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BACHELOR THESIS

Inconsistencies of Fine's Prism Models

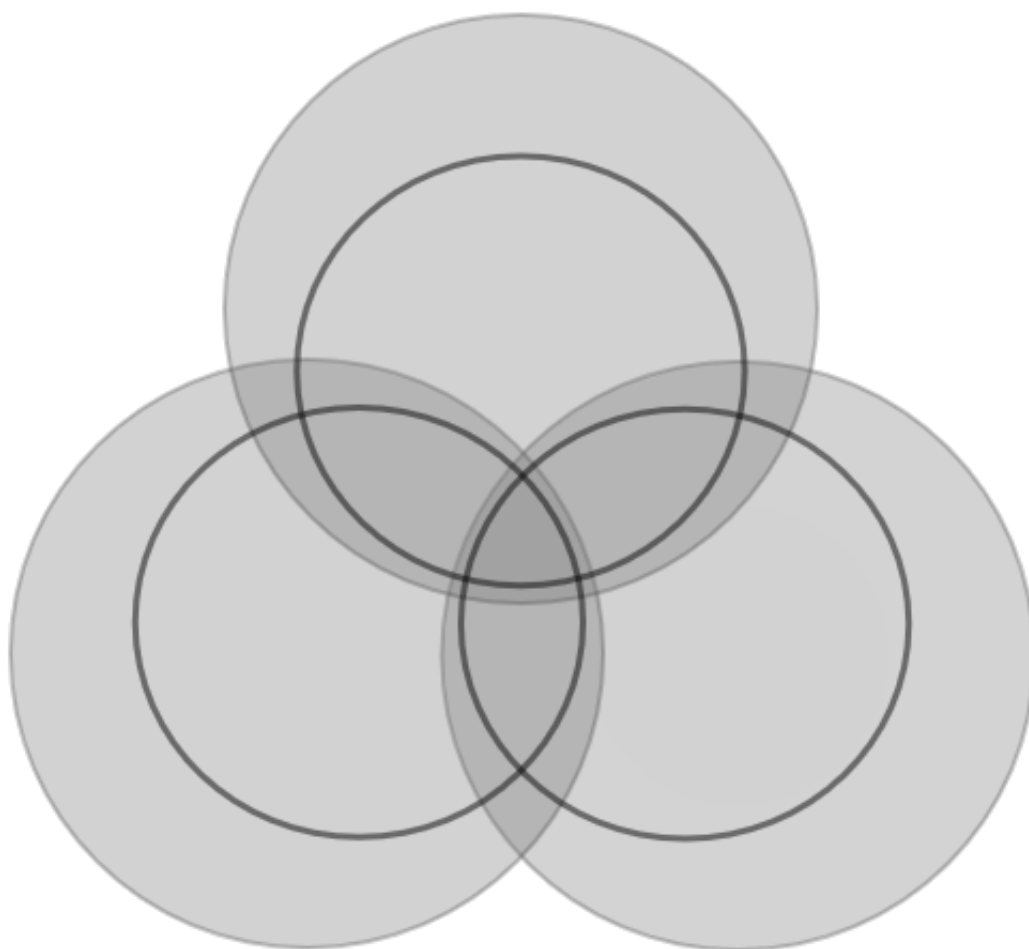
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Abstract

The concept of a prism model was introduced by Arthur Fine to be used for local and realistic descriptions of quantum statistics. Recently Feintzeig and Fletcher have shown that a wide class of generalizations of the classical probability model are incapable of replicating the quantum statistics for certain experiments. We employ a modification of their strategy to see under what conditions prism models cannot replicate quantum statistics. In doing so we differentiate two interpretations: one which we can show to be inconsistent in replicating the statistics of certain experiments, and one which commits to unavoidable rejection rates for certain experiments.



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1 Introduction

Whether there is more to know about a system such that more precise predictions can be made; admittedly, when confronted with an indeterministic theory of physical reality, this is a natural question to ask. And so it is that physicists and philosophers alike have pondered the existence of hidden variable theories since the development of quantum mechanics. In such theories one stipulates the existence of a hidden variable, a piece of knowledge crucial to determining the outcomes of experiments, so that, by not knowing it precisely, one is left with the probabilistic predictions of quantum mechanics, a guessing game. Simple as it sounds though, the unorthodox behaviour of quantum statistics makes it difficult to offer such theories without accomodating for the violation of “natural” principles, as has been shown by various so-called no-go theorems.

Most notable among these is Bell’s theorem and its generalizations (e.g.: Clauser et al. (1969), Bell (1971), Clauser and Horne (1974), Aspect (1983) and Mermin (1986)), which restrict the kinds of hidden variable theories for quantum mechanics that are consistent; to pass the Bell test a hidden variable theory must not be local¹. A particular version (Fine 1982) states that the statistics of a quantum mechanical experiment satisfy all Bell inequalities if and only if they may be given by a Kolmogorovian probability model satisfying weak constraints. A strong argument is made then, when, within the field of quantum mechanics, the violation of these inequalities are theoretically allowed and empirically realised, against the viability of such Kolmogorovian probability models in hidden variable theories.

Throughout the hystory of modern physics, there have been numerous suggestions of hidden variable theories that try to escape the implications of Bell’s theorem while still remaining local. In a recent paper by Benjamin H. Feintzeig and Samuel C. Fletcher (On Noncontextual, Non-Kolmogorovian Hidden Variable Theories; 12-08-2016) Feintzeig and Fletcher address a wide class of generalizations of the Kolmogorovian hidden variable model and manage to show their inconsistency, and with that debunk the viability of various of the proposed theories. It is with the success of their no-go theorem in mind, that we will address a type of hidden variable theory that has managed to slip past the scope of their no-go theorem and try to employ a somewhat altered version of their strategy as our own.

The elusive hidden variable theory in question was proposed by Arthur Fine (1981) and draws from the notion that the various derivations of the Bell inequalities invariably rely on background assumptions other than locality and realism. Hence, by manoeuvring in the degree of freedom provided by the withdrawal of these assumptions, Fine managed to construct probabilistic models that supposedly yield local and consistent hidden variable theories; he dubs them prism models.

¹The principle of locality states that objects cannot interact instantaneously over distance; their influence must be mediated through space.

2 Prism models

Bell's inequalities are usually derived by considering a quantum system of two entangled qubits². An example of such a system are two interacting particles exhibiting spin- $\frac{1}{2}$ that were subsequently separated in space. In quantum mechanics one can predict correlations between measurements of the individual particles' spins. According to Bell however, these correlations would satisfy the Bell inequalities if there was a local hidden variables theory that described the statistics of the experiment. As happens to be the case in quantum theory, violation of these inequalities is allowed, and in our example specifically, they are empirically shown to be violated; therefore a hidden variable theory cannot successfully explain the phenomenon.

Fine pointed out a flaw in the derivation of these equalities, which, according to him, if accounted for, may lead to a local hidden variable theory that is consistent. Suppose we were attempting to construct a hidden variable theory for our example and were to assign a hidden quality, a type, to our two particles; we cannot observe it directly, but the type of a particle determines the outcome of a measurement. So far we have described the usual framework of local hidden variables as discussed in the Bell literature. There is however, one assumption implicit in that literature that we have deliberately not included, namely that each particle type is suitable for every potential measurement within our experiment. In the absence of this assumption the derivation of the Bell inequalities fails and we are left with an unscathed model to support a hidden variable theory.

What does Fine mean then by an unmeasurable quality and why would it be a natural thing to suppose? Probabilities, realistically, should be grounded in physical properties; are the requisite properties not present, then there can be no question of probability (not even zero). It would be odd to try and assess the probability with which the viscosity of an electron is higher than a certain magnitude, and claiming the probability is simply zero is a fallacy, since an electron does not exhibit viscosity - the requisite physical properties are not present. Since we have our particle types code for certain physical properties, it would be a natural extension to let them code for those properties that allow measurement to be made as well.

It would appear that local probability models generating quantum correlation statistics require each observable to have an individual set of states for which a measurement of that observable is suitable; throughout the remainder of this paper we shall refer to this set as the *category* of an observable. With the addition of this property, we can now code all necessary physical properties for determining the outcome of a particular measurement in the particle type, or more generally, the hidden state governing phenomena. To exhibit this coding we shall now introduce the *response function*, whose significance lies in telling us exactly how a certain hidden state will respond to a measurement.

Consider, as we shall exclusively throughout the remainder of this paper, a binary observable O - a yes-no question for our system, and a hidden variable space \mathcal{X} such that for each $\lambda \in \mathcal{X}$ it is known precisely whether a measurement in this state will result in a yes-answer, a no-answer or whether the measurement fails to complete at all (what physically happens in this latter case, we shall address later). We are then set to define the response function of O , $R_O : \mathcal{X} \rightarrow \{1, 0, D\}$, in the following way:

$$R_O(\lambda) = \begin{cases} 1 & \text{if } \lambda \text{ yields a yes-measurement} \\ 0 & \text{if } \lambda \text{ yields a no-measurement} \\ D & \text{if } \lambda \text{ is unsuitable for measurement} \end{cases}$$

²A quantum-bit is a two-level quantum system, i.e.: a system described by a two-dimensional Hilbert space

Regardless of what actually happens when a hidden state is unsuitable for measurement (deficient), the relevant trait of this event is that a researcher would be unable to assess the measurement as positive, or negative. As such, if we interpret probability as the relative rate of occurrence over repeated trails, the researcher in question would count positive outcomes against negative outcomes and ignore deficient ones. We can express this mathematically through the conditional probability:

$$P(O) = P(R_O(\lambda) = 1 | R_O(\lambda) \neq D),$$

the probability that λ yields a positive measurement given that λ is measurable in the first place. Alternatively we can define the subsets of interest $A_O = \{\lambda \in \mathcal{X} : R_O(\lambda) = 1\}$ and $\sigma_O = \{\lambda \in \mathcal{X} : R_O(\lambda) \neq D\}$ that allow us to express the conditional probability as the quotient,

$$P(O) = \frac{P(\lambda \in A_O \cap \sigma_O)}{P(\lambda \in \sigma_O)} = \frac{P(\lambda \in A_O)}{P(\lambda \in \sigma_O)}$$

since, obviously, $A_O \subseteq \sigma_O$.

