+1}, \bar{e}, f^\uparrow_{h+1}(\bar{a}, a_{h+1}, \bar{x}_n^{h+2})).$$

We treat the two quantifiers separately.

$\exists$: From the construction of $\mathcal{W}^\exists(\mathcal{M}, \varphi)$ in Section 4 it follows that for all $a_{h+1} \in \mathcal{D}_{w_h}$ there exists a node $w_{h+1} \succcurlyeq w_h$ and an element $e_{h+1} \in \mathcal{D}_{w_{h+1}}$ such that tuple $(\bar{a}_{h+1}\colon \bar{e}_{h+1}\colon \bar{w}_{h+1})$ belongs to $\mathcal{W}_{h+1}^\exists(\mathcal{M}, \varphi)$. By the induction hypothesis,

$$\mathcal{M}_\varphi^\exists, w_{h+1} \nVdash \varphi^s_{(h+2)\exists}(\bar{a}_{h+1}, \bar{e}_{h+1}, f^\uparrow_{h+2}(\bar{a}_{h+1}, \bar{x}_n^{h+2})).$$

By (6), $e_{h+1} = \iota^\exists_{w_{h+1}}(f_{h+1}(\bar{a}_{h+1}))$, proving 1.

$\forall$: From the construction of $\mathcal{W}^\forall(\mathcal{M}, \varphi)$ in Section 4 it follows that for all $w_{h+1} \succcurlyeq w_h$ and all $a_{h+1} \in \mathcal{D}_{w_{h+1}}$ there is a $e_{h+1} \in \mathcal{D}_{w_{h+1}}$ such that $(\bar{a}_{h+1}\colon \bar{e}_{h+1}\colon \bar{w}_{h+1})$ belongs to $\mathcal{W}_{h+1}^\forall(\mathcal{M}, \varphi)$. By the induction hypothesis,

$$\mathcal{M}_\varphi^\forall, w_{h+1} \Vdash \varphi^s_{(h+2)\forall}(\bar{a}_{h+1}, \bar{e}_{h+1}, f^\uparrow_{h+2}(\bar{a}_{h+1}, \bar{x}_n^{h+2})).$$

By (6), $e_{h+1} = \iota^\forall_{w_{h+1}}(f_{h+1}(\bar{a}_{h+1}))$, which gives 2. □

5.4. Main theorem for standard prenex formulas

Theorem 2. *For any* $\Diamond$*-standard prenex sentence* φ*:*

1. *If* $\Diamond = \exists$ *and* $\mathcal{M} \nVdash \varphi$*, then* $\mathcal{M}_\varphi^\exists \nVdash \varphi^s_\exists$.
2. *If* $\Diamond = \forall$ *and* $\mathcal{M} \Vdash \varphi$*, then* $\mathcal{M}_\varphi^\forall \Vdash \varphi^s_\forall$.
3. *If* $\Diamond = \forall$*, then for all nodes* w *and sentences* χ *that do not contain symbols from* $\mathcal{F}_\varphi$*:*

$$\mathcal{M}_\varphi^\forall, w \Vdash \chi \textit{ if and only if } \mathcal{M}, w \Vdash \chi.$$

Proof. Statements 1 and 2 follow from Lemma 10 because $\mathcal{W}_0^\exists(\mathcal{M}, \varphi)$ is empty and $\varphi^s_\Diamond = \Diamond x_1 \dots x_n$
$\psi(\bar{x}, f_1(x_1), \dots, f_n(\bar{x})) = \varphi^s_{1\Diamond}$.

3. By Lemma 9, for any sentence χ that only contains terms in $\mathcal{D}_w^\forall$:

$$\mathcal{M}_\varphi^\forall, w \Vdash \chi \text{ if and only if } \mathcal{M}, w \Vdash \iota_w^\forall \chi.$$

Thus it suffices to show that

$$\mathcal{M}, w \Vdash \iota^{\forall}_w \chi \text{ if and only if } \mathcal{M}, w \Vdash \chi.$$

We prove this by induction on χ. If it is an atomic formula, then this follows from the fact that $\iota^{\forall}_w$ equals I_w on elements in $\mathcal{L}$ and is the identity on elements in $\mathcal{D}_w$. The other cases are analogous to those in the proof of Lemma 9. □

Corollary 2. *For any prenex sentence φ and any model $\mathcal{M}$, there are models $\mathcal{M}^e_\varphi$ and $\mathcal{M}^a_\varphi$ with the same frame as $\mathcal{M}$ such that*

1. *If $\mathcal{M} \nVdash \varphi$, then $\mathcal{M}^e_\varphi \nVdash \varphi^s_\exists$.*
2. *If $\mathcal{M} \Vdash \varphi$, then $\mathcal{M}^a_\varphi \Vdash \varphi^s_\forall$.*
3. *For all sentences χ that do not contain symbols from $\mathcal{F}_\varphi$ and all nodes w:*

$$\mathcal{M}^a_\varphi, w \Vdash \chi \textit{ if and only if } \mathcal{M}, w \Vdash \chi.$$

Proof. 1. Suppose $\mathcal{M} \nVdash \varphi$. Let $\chi = \varphi^\exists_{sf}$. Hence $\mathcal{M} \nVdash \chi$ and χ is a standard prenex sentence. By Theorem 2, $\mathcal{M}^\exists_\chi \nVdash \chi^s_\exists$. The proof of Lemma 1 implies that there is a model $\mathcal{M}'$ with the same frame as $\mathcal{M}^\exists_\chi$ (and thus the same frame as $\mathcal{M}$) such that $\mathcal{M}' \nVdash \varphi^s_\exists$. So we can take $\mathcal{M}'$ for $\mathcal{M}^e_\varphi$. The proof of 2 is analogous. For 3, use observation 3 of Theorem 2 as well as Lemma 1. □

Corollary 3. *In any (constant domain) well-founded tree logic L, for any conjunction φ of prenex sentences:*

$$\vdash_L \varphi \textit{ if and only if } \vdash_L \varphi^s \textit{ if and only if } \vdash_{LCD} \varphi^s.$$

Proof. Recall that $\varphi^s = \varphi^s_\exists$. That $\vdash_L \varphi$ implies $\vdash_L \varphi^s$ and $\vdash_{LCD} \varphi^s$ follows from the fact that $\vdash_{IQC} \varphi \to \varphi^s$. We prove the other direction by contraposition.

