Modular forms of varying weight. III

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1. Introduction

1.1 In [2] and [3] we studied, for the full modular group, families of real analytic modular forms parametrized by the weight. In this note we conclude this sequence of papers.

1.2 The spectral theory of automorphic forms has great number theoretic interest; see e.g. [10] or [8]. Usually one studies automorphic forms for a fixed weight and a fixed appropriate multiplier system. This sequence of papers started from the question how automorphic forms would vary if one changes the weight and the multiplier system.

This is not a new question. As I might have remarked already in the introduction of [2], the continuity of the spectral decomposition under changes of the weight, and the group, has been studied in [9]. There only continuity in the weight is considered, but the question of analyticity is raised in § 10, (iii) of [9].

Analytic perturbations of automorphic forms under changes of the underlying Riemann metric are considered in [4], § 5 and [15].

1.3 In these notes we consider only the full modular group. There is only one family v_r of multiplier systems in this case, obtained from the $2r$-th power of the Dedekind eta function. The parameter r runs through $\mathbb{C}/12\mathbb{Z}$; for the spectral theory we are interested in real values of r. The multiplier system v_r is suitable for weights $q \equiv r \bmod 2$.

In [2] and [3] we considered the weight $q = r$ only.

In [2] we showed that for $r \in (0, 12)$ all square integrable eigenfunctions of the Casimir operator of sl_2 depend analytically on r, and that the eigenvalues are analytic functions $(0, 12) \to \mathbb{R}$. One of these eigenvalues, $\lambda_\eta : r \mapsto \frac{1}{2} r \left(1 - \frac{1}{2} r\right)$, corresponds to the powers of the Dedekind eta function. All other ones are unknown.

The main theme in [3] is the meromorphic continuation of Eisenstein and Poincaré series in the complex variables r and s, where s parametrizes the eigenvalue

$$\frac{1}{4}-s^2.$$

1.4 One of the results of this paper concerns the extendability of eigenvalues across 0 or 12.

By multiplication by powers of the Dedekind eta function we see that all holomorphic cusp forms occur in families parametrized by the weight running through a halfline. The same holds for antiholomorphic cusp forms, and for all cusp forms derived from (anti)holomorphic ones by differential operators.

Our interest is in the other, real analytic, square integrable modular forms; we shall call them of continuous series type.

We prove that, except possibly for weight zero, all square integrable modular forms of continuous series type occur in families for which the weight varies in a real interval with endpoints in $12\mathbb{Z}$. The limit of the eigenvalue at the endpoints equals $\frac{1}{4}$.

Modular cusp forms of weight zero may occur as member of such families or they may arise from singularities of the Eisenstein series continued in two variables. I could not resolve the question whether both possibilities really occur.

In weight zero it is known that all modular cusp forms have an eigenvalue larger than $\frac{3\pi^2}{2}$. For $0<r<12$ we show that all eigenvalues of continuous series type are larger than $\frac{1}{4}$. This means that all exceptional eigenvalues are the known ones, coming from holomorphic or antiholomorphic cusp forms.

1.5 In these notes we have only considered the full modular group. Generalization of the results in [2] and [3] to more general groups might be possible under suitable circumstances. In this note we use much more information on the modular group:

1) Hecke operators and multiplicity one in weight zero.

2) The absence of exceptional eigenvalues in weight zero.

3) The explicit form of the Eisenstein series in weight zero.

It seems unlikely that the results in this paper give a good indication what to expect for other groups.

1.6 In section 2 we introduce some notations and formulate the results. In section 3 the reader will find some indications on the way the results are proved in sections 4—10.

2. Statement of results

We introduce some notations and formulate the main results of this note.

2. 1 Notation. $\mathfrak{h}$ is the upper half plane; on it acts the group $\Gamma = SL_2(\mathbb{Z})$. The standard fundamental domain of Γ is $F = \left\{ z \in \mathfrak{h} : |z| \geqq 1, |\mathrm{Re}\, z| \leqq \frac{1}{2} \right\}$. We write $z = x + iy$ with x, y real.

All multiplier systems for Γ are obtained as v_r, the $(2r)$-th power of the multiplier system of the Dedekind eta function; here $r \in \mathbb{C} \bmod 12\mathbb{Z}$.

For $q \in \mathbb{C}$ we shall use the following differential operators on $\mathfrak{h}$:

$$L_q = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iqy \frac{\partial}{\partial y},$$

$$E_q^{\pm} = \pm 2iy \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \pm q.$$

They satisfy

$$L_q = -\frac{1}{4} E_{q \mp 2}^{\pm} E_q^{\mp} - \frac{1}{4} q^2 \pm \frac{1}{2} q.$$

2. 2 Definition. Let $q, r, \lambda \in \mathbb{C}$ with $q \equiv r \bmod 2$. $A(q, r, \lambda)$ is the linear space of $f \in C^\infty(\mathfrak{h})$ satisfying

$(a)_{q,r}$: $f(\gamma z) = v_r(\gamma) e^{iq \arg(cz+d)} f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

$(e)_{q,\lambda}$: $L_q f = \lambda f$.

The elements of $A(q, r, \lambda)$ we call *modular forms* of weight q, eigenvalue λ and with multiplier system v_r.

Remark that we do not impose a growth condition at the cusp, contrary to usual practice (compare e.g. [18]).

2. 3 Examples. i) For each $r \in \mathbb{C}$ with $\mathrm{Re}\, r > 0$ we may define

$$\eta(z)^{2r} = e^{\frac{\pi i r z}{6}} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{2r}$$

as holomorphic modular form. One easily checks that

$$\eta_r(z) := y^{\frac{1}{2}r} \cdot \eta(z)^{2r}$$

defines $\eta_r \in A\left(r, r, \frac{1}{2} r \left(1 - \frac{1}{2} r\right)\right)$.

ii) The Poincaré series $P(\cdot, \infty, s, n)$, with $n \neq 0$, in [14], (3. 32), are elements of $A(0, 0, s - s^2)$ which grow exponentially in the cusp.

2. 4 The differential operators E_q^+ and E_q^- (see 2. 1) give linear maps

$$E_q^\pm : A(q, r, \lambda) \longrightarrow A(q \pm 2, r, \lambda),$$

the *Maass operators.*

Modular forms have a Fourier expansion with respect to x. The individual terms of the expansion are elements of the following spaces:

2. 5 Definition. Let $q, m, \lambda \in \mathbb{C}$.

$W(q, m, \lambda)$ is the linear space of $f \in C^\infty(\mathfrak{h})$ satisfying:

$(w)_m$: $f(x+iy) = e^{2\pi i m x} f(iy)$,

$(e)_{q,\lambda}$ as in 2. 2.

${}^\circ W(q, m, \lambda)$ is the subspace of $f \in W(q, m, \lambda)$ satisfying

$$|f(z)| \ll y^{-d} \quad \text{for} \quad y \to \infty \quad \text{for all} \quad d \in \mathbb{R}.$$

2. 6 Definition. For $\nu \in \mathbb{Z}$, $q, r, \lambda \in \mathbb{C}$, $q \equiv r \bmod 2$:

$$F_\nu = F_\nu(r) : A(q, r, \lambda) \longrightarrow W\left(q, \nu + \frac{r}{12}, \lambda\right)$$

is "taking the Fourier term of order $\nu + \frac{r}{12}$"

$$F_\nu(r) f(z) = \int_0^1 e^{-2\pi i\left(\nu + \frac{r}{12}\right)x'} f(x'+z)\, dx'.$$

2. 7 Definition. Let $q, r, \lambda \in \mathbb{C}$, $q \equiv r \bmod 2$.

$S(q, r, \lambda)$ is the subspace of $f \in A(q, r, \lambda)$ such that

$$F_\nu f \in {}^\circ W\left(q, \nu + \frac{r}{12}, \lambda\right) \text{ for all } \nu \in \mathbb{Z}.$$

Such functions we call *cusp forms.*

2. 8 The differential operators $E_q^\pm$ satisfy

$$E_q^\pm X(q, r \text{ or } m, \lambda) \subset X(q \pm 2, r \text{ or } m, \lambda)$$

for $X = A, S, W$ or ${}^\circ W$ and commute with F_ν.

From 2. 1 follows that $E_{q\mp 2}^\pm E_q^\mp$ is multiplication by $-4\lambda - q^2 \pm 2q$. So for general values of λ and q the Maass operators are bijections. For many purposes this enables us to study the case $q = r$ and deduce the other cases from it.

If $q, r \in \mathbb{R}$, then $|f|$ is Γ-invariant on $\mathfrak{h}$ for each $f \in A(q, r, \lambda)$. So we may consider square integrability of f.

2. 9 Definition. Let $q, r \in \mathbb{R}$, $q \equiv r \bmod 2$.

$L^2(q, r)$ is the Hilbert space of classes of functions f on $\mathfrak{h}$ satisfying $(a)_{q,r}$ and

$$\int_F |f(z)|^2 y^{-2}\, dx\, dy < \infty.$$

The scalar product is given by

$$\langle f_1, f_2\rangle = \int_F f_1(z)\,\overline{f_2(z)}\,y^{-2}\,dx\,dy.$$

One may show that $S(q, r, \lambda) \subset L^2(q, r)$ for all $\lambda \in \mathbb{C}$.

2. 10 L_q determines in $L^2(q, r)$ a self adjoint operator $A_{q,r}$ with lower bound $\frac{1}{2}|q|\left(1-\frac{1}{2}|q|\right)$; see e.g. [10], (1) on p. 370, or [2], § 5. Except for the case $q=r=\lambda=0$ the λ-eigenspace of $A_{q,r}$ coincides with $S(q, r, \lambda)$.

The self-adjointness of $A_{q,r}$ implies that

$$S(q, r, \lambda) = 0 \text{ for } q, r \in \mathbb{R},\ \lambda \notin \mathbb{R}.$$

2. 11 Notation. For $r \in \mathbb{R}$ define $\tau(r)$ and $\lambda_0(r)$ by $\tau(r) \in [0, 1]$,

$$\tau(r) \equiv \pm r \bmod 2$$

$$\lambda_0(r) = \frac{1}{2}\tau(r)\left(1-\frac{1}{2}\tau(r)\right).$$

2. 12 For fixed $r, \lambda \in \mathbb{R}$ we may consider

$$X = \bigoplus_{q \equiv r \bmod 2} S(q, r, \lambda).$$

The endomorphisms induced by the Maass operators generate an action of the Lie algebra of $SL_2(\mathbb{R})$ in X, which is the infinitesimal representation corresponding to a unitary representation of the universal covering group of $SL_2(\mathbb{R})$. See e.g. [17], or [1], 3. 4. This implies that $S(q, r, \lambda) \neq 0$ is only possible if

$$\lambda > \lambda_0(r), \text{ “continuous series”}$$

or

$$\lambda = \frac{1}{2}b\left(1-\frac{1}{2}b\right),\ b>0,\ b \equiv \pm r \bmod 2,\ \pm q \geqq b, \text{ “discrete series”}.$$

2. 13 The discrete series type cusp forms are fully determined by the classical holomorphic ones.

As in [1], propositions 4. 5. 7 and 4. 5. 8, we may reduce the study of $S\left(q, r, \frac{1}{2}b\left(1-\frac{1}{2}b\right)\right)$ to that of $S\left(\pm b, r, \frac{1}{2}b\left(1-\frac{1}{2}b\right)\right)$ by use of the Maass operators, and show that each $f \in S\left(\pm b, r, \frac{1}{2}b\left(1-\frac{1}{2}b\right)\right)$ satisfies $E^{\mp}_{\pm b} f = 0$. This implies that $f(z) = y^{\frac{1}{2}b}h(z)$, with h a holomorphic or antiholomorphic modular cusp form of weight b and multiplier system $v_{\pm r}$. We may assume that we had chosen $0 < \pm r \leqq 12$. So h or $\bar{h} \in \eta^{\pm 2r} \cdot M_{b \mp r}$, with M_k the space of entire modular forms of weight k and trivial multiplier system.

So all discrete series type cusp forms occur in families parametrized by the weight varying over a half line.

Our purpose is to study the continuous series type cusp forms. Our results are:

2. 14 Proposition. *Let* $l \in 2\mathbb{Z}$.

There are a countable set Λ_l *of analytic functions* $(0, 12) \to \mathbb{R}$ *and for each* $\lambda \in \Lambda_l$ *a finite number of analytic functions*

$$f_{\lambda,l}^i: (0, 12) \times \mathfrak{h} \to \mathbb{C} \qquad 1 \leqq i \leqq N_{\lambda,l}$$

such that for each $r \in (0, 12)$:

i) $f_{\lambda,l}^i(r, .) \in S(r+l, r, \lambda(r))$,

ii) $\{f_{\lambda,l}^i(r, .): \lambda \in \Lambda_l, 1 \leqq i \leqq N_{\lambda,l}\}$ *is a complete orthonormal system in* $H(r+l, r)$.

For the case $l=0$ this follows from proposition 2. 5 in [2]. The other cases follow easily from 2. 12.

2. 15 Theorem. i) *For* $l \in 2\mathbb{Z}$ *put*

$$\Lambda_l^d = \{\lambda \in \Lambda : \lambda(r) \leqq \lambda_0(r) \quad \text{for all} \quad r \in (0, 12)\}.$$

Then $\Lambda^c = \Lambda_l \backslash \Lambda_l^d$ *does not depend on* l.

ii) $\lambda(r) > \frac{1}{4}$ *for all* $\lambda \in \Lambda^c$ *and* $r \in (0, 12)$.

iii) $N_{\lambda,l} = 1$ *for all* $\lambda \in \Lambda^c$ *and* $l \in 2\mathbb{Z}$.

Most of part i) follows directly from use of the Maass operators. We have to exclude the possibility that $\lambda(r) = \mu(r)$ for some $r \in (0, 12)$ with $\lambda \in \Lambda^c$ and $\mu \in \Lambda_l^d$.

Part ii) implies that the only possible exceptional eigenvalues are those coming from discrete series type cusp forms. This simplifies estimates of sums of Kloosterman sums as discussed in [16], § 7, or [7], theorem 2. For a bit stronger estimate of $\lambda(r)$ for $0 < r < 12$ see section 10.

Part iii) is a kind of analytic multiplicity one result. To each eigenvalue of continuous series type corresponds a one-dimensional family of cuspidal eigenspaces. At points where the graphs of eigenvalues intersect, the dimension of the space of cusp forms exceeds 1, but within this space the various eigenfamilies keep their individuality.

2. 16 Notation. If $\lambda \in \Lambda^c$ we write $f_{\lambda,l}$ instead of $f_{\lambda,l}^1$.

Remark that the $f_{\lambda,l}$ are determined up to multiplication by analytic functions with absolute value 1. At this point we do not correlate the choice of $f_{\lambda,l}$ for various $l \in 2\mathbb{Z}$.

For $\lambda \in \Lambda^d$ we have seen in 2. 13 that λ and the corresponding eigenfamilies have analytic extensions to half lines. For continuous series type eigenvalues we have the following result.

2. 17 Proposition. *Let* $\lambda \in \Lambda^c$.

i) *If* $F_0(r) f_{\lambda,0}(r) = 0$ *for all* $r \in (0, 12)$, *then* λ *and all* $f_{\lambda,l}$ *have analytic extensions to* $(-12, 12)$. *For each* $r \in (-12, 0]$:

$$f_{\lambda,l}(r) \in S(r+l, r, \lambda(r)), \quad \|f_{\lambda,l}(r)\| = 1.$$

ii) *If* $F_0(r) f_{\lambda,0}(r) \neq 0$ *for some* $r \in (0, 12)$, *then*

a) λ *has no analytic extension to a neighbourhood of* 0.

b) $\lim_{r \downarrow 0} \lambda(r) = \frac{1}{4}$.

It will become clear that there are countably many $\lambda \in \Lambda^c$ satisfying ii). We could not prove the presence or absence of $\lambda \in \Lambda^c$ for which i) holds.

Extendable eigenvalues would for $l=0$ at $r=0$ give real analytic cusp forms of weight zero. Another possibility for weight zero cusp forms to arise is from singularities of the Eisenstein series considered as function of weight and eigenvalue jointly.

2. 18 In [3], proposition 2. 19 we showed that there is a unique meromorphic family $E = E^0$ of modular forms on some neighbourhood of $(-12, 12) \times \mathbb{C}$ in $\mathbb{C}^2$ such that

a) $E(r, s) \in A\left(r, r, \frac{1}{4} - s^2\right)$ wherever it is holomorphic.

b) $F_\nu(r)\, E(r, s) \in {}^\circ W\left(r, \nu + \frac{r}{12}, \frac{1}{4} - s^2\right)$ for all $\nu \in \mathbb{Z}$, $\nu \neq 0$.

c) $F_0(r)\, E(r, s) = \mu^0(r, s) + C(r, s)\, \mu^0(r, -s)$, with C a meromorphic function and $\mu^0(r, s)$ an explicitly given element of $W\left(r, \frac{r}{12}, \frac{1}{4} - s^2\right)$.

For $r = 0$, $\mathrm{Re}\, s > \frac{1}{2}$, $s \notin \frac{1}{2}\mathbb{Z}$ we showed that E is holomorphic at $(0, s)$ and that

$$E(0, s) = \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1}} \left(\frac{y}{|cz + d|^2} \right)^{\frac{1}{2} + s},$$

so we may call E the analytic continuation in two variables of the Eisenstein series.

If $s_0 \in i\mathbb{R}$ then C may be indeterminate at $(0, s_0)$, although $s \mapsto C(0, s)$ is known to be holomorphic at s_0.

2. 19 Proposition. *Let* $s_0 \in i\mathbb{R}$. *Equivalent are:*

i) $S\left(0, 0, \frac{1}{4} - s_0^2\right)$ *is not spanned by*

$\left\{ f_{\lambda,0}(0) \colon \lambda \in \Lambda^c,\ \lambda \text{ has an analytic extension to } (-12, 12),\ \lambda(0) = \frac{1}{4} - s_0^2 \right\}$.

ii) C *is not holomorphic at* $(0, s_0)$.