Knowing now the essential characteristics of a prism model, we are ready to compile these concepts in a more manageable mathematical model. For this we shall draw inspiration from the Kolmogorovian notion of a probability space wherein one considers a sample space \mathcal{X} , the collection of all states, and an event space Σ , a collection of measurable subsets of \mathcal{X} that signify events. To be sure we include all necessary features of an event space, like the meet or negation of events, we shall employ the notion of an algebra to guarantee them.

Definition 1. An *algebra* Σ for a nonempty set \mathcal{X} is a subset of the powerset $\mathcal{P}(\mathcal{X})$ such that

1. $\mathcal{X} \in \Sigma$
2. For all $A \in \Sigma : A^c = \mathcal{X} \setminus A \in \Sigma$
3. For all $A, B \in \Sigma : A \cup B \in \Sigma$

Note that $\emptyset \in \Sigma$, for all $A, B \in \Sigma, A \cap B = (A^c \cup B^c)^c \in \Sigma$ and consequently, for all $A, B \in \Sigma, A \setminus B = A \cap B^c \in \Sigma$.

Definition 2.³ A *classical (Kolmogorovian) probability space* is a triple (\mathcal{X}, Σ, u) , with Σ an algebra for the nonempty set \mathcal{X} and $u : \Sigma \rightarrow [0, 1]$, such that

1. $u(\mathcal{X}) = 1$
2. $u(\emptyset) = 0$
3. For all $A, B \in \Sigma : u(A \cap B) = 0 \implies u(A \cup B) = u(A) + u(B)$

³Usually the definition of a Kolmogorovian probability includes $A \cap B = \emptyset \implies u(A \cup B) = u(A) + u(B)$ as an axiom instead of 2. and 3. in our definition. These definitions are equivalent though, and the one given here shall be more convenient for our uses.

The function u is called the probability measure and signifies the probability of an event: in a hidden variable theory, a collection of hidden states that would cause a certain observation. In our prism model though, the probability of one observation is fixed by two events: finding a hidden state within those states that give a positive measurement and finding one within those states that will give a positive or negative measurement. In order to calculate the probability of an observation then, we require the probability measure of both events. Fortunately these events inhabit the same sample space \mathcal{X} and to suit the dualistic nature of one event in a prism model it will be enough to simply take our “event space” to be Σ^2 , the Cartesian product of an algebra of \mathcal{X} with itself.

Definition 3. A *prism model* is a triple $(\mathcal{X}, \Sigma^2, u)$, such that (\mathcal{X}, Σ, u) is a classical probability space.

Here u no longer signifies the probability of an event, but instead we find the probability of an event in Σ^2 by taking the quotient of the measure of its first argument and its second argument:

$$P(O) = \frac{u(A_O)}{u(\sigma(O))}, \text{ where } (A_O, \sigma(O)) \in \Sigma^2.$$

As of now, one might take issue with this definition since we allow it to include events that seem out of place. In compliance with the idea of a prism model, we should always have that if $(A_1, A_2) \in \Sigma^2$ then $A_1 \subseteq A_2$. In the following section this oddity shall be dealt with appropriately by equating the observables from our quantum statistics only with those events for which, indeed, the first argument is a subset of the second.

2.1 Prism model representations

Before we can employ a prism model to account for the statistics of a quantum mechanical experiment, we are first to give a mathematical expression of such an experiment. We present the “simple quantum mechanical experiment”, a term taken from Feintzeig and Fletcher.

Definition 4. A *simple quantum mechanical experiment* is a triple $(\mathcal{H}, \psi, \mathcal{O})$, where \mathcal{H} is a Hilbert space⁴, $\psi \in \mathcal{H}$ is a unit vector, and $\mathcal{O} = \{O_1, \dots, O_n\}$ is a finite set of projection operators on \mathcal{H} .

Here ψ represents the state of a system and the collection of projectors \mathcal{O} represents all possible measurements that can be made on ψ . Each operator in \mathcal{O} corresponds to a binary observable, a yes-no question, such that the probability of a “yes” measurement is given by $P(O_i) = \langle \psi | O_i \psi \rangle$. These measurements need not be compatible; in general the act of making one measurement may affect the outcome of another. This is the type of quantum mechanical we shall study exclusively throughout this paper and, therefore, it is important to lay down some basic terminology:

1. We shall call projectors $O_a, O_b \in \mathcal{O}$ *orthogonal* if

$$\langle \psi | O_a O_b \psi \rangle = \langle \psi | O_b O_a \psi \rangle = 0 \text{ for all unit vectors } \psi \in \mathcal{H}.$$

2. We shall say that a collection of n projectors $\mathcal{O}^n \subseteq \mathcal{O}$ *spans* \mathcal{H} if

$$\sum_{i=1}^n \langle \psi | \mathcal{O}_i^n \psi \rangle = 1 \text{ for all unit vectors } \psi \in \mathcal{H}.$$

⁴A hilbert space is a complete inner product space.

Furthermore, in the context of simple quantum mechanical experiments, we shall denote $P(O_i) = \langle \psi | O_i \psi \rangle$.

Theorem 1 (Fine 1982, Pitowsky 1989). The statistics of the outcomes of a simple quantum mechanical experiment satisfy all Bell inequalities if and only if the experiment has a Kolmogorovian probability space representation.

Since it is a well known fact that, for some quantum mechanical experiments, their statistics do not satisfy the Bell inequalities, we can pose as a corollary, that there exist simple quantum mechanical experiments with no classical probability space representation. In general, a hidden variable theory aims at providing a consistent representation for all quantum mechanical experiments. In this regard then, the viability of Kolmogorovian probability spaces in hidden variable theories is severely impaired.

To achieve our goal of creating a hidden variable representation, we have to equate all observables in our simple quantum mechanical experiment to the events of a prism model's event space. Additionally, while doing this we have to be careful to equate only to those events for which the first argument is a subset of the second.

Definition 5. Let $(\mathcal{H}, \psi, \mathcal{O})$ be a simple quantum mechanical experiment and $(\mathcal{X}, \Sigma^2, u)$ a prism model, then we call $E : \mathcal{O} \rightarrow \Sigma^2$ a *representation* of $(\mathcal{H}, \psi, \mathcal{O})$ on $(\mathcal{X}, \Sigma^2, u)$ if

$$P(O_i) = \frac{u(E_1(O_i))}{u(E_2(O_i))} \text{ and } E_1(O_i) \subseteq E_2(O_i) \text{ for all } O_i \in \mathcal{O}$$

And now, to conclude our search for a prism model representation quantum statistics, all there is left to do is to meld together the final product.

Definition 6. Let $(\mathcal{H}, \psi, \mathcal{O})$ be a simple quantum mechanical experiment, $(\mathcal{X}, \Sigma^2, u)$ a prism model and E a representation of $(\mathcal{H}, \psi, \mathcal{O})$ on $(\mathcal{X}, \Sigma^2, u)$, then we call $(\mathcal{X}, \Sigma^2, u, E)$ a *prism model representation* of $(\mathcal{H}, \psi, \mathcal{O})$.

Thus prism model representation for a simple quantum mechanical experiments have a pair of measurable sets corresponding to each individual projection operator in \mathcal{O} . As of yet this is a very basic notion of representation and some important features, like the inclusion of probabilities of joint measurements, shall be addressed in time.

3 Strategy

Let us, before we attempt to challenge the viability of prism model representations, take a quick look at the strategy used by Feintzeig and Fletcher in their paper. The engine behind Feintzeig and Fletcher's central argument is the Kochen-Specker theorem.

Theorem 2 (Kochen and Specker 1967). For any Hilbert space \mathcal{H} with $\dim(\mathcal{H}) \geq 3$ there is a finite collection of projection operators \mathcal{O} on it such that there is no function $f : \mathcal{O} \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every subset of \mathcal{O} whose elements are mutually orthogonal and span \mathcal{H} .

The intent of their argument is to show that there cannot be certain kinds of representations for simple quantum mechanical experiments with such a property. This is in essence a proof by existence: there exists a quantum mechanical experiment for which there are no such representations. To achieve this, we introduce the concept of a witness, whose existence is guaranteed by theorem 2 .

Definition 7. A *KS Witness* is a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O})$ such that $\dim(\mathcal{H}) \geq 3$ and there is no function $f : \mathcal{O} \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every subset of \mathcal{O} whose elements are mutually orthogonal and span \mathcal{H} .

We now have the necessary tools at our disposal to tackle various hidden variable theories, but, as for now, we shall demonstrate Feintzeig and Fletcher's strategy on a simple case: The Kolmogorovian hidden variable representation.

Definition 8. Let $(\mathcal{H}, \psi, \mathcal{O})$ be a simple quantum mechanical experiment and (\mathcal{X}, Σ, u) a Kolmogorovian probability space, then we call the quadruple $(\mathcal{X}, \Sigma, u, E)$, where $E : \mathcal{O} \rightarrow \Sigma$, a *Kolmogorovian hidden variable representation* if:

1. for all $O_i \in \mathcal{O} : P(O_i) = u(E(O_i))$
2. for all orthogonal $O_a, O_b \in \mathcal{O} : u(E(O_a) \cap E(O_b)) = 0$.