Suppose that $\nvdash \varphi$ and let $\mathcal{M}$ be a countermodel, that is, a model of L such that $\mathcal{M} \nVdash \varphi$. Then $\mathcal{M}^e_\varphi \nVdash \varphi^s$ by Corollary 2. Since the frame of $\mathcal{M}^e_\varphi$ is the same as the frame of $\mathcal{M}$, $\mathcal{M}^e_\varphi$ is a model of L as well. Thus $\nvdash_L \varphi^s$. As remarked at the beginning of Section 5.1, $\mathcal{M}^e_\varphi$ is a constant domain model. Therefore also $\nvdash_{LCD} \varphi^s$. □

Note that for IQC, the first equivalence of Corollary 2 can easily be obtained by using an appropriate sequent calculus for IQC, such as G3i from [14]. For most other well-founded tree logics no such calculus is known, so that we cannot apply such a syntactic argument. The statement that φ is derivable in IQC if and only if φ^s is derivable in $\mathsf{IQC_{CD}}$ seems to be a new insight.

5.5. Generalization

To apply the above technique to sentences that are propositional combinations of standard prenex sentences, we change the definition of the model $\mathcal{M}^\Diamond_\varphi$ such that besides the properties discussed above it also satisfies that for any prenex sentence χ independent of φ, in all nodes w: $\mathcal{M}^\Diamond_\varphi, w \Vdash \chi$ if and only if $\mathcal{M}, w \Vdash \chi$. In other words, the forcing of sentences that do not have predicates in common with φ remains unchanged when replacing $\mathcal{M}$ by $\mathcal{M}^\Diamond_\varphi$. This is not the case in the definition of $\mathcal{M}^\Diamond_\varphi$ given above. For it can be the case that $\mathcal{M} \Vdash \forall x P(x)$ and for some skolem function $f \in \mathcal{F}_\varphi$ and some node v and $a \in \mathcal{D}_v$, $\iota^\Diamond_v(a)$ is undefined, and whence $\mathcal{M}^\Diamond_\varphi \nVdash P(f(a))$ and thus $\mathcal{M}^\Diamond_\varphi \nVdash \forall x P(x)$.

To obtain a model with the desired property, we replace $\iota_w^\Diamond$ by a total function $\nu_w^\Diamond$ that is equal to $\iota_w^\Diamond$ on all elements of $\mathcal{D}_w^\Diamond$ that do not contain skolem functions in $\mathcal{F}_\varphi$, and maps all other elements to an arbitrary element in $\mathcal{D}_0$. Then forcing is defined as before for predicates that occur in φ and for predicates that do not occur in φ it is defined as before but with $\nu_w^\Diamond$ replacing $\iota_w^\Diamond$. The resulting model has the desired properties. This section contains the technical details behind this idea.

There is one additional change: in going from $\mathcal{M}$ to $\mathcal{M}_\varphi^\Diamond$ we want to replace only the submodel $\mathcal{M}_v$ by $(\mathcal{M}_v)_\varphi^\Diamond$ for some nodes v. Such a term model is denoted $\mathcal{M}_{v\varphi}^\Diamond = (W_v, R_v, I^{v\Diamond}, D^{v\Diamond}, \Vdash^{v\Diamond})$, where $I^{v\Diamond}$ and $D^{v\Diamond}$ simply are the restrictions of $I^\Diamond$ and $D^\Diamond$ to the frame of $\mathcal{M}_v$, and the forcing $\Vdash^{v\Diamond}$ is defined as follows. As before, we write $\Vdash$ for $\Vdash^{v\Diamond}$.

Choose an element d_0 in the domain at the root and let, for every node w and every number h, $g_w^h : (\mathcal{D}_w^\Diamond)^h \to \mathcal{D}_w^\Diamond$ denote the h-ary constant function that maps all elements to d_0. The auxiliary interpretation $\iota_w^\Diamond$ is replaced by the total function ν_w on $\mathcal{D}_w^\Diamond$. First put for every h-ary function $g \in \mathcal{L} \cup \mathcal{F}_\varphi$ and every element $a \in \mathcal{D}_w$:

$$\nu_w(g) \equiv_{def} \begin{cases} I_w(g) & \text{if } g \in \mathcal{L} \\ g_w^h & \text{if } g \in \mathcal{F}_\varphi \end{cases} \qquad \nu_w(a) \equiv_{def} a.$$

Then inductively extend ν_w to an interpretation of all terms in $\mathcal{D}_w^\Diamond$ in the usual way: for $t = g(s_1, \ldots, s_m)$ define

$$\nu_w(t) \equiv_{def} \nu_w(g)(\nu_w s_1, \ldots, \nu_w s_m).$$

Clearly, if $w \preccurlyeq v$, then $\nu_w(t) = \nu_v(t)$ for any term t. For a predicate $P(x_1, \ldots, x_m)$, forcing is defined for all nodes w and m terms $\bar{s} = s_1, \ldots, s_m \in \mathcal{D}_w^\Diamond$:

$$\mathcal{M}_{v\varphi}^\Diamond, w \Vdash P(\bar{s}) \equiv_{def} \begin{cases} \iota_w \bar{s} \in \mathcal{D}_w \text{ and } \mathcal{M}, w \Vdash P(\iota_w \bar{s}) & \text{if } P \text{ occurs in } \varphi \\ \mathcal{M}, w \Vdash P(\nu_w \bar{s}) & \text{otherwise.} \end{cases}$$

Here "P occurs in φ" means that the symbol P occurs in φ. That the above defines a model follows from the next lemma, which proof is analogous to that of Lemma 7.

