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A new approach to the spectral theory of the fourth moment of the Riemann zeta-function

By *Roelof W. Bruggeman* at Utrecht and *Yoichi Motohashi* at Tokyo

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Abstract. The aim of the present article is to exhibit a new proof of the explicit formula for the fourth moment of the Riemann zeta-function that was established by the second named author a decade ago. Our proof is new, particularly in that it dispenses altogether with the spectral theory of sums of Kloosterman sums that played a predominant rôle in the former proof. Our argument is, instead, built directly upon the spectral structure of the space $L^2(\Gamma\backslash G)$, with $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $G = \mathrm{PSL}_2(\mathbb{R})$. The discussion below thus seems to provide a new insight into the nature of the Riemann zeta-function, especially in its relation with automorphic forms over linear Lie groups that has been perceived by many.

The plan of the paper is as follows: In Section 1 we discuss salient points of the former proof and describe the explicit formula in a conventional fashion. In Section 2 its reformulation is presented in terms of automorphic representations occurring in $L^2(\Gamma\backslash G)$. With this, our motivation is precisely related. We then proceed to our new proof. In Section 3 we construct a Poincaré series over G whose value at the unit element is close to the non-diagonal part of the fourth moment in question. In Section 4 we develop an account of the Kirillov scheme, with which, in Section 5, projections of the Poincaré series into irreducible subspaces of $L^2(\Gamma\backslash G)$ are explicitly calculated in terms of the seed function. Then, in Section 6 a limiting procedure with respect to the seed is performed, and we reach a basic spectral expression. This ends in effect our proof of the explicit formula, since it remains to appeal to a process of analytic continuation, which is, however, the same as the corresponding part of the former proof, and can largely be omitted.

Notations are introduced where they are needed for the first time, and will continue to be effective thereafter. The parameters $C > 0$ and $\varepsilon > 0$ are assumed locally to be constants arbitrarily large and small, respectively. The dependency of implicit constants on them can be inferred from the context.

1. Explicit formula

We formulate the k-th moment of the Riemann zeta-function by

$$\mathcal{M}_k(\zeta; g) = \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + it\right)\right|^k g(t)\, dt, \tag{1.1}$$

where k is an arbitrary fixed number, and the weight function g is assumed, for the sake of simplicity but without much loss of generality, to be even, entire, real on $\mathbb{R}$, and of fast decay in any fixed horizontal strip. These quantities have been regarded as major subjects in Analytic Number Theory not only because they have important applications to a variety of classical problems such as the distribution of prime numbers but also as they are indispensable means to reveal the intriguing nature of the Riemann zeta-function. Laying stress upon the latter aspect, we shall deal with the case $k = 4$, the fourth moment, which has perhaps the richest history of investigations in the theory of mean values of the zeta-function.

The explicit formula for $\mathscr{M}_4(\zeta; g)$ appeared first in [17] and later in [19] with certain sophistication. It resulted from an attempt to generalize the method with which F. V. Atkinson [1] arrived at an explicit formula for the unweighted mean square. Atkinson's formula can indeed be reckoned as the first explicit result in the theory of mean values of zeta and L-functions. In retrospect, the novelty of his idea rests in that he saw a lattice structure in a certain non-diagonal expression which is similar to (1.6) below. That probably originated in the works by H. Weyl and J. G. van der Corput on the estimation of trigonometrical sums. To make his observation effective, Atkinson appealed to two fundamental implements, analytic continuation and the Poisson sum formula. Because of its essential relevance to our present purpose, we shall summarize Sections 4.1–4.3 of [19] and indicate how his observation extends to the fourth moment situation:

Atkinson's argument, when applied to $\mathscr{M}_2(\zeta; g)$, starts with the introduction of the function

$$(1.2) \qquad I(w_1, w_2; g) = \int_{-\infty}^{\infty} \zeta(w_1 - it)\zeta(w_2 + it)g(t)\,dt, \\ = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} a^{-w_1} b^{-w_2} \hat{g}\left(\frac{1}{2\pi}\log\frac{a}{b}\right),$$

where $(w_1, w_2) \in \mathbb{C}^2$ is in the region of absolute convergence, and

$$(1.3) \qquad \hat{g}(x) = \int_{-\infty}^{\infty} g(t)\exp(-2\pi ixt)\,dt.$$

A shift, either upward or downward, of the contour in (1.2) gives readily that $I(w_1, w_2; g)$ is meromorphic over the entire $\mathbb{C}^2$ and regular in a neighbourhood of the point $(1/2, 1/2)$, and that

$$(1.4) \qquad \mathscr{M}_2(\zeta; g) = I\left(\frac{1}{2}, \frac{1}{2}; g\right) + 2\pi \operatorname{Re} g\left(\frac{1}{2}i\right).$$

Thus $\mathscr{M}_2(\zeta; g)$ can be viewed as a special value of a meromorphic function of two variables. To attain a continuation of $I(w_1, w_2; g)$ that could be truly informative, the double sum in (1.2) is split into three parts according as $a = b$, $a < b$ and $a > b$ so that

$$(1.5) \qquad I(w_1, w_2; g) = \zeta(w_1 + w_2)\hat{g}(0) + I_1(w_1, w_2; g) + I_1(w_2, w_1; g),$$

where

$$(1.6)\qquad I_1(w_1,w_2;g)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^{w_1}(m+n)^{w_2}}\hat{g}\left(\frac{1}{2\pi}\log\left(1+\frac{n}{m}\right)\right).$$

Then $I_1(w_1,w_2;g)$ has to be continued analytically to a neighbourhood of $(1/2,1/2)$. This is achieved by regarding the inner sum as the one over the lattice $\mathbb{Z}$. An application of the Poisson sum formula yields a meromorphic continuation of $I_1(w_1,w_2;g)$ to the whole $\mathbb{C}^2$. Hence, the expression (1.5) holds throughout $\mathbb{C}^2$ as a relation of the four meromorphic functions. The specialization $(w_1,w_2)\to(1/2,1/2)$ is performed in the resulting spectral decomposition of the right side of (1.5), and via (1.4) the following explicit formula arises:

$$(1.7)\qquad \mathcal{M}_2(\zeta;g)=\int_{-\infty}^{\infty}\left[\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+2c_E-\log 2\pi\right]g(t)\,dt+2\pi\operatorname{Re}g\left(\frac{1}{2}i\right)$$
$$+4\sum_{n=1}^{\infty}d(n)\int_0^{\infty}\frac{1}{\sqrt{r(r+1)}}g_c\left(\frac{1}{2\pi}\log\left(1+\frac{1}{r}\right)\right)\cos(2\pi nr)\,dr,$$

where c_E is the Euler constant, d the divisor function, and $g_c=\operatorname{Re}\hat{g}$ the cosine Fourier transform of g (see [19], (4.1.16)).

Turning to the fourth moment, we consider the function

$$(1.8)\qquad J(w;g)=\int_{-\infty}^{\infty}\zeta(w_1-it)\zeta(w_2+it)\zeta(w_3+it)\zeta(w_4-it)g(t)\,dt$$
$$=\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\sum_{d=1}^{\infty}a^{-w_1}b^{-w_2}c^{-w_3}d^{-w_4}\hat{g}\left(\frac{1}{2\pi}\log\frac{ad}{bc}\right),$$

where $w=(w_1,w_2,w_3,w_4)\in\mathbb{C}^4$ is in the region of absolute convergence. As before, $J(w;g)$ is meromorphic over the entire $\mathbb{C}^4$, regular in a neighbourhood of the point $\mathrm{p}_{\frac{1}{2}}=(1/2,1/2,1/2,1/2)$, and

$$(1.9)\qquad \mathcal{M}_4(\zeta;g)=J(\mathrm{p}_{\frac{1}{2}};g)-2\pi\operatorname{Re}\left\{(c_E-\log 2\pi)g\left(\frac{1}{2}i\right)+\frac{1}{2}g'\left(\frac{1}{2}i\right)\right\}.$$

Analogously to (1.5), the quadruple sum in (1.8) is split into three parts according as $ad=bc$, $ad<bc$ and $ad>bc$ so that

$$(1.10)\qquad J(w;g)=\frac{\zeta(w_1+w_2)\zeta(w_1+w_3)\zeta(w_2+w_4)\zeta(w_3+w_4)}{\zeta(w_1+w_2+w_3+w_4)}\hat{g}(0)$$
$$+J_1(w;g)+J_1(w';g),$$

where $w'=(w_2,w_1,w_4,w_3)$. Our task is then to continue $J_1(w;g)$ analytically to a neighbourhood of $\mathrm{p}_{\frac{1}{2}}$. This is by no means straightforward. With $J_1(w;g)$, we have

$$(1.11)\qquad J_1(w;g)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_{w_1-w_4}(m)\sigma_{w_2-w_3}(m+n)}{m^{w_1}(m+n)^{w_2}}\hat{g}\left(\frac{1}{2\pi}\log\left(1+\frac{n}{m}\right)\right),$$

where $\sigma_a(n)$ is the sum of a-th powers of divisors of n. We face a far more involved expression than (1.6). It is thus remarkable that Atkinson's view on (1.6) extends to (1.11). Namely, the inner sum embraces again a lattice structure. Here the lattice is $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, a discrete subgroup of the Lie group $G = \mathrm{PSL}_2(\mathbb{R})$, as to be detailed in the next section.

This observation was made in [16] (see also [19], Section 4.2) but its proper exploitation was actuated only recently when the present work was commenced. In the former proof, instead, the Ramanujan expansion is employed to separate the variables m, n in the arithmetic factor $\sigma_{w_2-w_3}(m+n)$, and the Voronoï scheme is applied to the arising sum over m; namely, the functional equation for the Estermann zeta-function is invoked. In this way, $J_1(w;g)$ is transformed into a sum of Kloosterman sums. Then the Kloosterman-Spectral sum formula of N. V. Kuznetsov ([19], Theorems 2.3 and 2.5) plays a fundamental rôle; and a spectral decomposition of $J_1(w;g)$ emerges. This is the most salient point in the former proof, i.e., the assertion in [19], Section 4.5. The argument is, however, circuitous. In the present paper we aim to reach the same with a direct reasoning based on the above observation.

With the spectral decomposition of $J_1(w;g)$ thus obtained, the argument in [19], Section 4.6, establishes the existence of $J_1(w;g)$ as a meromorphic function over the entire $\mathbb{C}^4$. Hence, the expression (1.10) holds throughout $\mathbb{C}^4$ as a relation of the four meromorphic functions. The specialization $w \to \mathrm{p}_{\frac{1}{2}}$ gives rise to the explicit formula for $\mathcal{M}_4(\zeta;g)$.

We now define the basic spectral terms in the context of [19]: Let $\mathbb{H}$ be the hyperbolic upper half plane $\{z : z = x+iy, x \in \mathbb{R}, y > 0\}$ equipped with the invariant measure $d\mu(z) = dx\,dy/y^2$. Let $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4} : \kappa_j > 0, j \geqq 1\right\} \cup \{0\}$ be the discrete spectrum, arranged in non-decreasing order, of the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ acting on the Hilbert space $L^2(\Gamma\backslash\mathbb{H}, d\mu)$ composed of all Γ-automorphic functions that are square integrable over $\Gamma\backslash\mathbb{H}$ against $d\mu$. We denote by ψ_j an L^2-eigenfunction corresponding to λ_j, i.e., $\Delta\psi_j = \lambda_j\psi_j$. It has the Fourier expansion

$$\psi_j(z) = \sqrt{y} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \rho_j(n) K_{i\kappa_j}(2\pi|n|y) \exp(2\pi i n x), \tag{1.12}$$

where K_ν is the K-Bessel function of order ν. We may assume that the set $\{\psi_j : j \geqq 1\}$ forms an orthonormal system, and that each ψ_j is a simultaneous eigenfunction of all Hecke operators. The latter means that we have, for each positive integer n,

$$\frac{1}{\sqrt{n}} \sum_{\substack{a=1 \\ ad=n}}^{n} \sum_{b=1}^{d} \psi_j\left(\frac{az+b}{d}\right) = t_j(n)\psi_j(z), \tag{1.13}$$

with a certain real number $t_j(n)$, and also $\psi_j(-\bar{z}) = \epsilon_j\psi_j(z)$, with $\epsilon_j = \pm 1$. In particular, we have $\rho_j(1) \neq 0$, $\rho_j(n) = \rho_j(1)t_j(n)$, $\rho_j(-n) = \epsilon_j\rho_j(1)t_j(n)$. Then the Hecke series $H_j(s)$ associated with ψ_j is defined by

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}, \tag{1.14}$$

which converges absolutely for $\operatorname{Re} s > 1$, and continues to an entire function.

Each ψ_j is called a real analytic cusp form. We next introduce holomorphic cusp forms: If $\psi(z)$ is holomorphic throughout $\mathbb{H}$, $\psi(z)(dz)^\ell$ is Γ-invariant, with a certain fixed positive integer ℓ, and $\psi(i\infty)=0$, then ψ is called a holomorphic cusp form of weight 2ℓ. The linear space consisting of all such functions is a Hilbert space of finite dimension, in which the norm of ψ is defined to be the square root of the integral of $|y^\ell \psi(z)|^2$ over $\Gamma\backslash\mathbb{H}$ against the measure $d\mu$. Let then $\{\psi_{j,\ell}(z) : 1 \leqq j \leqq \vartheta(\ell)\}$ be an orthonormal basis of this Hilbert space. We have the Fourier expansion

$$(1.15)\qquad \psi_{j,\ell}(z) = \sum_{n=1}^{\infty} \rho_{j,\ell}(n) n^{\ell-\frac{1}{2}} \exp(2\pi i n z).$$

Similarly to (1.13) we may assume that for any positive integer n

$$(1.16)\qquad \frac{1}{\sqrt{n}} \sum_{\substack{a=1\\ ad=n}}^{n} \left(\frac{a}{d}\right)^{\ell} \sum_{b=1}^{d} \psi_{j,\ell}\left(\frac{az+b}{d}\right) = t_{j,\ell}(n)\psi_{j,\ell}(z),$$

with a real number $t_{j,\ell}(n)$ so that $\rho_{j,\ell}(1) \neq 0$ and $\rho_{j,\ell}(n) = \rho_{j,\ell}(1) t_{j,\ell}(n)$. As before, the Hecke series associated to $\psi_{j,\ell}$ is defined by

$$(1.17)\qquad H_{j,\ell}(s) = \sum_{n=1}^{\infty} t_{j,\ell}(n) n^{-s},$$

which is again an entire function.