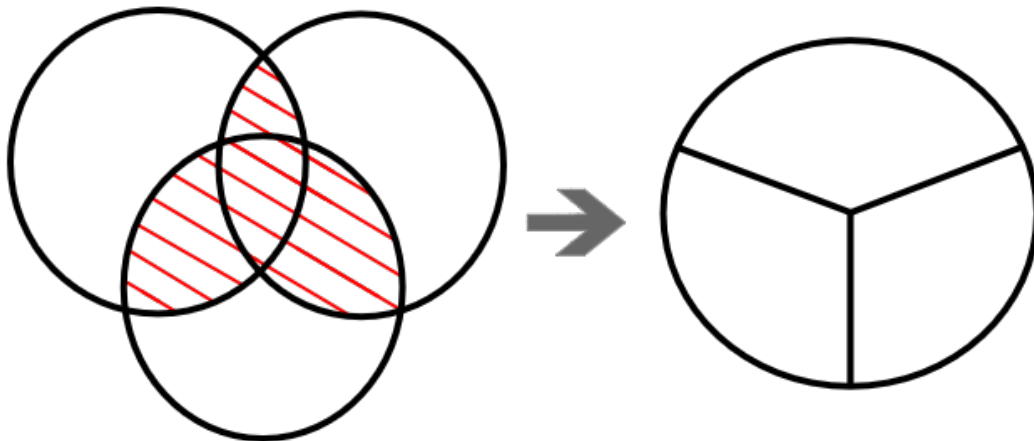
Note that, aside from the straightforward equating of observables to events (1), we have endowed the representation with an additional structure (2), namely that of subsequent measurement. In the statistics of simple quantum mechanical experiments the probability of subsequently measuring two positive outcomes for a pair of observables that correspond to orthogonal projection operators⁵ is zero. To account for this in our Kolmogorovian hidden variable representation, we set the probability measure for the meet of two events (their intersection) that represent such observables to zero. This additional structure is crucial to showing that no KS witness can have a Kolmogorovian hidden variable representation; for this procedure we will now give an intuitive rundown:

Consider a Kolmogorovian hidden variable representation of a simple quantum mechanical experiment with $\dim(\mathcal{H}) \geq 3$ and from that hidden variable representation a collection of events that represent mutually orthogonal operators $\{O_1, \dots, O_n\}$ that span \mathcal{H} . Condition 2 of a Kolmogorovian hidden variable representation allows us to ignore the intersections of these events and subtract them from our model without changing its behaviour, thereby allowing us to regard the events as disjoint. That in turn allows us to write:

$$u(\cup_{i=1}^n O_i) = \sum_{i=1}^n u(O_i) = \sum_{i=1}^n \langle \psi | O_i \psi \rangle = 1$$

⁵Mutually orthogonal projectors are compatible, so the order of their measurements does not matter.

So after subtracting all intersections we are left with a disjoint union of events whose probability measure is 1. In figure 1 this step is depicted for the 3 dimensional case in a Venn diagram.



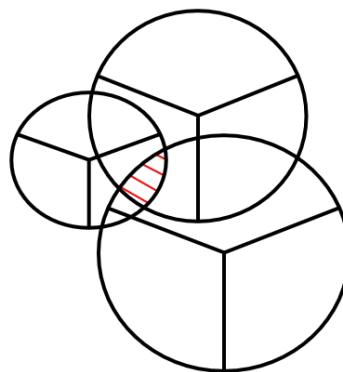
Figure“1: After subtracting the hatched null space from the left-hand side, we attain the disjoint union of sets on the right-hand side that has a probability measure of 1.

It is easy to check that $u(A \cup B) = u(A) + u(B) - u(A \cap B)$ holds for a Kolmogorovian probability space, and from that, to deduce that $u(A \cap B) = 1$ if $u(A) = u(B) = 1$, since $u(A \cup B)$ cannot be larger than 1. Then, if we were to repeat the above process of eliminating null spaces for all collections of operators that are mutually orthogonal and span \mathcal{H} and take the (finite) intersection of their unions, it would again have a probability measure of 1 and thus, more importantly, be nonempty. This step is depicted in figure 2 for the 3 dimensional case and three such collections of events in a Venn diagram.

This simple procedure has now shown us able to devise a function $f : \mathcal{O}_C \rightarrow \{0, 1\}$, namely, if we were to take any x in the nonempty aforementioned intersection, and consider its response function for each individual operator $O \in \mathcal{O}$:

$$f_x(O) = R_O(x) = \begin{cases} 1 & \text{if } x \in E(O) \\ 0 & \text{if } x \notin E(O) \end{cases}$$

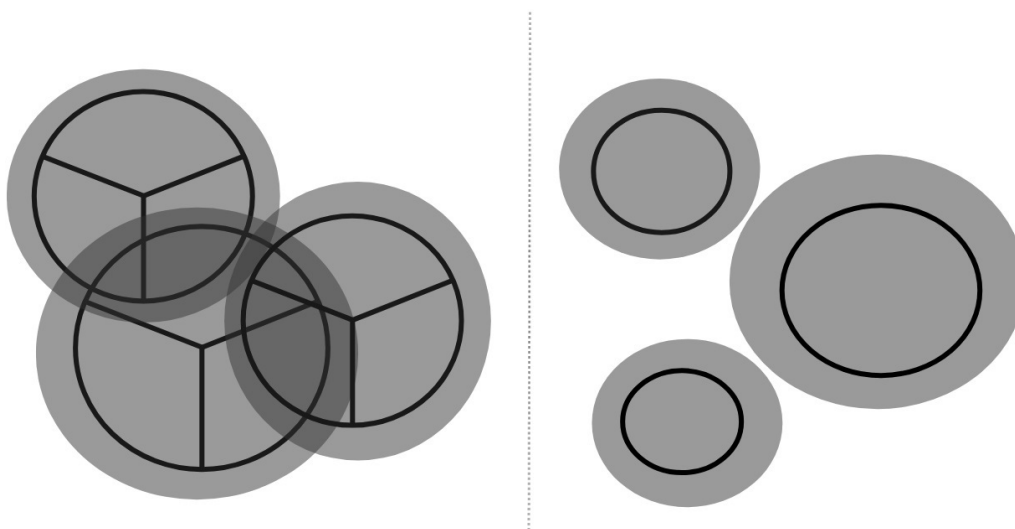
then f_x assigns 1 to exactly 1 element of every collection of operators that span \mathcal{H} and are mutually orthogonal. For indeed x lies in the intersection of the unions of all collections of events that represent such operators, but not, since we've excluded them, in the intersection of any two events that represent orthogonal operators. By following this procedure we can devise a function with these characteristics for any simple quantum mechanical experiment that is represented by a Kolmogoravian probability space and can conclude that no KS witness has a kolmogoravian hidden variable representation, thereby deeming it inconsistent.



Figure“2: The intersection of the unions of those events that represent mutually orthogonal operators that span \mathcal{H} is nonempty

Applying a same course of reasoning to prism model representations will bear complications, for division over categories allows events to give the desired probabilities for an experiment while being completely or partially isolated in the sample space. Unlike the Kolmogorovian model for example, the probability of an event is measured relative to their categories, so it is perfectly possible for some events to have probability 1 and still be disjoint from each other if their categories are disjoint.

Even if we were to suppose additional structure that would allow us to ignore intersections of events that represent orthogonal operators, we could simply avoid finding hidden variables that respond to each observable as desired, by grouping individual events, or perhaps⁶ unions of event-representations for those operators that span \mathcal{H} , in their own separate category and letting them exhibit their probabilistic properties relative to that category, bypassing any further interaction. In figure 3 these examples are depicted in Venn diagrams.



Figure“3: The grey shaded areas signify the categories of the sets they contain. In the left-hand side the union of disjoint sets are grouped in their respective categories and in the right-hand side individual sets are grouped within their own categories, potentially avoiding interaction.

This feat of freedom prism models provide is simultaneously their greatest strength, but also their greatest weakness. It allows us to account for non-classical statistics, but if we exploit it too much, like categorizing all observables separately, what are we then to make of successive observations - are they impossible? That is certainly not the case in nature, at least not for all observables. Is there then a minimum degree of moderation we should use - and what is it? To answer these questions we shall first explore notions of successive measurement, and see if with their help or under what circumstances, we can apply an extension of the above strategy to show the inconsistencies of prism models.

⁶We will see later that, as a corollary to theorem 4, this is actually not a viable way of avoiding inconsistency.

4 Successive measurements

It is desirable for a hidden variable theory to require some stability for its hidden variable; allowing the variable to be to capricious counteracts the intent of the theory - the potential for more accurate predictions. What use is knowing the current hidden state of a system if the act of measurement makes it reset? A weak constraint to assure some form of predictability is to require that a hidden variable remains stable throughout a series measurements represented by orthogonal operators; when conducting a measurement that is represented by an operator not orthogonal to the previous ones, the adjusting of the measuring apparatus may instead cause the hidden variable to change. In order to give a practical model, we shall adopt this constraint in our prism model representation.

Successive measurements is a delicate subject when introducing a concept like unmeasurability; it is here, where we are confronted unavoidably with the case of deficiency and forced to expand on its behaviour. To epitomize the general notion of successive measurement, we will introduce the main subjects of this section: P_{and} and P_{or} . Where the former is to be read as the probability we will measure positive for **all** observables in a set and the latter the probability we will measure positive for **at least one** observable in a set. Since we are dealing with quantum mechanics, where operators are not generally commutative, we should also accommodate for the order in which these measurements are held. Let us then introduce some notation:

Consider a collection of observables \mathcal{O} and an ordered subset $\mathcal{O}^n \subseteq \mathcal{O}$ of cardinality n , such that $\mathcal{O}_i^n \in \mathcal{O}^n$ signifies the i th element of \mathcal{O}^n . We write,

$P_{\text{or}}(\mathcal{O}^n) = P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n)$ the probability of measuring positively for at least 1 observable if measurements are conducted in the order $1, \dots, n$

and

$P_{\text{and}}(\mathcal{O}^n) = P_{\text{and}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n)$ the probability of measuring positively for all observables if measurements are conducted in the order $1, \dots, n$.

