3.4 Travelling to Hamburg

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3.4.1 Introduction

This section describes an example of a task that was designed and field-tested for the ICME13 conference. Its aim is to illustrate how the principles of Realistic Mathematics Education (RME) (see, e.g., Van den Heuvel-Panhuizen & Drijvers, 2014). can guide the design of a new task. Indeed, task design is a core element in setting up mathematics education according to an RME approach. This is one of the reasons why design-based research is an important research methodology in many studies on RME (Bakker & Van Eerde, 2015).

The task presented here involves setting up a graph that many students are not familiar with, because it displays one distance plotted against another. For several reasons, it makes sense to have students work on such less common graphs. First, graphs in mathematics education in almost all cases involve an independent variable, often called x, on the horizontal axis, and a dependent variable, for example y or $f(x)$, on the vertical axis. However, there are also other types of graphs than these common x-y graphs. In economics, the independent variable—not always x but also t for time—can also be plotted on the vertical axis rather than on the horizontal. In physics—think about phase diagrams—the independent variable may be a parameter that is not plotted on one of the axes. The latter case reflects the mathematical notion of parametric curve, in which the independent variable remains implicit. In short, students should be prepared for other types of graphs as well.

A second, more general reason to address non-typical types of graphs is the worldwide call for mathematical thinking and problem solving as overarching goals in mathematics education (Devlin, 2012; Doorman et al., 2007; Schoenfeld, 1992). If students are to be educated to become literate citizens and versatile professionals, they should be trained to deal with uncommon problem situations that invite flexibility. As such, mathematical thinking has become a core aim in the recent curriculum reform in the Netherlands (Drijvers, 2015; Drijvers, De Haan, & Doorman, submitted).

In this section, first the task will be presented together with some design considerations that led to its present form. Next, a brief sketch is provided of the results of the field test in school and of what the task brought to the fore at the ICME13 conference. As a task may need adaptation to the specific context in which it is used, its elaboration is described next for the purpose of in-service teacher training, including a three-dimensional perspective. Finally, the main points will be revisited in the conclusion section.

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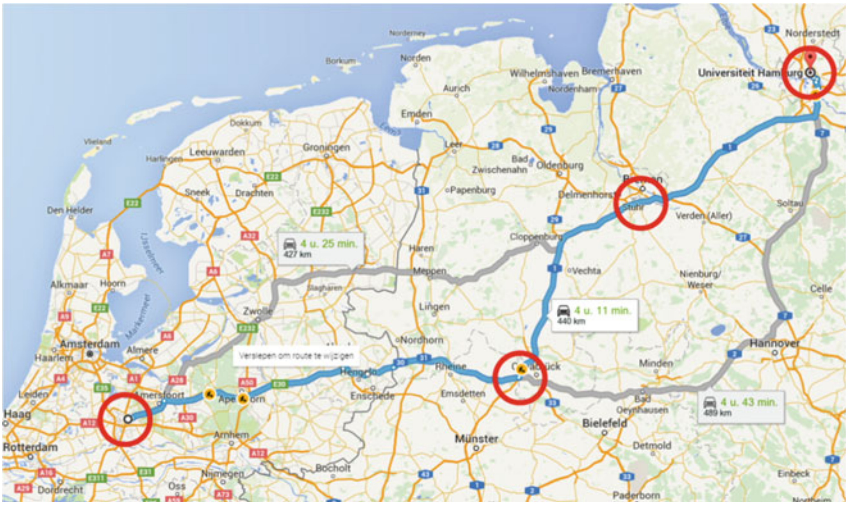


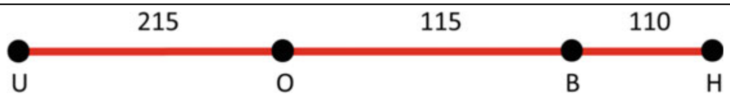
To attend the ICME13 conference, we travelled to Hamburg by car. To prepare for this, I consulted Google Maps. Here you see the result: we leave Utrecht, in the Netherlands. The distance to Hamburg is 440 km. After 215 km, we pass by Osnabrück. Another 115 and we pass by Bremen. Next, we need to drive the final 110 km to reach Hamburg.

Fig. 3.12 Setting the scene for the task

3.4.2 *Task Design*

Figure 3.12 shows the presentation of the task in the form of a picture, displayed through a data projector, and a suggested text that might be spoken by the teacher. As the problem situation is a somewhat personal story from 'real life', it is preferable to deliver the text orally rather than in written form. It is expected that the task becomes 'experientially real' through this form of presentation. In the task, the perspective taken is that of a participant in the ICME13 conference in Hamburg, Germany. Of course, this perspective could easily be adapted to other situations that are more relevant to the audience.