2. 20 In proposition 2. 17 we only considered extension of eigenvalues and eigenfamilies at 0. The full result we formulate in terms of holomorphic functions. It clearly implies the real analytic results.

2. 21 Theorem. *Put*

$$\Lambda^n = \{\lambda \in \Lambda^c : \lambda \text{ has no analytic extension to a neighbourhood of } 0\}.$$

i) *For each* $\lambda \in \Lambda^n$ *there is an open neighbourhood* Ω_λ *in* $\mathbb{C}$ *of* $(0, n_\lambda)$, *with* $n_\lambda \in 12\mathbb{Z}$, $n_\lambda \geqq 12$, *such that*

a) λ *has a holomorphic extension* $\lambda: \Omega_\lambda \to \mathbb{C}$.

b) *Each* $f_{\lambda,l}$ *with* $l \in 2\mathbb{Z}$, *has an analytic extension* $f_{\lambda,l}: \Omega_\lambda \times \mathfrak{h} \to \mathbb{C}$, *holomorphic in the first variable, such that* $f_{\lambda,l}(r) \in S(r+l, r, \lambda(r))$ *for all* $r \in \Omega_\lambda$.

c) λ *has no holomorphic extension to a neighbourhood of* n_λ.

d) $\lim_{r \downarrow 0} \lambda(r) = \frac{1}{4}, \quad \lim_{r \uparrow n_\lambda} \lambda(r) = \frac{1}{4}.$

ii) a) *For each* $\mu \in \Lambda^c$ *there are unique* $\lambda \in \Lambda^n$ *and* $\nu_\mu \in 12\mathbb{Z}$, $0 \leqq \nu_\mu < n_\lambda$ *such that*

$$\mu(r) = \lambda(r + \nu_\mu) \quad \text{for} \quad 0 < r < 12.$$

b) *One may take* $f_{\mu,l}(r) = f_{\lambda,l}(r + \nu_\mu)$ *for* $0 < r < 12$.

3. Indication of the proofs

First we prove a result, proposition 3. 1, which makes clearer the nature of Λ^c. After that we give some indications to help the reader find his way in the following sections.

We have already remarked that for eigenvalues larger than $\lambda_0(r)$ the Maass operators give bijections between spaces of square integrable modular forms of weights $q \equiv r \bmod 2$; see 2. 8. So i) of theorem 2. 15 follows from proposition 2. 14 as soon as we have the following result:

3. 1 Proposition. *Let* $l \in 2\mathbb{Z}$.

If $\lambda \in \Lambda_l$ *satisfies* $\lambda(r) > \lambda_0(r)$ *for some* $r \in (0, 12)$, *then* $\lambda(r) > \lambda_0(r)$ *for all* $r \in (0, 12)$.

For λ_0 see 2. 11.

Proof. From 2. 12 follows that if $\lambda(r) > \lambda_0(r)$ for some $r \in (0, 12)$, then $\lambda(r) \geqq \lambda_0(r)$ for all $r \in (0, 12)$. Suppose $\lambda(r_0) = \lambda_0(r_0)$ for some $r_0 \in (0, 12)$. Then $f^1_{\lambda,l}(r_0)$ is of discrete series type, hence it is a linear combination of some $f^i_{\mu,l}(r_0)$ with $\mu \in \Lambda^d_l$, see the discussion in 2. 13. This contradicts the orthonormality in proposition 2. 14.

3. 2 At this point we may remark that the analyticity of $\lambda \in \Lambda^c$ on $(0, 12)$ is equivalent to holomorphy of λ on some neighbourhood of $(0, 12)$ in $\mathbb{C}$. This neighbourhood may depend on λ; we denote it U_λ.

3. 3 To find ones way through the following sections one should keep in mind the following main themes:

1) Extendability of eigenvalues is governed by the vanishing of Fourier coefficients; proposition 2. 17 i) and ii) a).

2) Limit behaviour for $r \downarrow 0$ of non-extendable eigenvalues; proposition 2. 17 ii) b).

3) Estimates of eigenvalues; theorem 2. 15 ii). In propositions 10. 1 and 10. 2 stronger results are given.

4) Do all cusp forms of weight 0 occur in families of square integrable forms? Proposition 2. 19.

3. 4 *Theme* 1. As soon as one has got proposition 2. 17 i) and ii) a), one may fit things together to obtain theorem 2. 15 iii) and theorem 2. 21 except i) d). This is done in section 8.

In [2], proposition 2. 8, we already proved proposition 2. 17 i) for the case $l=0$; the other cases easily follow. The proof of proposition 2. 17 ii) a) requires much more work. It uses the meromorphic continuation in two variables of Eisenstein and Poincaré series as studied in [3].

3. 5 In section 5 we prove that for $0<r<12$ all families of square integrable modular forms of continuous series type with non-vanishing F_ν are described by singularities of the Poincaré series of order ν.

The Poincaré series cannot be continued meromorphically to $r=0$, but in the case $\nu=0$, in which we are interested, there is a relation with the Eisenstein series E. For $\nu=0$ the results of section 5 may be expressed with help of the coefficient C from the Fourier expansion of E. We do this in section 6. The proof of proposition 2. 17 ii) a) is completed in proposition 6. 14.

3. 6 *Theme* 2. In section 6 we use the results of section 5 and the known expression for $s \mapsto C(0, s)$ to prove $\liminf_{r \downarrow 0} \lambda(r) \geqq \frac{1}{4}$ for non-extendable λ. This line of thought is persued in section 7, where we prove that $\frac{1}{4}$ and ∞ are the only possible limit values. The latter possibility is excluded in section 10 by an estimate of $\lambda'(r)$ for $0<r\leqq 2$.

3. 7 *Theme* 4. In section 6 we already get some information on the relation between singularities of C and cusp forms of weight zero.

In section 9 we go deeply into the proofs in [2]. Analytic perturbation theory for linear operators combined with ideas of Colin de Verdière makes it possible to prove proposition 2. 19. We also obtain a formula for the derivatives of eigenvalues.

3. 8 *Theme* 3. Theorem 2. 15 ii) follows from explicit estimates of eigenvalues in propositions 10. 1 and 10. 2. In section 10 we obtain these results by integrating Fourier coefficients of modular forms to estimate eigenvalues and their derivatives.

3. 9 Notation. There are two notational regimes in the following sections.

In section 2 we considered modular forms of weight $q=r+l$ with $l \in 2\mathbb{Z}$. So we needed q, r and λ in the notation. This is also the point of view in sections 8 and 10.

In sections 5—7, 9 we stick to weight $q=r$ and denote the eigenvalue by $\frac{1}{4}-s^2$. So here we only need the parameters r and s. This corresponds to the point of view in [3]. These notations are introduced in section 4.

4. Sheaves of eigenfunctions of L_r

In the sections 4—7 we assume that the weight q equals the parameter r of the multiplier system. We parametrize the eigenvalue by $\lambda=\frac{1}{4}-s^2$.

4. 1 In this section we recall results from [3] and fix notations. These notations are a bit less general than those used in [3].

4. 2 Notation. Let $r, s \in \mathbb{C}$, $\nu \in \mathbb{Z}$.

$$A[r,s]=A\left(r,r,\frac{1}{4}-s^2\right), \quad \text{(see 2. 2)}$$

$$W^\nu[r,s]=W\left(r,\nu+\frac{r}{12},\frac{1}{4}-s^2\right), \quad \text{(see 2. 5)}$$

$${}^\circ W^\nu[r,s]={}^\circ W\left(r,\nu+\frac{r}{12},\frac{1}{4}-s^2\right),$$

$$A^\nu[r,s]=\{f\in A[r,s]: F_\mu(r)f\in {}^\circ W^\mu[r,s] \quad \text{for all} \quad \mu\in\mathbb{Z},\ \mu\neq\nu\},$$

$$S[r,s]=S\left(r,r,\frac{1}{4}-s^2\right), \quad \text{(see 2. 7)}$$

$$S^\nu[r,s]=\{f\in S[r,s]: F_\nu(r)f=0\}.$$

In [3] the last three spaces are denoted by respectively $A^{(\nu)}[r,s]$, $S^0[r,s]$ and $S^{(\nu)}[r,s]$.

4. 3 As $\dim {}^\circ W^\nu[r,s]=\begin{cases}0 & \text{if } \operatorname{Re}\left(\nu+\frac{r}{12}\right)=0\\ 1 & \text{otherwise}\end{cases}$ we have

$$S[0,s]=S^0[0,s]=S\left(0,0,\frac{1}{4}-s^2\right).$$

4. 4 Proposition (Maass-Selberg relation). *Let r, s, ν be as in 4. 2. There exists a nondegenerate bilinear form* Wr^ν *on $W^\nu[r,s]$ such that for all $f, g\in A^\nu[r,s]$:*

$$\mathrm{Wr}^\nu(F_\nu f, F_\nu g)=0.$$

See [3], proposition 7. 8 and definition 5. 3.

4. 5 Notation. On any complex analytic space V we denote by $\mathcal{O}$, resp. $\mathcal{M}$, the sheaf of holomorphic, resp. meromorphic functions on V. We write ${}^n\mathcal{O}$ and ${}^n\mathcal{M}$ if we want to emphasize the complex dimension n of V.

4. 6 Notation. $Y=\{(r,s)\in\mathbb{C}^2: -12<\operatorname{Re} r<12\}$,

$Y^*=\{(r,s)\in Y: \operatorname{Re} r\neq 0\}$,

$Y^+=\{(r,s)\in\mathbb{C}^2: 0<\operatorname{Re} r<12\}$.

4.7 Notation. In [3], 7.2 and 4.6, we associated to $A^\nu[r,s]$, or $A^{(\nu)}[r,s]$ in the notation of [3], an ${}^2\mathcal{O}$-module $\mathcal{A}^{(\nu)}$ on Y; here we shall denote it by $\mathcal{A}^\nu$. Sections of $\mathcal{A}^\nu$ are analytic functions in (r,s,z), $(r,s)\in Y$, $z\in\mathfrak{h}$, holomorphic in (r,s), giving an element of $A^\nu[r,s]$ for fixed (r,s). Similarly we defined $\mathcal{W}^\nu$ and ${}^\circ\mathcal{W}^\nu$, replacing A^ν by W^ν or ${}^\circ\mathcal{W}^\nu$.

4.8 These sheaves of $\mathcal{O}$-modules have an important property, discussed in [3], 4.13, 4.15. In the case of $\mathcal{A}^\nu$: if f is analytic on $U\times\mathfrak{h}$, $U\subset Y$, holomorphic in (r,s) and $f(r,s;.)\in A^\nu[r,s]$ for all (r,s) in a dense open subset of U, then $f\in\mathcal{A}^\nu(U)$.

$\mathcal{W}^\nu$ and ${}^\circ\mathcal{W}^\nu$ also have these properties.

In [3], lemma 4.15 ii) is incorrect; part i) suffices in all our applications. In 4.1 of [3] allow D to be a, not necessarily complete, subspace of the antidual.

4.9 Let $j\colon W\to Y$ be a holomorphic map between complex analytic spaces. $\mathcal{A}_j^\nu$ is the $\mathcal{O}$-module on W for which $f\in\mathcal{A}_j^\nu(U)$ means: $f\colon U\times\mathfrak{h}\to\mathbb{C}$ real analytic, holomorphic on U, $f(w,.)\in A^\nu[jw]$ for each $w\in U$.

$\mathcal{W}_j^\nu$ and ${}^\circ\mathcal{W}_j^\nu$ are defined similarly.

These sheaves on W have the properties described in 4.8.

Composition gives $j^\#\colon\mathcal{A}^\nu(U)\to\mathcal{A}_j^\nu(j^{-1}U)$, etc.

If $\dim W=1$, then we call $j^\#$ a *restriction map.*

4.10 Notation. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules without torsion; for instance one of the sheaves in 4.7 or 4.9. We defined in [3], 4.10, for $f\in\mathcal{M}\otimes\mathcal{F}(U)$:

the zero set

$$N(f)=\{p\in U\colon \text{ if } \varphi\in\mathcal{O}_p,\ \varphi\neq 0 \text{ and } \varphi f\in\mathcal{F}_p, \text{ then } (\varphi f)(p)=0\},$$

the singular set

$$\mathrm{Sing}(f)=\{p\in U\colon \text{ if } \varphi\in\mathcal{O}_p,\ \varphi\neq 0 \text{ and } \varphi f\in\mathcal{F}_p, \text{ then } \varphi(p)=0\},$$

and the set where f is *indeterminate*

$$\mathrm{Indet}(f)=N(f)\cap\mathrm{Sing}(f).$$

4.11 If $j\colon W\to Y$ is holomorphic, $\dim W=1$ and $\mathcal{F}$ is one of the sheaves on Y discussed in 4.7, then we may define

$$j^\# f\in{}^1\mathcal{M}\otimes\mathcal{F}(j^{-1}U) \quad\text{for}\quad f\in{}^2\mathcal{M}\otimes\mathcal{F}(U),$$

provided jW intersects $\mathrm{Sing}(f)$ in isolated points.

4. 12 Definition ([3], 3. 6). Let $p=(r_0, s_0)\in Y$, let $\chi\in\mathcal{O}(U)$ represent an irreducible element of $\mathcal{O}_p$. By a *local curve* through p along $N(\chi)$ we mean a holomorphic map $j\colon W\to Y$ on an open set $W\subset\mathbb{C}$, $0\in W$, such that:

i) $j0=p$,

ii) $jW\subset N(\chi)$,

iii) j is minimal, e.g. if $j=j_1\circ h$ with j_1 satisfying i) and ii) and h holomorphic, then $h(0)=0$ and $h'(0)\neq 0$.

Actually we are interested in the germ of j up to equivalence $j\mapsto j\circ h$ as in iii).

If j is not equivalent to $w\mapsto(r_0, s_0+w)$, then it is equivalent to $w\mapsto(r_0+w^q, s(w))$, with $q\geqq 1$, q integral, s holomorphic, $s(0)=s_0$; see [5], III, § 1. 5, p. 131.

4. 13 In [3], 7. 5, we remarked that the sheaf $\mathcal{S}$, built from S in the way discussed in 4. 7, is zero on a neighbourhood of $(-12, 12)\times\mathbb{C}$. On the other hand, there are many local curves j such that $\mathcal{S}_j\neq 0$.

4. 14 Lemma. *Let $p=(r_0, s_0)\in Y$, $r_0\in(-12, 12)$, $j\colon W\to Y$ a local curve through p. Let $\mathcal{F}$ be $\mathcal{S}^0$ if $r_0=0$ and $\mathcal{S}$ if $r_0=0$. Suppose that the stalk $(\mathcal{F}_j)_0$ is nonzero.*

Then $s_0\in\mathbb{R}\cup i\mathbb{R}$ and we may take j of the form $jw=(r_0+w^q, s(w))$ with $s\colon W\to\mathbb{C}$ holomorphic and $s(0)=s_0$

if $s_0\neq 0$, then $q=1$,

if $s_0=0$, then $q=1$ or 2,

if $q=2$, then s is not constant and $s(-w)=-s(w)$.

Proof. j cannot be "vertical", for if $jw=(r_0, s(w))$ then there would be nonzero cusp forms with weight r_0 and nonreal eigenvalue, compare 2. 10. So we may arrange that $jw=(r_0+w^q, s(w))$ as in 4. 12.

Put $W_{re}=\{w\in W\colon w^q\in\mathbb{R}\}$. The sheaf $\mathcal{F}_j$ has the properties mentioned in 4. 8, so we may remove zeros from sections of $\mathcal{F}_j$. This means that $S[jw]\neq 0$ for all $w\in W_{re}$, provided W is small enough. So $\frac{1}{4}-s(w)^2\in\mathbb{R}$ for all $w\in W_{re}$. This implies $s_0=s(0)\in\mathbb{R}\cup i\mathbb{R}$, and $\frac{1}{4}-s(w)^2=H(w^q)$ for some holomorphic function H. If $s_0\neq 0$ we have holomorphy of $\sqrt{\frac{1}{4}-H(w^q)}$ at $w=0$, so $q=1$ by minimality, see 4. 12.

Consider $s_0=0$. As $s(w)=H(w^q)^{\frac{1}{2}}$ is holomorphic, we see that $s(w)$ is a holomorphic function of $w^{\frac{q}{2}}$ if q is even, and of w^q if q is odd. By minimality we have $q=1$ as the only odd possibility. If q is even, then $q=2$ and $s(w)=h_+(w^2)+wh_-(w^2)$, with h_+ and h_- holomorphic. As $s(w^2)$ is holomorphic in w^2, we conclude that $h_+(w)\,h_-(w)=0$. If $h_-(w)=0$, then j would not be minimal, so $s(w)=wh_-(w^2)\neq 0$.

4.15 In [3], 2.7 and 2.13 we explicitly gave sections of the $\mathscr{W}^\nu$:

The exponentially decreasing Whittaker function led to

$$\omega^\nu \in \begin{cases} {}^\circ\mathscr{W}^\nu(Y) & \text{if } \nu \neq 0, \\ {}^\circ\mathscr{W}^\nu(Y^*) & \text{if } \nu = 0 \end{cases}$$

such that ${}^\circ\mathscr{W}^\nu = \mathcal{O} \cdot \omega^\nu$ on Y, or Y^*.