Furthermore, since these quantities are well defined for mutually orthogonal projectors in simple quantum mechanical experiments, we will immediately equate them to their correct expressions. Note that the order of measurement does not matter for mutually orthogonal projectors.

$$P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \sum_{i=1}^n \langle \psi | \mathcal{O}_i^n \psi \rangle$$

and

$$P_{\text{and}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = 0.$$

Where the elements of \mathcal{O}^n are mutually orthogonal, as we shall assume them to be for the remainder of this section. If, in addition to that, the elements of \mathcal{O}^n also span \mathcal{H} , we can write:

$$P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \sum_{i=1}^n \langle \psi | \mathcal{O}_i^n \psi \rangle = 1$$

To further ease future transcription when in the context of prism model representations let us also denote:

$$\sigma(\mathcal{O}^n) = \cap_{i=1}^n E_2(\mathcal{O}_i^n), \text{ the shared category for a set of observables}$$

We have seen in the previous section that the power of prism models comes from separating, to a certain degree at least, the categories of different observables. But, by doing this, we must expect to find hidden variables from outside the category of observables we are trying to measure. Since this scenario is bound to happen when taking successive measurements of observables whose categories are not the same or may not even intersect, it is important to develop a good understanding of the characteristics of such scenarios when attempting to provide prism model expressions for P_{and} and P_{or} . It is time to ask: what happens when we come across a hidden variable that responds deficiently to our measurement?

We shall provide two different interpretations for the concept of deficiency, a straightforward one, and one that shall attempt to relax the restrictions of the former. Naturally we will start with the straightforward one which we will dub a “hard ensemble interpretation” for its uncompromising nature. In this interpretation the case of deficiency when encountered will in some way be conveyed to the researcher by the measuring apparatus. Albeit by not responding in a certain amount of time, or by exploding or simply declaring “measurement failed” on a display, the important thing to take away is that we tried to measure, we failed, nothing left to do about it.

In the case of a single measurement this translates smoothly into the expression of probability we utilize in a prism model: a researcher takes multiple measurements and divides them over three piles; if a measurement is positive it goes to pile A, if it is negative it goes to pile B and if it is deficient it goes to pile C. If he were then to calculate the probability of measuring positive for an observable, he can only relate the amount of positive measurements (pile A) to those that were completed (pile A and B) and is to ignore those that were failed attempts at measuring (pile C).

Complications arise though, when he attempts to measure a series of observables and, by doing so, is bound to encounter and process partially deficient data. Suppose for example he is trying to measure an or-series of three observables, that is the binary observable that will be positive if at least one of the three observables has been measured positive, and, by doing this multiple times, he wants to find P_{or} of that series. What kind of measurements would he have to throw on pile C and write off as a failed attempt? A series of one positive, one negative and one deficient measurement, even though we have actually measured positively for one of the observables? This may seem unfair, but are we then to add a measurement of (negative, negative, deficient) to pile B? The answers to these questions are by no means evident; even ordinary logic is conflicted here. There may, perhaps, be a correct way of going about this, perhaps even multiple ones, our solution however is to accommodate for every choice of “pile distribution” that is to some degree sensible.

Retaining generality, we argue that any sensible procedure would have to include in pile A at least all those measurements for which at least one observable measured positive and the rest not deficient, and in pile B at least that measurement for which all observables measured negative. Expressing this in our prism model formalism, let \mathcal{O}^n be a set of observables and E their representation:

$$\begin{aligned} \text{Pile A} &= A_{\mathcal{O}^n} \cup (\sigma(\mathcal{O}^n) \cap \cup_{i=1}^n E_1(\mathcal{O}_i^n)) \\ \text{Pile B} &= B_{\mathcal{O}^n} \cup (\sigma(\mathcal{O}^n) \setminus \cup_{i=1}^n E_1(\mathcal{O}_i^n)) \\ \text{Pile A} + \text{Pile B} &= A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n} \cup \sigma(\mathcal{O}^n) \end{aligned}$$

Where $A_{\mathcal{O}^n}$ and $B_{\mathcal{O}^n}$, representing the freedom of the researcher to include other types of measurements, are disjoint from each other and from $\sigma(\mathcal{O}^n)$.

Similarly, for the “and” case, we will argue that pile A should at least include those measurements that have measured positive all throughout, and the pile B should include at least those that have measured successfully all throughout and for which at least one observable measured negative. Expressing this in our prism model formalism we write:

$$\begin{aligned} \text{Pile A} &= A_{\mathcal{O}^n} \cup \bigcap_{i=1}^n (E_1(\mathcal{O}_i^n)) \\ \text{Pile B} &= B_{\mathcal{O}^n} \cup (\sigma(\mathcal{O}^n) \setminus \bigcap_{i=1}^n E_1(\mathcal{O}_i^n)) \\ \text{Pile A} + \text{Pile B} &= A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n} \cup \sigma(\mathcal{O}^n) \end{aligned}$$

Where, again, $A_{\mathcal{O}^n}$ and $B_{\mathcal{O}^n}$ are disjoint from each other and from $\sigma(\mathcal{O}^n)$.

Given which hidden variables belong to which piles, calculating the probabilities P_{and} and P_{or} is straightforward:

$$P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \frac{\# \text{ Pile A}}{\# \text{ Pile A} + \# \text{ Pile B}} = \frac{u((\sigma(\mathcal{O}^n) \cap \bigcup_{i=1}^n E_1(\mathcal{O}_i^n)) \cup A_{\mathcal{O}^n})}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})}$$

and

$$P_{\text{and}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \frac{\# \text{ Pile A}}{\# \text{ Pile A} + \# \text{ Pile B}} = \frac{u(\bigcap_{i=1}^n E_1(\mathcal{O}_i^n) \cup A_{\mathcal{O}^n})}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})}$$

Before we add these structures to our prism model representation, let us make one last simplification:

$$0 = P_{\text{and}}(\mathcal{O}_a^n, \mathcal{O}_b^n) = \frac{u((E_1(\mathcal{O}_a^n) \cap E_1(\mathcal{O}_b^n)) \cup A_{\mathcal{O}^n})}{u((E_2(\mathcal{O}_a^n) \cap E_2(\mathcal{O}_b^n)) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})} \implies$$

$$u((E_1(\mathcal{O}_a^n) \cap E_1(\mathcal{O}_b^n)) \cup A_{\mathcal{O}^n}) = 0 \implies$$

$$u(E_1(\mathcal{O}_a^n) \cap E_1(\mathcal{O}_b^n)) + u(A_{\mathcal{O}^n}) = 0 \implies$$

$$u(E_1(\mathcal{O}_a^n) \cap E_1(\mathcal{O}_b^n)) = 0 \implies$$

$$P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \frac{u((\sigma(\mathcal{O}^n) \cap \bigcup_{i=1}^n E_1(\mathcal{O}_i^n)) \cup A_{\mathcal{O}^n})}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})} = \frac{u(A_{\mathcal{O}^n}) + \sum_{i=1}^n u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})}$$

Herewith we have completed the hard ensemble interpretation.

Definition 9. A *hard ensemble interpretation* of a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O})$ is a prism model representation of $(\mathcal{H}, \psi, \mathcal{O})$ with the additional structure:

$$P_{\text{or}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \frac{u(A_{\mathcal{O}^n}) + \sum_{i=1}^n u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})}$$

and

$$P_{\text{and}}(\mathcal{O}_1^n, \dots, \mathcal{O}_n^n) = \frac{u(\bigcap_{i=1}^n E_1(\mathcal{O}_i^n) \cup A_{\mathcal{O}^n})}{u(\sigma(\mathcal{O}^n) \cup A_{\mathcal{O}^n} \cup B_{\mathcal{O}^n})}$$

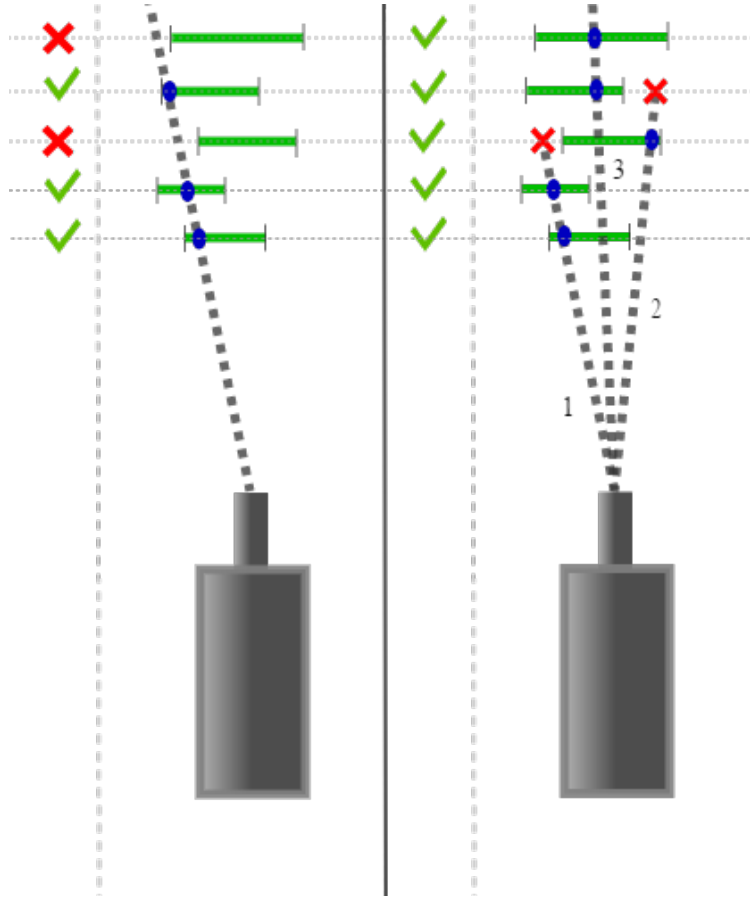
for all $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and $A_{\mathcal{O}^n}, B_{\mathcal{O}^n}$ disjoint from each other and from $\sigma(\mathcal{O}^n)$.