Lemma 11. *For any predicate P, if $v \preccurlyeq w \preccurlyeq u$, $\bar{s} \in \mathcal{D}_w^\Diamond$ and $\mathcal{M}_{v\varphi}^\Diamond, w \Vdash P(\bar{s})$, then $\mathcal{M}_{v\varphi}^\Diamond, u \Vdash P(\bar{s})$.*

Lemma 12. *For all formulas $\chi(\bar{z})$ that do not contain predicate symbols that occur in φ nor function symbols from $\mathcal{F}_\varphi$, for all $\bar{s} \in \mathcal{D}_w^\Diamond$ and all $v \preccurlyeq w$:*

$$\mathcal{M}_{v\varphi}^\Diamond, w \Vdash \chi(\bar{s}) \text{ if and only if } \mathcal{M}, w \Vdash \chi(\nu_w \bar{s}). \tag{9}$$

Proof. In the atomic case the lemma holds by definition, using that $\nu_w(t) = I_w(t)$ for all terms t in $\chi(\bar{z})$, because $\chi(\bar{z})$ does not contain symbols in $\mathcal{F}_\varphi$. We treat the quantifiers, the connectives are straightforward.

Suppose $\chi = \exists x \psi(x, \bar{s})$. The direction from right to left of (9) follows from the fact that $\nu_w(a) = a$ for any $a \in \mathcal{D}_w$. Therefore suppose $\mathcal{M}_{v\varphi}^\Diamond, w \Vdash \exists x \psi(x, \bar{s})$ and thus $\mathcal{M}_{v\varphi}^\Diamond, w \Vdash \psi(b, \bar{s})$ for some $b \in \mathcal{D}_w^\Diamond$. Thus $\mathcal{M}, w \Vdash \psi(\nu_w b, \nu_w \bar{s})$ by the induction hypothesis, and thus $\mathcal{M}, w \Vdash \exists x \psi(x, \nu_w \bar{s})$.

Suppose $\chi = \forall x \psi(x, \bar{s})$. Suppose $\mathcal{M}, w \Vdash \forall x \psi(x, \nu_w \bar{s})$ and consider a node $u \succcurlyeq w$ and $b \in \mathcal{D}_u^\Diamond$. Hence $\mathcal{M}, u \Vdash \psi(\nu_u b, \nu_u \bar{s})$. The induction hypothesis gives $\mathcal{M}_{v\varphi}^\Diamond, u \Vdash \psi(b, \bar{s})$, which proves $\mathcal{M}_{v\varphi}^\Diamond, w \Vdash \chi$. For the other direction, assume $\mathcal{M}_{v\varphi}^\Diamond, w \Vdash \chi$ and consider a node $u \succcurlyeq w$

and $b \in \mathcal{D}_u$. As $b \in \mathcal{D}_u^{\Diamond}$, we have $\mathcal{M}_{v\varphi}^{\Diamond}, u \Vdash \psi(b, \bar{s})$. Hence $\mathcal{M}, u \Vdash \psi(\nu_u b, \nu_u \bar{s})$ by the induction hypothesis. Since $\nu_u(b) = b$, we have $\mathcal{M}, u \Vdash \psi(b, \nu_u \bar{s})$. This proves $\mathcal{M}, w \Vdash \chi$. □

Theorem 3. *For any $\Diamond$-standard prenex sentence φ:*

1. *If $\Diamond = \exists$ and $\mathcal{M}, v \nVdash \varphi$, then $\mathcal{M}_{v\varphi}^{\exists}, v \nVdash \varphi_{\exists}^{s}$.*
2. *If $\Diamond = \forall$ and $\mathcal{M}, v \Vdash \varphi$, then $\mathcal{M}_{v\varphi}^{\forall}, v \Vdash \varphi_{\forall}^{s}$.*
3. *For all sentences χ that do not contain predicate symbols that occur in φ nor function symbols from $\mathcal{F}_\varphi$ and all nodes $w \succcurlyeq v$:*

$$\mathcal{M}, w \Vdash \chi \text{ if and only if } \mathcal{M}_{v\varphi}^{\Diamond}, w \Vdash \chi.$$

Proof. By Theorem 2 and Lemma 12. □

This theorem can be generalised to arbitrary prenex sentences in the same way that Theorem 2 can be generalised to Corollary 2.

Corollary 4. *For any prenex sentence φ and any model $\mathcal{M}$, there are models $\mathcal{M}_\varphi^e$ and $\mathcal{M}_\varphi^a$ with the same frame as $\mathcal{M}$ such that*

1. *If $\mathcal{M}, v \nVdash \varphi$, then $\mathcal{M}_{v\varphi}^{e}, v \nVdash \varphi_{\exists}^{s}$.*
2. *If $\mathcal{M}, v \Vdash \varphi$, then $\mathcal{M}_{v\varphi}^{a}, v \Vdash \varphi_{\forall}^{s}$.*
3. *For all sentences χ that do not contain predicate symbols that occur in φ nor function symbols from $\mathcal{F}_\varphi$ and all nodes $w \succcurlyeq v$:*

$$\mathcal{M}, w \Vdash \chi \text{ if and only if } \mathcal{M}_{v\varphi}^{e}, w \Vdash \chi \text{ if and only if } \mathcal{M}_{v\varphi}^{a}, w \Vdash \chi.$$

6. Combinations of standard prenex formulas

6.1. *Choice functions*

In this and the next section the results above are extended beyond prenex sentences. We are interested in propositional combinations $A(\varphi_1, \ldots, \varphi_k)$ of prenex sentences $\bar{\varphi}$ and describe classes of propositional formulas such that for any $A(\bar{p})$ in the class, $A(\bar{\varphi})$ belongs to the skolem class for all (independent) prenex predicate sentences $\bar{\varphi}$. We can assume that the propositional formula $A(p_1, \ldots, p_k)$ is rigid (Section 2), because any such propositional combination $A(\varphi_1, \ldots, \varphi_k)$ can be seen as the instantiation $B(\psi_1, \ldots, \psi_m)$ of a rigid formula $B(q_1, \ldots, q_m)$, where every ψ_i is one of the $\varphi_1, \ldots, \varphi_k$. Therefore, from now on, we restrict our attention to rigid propositional formulas.