With this, we may state the explicit formula for $\mathscr{M}_4(\zeta; g)$:

Theorem A. *It holds that*

$$(1.18)\qquad \mathscr{M}_4(\zeta; g) = \{\mathscr{M}_{4,0} + \mathscr{M}_{4,1} + \mathscr{M}_{4,2} + \mathscr{M}_{4,3}\}(\zeta; g),$$

where

$$(1.19)\qquad \mathscr{M}_{4,0}(\zeta; g) = \int_{-\infty}^{\infty} \sum_{\substack{p,q,u,v \geqq 0\\ pu+qv \leqq 4}} c(p,q,u,v) \operatorname{Re}\left\{ \left(\frac{\Gamma^{(p)}}{\Gamma}\right)^{u} \left(\frac{\Gamma^{(q)}}{\Gamma}\right)^{v} \left(\frac{1}{2}+it\right) \right\} g(t)\, dt$$
$$- 2\pi \operatorname{Re}\left\{ (c_E - \log 2\pi) g\left(\frac{1}{2}i\right) + \frac{1}{2} i g'\left(\frac{1}{2}i\right) \right\},$$

with effectively computable real absolute constants $c(p,q,u,v)$; *and*

$$(1.20)\qquad \mathscr{M}_{4,1}(\zeta; g) = \sum_{j=1}^{\infty} \alpha_j H_j\left(\frac{1}{2}\right)^3 \Lambda(i\kappa_j; g),$$

$$(1.21)\qquad \mathscr{M}_{4,2}(\zeta; g) = \sum_{\ell=1}^{\infty} \sum_{j=1}^{\vartheta(\ell)} \alpha_{j,\ell} H_{j,\ell}\left(\frac{1}{2}\right)^3 \Lambda\left(\ell - \frac{1}{2}; g\right),$$

$$(1.22)\qquad \mathscr{M}_{4,3}(\zeta; g) = \int_{(0)} \frac{\left(\zeta\left(\frac{1}{2}+\nu\right)\zeta\left(\frac{1}{2}-\nu\right)\right)^3}{\zeta(1+2\nu)\zeta(1-2\nu)} \Lambda(\nu; g) \frac{d\nu}{\pi i},$$

with

$$\alpha_j = \frac{|\rho_j(1)|^2}{\cosh \pi\kappa_j}, \quad \alpha_{j,\ell} = \frac{\Gamma(2\ell)}{2^{4\ell-1}\pi^{2\ell+1}} |\rho_{j,\ell}(1)|^2. \tag{1.23}$$

The contour in (1.22) *is the imaginary axis, and*

$$\begin{aligned}\Lambda(\nu; g) &= \int_0^\infty \frac{1}{\sqrt{r(r+1)}} g_c\left(\frac{1}{2\pi}\log\left(1+\frac{1}{r}\right)\right) \\ &\times \mathrm{Re}\left\{ r^{-\frac{1}{2}-\nu}\left(1-\frac{1}{\sin \pi\nu}\right) \frac{\Gamma\left(\frac{1}{2}+\nu\right)^2}{\Gamma(1+2\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+\nu; 1+2\nu; -\frac{1}{r}\right)\right\} dr,\end{aligned} \tag{1.24}$$

with F the hypergeometric function.

This is [19], Theorem 4.2, with a minor change of notation. The right side of (1.18) has a characteristic pertinent to the spectral structure of $L^2(\Gamma\backslash G)$ that is developed in the next section. It does not contain any trace of the use of Kloosterman sums. Then, a problem comes out: Find a way to reach (1.18) as directly as possible, especially without recourse to the reduction to the spectral theory of sums of Kloosterman sums. As has been indicated above, this is the principal motivation of the present work. In what follows we shall show an answer to this basic problem in the theory of the Riemann zeta-function. It is a realization of the programme given in [19], Section 4.2.

Remark. A. Ivić's lecture notes [12] give a thorough account of the theory of mean values of the Riemann zeta-function. Some major applications of Theorem A are given, and also in [19]. A recent notable contribution is in Ivić [13]. Those are, in a variety of ways, generalizations of consequences derived from the formula (1.7). A typical instance is the following assertion, which could be regarded as an analogue of the localized version of Atkinson's formula [1]: Let T tend to infinity and $T^{\frac{1}{2}}(\log T)^{-C} \leqq G \leqq T(\log T)^{-1}$. Then we have

$$\begin{aligned}&\frac{1}{\sqrt{\pi}G}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^4 \exp\left(-\left(\frac{t}{G}\right)^2\right) dt \\ &= \frac{\pi}{\sqrt{2T}}\sum_{j=1}^{\infty}\alpha_j H_j\left(\frac{1}{2}\right)^3 \kappa_j^{-\frac{1}{2}} \sin\left(\kappa_j \log\frac{\kappa_j}{4eT}\right)\exp\left(-\frac{1}{4}\left(\frac{G\kappa_j}{T}\right)^2\right) \\ &\quad + O((\log T)^{3C+9})\end{aligned} \tag{1.25}$$

(see [19], (5.1.44)). This makes it clear that the values of the zeta-function on the critical line are related to eigenvalues of the hyperbolic Laplacian.

2. Reformulation

Now, we make precise the lattice structure of the function $J_1(w;g)$: We define a function g_* on G by

$$(2.1)\qquad g_*(\mathrm{g}) = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\hat{g}\left(\frac{1}{2\pi}\log\frac{ad}{bc}\right)\iota(ad), \quad G \ni \mathrm{g} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where ι is the characteristic function of the negative reals, and the matrix is in the projective sense. Then a rearrangement gives

$$(2.2)\qquad J_1(w;g) = \frac{1}{4}\sum_{\substack{\mathrm{g}\in M_2(\mathbb{Z})\\ \det \mathrm{g}>0}} (\det \mathrm{g})^{-z_1-\frac{1}{2}} g_*\left(\frac{\mathrm{g}}{\sqrt{\det \mathrm{g}}}\right),$$

where w is in the region of absolute convergence, and

$$(2.3)\qquad z_1 = \frac{1}{2}(w_1+w_2+w_3+w_4-1).$$

Invoking Hecke's representatives for the quotient $SL_2(\mathbb{Z})\backslash M_2(\mathbb{Z})$, we have, in place of (2.2),

$$(2.4)\qquad J_1(w;g) = \frac{1}{2}\sum_{n=1}^{\infty} n^{-z_1-\frac{1}{2}}\sum_{d|n}\sum_{b \bmod d}\sum_{\gamma\in\Gamma} g_*\left(\gamma\begin{bmatrix} 1 & b/d \\ & 1 \end{bmatrix}\begin{bmatrix} \sqrt{n}/d & \\ & d/\sqrt{n} \end{bmatrix}\right).$$

The lattice structure is now evident.

Further, if we put

$$(2.5)\qquad \mathscr{P}g_*(\mathrm{g}) = \sum_{\gamma\in\Gamma} g_*(\gamma\mathrm{g}), \quad \mathrm{g}\in G,$$

and

$$(2.6)\qquad \mathscr{T} = \sum_{n=1}^{\infty} T_n n^{-z_1},$$

with the Hecke operators T_n which act from the left (cf. (1.13) and (1.16)), then (2.4) gives formally

$$(2.7)\qquad J_1(w;g) = \frac{1}{2}\mathscr{T}\mathscr{P}g_*(1),$$

where the argument on the right side is the unit element of G.

This is revealing. However, the Poincaré series $\mathscr{P}g_*$ is not really defined throughout G. In fact, the sum (2.5) diverges on a dense subset of G; see the discussion leading (3.5) below. Nevertheless, (2.7) itself is correct and demonstrates that $J_1(w; g)$ is an object closely related to the space of Γ-automorphic functions over G.

We now start pursuing this line of reasoning.

To begin with, we collect here elements of the theory of Γ-automorphic representations of G: We write

$$\text{(2.8)} \qquad \mathrm{n}[x] = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, \quad \mathrm{a}[y] = \begin{bmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{bmatrix}, \quad \mathrm{k}[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Let $N = \{\mathrm{n}[x] : x \in \mathbb{R}\}$, $A = \{\mathrm{a}[y] : y > 0\}$, and $K = \{\mathrm{k}[\theta] : \theta \in \mathbb{R}/\pi\mathbb{Z}\}$ so that $G = NAK$ be the Iwasawa decomposition of the Lie group G. We read it as

$$G \ni \mathrm{g} = \mathrm{nak} = \mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta];$$

throughout the sequel, the coordinate (x, y, θ) retains this definition. The Haar measures on the groups N, A, K, G are defined, respectively, by $d\mathrm{n} = dx$, $d\mathrm{a} = dy/y$, $d\mathrm{k} = d\theta/\pi$, $d\mathrm{g} = d\mathrm{n}\, d\mathrm{a}\, d\mathrm{k}/y$, with Lebesgue measures $dx, dy, d\theta$. The Lie algebra of G is spanned by

$$\text{(2.9)} \qquad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The universal enveloping algebra of G is denoted by $\mathcal{U}$. Its center is the polynomial ring on the Casimir element $\Omega = y^2(\partial_x^2 + \partial_y^2) - y\partial_x\partial_\theta$.

The space $L^2(\Gamma\backslash G)$ is composed of all left Γ-automorphic functions on G, vectors for short, which are square integrable over $\Gamma\backslash G$ against $d\mathrm{g}$. Elements of G act unitarily on vectors from the right. We have the orthogonal decomposition into invariant subspaces

$$\text{(2.10)} \qquad L^2(\Gamma\backslash G) = \mathbb{C}\cdot 1 \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^eL^2(\Gamma\backslash G).$$

Here ${}^0L^2$ is the cuspidal subspace spanned by vectors whose Fourier expansions with respect to the left action of N have vanishing constant terms. The subspace ${}^eL^2$ is spanned by integrals of Eisenstein series, as is to be detailed in Lemma 2 below. Note that invariant subspaces and Γ-automorphic representations of G are interchangeable concepts, and we refer to them in a mixed way.

The cuspidal subspace splits into irreducible subspaces:

$$\text{(2.11)} \qquad {}^0L^2(\Gamma\backslash G) = \overline{\bigoplus V}.$$

The Casimir operator becomes a constant multiplication in each V; that is,

$$\text{(2.12)} \qquad \Omega|_{V^\infty} = \left(\nu_V^2 - \frac{1}{4}\right)\cdot 1,$$

where V^∞ is the set of all smooth vectors in V. Under our present supposition that $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, we can restrict our attention to two cases: either $i\nu_V < 0$ or ν_V is equal to half a positive odd integer. According to the right action of K, the space V is decomposed into K-irreducible subspaces

$$(2.13)\qquad V = \overline{\bigoplus_{p=-\infty}^{\infty} V_p}, \quad \dim V_p \leqq 1.$$

If it is not trivial, V_p is spanned by a Γ-automorphic function on which the right translation by $\mathrm{k}[\theta]$ becomes the multiplication by the factor $\exp(2ip\theta)$. It is called a Γ-automorphic form of spectral parameter ν_V and weight $2p$.

Let us assume temporarily that V belongs to the unitary principal series, i.e., $i\nu_V < 0$, under our present situation. Then one can show that $\dim V_p = 1$ for all $p \in \mathbb{Z}$ and that there exists a complete orthonormal system $\{\varphi_p \in V_p : p \in \mathbb{Z}\}$ of V such that

$$(2.14)\qquad \varphi_p(\mathrm{g}) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathscr{A}^{\operatorname{sgn}(n)} \phi_p(\mathrm{a}[|n|]\mathrm{g}; \nu_V),$$

where $\phi_p(\mathrm{g}; \nu) = y^{\nu+\frac{1}{2}} \exp(2ip\theta)$, and

$$(2.15)\qquad \mathscr{A}^{\delta}\phi_p(\mathrm{g}; \nu) = \int_{-\infty}^{\infty} \exp(-2\pi i\delta x)\phi_p(\mathrm{wn}[x]\mathrm{g}; \nu)\,dx, \quad \mathrm{w} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

The $\mathscr{A}^{\delta}$ is a specialization of the Jacquet operator. This follows from a study of the Fourier expansion of φ_p coupled with the action of the Maass operators

$$\mathbf{E} = e^{2i\theta}(2iy\partial_x + 2y\partial_y - i\partial_\theta)$$

and $\overline{\mathbf{E}}$. It should be observed that the coefficients $\varrho_V(n)$ in (2.14) do not depend on the weight.

We note that

$$(2.16)\qquad \begin{aligned} \mathscr{A}^{\delta}\phi_p(\mathrm{g}; \nu) &= y^{\frac{1}{2}-\nu} \exp(2\pi i\delta x) \int_{-\infty}^{\infty} \frac{\exp(2\pi iy\xi)}{(\xi^2+1)^{\frac{1}{2}+\nu}} \left(\frac{\xi+i}{\xi-i}\right)^{\delta p} d\xi \cdot \exp(2pi\theta) \\ &= (-1)^p \pi^{\frac{1}{2}+\nu} \exp(2\pi i\delta x) \frac{W_{\delta p,\nu}(4\pi y)}{\Gamma\left(\delta p + \frac{1}{2} + \nu\right)} \exp(2pi\theta), \end{aligned}$$

where $W_{\lambda,\mu}(y)$ is the Whittaker function (see [24], Chapter XVI). The first line is valid for $\operatorname{Re}\nu > 0$, while the second defines $\mathscr{A}^{\delta}\phi_p$ for all $\nu \in \mathbb{C}$. In particular, we have the expansion

$$(2.17)\qquad \varphi_0(\mathrm{g}) = \frac{2\pi^{\frac{1}{2}+\nu_V}}{\Gamma\left(\frac{1}{2}+\nu_V\right)} \sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \varrho_V(n) K_{\nu_V}(2\pi|n|y) \exp(2\pi inx).$$

This corresponds to (1.12). Namely, on the identification

$$\varphi_0(\mathrm{g}) = \psi_j(x+iy), \tag{2.18}$$

we have

$$\nu_V = i\kappa_j, \quad \varrho_V(n) = \frac{\Gamma\left(\frac{1}{2}+\nu_V\right)}{2\pi^{\frac{1}{2}+\nu_V}}\rho_j(n). \tag{2.19}$$

Next, let us consider a V in the discrete series; that is, $\nu_V = \ell - \frac{1}{2}$, $1 \leqq \ell \in \mathbb{Z}$. We have, in place of (2.13),

$$\text{either} \quad V = \overline{\bigoplus_{p=\ell}^{\infty} V_p} \quad \text{or} \quad V = \overline{\bigoplus_{p=-\infty}^{-\ell} V_p}, \tag{2.20}$$

with $\dim V_p = 1$, corresponding to the holomorphic and the antiholomorphic discrete series. The involution $\omega : \mathrm{g} = \mathrm{nak} \mapsto \mathrm{n}^{-1}\mathrm{ak}^{-1}$ maps one to the other. In the holomorphic case, we have a complete orthonormal system $\{\varphi_p : p \geqq \ell\}$ in V such that

$$\varphi_p(\mathrm{g}) = \pi^{\frac{1}{2}-\ell}\left(\frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)}\right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\varrho_V(n)}{\sqrt{n}} \mathscr{A}^+\phi_p\left(\mathrm{a}[n]\mathrm{g}; \ell - \frac{1}{2}\right). \tag{2.21}$$

In particular, we have

$$\varphi_\ell(\mathrm{g}) = (-1)^\ell \frac{2^{2\ell}\pi^{\ell+\frac{1}{2}}}{\sqrt{\Gamma(2\ell)}} \exp(2i\ell\theta) y^\ell \sum_{n=1}^{\infty} \varrho_V(n) n^{\ell-\frac{1}{2}} \exp\big(2\pi i n(x+iy)\big), \tag{2.22}$$

which corresponds to (1.15). Thus, on the identification

$$\varphi_\ell(\mathrm{g}) = y^\ell \exp(2i\ell\theta)\psi_{j,\ell}(x+iy), \tag{2.23}$$

we have

$$\nu_V = \ell - \frac{1}{2}, \quad \varrho_V(n) = (-1)^\ell \frac{\sqrt{\Gamma(2\ell)}}{2^{2\ell}\pi^{\ell+\frac{1}{2}}} \rho_{j,\ell}(n). \tag{2.24}$$

On the other hand, if V is in the antiholomorphic discrete series, then we have a complete orthonormal system $\{\varphi_p : p \leqq -\ell\}$ in V such that

$$\varphi_p(\mathrm{g}) = \pi^{\frac{1}{2}-\ell}\left(\frac{\Gamma(|p|+\ell)}{\Gamma(|p|-\ell+1)}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{-1} \frac{\varrho_V(n)}{\sqrt{|n|}} \mathscr{A}^-\phi_p\left(\mathrm{a}[|n|]\mathrm{g}; \ell - \frac{1}{2}\right), \tag{2.25}$$

with $\varrho_V(n) = \varrho_{V^*}(-n)$, where $V^* = \{\varphi\omega : \varphi \in V\}$ is in the holomorphic discrete series.