A foreseeable problem we may encounter when adopting the hard ensemble interpretation, is that the likelihood of failing a measurement may be unrealistically high if we try to model a very complex experiment or, perhaps, it may be impossible to measure two observables successfully in a row. Separating the categories of observables to some extent in order to escape no-go theorems, means the effective amount of hidden variables that are unmeasurable for some observables will increase by the same extent. This implicit tendency towards failure will remain no matter how much we refine our measuring apparatus - an unappealing lookout for physicists, but also a potential basis for empirical refutation. We are determined though, to fully explore the promises of prism models for describing quantum statistics, and therefore, since this tendency may prove troublesome for the viability of prism model representations in hidden variable theories, we shall introduce a second interpretation of deficiency that is not prone to inherent failure of measurements. In short, we will give our hidden variable the chance to adapt when it is caught deficient for some observable, and will dub this interpretation appropriately: a soft ensemble interpretation.

The idea behind the soft ensemble interpretation is that if we encounter a hidden state that is suitable for our measurement, it will remain fixed and determine the outcome as usual, but if we encounter one that is deficient, instead of remaining fixed we will assure an outcome by allowing the variable to reset - much like we allow a hidden variable to change if we switch the measuring apparatus for non-orthogonal observables. This way, our prism model will always provide for successful measurements (inefficiency of measuring apparatus may still cause failed measurements); in contrast to the hard ensemble interpretation we will put successful measureability over stability.

A smooth transition to our already established expression for probability is also present for this interpretation: if we were to take a measurement and the hidden variable is not suitable, we would not even notice and it will immediately reset. Therefore we will be sure to eventually find one among those states that are suitable and the probability of finding one that responds positive is the conditional probability of finding a positively responding state given that we found one that is suitable.

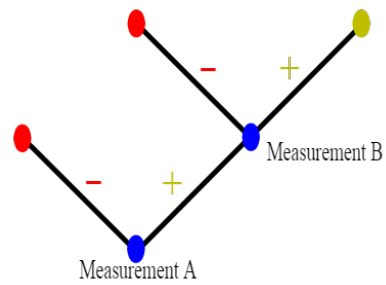
A depiction of the soft ensemble mechanisms in contrast to the hard ensemble interpretation is given in figure 4, there our system is depicted as a shooting device firing hidden variables at our measuring apparatus. In the hard ensemble interpretation this device is only capable of firing one shot at a time, but in the soft ensemble interpretation it behaves more like a machine gun, that, when missing a target, can immediately attempt a second shot until the target is hit without even knowing multiple shots have been fired.



Figure“4: A system providing hidden states for a series of measurements. The green areas signify the categories of each measurements. On the left-hand side the process is governed by the hard ensemble interpretation: only one state is provided and failed measurements are bound to occur. On the right-hand side the process is governed by the soft ensemble interpretation: as soon as a state is unsuitable for measurement a new one is provided immediately - measurements are never doomed to fail.

In order to retain the highest degree of generality we will only add a P_{and} structure to our representation, similar to the Kolmogorovian hidden variable representation of the previous section, and we will later see this is sufficient for the deduction of a potent theorem. Since the soft ensemble interpretation always provides for successful measurement, we don't have to worry about partial deficiency and can find an expression for P_{and} through well-established probability laws. We must ask ourselves: what is the chance of our first measurement being positive and then, subsequently, our second one also being positive? Formulating this mathematically: $P_{\text{and}}(O_a, O_b) = P(O_a)P(O_b|O_a)$. From there we can easily express this conditional probability by partitioning the event of finding a hidden state that responds positive to O_a into the cases where this state is either originally deficient for O_b (D) or it is not (M), then applying the law of total probability:

$$P(O_b|O_a) = P(O_b|D \wedge O_a)P(D|O_a) + P(O_b|M \wedge O_a)P(M|O_a).$$



Figure“5: Two consecutive measurements are made. P_{and} is the ratio of cases for which they are both positive (green dot).

And since $P(O_b|D \wedge O_a) = P(O_b)$ in the soft ensemble interpretation (the hidden variable resets when it is unmeasurable), we find:

$$P(O_b|O_a) = P(O_b)P(D|O_a) + P(O_b|M \wedge O_a)P(M|O_a) = \frac{u(E_1(O_b))u(E_1(O_a) \setminus E_2(O_b))}{u(E_2(O_b))u(E_1(O_a))} + \frac{u(E_1(O_b) \cap E_1(O_a))u(E_2(O_b) \cap E_1(O_a))}{u(E_2(O_b) \cap E_1(O_a))u(E_1(O_a))}.$$

And multiplying by the probability of the initial positive measurement:

$$P_{\text{and}}(O_a, O_b) = P(O_a)P(O_b|O_a) = \frac{E_1(O_a)}{E_2(O_a)} \left(\frac{u(E_1(O_b))u(E_1(O_a) \setminus E_2(O_b))}{u(E_2(O_b))u(E_1(O_a))} + \frac{u(E_1(O_b) \cap E_1(O_a))}{u(E_1(O_a))} \right) = \frac{u(E_1(O_b))u(E_1(O_a) \setminus E_2(O_b))}{u(E_2(O_a))u(E_2(O_b))} + \frac{u(E_1(O_b) \cap E_1(O_a))}{u(E_2(O_a))}.$$

With this result we are set to implement the P_{and} structure in our representation.

Definition 10. A *soft ensemble interpretation* of a simple quantum mechanical experiment $(\mathcal{H}, \psi, \mathcal{O})$ is a prism model representation of $(\mathcal{H}, \psi, \mathcal{O})$ with the additional structure:

$$P_{\text{and}}(O_a, O_b) = \frac{u(E_1(O_b))u(E_1(O_a) \setminus E_2(O_b))}{u(E_2(O_a))u(E_2(O_b))} + \frac{u(E_1(O_b) \cap E_1(O_a))}{u(E_2(O_a))}$$

for all $O_a, O_b \in \mathcal{O}$ with $O_a \perp O_b$.

We have introduced the soft ensemble interpretation as a more flexible alternative to the hard ensemble interpretation, but this flexibility only applies to the way the interpretation deals with deficiency. In no way have we posed that the soft ensemble interpretation is less prone to inconsistency when it comes down to representing quantum statistics; one may even say that, by demanding it to always deliver for successful measurements, we are putting stronger restrictions on it than on the hard ensemble interpretation.

5 Central theorems

The proposed prism model representations are not intended by themselves as definite models for hidden variable theories. In practice actual proposed hidden variable theories often add more structure than is demanded by our representations. Rather, they are intended as a broad framework encompassing many potential, more interesting hidden variable theories utilizing prism models. Therefore the theorems stated in this paper for the general representations bear on all such further specializations.

The first main result touches on the hard ensemble interpretation and offers a specific constraint concerning the dilemma of outweighing the separation of categories against the increasing likelihood of failing measurements. The enabler of this result shall be the previously introduced KS witness, the same type of experimental setup that has proven troublesome for many other types of probability representations.

Theorem 3. No KS witness $(\mathcal{H}, \psi, \mathcal{O})$ has a hard ensemble interpretation $(\mathcal{X}, \Sigma^2, u, E)$ satisfying $u(\sigma(\mathcal{O})) > 0$.

Proof:

Let $(\mathcal{H}, \psi, \mathcal{O})$ be a KS witness, $(\mathcal{X}, \Sigma^2, u, E)$ a hard ensemble interpretation of $(\mathcal{H}, \psi, \mathcal{O})$ and $u(\sigma(\mathcal{O})) > 0$, our goal is to derive a contradiction by constructing a function that assigns 1 to exactly one element of every subset of \mathcal{O} whose elements are mutually orthogonal and span \mathcal{H} .