Throughout this section, $A(p_1, \ldots, p_k)$ is a rigid propositional formula and $\bar{\varphi} = \varphi_1, \ldots, \varphi_k$ are predicate prenex sentences. Analogous to the simpler case, we prove that $\vdash A(\bar{\varphi})^s$ implies $\vdash A(\bar{\varphi})$ by showing how from a countermodel to $A(\bar{\varphi})$ a countermodel to $A(\bar{\varphi})^s$ can be obtained. We prove that given $\mathcal{M} \nVdash A(\bar{\varphi})$ we can choose for every atom p_i a node $\delta(p_i)$ in $\mathcal{M}$ and construct a model $\mathcal{M}_{\bar{\varphi}}$ with the same frame as $\mathcal{M}$ such that

$$\begin{cases} \mathcal{M}, \delta(p_i) \Vdash \varphi_i \text{ and } \mathcal{M}_{\bar{\varphi}}, \delta(p_i) \Vdash \varphi_i^s & \text{if } p_i \text{ occurs negatively in } A, \\ \mathcal{M}, \delta(p_i) \nVdash \varphi_i \text{ and } \mathcal{M}_{\bar{\varphi}}, \delta(p_i) \nVdash \varphi_i^s & \text{if } p_i \text{ occurs positively in } A. \end{cases}$$

Then we show how this implies $\mathcal{M}_{\bar{\varphi}} \nVdash A(\bar{\varphi})^s$. The rest of this section provides the technical details behind this idea.

We use the convention that for predicate sentences $\varphi_1, \ldots, \varphi_k$, $\mathcal{M} \Vdash X(\bar{\varphi})$ denotes that $\mathcal{M} \Vdash A(\bar{\varphi})$ for all $A \in X$ and $\mathcal{M} \nVdash X(\bar{\varphi})$ that $\mathcal{M} \nVdash A(\bar{\varphi})$ for all $A \in X$. Whence $\mathcal{M} \nVdash X(\bar{\varphi})$ is not the negation of the statement $\mathcal{M} \Vdash X(\bar{\varphi})$. Furthermore, we use the convention that given a sequence of predicate sentences $\bar{\varphi} = \varphi_1, \ldots, \varphi_k$ and a propositional formula A which atoms are among $p_1, \ldots, p_k$, the expression $A(\bar{\varphi})$ denotes the formula that is the result of replacing p_i, if it occurs in A, by φ_i. For example, for $n = 3$ and $A(p_1, p_2, p_3) = (p_2 \vee p_3)$, $A(\varphi_1, \varphi_2, \varphi_3) = \varphi_2 \vee \varphi_3$.

6.2. Countermodels

Given two sets X and Y, $\mathcal{O}(X, Y)$ denotes the set of atoms that occur in X or Y, and $\mathcal{O}^p(X, Y)$ ($\mathcal{O}^n(X, Y)$) denotes the set of atoms that occur positively in X or negatively in Y (negatively in X or positively in Y). We only use this notion in case X and Y have no atoms in common.

Lemma 13. *Suppose that $\mathcal{M}$ is a model and X^+ and X^- are sets of rigid propositional formulas in the variables $p_1, \ldots, p_k$ such that no implication occurs positively in a formula in X^+ or negatively in a formula in X^-, no atom occurs in more than one formula in $X^+ \cup X^-$, and no atom occurs more than once in any formula in $X^+ \cup X^-$.*

Then there exists a choice function $\delta = \delta[\mathcal{M}, X^+, X^-] : \mathcal{O}(X^+, X^-) \to W_{\mathcal{M}}$ such that for all models $\mathcal{N}$ with the same frame as $\mathcal{M}$ and all predicate sentences $\bar{\varphi} = \varphi_1, \ldots, \varphi_k$ and $\bar{\psi} = \psi_1, \ldots, \psi_k$, where δp_i is short for $\delta(p_i)$: If $\mathcal{M} \Vdash X^+(\bar{\varphi})$ and $\mathcal{M} \nVdash X^-(\bar{\varphi})$, then the following holds.

1. *For all $p_i \in \mathcal{O}^p(X^+, X^-) : \mathcal{M}, \delta p_i \Vdash \varphi_i$.*
2. *For all $p_i \in \mathcal{O}^n(X^+, X^-) : \mathcal{M}, \delta p_i \nVdash \varphi_i$.*
3. *For all $A \in X^+$: if ($\mathcal{M}, \delta p_i \Vdash \varphi_i \Leftrightarrow \mathcal{N}, \delta p_i \Vdash \psi_i$) for all $p_i \in \mathcal{O}(A)$, then $\mathcal{N} \Vdash A(\bar{\psi})$.*
4. *For all $A \in X^-$: if ($\mathcal{M}, \delta p_i \Vdash \varphi_i \Leftrightarrow \mathcal{N}, \delta p_i \Vdash \psi_i$) for all $p_i \in \mathcal{O}(A)$, then $\mathcal{N} \nVdash A(\bar{\psi})$.*

Proof. Let X^+, X^-, $\mathcal{M}$, $\mathcal{N}$, $\bar{\varphi}$, and $\bar{\psi}$ be as in the lemma. The proof is with induction to the sum of the number of connectives in X^+ and in X^-.

