Treffers sees students' own productions and constructions not only as a didactical tool, but also as a goal of mathematics education. Naturally this is true, since producing and constructing mathematics yourself is in essence mathematising. This ability is even more important in modern society than it already was in the early days of Wiskobas.

> Treffers: My general recommendation for the future of mathematics education is: enlarge the role of students' own productions and own constructions, in practice, in problem solving, and in the combination of the two.

3.3 Contexts to Make Mathematics Accessible and Relevant for Students—Jan de Lange's Contributions to Realistic Mathematics Education

Michiel Doorman[3]

3.3.1 Introduction

De Lange has been working at Utrecht University for 40 years. He led the Freudenthal Institute from 1981 and as Professor/Director from 1989 until 2005. He started as a mathematician and initially was more interested in upper secondary education. His most recent interests lie in the study of talents and competencies such as the scientific reasoning of very young children. De Lange worked on the theoretical basis of assessment design, carrying it through to practical impact in the Netherlands and internationally as chair of the PISA Mathematics Expert Group (Fig. 3.7).

De Lange's contributions to the ideas underpinning Realistic Mathematics Education (RME) are strongly connected to the role of contexts in mathematical problems. In traditional mathematics education, contexts are included in textbooks as word problems or as applications at the end of a chapter. These contexts play hardly any role in students' learning processes. Word problems are mostly short storylines presenting a mathematical problem that has a straightforward solution. Applications at the end of a chapter help students to experience how the acquired mathematical procedures can be applied in a context outside mathematics.

In RME, context problems have a more central role in students' learning process from the very start onwards. These problems are presented in a situation that can be experienced as realistic by the students and do not have a straightforward solution procedure. On the contrary, students are invited to mathematise the situation and to invent and create a solution. Ideally, students' intuitive and informal solutions anticipate the topics of the chapter and provide opportunities for the teacher to connect these topics to the students' current reasoning. In RME, such context problems are

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Fig. 3.7 Jan de Lange during the interview

expected to have a central role in the guided reinvention of mathematics by the students themselves.

This understanding of the potential of context problems was developed during the 1970s and 1980s and was largely inspired by the work of De Lange. His contributions to the Dutch didactic tradition consisted of developing a large collection of teaching units used mainly in innovation-oriented curriculum projects in the Netherlands and in the USA. In preparation for the Dutch contribution to ICME13 Thematic Afternoon session on European Didactic Traditions, De Lange was interviewed to reflect on his work and specifically on the importance of contexts in mathematics education.

3.3.2 Using a Central Context for Designing Education

After De Lange graduated in the 1970s he started his career as a mathematics teacher. Soon he discovered that students reacted quite differently to the topics that he tried to address in his lessons. Most surprising for him was that they did not recognize mathematics in the world around them. After De Lange moved to the Freudenthal Institute, one of his ambitions was to find contexts that could be used to make mathematics accessible. A teacher in lower secondary school asked him if he could do something for her students who had problems with trigonometric ratios. De Lange designed a unit intended for a couple of weeks of teaching.

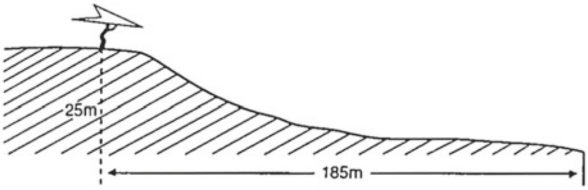
> De Lange: I started to work on one of my hobbies, planes, and I wrote a little booklet. It is called *Flying Through Maths*. It is about all kinds of different mathematics, all in one context. It is about glide ratios, vectors and sine and cosine.

All the mathematics in this teaching unit is presented in the context of planes and flying, which approach is later referred to by De Lange as an example of 'central context design' (De Lange, 2015). This means that the same context is used to

GLIDE RATIO

The glide ratio is used to compare the performances of gliders or hang gliders. The first hang glider had a glide ratio of 1:4 (one to four). The improved version had a glide ratio of 1:7.