Let us denote $N_{\mathcal{O}^n} = \sigma(\mathcal{O}^n) \setminus \cup_{i=1}^n E_1(\mathcal{O}_i^n)$ for $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and span \mathcal{H} , i.e. the set of hidden variables that would yield a “no” measurement for all observables in an exhaustive collection. Since (\mathcal{X}, Σ, u) is a classical probability space we can write:

$$u(\sigma(\mathcal{O}^n)) = u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) + u(\sigma(\mathcal{O}^n) \setminus \cup_{i=1}^n E_1(\mathcal{O}_i^n)).$$

And rewriting:

$$u(N_{\mathcal{O}^n}) = u(\sigma(\mathcal{O}^n)) - u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)).$$

Furthermore, by the hard ensemble behaviour of $(\mathcal{X}, \Sigma^2, u, E)$ we can write:

$$P_{\text{or}}(\mathcal{O}^n) = \frac{u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) + u(A_{\mathcal{O}^n})}{u(\sigma(\mathcal{O}^n)) + u(A_{\mathcal{O}^n}) + u(B_{\mathcal{O}^n})} = \sum_{i=1}^n \langle \psi | \mathcal{O}_i^n \psi \rangle = 1.$$

Then rewriting:

$$u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) + u(A_{\mathcal{O}^n}) = u(\sigma(\mathcal{O}^n)) + u(A_{\mathcal{O}^n}) + u(B_{\mathcal{O}^n})$$

and

$$u(B_{\mathcal{O}^n}) + u(\sigma(\mathcal{O}^n)) - u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) = 0.$$

Since both $u(B_{\mathcal{O}^n}) \geq 0$ and $u(\sigma(\mathcal{O}^n)) - u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) \geq 0$ we must have

$$u(N_{\mathcal{O}^n}) = u(\sigma(\mathcal{O}^n)) - u(\cup_{i=1}^n E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) = 0.$$

Let us denote $T_{\mathcal{O}^n} = \cup\{E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n) : \mathcal{O}_i^n, \mathcal{O}_j^n \in \mathcal{O}^n \text{ and } i \neq j\}$ for $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and span \mathcal{H} , i.e. the set of hidden variables that would yield a “yes” measurement for both observables of some nonequal pair of a collection of contradicting observables. Then, since $(\mathcal{X}, \Sigma^2, u, E)$ is a hard ensemble interpretation, we have for $\mathcal{O}_i^n, \mathcal{O}_j^n \in \mathcal{O}^n$ with $i \neq j$:

$$P_{\text{and}}(\mathcal{O}_i^n, \mathcal{O}_j^n) = \frac{u(E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n) \cup A'_{\mathcal{O}^n})}{u(E_2(\mathcal{O}_i^n) \cap E_2(\mathcal{O}_j^n) \cup A'_{\mathcal{O}^n} \cup B'_{\mathcal{O}^n})} = 0.$$

Then because $u(E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n) \cup A'_{\mathcal{O}^n}) = 0 \implies u(E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n)) = 0$ and consequently, $T_{\mathcal{O}^n} \leq \sum\{u(E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n)) : \mathcal{O}_i^n, \mathcal{O}_j^n \in \mathcal{O}^n \text{ and } i \neq j\} = 0$, we find

$$T_{\mathcal{O}^n} = 0.$$

We can now define the reduced shared category $\acute{\sigma}(\mathcal{O}^n) = \sigma(\mathcal{O}^n) \setminus (N_{\mathcal{O}^n} \cup T_{\mathcal{O}^n})$ and use it to define:

$B_{\mathcal{O}} = \cap\{\sigma(\mathcal{O}^n) : \mathcal{O}^n \subseteq \mathcal{O} \text{ whose elements are mutually orthogonal and span } \mathcal{H}\}$

and

$\acute{B}_{\mathcal{O}} = \cap\{\acute{\sigma}(\mathcal{O}^n) : \mathcal{O}^n \subseteq \mathcal{O} \text{ whose elements are mutually orthogonal and span } \mathcal{H}\}$.

We claim that $\acute{B}_{\mathcal{O}}$ is nonempty. We can write:

$$u(\acute{B}_{\mathcal{O}}) \geq u(B_{\mathcal{O}}) - u(\cup\{N_{\mathcal{O}^n} \cup T_{\mathcal{O}^n} : \mathcal{O}^n \subseteq \mathcal{O} \text{ whose elements are mutually orthogonal and span } \mathcal{H}\}) \geq u(B_{\mathcal{O}}) - \sum\{u(N_{\mathcal{O}^n}) + u(T_{\mathcal{O}^n}) : \mathcal{O}^n \subseteq \mathcal{O} \text{ whose elements are mutually orthogonal and span } \mathcal{H}\} = u(B_{\mathcal{O}}).$$

Indeed we see that the set $\acute{B}_{\mathcal{O}}$ is nonempty:

$$u(\sigma(\mathcal{O})) > 0 \implies u(B_{\mathcal{O}}) > 0 \implies u(\acute{B}_{\mathcal{O}}) > 0 \implies \acute{B}_{\mathcal{O}} \neq \emptyset.$$

We are now set the derive our contradiction:

$\acute{B}_{\mathcal{O}}$ is nonempty, so we can consider an element $x \in \acute{B}_{\mathcal{O}}$ and a corresponding function

$$\begin{aligned} f_x : \mathcal{O} &\rightarrow \{0, 1\} \\ O &\mapsto R_O(x). \end{aligned}$$

We claim f_x assigns 1 to exactly one element of every subset of \mathcal{O} whose elements are mutually orthogonal and span \mathcal{H} . Before we prove this let us note that f_x is defined correctly; $x \in \acute{B}_{\mathcal{O}} \implies x \in E_2(O)$ for all $O \in \mathcal{O}$ so indeed $R_O(x)$ will only either be equal to 1 or to 0.

Now let $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and span \mathcal{H} :

Suppose $f_x(O) = 0$ for all $O \in \mathcal{O}^n$. Then $x \in E_2(O) \setminus E_1(O)$ for all $O \in \mathcal{O}^n$ or equivalently $x \in N_{\mathcal{O}^n}$, this is a contradiction, so $f_x(O) = 1$ for at least one $O \in \mathcal{O}^n$.

Suppose $O_i, O_j \in \mathcal{O}^n, O_i \neq O_j$ and $f_x(O_i) = f_x(O_j) = 1$. Then $x \in E_1(O_i) \cap E_2(O_j)$ and $x \in T_{\mathcal{O}^n}$, this is a contradiction, so $f_x(O) = 1$ for at most 1 one $O \in \mathcal{O}^n$.

We see that f_x indeed has the desired characteristics; this is a contradiction and our proof is complete. □

Theorem 3 shows that there are certain types of experiments that cannot be modeled by a hard ensemble interpretation in which there is a nonzero probability of finding a hidden state that is suitable for all measurements. In itself this is not a strict no-go result, but it implies severe limitations on physical observation that may make a compelling case against the viability of the hard ensemble interpretation.

For the soft ensemble interpretation we promise an even more potent result: in the case of a special KS witness we can deduce an absolute contradiction, thereby rendering the soft ensemble interpretation inconsistent. Before we state this theorem however, we have to develop some further notions; note, for example, that we have made our promise exclusively for the case of a special KS witness. We require namely, for the deduction of our theorem, that no observable in our experiment has probability zero - a quality we will dub saturation.

Definition 11. We call a KS witness $(\mathcal{H}, \psi, \mathcal{O})$ *saturated* if $\langle \psi | O \psi \rangle \neq 0$ for all $O \in \mathcal{O}$.

Let there be no question whether such an KS witness exists; since we are dealing with a finite amount of projection operators, we should always be able to choose a unit vector that is not perpendicular to any of them. Further proof though, is needed for our next theorem, in which we will state a crucial property for any KS witness. Namely that, at its core, the sets of mutually orthogonal operators that span \mathcal{H} are sequentially interconnected.

Theorem 4. Every KS witness has a nonempty subcollection $\mathcal{C} \subseteq \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$ such that, if we denote $\mathcal{O}_c = \{O \in \mathcal{O} : \text{there exists } \mathcal{O}^n \in \mathcal{C} \text{ such that } O \in \mathcal{O}^n\}$:

1. There is no function $f : \mathcal{O}_c \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every element of \mathcal{C} .
2. For all $\mathcal{O}^n, \hat{\mathcal{O}}^n \in \mathcal{C}$ there exists a sequence $\{\mathcal{O}^{n,i}\}_{i \in \{1, \dots, k\}} \subseteq \mathcal{C}$ for which $\mathcal{O}^n = \mathcal{O}^{n,1}, \hat{\mathcal{O}}^n = \mathcal{O}^{n,k}$ and for all $i \in \{1, \dots, k-1\}$ there exists a connecting operator O_c such that $O_c \in \mathcal{O}^{n,i}$ and $O_c \in \mathcal{O}^{n,i+1}$.

Proof:

Let $(\mathcal{H}, \psi, \mathcal{O})$ be a KS witness, denote $\mathcal{O}_A = \{O \in \mathcal{O} : \text{there exists } \mathcal{O}^n \in A \text{ such that } O \in \mathcal{O}^n\}$ for $A \subset \mathcal{P}(\mathcal{O})$, subsets of the power set of \mathcal{O} , and suppose there is no nonempty subcollection $\mathcal{C} \subseteq \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$ for which:

1. There is no function $f : \mathcal{O}_c \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every element of \mathcal{C} .
2. For all $\mathcal{O}^n, \hat{\mathcal{O}}^n \in \mathcal{C}$ there exists a sequence $\{\mathcal{O}^{n,i}\}_{i \in \{1, \dots, k\}} \subseteq \mathcal{C}$ for which $\mathcal{O}^n = \mathcal{O}^{n,1}, \hat{\mathcal{O}}^n = \mathcal{O}^{n,k}$ and for all $i \in \{1, \dots, k-1\}$ there exists a connecting operator O_c such that $O_c \in \mathcal{O}^{n,i}$ and $O_c \in \mathcal{O}^{n,i+1}$.

Define the following equivalence relation on $\{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$:

$\mathcal{O}^n \sim \mathcal{O}^k$ if there exists a sequence $\{\mathcal{O}^{n,i}\}_{i \in \{1, \dots, k\}} \subseteq \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$ for which $\mathcal{O}^n = \mathcal{O}^{n,1}$, $\mathcal{O}^k = \mathcal{O}^{n,k}$ and for all $i \in \{1, \dots, k-1\}$ there exists a connecting operator O_c such that $O_c \in \mathcal{O}^{n,i}$ and $O_c \in \mathcal{O}^{n,i+1}$.