In case that number is zero, all formulas in $X^+ \cup X^-$ are atoms. We let δp_i be the root of $\mathcal{M}$. Since any formula $A \in X^+ \cup X^-$ is an atom p_i, $A(\bar{\varphi}) = \varphi_i$ and $A(\bar{\psi}) = \psi_i$ for some i. Thus 3 and 4 clearly hold. That 1 and 2 hold follows from the fact that in this case $\mathcal{O}^p(X^+, X^-) = X^+$ and $\mathcal{O}^n(X^+, X^-) = X^-$. This completes the atomic case.

For the induction step, assume that at least one of X^+ and X^- contains at least one connective. We treat conjunction and implication; the case that $X^+ \cup X^-$ contains a disjunction is similar to the conjunction case.

Suppose $A_1 \wedge A_2 \in X^+$. Let $Y = \{A_1, A_2\} \cup (X^+ \backslash \{A_1 \wedge A_2\})$ and define $\delta = \delta[\mathcal{M}, X^+, X^-]$ to be $\delta[\mathcal{M}, Y, X^-]$. Note that Y contains no positive implications because X^+ does not. Therefore by the induction hypothesis 1–4 hold for $\delta[\mathcal{M}, Y, X^-]$. We have to show that 1–4 hold for $\delta[\mathcal{M}, X^+, X^-]$ as well. That 1 and 2 hold follows from the fact that $\mathcal{O}^p(X^+, X^-) = \mathcal{O}^p(Y, X^-)$ and $\mathcal{O}^n(X^+, X^-) = \mathcal{O}^n(Y, X^-)$. Assume that the antecedent of 3 holds for some $A \in X^+$. If $A \in Y$, then $\mathcal{N} \Vdash A(\bar{\psi})$ by the induction hypothesis. If $A = A_1 \wedge A_2$, then since $\mathcal{N} \Vdash A_i(\bar{\psi})$ holds by the induction hypothesis for $i = 1,2$, $\mathcal{N} \Vdash A(\bar{\psi})$ follows. Property 4 holds by the induction hypothesis.

Next, suppose $A_1 \wedge A_2 \in X^-$ and define

$$Y = (X^- \backslash \{A_1 \wedge A_2\}) \cup \{A_i \mid i = 1, 2 \text{ and } \mathcal{M} \nVdash A_i(\bar{\varphi})\}.$$

Define $\delta = \delta[\mathcal{M}, X^+, X^-]$ to be $\delta[\mathcal{M}, X^+, Y]$. Note that Y contains no negative implications because X^- does not. Therefore by the induction hypothesis 1–4 hold for $\delta[\mathcal{M}, X^+, Y]$. We show that 1–4 hold for $\delta[\mathcal{M}, X^+, X^-]$ as well. That 1 and 2 hold follows from the fact that $\mathcal{O}^p(X^+, Y) \subseteq \mathcal{O}^p(X^+, X^-)$ and $\mathcal{O}^n(X^+, Y) \subseteq \mathcal{O}^n(X^+, X^-)$. Property 3 holds by the induction hypothesis. For 4, assume that its antecedent holds for some $A \in X^-$. If $A \in Y$, then $\mathcal{N} \nVdash A(\bar{\psi})$ by the induction hypothesis. If $A = A_1 \wedge A_2$, then since $\mathcal{N} \nVdash A_i(\bar{\psi})$ holds by the induction hypothesis for at east one $i \in \{1, 2\}$, $\mathcal{N} \nVdash A(\bar{\psi})$ follows.

The case that remains is implication. Thus consider $A_1 \to A_2 \in X^-$ (recall that no implication occurs positively in X^+). Because we have $\mathcal{M} \nVdash (A_1 \to A_2)(\bar{\varphi})$, there is a node w such that $\mathcal{M}, w \Vdash A_1(\bar{\varphi})$ and $\mathcal{M}, w \nVdash A_2(\bar{\varphi})$. Let $Y = X^- \backslash \{A_1 \to A_2\}$, $Z^+ = \{A_1\}$, $Z^- = \{A_2\}$, and define $\delta = \delta[\mathcal{M}, X^+, X^-]$ as follows:

$$\delta p_i \equiv_{def} \begin{cases} \delta[\mathcal{M}_w, Z^+, Z^-](p_i) & \text{if } p_i \text{ occurs in } A_1 \text{ or } A_2 \\ \delta[\mathcal{M}, X^+, Y](p_i) & \text{otherwise.} \end{cases}$$

This function is well-defined because of the assumptions on X^+ and X^-. By the induction hypothesis, 1–4 hold for $\delta[\mathcal{M}_w, Z^+, Z^-]$ and $\delta[\mathcal{M}, X^+, Y]$. We show it holds for $\delta[\mathcal{M}, X^+, X^-]$. 1 and 2 follow from the fact that $\mathcal{O}^p(X^+, X^-) = \mathcal{O}^p(X^+, Y) \cup \mathcal{O}^p(Z^+, Z^-)$ and $\mathcal{O}^n(X^+, X^-) = \mathcal{O}^n(X^+, Y) \cup \mathcal{O}^n(Z^+, Z^-)$. Property 3 holds by the induction hypothesis on $\delta[\mathcal{M}, X^+, Y]$. Assume that the antecedent of 4 holds for some $A \in X^-$. If A is not equal to $A_1 \to A_2$, then $\mathcal{N} \nVdash A(\bar{\psi})$ follows from the induction hypothesis on $\delta[\mathcal{M}, X^+, Y]$. If A is equal to $A_1 \to A_2$, we have to show that $\mathcal{N} \nVdash (A_1 \to A_2)(\bar{\psi})$. Note that no implication occurs positively in Z^+ or negatively in Z^-. Therefore, by the induction hypothesis: if for all $p_i \in \mathcal{O}(Z^+, Z^-)$ we have

$$\mathcal{M}_w, \delta[\mathcal{M}_w, Z^+, Z^-](p_i) \Vdash \varphi_i \text{ if and only if } \mathcal{N}_w, \delta[\mathcal{M}_w, Z^+, Z^-](p_i) \Vdash \psi_i, \tag{10}$$

then $\mathcal{N}_w \Vdash A_1(\bar{\psi})$ and $\mathcal{N}_w \nVdash A_2(\bar{\psi})$, and thus $\mathcal{N} \nVdash (A_1 \to A_2)(\bar{\psi})$. Since the antecedent of 4 implies (10) for all $p_i \in \mathcal{O}(Z^+, Z^-)$, this completes the proof. □