With this, (1.13) and (1.16) are extended to

$$T_n|_V = t_V(n) \cdot 1, \tag{2.26}$$

for any V, where $t_V(n)$ is newly defined to be equal to either $t_j(n)$ or $t_{j,\ell}(n)$, as specified above. Thus, for any non-zero integer n,

$$\varrho_V(n) = \varrho_V(\operatorname{sgn}(n))t_V(|n|). \tag{2.27}$$

We have here the obvious convention that $\varrho_V(-1) = 0$ and $\varrho_V(1) = 0$ for V in the holomorphic and antiholomorphic discrete series, respectively. If V is in the unitary principal series, then $\varrho_V(-1) = \epsilon_V \varrho_V(1)$ following the assertion adjacent (1.13), but with ϵ_j being denoted by ϵ_V.

We observe that (2.19), (2.24), and (2.27) translate (1.23) into

$$|\varrho_V(1)|^2 + |\varrho_V(-1)|^2 = \frac{1}{2}\alpha_j \quad \text{or} \quad \frac{1}{2}\alpha_{j,\ell}, \tag{2.28}$$

according to the series to which V belongs. Lemmas 2.3 and 2.4 of [19] yield

$$\sum_{\substack{V \\ |\nu_V| \leqq N}} |\varrho_V(\pm 1)|^2 \ll N^2, \tag{2.29}$$

as N tends to infinity, with the implied constant being absolute. Those lemmas give also

$$t_V(n) \ll n^{\frac{1}{4}+\varepsilon}. \tag{2.30}$$

See the remark at the end of the present section.

Replacing (1.14) and (1.17), we associate the Hecke series

$$H_V(s) = \sum_{n=1}^{\infty} t_V(n) n^{-s} \tag{2.31}$$

to each V, in both types of cuspidal representations. This continues to an entire function, satisfying the functional equation

$$\begin{aligned} H_V(s) = {} & 2^{2s-1}\pi^{2(s-1)}\Gamma(1-s+\nu_V)\Gamma(1-s-\nu_V) \\ & \times \{\epsilon_V \cos \pi\nu_V - \cos \pi s\} H_V(1-s), \end{aligned} \tag{2.32}$$

where ϵ_V is left undefined for V in the discrete series, since $\cos \pi\nu_V = 0$ there. In particular, $H_V(s)$ is of polynomial growth in both s and ν_V in any fixed vertical strip in the s-plane (see [19], Chapter 3).

Now, let us reformulate the explicit formula (1.18) in terms of Γ-automorphic representations of G. To this end, we introduce the Bessel function of representations of G, in the sense of [10]:

$$j_\nu(u) = \pi \frac{\sqrt{|u|}}{\sin \pi\nu} \left(J_{-2\nu}^{\operatorname{sgn}(u)}(4\pi\sqrt{|u|}) - J_{2\nu}^{\operatorname{sgn}(u)}(4\pi\sqrt{|u|}) \right), \tag{2.33}$$

where $J_\nu^+ = J_\nu$ and $J_\nu^- = I_\nu$ with the ordinary notation for Bessel functions. Also we put

$$\Theta(\nu; g) = \int_0^\infty \frac{1}{\sqrt{r(r+1)}} g_c\left(\frac{1}{2\pi}\log\left(1+\frac{1}{r}\right)\right)\Xi(r;\nu)\,dr, \tag{2.34}$$

with

$$\Xi(r;\nu) = \int_{\mathbf{R}^\times} j_0(-u) j_\nu\left(\frac{u}{r}\right)\frac{d^\times u}{\sqrt{|u|}}, \quad d^\times u = \frac{du}{|u|}. \tag{2.35}$$

Then Theorem A takes the following new form:

Theorem B. *We have*

$$\mathcal{M}_4(\zeta; g) = \{\mathcal{M}_4^{(r)} + \mathcal{M}_4^{(c)} + \mathcal{M}_4^{(e)}\}(\zeta; g). \tag{2.36}$$

Here $\mathcal{M}_4^{(r)} = \mathcal{M}_{4,0}$, *and*

$$\mathcal{M}_4^{(c)}(\zeta; g) = \sum_V (|\varrho_V(1)|^2 + |\varrho_V(-1)|^2) H_V\left(\frac{1}{2}\right)^3 \Theta(\nu_V; g), \tag{2.37}$$

$$\mathcal{M}_4^{(e)}(\zeta; g) = \int_{(0)} \frac{\left(\zeta\left(\frac{1}{2}+\nu\right)\zeta\left(\frac{1}{2}-\nu\right)\right)^3}{\zeta(1+2\nu)\zeta(1-2\nu)}\Theta(\nu; g)\frac{d\nu}{2\pi i}. \tag{2.38}$$

The variable V runs over a maximal orthogonal system of Hecke-invariant cuspidal Γ-automorphic representations of G.

Proof. This is a corrected version of the reformulation made in [20] without details. We shall afterward prove this directly using the spectral expansion of smooth vectors, i.e., Lemma 2 below, and the Kirillov scheme developed in Section 4. The latter will reveal the reason why the integral transform (2.34) comes up in the explicit formula. Here we show briefly how (2.36) follows from (1.18). This might appear redundant but seems to have its own interest. Also it provides us with an opportunity to make certain preparation for our later discussion.

We observe two Mellin transforms: If $|\mathrm{Re}\,\nu| - \frac{1}{2} < \mathrm{Re}\,s$, then

$$\int_{-\infty}^0 j_\nu(u)|u|^{s-1}\,du = \frac{1}{\pi}(2\pi)^{-2s}\cos\pi\nu\,\Gamma\left(s+\frac{1}{2}+\nu\right)\Gamma\left(s+\frac{1}{2}-\nu\right); \tag{2.39}$$

and if $|\mathrm{Re}\,\nu| - \frac{1}{2} < \mathrm{Re}\,s < -\frac{1}{4}$, then

$$\int_0^\infty j_\nu(u)u^{s-1}\,du = -\frac{1}{\pi}(2\pi)^{-2s}\sin\pi s\,\Gamma\left(s+\frac{1}{2}+\nu\right)\Gamma\left(s+\frac{1}{2}-\nu\right). \tag{2.40}$$

The former follows from (8) of [23], Section 13.21, and the latter from (1) of [23], Section 13.24. Both integrals are absolutely convergent in the respective ranges, because of (2.45) below.

To compute the following integral, we replace the factor $j_\nu(u/r)$ using the Mellin inversion of (2.39) with a contour (β), i.e., the vertical line $\mathrm{Re}\, s = \beta$. It is to be chosen so that the resulting double integral is absolutely convergent. Then we may exchange the order of integration and use (2.40) for the inner integral which is a Mellin transform of $j_0(u)$, $u > 0$. In this way, we have, for $r > 0$ and, e.g., for $-1/4 < \beta < -|\mathrm{Re}\,\nu|$,

$$(2.41)\qquad \int_{-\infty}^{0} j_0(-u) j_\nu\left(\frac{u}{r}\right)\frac{du}{|u|^{\frac{3}{2}}}$$
$$= \frac{\cos(\pi\nu)}{\pi^2 i}\int_{(\beta)} \cos\pi s\,\Gamma\left(s+\frac{1}{2}+\nu\right)\Gamma\left(s+\frac{1}{2}-\nu\right)\Gamma(-s)^2 r^s\,ds$$
$$= r^{-\frac{1}{2}+\nu}\frac{\Gamma\left(\frac{1}{2}-\nu\right)^2}{\Gamma(1-2\nu)} F\left(\frac{1}{2}-\nu,\frac{1}{2}-\nu;1-2\nu;-\frac{1}{r}\right)$$
$$+ \text{ the same expression but with } \nu \mapsto -\nu.$$

For the second equality see e.g. [19], pp. 119–120. Similarly but exchanging the rôles of the factors $j_0(-u)$ and $j_\nu(u/r)$, we have, on the same condition,

$$(2.42)\qquad \int_{0}^{\infty} j_0(-u) j_\nu\left(\frac{u}{r}\right)\frac{du}{u^{\frac{3}{2}}}$$
$$= \frac{1}{\pi^2 i}\int_{(\beta)} \cos\pi s\,\Gamma(-s+\nu)\Gamma(-s-\nu)\Gamma\left(s+\frac{1}{2}\right)^2 r^{-s-\frac{1}{2}}\,ds$$
$$= \frac{r^{-\frac{1}{2}+\nu}}{\sin\pi\nu}\frac{\Gamma\left(\frac{1}{2}-\nu\right)^2}{\Gamma(1-2\nu)} F\left(\frac{1}{2}-\nu,\frac{1}{2}-\nu;1-2\nu;-\frac{1}{r}\right)$$
$$+ \text{ the same expression but with } \nu \mapsto -\nu.$$

Thus we find that $\Theta(\nu;g) = 2\Lambda(\nu;g)$ for $\nu \in i\mathbb{R}$. Then (2.38) is immediate. Also, taking into account the first case of (2.28) we obtain (2.37) if V is restricted to the unitary principal series.

On the other hand, if ℓ is a positive integer, then we have $j_{\ell-\frac{1}{2}}(u) = 0$ for $u < 0$ and $j_{\ell-\frac{1}{2}}(u) = (-1)^\ell 2\pi\sqrt{u}J_{2\ell-1}(4\pi\sqrt{u})$ for $u > 0$. This gives, for $-\ell < \mathrm{Re}\, s < -1/4$,

$$(2.43)\qquad \int_0^\infty j_{\ell-\frac{1}{2}}(u)u^{s-1}\,du = (-1)^\ell(2\pi)^{-2s}\frac{\Gamma(s+\ell)}{\Gamma(\ell-s)},$$

and

$$(2.44)\qquad \int_0^\infty j_0(-u) j_{\ell-\frac{1}{2}}\left(\frac{u}{r}\right)\frac{du}{u^{\frac{3}{2}}} = 2(-r)^{-\ell}\frac{\Gamma(\ell)^2}{\Gamma(2\ell)}F\left(\ell,\ell;2\ell;-\frac{1}{r}\right).$$

Thus we have $\Theta\left(\ell-\frac{1}{2};g\right)=\Lambda\left(\ell-\frac{1}{2};g\right)$ if ℓ is even. However, the same does not hold for all ℓ, since $\Lambda\left(\ell-\frac{1}{2};g\right)=0$ if ℓ is odd. Nevertheless, (2.32) gives $H_V\left(\frac{1}{2}\right)=0$ if $\nu_V = 2\ell+\frac{1}{2}$. That is, such V are irrelevant. With this, on noting (2.20) and the second case of (2.28), we end the proof.

It should be remarked that if ν is bounded and $|\operatorname{Re}\nu|<\frac{1}{2}$, then

$$(2.45)\qquad j_\nu(u) \ll \begin{cases} u^{\frac{1}{4}} & \text{if } u>1,\\ |u|^{-\frac{1}{4}} & \text{if } u<-1,\\ |u|^{\frac{1}{2}-|\operatorname{Re}\nu|-\varepsilon} & \text{if } |u|\leqq 1.\end{cases}$$

Also, for integral $\ell \geqq 1$

$$(2.46)\qquad j_{\ell-\frac{1}{2}}(u) \ll \min(u^{\frac{1}{4}}, u^{\ell}),\quad u>0.$$

In fact, if $|u|$ is small, both follow from power series expansions of Bessel functions. If u is negative, then $j_\nu(u) = 4\sqrt{|u|}\cos\pi\nu K_{2\nu}(4\pi\sqrt{|u|})$, which decays exponentially. If u tends to $+\infty$, (2.46) is a consequence of a well-known asymptotic expansion for J-Bessel functions (see [23], Section 7.21). To treat the remaining case, we note that the formula (12) in [23], p. 180 gives that for $u>0$, $|\operatorname{Re}\nu|<1/2$,

$$(2.47)\qquad j_\nu(u) = 4\sqrt{u}\int_0^\infty \cos(4\pi\sqrt{u}\cosh\xi)\cosh(2\nu\xi)\,d\xi.$$

If u is large, then we divide the integral at $\xi = u^{-\frac{1}{4}}$. We bound the integrand trivially for smaller ξ; otherwise we integrate in part with respect to the cosine factor. This ends the proof of (2.45)–(2.46).

Remark. The proofs of (2.29)–(2.30) in [19] might appear to come rather close to the spectral theory of sums of Kloosterman sums. A closer examination will, however, reveal that the proofs depend only on a non-trivial bound for individual Kloosterman sums. T. Estermann's elementary bound should work fine for (2.29). As to (2.30), A. Weil's bound is used, but it can be replaced by Estermann's, too, although a weaker bound results. In fact, the exponent in (2.30) is not much relevant to our discussion. It should be worth stressing that our proof, developed below, of Theorem B depends on the theory of Kloosterman sums solely via (2.29)–(2.30).

The term $\mathcal{M}_{4,2}$ in (1.18) is there because the Kloosterman-Spectral sum formula is employed in the former proof and it contains a contribution coming from holomorphic cusp forms. In (2.36) this part is replaced by the contribution of irreducible representations belonging to the discrete series. Concerning the sum formula, the notes [3] by the first named author arose from the wish to understand various terms by means of the theory of automorphic representations; see also [4]. In this respect, our present work shares motivations with [3] to a considerable extent. Besides, we have been inspired by J. W. Cogdell and I. Piatetski-Shapiro [8].

3. Poincaré series

We now start a proof of Theorem B by appealing to the spectral theory of $L^2(\Gamma\backslash G)$. In this section we fix the Poincaré series on which our discussion is to be developed.