The section $\mu^\nu \in \mathscr{M} \otimes \mathscr{W}^\nu(Y)$ is in general exponentially increasing. For (r, s) in a dense open subset of Y the functions $\mu^\nu(r, s)$ and $\mu^\nu(r, -s)$ form a basis of $W^\nu[r, s]$.

$$\omega^\nu(r, s) = v^\nu(r, s)\, \mu^\nu(r, s) + v^\nu(r, -s)\, \mu^\nu(r, -s),$$

with

$$v^\nu(r, s) = \left(4\pi\varepsilon\left(\nu + \frac{1}{2}r\right)\right)^{\frac{1}{2}+s} \Gamma(-2s)\, \Gamma\left(\frac{1}{2} - s - \frac{1}{2}\varepsilon r\right)^{-1},$$

$$\varepsilon = \operatorname{sign} \operatorname{Re}\left(\nu + \frac{1}{12} r\right).$$

4.16 Near $\left(0, \frac{1}{2}\right)$ we shall also use the basis μ^0, ζ, with

$$\zeta(r, s) = \mu^0(r, -s) + w(r, s)\, \mu^0(r, s),$$

$$w(r, s) = \frac{\pi r}{3} \frac{\Gamma(-2s)\, \Gamma\left(\frac{1}{2} + s - \frac{1}{2} r\right)}{\Gamma(2s)\, \Gamma\left(\frac{1}{2} - s - \frac{1}{2} r\right)}.$$

See [3], 5.7 iii).

5. Cusp forms and singularities of Poincaré series

The main result of [3] is the analytic continuation in two variables of Eisenstein and Poincaré series. In this section we relate singularities of the meromorphically continued Poincaré series to local curves j for which $\mathscr{S}_j \neq 0$.

Here we take curves in Y^*. The case that j passes through points $(0, s)$ we shall consider in section 6.

We shall apply the propositions 5.6 and 5.7 with $\nu = 0$ only.

5.1 Proposition ([3], 2.20). *Let* $\nu \in \mathbb{Z}$.

i) *There exist a neighbourhood* $Y^*(\nu)$ *of* $((-12, 12) \setminus \{0\}) \times \mathbb{C}$ *in* Y^* *and a unique*

$$P^\nu \in \mathscr{M} \otimes \mathscr{A}^\nu(Y^*(\nu))$$

such that

$$F_\nu P^\nu(r, s) = \mu^\nu(r, s) + D^\nu_\nu(r, s)\, \omega^\nu(r, s)$$

for some $D^\nu_\nu \in \mathscr{M}(Y^*(\nu))$.

ii) (Functional equations)

a) $v^\nu(r,-s)\,P^\nu(r,-s) = -v^\nu(r,s)\,P^\nu(r,s)$,

b) $P^\nu(r,s;-\bar{z}) = P^{-\nu}(-r,s;z) = \overline{P^\nu(\bar{r},\bar{s};\bar{z})}$.

The identities in ii) are identities of meromorphic functions. For $\lambda \in \mathbb{Z}$, $\lambda \neq \nu$ we have $F_\lambda P^\nu = D_\lambda^\nu \omega^\lambda$ with some $D_\lambda^\nu \in \mathcal{M}(Y^*(\nu))$. For v^ν see 4. 15.

5. 2 Notation. For $r \in \mathbb{C}$, $|\mathrm{Re}\, r| < 12$, put $j_r\colon \mathbb{C} \to Y$, $j_r s = (r,s)$.

In the following lemma we use $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5. 3 Lemma. *Let $\nu \in \mathbb{C}$, $r \in (-12, 12)$.*

i) *The Poincaré series, or Eisenstein series if $r = \nu = 0$,*

$$P_r^\nu(s;z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v_r(\gamma)^{-1}\, e^{-ir\arg(cz+d)}\, \mu^\nu(r,s;\gamma z)$$

converges absolutely for $\mathrm{Re}\, s > \frac{1}{2}$, uniformly for (r,s,z) in compact sets.

ii) *Let $r \neq 0$.*

a) $j_r^{\#} P^\nu \in {}^1\mathcal{M} \otimes \mathcal{A}_{j_r}^\nu(\mathbb{C})$.

b) *If $\mathrm{Re}\, s > \frac{1}{2}$, then $P^\nu \in \mathcal{A}_{(r,s)}^\nu$ and $P^\nu(r,s) = P_r^\nu(s)$.*

iii) *Let $r \neq 0$, $\mathrm{Re}\, s_0 > -\frac{1}{2}$. Suppose $s_0 \in \mathrm{Sing}(j_r^{\#} P^\nu)$. Then*

a) $s_0 \in i\mathbb{R} \cup \left[\frac{1}{2}\tau(r) - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\tau(r)\right]$,

b) *$s \mapsto (s-s_0)^q j_r^{\#} P^\nu(s)$ is holomorphic at s_0 with $q = 1$ if $s_0 \neq 0$ and $q = 1$ or 2 if $s_0 = 0$.*

For $\tau(r)$ see 2. 11.

Assertion ii) a) is not trivial, see 4. 11. Part ii) b) shows that P^ν is indeed the analytic continuation in two variables of a Poincaré series.

Proof. Parts i) and ii) follow from [3], proposition 2. 23. The proof of part iii) we borrow from [4], théorème 3.

Take $\psi \in C^\infty(0, \infty)$ with $0 \leqq \psi \leqq 1$, $\psi(y) = 0$ if $y \leqq 2$, $\psi(y) = 1$ if $y \geqq 3$. Define for $\mathrm{Re}\, s > -\frac{1}{2}$ and $z \in F$

$$h(s, z) = \psi(y)\, \mu^{\nu}(r, s; z).$$

We may extend $h(s, .)$ to $\mathfrak{h}$ by imposing $(a)_{r,r}$, see 2.2. The function $s \mapsto h(s)$ is holomorphic in distribution sense. Moreover,

$$H(s) = \left(L_r - \frac{1}{4} + s^2\right) h(s)$$

defines $H: \left\{s \in \mathbb{C}: \mathrm{Re}\, s > -\frac{1}{2}\right\} \to L^2(r, r)$, holomorphic in L^2-sense.

As the operator $A_{r,r}$, discussed in 2.10, has compact resolvent, we get a meromorphic map g on $\mathrm{Re}\, s > -\frac{1}{2}$ in L^2-sense by

$$g(s) = -\left(A_{r,r} - \frac{1}{4} + s^2\right)^{-1} H(s).$$

Now

$$Q(s) = h(s) + g(s)$$

is meromorphic on $\mathrm{Re}\, s > -\frac{1}{2}$ in distribution sense and satisfies

$$\left(L_{r,r} - \frac{1}{4} + s^2\right) Q(s) = 0.$$

As $L_{r,r}$ is elliptic, this implies that $Q(s)$ satisfies $(a)_{r,r}$ and $(e)_{r, \frac{1}{4} - s^2}$ where it is holomorphic. By considering the Fourier expansions we see that it differs from $j_r^{\#} P^{\nu}(s)$ by an L^2-meromorphic function. The discreteness of the spectrum of $A_{r,r}$ shows that this function vanishes.

So all poles in $\mathrm{Re}\, s > -\frac{1}{2}$ of $j_r^{\#} P^{\nu}(s) = Q(s)$ come from the resolvent $\left(A_{r,r} - \frac{1}{4} + s^2\right)^{-1}$, and have at most first order in $\frac{1}{4} - s^2$. This gives iii) b). The poles can only occur if $\lambda = \frac{1}{4} - s^2 \in \mathbb{R}$ satisfies the conditions in 2.12. Using ii) b) we obtain iii) a).

5.4 Lemma. *Let $r \in (-12, 12)$, $r \neq 0$, $\mathrm{Re}\, s_0 > -\frac{1}{2}$, $\nu \in \mathbb{Z}$. Put*

$$f = \lim_{s \to s_0} (s - s_0)^q j_r^{\#} P^{\nu}(s),$$

with q minimal such that the limit exists.

i) $f \in A^\nu[r, s_0]$, $f \neq 0$.

ii) *If* $f \notin S[r, s_0]$, *then* $S[r, s_0] = S^\nu[r, s_0]$.

iii) *If* $f \in S[r, s_0]$, *then* $S[r, s_0] = \mathbb{C} \cdot f \oplus S^\nu[r, s_0]$ *(orthogonal decomposition). If in addition* $|\mathrm{Re}\, s_0| < \frac{1}{2}(1 - \tau(r))$, *then* $1 \leqq q \leqq 2$, *and* $q = 1$ *if* $s_0 \neq 0$.

Proof. Part i) is clear from 4.8. Suppose f is a cusp form. In lemma 8.4 in [3] we see that f is orthogonal to $S^\nu[r, s_0]$; as $S^\nu[r, s_0]$ has codimension at most 1 in $S[r, s_0]$, we get the orthogonal decomposition in iii). We have supposed that $F_\nu f \in {}^\circ W^\nu[r, s_0]$. On the other hand

$$F_\nu f = \lim_{s \to s_0} (s - s_0)^q \left(\mu^\nu(r, s) + D_\nu^\nu(r, s)\, \omega^\nu(r, s)\right).$$

From 5.7 ii) in [3] we know that under the condition $|\mathrm{Re}\, s_0| < \frac{1}{2}(1 - \tau(r))$, both μ^ν and ω^ν are holomorphic at (r, s_0) and form a basis of $\mathscr{W}^\nu$ near (r, s_0). So $q \geqq 1$, and iii) follows from iii) of lemma 5.3.

Let $h \in S[r, s_0]$. The Maass-Selberg relation, 4.4, implies that $F_\nu f$ and $F_\nu h$ are proportional. So if $F_\nu h \neq 0$ then $F_\nu f \in {}^\circ W^\nu[r, s_0]$, which shows $f \in S[r, s_0]$. This proves ii).

5.5 $\mathrm{Sing}(P^\nu)$ is an analytic subset of $Y^*(\nu)$ of dimension one. Let $p \in \mathrm{Sing}(P^\nu)$. The germ of $\mathrm{Sing}(P^\nu)$ at p may be described as $\bigcup_{i=1}^{t} N(\chi_i) = N(\chi_1 \cdots \chi_t)$, with $\chi_1, \chi_2, \ldots, \chi_t$ different irreducible elements of ${}^2\mathcal{O}_p$, all relatively prime to each other. We may assume that all χ_i live on the same neighbourhood U of p. The irreducible components $N(\chi_i)$ intersect each other discretely, i.e. only in p if U has been chosen small enough.

5.6 Proposition. *Let* $\nu \in \mathbb{Z}$, $p = (r_0, s_0)$, $r_0 \in (-12, 12)$, $r_0 \neq 0$, $\mathrm{Re}\, s_0 > -\frac{1}{2}$. *Suppose* $p \in \mathrm{Sing}(P^\nu)$. *Let* $\chi \in {}^2\mathcal{O}(U)$ *represent an irreducible element of* ${}^2\mathcal{O}_p$ *such that* $N(\chi) \subset \mathrm{Sing}(P^\nu)$. *Take a local curve* $j\colon W \to Y^*(\nu)$ *through* p *along* $N(\chi)$ *such that* jW *intersects other irreducible components of the germ of* $\mathrm{Sing}(P^\nu)$ *only in* p.

Take $n \in \mathbb{N}$ *minimal such that* $\chi^n P^\nu$ *is holomorphic at all points of* $j(W \setminus \{0\})$.

i) $n = 1$ *or* 2. *If* χ *is not equivalent to* $(r, s) \mapsto s$ *then* $n = 1$.

ii) $j^*(\chi^n P^\nu) \in (\mathscr{S}_j)(W \setminus \{0\})$ *is nonzero.*

iii) $\chi^n P^\nu(jw) \perp S^\nu[jw]$ *for all* $w \in W \setminus \{0\}$ *such that* $jw \in (-12, 12) \times \mathbb{C}$.

Proof. By ii) of lemma 5. 3 the germ of χ cannot be equivalent to $(r, s) \mapsto r - r_0$. So we may assume that $jw = (r_0 + w^q, s(w))$, as in 4. 12. By minimality of n we have $j^*(\chi^n P^\nu) \neq 0$, as $n \geqq 1$ it is cuspidal. This gives ii).

The orthogonality in iii) follows from lemma 5. 4 iii).

Now we take $w \in W \setminus \{0\}$ such that $r = r_0 + w^q \in (-12, 12) \setminus \{0\}$, and $(\chi^n P^\nu)(jw) \neq 0$. Put $s_1 = s(w)$. On $\mathbb{C}$ we have

$$j_r^\#(\chi^n P^\nu) = (j_r^\# \chi)^n (j_r^\# P^\nu).$$

Put $m = n$ (order of $j_r^\# \chi$ at s_1). Lemma 5. 4 implies that $m = 1$ if $s_1 \neq 0$ and $m = 1$ or 2 if $s_1 = 0$. This implies $n = 1$ or 2 and $n = 1$ if $s(\,.\,)$ is not the constant 0.

5. 7 Proposition. *Let* $\nu \in \mathbb{Z}$, $p = (r_0, s_0)$, $r_0 \in (-12, 12)$, $r_0 \neq 0$,

$$|\mathrm{Re}\, s_0| < \frac{1}{2}(1 - \tau(r_0)).$$

Suppose that $j\colon W \to Y^*$ *is a local curve through* p *such that*

$$(\mathcal{S}_j)_p \neq (\mathcal{S}_j^\nu)_p.$$

Then

i) $jW \subset \mathrm{Sing}(P^\nu)$.

ii) *Choose* $\chi \in \mathcal{O}_p$ *irreducible such that* $jW \subset N(\chi)$ *and* $n \geqq 1$ *as in proposition* 5. 6. *Then*

$$j^*(\chi^n D_\nu^\nu) \in {}^1\mathcal{M}(W)$$

is nonzero, and

$$({}^1\mathcal{M} \otimes \mathcal{S}_j)_0 = {}^1\mathcal{M}_0 \cdot j^*(\chi^n P^\nu) \oplus ({}^1\mathcal{M} \otimes \mathcal{S}_j^\nu)_0.$$

Proof. j has the form given in lemma 4. 14:

$$jw = (r_0 + w^q, s(w)).$$

There are infinitely many $w \in W$ such that $w^q \in \mathbb{R}$ and $S[jw] \neq S^\nu[jw]$. Lemma 5. 4 shows that $jw \in \mathrm{Sing}(P^\nu)$ for those w. This implies i).

Let $f \in ({}^1\mathcal{M} \otimes \mathcal{S}_j)(W_1)$, $0 \in W_1 \subset W$. Proposition 5. 6 iii) implies that $j^*(\chi^n D_\nu^\nu) \in {}^1\mathcal{M}(W)$ is nonzero; so $f - m \cdot j^*(\chi^n P^\nu) \in {}^1\mathcal{M} \otimes \mathcal{S}_j^\nu(W_1)$ for some $m \in \mathcal{M}(W_1)$. Furthermore

$$m \cdot j^*(\chi^n P^\nu) \in {}^1\mathcal{M} \otimes \mathcal{S}_j^\nu(W_1)$$

clearly implies $m = 0$.

6. Singularities of the Eisenstein series

The $P^\nu(r, s)$ of proposition 5.1 are not defined for $\operatorname{Re} r = 0$. In proposition 2.19 of [3] functions E^ν are considered which are meromorphic at $(0, s)$ as well. We shall need the case $\nu = 0$ only. As the formulas for E^ν are different for $\nu = 0$ and $\nu \neq 0$, we discuss the former case only, and simplify the notation.

The main results in this section are proposition 6.5, which considers singularities of E at $(0, s_0)$ with $\frac{1}{4} - s_0^2 \geqq 0$, $\operatorname{Re} s_0 \geqq 0$, and propositions 6.11 and 6.14, which give part of proposition 2.17 ii).

6.1 Proposition ([3], 2.19 and 2.21). i) *There exist a neighbourhood* $Y(0)$ *of* $(-12, 12) \times \mathbb{C}$ *in* Y *and a unique section*

$$E \in \mathcal{M} \otimes \mathcal{A}^0(Y(0))$$

such that

$$F_0 E(r, s) = \mu^0(r, s) + C(r, s)\, \mu^0(r, -s)$$

with $C \in \mathcal{M}(Y(0))$.

ii) a) $E(r, -s) = C(r, -s)\, E(r, s)$,
$C(r, s)\, C(r, -s) = 1$.

b) $E(r, s, -\bar{z}) = E(-r, s; z) = \overline{E(\bar{r}, \bar{s}; z)}$,
$C(r, s) = C(-r, s) = \overline{C(\bar{r}, \bar{s})}$.

iii) *On* $Y(0) \cap Y^*(0)$:

a) $P^0(r, s) = v^0(r, -s)\,(v^0(r, -s) - v^0(r, s)\, C(r, s))^{-1} E(r, s)$,
$D_0^0(r, s) = (v^0(r, -s)\, C(r, -s) - v^0(r, s))^{-1}$.

b) $E(r, s) = (1 + v^0(r, s)\, D_0^0(r, s))^{-1} P^0(r, s)$,
$C(r, s) = v^0(r, -s)\, D_0^0(r, s)\,(1 + v^0(r, s)\, D_0^0(r, s))^{-1}$.

Remark. In [3] we wrote E^0 and C_0^0 instead of E and C.

6.2 Proposition ([3], 2.23). a) $j_0^\# E \in {}^1\mathcal{M} \otimes \mathcal{A}^0_{j_0}(\mathbb{C})$; *for* $\operatorname{Re} s > \frac{1}{2}$ *it is given by* $P_0^0(s)$ *in* i) *of lemma* 5.3.

b) *If* $\operatorname{Re} s > \frac{1}{2}$, $s \notin \frac{1}{2}\mathbb{Z}$, *then* E *is holomorphic at* $(0, s)$.

Note that $j_0^\# E$ may be holomorphic at s and nevertheless E indeterminate at $(0, s)$. (Remember $j_0^\# E\colon s \mapsto E(0, s)$.)

For $s = \frac{1}{2} l$, $l \geqq 2$, see [3], 7. 19.

6. 3 Remark. $j_0^\# E$ is the usual Eisenstein series for the modular group; see e.g. [10], proposition 8. 6, p. 65 and p. 76. It is known that

$$j_0^\# C(s) = \Lambda(2s)/\Lambda(2s+1),$$

with

$$\Lambda(a) = \pi^{-\frac{1}{2}a}\,\Gamma\left(\frac{1}{2}a\right)\zeta(a),$$

ζ the Riemann zeta function.