Lemma 14. *For any propositional rigid nni formula $A(p_1, \ldots, p_k)$, any model $\mathcal{M}$ and predicate prenex sentences $\bar{\varphi} = \varphi_1, \ldots, \varphi_k$ such that $\mathcal{M} \nVdash A(\bar{\varphi})$, there exists a function $\delta = \delta[\mathcal{M}, \varnothing, \{A\}] : \mathcal{O}(\varnothing, \{A\}) \to W_{\mathcal{M}}$ such that*

1. *If p_i occurs negatively in A, then $\mathcal{M}, \delta p_i \Vdash \varphi_i$,*
2. *If p_i occurs positively in A, then $\mathcal{M}, \delta p_i \nVdash \varphi_i$.*

And for any model $\mathcal{N}$ with the same frame as $\mathcal{M}$, $\mathcal{N} \nVdash A(\bar{\varphi})^s$ holds whenever the following two statements hold:

3. *If p_i occurs negatively in A, then $\mathcal{N}, \delta p_i \Vdash (\varphi_i)^s_\forall$,*
4. *If p_i occurs positively in A, then $\mathcal{N}, \delta p_i \nVdash (\varphi_i)^s_\exists$.*

Proof. Let $\delta = \delta[\mathcal{M}, \varnothing, \{A\}]$ be the function whose existence is proved in Lemma 13, where $\psi_i = (\varphi_i)^s_\exists$ if p_i occurs positively in A and $\psi_i = (\varphi_i)^s_\forall$ otherwise. Since $A(\bar{\varphi})^s = A(\bar{\psi})$, Lemma 13 implies $\mathcal{N} \nVdash A(\bar{\varphi})^s$ if statements 3–4 of this lemma hold. □

6.3. *Combinations of independent formulas*

Recall that sentences $\varphi_1, \ldots, \varphi_k$ are *independent* if no two sentences have a predicate symbol in common, and a formula is *nni (no negative implications)* if no implication occurs negatively in the formula.

Theorem 4. *In any well-founded tree complete intermediate logic L, for any propositional formula $A(p_1, \dots, p_k)$ that is equivalent to a rigid nni formula, and all independent predicate prenex sentences $\bar{\varphi} = \varphi_1, \dots, \varphi_k$:*

$$\vdash_L A(\bar{\varphi}) \text{ if and only if } \vdash_L A(\bar{\varphi})^s.$$

Proof. Suppose $\mathcal{M} \nVdash A(\bar{\varphi})$. We assume without loss of generality that A itself is a rigid nni formula. We use in this proof the following abbreviation:

$$\psi_i \equiv_{def} \begin{cases} (\varphi_i)^s_\exists & \text{if } p_i \text{ occurs positively in } A(p_1, \dots, p_k) \\ (\varphi_i)^s_\forall & \text{if } p_i \text{ occurs negatively in } A(p_1, \dots, p_k). \end{cases}$$

Because $A(\bar{\varphi})^s = A(\bar{\psi})$, we have to show that $\mathcal{N} \nVdash A(\bar{\psi})$.

By Lemma 14, there is a function $\delta = \delta[\mathcal{M}, \varnothing, \{A\}] : \mathcal{O}(\varnothing, \{A\}) \to W_{\mathcal{M}}$ such that 1–2 hold and for any model $\mathcal{N}$ with the same frame as $\mathcal{M}$ we have $\mathcal{N} \nVdash A(\bar{\psi})$ if properties 3–4 of Lemma 14 hold. We write w_i for $\delta(p_i)$.

Using the notation from Corollary 4, define $\mathcal{M}_0 = \mathcal{M}$ and for all $0 < i \leq k$:

$$\mathcal{M}_i \equiv_{def} \begin{cases} (\mathcal{M}_{i-1})^e_{w_i\varphi_i} & \text{if } p_i \text{ occurs positively in } A(p_1, \dots, p_k) \\ (\mathcal{M}_{i-1})^a_{w_i\varphi_i} & \text{if } p_i \text{ occurs negatively in } A(p_1, \dots, p_k). \end{cases}$$

Let $\mathcal{N} \equiv_{def} \mathcal{M}_k$. Note that $\mathcal{N}$ has the same frame as $\mathcal{M}$. Therefore, to show that $\mathcal{N} \nVdash A(\bar{\psi})$ it suffices to show for all $1 \leq i \leq k$:

$$\mathcal{M}, w_i \Vdash \varphi_i \Leftrightarrow \mathcal{N}, w_i \Vdash \psi_i.$$

Consider an atom p_i. Because for no $i \neq j$, sentences ψ_i and ψ_j have a skolem function in common, and the $\varphi_1, \dots, \varphi_k$ are independent, Corollary 4 implies that for all h, j such that $0 \leq h < i < j \leq k$:

$$\mathcal{M}_h, w_i \Vdash \varphi_i \Leftrightarrow \mathcal{M}_i, w_i \Vdash \varphi_i \tag{11}$$

$$\mathcal{M}_i, w_i \Vdash \varphi_i \Leftrightarrow \mathcal{M}_{i+1}, w_i \Vdash \psi_i \tag{12}$$

$$\mathcal{M}_j, w_i \Vdash \psi_i \Leftrightarrow \mathcal{M}_{i+1}, w_i \Vdash \psi_i. \tag{13}$$

That (11) and (13) hold follows immediately from 3 in Corollary 4. For (12), this follows from the fact that in case $\mathcal{M}_i, w_i \Vdash \varphi_i$, atom p_i occurs negatively in A, and whence $\psi = (\varphi_i)^s_\forall$ and $\mathcal{M}_{i+1}, w_i \Vdash \psi_i$. Similar reasoning applies to the case that $\mathcal{M}_i, w_i \nVdash \varphi_i$.