Figure 3.13 shows how the task presentation might continue. It shows a schematisation of the problem situation, in which the 'noise' of the real map has been removed. In the text that might be spoken, this schematisation and the underlying mental step of representing the highway as a line segment are explicitly addressed. Depending on the audience and the intended goal of the task, of course, one might consider leaving this step up to the student and to reduce guidance at the benefit of opportunities for guided reinvention. For the field tests addressed in the next section, it was decided not to do so, due to the expected level of the students and the time constraints. The text in Fig. 3.13 ends with the problem statement. Students are invited to use their worksheet, which contained two coordinate systems like the one shown in Fig. 3.14.



If I depict the highways as straight-line segments, I can globally represent the situation in this way. U stands for Utrecht, 215 to O (Osnabrück), 115 to B (Bremen), and 110 to H (Hamburg). Now I am wondering the following. During the trip, while driving, my distances to both Osnabrück and Bremen constantly change. What would a graph look like, if I put the distance to Bremen vertically, and my distance to Osnabrück horizontally? Could you please take a few minutes to sketch the graph in which the distance to B during the trip is vertically plotted against the distance to O on the horizontal axis? Please use your worksheet!

Fig. 3.13 Schematising the situation and posing the problem

Fig. 3.14 Coordinate system that is shown on the worksheet

One might wonder if this is a realistic task. Who would be so silly as to raise this question? Well, we were. We were three mathematicians who were bored during the long trip to Hamburg. How to "sell" this to students? A possible approach could be to "play the card of the strange mathematician", but this should not be exaggerated. Our experience is that students may be intrigued by such problem situations, even if the question itself is not solving a 'real' problem. Also, even if one might ask "Why do we need to know this?", the problem situation is sound in the mathematical sense, and has possibilities for applications in mathematics and science, as explained in Sect. 3.4.1.

After the task is presented, students can start to work in pairs, in small groups, or individually. As it is expected that there will be quite a bit of differentiation in the class—some students might solve this task in a minute, whereas others may not have a clue where to start—the students who finish quickly can be provided orally with an additional task:

> If you feel you are doing well, please think of a question that you might use to help a peer who doesn't know how to start, a question that might serve as a scaffold.

Also, some scaffolding questions are prepared that may serve as a hint to react to students who have difficulties with the task and raise their hands for help. For example: How can you make a start? Do you know a similar but easier problem?

Does this resemble a problem that you have seen in the past? Where are you in the *O*-*B* plane when you are leaving Utrecht? And when you arrive in Hamburg? And when you pass by Osnabrück?

To be effective, such a class activity needs a whole-class wrap-up. It might start with the question of how to help peers to make a start, or how you started yourself. For example, one can consider finding the position in the plane for the special moments of leaving Utrecht and arriving in Hamburg. This leads to the points $(O, B) = (215, 330)$ and $(O, B) = (225, 110)$. How about, when passing Osnabrück and Bremen? Another option is to imagine what happens in between *O* and *B*: the distance to *O* increases as much as the distance to B decreases. How does this affect the graph? A natural question that emerges, is whether the driving speed should be constant, and if it matters at all. Would the graph look different if you walk from Utrecht to Hamburg rather than driving (not recommended, of course)? In an advanced class, with many students coming up with a sensible graph, it might be interesting to show an animation in a dynamic geometry environment, which in its turn may invite setting up parametric equations. Of course, how far one can go in such a wrap-up largely depends on the students' progress. If needed, postponing the presentation of the results to the next lesson may be an appropriate 'cliff-hanger teaching strategy'. The expertise and the experience of the teacher in leading the whole-class wrap-up are decisive in making the task work in class.

In retrospective, the following considerations guided the design of this task:

– To make the problem situation come alive for the audience at the ICME conference, the trip to Hamburg was chosen as a point of departure. The 'experientially real' criterion was decisive.
– In the beginning, there was some hesitation on whether to travel by train or by car. The advantage of the train would have been that, contrary to cars driving on highways, trains do pass through the city centres. However, it was estimated that the car version would be more recognizable to the audience, particularly in combination with the Google Maps image and driving directions. As an aside, the designers of the tasks did not travel by car to Hamburg themselves; the story is based on another car trip. The point in designing this task is not the truth of the story behind it, but its experiential reality and mathematical soundness.
– How openly to phrase the problem? When designing the problem different versions came up, with different levels of support. Indeed, the version we had in mind might be quite a surprising challenge to students, but it was expected that through the scaffolding hints mentioned above, it would be possible to have the students start.
– How to present the problem? It was decided to present the task orally to the class as a whole, supported by slides displayed through a data projector. The idea here was that this would enhance the personal character of the problem situation. Also, such an oral whole-class introduction is expected to provide a collaborative setting, while working on a shared problem. Finally, an oral presentation can be a welcome change after many textbook-driven activities.