In place of (2.1) we put

$$f_{\psi\tau}(\mathrm{g}) = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi\left(\frac{ad}{bc}\right)\tau(ad), \quad G \ni \mathrm{g} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.1}$$

where

$$\psi^{(l)}(x) \ll \min(|x|^B, |x|^{-B}), \tag{3.2}$$

and

$$\tau(x) = 0, \quad x > 0; \quad \tau^{(l)}(x) \ll \min(|x|^B, |x|^{-B}), \tag{3.3}$$

for each l and any constant $B > 0$. In other words, it is assumed that all derivatives of $\psi(x)$ and $\tau(x)$ decay faster than any power of $|x|$ as x tends to 0 and infinity. The specialization $\psi(x) = \hat{g}\left(\frac{1}{2\pi}\log|x|\right)$ naturally satisfies (3.2). We consider the Poincaré series $\mathscr{P}f_{\psi\tau}(\mathrm{g})$ following (2.5). Ignoring the convergence issue temporarily, we have, as (2.7),

$$\frac{1}{2}\mathscr{T}\mathscr{P}f_{\psi\tau}(1) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_{w_1-w_4}(m)\sigma_{w_2-w_3}(m+n)}{m^{w_1}(m+n)^{w_2}}\psi\left(\frac{m}{m+n}\right)\tau\left(-\frac{m}{n}\right). \tag{3.4}$$

One may take the limit as τ tends to $\imath$ the characteristic function of the negative reals. Then the result is indeed comparable with (2.7) (see also (1.11)). However, in general the series $\mathscr{P}f_{\psi\tau}(\mathrm{g})$ does not converge for all g. To see this, let $\gamma_0 \in \Gamma$ be a hyperbolic element, and $\mathrm{g}_0 \in G$ be such that $\mathrm{g}_0^{-1}\gamma_0\mathrm{g}_0 = \mathrm{a}[\lambda]$ with $\lambda > 1$. Then, for any integer n, we have $f_{\psi\tau}(\gamma_0^n\mathrm{g}_0) = f_{\psi\tau}(\mathrm{g}_0)\lambda^{\frac{1}{2}n(w_2+w_4-w_1-w_3)}$, which obviously implies the divergence of $\mathscr{P}f_{\psi\tau}$ at g_0, provided $f_{\psi\tau}(\mathrm{g}_0) \neq 0$. Hence $\mathscr{T}\mathscr{P}f_{\psi\tau}$ is not well-defined. To overcome this difficulty, we introduce the modification

$$f_{\psi\tau\eta} = |a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi\left(\frac{ad}{bc}\right)\tau(ad)\eta\left(\frac{d}{c}\right), \tag{3.5}$$

with an η satisfying

$$\eta(-x) = \eta(x); \quad \eta^{(l)}(x) \ll \min(|x|^B, |x|^{-B}), \tag{3.6}$$

as before. Note that $f_{\psi\tau\eta}$ is left A-equivariant:

$$f_{\psi\tau\eta}(\mathrm{a}[y]\mathrm{g}) = y^{z_2-\frac{1}{2}} f_{\psi\tau\eta}(\mathrm{g}), \quad z_2 = \frac{1}{2}(w_3 + w_4 - w_1 - w_2 + 1). \tag{3.7}$$

Lemma 1. *Let $f = f_{\psi\tau\eta}$, with bounded w in the domain*

$$\mathrm{Re}(w_3 + w_4) > 2 + \mathrm{Re}(w_1 + w_2) > 4. \tag{3.8}$$

Then the Poincaré series $\mathcal{P}f$ converges absolutely and uniformly to an infinitely differentiable Γ-automorphic function on G. The same holds for $\mathcal{T}\mathcal{P}f$, and in particular

$$\frac{1}{2}\mathcal{T}\mathcal{P}f(1) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \psi\left(\frac{m}{m+n}\right)\tau\left(-\frac{m}{n}\right) \sum_{\substack{ad=m \\ bc=m+n}} a^{-w_1} b^{-w_2} c^{-w_3} d^{-w_4} \eta\left(\frac{d}{c}\right), \tag{3.9}$$

with positive integers a, b, c, d.

Remark. Throughout the sequel, this definition for f will be retained. The condition $\mathrm{Re}(w_3 + w_4) > 2 + \mathrm{Re}(w_1 + w_2) > 3$ is sufficient for the convergence of $\mathcal{P}f$, but $\mathcal{T}\mathcal{P}f$ requires (3.8). The heuristic identity (3.4) can be understood to be the limit of (3.9) as η tends to the characteristic function of $\mathbb{R}^\times$. Also, the limiting procedure mentioned above with respect to τ is to be considered. We shall perform these with an explicit choice of τ and η, in the final section. The infinite differentiability is required because of our appeal to Lemma 2 below. However, one may take a conceptually simpler approach, and then this stringent condition can be avoided, at the cost of an extra estimation procedure. See the remark at the end of this section. In particular, the smoothness condition on ψ, τ, η introduced above could considerably be relaxed, although it is irrelevant to our present purpose.

Proof. Let $G = AN \sqcup AN\mathrm{w}N$ be the Bruhat decomposition of G, where w, A, N are defined above. We have, for $\sin\theta \neq 0$, i.e., in the big cell,

$$\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta] = \begin{bmatrix} \dfrac{\sqrt{y}}{\sin\theta} & \\ & \dfrac{\sin\theta}{\sqrt{y}} \end{bmatrix} \mathrm{n}\left[\frac{x}{y}\sin^2\theta - \sin\theta\cos\theta\right]\mathrm{wn}[-\cot\theta]. \tag{3.10}$$

Since f vanishes on the small cell, we can restrict ourselves to the case $\sin\theta \neq 0$. We have

$$f(\mathrm{g}) \ll y^{\mathrm{Re}\, z_2 - \frac{1}{2}} \left|\frac{x}{y} - \cot\theta\right|^{-\mathrm{Re}\, w_1} \left|\frac{x}{y} + \tan\theta\right|^{-\mathrm{Re}\, w_2} \tag{3.11}$$

$$\times |\sin\theta|^{-\mathrm{Re}(w_1+w_3)} |\cos\theta|^{-\mathrm{Re}(w_2+w_4)} \left|\psi\left(\frac{\frac{x}{y} - \cot\theta}{\frac{x}{y} + \tan\theta}\right)\eta(-\cot\theta)\right|.$$

We claim that if $\mathrm{Re}\, w_j$ are all bounded then

$$|\sin\theta|^{-w_1-w_3}|\cos\theta|^{-w_2-w_4}\eta(-\cot\theta) \ll 1, \tag{3.12}$$

and if moreover $\mathrm{Re}(w_1+w_2) \geqq 0$ then

$$(3.13) \qquad \left|\frac{x}{y}-\cot\theta\right|^{-w_1}\left|\frac{x}{y}+\tan\theta\right|^{-w_2}\psi\left(\frac{\frac{x}{y}-\cot\theta}{\frac{x}{y}+\tan\theta}\right) \ll \min\left(1,\left|\frac{x}{y}\right|^{-\mathrm{Re}(w_1+w_2)}\right).$$

To prove (3.12) we may assume that $|\cos\theta| \leqq |\sin\theta|$, and consequentially

$$|\cos\theta| \leqq 1/\sqrt{2} \leqq |\sin\theta|.$$

Then, by (3.6) the left side is $\ll |\sin\theta|^{-C-\mathrm{Re}(w_1+w_3)}|\cos\theta|^{C-\mathrm{Re}(w_2+w_4)} \ll 1$. To prove (3.13) we need to consider the two cases $|x/y|<2$ and $|x/y|\geqq 2$ separately. In the first case we note that $|x/y-\cot\theta|+|x/y+\tan\theta| \geqq |\cot\theta+\tan\theta| \geqq 2$. Thus we may assume, for instance, that $X=|x/y-\cot\theta|\geqq 1$. We have, with $Y=|x/y+\tan\theta|$, that either $Y\geqq X\geqq 1$ or $X\geqq Y\geqq 1$ or $X\geqq 1\geqq Y$. The left side of (3.13) is, by (3.2),

$$(3.14) \qquad \ll X^{-\mathrm{Re}\,w_1}Y^{-\mathrm{Re}\,w_2}\min\left(\left(\frac{X}{Y}\right)^C,\left(\frac{Y}{X}\right)^C\right) \leqq 1,$$

provided $\mathrm{Re}(w_1+w_2)\geqq 0$. In the remaining case the left side of (3.13) is

$$(3.15) \qquad \ll \left|\frac{x}{y}\right|^{-\mathrm{Re}(w_1+w_2)} X_1^{-\mathrm{Re}\,w_1}Y_1^{-\mathrm{Re}\,w_2}\min\left(\left(\frac{X_1}{Y_1}\right)^C,\left(\frac{Y_1}{X_1}\right)^C\right),$$

where $X_1=|1-(y/x)\cot\theta|$, $Y_1=|1+(y/x)\tan\theta|$. We may assume, for instance, that $|\tan\theta|\leqq 1$. Then we have $1/2\leqq Y_1\leqq 3/2$, and thus (3.15) is $\ll |x/y|^{-\mathrm{Re}(w_1+w_2)}$, which gives (3.13). Summing up, we have

$$(3.16) \qquad f(\mathrm{g}) \ll y^{\mathrm{Re}\,z_2-\frac{1}{2}}\min\left(1,\left|\frac{x}{y}\right|^{-\mathrm{Re}(w_1+w_2)}\right),$$

provided $\mathrm{Re}\,w_j$ are all bounded and $\mathrm{Re}(w_1+w_2)\geqq 0$. It should be remarked that this is proved without taking into account the effect of the factor $\tau(ad)$.

The bound (3.16) gives

$$(3.17) \qquad \sum_{\mu\in\Gamma_\infty}|f(\mu\mathrm{g})| \ll (1+y)y^{\mathrm{Re}\,z_2-\frac{1}{2}}, \quad \Gamma_\infty=\Gamma\cap N,$$

if $\mathrm{Re}(w_1+w_2)>1$, and it follows that

$$(3.18) \qquad \mathscr{P}f(\mathrm{g}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{\mu\in\Gamma_\infty} f(\mu\gamma\mathrm{g})$$

is absolutely convergent for any g if (3.8) holds. In fact this is the result of comparing (3.18) with the Eisenstein series

$$(3.19) \qquad E_p(\mathrm{g};\nu) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\phi_p(\gamma\mathrm{g};\nu),$$

which converges absolutely for $\mathrm{Re}\,\nu > 1/2$. The assertion on the convergence of $\mathcal{T}\mathcal{P}f$ is now immediate, and the formula (3.9) follows readily.

It remains to prove that $\mathcal{P}f$ is infinitely differentiable throughout G. We may restrict ourselves to the case where none of the elements of the matrix g is equal to 0, for $\tau(ad) \neq 0$ implies this. The boundary situation can be discussed likewise. Computing it explicitly, we see that $\mathbf{X}f(\mathrm{g}) = (d/dt)_{t=0} f(\mathrm{g}\exp(\mathbf{X}t))$, with $\mathbf{X}$ as in (2.9) and g as in (3.1), is a linear combination of the five functions

$$(3.20)\qquad \begin{aligned} &a|a|^{-w_1}|b|^{-w_2-1}|c|^{-w_3}|d|^{-w_4}\psi(ad/(bc))\tau(ad)\eta(d/c),\\ &c|a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4-1}\psi(ad/(bc))\tau(ad)\eta(d/c),\\ &\frac{a}{b^2c}|a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi'(ad/(bc))\tau(ad)\eta(d/c),\\ &ac|a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi(ad/(bc))\tau'(ad)\eta(d/c),\\ &|a|^{-w_1}|b|^{-w_2}|c|^{-w_3}|d|^{-w_4}\psi(ad/(bc))\tau(ad)\eta'(d/c). \end{aligned}$$

They are majorized by the right side of (3.16) because of (3.2), (3.3) and (3.6), and $\mathcal{P}\mathbf{X}f$ is absolutely and uniformly convergent throughout G, provided (3.8). That is, $\mathbf{X}\mathcal{P}f = \mathcal{P}\mathbf{X}f$. Similarly, one may show the same for $\mathcal{P}\mathbf{Y}f$ and $\mathcal{P}\mathbf{H}f$. This procedure can be repeated indefinitely, by virtue of the fast decay of the derivatives of ψ, τ, η. Hence, for any $\mathbf{u} \in \mathcal{U}$ we have proved that $\mathbf{u}\mathcal{P}f = \mathcal{P}\mathbf{u}f$ in the pointwise sense, if (3.8) holds. We end the proof of the lemma.

More precisely, we have, for any fixed $\mathbf{u}$,

$$(3.21)\qquad \begin{aligned} \mathbf{u}\mathcal{P}f(\mathrm{g}) &= \sum_{\mu\in\Gamma_\infty} \mathbf{u}f(\mu\mathrm{g}) + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\notin\Gamma_\infty}} \sum_{\mu\in\Gamma_\infty} \mathbf{u}f(\mu\gamma\mathrm{g})\\ &= \sum_{\mu\in\Gamma_\infty} \mathbf{u}f(\mu\mathrm{g}) + O(y^{\frac{3}{2}-\mathrm{Re}\,z_2}), \end{aligned}$$

for sufficiently large y, provided w is as in Lemma 1. By the Poisson sum formula

$$(3.22)\qquad \sum_{\mu\in\Gamma_\infty} \mathbf{u}f(\mu\mathrm{g}) = \int_{-\infty}^{\infty} \mathbf{u}f(\mathrm{n}[u]\mathrm{g})\,du + O(y^{\frac{3}{2}-\mathrm{Re}\,z_2}),$$

for sufficiently large y. To see this, we use that $\mathbf{u}f(\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta]) = y^{z_2-\frac{1}{2}}\mathbf{u}f(\mathrm{n}[x/y]\mathrm{k}[\theta])$, and hence

$$(3.23)\qquad \begin{aligned} &\int_{-\infty}^{\infty} \mathbf{u}f(\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta])\exp(-2\pi imx)\,dx\\ &\qquad = y^{z_2+\frac{1}{2}} \int_{-\infty}^{\infty} \mathbf{u}f(\mathrm{n}[x]\mathrm{k}[\theta])\exp(-2\pi imyx)\,dx, \end{aligned}$$

which is $\ll (|m|y+1)^{-C}$ via integration in parts. The relations (3.21)–(3.23) show that $\mathbf{u}\mathcal{P}f$

is not in $L^2(\Gamma\backslash G)$, in general. Because of this, we subtract a Γ-invariant function from $\mathbf{u}\mathscr{P}f$ to have a square integrable function (an automorphic regularization): We put

$$\mathscr{P}_0 f = \mathscr{P}f - \mathscr{P}_\infty f, \tag{3.24}$$

where

$$\mathscr{P}_\infty f(\mathrm{g}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{-\infty}^{\infty} f(\mathrm{n}[u]\gamma\mathrm{g})\, du, \tag{3.25}$$

the convergence of which follows from (3.23) with $m = 0$, $\mathbf{u} = 1$. An examination of the above shows readily that for each $\mathbf{u} \in \mathscr{U}$

$$\mathbf{u}\mathscr{P}_0 f \in L^2(\Gamma\backslash G), \tag{3.26}$$

provided (3.8).