Reflexivity, transitivity and symmetry can easily be proven by taking the sequence of two times the same operator, joining two sequences and reversing a sequence respectively. Therefore then we can look at the partition \mathcal{P} of $\{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$ by \sim into its equivalence classes. We have then:

$$A, B \in \mathcal{P} \text{ and } A \neq B \implies A \cap B = \emptyset,$$

$$\cup \mathcal{P} = \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\},$$

$$A \in \mathcal{P} \implies \text{For all } a, b \in A : a \sim b,$$

$$A \in \mathcal{P} \implies \text{For all } a \in A : \text{if } a \sim b \text{ then } b \in A.$$

By assumption then, for all $A \in \mathcal{P}$ there exists a function $f_A : \mathcal{O}_A \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every element of A . Furthermore we can see that if $A, B \in \mathcal{P}$ and $A \neq B$ we have $\mathcal{O}_A \cap \mathcal{O}_B = \emptyset$, since if it were not the case, there would exist elements $a \in A$ and $b \in B$ for which $a \sim b$ and thus $b \in A$ and $b \in B$, which contradicts $A \cap B = \emptyset$. We can now define a function $f : \mathcal{O} \rightarrow \{0, 1\}$:

$$f(O) = \begin{cases} f_A(O) & \text{if } O \in \mathcal{O}_A \text{ for some } A \in \mathcal{P} \\ 0 & \text{otherwise} \end{cases}$$

We claim that f is well defined; indeed we have shown that for all $A, B \in \mathcal{P}$ and $A \neq B$ we have $\mathcal{O}_A \cap \mathcal{O}_B = \emptyset$, so, for all $O \in \mathcal{O}$, $f(O)$ is either equal to 0 or to $f_A(O)$ for exactly one $A \in \mathcal{P}$. And since $f(O) = f_A(O)$ for all $O \in \mathcal{O}_A$, f will assign 1 to exactly one element of every element of A , for all $A \in \mathcal{P}$. And since $\cup \mathcal{P} = \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$, f will assign 1 to exactly one element of every element of $\{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$. This contradicts our choice of a KS witness and so our assumption must be false and our proof is complete. □

With the addition of theorem 4 we are set to state the second main result of this paper, in which we will reveal the unconditional inconsistency of the soft ensemble interpretation.

Theorem 5. No saturated KS witness $(\mathcal{H}, \psi, \mathcal{O})$ has a soft ensemble interpretation $(\mathcal{X}, \Sigma^2, u, E)$.

Proof:

For the sake of deriving a contradiction let $(\mathcal{H}, \psi, \mathcal{O})$ be a saturated KS witness and $(\mathcal{X}, \Sigma^2, u, E)$ a soft ensemble interpretation of $(\mathcal{H}, \psi, \mathcal{O})$. Consider two orthogonal operators $O_a, O_b \in \mathcal{O}$, then, since $P_{\text{and}}(O_a, O_b) = 0$ and $u(E_1(O_b)) > 0$, we can deduce two important properties for our model:

$$\begin{aligned} & \frac{u(E_1(\mathcal{O}_b^n))u(E_1(\mathcal{O}_a^n) \setminus E_2(\mathcal{O}_b^n))}{u(E_2(\mathcal{O}_a^n))u(E_2(\mathcal{O}_b^n))} + \frac{u(E_1(\mathcal{O}_b^n) \cap E_1(\mathcal{O}_a^n))}{u(E_2(\mathcal{O}_a^n))} = 0 \\ \implies & \\ & u(E_1(\mathcal{O}_a^n) \setminus E_2(\mathcal{O}_b^n)) = 0 \end{aligned} \tag{1}$$

and

$$u(E_1(\mathcal{O}_b^n) \cap E_1(\mathcal{O}_a^n)) = 0 \tag{2}$$

Let us again denote $T_{\mathcal{O}^n} = \cup\{E_1(\mathcal{O}_i^n) \cap E_1(\mathcal{O}_j^n) : \mathcal{O}_i^n, \mathcal{O}_j^n \in \mathcal{O}^n \text{ and } i \neq j\}$ and $N_{\mathcal{O}^n} = \sigma(\mathcal{O}^n) \setminus \cup_{i=1}^n E_1(\mathcal{O}_i^n)$ for $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and span \mathcal{H} . Property (2) makes it evident that $u(T_{\mathcal{O}^n}) = 0$ always holds and we will now show the same for $N_{\mathcal{O}^n}$:

Consider $\mathcal{O}^n \subseteq \mathcal{O}$ whose elements are mutually orthogonal and span \mathcal{H} , then using property (1) we can write:

$$u(E_1(\mathcal{O}_i^n) \setminus \sigma(\mathcal{O}^n)) = u(\cup_{j=1}^n E_1(\mathcal{O}_i^n) \setminus E_2(\mathcal{O}_j^n)) \leq \sum_{j=1}^n u(E_1(\mathcal{O}_i^n) \setminus E_2(\mathcal{O}_j^n)) = 0.$$

And furthermore,

$$\begin{aligned} \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} &= \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n)) - u(E_1(\mathcal{O}_i^n) \setminus \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} = \\ \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n))}{u(\sigma(\mathcal{O}^n))} &\geq \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n))}{u(E_2(\mathcal{O}_i^n))} = 1. \end{aligned}$$

so,

$$\sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} \geq 1. \tag{3}$$

On the other hand we have for $a \neq b$:

$$E_1(\mathcal{O}_a^n \cap \mathcal{O}_b^n \cap \sigma(\mathcal{O}^n)) \leq u(E_1(\mathcal{O}_a^n \cap \mathcal{O}_a^n)) = 0.$$

Then by axiom 3 of a Kolmogorovian probability space:

$$u(\cup_{i=1}^n (E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))) = \sum_{i=1}^n u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))$$

Furthermore, since $\cup_{i=1}^n (E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)) \subseteq \sigma(\mathcal{O}^n)$, we can write:

$$\begin{aligned}
& \frac{u(\cup_{i=1}^n (E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n)))}{\sigma(\mathcal{O}^n)} \leq 1 \\
& \implies \\
& \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} \leq 1
\end{aligned} \tag{4}$$

Combining the results (3) and (4) leaves us to conclude:

$$\begin{aligned}
1 & \geq \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} \geq 1 \implies \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))}{u(\sigma(\mathcal{O}^n))} = 1 \implies \\
& u(\sigma(\mathcal{O}^n)) = \sum_{i=1}^n u(E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))
\end{aligned}$$

From this we can now see that $u(N_{\mathcal{O}^n}) = 0$:

$$u(\sigma(\mathcal{O}^n)) = u(N_{\mathcal{O}^n}) + \sum_{i=1}^n u((E_1(\mathcal{O}_i^n) \cap \sigma(\mathcal{O}^n))) = u(N_{\mathcal{O}^n}) + u(\sigma(\mathcal{O}^n)) \implies u(N_{\mathcal{O}^n}) = 0.$$

but, moreover, since $u(\sigma(\mathcal{O}^n)) \leq u(E_2(\mathcal{O}_i^n))$ for all $i \in \{1, \dots, n\}$,

$$\begin{aligned}
\sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n))}{u(\sigma(\mathcal{O}^n))} &= \sum_{i=1}^n \frac{u(E_1(\mathcal{O}_i^n))}{u(E_2(\mathcal{O}_i^n))} \implies \\
u(\sigma(\mathcal{O}^n)) &= u(E_2(\mathcal{O}_i^n)) \text{ for all } i \in \{1, \dots, n\}
\end{aligned} \tag{5}$$

We will apply this result in our proof that $\dot{B}_{\mathcal{C}}$ is nonempty if we make the following definitions:

Let $\mathcal{C} \subseteq \{\mathcal{O}^n \subseteq \mathcal{O} : \text{whose elements are mutually orthogonal and span } \mathcal{H}\}$ be a nonempty subcollection such that, if we denote $\mathcal{O}_{\mathcal{C}} = \{O \in \mathcal{O} : \text{there exists } \mathcal{O}^n \in \mathcal{C} \text{ such that } O \in \mathcal{O}^n\}$:

1. There is no function $f_x : \mathcal{O}_{\mathcal{C}} \rightarrow \{0, 1\}$ that assigns 1 to exactly one element of every element of \mathcal{C} .
2. For all $\mathcal{O}^n, \hat{\mathcal{O}}^n \in \mathcal{C}$ there exists a sequence $\{\mathcal{O}^{n,i}\}_{i \in \{1, \dots, k\}} \subseteq \mathcal{C}$ for which $\mathcal{O}^n = \mathcal{O}^{n,1}, \hat{\mathcal{O}}^n = \mathcal{O}^{n,k}$ and for all $i \in \{1, \dots, k-1\}$ there exists a connecting operator O_c such that $O_c \in \mathcal{O}^{n,i}$ and $O_c \in \mathcal{O}^{n,i+1}$.