- It was decided to provide the students with the crucial linear representation (Fig. 3.13) and, in this way, give away the first schematisation step. Other choices can make perfect sense here. All depends on the level of the students, their preliminary knowledge, the time available, and the learning goals.
- To deal with student differences in this task, a second layer was built in, namely, that of thinking of hints for peers. In this way, students who finished the task quickly were invited to put themselves in the place of their slower peers, and, as a consequence, reflect on the thinking process needed to solve the task.

3.4.3 Field Tests

To prepare the activity for the ICME13 conference, the task was field-tested in a bilingual class in a rural school in the Netherlands. The students, 13- to 14-year olds, took part in the pre-university stream within secondary education. The pilot took one 50-minute lesson. After the oral introduction, students went to work. The question needed to be repeated once or twice. Also, we had a short whole-class discussion after the first tentative graphs, and invited the students to sketch a second one afterwards.

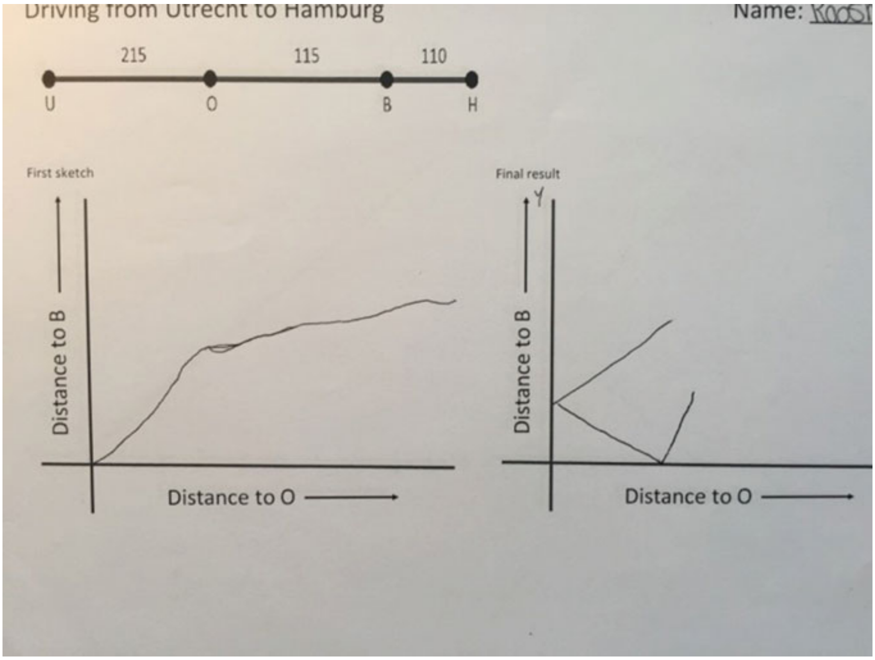


Fig. 3.15 Graphs by Student 1

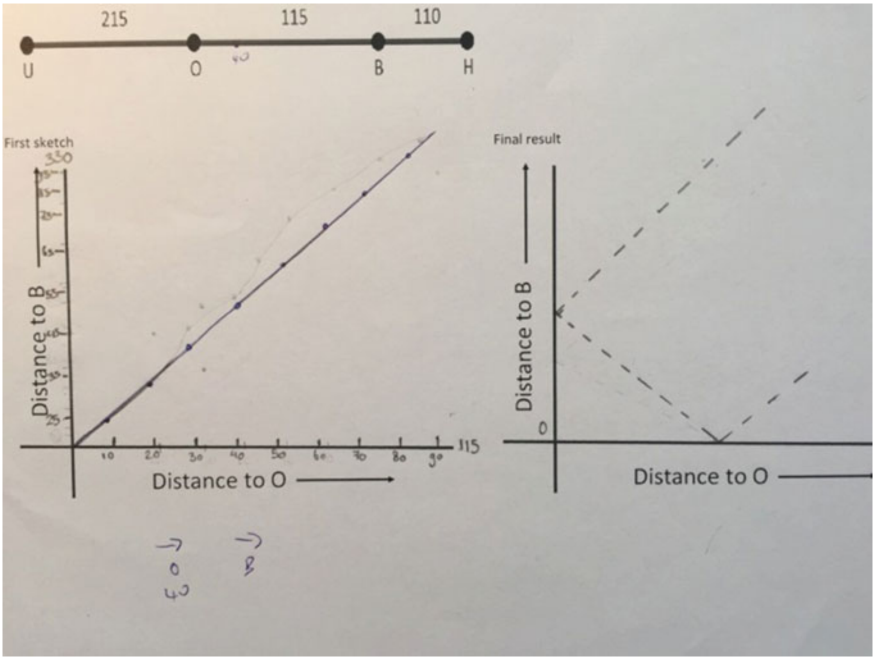


Fig. 3.16 Graphs by Student 2

Figure 3.15 shows the work of Student 1 who initially seemed to identify the graph with the map, not an uncommon phenomenon while introducing graphs.