Observe that

$$\int_{-\infty}^{\infty} f(\mathrm{n}[u]\mathrm{g})\, du = \sum_{p=-\infty}^{\infty} f_p \phi_p(\mathrm{g}; z_2), \tag{3.27}$$

with $f_p \ll (|p|+1)^{-C}$. Hence

$$\mathscr{P}_\infty f(\mathrm{g}) = \sum_{p=-\infty}^{\infty} f_p E_p(\mathrm{g}; z_2). \tag{3.28}$$

Also, a computation shows that

$$\int_{-\infty}^{\infty} f(\mathrm{n}[u]\mathrm{g})\, du = |c|^{w_2-w_3-1}|d|^{w_1-w_4-1}\eta\left(\frac{d}{c}\right) \times \int_0^{\infty} \frac{1}{x^{w_1}(x+1)^{w_2}}\psi\left(\frac{x}{x+1}\right)\tau(-x)\, dx, \tag{3.29}$$

with g as in (3.1). In particular,

$$\mathscr{P}_\infty f(1) = 2\hat{\psi}_\tau(0)\sum_{c=1}^{\infty}\sum_{\substack{d=1\\(c,d)=1}}^{\infty} c^{w_2-w_3-1}d^{w_1-w_4-1}\eta\left(\frac{d}{c}\right), \tag{3.30}$$

anticipating (5.20).

We are about to invoke the point-wise spectral expansion in $L^2(\Gamma\backslash G)$, which is to be applied to $\mathscr{P}_0 f$. Then we need to quote the Fourier expansion of the Eisenstein series:

$$E_p(\mathrm{g};\nu) = \phi_p(\mathrm{g};\nu) + d_p(\nu)\phi_p(\mathrm{g};-\nu) + \frac{1}{\zeta(1+2\nu)}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} |n|^{-\frac{1}{2}-\nu}\sigma_{2\nu}(|n|)\mathscr{A}^{\mathrm{sgn}(n)}\phi_p(\mathrm{a}[|n|]\mathrm{g};\nu), \tag{3.31}$$

provided the right side is finite, where

$$d_p(\nu) = \frac{(-1)^p \pi \Gamma(2\nu)\zeta(2\nu)}{2^{2\nu-1}\zeta(2\nu+1)\Gamma\left(\frac{1}{2}+\nu+p\right)\Gamma\left(\frac{1}{2}+\nu-p\right)}. \tag{3.32}$$

We have also the functional equation

$$E_p(g;\nu) = d_p(\nu)E_p(g;-\nu), \quad d_p(\nu)d_p(-\nu) = 1. \tag{3.33}$$

The proof of (3.31)–(3.33) is practically the same as that of Lemma 1.2 of [19].

Making the assertions (2.10), (2.11), and (2.13) precise, we have

Lemma 2. *Let ϖ_V be the orthogonal projection to V, and ϖ_E to ${}^eL^2(\Gamma\backslash G)$. Let h be a vector such that $\mathbf{u}h \in L^2(\Gamma\backslash G)$ for any $\mathbf{u} \in \mathscr{U}$. Then the spectral decomposition*

$$h(\mathrm{g}) = \frac{3}{\pi}\langle h, 1\rangle_{\Gamma\backslash G} + \sum_V \varpi_V h(\mathrm{g}) + \varpi_E h(\mathrm{g}) \tag{3.34}$$

converges absolutely for each $\mathrm{g} \in G$. Similarly

$$\varpi_V h(\mathrm{g}) = \sum_{p=-\infty}^{\infty} \langle h, \varphi_p\rangle_{\Gamma\backslash G}\varphi_p(\mathrm{g}), \tag{3.35}$$

where φ_p are as above together with an obvious convention for V in the discrete series. Also

$$\varpi_E h(\mathrm{g}) = \sum_{p=-\infty}^{\infty} \int_{(0)} e_p(h;\nu)E_p(\mathrm{g};\nu)\frac{d\nu}{4\pi i}, \tag{3.36}$$

with

$$e_p(h;\nu) = \int_{\Gamma\backslash G} h(\mathrm{g})\overline{E_p(\mathrm{g};\nu)}\,d\mathrm{g}. \tag{3.37}$$

Proof. This assertion is taken from Section 1.2 of [8], which is based on [9]. Other approaches are possible. See the remark at the end of this section.

Hence, we have, by (3.26), a pointwise spectral decomposition of $\mathscr{P}_0 f$. We may put the result as

$$\mathscr{P}f(\mathrm{g}) = \mathscr{P}_\infty f(\mathrm{g}) + \sum_V \varpi_V \mathscr{P}_0 f(\mathrm{g}) + \varpi_E \mathscr{P}_0 f(\mathrm{g}), \tag{3.38}$$

with

$$\varpi_E \mathscr{P}_0 f(\mathrm{g}) = \sum_{p=-\infty}^{\infty} \int_{(0)} e_p^{(1)}(\mathscr{P}_0 f;\nu)E_p(\mathrm{g};\nu)\frac{d\nu}{4\pi i}. \tag{3.39}$$

Here, $e_p^{(1)}$ is the part of e_p corresponding to the third term on the right of (3.31). The identity (3.38) depends on the fact that $\langle \mathscr{P}_0 f, 1\rangle_{\Gamma\backslash G} = 0$ and $e_p(\mathscr{P}_0 f; \nu) = e_p^{(1)}(\mathscr{P}_0 f; \nu)$ for all p. Both follow readily from the definition (3.24) of $\mathscr{P}_0 f$.

The last sum can be taken inside the integral because of absolute convergence (see the remark given after (5.33) below). Applying the Hecke operator to (3.38) and (3.39) termwise, and invoking (2.26) and (3.28), we get, on (3.8),

$$\mathscr{T}\mathscr{P}f(\mathrm{g}) = \mathscr{T}\mathscr{P}_\infty f(\mathrm{g}) + \sum_V \mathscr{T}\varpi_V \mathscr{P}_0 f(\mathrm{g}) + \mathscr{T}(\varpi_E^{(0)} + \varpi_E^{(1)})\mathscr{P}_0 f(\mathrm{g}), \tag{3.40}$$

where

$$\begin{aligned} \mathscr{T}\mathscr{P}_\infty f(\mathrm{g}) &= \zeta(w_1 + w_2 - 1)\zeta(w_3 + w_4)\mathscr{P}_\infty f(\mathrm{g}), \\ \mathscr{T}\varpi_V \mathscr{P}_0 f(\mathrm{g}) &= H_V(z_1)\varpi_V \mathscr{P}_0 f(\mathrm{g}), \\ \mathscr{T}\varpi_E^{(j)}\mathscr{P}_0 f(\mathrm{g}) &= \int_{(0)} \zeta(z_1 + \nu)\zeta(z_1 - \nu)\mathscr{E}^{(j)}(\mathscr{P}_0 f; \mathrm{g}, \nu)\frac{d\nu}{4\pi i}, \end{aligned} \tag{3.41}$$

with z_1 as in (2.3). Here we have used $T_n E_p(\mathrm{g};\nu) = n^{-\nu}\sigma_{2\nu}(n)E_p(\mathrm{g};\nu)$, and have put

$$\mathscr{E}^{(j)}(\mathscr{P}_0 f; \mathrm{g}, \nu) = \sum_{p=-\infty}^{\infty} e_p^{(1)}(\mathscr{P}_0 f; \nu)E_p^{(j)}(\mathrm{g};\nu), \tag{3.42}$$

where $E_p^{(0)}$ is the sum of the first two terms on the right of (3.31) and $E_p^{(1)}$ the rest. Observe that the three Hecke series in (3.41), i.e., $H_V(z_1)$ and its analogues, are all absolutely bounded under (3.8).

In view of (3.9), we use (3.40) when g is the unit element. Namely, our original problem has been reduced to computing the quantities $\varpi_V \mathscr{P}_0 f(1)$ and $\mathscr{E}^{(j)}(\mathscr{P}_0 f; 1, \nu)$. Note that we have already (3.30).

Remark. The spectral decomposition in $L^2(\Gamma\backslash G)$ can be derived, via the Fourier expansion with respect to the right action of K, from that in $L^2(\Gamma\backslash \mathbb{H})$, thus for instance, from a minor extension of [19], Chapter 1. Naturally, the pointwise convergence in (3.38) is crucial for our purpose. Thus, it should be stressed that there is a way, based on explicit estimation, to achieve the same without recourse to Lemma 2 but rather starting with the convergence in $L^2(\Gamma\backslash G)$. The necessary uniform estimate is in fact provided by the discussion of either Section 5 or 6. This indicates that we may relax considerably the stringent decay condition on derivatives of ψ, τ, η imposed above.

4. Big cell

The unfolding argument reduces our task further to an application of the harmonic analysis in the big cell of the Bruhat decomposition, as is to be seen in the next section. Hence we collect here fundamentals in this context, which may be termed the Kirillov scheme collectively.

We first extend (2.15) by

$$\mathscr{A}^\delta\phi = \sum_{p=-\infty}^{\infty} c_p \mathscr{A}^\delta \phi_p, \quad \phi = \sum_{p=-\infty}^{\infty} c_p \phi_p, \tag{4.1}$$

where $\phi_p(\mathrm{g}) = \phi_p(\mathrm{g}; \nu)$, and ϕ is smooth, i.e., c_p decays faster than any negative power of $|p|$ as p tends to infinity. Note that we occasionally omit to mention ν. We shall show that (4.1) exists for any ν. For this and other purposes, the following estimates will be useful; bounds up to (4.5) are all uniform for p and $|\mathrm{Re}\,\nu| < 1/2$.

The first line of (2.16) gives

$$\begin{aligned}\mathscr{A}^\delta\phi_p(\mathrm{a}[y]) &= \mathscr{A}^\delta\phi_0(\mathrm{a}[y]) + y^{\frac{1}{2}-\nu} \int_{-\infty}^{\infty} \frac{\exp(2\pi i y\xi)}{(\xi^2+1)^{\frac{1}{2}+\nu}} \left(\left(\frac{\xi+i}{\xi-i}\right)^{\delta p} - 1 \right) d\xi \\ &= \frac{2\pi^{\frac{1}{2}+\nu}}{\Gamma\left(\frac{1}{2}+\nu\right)} y^{\frac{1}{2}} K_\nu(2\pi y) + O\big(y^{\frac{1}{2}-\mathrm{Re}\,\nu}(|p|+1)\big).\end{aligned} \tag{4.2}$$

By the power series expansion for K_ν, we get, as $y \to 0^+$,

$$\mathscr{A}^\delta\phi_p(\mathrm{a}[y]) \ll (|p|+|\nu|+1) y^{\frac{1}{2}-|\mathrm{Re}\,\nu|-\varepsilon}. \tag{4.3}$$

On the other hand, we have, by integration in parts,

$$\mathscr{A}^\delta\phi_p(\mathrm{a}[y]) = \frac{y^{-\frac{1}{2}-\nu}}{2\pi i} \int_{-\infty}^{\infty} \frac{((1+2\nu)\xi + 2\delta p i)\exp(2\pi i y\xi)}{(\xi^2+1)^{\frac{3}{2}+\nu}} \left(\frac{\xi+i}{\xi-i}\right)^{\delta p} d\xi. \tag{4.4}$$

Shifting the contour to $\mathrm{Im}\,\xi = (|\nu|+|p|+1)^{-1}$, we see that

$$\mathscr{A}^\delta\phi_p(\mathrm{a}[y]) \ll (|p|+|\nu|+1) y^{-\frac{1}{2}-\mathrm{Re}\,\nu} \exp\left(-\frac{y}{|\nu|+|p|+1}\right). \tag{4.5}$$

Repeating integration in parts in (4.4), we find that (4.1) converges in any fixed vertical strip of the ν-plane. Note that

$$\mathscr{A}^\delta\phi(\mathrm{g}) = \int_{-\infty}^{\infty} \exp(-2\pi\delta i x)\phi(\mathrm{wn}[x]\mathrm{g})\, dx, \tag{4.6}$$

for those ν in the domain where the integral converges uniformly. In fact the equality holds at least for $\mathrm{Re}\,\nu > 0$, and the assertion follows by analytic continuation.

We then define the Kirillov operator $\mathscr{K}$ by

$$\mathscr{K}\phi(u) = \mathscr{A}^{\mathrm{sgn}(u)}\phi(\mathrm{a}[|u|]), \quad u \in \mathbb{R}^\times. \tag{4.7}$$

This concept will play a crucial rôle in our argument, via the following three lemmas:

Lemma 3. *Let ϕ be smooth as in* (4.1). *We have, with the right translation R,*

$$\mathscr{K}R_{\mathrm{n}[x]}\phi(u) = \exp(2\pi iux)\mathscr{K}\phi(u), \quad \mathscr{K}R_{\mathrm{a}[y]}\phi(u) = \mathscr{K}\phi(uy). \tag{4.8}$$

Also, if $|\mathrm{Re}\,\nu| < \frac{1}{2}$, *then*

$$\mathscr{K}R_{\mathrm{w}}\phi(u) = \int_{\mathbf{R}^\times} j_\nu(u\lambda)\mathscr{K}\phi(\lambda)\, d^\times\lambda, \tag{4.9}$$

where j_ν is defined by (2.33).

Proof. The explicit description (4.9) of the action of the Weyl element is probably due to N. Ja. Vilenkin (see Section 7 of [21], Chapter VII, as well as the formula (17) of [22], p. 454). A rigorous proof can be found in [20], Theorem 2, which is developed in the context of automorphy but in fact asserts the above. It is shown there that the function

$$\Gamma_p(s) = \int_0^\infty \mathscr{A}^+\phi_p(\mathrm{a}[y])\, y^{s-\frac{3}{2}}\, dy \tag{4.10}$$

continues meromorphically to $\mathbb{C}$, and satisfies the Jacquet-Langlands local functional equation

$$\begin{aligned}(-1)^p\Gamma_p(s) = {} & 2^{1-2s}\pi^{-2s}\Gamma(s+\nu)\Gamma(s-\nu)\\ & \times \big(\cos\pi s\Gamma_p(1-s) + \cos\pi\nu\Gamma_{-p}(1-s)\big).\end{aligned} \tag{4.11}$$

The Mellin inversion of this coupled with (2.39)–(2.40) gives (4.9) for $\phi = \phi_p$. A combination of (2.45), (4.3), and (4.5) yields the necessary analytic continuation in ν, and the extension to smooth ϕ.

Lemma 4. *Let $\nu \in i\mathbb{R}$, and introduce the Hilbert space*

$$U_\nu = \overline{\bigoplus_{p=-\infty}^{\infty} \mathbb{C}\phi_p}, \quad \phi_p(\mathrm{g}) = \phi_p(\mathrm{g};\nu), \tag{4.12}$$

equipped with the norm

$$\|\phi\|_{U_\nu} = \sqrt{\sum_{p=-\infty}^{\infty} |c_p|^2}, \quad \phi = \sum_{p=-\infty}^{\infty} c_p\phi_p. \tag{4.13}$$

Then the operator $\mathscr{K}$ is a unitary map from U_ν onto $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$.