6. 4 Facts. i) $j_0^\# C$ is holomorphic at each $s \in \mathbb{C}$ with $\operatorname{Re} s \geqq 0$, $s \neq \frac{1}{2}$.

ii) $j_0^\# C(s) < 0$ for $s \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

iii) $j_0^\# C(s) \neq 0$ for $s \in i\mathbb{R} \cup \left(0, \frac{1}{2}\right)$.

iv) a) $j_0^\# C(0) = -1$,

b) $\lim_{s \to 0} s j_0^\# C(s)\,(1 + j_0^\# C(s))^{-1}$ exists and is nonzero.

See [20]; II, theorem 2. 1 and III, 3. 8 for i); II, 2. 6 and II, (2. 7. 1) for ii); III, 3. 8 for iii). To obtain iv) write $j_0^\# C(s) = \Lambda(+2s)\,\Lambda(-2s)^{-1}$ and use [20], II, end of 2. 12.

6. 5 Proposition. i) *Let* $s_0 \in \mathbb{C}$, $0 \leqq \operatorname{Re} s_0 \leqq \frac{1}{2}$, $s_0 \neq \frac{1}{2}$. *Suppose* $(0, s_0) \in \operatorname{Sing}(C)$. *Then*

a) $(0, s_0) \in \operatorname{Indet}(C)$,

b) $S^0[0, s_0] \neq 0$,

c) $s_0 \in i\mathbb{R}$, $s_0 \neq 0$.

ii) *Define* $c_1(r, s) = \dfrac{C(r, -s) - w(r, s)}{\frac{1}{2} - s - \frac{1}{2} r}$.

Then c_1 *is holomorphic at* $\left(0, \frac{1}{2}\right)$,

$$c_1\left(0, \frac{1}{2}\right) = -\frac{\pi}{3}.$$

Proof of i) a) and i) c). If $(0, s_0) \notin \mathrm{Indet}(C)$, then $s_0 \in \mathrm{Sing}(j_0^{\#} C)$, so 6. 4 i) implies i) a). To get i) c) from i) b) remark that $\frac{1}{4} - s_0^2 \geqq \frac{3\pi^2}{2}$, see e.g. [10], proposition 2. 1, p. 511.

The proof of i) b) and ii) will be given in a sequence of lemmas.

We let $s_0 \in \mathbb{C}$, $0 \leqq \mathrm{Re}\, s_0 \leqq \frac{1}{2}$; the case $s_0 = \frac{1}{2}$ corresponds to ii). We use the notations:

$$G(r, s) = \begin{cases} E(r, s) & \text{if } s_0 \neq 0, \frac{1}{2}, \\ \{1 + C(r, s)\}^{-1} E(r, s) & \text{if } s_0 = 0, \\ C(r, -s)\, E(r, s) & \text{if } s_0 = \frac{1}{2}. \end{cases}$$

$$c(r, s) = \begin{cases} C(r, s) & \text{if } s_0 \neq 0, \frac{1}{2}, \\ s\, C(r, s)\, \{1 + C(r, s)\}^{-1} & \text{if } s_0 = 0, \\ C(r, -s) - w(r, s) & \text{if } s_0 = \frac{1}{2}. \end{cases}$$

$$\lambda(r, s) = \begin{cases} \mu^0(r, -s) & \text{if } s_0 \neq 0, \frac{1}{2}, \\ s^{-1}\{\mu^0(r, -s) - \mu^0(r, s)\} & \text{if } s_0 = 0, \\ \mu^0(r, s) & \text{if } s_0 = \frac{1}{2}. \end{cases}$$

$$v(r, s) = \begin{cases} \mu^0(r, s) & \text{if } s_0 \neq \frac{1}{2}, \\ \zeta(r, s) & \text{if } s_0 = \frac{1}{2}. \end{cases}$$

Remark that λ and v are meromorphic sections of $\mathscr{W}^0$, and that G is one of $\mathscr{A}^0$. (For ζ and w see 4. 16.)

6. 6 Lemma. i) *$v(r, s)$ and $\lambda(r, s)$ form a basis of $W^0[r, s]$ for all (r, s) in some neighbourhood U of $(0, s_0)$.*

ii) *G is the unique section of $\mathscr{M} \otimes \mathscr{A}^0(U)$ satisfying $F_0 G \in v + \mathscr{M}(U) \cdot \lambda$.*

iii) *$F_0 G = v + c\lambda$.*

Proof. For part i) consult [3], 5. 7. The uniqueness in ii) follows from i) in proposition 6. 1. In the case $s_0 = 0$ one may dispel any worry about $1 + C$ being identically zero by consulting 6. 3.

6.7 Lemma. *Let $p=(0, s_0)$. Take $\psi \in \mathcal{O}_p$, $\psi \neq 0$. Equivalent are:*

i) *ψG is holomorphic at p.*

ii) *ψc is holomorphic at p.*

Proof. The implication i) $\Rightarrow$ ii) follows from the holomorphy of v and λ at p, see [3], 4.24 and 5.7.

Assume ii) and suppose that $p \in \operatorname{Sing}(\psi G)$. Choose a local curve $j: W \to Y$ through p along an irreducible component of the germ of $\operatorname{Sing}(\psi G)$ at p such that jW does not intersect other components of the germ of $\operatorname{Sing}(\psi G)$ outside p. Let $\chi \in \mathcal{O}_p$ be irreducible such that $jW \subset N(\chi)$. Choose $n \geqq 1$ minimal such that $\chi^n \psi G$ is holomorphic at all points of $j(W \setminus \{0\})$.

As $j^*(\chi^n \psi c)=0$, we have $j^*(\chi^n \psi G) \in (\mathcal{S}_j^0)(W \setminus \{0\})$. Further $j^*(\chi^n \psi G) \neq 0$ by minimality of n. Lemma 4.14 shows that there exists $q=(r_1, s_1)$, $r_1 \neq 0$, $r_1 \in (-12, 12)$, $q=jw_1$, $w_1 \in W \setminus \{0\}$, with $s_1 \in i\mathbb{R} \cup \left(0, \frac{1}{2}\right)$ and $(\chi^n \psi G)(q) \neq 0$. On a neighbourhood of q we have $\chi^n \psi G = \mu P^0$, with μ meromorphic, $\mu \neq 0$; see iii) b) in proposition 6.1. Lemma 5.4 iii) implies that $(\chi^n \psi G)(q)$ is orthogonal to itself.

6.8 Lemma. i) *If $s_0 \neq \frac{1}{2}$ and $(0, s_0) \in \operatorname{Sing}(c)$, then $S^0[0, s_0] \neq 0$.*

ii) *c is holomorphic at $\left(0, \frac{1}{2}\right)$.*

Proof. Suppose $p=(0, s_0) \in \operatorname{Sing}(c)$. Take $\psi \in \mathcal{O}_p$, $\psi \neq 0$, minimal with respect to divisibility in $\mathcal{O}_p$, such that ψG and ψc are holomorphic at p. Take an irreducible factor χ of ψ in $\mathcal{O}_p$ and take a local curve $j: W \to Y$ through p along $N(\psi)$.

Take $m \in \mathbb{Z}$ maximal such that

$$f = \lim_{w \to 0} w^{-m} \cdot j^*(\psi G)(w)$$

exists; then $f \in A^0[0, s_0]$ is nonzero, and $F_0 f$ is a multiple of $\lambda(0, s_0)$. On the other hand the Maass-Selberg relation, 4.4, implies that $F_0 f$ is a multiple of

$$F_0 j_0^* G(s_0) = v(0, s_0) + (j_0^* c)(s_0) \cdot \lambda(0, s_0) \quad \text{if} \quad s_0 \neq \frac{1}{2},$$

$$F_0 1 = v(0, s_0) \quad \text{if} \quad s_0 = \frac{1}{2}.$$

So we conclude that $f \in S^0[0, s_0]$. This gives i), and also ii), for $S^0\left[0, \frac{1}{2}\right] = 0$.

6.9 *Proof* of i) b) in proposition 6.5. In the case $s_0 \neq 0$ the assertion is given by lemma 6.8.

Let $s_0=0$. By [10], proposition 2. 1, p. 511, we know that $S^0[0,s]\neq 0$ only if $\frac{1}{4}-s^2\geqq\frac{3\pi^2}{2}$. So lemma 6. 8 gives holomorphy of c at $(0,0)$. By iv) b) in 6. 4 we get $c(0,0)\neq 0$. As

$$C(r,s)=-c(r,s)\{c(r,s)-s\}^{-1}$$

the holomorphy of C at $(0,0)$ follows.

6. 10 *Proof* of ii) in proposition 6. 5.

Remark that for $0<r<1$:

$$\omega^0\left(r,\frac{1}{2}(1-r)\right)=\zeta\left(r,\frac{1}{2}(1-r)\right)\cdot\left(\frac{\pi r}{3}\right)^{\frac{1}{2}r}.$$

So for η_r as in 2. 3:

$$F_0\eta_r\in\mathbb{C}\cdot\zeta\left(r,\frac{1}{2}(1-r)\right),$$

$$F_0\eta_r\neq 0.$$

From the discussion in 2. 13 follows that $S^0\left[r,\frac{1}{2}(1-r)\right]=0$, so by iii) in lemma 5. 4 and iii) a) and ii) a) in proposition 6. 1:

$$\eta_r(r)=a(r)\lim_{s\to\frac{1}{2}(1-r)}\left(s-\frac{1}{2}(1-r)\right)^q \cdot\frac{v^0(r,-s)}{v^0(r,-s)\,C(r,-s)-v^0(r,s)}\cdot G(r,s)$$

with $a(r)\in\mathbb{C}$ and $q\in\mathbb{Z}$. So, as $F_0\eta_r$ is a multiple of $\zeta\left(r,\frac{1}{2}(1-r)\right)$, we obtain

$$a(r)\cdot\lim_{s\to\frac{1}{2}(1-r)}\frac{\left(s-\frac{1}{2}(1-r)\right)^q}{C(r,-s)-\dfrac{v^0(r,s)}{v^0(r,-s)}}\neq 0,$$

$$a(r)\cdot\lim_{s\to\frac{1}{2}(1-r)}\frac{\left(s-\frac{1}{2}(1-r)\right)^q c(r,s)}{C(r,-s)-\dfrac{v^0(r,s)}{v^0(r,-s)}}=0.$$

So $s \mapsto c(r, s)$ has a zero at $\frac{1}{2}(1-r)$ for each $r \in (0, 1)$. So

$$c(r, s) = \left(\frac{1}{2} - s - \frac{1}{2} r\right)^m c_1(r, s)$$

with c_1 holomorphic near $\left(0, \frac{1}{2}\right)$ and $m \geqq 1$.

From 6. 3 follows: $\lim_{s \to \frac{1}{2}} \left(\frac{1}{2} - s\right)^{-1} c(0, s) = -\frac{\pi}{3}$, so $m=1$ and $c_1\left(0, \frac{1}{2}\right) = -\frac{\pi}{3}$.

6. 11 Proposition. *Define Λ^0 as the set of $\lambda \in \Lambda^c$ satisfying $F_0(r) f^i_{\lambda,0}(r) \neq 0$ for some $r \in (0, 12)$, for some $i \in [1, N_{\lambda,0}]$. If $\lambda \in \Lambda^0$, then $\liminf_{r \downarrow 0} \lambda(r) \geqq \frac{1}{4}$.*

We need a lemma.

6. 12 Lemma. *Put* $B(r, s) = \dfrac{\Gamma(1+2s)\, \Gamma\left(\frac{1}{2} - s - \frac{1}{2} r\right)}{\Gamma(1-2s)\, \Gamma\left(\frac{1}{2} + s - \frac{1}{2} r\right)} \cdot C(r, -s)$. *If* $\lambda \in \Lambda^0$ *and* $s: r \mapsto \sqrt{\frac{1}{4} - \lambda(r)}$ *is holomorphic at $r_0 \in (0, 1)$, then*

$$r \mapsto \left(\frac{\pi r}{3}\right)^{2s(r)} + B(r, s(r))$$

is identically zero on a neighbourhood of r_0.

Proof. By proposition 5. 7 the local curve $j: r - r_0 \mapsto (r, s(r))$ runs along a component of $\operatorname{Sing}(D^0_0)$, so

$$j^* \big((r, s) \mapsto v^0(r, -s)\, C(r, -s) - v^0(r, s)\big)$$

is zero. The explicit expression for v^0 in 4. 15 gives the lemma.

6. 13 *Proof* of proposition 6. 11. Consider $r_0 \in (0, 1)$ and $s_0 \in \left(0, \frac{1}{2}(1-r_0)\right)$ such that $\lambda(r_0) = \frac{1}{4} - s_0^2$. Now we apply lemma 6. 12 with $s(r_0) = s_0$.

From 6. 4 follows that $B(r, s)$ is negative for $r=0$ and $0 \leqq s < \frac{1}{2}$. Now suppose $\liminf_{r \downarrow 0} \lambda(r) = \frac{1}{4} - \alpha^2$ with $\alpha \in \left(0, \frac{1}{2}\right)$. Choose $\beta \in (0, \alpha)$. There are $\varepsilon_1 < 1$ and $\delta > 0$ such that $B < -\delta$ on $[0, \varepsilon_1] \times [\beta, \alpha]$. Take $\varepsilon_2 \in (0, \varepsilon_1)$ such that $\left(\frac{\pi \varepsilon_2}{3}\right)^{2\beta} < \delta$. Then for $(r, s) \in [0, \varepsilon_2] \times [\alpha, \beta]$:

$$\left(\frac{\pi r}{3}\right)^{2s} < \delta.$$

So the relation in lemma 6. 12 cannot be satisfied in this region.

Suppose $\liminf_{r\downarrow 0} \lambda(r)=0$. So we may choose (r_n, s_n) tending to $\left(0,\frac{1}{2}\right)$ satisfying $0<r_n<1$ and $0<s_n<\frac{1}{2}(1-r_n)$, for which

$$\left(\frac{\pi r_n}{3}\right)^{2s_n}+B(r_n, s_n)=0.$$

Now use ii) of proposition 6. 5 to obtain

$$c_1(r, s)=\frac{\Gamma(2-2s)\,\Gamma\left(\frac{1}{2}+s-\frac{1}{2}r\right)}{\Gamma(1+2s)\,\Gamma\left(\frac{3}{2}-s-\frac{1}{2}r\right)}\cdot\frac{B(r, s)+\frac{\pi r}{3}}{1-2s}.$$

So we would obtain

$$c_1\left(0,\frac{1}{2}\right)=\lim_{n\to\infty}\frac{\left(\frac{\pi r_n}{3}\right)^{2s_n}-\frac{\pi r_n}{3}}{2s_n-1}$$

$$=\lim_{n\to\infty}\left(\frac{\pi r_n}{3}\right)^{\sigma_n}\cdot\ln\left(\frac{\pi r_n}{3}\right)\qquad(2s_n\leqq\sigma_n\leqq 1)$$

$$=0,$$

in contradiction to $c_1\left(0,\frac{1}{2}\right)=-\frac{\pi}{3}$.

6. 14 Proposition. *If $\lambda\in\Lambda^0$, then λ has no holomorphic extension to a neighbourhood of* 0.

For Λ^0 see proposition 6. 11.

Proof. Suppose λ has a holomorphic extension at 0. Write $\lambda(0)=\frac{1}{4}-s_0^2$ with $\operatorname{Re}s_0=0$, see proposition 6. 11. Let j be a local curve through $(0, s_0)$ along the zero set of $(r, s)\mapsto\frac{1}{4}-s^2-\lambda(r)$. By 4. 12 we may assume $j(v)=(v^q, s_0+h(v))$, with $h(0)=0$.

By the definition of B in 6. 12 we see that

$$v\mapsto -B(v^q, s_0+h(v))$$

is meromorphic at 0, and by lemma 6. 12 given by

$$v\mapsto\left(\frac{\pi v^q}{3}\right)^{2s_0+h(v)}\quad\text{for}\quad\operatorname{Re}v^q>0.$$

This is only possible if $h(\,.\,)$ is constant, so $h=0$ and $q=1$ (see 4. 12). But then $2s_0 \in \mathbb{Z}$, so $s_0=0$. This means that $\lambda(r)=\frac{1}{4}$ for all $r\in(0,12)$, which is excluded by proposition 3. 1.

7. Non-extendable eigenvalues

For $\lambda \in \Lambda^0$, as defined in proposition 6. 11, we have already seen that $\liminf_{r\downarrow 0} \lambda(r) \geqq \frac{1}{4}$, see proposition 6. 11. In this section we continue the use of lemma 6. 12 to prove:

7. 1 Proposition. *If $\lambda\in\Lambda^0$, then either $\lim_{r\downarrow 0}\lambda(r)=\frac{1}{4}$ or $\lim_{r\downarrow 0}\lambda(r)=\infty$.*

We shall use:

7. 2 Lemma. *Let $\psi: D\to\mathbb{C}$, $D=\{z\in\mathbb{C}: |z|<\delta\}$, $\delta>0$, be real analytic and suppose that 0 is the only zero of ψ on D. Let $\alpha, \beta\in\mathbb{R}$ satisfy $0<\beta-\alpha<2\pi$; put $s=\{re^{i\phi}\in D: 0<r, \alpha<\phi<\beta\}$. Then each continuous real valued function a on S such that $\psi=|\psi|e^{ia}$ on S, is bounded.*

Proof. Each argument a is continuous and bounded outside each neighbourhood of 0 in S. So we may replace D by a smaller neighbourhood of 0 in the course of the proof. After replacing ψ by $u\cdot\psi$, with $u\in\mathbb{C}$, $u\neq 0$, we may assume that $\varrho=\operatorname{Re}\psi$ is nonzero on $i\mathbb{R}\cap D$. We shall prove the lemma for the case $\alpha=-\frac{1}{2}\pi$, $\beta=\frac{1}{2}\pi$. The general case then follows immediately.