The second sketch made by Student 2, shown in Fig. 3.16, is much better, even if the first and the last part of the graph are not parallel. Student 2's first graph was linear, suggesting a proportional increase of both distances. Clearly, this student did not have a correct mental image of the problem situation at the start. After the whole-class interruption and maybe some discussion with peers, the second graph was close to perfect.

In the whole-class wrap-up, Student 2 explained his initial reasoning, but was interrupted by Student 3, who introduced the notion of linearity.

Student 2: First, I had like this, but I thought, you can't be in the origin at the same time, you can't be in *B* and in *O* at the same time ….

Teacher: Yeah, you cannot.

Student 2: So, I thought like, maybe they, yeah, I don't know, I can't really explain it.

Student 3: It's a linear formula.

Teacher: Wow, how come? Why is it…. Please, explain.

Student 3: Well, ehm, since there isn't, eh yes, since the amount added always is the same, the first step, it's a linear formula.

This short one-lesson intervention confirmed the initial expectations, that the problem situation was rich and could give rise to interesting discussions.

During the ICME13 conference in Hamburg, the task was piloted again in a similar way with an audience of about 250 attendants. Of course, individual help was hard to deliver in this large-scale setting. Still, in comparison to the field test in class, similar patterns could be observed. Also, the need for level differentiation was bigger than in the secondary class, due to the heterogeneity of the ICME audience. It was surprising how mathematics teachers, researchers and educators have their schemes for graphing, and can get quite confused once these schemes are challenged by new situations.

3.4.4 Possible Task Extensions

As already mentioned, guided reinvention, meaning and experiential reality are subtle matters. To be able to deal with this subtlety appropriately in a setting with students of different levels, a good task should provide teachers with opportunities to simplify the task, to provide variations, and to deepen and extend the task. A straightforward way to simplify the task is to leave out one of the two cities between Utrecht and Hamburg, or even to leave out both and ask for the graph of the distance to Hamburg against the distance to Utrecht. These might be appropriate first steps towards solving the original problem. As variations, one may consider similar situations, such as the already mentioned trip by train, or a bike ride from home to school.

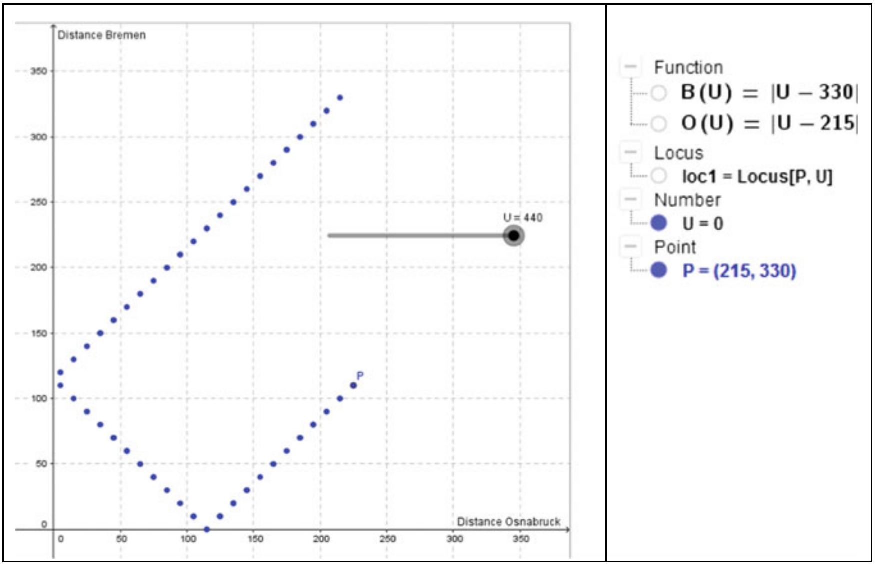


Fig. 3.17 Animation in Geogebra (left) and the underlying function definitions (right)