Proof. This seems to stem from A. A. Kirillov [15]. A proof of the unitaricity is given in [20], Theorem 1, though disguised in the context of automorphy. It depends on the following integral formula: For any $\alpha, \beta \in \mathbb{C}$ and $|\mathrm{Re}\,\nu| < 1/2$

$$(4.14)\qquad \int_0^\infty W_{\alpha,\nu}(u)W_{\beta,\nu}(u)\frac{du}{u}=\frac{\pi}{(\alpha-\beta)\sin(2\pi\nu)}$$
$$\times\left[\frac{1}{\Gamma\left(\frac{1}{2}-\alpha+\nu\right)\Gamma\left(\frac{1}{2}-\beta-\nu\right)}-\frac{1}{\Gamma\left(\frac{1}{2}-\alpha-\nu\right)\Gamma\left(\frac{1}{2}-\beta+\nu\right)}\right]$$

(see [11], formula 7.611(3)). The proof in [20] of this employs the Whittaker differential equation. Here we shall show the surjectivity of the map. Thus, let us assume that $\nu\in i\mathbb{R}$, and that a smooth function k, compactly supported on $\mathbb{R}^\times$, is orthogonal to all $\mathscr{K}\phi_p$. Multiply (4.4) by k and integrate, change the order of integration, and undo the integration in parts with respect to the outer integral. We have

$$(4.15)\qquad 0=\int_{\mathbb{R}^\times} k(u)\overline{\mathscr{K}\phi_p(u)}\,d^\times u$$
$$=\int_{-\infty}^{\infty}\frac{1}{(\xi^2+1)^{\frac{1}{2}+\nu}}\left(\frac{\xi+i}{\xi-i}\right)^p\int_{-\infty}^{\infty}k(u)|u|^{-\frac{1}{2}+\nu}\exp(-2\pi iu\xi)\,du\,d\xi.$$

Observe that the system $\{((\xi+i)/(\xi-i))^p : p\in\mathbb{Z}\}$ is complete orthonormal in the space $L^2(\mathbb{R},(\pi(\xi^2+1))^{-1}\,d\xi)$. Hence the Fourier transform of $k(u)|u|^{-\frac{1}{2}+\nu}$ vanishes identically, whence the assertion.

Next, we consider the complementary series or the situation with $-1/2<\nu<1/2$. This is included here only for the sake of completeness; such a representation of G does not occur in $L^2(\Gamma\backslash G)$. Obviously, Lemma 2 remains valid. The definition (4.12) is the same, but (4.13) has to be replaced by the norm

$$(4.16)\qquad \sqrt{\pi^{2\nu}\sum_{p=-\infty}^{\infty}\frac{\Gamma\left(p+\frac{1}{2}-\nu\right)}{\Gamma\left(p+\frac{1}{2}+\nu\right)}|c_p|^2}\,.$$

With this, the above proof extends readily, and Lemma 4 holds for these ν as well.

On the other hand, in dealing with the holomorphic discrete series, (4.12) needs to be replaced by the Hilbert space

$$(4.17)\qquad D_\ell=\overline{\bigoplus_{p=\ell}^{\infty}\mathbb{C}\phi_p},\quad \phi_p(g)=\phi_p\left(g;\ell-\frac{1}{2}\right),\quad 1\le\ell\in\mathbb{Z},$$

equipped with the norm

$$(4.18)\qquad \|\phi\|_{D_\ell}=\sqrt{\pi^{2\ell-1}\sum_{p=\ell}^{\infty}\frac{\Gamma(p-\ell+1)}{\Gamma(p+\ell)}|c_p|^2},\quad \phi=\sum_{p=\ell}^{\infty}c_p\phi_p.$$

Since $\mathscr{A}^-$ annihilates D_ℓ, we are concerned with $\mathscr{A}^+$ only. The expression (2.16), $\delta = +$, holds without changes. With this, the operator $\mathscr{K}$ is defined as before.

Lemma 5. *The operator $\mathscr{K}$ is a unitary map from D_ℓ onto $L^2((0,\infty), \pi^{-1}d^\times)$. Also, for any smooth $\phi \in D_\ell$, we have (4.9) with $\nu = \ell - \frac{1}{2}$. The analogue for the antiholomorphic discrete series is obtained by applying the involution ω.*

Proof. The third assertion is immediate. As to the unitaricity of $\mathscr{K}$, it is proved with a minor change of the above argument. In fact, the Whittaker function $W_{p,\ell-\frac{1}{2}}(u)$ $(p \geqq l)$ is a product of $\exp(-u/2)u^\ell$ and a polynomial on u of degree $p - \ell$, as (2.16) implies. Thus the proof of (4.14) in [20] can be carried out also for the product $W_{p,\ell-\frac{1}{2}}(u)\, W_{q,\ell-\frac{1}{2}}(u)$ with integers p, q, although the condition on $\operatorname{Re}\nu$ is violated. The result is equal to the limit of (4.14) as (α, β, ν) tends to $\left(p, q, \ell - \frac{1}{2}\right)$. As to the surjectivity, we argue as follows: Let k be smooth and compactly supported on $(0,\infty)$. If k is orthogonal to all $\mathscr{K}\phi_p$, $\ell \leqq p$, then we have, by the remark just made on $W_{p,\ell-\frac{1}{2}}(u)$,

$$\int_0^\infty k(u)\exp(-2\pi u)u^p \frac{du}{u} = 0, \quad \ell \leqq p. \tag{4.19}$$

This implies that the Fourier transform of $k(u)\exp(-2\pi u)u^{\ell-1}$ vanishes identically; in fact it suffices to expand the additive character into a power series and integrate termwise. Hence $k \equiv 0$. The counterpart of (4.9), with $\phi = \phi_p$, can be proved in much the same way as before. Its extension to any smooth ϕ is immediate with (2.46) and

$$\mathscr{K}\phi_p(u) = \mathscr{A}^+\phi_p(\mathrm{a}[u]) \ll \min(u, |p|+1)u^{-\ell}, \quad u > 0, \tag{4.20}$$

uniformly in p. This comes from (2.16) and (4.4).

Remark. The identity (4.9) is crucial for our purpose. In a context related to ours, this is given in [8], Theorem 4.1, without proof nor attribution. The formula seems to have appeared in print for the first time in [21], thus our partial attribution above, but the argument there lacks an adequate discussion on the convergence issue; the same can be said about [22]. A rigorous proof is given in [20], which is outlined above; the argument depends on the Mellin transform, and thus is different from those in [21], [22]. On the other hand, the recent article [2] provides an independent proof along the line of [21]. The present authors thank M. Baruch for this information. We stress that Lemmas 3 and 4 extend to $\mathrm{PSL}_2(\mathbb{C})$; see [6], Part XIII, which itself is an extension of [20]. The corresponding surjectivity assertion is not discussed in [6], Part XIII, but the above argument starting with (4.15) should extend to $\mathrm{PSL}_2(\mathbb{C})$ on the basis of the discussion in [7], Section 5.

5. Projections

We are now ready to deal with the task encountered at the end of Section 3 or the explicit computation of $\varpi_\nu \mathscr{P}_0 f(1)$ and $\mathscr{E}^{(j)}(\mathscr{P}_0 f; 1, \nu)$ in terms of ψ, τ, η. The basic implement is the Kirillov scheme. The condition (3.8) is imposed throughout the present section.

Let us first consider V in the unitary principal series, so that $\nu_V \in i\mathbb{R}$. Let φ_p be as in (2.14). Since Eisenstein series are orthogonal to any cuspidal element, we have

$$(5.1)\qquad \langle \mathscr{P}_0 f, \varphi_p \rangle_{\Gamma\backslash G} = \langle \mathscr{P} f, \varphi_p \rangle_{\Gamma\backslash G} = \int_G f(\mathrm{g})\overline{\varphi_p(\mathrm{g})}\, d\mathrm{g}.$$

The unfolding procedure in the second line is justified by (3.16) and the exponential decay of φ_p as $y \to \infty$. The latter follows from (2.30) and (4.5). We have

$$(5.2)\qquad \langle \mathscr{P}_0 f, \varphi_p \rangle_{\Gamma\backslash G} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\overline{\varrho_V(n)}}{\sqrt{|n|}} \int_G f(\mathrm{g})\overline{\mathscr{A}^{\operatorname{sgn}(n)}\phi_p(\mathrm{a}[|n|]\mathrm{g})}\, d\mathrm{g} = H_V(z_2)\big(\overline{\varrho_V(1)}\Phi_p^{+} + \overline{\varrho_V(-1)}\Phi_p^{-}\big) f(\nu_V),$$

where (2.27) and (3.7) have been used, and

$$(5.3)\qquad \Phi_p^{\delta} f(\nu) = \int_G f(\mathrm{g})\overline{\mathscr{A}^{\delta}\phi_p(\mathrm{g};\nu)}\, d\mathrm{g}.$$

The absolute convergence that is necessary to have the first line of (5.2) follows from that of (5.3), which in turn results from (2.30) and (4.5). By (3.35) or gathering together the projections of $\mathscr{P}_0 f$ to V_p, we have now

$$(5.4)\qquad \varpi_V \mathscr{P}_0 f(\mathrm{g}) = \frac{1}{2}\big(|\varrho_V(1)|^2 + |\varrho_V(-1)|^2\big) H_V(z_2) \times \sum_{n=1}^{\infty} \frac{t_V(n)}{\sqrt{n}} \big(\mathscr{B}^{(+,+)} + \mathscr{B}^{(-,-)} + \epsilon_V \mathscr{B}^{(+,-)} + \epsilon_V \mathscr{B}^{(-,+)}\big) \times f(\mathrm{a}[n]\mathrm{g};\nu_V),$$

with

$$(5.5)\qquad \mathscr{B}^{(\delta_1,\delta_2)} f(\mathrm{g};\nu) = \sum_{p=-\infty}^{\infty} \Phi_p^{\delta_1} f(\nu) \mathscr{A}^{\delta_2}\phi_p(\mathrm{g};\nu) = \exp(2\pi i \delta_2 x) \sum_{p=-\infty}^{\infty} \Phi_p^{\delta_1} f(\nu) \mathscr{A}^{\delta_2}\phi_p(\mathrm{a}[y]) \exp(2ip\theta).$$

We shall prove that the right side of (5.4) converges absolutely to a continuous function in V.

To this end, we show the bound

$$(5.6)\qquad \Phi_p^{\delta} f(\nu) \ll (|p| + |\nu| + 1)^{-C}, \quad |\operatorname{Re}\nu| < \frac{1}{2}.$$

A combination of (4.5) and (5.6) yields

$$\mathscr{B}^{(\delta_1,\delta_2)}f(\mathrm{g};\nu) \ll y^{\frac{1}{2}-|\mathrm{Re}\,\nu|-\varepsilon}\big((y+1)(|\nu|+1)\big)^{-C}, \tag{5.7}$$

in the same region of ν, whence the above claim on (5.4). To prove (5.6), observe, as in the proof of Lemma 1, that the function $\mathbf{u}f$ is bounded by the right side of (3.16), for any given $\mathbf{u} \in \mathscr{U}$. Thus, the second line of (2.16) and (4.5) give

$$\begin{aligned}\Phi_p^\delta \mathbf{u}f &\ll \int_0^\infty y^{\mathrm{Re}\,z_2-\frac{3}{2}}|\mathscr{A}^\delta\phi_p(\mathrm{a}[y])|\,dy \\ &\ll (|p|+|\nu|+1)^{\mathrm{Re}\,z_2-\mathrm{Re}\,\nu}.\end{aligned} \tag{5.8}$$

Since $\mathscr{A}^\delta$ is an intertwining operator with respect to the action of the elements of $\mathscr{U}$, we have, for any positive integer q,

$$\Phi_p^\delta(\Omega+i\partial_\theta^2)^q f = \left(\nu^2-\frac{1}{4}-4ip^2\right)^q \Phi_p^\delta f, \tag{5.9}$$

by integration in parts, which can be justified with (5.8). This obviously gives (5.6).

We may now look at $\mathscr{B}^{(\delta_1,\delta_2)}f(\mathrm{a}[y];\nu)$ closer with the Kirillov scheme: We assume that $\nu \in i\mathbb{R}$. We have

$$\mathscr{B}^{(\delta_1,\delta_2)}f(\mathrm{a}[y];\nu) = \sum_{p=-\infty}^{\infty}\Phi_p^{\delta_1}f(\nu)\mathscr{K}\phi_p(\delta_2 y) = \mathscr{K}\mathscr{L}^{\delta_1}f(\delta_2 y), \tag{5.10}$$

where

$$\mathscr{L}^\delta f = \sum_{p=-\infty}^{\infty}\Phi_p^\delta f(\nu)\phi_p \tag{5.11}$$

is a smooth element in U_ν, because of (5.6). The unitaricity assertion in Lemma 4 gives

$$\Phi_p^\delta f = \langle \mathscr{L}^\delta f, \phi_p\rangle_{U_\nu} = \frac{1}{\pi}\int_{\mathbb{R}^\times}\mathscr{K}\mathscr{L}^\delta f(u)\overline{\mathscr{K}\phi_p(u)}\,d^\times u. \tag{5.12}$$

This and the surjectivity assertion there imply that if one can transform (5.3) into

$$\Phi_p^\delta f = \frac{1}{\pi}\int_{\mathbb{R}^\times} Y^\delta(u)\overline{\mathscr{K}\phi_p(u)}\,d^\times u \tag{5.13}$$

then it should follow that

$$\mathscr{B}^{(\delta_1,\delta_2)}f(\mathrm{a}[y]) = Y^{\delta_1}(\delta_2 y). \tag{5.14}$$

Note that we have used implicitly a simple continuity argument, which will be made explicit at (5.22).