4. Define what is meant by a glide ratio.



Otto Lilienthal made about 2500 flights. On one of his flights from the Rhinower Mountains near Berlin, he started from a height of 25 meters and flew a distance of 185 meters. On his next flight he changed his glider a bit, and then started at a height of 20 meters to reach 155 meters.

5. What were the glide ratios of Otto Lilienthal's two gliders? Was the second glider better than the first one?

Fig. 3.8 *Glider* problem from the booklet *Flying Through Maths* (De Lange, 1991, p. 7)

introduce students into various mathematical concepts. One of the concepts that was presented in this flying context was the glide ratio (Fig. 3.8).

The context of flying is used to encourage students to reason about covering distances when gliding from a certain height. By comparing different flights, students are expected to come up with some thinking about the glide ratio, i.e., the ratio between the distance covered and the starting height. This glide ratio plays a role in the context problems in the beginning of a chapter on slopes. In this way, the glide ratio is meaningful for students. They can use it for solving problems that they can experience as real problems. Later in the chapter this glide ratio is generalized to triangles and as a measure that can be used to calculate or compare slopes.

3.3.3 Contexts for Introducing and Developing Concepts

RME brought about a new perspective on the use of contexts in mathematics education. Contexts are not only considered as an area for application learned mathematics, but also have an important role in the introduction and development of mathematical concepts.

> De Lange: Applications is one thing. In the traditional textbooks, it was the end of the book. You started first with learning mathematics, and then you got the applications of mathematics. Through our theory developed at the Freudenthal Institute in the 70s, we changed that to developing concepts through context. So, you had to be very careful, because if the context is not very suited for the concept development, you are riding the wrong train on the wrong

track. But I think in general we can say for a lot of concepts we found very nice contexts to start with.

An example is the context of exponential growth that can support students in developing the logarithm-concept (De Lange, 1987). In this context, the growth of water plants, students first calculate the exponential increase of the area covered by these water plants with the growth factor and the number of growing weeks. At a certain moment, the question in the context is reflected. The question is no longer what the area is after a certain number of weeks, but how many weeks are needed to get an area that is 10 times as much? Students will experience that this is independent of the starting situation and can be estimated by repeating the growth factor (e.g., with a growth factor of 2 this is a bit more than 3 weeks). After these introductory tasks, ${}^{2}\log 10$ is defined as the time needed to get 10 times the area of water plants when the growth factor per week is 2. With this context and the concrete contextual language in mind, students can develop basic characteristics of logarithmic relations such as ${}^{2}\log 3 + 1 = {}^{2}\log 6$ as follows: with this 1 extra week, you get 2 times more than 3, which equals 6. Similarly, ${}^{2}\log 6 + {}^{2}\log 2$ has to equal ${}^{2}\log 12$, as ${}^{2}\log 6$ is the number of weeks to get 6 times as much, and ${}^{2}\log 2$ is the number of weeks to get 2 times as much. The time needed to first get 6 times as much followed by the time needed to get 2 times as much has to be equal to the time needed to get 12 times as much.

With such a context problem, a concept is not only explored, but also more or less formalized. Such a concrete foundation is important because it offers opportunities for students in the future to reconstruct the procedure and meaning of the abstract calculation procedures by themselves.

3.3.4 Relevant Mathematics Education

The aforementioned examples show the potential of contexts for learning mathematics and for making that learning process meaningful and relevant for students. This approach to mathematics education connects to the RME instructional theory in which the learning of mathematics is interpreted as extending your common sense reasoning about the world around you. Hence, De Lange emphasises in his reflection on educational design that designers need to find contexts by meeting the real world outside mathematics and experience the potential of contexts by going to real classrooms (De Lange, 2015). Observing authentic classroom activities is crucial. He stresses the importance of direct observations without using video in order to observe much more. In such a direct observation, one is able to look at the students' notes, one has the possibility to participate with their work, and one can ask questions in order to understand why students do what they do.