By Lemma 14, it suffices to show that if p_i occurs negatively (positively) in A, then $\mathcal{N}, w_i \Vdash \psi_i$ ($\mathcal{N}, w_i \nVdash \psi_i$). In the first case, $\mathcal{M}_{i+1}, w_i \Vdash \psi_i$ by property 1 of Lemma 14, (11), (12) and Corollary 4, and in the second case $\mathcal{M}_{i+1}, w_i \nVdash \psi_i$ by property 2 of Lemma 14, (11), (12) and Corollary 4. By (13) it follows that $\mathcal{N}, w_i \Vdash \psi_i$ in the first case and $\mathcal{N}, w_i \nVdash \psi_i$ in the second case, which is what had to be shown. □

6.4. Combinations of general formulas

Theorem 5. *In any well-founded tree complete intermediate logic L, for any propositional formula A that is equivalent to a rigid nni formula and all predicate prenex sentences $\bar{\varphi} = \varphi_1, \dots, \varphi_k$ such that for any positive occurrence of p_i in A, φ_i is quantifier free:*

$$\vdash_L A(\bar{\varphi}) \text{ if and only if } \vdash_L A(\bar{\varphi})^s.$$

Proof. Without loss of generality we assume that A itself is a rigid nni formula as in the lemma. Suppose $\mathcal{M} \not\Vdash A(\bar{\varphi})$. For $i = 1, \ldots, k$, let

$$\psi_i \equiv_{def} \begin{cases} (\varphi_i)^s_\forall & \text{if } p_i \text{ occurs negatively in } A \\ \varphi_i & \text{if } p_i \text{ occurs positively in } A. \end{cases}$$

Clearly, $A(\bar{\varphi})^s = A(\bar{\psi})$.

By Lemma 14 there is a function $\delta = \delta[\mathcal{M}, \varnothing, \{A\}] : \mathcal{O}(\varnothing, \{A\}) \to W_\mathcal{M}$ such that 1–2 hold and for any model $\mathcal{N}$ with the same frame as $\mathcal{M}$ we have $\mathcal{N} \not\Vdash A(\bar{\psi})$ if properties 3–4 of Lemma 14 hold. We denote $\delta(p_i)$ with w_i.

Using the notation from Corollary 2, define models $\mathcal{M}_i$ for $0 < i \leq k$: $\mathcal{M}_0 \equiv_{def} \mathcal{M}$ and $\mathcal{N} \equiv_{def} \mathcal{M}_k$ and

$$\mathcal{M}_i \equiv_{def} \begin{cases} (\mathcal{M}_{i-1})^a_{w_i \varphi_i} & \text{if } p_i \text{ occurs negatively in } A \\ \mathcal{M}_{i-1} & \text{if } p_i \text{ occurs positively in } A. \end{cases}$$

Note that $\mathcal{N}$ has the same frame as $\mathcal{M}$. Therefore, to show that $\mathcal{N} \not\Vdash A(\bar{\varphi})^s$ it suffices to show that

$$\begin{cases} \mathcal{N}, w_i \Vdash \psi_i & \text{if } p_i \text{ occurs negatively in } A \\ \mathcal{N}, w_i \not\Vdash \psi_i & \text{if } p_i \text{ occurs positively in } A. \end{cases}$$

Because for $i \neq j$, sentences $(\varphi_i)^s_\forall$ and $(\varphi_j)^s_\forall$ have no skolem functions in common, Corollary 4 implies that for all $0 \leq h < i < j \leq k$:

$$\mathcal{M}_h, w_i \Vdash \varphi_i \Leftrightarrow \mathcal{M}_i, w_i \Vdash \varphi_i$$
$$\mathcal{M}_i, w_i \Vdash \varphi_i \Rightarrow \mathcal{M}_{i+1}, w_i \Vdash \psi_i$$
$$\mathcal{M}_j, w_i \Vdash \psi_i \Leftrightarrow \mathcal{M}_{i+1}, w_i \Vdash \psi_i.$$

As in the last paragraph of Theorem 4, this implies what we had to show. □

The following corollary is a strengthening of the previous theorem that contains a restriction on A rather than the φ_i.

Corollary 5. *In any well-founded tree complete intermediate logic L, for any propositional formula $A(p_1, \ldots, p_k)$ equivalent to a rigid nni formula in which no atom occurs positively and all predicate prenex sentences $\bar{\varphi} = \varphi_1, \ldots, \varphi_k$:*

$$\vdash_L A(\bar{\varphi}) \text{ if and only if } \vdash_L A(\bar{\varphi})^s.$$

Theorem 6. *In any well-founded tree complete intermediate logic L, for any predicate prenex sentences $\bar{\varphi} = \varphi_0, \varphi_1, \ldots, \varphi_k$ such that φ_0 is independent of any of the $\varphi_1, \ldots, \varphi_k$:*

$$\vdash_L \bigwedge_{i=1}^k \varphi_i \to \varphi_0 \text{ if and only if } \vdash_L (\bigwedge_{i=1}^k \varphi_i \to \varphi_0)^s.$$