We may write (5.3) as

$$\Phi_p^\delta f = \int_0^\infty u^{z_2-\frac{3}{2}} \int_{NwN} f(\mathrm{g})\overline{R_{\mathrm{g}}\mathscr{A}^\delta\phi_p(\mathrm{a}[u])}\, d\dot{\mathrm{g}}\, du. \tag{5.15}$$

Here $\mathrm{g} = \mathrm{n}[x_1]\mathrm{wn}[x_2]$ and $d\dot{\mathrm{g}} = \pi^{-1}\, dx_1\, dx_2$; the formula (3.10) gives the Jacobian for this change of variables. We observe

$$\begin{aligned} R_{\mathrm{g}}\mathscr{A}^\delta\phi_p(\mathrm{a}[u]) &= R_{\mathrm{wn}[x_2]}\mathscr{A}^\delta\phi_p(\mathrm{n}[x_1u]\mathrm{a}[u]) \\ &= \exp(2\pi i\delta x_1 u)R_{\mathrm{wn}[x_2]}\mathscr{A}^\delta\phi_p(\mathrm{a}[u]) \\ &= \exp(2\pi i\delta x_1 u)\mathscr{A}^\delta R_{\mathrm{w}}R_{\mathrm{n}[x_2]}\phi_p(\mathrm{a}[u]). \end{aligned} \tag{5.16}$$

By Lemma 3

$$\begin{aligned} \mathscr{A}^\delta R_{\mathrm{w}}R_{\mathrm{n}[x_2]}\phi_p(\mathrm{a}[u]) &= \mathscr{K}R_{\mathrm{w}}R_{\mathrm{n}[x_2]}\phi_p(\delta u) \\ &= \int_{\mathbf{R}^\times} j_\nu(\delta u\lambda)\mathscr{K}R_{\mathrm{n}[x_2]}\phi_p(\lambda)\, d^\times\lambda \\ &= \int_{\mathbf{R}^\times} \exp(2\pi i x_2\lambda)j_\nu(\delta u\lambda)\mathscr{K}\phi_p(\lambda)\, d^\times\lambda. \end{aligned} \tag{5.17}$$

Thus

$$\begin{aligned} \Phi_p^\delta f = \frac{1}{\pi}\int_0^\infty u^{z_2-\frac{3}{2}} &\int_{\mathbf{R}^2} f(\mathrm{n}[x_1]\mathrm{wn}[x_2])\exp(-2\pi i\delta x_1 u) \\ &\times \int_{\mathbf{R}^\times}\exp(-2\pi i x_2\lambda)j_\nu(\delta u\lambda)\overline{\mathscr{K}\phi_p(\lambda)}\, d^\times\lambda\, dx_1\, dx_2\, du, \end{aligned} \tag{5.18}$$

where we have used that $j_\nu = j_{-\nu}$. Applying change of variables $x_1 \to \dfrac{x_1}{x_2}$, $x_2 \to -x_2$, we have

$$\begin{aligned} \Phi_p^\delta f = \frac{1}{\pi}\int_0^\infty u^{z_2-\frac{3}{2}} &\int_{\mathbf{R}^\times} |x_2|^{w_1-w_4}\hat{\psi}_\tau\left(\delta\frac{u}{x_2}\right)\eta(x_2) \\ &\times \int_{\mathbf{R}^\times}\exp(2\pi i\lambda x_2)j_\nu(\delta u\lambda)\overline{\mathscr{K}\phi_p(\lambda)}\, d^\times\lambda\, d^\times x_2\, du, \end{aligned} \tag{5.19}$$

with

$$\hat{\psi}_\tau(u) = \int_0^\infty \frac{1}{x_1^{w_1}(x_1+1)^{w_2}}\psi\left(\frac{x_1}{x_1+1}\right)\tau(-x_1)\exp(-2\pi i u x_1)\, dx_1. \tag{5.20}$$

Here (3.3) and (3.6) have been used. The triple integral in (5.19) converges absolutely. In fact, a multiple application of integration in parts gives, for each l,

$$\left(\frac{d}{du}\right)^l \hat{\psi}_\tau(u) \ll (|u|+1)^{-C}, \tag{5.21}$$

because of (3.3). A combination of (2.45), (4.3), (4.5), and (5.21) yields that the integral whose integrand is the absolute value of that in (5.19) is $\ll |p|+1$, with the implied constant depending on ν. Hence we have, for any smooth $\phi \in U_\nu$,

$$(5.22) \qquad \langle \mathscr{L}^\delta f, \phi \rangle_{U_\nu} = \frac{1}{\pi} \int_{\mathbf{R}^\times} \left\{ \int_0^\infty u^{z_2 - \frac{3}{2}} \int_{\mathbf{R}^\times} |x_2|^{w_1 - w_4} \right. \\ \left. \times\, j_\nu(\delta u \lambda) \hat{\psi}_\tau\left(\delta \frac{u}{x_2}\right) \eta(x_2) \exp(2\pi i \lambda x_2)\, d^\times x_2\, du \right\} \overline{\mathscr{K}\phi(\lambda)}\, d^\times \lambda.$$

Via (5.10), Lemma 4 now gives rise to

$$(5.23) \qquad \mathscr{B}^{(\delta_1,\delta_2)} f(\mathrm{a}[y];\nu) = \mathscr{B}^{\delta_1\delta_2} f(\mathrm{a}[y];\nu),$$

with

$$(5.24) \qquad \mathscr{B}^\delta f(\mathrm{a}[y];\nu) = \int_0^\infty u^{z_2 - \frac{3}{2}} j_\nu(\delta u y) \\ \times \int_{\mathbf{R}^\times} |x_2|^{w_1 - w_4} \hat{\psi}_\tau\left(\delta \frac{u}{x_2}\right) \eta(x_2) \exp(2\pi i y x_2)\, d^\times x_2\, du.$$

Inserting this into (5.4), we obtain, via (3.41),

Lemma 6. *If V is in the unitary principal series, then*

$$(5.25) \qquad \mathscr{T}\varpi_V \mathscr{P}_0 f(1) = \left(|\varrho_V(1)|^2 + |\varrho_V(-1)|^2\right) H_V(z_1) H_V(z_2) \\ \times \sum_{n=1}^\infty \frac{t_V(n)}{\sqrt{n}} (\mathscr{B}^+ + \epsilon_V \mathscr{B}^-) f(\mathrm{a}[n];\nu_V),$$

provided (3.8).

Next, we treat the discrete series. We assume that V is in the holomorphic discrete series, having the complete orthonormal system $\{\varphi_p : p \geqq \ell\}$ with φ_p given in (2.21). The relation (5.1) extends as it is; the unfolding procedure depends on the observation on the Whittaker function $W_{p,\ell-\frac{1}{2}}$ made in the proof of Lemma 5. Then, (5.2), with an obvious interpretation of (5.3), is replaced by

$$(5.26) \qquad \langle \mathscr{P} f, \varphi_p \rangle_{\Gamma\backslash G} = \pi^{\frac{1}{2}-\ell}\, \overline{\varrho_V(1)} H_V(z_2) \left(\frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \right)^{\frac{1}{2}} \Phi_p^+ f\left(\ell - \frac{1}{2}\right),$$

and (5.4) by

$$(5.27) \qquad \varpi_V \mathscr{P}_0 f(\mathrm{g}) = |\varrho_V(1)|^2 H_V(z_2) \sum_{n=1}^\infty \frac{t_V(n)}{\sqrt{n}} \mathscr{B} f\left(\mathrm{a}[n]\mathrm{g}; \ell - \frac{1}{2}\right).$$

Here

$$(5.28) \qquad \mathscr{B} f\left(\mathrm{g}; \ell - \frac{1}{2}\right) = \pi^{1-2\ell} \sum_{p=\ell}^\infty \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \Phi_p^+ f\left(\ell - \frac{1}{2}\right) \mathscr{A}^+ \phi_p\left(\mathrm{g}; \ell - \frac{1}{2}\right),$$

which replaces (5.5). The $\mathscr{B}f$ exists as a continuous function in V. On noting (4.20), this follows from $\Phi_p^+ f\left(\ell - \frac{1}{2}\right) \ll (|p|+1)^{-C}$. To get the latter, we observe that for $\mathbf{u} = \partial_\theta^q \in \mathscr{U}$, with any positive integer q,

$$\langle \mathscr{P}\mathbf{u}f, \varphi_p\rangle_{\Gamma\backslash G} = \langle \mathbf{u}\mathscr{P}f, \varphi_p\rangle_{\Gamma\backslash G} = (-1)^q \langle \mathscr{P}f, \bar{\mathbf{u}}\varphi_p\rangle_{\Gamma\backslash G}. \tag{5.29}$$

We use (5.26) on the right side and $|\langle \mathscr{P}\mathbf{u}f, \varphi_p\rangle_{\Gamma\backslash G}| \leqq \|\mathscr{P}_0\mathbf{u}f\|_{\Gamma\backslash G}$ on the left, which confirms the claim. We have actually proved that $\varpi_V \mathscr{P}_0 f$ exists as a continuous function in V.

We prove an extension of (5.24). This is now easy: We put, in place of $\mathscr{L}^\delta f$,

$$\mathscr{L}f = \pi^{1-2\ell} \sum_{p=\ell}^{\infty} \frac{\Gamma(p+\ell)}{\Gamma(p-\ell+1)} \Phi_p^+ f\left(\ell - \frac{1}{2}\right)\phi_p, \tag{5.30}$$

which is a smooth element in D_ℓ. Then, we can proceed much like (5.10)–(5.22), relying on Lemma 5 and (2.46), (4.20). Thus, we have

$$\mathscr{B}f(\mathrm{a}[y]) = \mathscr{B}^+ f\left(\mathrm{a}[y]; \ell - \frac{1}{2}\right), \tag{5.31}$$

with an extended use of notation.

Hence, taking into account the antiholomorphic discrete series as well, we have

Lemma 7. *If V is in the discrete series, then*

$$\begin{aligned} &\mathscr{T}\varpi_V \mathscr{P}_0 f(1) \\ &\quad = \left(|\varrho_V(1)|^2 + |\varrho_V(-1)|^2\right) H_V(z_1) H_V(z_2) \sum_{n=1}^{\infty} \frac{t_V(n)}{\sqrt{n}} \mathscr{B}^+ f(\mathrm{a}[n]; \nu_V), \end{aligned} \tag{5.32}$$

provided (3.8).

We now turn to the contribution of Eisenstein series. We see readily that

$$e_p^{(1)}(\mathscr{P}_0 f; \nu) = \frac{\zeta(z_2+\nu)\zeta(z_2-\nu)}{\zeta(1-2\nu)} (\Phi_p^+ + \Phi_p^-) f(\nu). \tag{5.33}$$

This and (5.6) confirm our claim on the convergence of (3.39) that is made prior to (3.40). The discussion of $\mathscr{E}^{(1)}(\mathscr{P}_0 f; \mathrm{a}[y], \nu)$ is obviously analogous to that of $\varpi_V \mathscr{P}_0 f(\mathrm{a}[y])$ with V in the unitary principal series. Hence, it suffices to state

Lemma 8. *Let $Z(\nu) = \zeta(z_1+\nu)\zeta(z_1-\nu)\zeta(z_2+\nu)\zeta(z_2-\nu)$. Then*

$$\begin{aligned} &\mathscr{T}\varpi_E^{(1)} \mathscr{P}_0 f(1) \\ &\quad = \int_{(0)} \frac{Z(\nu)}{\zeta(1+2\nu)\zeta(1-2\nu)} \left\{ \sum_{n=1}^{\infty} \frac{\sigma_{2\nu}(n)}{n^{\frac{1}{2}+\nu}} (\mathscr{B}^+ + \mathscr{B}^-) f(\mathrm{a}[n]; \nu) \right\} \frac{d\nu}{2\pi i}, \end{aligned} \tag{5.34}$$

provided (3.8).

As to $\mathscr{E}^{(0)}(\mathscr{P}_0 f; \mathrm{a}[y], \nu)$, we observe that the functional equation (3.33) implies the relation $d_p(\nu)e_p(\cdot\,;\nu) = e_p(\cdot\,;-\nu)$. Thus

$$\mathscr{E}^{(0)}(\mathscr{P}_0 f; 1, \nu) = \mathscr{D}(\mathscr{P}_0 f; \nu) + \mathscr{D}(\mathscr{P}_0 f; -\nu), \tag{5.35}$$

where

$$\mathscr{D}(\mathscr{P}_0 f; \nu) = \frac{\zeta(z_2+\nu)\zeta(z_2-\nu)}{\zeta(1-2\nu)}(\mathscr{C}^+ + \mathscr{C}^-)f(\nu), \tag{5.36}$$

with

$$\mathscr{C}^\delta f(\nu) = \sum_{p=-\infty}^{\infty} \Phi_p^\delta f(\nu). \tag{5.37}$$

To compute this, we put

$$\mathscr{C}_\delta f(\nu) = \sum_{p=-\infty}^{\infty} \int_G f(\mathrm{g}) \mathscr{A}^{-\delta}\phi_{-p}(\mathrm{g}; -\nu)\, d\mathrm{g}, \tag{5.38}$$

which is regular for $|\mathrm{Re}\,\nu| < 1/2$, since the integral satisfies the same bound as (5.6). We have $\mathscr{C}_\delta f(\nu) = \mathscr{C}^\delta f(\nu)$ on the imaginary axis. Let us suppose $-1/2 < \mathrm{Re}\,\nu < 0$. In (5.38), use the first line of (2.16), but with the contour $\mathrm{Im}\,\xi = 1/2$, so that the quadruple integral converges absolutely; here we need (3.16). Take the integral over K innermost, and apply integration in parts many times, while noting that $\partial_\theta^q f$ with any fixed q still satisfies the bound (3.16). We see now that the sum over p can be taken inside the first triple integral. Then we may shift the ξ-contour back to $\mathbb{R}$. Undoing integration in parts, we get

$$\begin{aligned}\mathscr{C}_\delta f(\nu) = \int_0^\infty y^{-\frac{3}{2}+\nu} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\big(2\pi i(y\xi - \delta x)\big)}{(\xi^2+1)^{-\nu+\frac{1}{2}}} \\ \times \sum_{p=-\infty}^{\infty} \left(\frac{\xi+i}{\xi-i}\right)^{\delta p} \int_0^\pi f(\mathrm{n}[x]\mathrm{a}[y]\mathrm{k}[\theta]) \exp(-2pi\theta)\frac{d\theta}{\pi}\, d\xi\, dx\, dy,\end{aligned} \tag{5.39}$$

and thus

$$\mathscr{C}_\delta f(\nu) = \int_0^\infty y^{z_2-1+\nu} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp\big(2\pi i y(\xi - \delta x)\big)}{(\xi^2+1)^{-\nu+\frac{1}{2}}} f(\mathrm{n}[x]\mathrm{k}_\xi)\, d\xi\, dx\, dy, \tag{5.40}$$

with

$$\mathrm{k}_\xi = \frac{1}{\sqrt{\xi^2+1}}\begin{bmatrix}\xi & \delta \\ -\delta & \xi\end{bmatrix} \in K. \tag{5.41}$$

The last double integral over (ξ, x) converges absolutely. We take the x-integral innermost, and perform the change of variable $x \mapsto \delta\xi + \delta x(\xi + \xi^{-1})$, getting

$$(5.42)\qquad \mathscr{C}_\delta f(v) = \int_0^\infty y^{z_2-1+v} \int_{-\infty}^{\infty} |\xi|^{w_1-w_4-1}(\xi^2+1)^{z_2+v}\hat{\psi}_\tau\left(\left(\xi+\frac{1}{\xi}\right)y\right)\eta(\xi)\,d\xi\,dy$$
$$= \left((\hat{\psi}_\tau)^+ + (\hat{\psi}_\tau)^-\right)(z_2+v)\eta^*(z_3+v),$$

where $(\hat{\psi}_\tau)^\pm$ and η^* are Mellin transforms on $(0,\infty)$ of $\hat{\psi}_\tau(\pm\cdot)$ and η, respectively; and

$$(5.43)\qquad z_3 = \frac{1}{2}(w_1+w_3-w_2-w_4+1).$$

In deducing (5.42), we have used (3.6) and (5.21). The second line of (5.42) is a regular function of v in a neighbourhood of the imaginary axis; thus, it is equal to $\mathscr{C}^\delta f(v)$ if $v \in i\mathbb{R}$.

Summing up, we obtain

Lemma 9.

$$(5.44)\qquad \mathscr{T}\varpi_E^{(0)}\mathscr{P}_0 f(1) = \int_{(0)} \frac{Z(v)}{\zeta(1-2v)}\left((\hat{\psi}_\tau)^+ + (\hat{\psi}_\tau)^-\right)(z_2+v)\eta^*(z_3+v)\frac{dv}{\pi i},$$

provided (3.8).