Identify D with a subset of $\mathbb{R}^2$ by $z\mapsto(\operatorname{Re} z, \operatorname{Im} z)$. As ϱ is real analytic we may extend it holomorphically to a neighbourhood W of 0 in $\mathbb{C}^2$. Take

$$W=\{(x,y)\in\mathbb{C}^2: |x|<\delta_1, |y|<\delta_1\}.$$

If δ_1 is small enough, then the irreducible components in W of the germ of $N(\varrho)$ at 0 intersect each other only in 0. Furthermore, each irreducible component is the image of a map $j: u\mapsto(u^q, h(u))$ on a suitable neighbourhood U of 0 in $\mathbb{C}$, with $q\geqq 1$, q integral, h holomorphic, $h(0)=0$, $h'(u)\neq 0$ for all $u\in U$; see 4. 12. We can exclude $\{(u,0): u\in U\}$ as a component by an earlier assumption. By taking δ_1 small enough we arrange that $N(\varrho)\cap\mathbb{R}^2$ does not contain isolated points different from 0. So $N(\varrho)\cap\mathbb{R}^2$ is the union of finitely many sets of the form

$$\{(\varepsilon_1 r^q, k(r)): 0<r<\delta_2\},$$

with $q\geqq 1$, $\varepsilon_1=1$ or -1, k real analytic, $k'(r)\neq 0$ for all $r\in(0,\delta_2)$. If δ_1 is small enough these sets are disjoint. If $\varepsilon_1=1$ such a set is contained in S, if $\varepsilon_1=-1$ it does not meet S. So on $S\cap W$ the zero set of $\varrho=\operatorname{Re}\psi$ has finitely many connected components, and a can take only finitely many values in $\pi\mathbb{Z}$ on $S_1=S\cap W\cap N(\varrho)$. As $S\cap W$ is connected this shows that a is bounded.

7.3 Notations. We use $B(.,.)$ as in lemma 6.12. We fix $\lambda \in \Lambda^0$ and define $t\colon (0,1) \to \mathbb{R}$ by

$$t(r) = \begin{cases} -\sqrt{\frac{1}{4} - \lambda(r)} \in \left(\frac{1}{2}(r-1), 0\right) & \text{if} \quad \lambda(r) < \frac{1}{4}, \\ \sqrt{\lambda(r) - \frac{1}{4}} \in [0, \infty) & \text{if} \quad \lambda(r) \geqq \frac{1}{4}. \end{cases}$$

Further

$$L = \lim_{r \downarrow 0} t(r) \quad \text{if it exists,}$$

$$LS = \limsup_{r \downarrow 0} t(r), \qquad LI = \liminf_{r \downarrow 0} t(r).$$

7.4 We know already $LI \geqq 0$, and want to prove that L exists and equals 0 or ∞. We shall do this by proving:

i) If $LI < \xi$ with $\xi > 0$, then $LS \leqq \xi$ (lemma 7.8).

ii) If $0 < LS < \infty$, then $LI < LS$ (lemma 7.9).

7.5 Lemma. *Put* $X = \left[0, \frac{1}{2}\right] \times i[0, \infty) \subset Y(0)$,

$$X_e = X \cap \operatorname{Sing}(B), \quad X_g = X \setminus X_e.$$

i) $|B(r, s)| = 1$ *for all* $(r, s) \in X_g$.

ii) X_e *is discrete in* X.

For $Y(0)$ see 6.1.

Proof. Remark that $\operatorname{Sing}(B) \cap X = \operatorname{Sing}\left(\frac{1}{C}\right) \cap X$. For $(r, s) \in X_g$:

$$|C(r, s)|^2 = C(r, s)\, C(r, -s) = 1,$$

see proposition 6.1 ii); the Γ-factors give a contribution of absolute value 1 as well. So i) follows.

To prove ii) it is sufficient to show that $X \cap \operatorname{Sing}(B) \subset \operatorname{Indet}(B)$. Let $p \in X_e$, ψ holomorphic at p, nonzero, such that ψB is holomorphic at p. If $\psi B(p) \neq 0$, then $|\psi B(q)| > \delta > 0$ for all $q \in X_g$, q near p; so $|\psi(q)| > \delta$ for those q. This implies $\psi(p) \neq 0$. But if ψ were invertible as holomorphic function, then it could not be used to make ψB holomorphic. So $\psi B(p) = 0$ for all such ψ.

7.6 If Z is a simply connected open subset of X_g, then we may choose a continuous function A on Z such that $B = e^{iA}$ on Z. If in addition the closure $\bar{Z}$ is compact and contained in X, then $\bar{Z}$ may contain only a finite number of points of X_e. Lemma 7.2 ensures that A stays bounded near those boundary points in X_e. So we have:

7.7 Lemma. *Let* $Z = (0, \zeta) \times i(\lambda, \mu)$ *with* $0 < \zeta < \frac{1}{2}$ *and* $0 < \lambda < \mu$. *If* $Z \cap X_e = \emptyset$, *then there exists a bounded continuous function* $A\colon Z \to \mathbb{R}$ *such that* $B = e^{iA}$ *on* Z.

7.8 Lemma. *Let $\xi>0$. If $LI<\xi$, then $LS\leqq\xi$.*

Proof. Suppose $LI<\xi<LS$. Choose in lemma 7.7: $\zeta=\frac{1}{2}$ and $LI<\lambda<\mu<LS$, such that $Z=\left(0,\frac{1}{2}\right)\times i(\lambda,\mu)$ satisfies $Z\cap X_e=\emptyset$; see ii) of lemma 7.5. By continuity the graph of it crosses Z infinitely often. Considering downward crossings only we find a sequence of disjoint intervals $(\lambda_n,\varrho_n)\subset\left(0,\frac{1}{2}\right)$, such that

$$\lambda_n\downarrow 0,\quad \varrho_n\downarrow 0,\quad t(\lambda_n)=\mu,\quad t(\varrho_n)=\lambda,$$
$$(r,it(r))\in Z\quad\text{for all}\quad r\in(\lambda_n,\varrho_n).$$

By lemma 6.12, with A chosen as in lemma 7.7, there are $l_n\in\mathbb{Z}$ such that

$$2t(r)\ln\left(\frac{\pi r}{3}\right)+\pi+2\pi l_n+A(r,it(r))=0\quad\text{for}\quad r\in(\lambda_n,\varrho_n).$$

So there are $C>0$ and $D_n\in\mathbb{R}$ such that

$$D_n\leqq 2t(r)\ln\left(\frac{\pi r}{3}\right)\leqq D_n+C\quad\text{for}\quad r\in(\lambda_n,\varrho_n).$$

By continuity of t:

$$D_n\leqq 2\mu\ln\left(\frac{\pi\lambda_n}{3}\right),$$
$$2\lambda\ln\left(\frac{\pi\varrho_n}{3}\right)\leqq D_n+C.$$

So $-C\leqq 2(\lambda-\mu)\ln\left(\frac{\pi\lambda_n}{3}\right)$, in contradiction to $\lambda_n\downarrow 0$.

7.9 Lemma. *If $0<LS<\infty$, then $LI<LS$.*

Proof. Suppose $0<LI=LS<\infty$. Take $0<\lambda<LS<\mu<\infty$ and $\zeta\in\left(0,\frac{1}{2}\right)$ such that

$$\lambda<t(r)<\mu\quad\text{for all}\quad r\in(0,\zeta)$$

and such that $Z\cap X_e=\emptyset$ for $Z=(0,\zeta)\times(\lambda,\mu)$. By the lemmas 6.12 and 7.7 we know that $2t(r)\ln\left(\frac{\pi r}{3}\right)$ is bounded for $r\in(0,\zeta)$. On the other hand for $r\in(0,\zeta)$:

$$-\infty=\lim_{r\downarrow 0}2\lambda\ln\left(\frac{\pi r}{3}\right)\geqq\liminf_{r\downarrow 0}2t(r)\ln\left(\frac{\pi r}{3}\right).$$

8. Proofs of global results

We now may prove theorem 2. 15 iii), proposition 2. 17 i) and ii) a), and most of theorem 2. 21. The key results are proposition 2. 8 in [2] and proposition 6. 14.

8. 1 We return to the notations introduced in section 2. As in 3. 2 we extend the $\lambda \in \Lambda_l$ as holomorphic functions on some open set U_λ in $\mathbb{C}$ containing (0, 12) such that we also have analytic extensions

$$f^i_{\lambda,l}\colon U_\lambda \times \mathfrak{h} \to \mathbb{C},$$

holomorphic in the first variable. This has been discussed in [2], 3. 4 and proposition 3. 6 for the case $l=0$. Holomorphy in L^2-sense and pointwise holomorphy coincide in this case.

8. 2 In proposition 3. 1 we have seen that it makes sense to consider $\Lambda^c = \Lambda_l \setminus \Lambda^d_l$. We already introduced

$$\Lambda^n = \{\lambda \in \Lambda^c\colon\ \lambda \text{ not extendable to a neighbourhood of } 0\},$$
$$\Lambda^0 = \{\lambda \in \Lambda^c\colon\ F_0(r) f^i_{\lambda,0}(r) \neq 0 \text{ for some } r \in (0, 12) \text{ for some } i\}.$$

We put $\Lambda^e = \Lambda^c \setminus \Lambda^n$.

In proposition 6. 14 we proved $\Lambda^0 \subset \Lambda^n$.

8. 3 Proposition. *Let* $\lambda \in \Lambda^c$.

i) $\Lambda^c \setminus \Lambda^0 \subset \Lambda^e$.

ii) *If* $N_{\lambda,0} \geqq 2$, *then* $\lambda \in \Lambda^e$.

Proof. i) is part of proposition 2. 8 in [2], and from this proposition ii) follows as well, as soon as we construct a nontrivial holomorphic linear combination f of the $f^i_{\lambda,0}$ with $F_0(r) f(r) = 0$ on some interval $J \subset (0, 12)$.

${}^\circ W\left(r, \frac{r}{12}, \frac{1}{4} - s^2\right)$ is a one-dimensional space with basis element $\omega^0(r, s)$, as discussed in [3], 2. 7—2. 9; consult [3], 2. 13 to see that $W^0[r, s]$ in [3] coincides with ${}^\circ W\left(r, \frac{r}{12}, \frac{1}{4} - s^2\right)$ defined in 2. 5. So

$$F_0(r) f^i_{\lambda,0}(r) = a_i(r)\, \omega^0\left(r, \sqrt{\frac{1}{4} - \lambda(r)}\right).$$

From proposition 3. 1 follows that $\lambda(r)$ is not identically equal to $\frac{1}{4}$. So we may pick an interval $J \subset (0, 12)$ on a neighbourhood of which $f^i_{\lambda,0}$ and $\omega^0\left(r, \sqrt{\frac{1}{4} - \lambda(r)}\right)$ are holomorphic in r. Then [3], 5. 7, implies that the a_i are also holomorphic near J. The construction of an f as desired now is easy.

8. 4 Definition. By $\mathcal{O}$ we denote in this section the sheaf of holomorphic functions on $\mathbb{C}$.

For $l \in 2\mathbb{Z}$, $\lambda \in \mathcal{O}(U)$ we denote by $\mathcal{S}_{\lambda,l}$ the $\mathcal{O}$-module on U with as sections on $V \subset U$ analytic functions $f: V \times \mathfrak{h} \to \mathbb{C}$, holomorphic in the first variable such that $f(r, .) \in S(r+l, r, \lambda(r))$ for all $r \in V$. This notation differs from the one used in sections 4—7. Here it is convenient to take r as variable.

By $\mathcal{S}^0_{\lambda,l}$ we denote the submodule of $\mathcal{S}_{\lambda,l}$ determined by $F_0(r) f(r) = 0$ for all $r \in V$.

8. 5 Proposition. *Let $\lambda \in \Lambda^e$. There exists a neighbourhood $\tilde{U}_\lambda$ of $(-12, 12)$ such that λ has an extension $\lambda \in \mathcal{O}(\tilde{U}_\lambda)$. We may assume that the $f^i_{\lambda,0}$ have extensions $f^i_{\lambda,0} \in \mathcal{S}^0_{\lambda,0}(\tilde{U}_\lambda)$, $1 \leqq i \leqq N_{\lambda,0}$, which at $r \in (-12, 12)$ generate the stalks $(\mathcal{S}^0_{\lambda,0})_r$ as $\mathcal{O}_r$-modules, such that for $r \in (-12, 12)$ the $f^i_{\lambda,0}(r)$ form an orthonormal system.*

Proof. Proposition 6. 14 implies that the $f^i_{\lambda,0} \in \mathcal{S}^0_{\lambda,0}(U_\lambda)$. Again proposition 2. 8 in [2] shows the existence of an extension $\lambda \in \mathcal{O}(\tilde{U}_\lambda)$ as stated in the proposition, and moreover, of sections $\psi_1, \ldots, \psi_m$ of $\mathcal{S}^0_{\lambda,0}$ on $\tilde{U}_\lambda$, generating the stalks at real points. The proof of this proposition in 7. 13 of [2] shows that the ψ_i may be taken orthonormal at real r. So their number equals $N_{\lambda,0}$ and they may serve as $f^i_{\lambda,0}$.

8. 6 In proposition 8. 5 we use weight r only. It is convenient to relate this to other weights:

8. 7 Lemma. *Let $\lambda \in \Lambda^c$.*

i) *$N_\lambda = N_{\lambda,l}$ does not depend on $l \in 2\mathbb{Z}$.*

ii) *If $\lambda \in \Lambda^e$ we may assume the $f^i_{\lambda,l}$ to be extended to $f^i_{\lambda,l} \in \mathcal{S}^0_{\lambda,l}(\tilde{U}_\lambda)$, orthonormal at $r \in (-12, 12)$.*

Proof. i) follows from proposition 3. 1 and the discussion in 2. 8.

If the $f^i_{\lambda,0} \in \mathcal{S}^0_{\lambda,0}(\tilde{U}_\lambda)$ have been chosen we determine all $f^i_{\lambda,l}$ by

$$E^+_{r+l} f^i_{\lambda,l}(r) = -2\sqrt{\lambda(r) + \frac{1}{2}(r+l) + \frac{1}{4}(r+l)^2}\, f^i_{\lambda,l+2}(r),$$

$$E^-_{r+l} f^i_{\lambda,l}(r) = +2\sqrt{\lambda(r) + \frac{1}{2}(r+l-2) + \frac{1}{4}(r+l-2)^2}\, f^i_{\lambda,l-2}(r).$$

If we have chosen U_λ or $\tilde{U}_\lambda$ small enough, then the root expressions depend holomorphically on r, use proposition 3. 1. The root expressions are positive for real r; the orthonormality is preserved by the adjointness of $E^\pm_q$ and $-E^\mp_{q\pm 2}$.

8. 8 Lemma. *Let $\lambda \in \Lambda^e$.*

i) *There is a unique $\mu \in \Lambda^c$ such that $\mu(r) = \lambda(r-12)$ for all $r \in U_\mu \cap (\tilde{U}_\lambda + 12)$.*

ii) *$N_\lambda = N_\mu$.*

iii) *For each $\nu \in \mathbb{Z}$ the following conditions are equivalent:*

a) *$F_{\nu+1}(r) f^i_{\lambda,0}(r) = 0$ for all $r \in (0, 12)$ and all $i \in [1, N_\lambda]$,*

b) *$F_\nu(r) f^i_{\mu,0}(r) = 0$ for all $r \in (0, 12)$ and all $i \in [1, N_\mu]$.*

Proof. $S(r, r, \lambda(r-12)) = S((r-12)+12, r-12, \lambda(r-12))$ for all $r \in \tilde{U}_\lambda + 12$. So the $r \mapsto f^i_{\lambda,12}(r-12)$ are sections in $\mathcal{S}^0_{\mu,0}(\tilde{U}_\lambda + 12)$, with $\mu(r) = \lambda(r-12)$. On $(0, 12)$ the $r \mapsto f^i_{\lambda,12}(r-12)$ form analytic families of cusp forms. If $\mu \notin \Lambda_0$, then the graph of μ would intersect the union of the graphs of all $\kappa \in \Lambda_0$ in only countably many points. This gives a contradiction to proposition 2.14. From proposition 3.1 follows that $\mu \in \Lambda^c$. This gives i).

The orthonormality of the $f^i_{\lambda,12}(r-12)$ for $0 < r < 12$ shows that $N_\mu \geqq N_\lambda$. By proposition 8.5 we may near $r \in (0, 12)$ express the $f^i_{\mu,0}(r)$ in the $f^i_{\lambda,0}(r-12)$; this gives $N_\mu \leqq N_\lambda$.

As $F_\nu(r) = F_{\nu+1}(r-12)$, we may rewrite a) in iii) as

a$_1$) $F_\nu(r+12) f^i_{\lambda,0}(r) = 0$ for all $r \in (0, 12)$ and $i \in [1, N_\lambda]$.

As $F_\nu(r+12)$ commutes with the Maass operators, see 2.4, we see that a$_1$) is equivalent to

a$_2$) $F_\nu(r+12) f^i_{\lambda,12}(r) = 0$ for all $r \in (0, 12)$ and $i \in [1, N_\lambda]$.

In proposition 4.24 of [3] we have seen that the Fourier coefficients are holomorphic in r; so we may replace $(0, 12)$ in a$_2$) by $(-12, 0)$. As the $f^i_{\lambda,12}(r-12)$ and the $f^i_{\mu,0}(r)$ may be expressed in each other, equivalence to b) follows.

8.9 Definition. For $\lambda \in \Lambda^e$ define $\varepsilon\lambda \in \Lambda^c$ by $\varepsilon\lambda(r) = \lambda(r-12)$.

Lemma 8.8 i) shows that $\varepsilon: \Lambda^e \to \Lambda^c$ is a function, it is clearly injective. Its inverse may be described in the following way:

8.10 Definition. $Jf(z) = f(-\bar{z})$ for $f: \mathfrak{h} \to \mathbb{C}$.

This is the operator X in [13], IV, (18), p. 179. Remark that $JS(q, r, \mu) = S(-q, -r, \mu)$ for $q, r, \mu \in \mathbb{C}$.