The existence of \mathcal{C} is guaranteed by theorem 4. Then once again we will define the reduced shared category $\acute{\sigma}(\mathcal{O}^n) = \sigma(\mathcal{O}^n) \setminus (N_{\mathcal{O}^n} \cup T_{\mathcal{O}^n})$ and use it to define:

$$B_{\mathcal{C}} = \cap \{\sigma(\mathcal{O}^n) : \mathcal{O}^n \in \mathcal{C}\} \text{ and } \dot{B}_{\mathcal{C}} = \cap \{\acute{\sigma}(\mathcal{O}^n) : \mathcal{O}^n \in \mathcal{C}\}$$

We claim that $\dot{B}_{\mathcal{C}}$ is nonempty. We can write:

$$u(\dot{B}_{\mathcal{C}}) \geq u(B_{\mathcal{C}}) - u(\cup \{N_{\mathcal{O}^n} \cup T_{\mathcal{O}^n} : \mathcal{O}^n \in \mathcal{C}\}) \geq u(B_{\mathcal{C}}) - \sum \{u(N_{\mathcal{O}^n}) + u(T_{\mathcal{O}^n}) : \mathcal{O}^n \in \mathcal{C}\} = u(B_{\mathcal{C}}).$$

We will show that $u(B_{\mathcal{C}}) > 0$ by an inductive argument:

Note that, since \mathcal{O} is finite, \mathcal{C} is finite and we can construct an all encompassing sequence $\{\mathcal{O}^{n,i}\}_{i \in \{1, \dots, k\}} \subseteq \mathcal{C}$ by ordering \mathcal{C} and joining the connecting sequences of all consecutive elements. Then $\sigma(\cup_{i \in \{1, \dots, k\}} \mathcal{O}^{n,i}) = B_{\mathcal{C}}$, and for all $i \in \{1, \dots, k-1\}$ there exists a connecting operator O_c such that $O_c \in \mathcal{O}^{n,i}$ and $O_c \in \mathcal{O}^{n,i+1}$. To found our argument let us note that

$$u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j})) = u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \cap \sigma(\mathcal{O}^{n,i+1})) \text{ for all } i \in \{1, \dots, k-1\}.$$

To see this let $i \in \{1, \dots, k-1\}$ and $O_c \in \mathcal{O}^{n,i}, O_c \in \mathcal{O}^{n,i+1}$ the connecting operator. Then $\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \cup \sigma(\mathcal{O}^{n,i+1}) \subseteq E_2(O_c)$ and we can write:

$$u(E_2(O_c)) \geq u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \cup \sigma(\mathcal{O}^{n,i+1})) = u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \setminus \sigma(\mathcal{O}^{n,i+1})) + u(\sigma(\mathcal{O}^{n,i+1})) = u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \setminus \sigma(\mathcal{O}^{n,i+1})) + u(E_2(O_c)).$$

Since $u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \setminus \sigma(\mathcal{O}^{n,i+1})) \geq 0$ we can conclude $u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \setminus \sigma(\mathcal{O}^{n,i+1})) = 0$ and we find as desired:

$$u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j})) = u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \cap \sigma(\mathcal{O}^{n,i+1})) + u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \setminus \sigma(\mathcal{O}^{n,i+1})) = u(\cap_{j=1}^i \sigma(\mathcal{O}^{n,j}) \cap \sigma(\mathcal{O}^{n,i+1})).$$

We can now conclude, by principle of induction, that $u(\sigma(\mathcal{O}^{n,1})) = u(\cap_{i=1}^k \sigma(\mathcal{O}^{n,i})) = u(\sigma(\cup_{i \in \{1, \dots, k\}} \mathcal{O}^{n,i})) = u(B_{\mathcal{O}})$ and, since from result (5) together with saturation $u(\sigma(\mathcal{O}^{n,1})) = u(E_2(O)) \geq u(E_1(O)) > 0$ for any $O \in \mathcal{O}^{n,1}$, we find $u(B_{\mathcal{O}}) > 0$.

Indeed we see that the set $\dot{B}_{\mathcal{C}}$ is nonempty:
 $u(B_{\mathcal{C}}) > 0 \implies u(\dot{B}_{\mathcal{C}}) > 0 \implies \dot{B}_{\mathcal{C}} \neq \emptyset$.

We are now set the derive our contradiction:

$\dot{B}_{\mathcal{C}}$ is nonempty, so we can consider an element $x \in \dot{B}_{\mathcal{C}}$ and a corresponding function

$$\begin{aligned} f_x : \mathcal{O}_{\mathcal{C}} &\rightarrow \{0, 1\} \\ O &\mapsto R_O(x). \end{aligned}$$

We claim f_x assigns 1 to exactly one element of every element of \mathcal{C} . Before we prove this let us note that f_x is defined correctly; $x \in \dot{B}_{\mathcal{C}} \implies x \in E_2(O)$ for all $O \in \mathcal{C}$ so indeed $R_O(x)$ will only either be equal to 1 or to 0.

Now let $\mathcal{O}^n \in \mathcal{C}$:

Suppose $f_x(O) = 0$ for all $O \in \mathcal{O}^n$. Then $x \in E_2(O) \setminus E_1(O)$ for all $O \in \mathcal{O}^n$ or equivalently $x \in N_{\mathcal{O}^n}$, this is a contradiction, so $f_x(O) = 1$ for at least one $O \in \mathcal{O}^n$.

Suppose $O_i, O_j \in \mathcal{O}^n, O_i \neq O_j$ and $f_x(O_i) = f_x(O_j) = 1$. Then $x \in E_1(O_i) \cap E_2(O_j)$ and $x \in T_{\mathcal{O}^n}$, this is a contradiction, so $f_x(O) = 1$ for at most 1 one $O \in \mathcal{O}^n$.

We see that f_x indeed has the desired characteristics; this is a contradicts property (1) of \mathcal{C} and our proof is complete.

□

Theorem 5 can be interpreted as a strict no-go result for the soft ensemble interpretation, for indeed in contrast to the hard ensemble interpretation, there exist quantum mechanical experiments that can, under no circumstances, be described by it.

6 Conclusion and further directions

Fine's prism models have the advantage of avoiding Bell's theorem, but to further test their viability in hidden variable theories, we have formalized Fine's concept of prism models in a concise mathematical expression, much like a Kolmogorovian probability space, but with additional degrees of freedom. To test whether the inclusion of deficiency allows prism models to represent quantum statistics, we have subjected them to a strategy for deriving a no-go result inspired by Feintzeig and Fletcher utilizing the Kochen and Specker theorem.

We have done so for two different interpretations of the prism model representations concerning the case of encountering deficient hidden states - the hard ensemble interpretation and the soft ensemble interpretation. For both we were able to modify Feintzeig and Fletcher's strategy and have derived potent results in the form of theorem 3 and theorem 5.

Theorem 5 can be interpreted as a strict no-go result for the soft ensemble interpretation; there exists a quantum mechanical experiment that cannot be represented by a soft ensemble interpretation. As it turned out, in our attempt to relax the hard ensemble interpretations uncompromising tendency towards failing measurements, we have constricted the soft ensemble interpretation too much for it to accomodate quantum statistics, by demanding suitable hidden states for all measurements.

Theorem 3 is somewhat more subtle and only delivers a partial no-go result; a contradiction can only be derived under a certain condition, rendering a specific group of hard ensemble interpretations unaffected. We claim that this group however, must exhibit such properties that severely limit the possibility of successful measurements in certain experiments, and may even, by exceeding these limits empirically, be shown inconsistent yet.

Rejection rates for the hard ensemble interpretation.

We have already raised the potential issue of a hard ensemble interpretation bringing about implicit failure of measurements, but in order to make tangible claims about this issue, we shall first further investigate how the result of Theorem 3 translates to observable rejection properties of KS witnesses. Theorem 3 rules out the existence of hard ensemble interpretations for KS witnesses for which the shared category $\sigma(\mathcal{O})$ is greater than zero, but what about those for which it is equal to zero? While they may technically exist, they do imply minimal rejection rates that could be violated empirically; let us formulate such an implication.

In addition to the shared category we will introduce the total category $\mathcal{C} = \cup_{i \in I} (E_2(O_i))$ and immediately note that by theorem 3: $u(\mathcal{C}) = u(\mathcal{C} \setminus \sigma(\mathcal{O})) + u(\sigma(\mathcal{O})) = u(\mathcal{C} \setminus \sigma(\mathcal{O}))$. Then for every hidden variable $\lambda \in u(\mathcal{C} \setminus \sigma(\mathcal{O}))$, there should be an operator $O \in \mathcal{O}$ such that $\lambda \in \mathcal{C} \setminus E_2(O)$, since otherwise λ would be part of the shared category. If we then take $I = \{1, \dots, n\}$ to be the indexing set of all operators $O_i \in \mathcal{O}$, we can write:

$\mathcal{C} \setminus \sigma(\mathcal{O}) = \cup_{i \in I} (\mathcal{C} \setminus E_2(O_i))$ - those hidden states in \mathcal{C} that are deficient for at least one $O_i \in \mathcal{O}$.

And from there $u(\mathcal{C}) = u(\mathcal{C} \setminus \sigma(\mathcal{O})) = u(\cup_{i \in I} (\mathcal{C} \setminus E_2(O_i))) \leq \sum_{i \in I} u(\mathcal{C} \setminus O_i)$ gives us,

$$u(\mathcal{C}) \leq \sum_{i \in I} u(\mathcal{C} \setminus O_i).$$

Consider $u(\mathcal{C}) = 1$, in which case the probability of a succesfull measurement should be the highest for all observables, since their is no chance of finding a hidden state for which all observables are deficient. A sum of n positive terms adding up to at least 1 must have an average value of at least $\frac{1}{n}$ for its terms. So on average the value $u(\mathcal{C} \setminus E_2(O_i))$, the probability of failing a measurement of O_i , is larger or equal to $\frac{1}{n}$.

We are justified to claim that if the statistics of a KS witness could be described by a hard ensemble interpretation, we would find an average rejection rate over all observables in the experimental setup of at least $\frac{1}{n}$, with n the total number of observables, **regardless of the efficiency of the measuring apparatus**. This claim can be tested empirically by measuring an ensemble of KS witnesses for each of their observables; if a lower rejection rate is found, the experiment cannot be modeled by the hard ensemble interpretation.

Other than a basis for empirical refutation, theorem 3 provides a footing from which further potential no-go theorems can be derived. If one manages to show that under certain conditions the shared category of a hard ensemble interpretation is greater than zero for a KS witness, then through reference to theorem 3 a no-go theorem is constructed for that case. Further research exploring the viability of prism models in hidden variable theories can be done by proposing and studying other types of deficiency interpretations; perhaps the circumvention of implicit rejection rates is still possible if we modify the soft ensemble interpretations behaviour for deficiency. We could for example consider, instead of a hidden variable resetting, a hidden variable partially resetting, but enough to still yield a successful measurement. Alternatively, the formalization of a general prism model representation may give way to the application of other no-go theorems, like perhaps the recent PBR theorem (Pusey et al., 2012) or an extension thereof. It may be interesting to compare what kind of implications different no-go theorems pose and what could be gained by combining them.

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