Proof. The direction from left to right is clear. The other direction is proved by contraposition. Let $\mathcal{M}$ be a model such that $\mathcal{M} \Vdash \varphi_i$ for all $i > 0$ and $\mathcal{M} \not\Vdash \varphi_0$. Using the notation from Corollary 4, define

$$\mathcal{M}_0 = \mathcal{M} \quad \mathcal{M}_i \equiv_{def} (\mathcal{M}_{i-1})^a_{\varphi_i} \quad \mathcal{N} = (\mathcal{M}_k)^e_{\varphi_0}.$$

By Theorem 2, $\mathcal{M}_k \Vdash (\varphi_i)^s_\forall$ for all $1 \leq i \leq k$. Corollary 4 implies that $\mathcal{N} \not\Vdash (\varphi_0)^s_\exists$ and $\mathcal{N} \Vdash (\varphi_i)^s_\forall$ for all $i > 0$. This proves that $\mathcal{N} \not\Vdash (\bigwedge_{i=1}^k \varphi_i \to \varphi_0)^s$. □

Table 1
Examples of formulas in the skolem class of IQC.

$A(p_1, \dots, p_k)$	$\varphi_1, \dots, \varphi_k$	Instance
$\bigwedge_{i=1}^{k} p_i$	prenex sentences	(Corollary 2)
$\bigwedge_{i=1}^{l} p_i \to \bigvee_{j=l+1}^{k} \neg p_j$	prenex sentences	(Corollary 5)
$\bigwedge_{i=1}^{l} p_i \to \bigvee_{j=l+1}^{k} p_j$	independent prenex sentences	(Theorem 4)
$\bigwedge_{i=2}^{k} p_i \to p_1$	φ_1 is independent of φ_i for all $2 \le i \le k$	(Theorem 6)
$\bigwedge_{i=1}^{l} p_i \to \bigvee_{j=l+1}^{k} p_j$	φ_j is quantifier-free for all $l+1 \le j \le k$	(Theorem 5).

7. Conclusion

The following results have been obtained for any well-founded tree complete intermediate logic L and any formula $A(p_1, \dots, p_k)$ that is equivalent to a rigid nni propositional formula. Given any prenex sentences $\bar{\varphi} = \varphi_1, \dots, \varphi_k$, the formula $A(\bar{\varphi})$ is in the skolem class of L if one of the following cases hold:

(Corollary 2) A is a conjunction of atoms;

(Corollary 2) A is a disjunction of conjunctions of atoms and L has the disjunction property;

(Theorem 4) $\varphi_1, \dots, \varphi_k$ are independent;

(Corollary 5) no atom occurs positively in A;

(Theorem 5) if p_i occurs positively in A, then φ_i is quantifier-free;

(Theorem 6) $A = (\bigwedge_{i=1}^{k} p_i \to p_0)$ and φ_0 is independent of all φ_i with $i > 1$.

Moreover, if A does not contain negative occurrences of atoms, then

$$\vdash_L A(\bar{\varphi}) \Leftrightarrow \vdash_{LCD} A(\bar{\varphi})^s.$$

The proof is analogous to that of Corollary 2, as the only models that occur in the proof besides the original model $\mathcal{M}$ are of the form $\mathcal{M}^{\exists}_{\varphi_i}$, and thus constant domain models. Because $L \subseteq L_{CD}$, the equivalence above implies that $A(\bar{\varphi})$ is in the skolem class of L if and only if it is in the skolem class of L_{CD}.

7.1. Positive examples

Table 1 contains various instances of the results above. It indicates, for various $A(\bar{p})$, for which $\bar{\varphi}$ the formula $A(\bar{\varphi})$ belongs to the skolem class. It is understood, but not explicitly indicated, that in the table any conjunction of the indicated A also belongs to the skolem class for the indicated prenex sentences. And similarly for disjunction in case the intermediate logic has the disjunction property.

7.2. Counterexamples

In this section we show that the results from Section 6 cannot be further improved, at least not in terms of the propositional formula A that occurs in the theorems.

Table 2
$A(\varphi_1, \varphi_2, \varphi_3)$ does not belong to the skolem class of IQC.

Formula $A(p_1, p_2, p_3)$	Prenex instances $\varphi_1, \varphi_2, \varphi_3$
$\neg\neg p_1$	$\varphi_1 = \forall x(Px \vee \neg Px)$
$\neg p_1 \to p_2$	$\varphi_1 = \forall x(Px \vee \neg Px)$ $\varphi_2 = \bot$
$(p_1 \to p_2) \to p_3$	$\varphi_1 = \forall x(Px \vee \neg Px)$ $\varphi_2 = \varphi_3 = \bot$
$p_1 \to p_2 \vee p_3$	$\varphi_1 = \forall x(Px \vee \psi)$ $\varphi_2 = \forall x Px$ $\varphi_3 = \psi$.

To see why this is the case for Theorem 4, observe that the first three formulas of Table 2 are the simplest formulas that are not equivalent, over IQC, to a formula in which no implication occurs negatively. For $A(\bar{p})$ being one of these three formulas, the table provides independent $\varphi_1, \varphi_2, \varphi_3$ such that $A(\bar{\varphi})$ does not belong to the skolem class of IQC.

The rigid nni formulas with at most two atoms that are not covered by Corollary 2 or Corollary 5 is (equivalent to a renaming of) $p_1 \to p_2$. Although any instance $\varphi_1 \to \varphi_2$ is in the skolem class provided φ_1, φ_2 are independent, we do not know whether the same holds for arbitrary prenex sentences φ_1, φ_2. However, already the implication $(p_1 \to p_2 \vee p_3)$ does not have this property, as the last line in Table 2 shows. Note that we cannot use $\varphi_1 = \forall x(Px \vee \psi)$ and $\varphi_2 = \forall x Px \vee \psi$ as a counterexample for $p_1 \to p_2$ because φ_2 is not a prenex formula.

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