Exploiting the real world guided De lange towards a wide variety of original and surprising contexts. One example is the art of ballooning. Flying a balloon depends completely on the strength and direction of the wind and the change of the wind

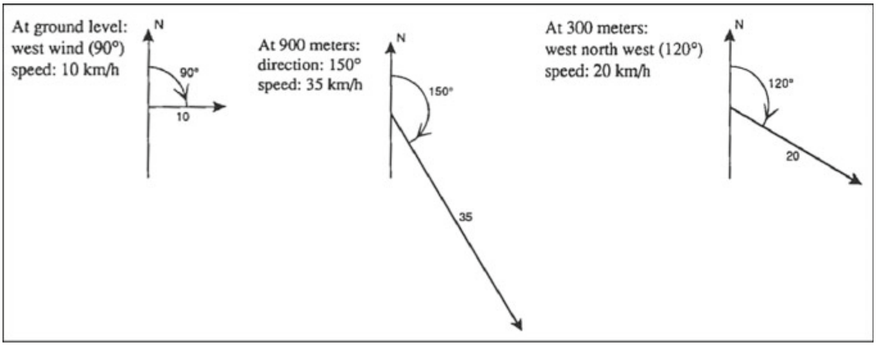


Fig. 3.9 Ballooning information from the booklet *Flying Through Maths* (De Lange, 1991, p. 34)

> In the Netherlands, one out of four high school students has used drugs (at least once). Design a study to find out how many students at your school have used drugs. Do you get a representative sample by using your school when you want to say something about the use of drugs at all schools in your city?

Fig. 3.10 *Drug use* problem (De Lange & Verhage, 1992, p. 12)

speed along different altitudes. The following example (Fig. 3.9) is also taken from the booklet *Flying Through Math* and is typical for a situation in which you are supposed to travel with a balloon from one spot to a target. The task for the students was to determine what happens when a balloon starts from Albuquerque and flies the first half hour at 300 m, then an hour at 900 m and finally, a half hour at 300 m.

In this context, not all information is available. Remaining questions are: How much time is needed to land? What happens when you go from one altitude to another and how much time does that take? The task becomes a real problem solving task for the students.

Through being in real classrooms De Lange could also look for contexts that trigger interest in students. In choosing these contexts, he was not afraid of using controversial situations. This can be recognized in a task for 15-year olds about drug use (see Fig. 3.10), for example.

In the interview, De Lange emphasised again that contexts serve many important roles in the teaching and learning of mathematics. They support conceptual development, can be motivating and raise interest, and also teach students how to apply mathematics.

> De Lange: We should be aware that contexts have to be mathematised. This means that we should be aware of what is the relevant mathematics in the contexts, which concepts plays an important role, and can the contexts serve as the starting point of modelling cycles. So, what you actually see is, that in the first phase of learning from context to concept, you use things, you do things, which are exactly the same as using the concept in a problem-solving activity.

In a certain country, the national defence budget is $30 million for 1980. The total budget for that year is $500 million. The following year the defence budget is $35 million, while the total budget is $605 million. Inflation during the period covered by the two budgets was 10 percent.

a. You are invited to give a lecture for a pacifist society. You intend to explain that the defence budget decreased over this period. Explain how you could do this.
b. You are invited to lecture to a military academy. You intend to explain that the defence budget increased over this period. Explain how you would do this.

Fig. 3.11 *Military-budget* problem (De Lange, 1987, p. 87)

The examples of problems discussed above all illustrate how contexts can be used for designing relevant and meaningful mathematics education. What can also be recognized is that in the 1980s designers like De Lange already anticipated what we now call 21st century skills. A nice example of this is the *Military-budget* problem (Fig. 3.11), which some thirty years ago was designed by De Lange to stimulate students to become mathematically creative and critical.

3.3.5 Conclusion

Creative designers like De Lange, people who are able to convince others of the limitations of many textbooks and who are able to translate general educational ideas into original and attractive resources for students, are of crucial importance for realising meaningful and relevant mathematics education. In 2011, he was awarded the *ISDDE Prize for Excellence in Design for Education*. Malcolm Swan wrote on behalf of the prize committee: “He has a flair for finding fresh, beautiful, original, contexts for students and shows humour in communicating them.” Without De Lange’s contributions many ideas in the Dutch didactic tradition would have been less well articulated, less well illustrated, and less influential in the world outside the Dutch context.