So if $\lambda \in \Lambda^c$, then the $r \mapsto Jf^i_{\lambda,-12}(12-r)$ are sections of $\mathcal{S}_{\mu,0}$ for $\mu(r) = \lambda(12-r)$. Hence $\mu \in \Lambda^c$.

8.11 The resulting map $\Lambda^c \to \Lambda^c$ we denote by ι:

$$\iota\lambda(r) = \lambda(12-r).$$

8.12 Clearly $\iota\varepsilon\iota: \varepsilon\Lambda^e \to \Lambda^c$ is the inverse of $\varepsilon: \Lambda^e \to \varepsilon\Lambda^e \subset \Lambda^c$.

ε and its inverse $\iota\varepsilon\iota$ together describe how far $\lambda \in \Lambda^c$ may be extended analytically. Each $\lambda \in \Lambda^c$ occurs in a sequence

$$\ldots, \lambda_2, \lambda_1, \lambda = \lambda_0, \lambda_{-1}, \lambda_{-2}, \ldots$$

with $\varepsilon\lambda_i = \lambda_{i+1}$. The sequence is finite if and only if $\lambda_i \in \Lambda^n$ for some $i \geqq 0$ and $\lambda_i \in \iota\Lambda^n$ for some $i \leqq 0$.

We shall see that indeed for $\lambda \in \Lambda^c$ the corresponding sequence is finite.

8. 13 Lemma. *Let* $f \in S(0, 0, \mu)$, $\mu \in \mathbb{R}$.

If $F_\nu(0)f = 0$ *for all* $\nu \geqq 0$, *then* $f = 0$.

Proof. Consider the Hecke operators acting in $S(0, 0, \mu)$:

$$T_p h(z) = h(pz) + \sum_{b=0}^{p-1} h\left(\frac{z+b}{p}\right), \qquad p \text{ prime}.$$

See e.g. [13], V, § 3.

The T_p act on the Fourier coefficients of positive and negative order separately; so we may suppose that f is a simultaneous eigenfunction of the T_p. Now consider $f + Jf$ and $f - Jf$; these are simultaneous eigenfunctions for all T_p and for J. The eigenvalues for each T_p are the same, but for J they differ. To each simultaneous eigenfunction is associated a representation of $GL_2(\mathbb{A})$ in $L^2(Z_{\mathbb{A}} GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$, with Z the center of GL_2 and $\mathbb{A}$ the adele ring of $\mathbb{Q}$; see § 5 of [6]. The finite components of this representation are determined by the eigenvalues of the T_p, the infinite one by $\frac{1}{4} - s^2$ and the eigenvalue of J. Proposition 6. 22 in [6] states that the finite components determine the infinite one. So our assumptions on f would violate multiplicity one.

8. 14 Lemma. *Let* $\lambda \in \Lambda^c$.

i) *There exists* $m \in \mathbb{Z}$, $m \geqq 0$, *such that* $\varepsilon^m \lambda \in \Lambda^n$.

ii) *There exists* $k \in \mathbb{Z}$, $k \geqq 0$, *such that* $(\iota\varepsilon\iota)^k \lambda \in \iota\Lambda^n$.

iii) $N_\lambda = 1$.

Proof. Suppose $\varepsilon^m \lambda \in \Lambda^e$ for all $m \geqq 0$. In proposition 6. 14 we have seen that $\Lambda^0 \subset \Lambda^n$, so

$$F_0(r)\, f^i_{\varepsilon^m \lambda, 0}(r) = 0 \quad \text{for all} \quad r \in (0, 12) \quad \text{and} \quad i \in [1, N_{\varepsilon^m \lambda}].$$

This implies by iii) of lemma 8. 8:

$$F_m(r)\, f^i_{\lambda, 0}(r) = 0 \quad \text{for all} \quad r \in (0, 12) \quad \text{and} \quad i \in [1, N_\lambda].$$

By analyticity of the Fourier coefficients we get

$$F_m(0)\, f^1_{\lambda, 0}(0) = 0 \quad \text{for all} \quad m \geqq 0.$$

So $f^1_{\lambda, 0}(0) = 0$ by lemma 8. 13, in contradiction to ii) of lemma 8. 7. This gives i).

We get ii) by applying i) to $\iota\lambda$.

By ii) of proposition 8. 3 we know $N_{\varepsilon^m \lambda} = 1$ if $m \geqq 0$ is minimal such that $\varepsilon^m \lambda \in \Lambda^n$. Now iii) follows from ii) of lemma 8. 8.

8. 15 *Proof* of theorem 2. 15 iii). See iii) in lemma 8. 14.

8. 16 *Proof* of proposition 2. 17 i) and ii) a). See proposition 8. 3 i) and lemma 8. 7 ii) to obtain i). Part ii) a) has already been shown in proposition 6. 14.

8. 17 *Proof* of theorem 2. 21, except i) d).

In 8. 12 and lemma 8. 14 we have seen that each element of Λ^c occurs in a maximal sequence $\lambda_n, \ldots, \lambda_2, \lambda_1$, with $\varepsilon\lambda_i = \lambda_{i+1}$ for $1 \leqq i \leqq n-1$. This means that $\lambda_n \in \Lambda^n$ has a holomorphic extension to some neighbourhood Ω, depending on λ_n, of $(0, 12n)$, which we may choose simply connected, and

$$\lambda_i(r) = \lambda_n(r + 12n - 12i), \quad 1 \leqq i \leqq n.$$

This shows i) a) and ii) a) of theorem 2. 21. The maximality of the sequence gives $\iota\lambda_1 \in \Lambda^n$, so i) c) of the theorem follows.

8. 18 Define $t_{i,l}$ and V_i for $1 \leqq i \leqq n$, $l \in 2\mathbb{Z}$ by

$$V_n = U_{\lambda_n}, \quad t_{n,l} = f_{\lambda_n,l},$$
$$V_i = \tilde{U}_{\lambda_i} + 12(n-i), \quad t_{i,l}(r) = f_{\lambda_i, l+12(n-i)}(r - 12(n-i)), \quad 1 \leqq i \leqq n-1.$$

We may assume that all V_i and $V_i \cap V_{i+1}$ are simply connected, that

$$V_i \subset \{r \in \mathbb{C}\colon 12(n-i-1) < \operatorname{Re} r < 12(n-i+1)\}$$

and that $\Omega_{\lambda_n} = \bigcup_{i=1}^{n} V_i$.

From the lemmas 8. 7 and 8. 8 follows that

$$t_{i,l} = \psi_{i,l} t_{i+1,l} \quad \text{on } V_i \cap V_{i+1}, \quad 1 \leqq i \leqq n-1,$$

with $\psi_{i,l} \in \mathcal{O}(V_i \cap V_{i+1})^*$. (See 8. 4 for $\mathcal{O}$.)

We have $H^1(\Omega_{\lambda_n}, \mathcal{O}^*) \cong H^2(\Omega_{\lambda_n}, \mathbb{Z}) = 0$, see e.g. [11], p. 181. This implies the existence of $\phi_{i,l} \in \mathcal{O}(V_i)^*$ such that $\phi_{i,l}\psi_{i,l} = \phi_{i+1,l}$ on $V_i \cap V_{i+1}$. So we may define g_l on Ω_{λ_n} by

$$g_l = \phi_{i,l} t_{i,l} \quad \text{on} \quad V_i.$$

This g_l is nonzero at each $r \in (0, 12n)$. Its norm in $L(r, r)$ is positive and is real analytic in r. So $r \mapsto \|g_l(r)\|^{-1}$ has a holomorphic extension to some neighbourhood of $(0, 12n)$. So take Ω_{λ_n} small enough and take $f_{\lambda_n,l}$ in i) b) equal to $\|g_l\|^{-1} g_l$. Part ii) b) is clear from the construction.

8. 19 This glueing argument could have been avoided by taking a more global viewpoint in [2], at the cost of some notational and conceptual complications.

9. Perturbation theory

We still have to prove theorem 2. 15 ii), theorem 2. 21 i) d) and proposition 2. 19.

In this section we prove proposition 2. 19. To do this we have to go back to the proofs in [2]. We also prove a result on the derivatives of eigenvalues, which we need in section 10.

9. 1 We work in the case $q=r$ only, so we use the notations of section 4.

9. 2 The main idea in [2] is to apply analytic perturbation theory to the operator $A_r = A_{r,r}$ in $L^2(r, r)$. To do this we had to map these spaces unitarily onto a fixed Hilbert space.

In [2], section 3, this was done by a map

$$L^2(r, r) \to L^2(0, 0): f \mapsto e^{-irt} f$$

with t a suitable real valued function on $\mathfrak{h}$. The differential operators L_r and $E_r^{\pm}$ correspond under this transformation to operators $L(r)$ and $E^{\pm}(r)$ described in [2], 3. 4 and 3. 11.

9. 3 If we are interested in $r \in (0, 12)$, we may study the sesquilinear form

$$s(r)\,[\phi, \psi] = -\frac{1}{4} r^2 \langle \phi, \psi \rangle + \frac{1}{8} \sum_{\pm} \langle E^{\pm}(r)\phi, E^{\pm}(r)\psi \rangle$$

in $L^2(0, 0)$. But if $r \in (-12, 12)$, then we need to work in a subspace ${}^aH$ of $L^2(0, 0)$ defined by requiring the Fourier coefficient of order zero to vanish above the level $y=a$ for some $a>5$.

In ${}^aH$ the form $s(r)$ is used to construct a selfadjoint holomorphic family of operators ${}^aA(\,.\,)$ on a neighbourhood ${}^aW$ of $(-12, 12)$.

Eigenfunctions of ${}^aA(r)$ in ${}^aH$ with eigenvalue $\frac{1}{4} - s^2$ are of the form $e^{-irt}.\,{}^{(a)}f$ with $f \in A^0[r, s]$ such that $F_0 f(ia) = 0$. Here we use the truncation

$${}^{(a)}h(z) = \begin{cases} h(z) & \text{if} \quad z \in F,\ y \leqq a, \\ h(z) - F_0\, h(z) & \text{if} \quad z \in F,\ y > a, \end{cases}$$

${}^{(a)}h$ extended to $\mathfrak{h}$ in such a way that it satisfies $(a)_{r,r}$ in 2. 2. This correspondence follows from [2], 6. 13, 6. 14 and the proof of 7. 8. In particular, $S^0[r, s]$ corresponds to a subspace of $\ker\left({}^aA(r) - \frac{1}{4} + s^2\right)$.

${}^aA$ has compact resolvent. This implies that we may use strong results from holomorphic perturbation theory.

9. 4 In the situation of proposition 2. 19 we are interested in $s_0 \in i\mathbb{R}$, $\frac{1}{2} - s_0^2 > \frac{1}{4}$, so $s_0 \neq 0$; cf. proposition 6. 5.

In [12], VII, § 3. 5 and § 4. 7, remark 4. 22 we see that, if ${}^aW$ is small enough, there are holomorphic functions $\lambda_1, \ldots, \lambda_m$ on ${}^aW$ and $f_1, \ldots, f_m$ holomorphic on ${}^aW$ with values in ${}^aH$, such that

a) $({}^aA(r) - \lambda_i(r))\, f_i(r) = 0 \qquad i = 1, \ldots, m, \quad r \in {}^aW,$

b) $\lambda_i(0) = \frac{1}{4} - s_0^2 \qquad i = 1, \ldots, m,$

c) $f_1(r), \ldots, f_m(r)$ form an orthonormal system for each $r \in {}^aW \cap \mathbb{R}$,

d) there exists $\varepsilon > 0$ such that if $r \in {}^aW \cap \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfies $\left|\lambda - \frac{1}{4} + s_0^2\right| < \varepsilon$, then $\{f_i(r) : \lambda_i(r) = \lambda\}$ is a basis of $\ker({}^aA(r) - \lambda)$.

As $s_0 \neq 0$, we may assume that $\lambda_i(r) = \frac{1}{4} - s_i(r)^2$ with $s_i(0) = s_0$ and s_i holomorphic on ${}^aW$.

Application of [2], lemma 7. 14 shows that we may rearrange the f_i with $\lambda_i = \lambda$ for some fixed function λ in such a way that at most one of them has non-vanishing Fourier coefficient $F_0 f_i$.

Define $j_i: {}^aW \to Y$ by $j_i(r) = (r, s_i(r))$. We arrive at:

$$h_l \in (\mathscr{A}^0_{j_l})({}^aW), \quad F_0 h_l \neq 0, \quad F_0 h_l(ia) = 0 \quad \text{for} \quad l = 1, \ldots, u,$$
$$g_l \in (\mathscr{S}^0_{j_l})({}^aW), \quad l = u+1, \ldots, m$$

such that

$$f_l = \begin{cases} e^{-irt \cdot (a)} h_l & \text{if} \quad 1 \leqq l \leqq u, \\ e^{-irt \cdot (a)} g_l & \text{if} \quad u+1 \leqq l \leqq m. \end{cases}$$

9. 5 Definition. For $s \in i\mathbb{R}$ define $S^f\{s\}$ and $S^e\{s\}$ by

i) $S^f\{s\}$ is the subspace of $S^0[0, s]$ spanned by

$$\left\{f_{\lambda,0}(0) : \lambda \in \Lambda^e,\ \lambda(0) = \frac{1}{4} - s^2\right\}.$$

ii) $S^0[0, s] = S^f\{s\} \oplus S^e\{s\}$ (orthogonal decomposition).

9. 6 We want to prove that $S^e\{s_0\} \neq 0$ is equivalent to $(0, s_0) \in \operatorname{Sing}(C)$.

Remark that the λ_l with $u+1 \leqq l \leqq m$ in 9. 4 are just the $\lambda \in \Lambda^e$ with $\lambda(0) = \frac{1}{4} - s_0^2$. So we have i) of:

9. 7 Lemma. i) *$g_{u+1}(0), \ldots, g_m(0)$ is an orthonormal basis of $S^f\{s_0\}$.*

ii) *$S^e\{s_0\}$ is contained in the space spanned by $h_1(0), \ldots, h_u(0)$.*

Proof of ii). $S^0[0, s]$ is an eigenspace of A_0 in $L^2(0, 0)$ and is contained in the corresponding eigenspace of ${}^aA(0)$.

9. 8 Lemma. *Equivalent are:*

a) *$u \geqq 1$ for all choices of $a > 5$.*

b) $(0, s_0) \in \operatorname{Sing}(C)$.

c) $(0, s_0) \in \operatorname{Sing}(E)$.

Proof. b) $\Leftrightarrow$ c) by lemma 6. 7. We prove b) $\Rightarrow$ a) and a) $\Rightarrow$ c).

Suppose b). Proposition 6.5 implies $(0, s_0) \in \operatorname{Indet}(C)$. So we may take a local curve j through $(0, s_0)$ along a component of the zero set of

$$(r, s) \mapsto \mu^0(r, s; ia) + C(r, s)\, \mu^0(r, -s; ia).$$

Remark that $\mu^0(0, w; iy) = y^{\frac{1}{2}+w}$, see [3], 2.13.

Now $f = {}^{(a)}(j^* E)$ corresponds to a family of eigenfunctions of ${}^aA$, holomorphic on a neighbourhood of 0. As in the proof of lemma 4.14 we conclude that j may be taken $j: w \mapsto (w, s(w))$ with s holomorphic and $s(0) = s_0$. So f may be expressed in the functions discussed in 9.4; as $F_0 f$ is not identically zero, we need one h_l. Hence a).

Suppose a). Take $a > 5$ such that

$$a^{\frac{1}{2}+s_0} + j_0^* C(s_0) \cdot a^{\frac{1}{2}-s_0} \neq 0,$$

with $j_0: s \mapsto (0, s)$; cf. 6.4. If $(0, s_0) \notin \operatorname{Sing}(E)$, then we get

$$F_0(r)\, E(r, s; ia) \neq 0 \quad \text{for} \quad (r, s) \quad \text{near} \quad (0, s_0).$$

For $1 \leqq l \leqq u$ we have $F_0(r)\, h_l(r) = c_l(r)\, F_0(r)\, E(r, s_l(r))$, provided r near 0; the $c_l(r)$ are in $\mathbb{C}$. As $F_0(r)\, h_l(r; ia) = 0$, we get $c_l = 0$. So $F_0 h_l = 0$, in contradiction to the arrangements in 9.4.

9.9 We now know (see 9.7 and 9.8):

$$S^e\{s_0\} \neq 0 \Rightarrow u \geqq 1 \quad \text{for all } a > 5 \Leftrightarrow (0, s_0) \in \operatorname{Sing}(C).$$

By the remark in 9.6 we have proved proposition 2.19 as soon as we show:

$$(0, s_0) \in \operatorname{Sing}(C) \Rightarrow S^e\{s_0\} \neq 0.$$

9.10 We now assume $(0, s_0) \in \operatorname{Sing}(C)$ and take $a > 5$ as assumed in the proof of lemma 9.8; this leaves a lot of freedom for a. We fix $\psi \in \mathcal{O}_{(0, s_0)}$, $\psi \neq 0$, minimal such that ψC is holomorphic at $(0, s_0)$.

Our aim is to prove that $h_1(0), \ldots, h_u(0)$ are elements of $S^e\{s_0\}$. We do this by expressing the h_l in E.

9.11 Lemma. *Put $\chi_l(r, s) = s_l(r) - s$ and $j_l: r \mapsto (r, s_l(r))$ for $1 \leqq l \leqq u$. There is a neighbourhood U of $(0, s_0)$ such that*

$$N(\chi_l) \cap \operatorname{Sing}(C) \cap U = \{(0, s_0)\}.$$

Proof. For U small enough either $N(\chi_l) \cap U \subset \operatorname{Sing}(C)$ or the intersection contains $(0, s_0)$ only.

Suppose $N(\chi_l) \cap U \subset \operatorname{Sing}(C)$. For the local curve j_l along $N(\chi_l)$ we know that $F_0\, j_l^*(\psi E)$ is proportional to $F_0 h_l$. This implies, for r near 0:

$$j_l^*(\psi C)(r) \cdot \mu^0(r, -s_l(r); ia) = 0.$$

Hence $j_l^*(\psi C) = 0$. So χ_l divides ψC in $\mathcal{O}_{(0, s_0)}$; by supposition χ_l also divides ψ. This contradicts the minimality of ψ.