As students may come up with different graphs and will make all kinds of gestures while explaining their reasoning, it might be convenient to show the graph through an animation in a dynamic geometry system. The left screen of Fig. 3.17 shows such an example in Geogebra, using a slider bar to move the point. This may help to illustrate the resulting graph. In the meantime, however, this raises a deeper question: How can you make this animation, which equations and definitions are needed? The right screen in Fig. 3.17 provides the answer. The following definitions were used:

• Distance to Utrecht: U	(Independent variable)
• Distance to Osnabrück: $O(U) = \|U - 215\|$	(Dependent variable)
• Distance to Bremen $B(U) = \|U - 330\|$	(Dependent variable)
• Point in the plane: $P(U) = (O(U), B(U))$	

In this way, we take a mathematical perspective and the problem forms a gateway to the fascinating world of parametric curves.

As a final extension, also a third city between Utrecht and Hamburg can be considered. For example, Cloppenburg is about in the middle of Osnabrück and Bremen: Osnabrück–Cloppenburg is 60 km, and Cloppenburg–Bremen is 55 km. Can you plot a graph, indicating how the distances to Osnabruck, Cloppenburg and Bremen co-vary during the trip? Note that this task, in line with its higher level, is phrased in a somewhat more abstract way. Of course, the graph in this case will be in three dimensions rather than in two. Again, an animation can be built in Geogebra (Fig. 3.18). Rotating the graph shows a familiar form (Fig. 3.19) and in a natural way raises new, interesting questions, such as on the angle between the trajectory and the planes. This latter extension to the third dimension was used in a teacher professional development course, in which the participants found the two-dimensional case relatively easy, but were intrigued by the problem situation.

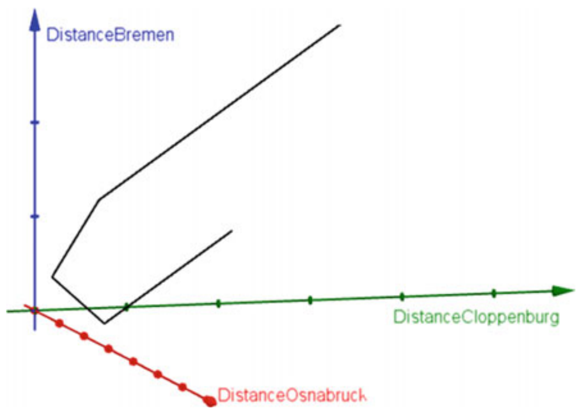


Fig. 3.18 3D graph in Geogebra

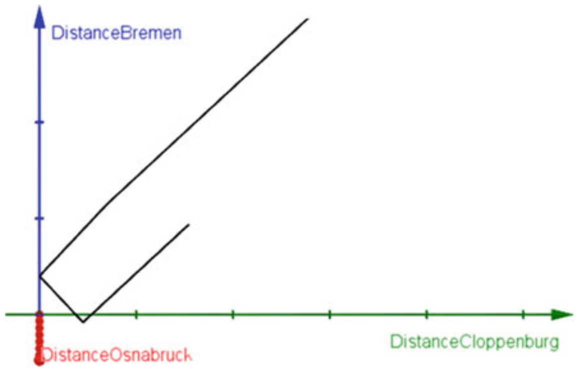


Fig. 3.19 3D graph in Geogebra seen from one of the axes

3.4.5 Conclusion

This example on task design according to RME principles revealed that for both students in Dutch secondary school and for participants of the ICME13 conference, it was hard to have the flexibility to refrain from the conventional time-distance graph paradigm and to open the horizon towards distance-distance graphs. This type of mathematical flexibility, needed in this unconventional and non-routine task, is core in problem solving, and at the heart of what RME sees as an essential value in mathematics education. The point of departure is 'realistic' in the sense that both target groups could imagine the situation and seemed to perceive it as realistic. What makes the task suitable from an RME perspective is that it can be used in different variations, appropriate for different levels of students and for different mathematical learning goals. Also, there are different, more and less mathematical, approaches and solution strategies, as well as follow-up questions. Finally, the somewhat surprising character of the task, may lead to the kind of lively and mathematically interesting interactions among students and between students and their teacher that are so important in the co-construction of mathematical meaning.

These task characteristics are central in RME and reflect the approach to mathematics education in the Dutch didactic tradition. To design tasks that elicit genuine mathematical activity in students is a challenge, not only in the Netherlands but in the mathematics education community world-wide!

3.5 Voices from Abroad

The chapter concludes with five sections which give a flavour of the international life of RME. From the beginning of the development of RME, mathematics educators all over the world were interested in it. This led to cooperation with a large number of countries where RME ideas and materials were tried out, discussed and adapted.