9. 12 Lemma. *There exist a neighbourhood V of 0 in $\mathbb{C}$ and holomorphic functions $\gamma_1, \ldots, \gamma_u$ on $V \setminus \{0\}$ such that*

$$h_l = \gamma_l \cdot j_l^* E \quad \text{on} \quad V \setminus \{0\}, \qquad 1 \leqq l \leqq u.$$

Proof. As $j_l^* E$ is holomorphic on $V \setminus \{0\}$ the Maass-Selberg relation, 4. 4, implies that $F_0 h_l$ and $F_0 j_l^* E$ are proportional. So

$$F_0(r)\, h_l(r) = \gamma_l(r) \cdot F_0(r)\, j_l^* E(r) \quad \text{for} \quad r \in V \setminus \{0\},$$

with $\gamma_l(r) \in \mathbb{C}$. By holomorphy of the Fourier coefficients we know that γ_l is meromorphic on V, cf. proposition 4. 24 in [3]. So by making V smaller we get γ_l holomorphic on $V \setminus \{0\}$.

On $V \setminus \{0\}$ we may express $\phi_l = h_l - \gamma_l \cdot j_l^* E$ as holomorphic linear combination of some g_i. So for $r \in \mathbb{R} \cap V$, $r \neq 0$:

$$\begin{aligned} 0 &= \langle e^{-irt} \cdot {}^{(a)}\phi_l(r),\, e^{-irt} \cdot {}^{(a)}h_l(r) \rangle \quad \text{see } 9.\,4 \\ &= \langle \phi_l(r),\, {}^{(a)}h_l(r) \rangle \quad \text{in} \quad L^2(r, r) \\ &= \gamma_l(r) \cdot \langle \phi_l(r),\, {}^{(a)}E(r, s_l(r)) \rangle + \langle \phi_l(r), \phi_l(r) \rangle. \end{aligned}$$

By lemma 8. 3 in [3]:

$$\langle \phi_l(r),\, {}^{(a)}E(r, s_l(r)) \rangle = 0.$$

So $\phi_l(r) = 0$ for infinitely many r, so $\phi_l = 0$.

9. 13 Lemma. *The γ_l and $\gamma_l \cdot j_l^* C$, $1 \leqq l \leqq u$, have a holomorphic extension to* 0.

The γ_l have been defined in 9. 12.

Proof. For $r \in V \setminus \{0\}$:

$$F_0 h_l(r) = \gamma_l(r) \cdot \mu^0(r, s_l(r)) + \gamma_l(r) \cdot j_l^* C(r) \cdot \mu^0(r, -s_l(r)).$$

As $\mu^0(r, s)$ and $\mu^0(r, -s)$ form a basis of $W^0[r, s]$ near p, see [3], 5. 4, 5. 7, we see that γ_l and $\gamma_l \cdot j_l^* C$ extend holomorphically to 0.

9. 14 Lemma. $\gamma_l(0) = 0$, *$j_l^* C$ is holomorphic in* 0, *for* $1 \leqq l \leqq u$.

Proof. For $r \in V \cap \mathbb{R}$, $r \neq 0$, we have $\frac{1}{4} - s_l(r)^2 \in \mathbb{R}$, for

$$r \mapsto \frac{1}{4} - s_l(r)^2$$

is an eigenvalue of a selfadjoint family of operators. As $s_0 \in i\mathbb{R}$, $s_0 \neq 0$, we see

$$|C(r, s_l(r))|^2 = C(r, s_l(r))\, C(r, -s_l(r)) = 1,$$

see proposition 6. 1, ii). So $j_l^* C$ cannot be singular at 0.

Suppose $\gamma_l(0) \neq 0$. The Maass-Selberg relation, 4. 4, implies that for some $\alpha_l \in \mathbb{C}$:

$$F_0 h_l(0, iy) = \alpha_l (y^{\frac{1}{2}+s_0} + j_0^\# C(s_0) y^{\frac{1}{2}-s_0}),$$

with $j_0 : s \mapsto (0, s)$. On the other hand

$$F_0 h_l(0, iy) = \gamma_l(0) \cdot (y^{\frac{1}{2}+s_0} + j_l^\# C(0) y^{\frac{1}{2}-s_0}).$$

Now remember the choice of a in the proof of lemma 9. 8. Taking $y = a$ in the first relation gives $\alpha_l = 0$. So $F_0 h_l(0) = 0$ and the second relation gives $\gamma_l(0) = 0$.

9. 15 *Proof* of proposition 2. 19. As discussed in 9. 6 and 9. 9 we have to check that

$$S^e\{s_0\} \neq 0.$$

But lemma 9. 14 implies $F_0 h_l(0) = 0$, so the $h_l(0)$ all are elements of $S^0[0, s_0]$, orthogonal to the $g_i(0)$ which span $S^f\{s_0\}$. So $h_1(0), \ldots, h_u(0)$ form a non-empty orthonormal basis of $S^e\{s_0\}$.

9. 16 Proposition. *Put*

$$e_\pm(z) = -\frac{3}{\pi} + y - 24 \sum_{n=1}^{\infty} \sigma_1(n) y e^{-2\pi n y \pm 2\pi i n x}.$$

For $\lambda \in A^c$ and $r \in (0, 12)$:

$$\frac{1}{2} r + \lambda'(r) = \frac{\pi}{12} \langle E_r^- f_{\lambda,0}(r), e_- \cdot f_{\lambda,0}(r) \rangle$$

$$- \frac{\pi}{12} \langle E_r^+ f_{\lambda,0}(r), e_+ \cdot f_{\lambda,0}(r) \rangle.$$

Remarks. $\sigma_a(n) = \sum_{d|n} d^a$. $e_\pm$ is the value at $s = \frac{1}{2}$ of the continuation of the Eisenstein series in weight ± 2:

$$\frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} e^{\mp 2i \arg(cz+d)} \cdot \left(\frac{y}{|cz+d|^2} \right)^{\frac{1}{2}+s}.$$

So $e_\pm \in A[\pm 2, 0, 0]$. The $f_{\lambda,0}(r)$ are quickly decreasing at the cusp, so the scalar products in the proposition make sense.

9. 17 Lemma. *Let $\tau \in C^\infty(\Gamma \backslash \mathfrak{h})$ be real-valued and let τ, τ_x and τ_y be exponentially decreasing for $y \to \infty$. Then for $\lambda \in A^c$ and $r \in (0, 12)$:*

$$\mathrm{Re}[\langle E_r^- f_{\lambda,0}(r), i(E_0^- \tau) \cdot f_{\lambda,0}(r) \rangle - \langle E_r^+ f_{\lambda,0}(r), i(E_0^+ \tau) \cdot f_{\lambda,0}(r) \rangle] = 0.$$

Remarks. The conditions on τ ensure convergence of the scalar products.

We shall use this lemma to connect proposition 9. 16 to the perturbational results in [2].

Proof. As $f_{\lambda,0}(r)$ is quickly decreasing, we may use $-E^{\mp}_{r\pm 2}$ as adjoint of $E^{\pm}_r$. Further we use:

$$E^{\pm}_r(fg) = g\cdot E^{\pm}_r f + f\cdot E^{\pm}_0 g.$$

We obtain, with $f(r)=f_{\lambda,0}(r)$:

$$\begin{aligned}\sum_{\pm}\mp\langle E^{\pm}_r f(r), (E^{\pm}_0\tau)\cdot f(r)\rangle &= \sum_{\pm}\mp\langle E^{\pm}_r f(r), E^{\pm}_r\{\tau\cdot f(r)\} - \tau\cdot E^{\pm}_r f(r)\rangle\\ &= \sum_{\pm}\pm\langle\{-4\lambda(r)-r^2\mp 2r\}\, f(r), \tau\cdot f(r)\rangle\\ &\quad +\sum_{\pm}\pm\langle E^{\pm}_r f(r), \tau\cdot E^{\pm}_r f(r)\rangle,\end{aligned}$$

which is real as τ is real. We used:

$$L_r = -\frac{1}{4}E^{\mp}_{r\pm 2}E^{\pm}_r - \frac{1}{4}r^2 \mp \frac{1}{2}r.$$

9. 18 Some functions used in [2] are: t, occurring in the transformation $f\mapsto e^{-irt}f$, and $b_{\pm} = 2iyt_y \mp 2yt_x \pm 1 = \pm 1 + i(E^{\pm}_0 t)$.

We may check in lemma 3. 2 of [2] that

$$t = 2\,\mathrm{Im}\log\eta + \tau,$$

with η the Dedekind eta function and $\tau\in C^{\infty}(\Gamma\backslash\mathfrak{h})$ satisfying the assumptions in lemma 9. 17. From

$$\log\eta(z) = \frac{\pi i z}{12} - \sum_{n\geq 1}\sigma_{-1}(n)e^{2\pi i n z}$$

we see that $e_{\pm} = -\frac{3}{\pi}\mp\frac{3i}{\pi}E^{\pm}_0(2\,\mathrm{Im}\log\eta)$. So $b_{\pm} = \mp\frac{\pi}{3}e_{\pm} + iE^{\pm}_0\tau$, and a computation shows that

$$E^{\mp}_{\pm 2}e_{\pm} = \frac{6}{\pi}.$$

9. 19 *Proof* of proposition 9. 16. We use section 6 of [2] with $B=\emptyset$; so we need no truncation, cf. 6. 15 in [2]. Let $\phi(r) = e^{-irt}f_{\lambda,0}(r) = e^{-irt}f(r)$. So ϕ is an eigenfamily for the eigenvalue λ of the family of operators associated to the sesquilinear form

$$s(r)\,[\psi,\chi] = -\frac{1}{4}r^2\langle\psi,\chi\rangle + \frac{1}{8}\sum_{\pm}\langle E^{\pm}(r)\psi, E^{\pm}(r)\chi\rangle.$$

As shown in [2], 6. 9—6. 12, the derivative of s is given by

$$s'(r)\,[\psi,\chi] = \frac{1}{8}\sum_{\pm}\{\langle E^{\pm}(r)\psi, e^{\mp 2it}b_{\pm}\chi\rangle + \langle e^{\mp 2it}b_{\pm}\psi, E^{\pm}(r)\chi\rangle\} - \frac{1}{2}r\langle\psi,\chi\rangle,$$

so if we apply [12], VII, § 4. 6, (4. 44) we obtain for those $r \in (0, 12)$ for which $\lambda(r)$ has multiplicity one:

$$\begin{aligned}
\lambda'(r) &= -\frac{1}{2} r \langle \phi(r), \phi(r) \rangle \\
&\quad + \frac{1}{8} \sum_{\pm} \{ \langle E^{\pm}(r)\, \phi(r), e^{\mp 2it} b_{\pm}\, \phi(r) \rangle \\
&\quad + \langle e^{\mp 2it} b_{\pm}\, \phi(r), E^{\pm}(r)\, \phi(r) \rangle \} \\
&= -\frac{1}{2} r + \frac{1}{8} \sum_{\pm} \{ \langle e^{-i(r\pm 2)t} E_r^{\pm} f(r), e^{-i(r\pm 2)t} b_{\pm} f(r) \rangle \\
&\quad + \langle e^{-i(r\pm 2)t} b_{\pm} f(r), e^{-i(r\pm 2)t} E_r^{\pm} f(r) \rangle \} \qquad \text{(see [2], 3. 11)} \\
&= \frac{1}{4} \sum_{\pm} \operatorname{Re} \langle E_r^{\pm} f(r), b_{\pm} f(r) \rangle - \frac{1}{2} r \\
&= \frac{\pi}{12} \sum_{\pm} \{ \mp \operatorname{Re} \langle E_r^{\pm} f(r), e_{\pm} f(r) \rangle \} - \frac{1}{2} r \\
&\quad + \frac{1}{4} \operatorname{Re} \sum_{\pm} \langle E_r^{\pm} f(r), i(E_0^{\pm} \tau) \cdot f(r) \rangle \\
&= -\frac{1}{2} r + \operatorname{Re} \frac{\pi}{12} \{ \langle E_r^{-} f(r), e_{-} f(r) \rangle - \langle E_r^{+} f(r), e_{+} f(r) \rangle \} \qquad \text{(see lemma 9. 17.)}.
\end{aligned}$$

Remark that

$$\begin{aligned}
\overline{\langle E_r^{-} f(r), e_{-} \cdot f(r) \rangle} &= \langle e_{-} \cdot f(r), E_r^{-} f(r) \rangle \\
&= -\langle E_{r-2}^{+} (e_{-} \cdot f(r)), f(r) \rangle = -\langle e_{-} \cdot E_r^{+} f(r) + f(r) \cdot E_{-2}^{+} e_{-}, f(r) \rangle \\
&= -\langle E_r^{+} f(r), e_{+} \cdot f(r) \rangle - \frac{6}{\pi}, \qquad \text{see 9. 18.}
\end{aligned}$$

So we may omit Re and obtain proposition 9. 16 for infinitely many $r \in (0, 12)$ forming a dense subset.

9. 20 We conclude the proof by continuity. By [2], lemma 6. 8, it is sufficient to show that $r \mapsto \phi(r) = e^{-irt} f(r)$ is continuous with respect to the norm

$$\|\chi\|_s^2 = \sum_{\pm} \|E^{\pm}(r_0)\chi\|^2$$

for $r_0 \in (0, 12)$. We even prove holomorphy:

For ζ large the resolvent $R(-\zeta, r) = (A(r) + \zeta)^{-1}$ gives a bounded holomorphic family of operators

$$\{L^2(0, 0) \textit{ with } \|.\|\} \to \{\operatorname{dom}(s(r_0)) \textit{ with } \|.\|_s\},$$

as follows from the reasoning around (4. 6) in [12], VII, § 4. 2. So

$$r \mapsto \phi(r) = (\lambda(r) + \zeta)\, R(-\zeta, r)\, \phi(r)$$

is holomorphic in the norm $\|\,.\,\|_s$.

9. 21 Remark. In the case $0<r<12$ the proofs in section 6 of [2] could have been given using $t = 2\,\mathrm{Im}\log\eta$. But the Fourier series expansion would be jumbled, which would be disastrous if we take $B \neq \emptyset$ in [2], section 6.

10. Estimates of eigenvalues

Now we prove the results in 2. 15 ii), 2. 17 ii) b) and 2. 21 i) d) by giving estimates of λ and λ' on (0, 12) for all $\lambda \in \Lambda^c$. We shall need weights $q = r + l$ for various $l \in 2\mathbb{Z}$, so we use the notations introduced in section 2.

Our main results are propositions 10. 1 and 10. 2.

10. 1 Proposition. *Put* $y_0 = \frac{1}{2}\sqrt{3}$ *and*

$$\mu_0(r) = \max_{1 \leqq n \leqq 5} \left[\frac{1}{8}\left\{ 2\pi y_0 - \left| 2\pi y_0 \left(1 - \frac{1}{6} r\right) + r - 2n \right| \right\}^2 - \frac{1}{4}\{r - 2n\}^2 \right].$$

For each $r \in (0, 12)$ *and* $\lambda \in \Lambda^c$:

$$\lambda(r) \geqq \mu_0(r).$$

This improves the estimate $\lambda(r) > \lambda_0(r)$, for $r \in (r_1, 12 - r_1)$ with

$$r_1 = \frac{1}{6}\{-4\pi y_0 + (32\pi^2 y_0^2 - 96\pi y_0 + 144)^{\frac{1}{2}}\} \left(1 - \frac{1}{3}\pi y_0\right)^{-2} \approx 1.0998.$$

(For λ_0 see 2. 11.)

10. 2 Proposition. *Let* $\varepsilon_0 = 24\, y_0\, e^{-2\pi y_0}(1 - e^{-2\pi y_0})^{-3} \approx 0.0912$,

$$\varepsilon_1 = \left[4\varepsilon_0^2 + \left\{ 2\varepsilon_0 + 2\left(\frac{3}{\pi} - y_0\right) \right\}^2 \right]^{\frac{1}{2}} \approx 0.4039,$$

$$\delta(\lambda, r) = \max_{1 \leqq n \leqq 5} \left[-\frac{1}{2} r + n - \frac{1}{12}\pi\varepsilon_1 \{4\lambda + (r - 2n)^2\}^{\frac{1}{2}} - \{4\lambda + (r - 2n)^2\}\,(12 - r)^{-1} \right]$$

for $\lambda > 0$ *and* $0 \leqq r \leqq 2$. *Define* μ_1 *as the solution on* [0, 2] *of*

$$\mu'(r) = \delta(\mu(r), r)$$

with initial value $\mu(0) = \frac{1}{4}$.

Then $\lambda(r) \geqq \mu_1(r)$ *for all* $\lambda \in \Lambda^c$ *and all* $r \in (0, 2]$.

Remarks. In these propositions we have $1 \leqq n \leqq 5$. This is related to the absence of discrete series type cusp forms in the weights $q = r + l$ with $0 < r < 12$ and $l = -10, -8, \ldots, -2$.

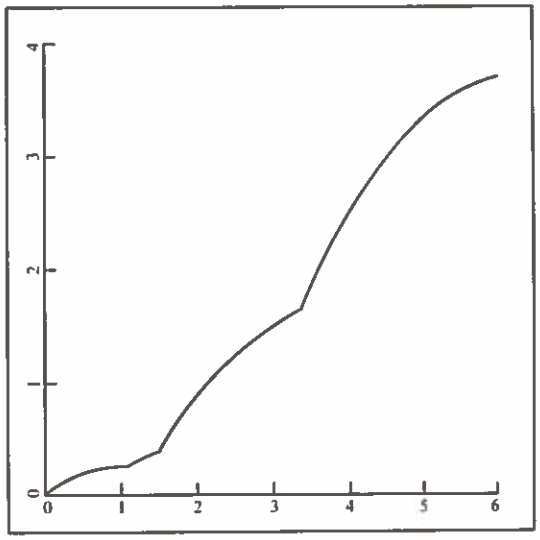

Figure 1

In figure 1 *the graph of* $r \mapsto \max(\lambda_0(r), \mu_0(r))$ is drawn. We consider only $0 \leqq r \leqq 6$, in view of 8. 11.

In figure 2 one finds the graphs of μ_1 and $\max(\lambda_0, \mu_0)$ on $[0, 2]$. The graph of μ_1 has been obtained by numerical integration with steplength 0.002.

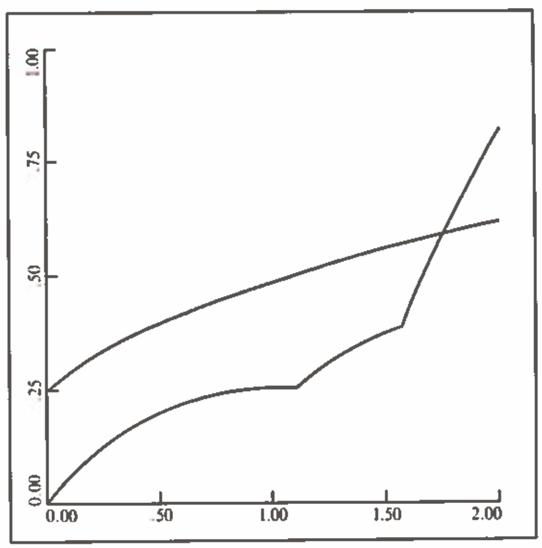

Figure 2

10. 3 *Proof* of proposition 2. 15 ii) from propositions 10. 1 and 10. 2. By the symmetry $r \mapsto 12-r$ discussed in 8. 10, we need only consider $r \in (0, 6]$. If we take $n=2$ in the expression defining μ_0, we see that for $r \in [2, 6]$:

$$\mu_0(r) \geqq \frac{1}{8}\left\{4 - r\left(1 - \frac{1}{3}\pi y_0\right)\right\}^2 - \frac{1}{4}(r-4)^2$$

$$\geqq \textit{the value at } r = 6 > \frac{1}{4}.$$

By proposition 10. 2 it is sufficient to show that $\mu_1(r) > \frac{1}{4}$ for $r \in (0, 2]$. To obtain this we show that $\delta\left(\frac{1}{4}, r\right) > 0$ for $0 \leqq r \leqq 2$.

For $0 \leqq r \leqq 2$, $1 \leqq n \leqq 5$, $q = r - 2n$:

$$\delta\left(\frac{1}{4}, r\right) \geqq -\frac{1}{2} q - \frac{1}{12} \pi \varepsilon_1 \left(1 + \frac{1}{2} q^2\right) - \frac{1}{10}(1 + q^2)$$
$$= -\left(\frac{1}{24} \pi \varepsilon_1 + \frac{1}{10}\right) q^2 - \frac{1}{2} q - \left(\frac{1}{12} \pi \varepsilon_1 + \frac{1}{10}\right).$$

As the polynomial in the right hand side is positive for $-\frac{5}{2} \leqq q \leqq -\frac{1}{2}$, it suffices to take $n = 1$ and 2 to get $\delta\left(\frac{1}{4}, r\right) > \frac{1}{4}$ for $0 \leqq r \leqq 2$.

10. 4 We shall deduce proposition 10. 2 from the following lemma.

10. 5 Lemma. *Let δ be as in proposition* 10. 2. *For each $\lambda \in \Lambda^c$:*

$$\lambda'(r) \geqq \delta(\lambda(r), r) \quad \textit{for} \quad r \in (0, 2].$$

10. 6 *Proof* of proposition 2. 17 ii) b) and theorem 2. 21 i) d). By the discussion in 8. 9—8. 11 it is clear how to derive 2. 21 i) d) from 2. 17 ii) b). We prove the latter result from lemma 10. 5.

By proposition 7. 1, 8. 2 and proposition 8. 3 it is sufficient to exclude the possibility that $\lim_{r \downarrow 0} \lambda(r) = \infty$ for some $\lambda \in \Lambda^n$.

Suppose $\lambda(r) \geqq \mu_2$ for all $r \in (0, r_1]$ for some $r_1 \in (0, 2)$ and some $\mu_2 > 0$. If μ_2 is large enough, then $\delta(\mu, r) \geqq -a\mu$ for all $\mu \geqq \mu_2$ and $r \in [0, 2]$ for some $a > 0$. So by lemma 10. 5

$$\lambda(r) \leqq \lambda(r_1) e^{a(r_1 - r)} \quad \text{for all} \quad r \in (0, r_1].$$

10. 7 *Proof* of proposition 10. 2 from lemma 10. 5. For $\lambda \in \Lambda^n$ we have $\lim_{r \downarrow 0} \lambda(r) = \frac{1}{4}$ (from proposition 2. 17 ii) b)) and for $\lambda \in \Lambda^e$ we know $\lim_{r \downarrow 0} \lambda(r) = \lambda(0) \geqq \frac{3\pi^2}{2} > \frac{1}{4}$, see [10], proposition 2. 1, p. 511.

By lemma 10. 5 we have for $\lambda \in \Lambda^c$ on (0, 2]:

$$\lambda(r) \geqq \mu_\lambda(r)$$

with μ_λ the solution of

$$\begin{cases} \mu'(r) = \delta(\mu(r), r), \\ \mu(0) = \lim_{r \downarrow 0} \lambda(r). \end{cases}$$

Indeed, suppose $\lambda < \mu_\lambda$ on $(a, b] \subset (0, 2]$, then we may take a such that either $a = 0$ or $\lambda(a) = \mu_\lambda(a)$. Then we would get

$$0 > \lambda(b) - \mu_\lambda(b) \geqq \int_a^b \{\delta(\lambda(r), r) - \delta(\mu_\lambda(r), r)\} dr \geqq 0,$$

for $\delta(u, r)$ is decreasing in u.

If $\lambda \in \Lambda^n$ then $\mu_\lambda = \mu_1$, so the assertion of proposition 10. 2 follows. For the case $\lambda \in \Lambda^e$ use $\lambda(0) > \frac{1}{4}$ and the monotony of the solutions of the differential equation in the initial value.

10. 8 The only task left is to prove proposition 10. 1 and lemma 10. 5.

10. 9 Lemma. *Let $r \in (0, 12)$, $\lambda \in \Lambda^c$, then for all $l \in 2\mathbb{Z}$*

$$\frac{1}{2}(r + l) + \lambda'(r) = \frac{\pi}{12} \sum_{\pm} \mp \langle E^{\pm}_{r+l} f_{\lambda,l}(r), e_{\pm} \cdot f_{\lambda,l}(r) \rangle.$$

Proof. By proposition 9. 16 it is sufficient to prove that

$$D_l = -\frac{1}{2} l + \frac{\pi}{12} \sum_{\pm} \mp \langle E^{\pm}_{r+l} f_{\lambda,l}(r), e_{\pm} \cdot f_{\lambda,l}(r) \rangle$$

does not depend on $l \in 2\mathbb{Z}$.

We use the notation $f_{\lambda,l}(r) = f_l$. As in the proof of lemma 8. 7 we may assume that the f_l have been chosen in such a way that

$$E^+_{r+l} f_l = -2 v_l f_{l+2}, \quad E^-_{r+l} f_l = +2 v_{l-2} f_{l-2},$$

with $v_l = \sqrt{\lambda(r) + \frac{1}{2}(r + l) + \frac{1}{4}(r + l)^2}$. So

$$D_l = -\frac{1}{2} l + \frac{\pi}{6} \cdot v_{l-2} \overline{a_{l-2}} + \frac{\pi}{6} \cdot v_l a_l$$

with $a_l = \langle f_{l+2}, e_+ \cdot f_l \rangle$; we used $\overline{e_-} = e_+$. So we want to prove:

$$v_l \overline{a_l} + v_{l+2} a_{l+2} - v_{l-2} \overline{a_{l-2}} - v_l a_l = \frac{6}{\pi}.$$

Using the adjointness of E^+_q and $-E^-_{q+2}$ we see

$$\begin{aligned} -2 v_l a_l &= \langle E^+_{r+l} f_l, e_+ \cdot f_l \rangle \\ &= -\langle f_l, (E^-_2 e_+) \cdot f_l + e_+ \cdot E^-_{r+l} f_l \rangle \\ &= -\frac{6}{\pi} - 2 v_{l-2} a_{l-2}, \quad \text{see 9. 18.} \end{aligned}$$

This completes the proof.

10. 10 To prove proposition 10. 1 and lemma 10. 5 we now give estimates in the spirit of the proof of the well known fact that for $r=0$ all continuous series eigenvalues are larger than $\frac{3\pi^2}{2}$; cf. [10], proposition 2. 1, p. 511. In fact, proposition 10. 1 is obtained by applying the same proof. For lemma 10. 5 more complicated estimates are needed.

10. 11 We fix $r \in (0, 12)$, $\lambda \in \Lambda^c$, $l \in 2\mathbb{Z}$. We write $q=r+l$, $f=f_{\lambda,l}(r)$ and $\mu=\lambda(r)$. The adjointness of $E_q^{\pm}$ and $-E_{q\pm 2}^{\mp}$ implies

$$2(4\mu+q^2)=\|E_q^+ f\|^2+\|E_q^- f\|^2.$$

10. 12 Notation. For functions χ on $\mathfrak{h}$ put

$$N_0(\chi)=\int\limits_{|x|\leqq\frac{1}{2},\, y\geqq y_0} |\chi|^2\, d\mu, \qquad y_0=\frac{1}{2}\sqrt{3},$$

$$N_1(\chi)=\int\limits_{|x|\leqq\frac{1}{2},\, y\geqq 1} |\chi|^2\, d\mu,$$

$$N_F(\chi)=\int\limits_F |\chi|^2\, d\mu,$$

with $F=\left\{z\in\mathbb{C}\colon |z|\geqq 1,\, |x|\leqq\frac{1}{2}\right\}$ the standard fundamental domain of Γ and $d\mu(z)=y^{-2}dx\wedge dy$.

So $N_1(\chi)\leqq N_F(\chi)\leqq N_0(\chi)$ and $N_0(\chi)\leqq 2N_F(\chi)$ if $|\chi|$ is Γ-invariant.

10. 13 Put $g=2yf_y$, $h=qf+2iyf_x$; so $E_q^{\pm}f=g\pm h$. Hence

$$4\mu+q^2=N_F(g)+N_F(h)\leqq N_0(g)+N_0(h)\leqq 2(4\mu+q^2).$$

10. 14 The Fourier expansion of f is:

$$\sum_n{}^{*} f_n(y)e^{2\pi inx}$$

where $\sum_n^{*}$ denotes summation over $n\equiv\frac{1}{12}r \bmod 1$.

The f_n are analytic in y and for each $a>0$:

$$\int_a^\infty |f_n(y)|^2 y^{-2}\, dy<\infty.$$

10. 15 Put

$$A(n)=\int_{y_0}^\infty |f_n(y)|^2 y^{-2}\, dy,\quad B(n)=\int_{y_0}^\infty (q-4\pi ny)^2\, |f_n(y)|^2 y^{-2}\, dy.$$

Then

$$1\leqq\sum_n{}^{*} A(n)=N_0(f)\leqq 2,$$

$$\sum_n{}^{*} B(n)=N_0(h)\leqq 2(4\mu+q^2).$$

10. 16 Lemma. *Put* $\alpha_l(r) = \inf_{y \geqq y_0,\, n \equiv \frac{r}{12} \bmod 1} |q - 4\pi n y|$.

i) $\alpha_l(r) > 0 \Leftrightarrow -l - 4\pi y_0 < \left(1 - \frac{1}{3}\pi y_0\right) r < -l$.

ii) *If* $\alpha_l(r) > 0$ *then*

$$-10 \leqq l \leqq -2, \quad \alpha_l(r) = 2\pi y_0 - \left| 2\pi y_0 \left(1 - \frac{1}{6} r\right) + q \right|.$$

iii) $\alpha_l(r)^2 \leqq 2(4\mu + q^2)$.

Proof. $\alpha_l(r) = \max\left\{0, \min\left(q - 4\pi\left(\frac{r}{12} - 1\right) y_0, -q + \frac{1}{3}\pi r y_0\right)\right\}$; from this i) and ii) follow easily. For iii) remark that $B(n) \geqq \alpha_l(r)^2 A(n)$ for all $n \equiv \frac{r}{12} \bmod 1$ and

$$\alpha_l(r)^2 \leqq \alpha_l(r)^2 \sum_n{}^{*} A(n) \leqq \sum_n{}^{*} B(n) \leqq 2(4\mu + q^2);$$

see 10. 15.

10. 17 *Proof* of proposition 10. 1. Lemma 10. 16 iii) implies

$$\lambda(r) \geqq \frac{1}{8}\alpha_l(r)^2 - \frac{1}{4}(r + l)^2,$$

and by ii) only $l \in [-10, -2]$ may be interesting. So take the maximum over these values.

10. 18 Lemma. *Put* $p(z) = y - 3\pi^{-1} - e_+(z)$; *then* $e_-(z) = y - 3\pi^{-1} - \overline{p(z)}$ *and for all* $z \in F$ *and* ε_0 *as in proposition* 10. 2:

$$|p(z)| \leqq \varepsilon_0.$$

Proof. $e_\pm(z) = y - \frac{3}{\pi} - 24y \sum_{m=1}^{\infty} \sigma_1(m) e^{2\pi i m(\pm x + iy)}$.

So the first statement is clear and the latter one follows from

$$\sum_{m=1}^{\infty} \sigma_1(m) q^m = \sum_{a,b \geqq 1} a q^{ab} \leqq \sum_{b \geqq 1} q^b (1 - q^b)^{-2} \leqq q(1 - q)^{-3}$$

for all $q \in (0, 1)$.

10. 19 *Proof* of lemma 10. 5. In view of proposition 9. 16 and lemma 10. 9 it is sufficient to prove for $-10 \leqq l \leqq -2$:

$$X := \sum_{\pm} \mp \langle E_q^{\pm} f, e_\pm f \rangle \geqq -\varepsilon_1 (4\mu + q^2)^{\frac{1}{2}} - \frac{\pi^{-1}(4\mu + q^2)}{1 - \frac{1}{12} r}.$$

With g and h as in 10. 13:

$$X = \int_F [-g(p - \bar{p}) + h(p + \bar{p}) + 2h(3\pi^{-1} - y)] \bar{f}\, d\mu.$$

By lemma 10. 18:

$$\left|\iint_F g\bar{f}(\bar{p}-p)\,d\mu\right| \leqq 2\varepsilon_0 N_F(g)^{\frac{1}{2}}, \quad \left|\iint_F h\bar{f}(p+\bar{p})\,d\mu\right| \leqq 2\varepsilon_0 N_F(h)^{\frac{1}{2}}.$$

Further

$$2\int_F (3\pi^{-1}-y)h\bar{f}\,d\mu = X_1 + X_2$$

corresponding to $F = F_1 \cup F_2$ with $F_1 = \{z \in F\colon y \leqq 1\}$ and $F_2 = \{z \in F\colon y \geqq 1\}$.

$$|X_1| \leqq 2(3\pi^{-1} - y_0)\, N_F(h)^{\frac{1}{2}}$$

So $|X - X_2| \leqq 2\varepsilon_0 N_F(g)^{\frac{1}{2}} + 2(\varepsilon_0 + 3\pi^{-1} - y_0)\, N_F(h)^{\frac{1}{2}}$. As $N_F(g) + N_F(h) = 4\mu + q^2$ we obtain

$$|X - X_2| \leqq \varepsilon_1 (4\mu + q^2)^{\frac{1}{2}}, \text{ with } \varepsilon_1 \text{ as in proposition 10. 2.}$$

Now we consider X_2 for $0 < r \leqq 2$ and $-10 \leqq l \leqq -2$:

$$X_2 = 2 \sum_n{}^{\bullet} \int_1^\infty (3\pi^{-1} - y)\,(q - 4\pi n y)\, |f_n(y)|^2 y^{-2}\, dy.$$

All terms with $n > 0$ are non-negative, so

$$X_2 \geqq 2 \sum_{n<0}{}^{\bullet} \int_1^\infty \left\{\frac{1}{4\pi n}(q - 4\pi n y)^2 + \left(\frac{3}{\pi} - \frac{q}{4\pi n}\right)(q - 4\pi n y)\right\} |f_n(y)|^2 y^{-2}\, dy$$

$$\geqq \frac{1}{2\pi\left(-1 + \frac{r}{12}\right)} \sum_{n<0}{}^{\bullet} B(n) + 2 \sum_{n<0}{}^{\bullet} \left(\frac{3}{\pi} - \frac{q}{4\pi n}\right) \int_1^\infty (q - 4\pi n y)\, |f_n(y)|^2 y^{-2}\, dy,$$

with $B(\,.\,)$ as in 10. 15. For $n < 0$ and $y \geqq 1$:

$$q - 4\pi n y \geqq r - 10 - 4\pi\left(-1 + \frac{r}{12}\right) > 4\pi - 10 + \left(1 - \frac{1}{3}\pi\right) 2 > 0,$$

$$\frac{3}{\pi} - \frac{q}{4\pi n} = \frac{q - 12n}{4\pi |n|} \geqq \frac{r - 10 + 12 - r}{4\pi |n|} > 0.$$

So

$$X_2 \geqq \frac{6}{\pi(r-12)} \sum_{n<0}{}^{\bullet} B(n) \geqq \frac{6}{\pi(r-12)} \sum_n{}^{\bullet} B(n) \geqq \frac{-12}{\pi}\, \frac{4\mu + q^2}{12 - r}$$

see 10. 15. Finally

$$X \geqq X_2 - |X - X_2| \geqq -\pi^{-1}(4\mu + q^2)\left(1 - \frac{r}{12}\right)^{-1} - \varepsilon_1 (4\mu + q^2)^{\frac{1}{2}}.$$

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