6. Limit

Collecting (3.30), (3.40), (3.41), and Lemmas 6–9, we arrive at a spectral decomposition of $\mathscr{T}\mathscr{P}f(1)$. Thus a major part of our proof of Theorem B has been finished. It remains for us to discuss the behaviour of this decomposition as η tends to the characteristic function of $\mathbb{R}^\times$, and τ to that of negative reals. This limiting procedure is our main task in the present section. Our basic implement here is the Mellin transform.

To facilitate various convergence issues, we restrict w by

(6.1)
$$\left\{w : \text{bounded and } \operatorname{Re} w_3 > \operatorname{Re} w_2 + 3 > 4, \frac{3}{2} + \operatorname{Re} w_1 > \operatorname{Re} w_4 > \operatorname{Re} w_1 + 1 > 2\right\},$$

which is obviously contained in (3.8). Also we set

$$(6.2)\qquad \eta(x) = \exp\left(-\frac{1}{2}\xi_1\left(|x|+\frac{1}{|x|}\right)\right),\quad x\in\mathbb{R}^\times,$$

and

$$(6.3)\qquad \tau(x)=0,\quad x\geqq 0;\quad \tau(x) = \exp\left(-\frac{1}{2}\xi_2\left(|x|+\frac{1}{|x|}\right)\right),\quad x<0,$$

where both $\xi_1, \xi_2 > 0$ are supposed to tend to 0. It is immediate that (3.9) implies

$$(6.4)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\mathscr{T}\mathscr{P}f(1)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sigma_{w_1-w_4}(m)\sigma_{w_2-w_3}(m+n)}{m^{w_1}(m+n)^{w_2}}\psi\left(\frac{m}{m+n}\right).$$

Via this relation a spectral decomposition of the right side member arises.

We consider first the contribution of the unitary principal series representations. To this end, we transform (5.24) into a Mellin inverse integral: Put

$$(6.5)\qquad \psi_\tau^*(s)=\int_0^\infty \frac{1}{(x+1)^{w_2}}\psi\left(\frac{1}{x+1}\right)\tau(-x)x^{s-1}\,dx,$$

which is regular and of fast decay in any vertical strip in the s-plane.

Lemma 10. *We have, for $\nu \in i\mathbb{R}$,*

$$(6.6)\qquad \mathscr{B}^\delta f(\mathrm{a}[y];\nu)=y^{-z_3+\frac{1}{2}}\int_{(0)}\eta^*(s_1)\Psi_\tau^\delta(s_1;\nu)(2\pi y)^{-s_1}\,ds_1,$$

where η^* is as in (5.42), *and*

$$(6.7)\qquad \Psi_\tau^\delta(s_1;\nu)=-4(2\pi)^{w_2-w_3-3}\int_{(\alpha)}\psi_\tau^*(s_2)\cos\left(\frac{1}{2}\pi\big((1+\delta)(z_1-s_2)+(1-\delta)\nu\big)\right)$$
$$\times\cos\left(\frac{1}{2}\pi\big(s_1+s_2+1-w_2-w_4-\delta(s_2+1-w_1-w_2)\big)\right)$$
$$\times\Gamma(z_1-s_2+\nu)\Gamma(z_1-s_2-\nu)\Gamma(s_2+1-w_1-w_2)$$
$$\times\Gamma(s_1+s_2+1-w_2-w_4)\,ds_2.$$

Here α is to satisfy $\mathrm{Re}\,z_1>\alpha>\mathrm{Re}(w_1+w_2)-1$. *Such an α exists if* (6.1) *holds.*

Proof. We first transform (5.20) into a Mellin inverse integral. To this end, we invert (6.5) and have

$$(6.8)\qquad \frac{1}{(x+1)^{w_2}}\psi\left(\frac{x}{x+1}\right)\tau(-x)=\frac{1}{2\pi i}\int_{(\alpha)}\psi_\tau^*(s)x^{s-w_2}\,ds,$$

with an arbitrary α. Here we have used $\tau(x)=\tau(1/x)$. Multiply both sides by the factor $x^{-w_1}\exp(-ax-2\pi iux)$, $a>0$, and integrate over $(0,\infty)$. The left side converges uniformly for $a \geqq 0$. Moving the contour (α) to the right if necessary, the double integral on the right side is seen to converge absolutely, provided $a>0$. Exchange the order of integration, compute the inner integral explicitly, and observe that the resulting integral is uniformly convergent for $a \geqq 0$, because of the fast decay of ψ_τ^*. Thus we have, for $u\in\mathbb{R}^\times$,

$$(6.9)\qquad \hat{\psi}_\tau(u) = \frac{1}{2\pi i}\int_{(\alpha)} \psi_\tau^*(s)(2\pi|u|)^{w_1+w_2-1-s}\Gamma(s+1-w_1-w_2)$$
$$\times \exp\left(-\frac{1}{2}\pi i\,\mathrm{sgn}(u)(s+1-w_1-w_2)\right) ds,$$

with any $\alpha > \mathrm{Re}(w_1+w_2)-1$.

To the inner integral of (5.24) we apply a similar procedure: Multiply the integrand by the factor $\exp(-a|x|)$, $a>0$, replace $\hat{\psi}_\tau$ by (6.9), and exchange the order of integration. We get the expression

$$(6.10)\qquad \frac{1}{2\pi i}\int_{(\alpha)} \psi_\tau^*(s_2)(2\pi u)^{w_1+w_2-1-s_2}\Gamma(s_2+1-w_1-w_2)$$
$$\times \sum_{\pm}\exp\left(\pm\frac{1}{2}\pi i\delta(s_2+1-w_1-w_2)\right)$$
$$\times \int_0^\infty x^{s_2-w_2-w_4}\exp\big(-(a\pm 2\pi i y)x\big)\eta(x)\,dx\,ds_2.$$

On noting that $\eta(x)=\eta(1/x)$, use the Mellin inverse of η^*. Because of the uniform convergence for $a \geqq 0$, we see that the integral in question is equal to

$$(6.11)\qquad -\frac{1}{2\pi^2}\int_{(0)} \eta^*(s_1)\int_{(\alpha)} \psi_\tau^*(s_2)(2\pi u)^{w_1+w_2-1-s_2}(2\pi y)^{w_2+w_4-1-s_1-s_2}$$
$$\times \Gamma(s_2+1-w_1-w_2)\Gamma(s_1+s_2+1-w_2-w_4)$$
$$\times \cos\left(\frac{1}{2}\pi\big(s_1+s_2+1-w_2-w_4-\delta(s_2+1-w_1-w_2)\big)\right) ds_2\,ds_1,$$

provided (6.1). Let us assume temporarily that α is such that $0<\mathrm{Re}\,z_1-\alpha<1/4$ as well as $\alpha>\mathrm{Re}(w_1+w_2)-1$, which does not conflict with (6.1). We insert (6.11) into (5.24). The resulting triple integral converges absolutely, because of (2.45). We take the u-integral innermost and invoke (2.39)–(2.40) on noting we have presently $\nu\in i\mathbb{R}$. In the result we can drop the condition $\mathrm{Re}\,z_1-\alpha<1/4$. This ends the proof.

The representation (6.7) shows that $\Psi_\tau^\delta(s_1;\nu)$ is in fact regular with respect to ν in a neighbourhood of the imaginary axis. A shift to the far right of the contour in (6.7) gives

$$(6.12)\qquad \Psi_\tau^\delta(s_1;\nu) \ll \big((|s_1|+1)/(|\nu|+1)\big)^C,$$

uniformly as $|\mathrm{Re}\,\nu|, |\mathrm{Re}\,s_1|<\varepsilon$, provided (6.1) holds. This is a consequence of the fast decay of ψ_τ^*.

Let us now take (6.2)–(6.3) into account precisely, and consider the limit of $\mathscr{B}^\delta(\mathrm{a}[y];\nu)$ as ξ_1,ξ_2 tend to 0: We claim first that

$$(6.13)\qquad \mathscr{B}^\delta f(\mathrm{a}[y];\nu) = 2\pi i y^{-z_3+\frac{1}{2}}\Psi_\tau^\delta(0;\nu) + O\big(y^{-\mathrm{Re}\,z_3+\frac{1}{2}}(\xi_1 y)^\varepsilon(|\nu|+1)^{-C}\big),$$

uniformly as $\xi_1 \to 0^+$. To confirm this, we shift the contour in (6.6) to (ε), and note that, since (6.2) implies $\eta^*(s_1) = 2K_{s_1}(\xi_1)$,

$$(6.14) \qquad \eta^*(s_1) = \frac{\pi}{\sin \pi s_1}\left(I_{-s_1}(\xi_1) - I_{s_1}(\xi_1)\right)$$
$$= \frac{\pi}{\sin \pi s_1}\frac{1}{\Gamma(1-s_1)}\left(\frac{\xi_1}{2}\right)^{-s_1} + O\left(\xi_1^{\mathrm{Re}\, s_1}\exp(-|s_1|)\right),$$

with the implied constant being absolute. The bound (6.12) yields that the error-term contributes negligibly; and as to the main term it suffices to shift the contour to $(-\varepsilon)$.

We insert (6.13) into (5.25), and get

$$(6.15) \qquad \mathcal{T}\varpi_V\mathcal{P}_0 f(1) = 2\pi i\left(|\varrho_V(1)|^2 + |\varrho_V(-1)|^2\right)H_V(z_1)H_V(z_2)H_V(z_3)$$
$$\times (\Psi_\tau^+ + \epsilon_V\Psi_\tau^-)(0;\nu_V) + O\left(|\varrho_V(1)|^2\xi_1^\varepsilon(|\nu_V|+1)^{-C}\right),$$

in which we have used the fact that (6.1) implies $\mathrm{Re}\, z_3 > 5/4$. Because of (2.29), we find that

$$(6.16) \qquad \lim_{\xi_1\to 0^+}\sum_V \mathcal{T}\varpi_V\mathcal{P}_0 f(1) = 2\pi i\sum_V\left(|\varrho_V(1)|^2 + |\varrho_V(-1)|^2\right)$$
$$\times H_V(z_1)H_V(z_2)H_V(z_3)(\Psi_\tau^+ + \epsilon_V\Psi_\tau^-)(0;\nu_V),$$

with V running over all irreducible representations in the unitary principal series.

Next, we observe that for $\mathrm{Re}\, s > 0$

$$(6.17) \qquad \psi_\tau^*(s) = \frac{1}{\pi i}\int_{(0)} K_\mu(\xi_2)\psi^*(s-\mu)\,d\mu.$$

Here ψ^* is defined to be the right side of (6.5) without the factor $\tau(-x)$. It is regular and of fast decay for $\mathrm{Re}\, s > 0$. We have

$$(6.18) \qquad \psi_\tau^*(s) = I_0(\xi_2)\psi^*(s) + \psi_\tau^{**}(s),$$

with

$$(6.19) \qquad \psi_\tau^{**}(s) = -\frac{1}{2i}\int_{(\varepsilon)} I_\mu(\xi_2)\left(\psi^*(s+\mu) + \psi^*(s-\mu)\right)\frac{d\mu}{\sin \pi\mu}.$$

For $\mathrm{Re}\, s > 0$ the $\psi_\tau^{**}(s)$ is regular and $\ll \xi_2^\varepsilon(|s|+1)^{-C}$. This gives readily

$$(6.20) \qquad \Psi_\tau^\delta(0;\nu) = \Psi^\delta(\nu) + O\left(\xi_2^\varepsilon(|\nu|+1)^{-C}\right),$$

where $\Psi^\delta(\nu)$ is defined by (6.7) with $s_1 = 0$ and ψ^* in place of ψ_τ^*.

Hence, we have proved, as a consequence of Lemmas 6 and 10,

Lemma 11. *Let* (6.1) *hold. Then, with V running over all irreducible representations in the unitary principal series, we have*

$$(6.21)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\sum_V \mathscr{T}\varpi_V\mathscr{P}_0 f(1) = \pi i\sum_V \left(|\varrho_V(1)|^2+|\varrho_V(-1)|^2\right) \times H_V(z_1)H_V(z_2)H_V(z_3)(\Psi^+ + \epsilon_V\Psi^-)(\nu_V).$$

Here

$$(6.22)\qquad \Psi^\delta(\nu) = -4(2\pi)^{w_2-w_3-3}\int_{(\alpha)}\psi^*(s)\cos\left(\frac{1}{2}\pi((1+\delta)(z_1-s)+(1-\delta)\nu)\right) \times\cos\left(\frac{1}{2}\pi(s+1-w_2-w_4-\delta(s+1-w_1-w_2))\right) \times\Gamma(z_1-s+\nu)\Gamma(z_1-s-\nu)\Gamma(s+1-w_1-w_2) \times\Gamma(s+1-w_2-w_4)\,ds,$$

with $\operatorname{Re} z_1 > \alpha > \operatorname{Re}(w_1+w_2)-1$ *and*

$$(6.23)\qquad \psi^*(s) = \int_0^\infty \frac{x^{s-1}}{(x+1)^{w_2}}\psi\left(\frac{1}{x+1}\right)dx.$$

We compare (6.21) with (4.5.5) and (4.5.9) of [19], and (6.22) with (4.4.16)–(4.4.17) there. To facilitate it, we give the table:

$$(6.24)\qquad (w_1,w_2,w_3,w_4)\mapsto(u,w,z,v),\quad |\varrho_V(\pm 1)|^2\mapsto\frac{1}{4}\alpha_j,\quad \Psi^\delta\mapsto\frac{2}{\pi i}\Phi_\delta,\quad \psi^*\mapsto\tilde{g},$$

where on the left are our present objects and on the right those corresponding in Sections 4.3–4.4 of [19]. Likewise $\epsilon_V\mapsto\epsilon_j$, $H_V\mapsto H_j$. Note that (4.4.16) in [19] is to be corrected: the second ξ on the right side should have the opposite sign. We find that the agreement is perfect. This ends the treatment of the unitary principal series.

Concerning the contribution of the discrete series, we return to Lemma 7. We observe that Lemma 10 holds for $\mathscr{B}^+\left(\mathrm{a}[y];\ell-\frac{1}{2}\right)$ with $\mathbb{Z}\ni\ell\geqq 1$ as well. In fact, the necessary change in the proof takes place only after (6.11). The use of (2.39)–(2.40) is replaced by that of (2.43). This and the argument leading to Lemma 11 yield the following assertion, which in view of (2.20) and (2.28) coincides with (4.5.6) of [19]:

Lemma 12. *Let* (6.1) *hold. Then, with V running over all irreducible representations in the discrete series, we have*

$$(6.25)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\sum_V \mathscr{T}\varpi_V\mathscr{P}_0 f(1)$$
$$= \pi i\sum_V\left(|\varrho_V(1)|^2+|\varrho_V(-1)|^2\right)H_V(z_1)H_V(z_2)H_V(z_3)\Psi^+(\nu_V),$$

where

$$(6.26)\qquad \Psi^+(\nu_V) = 2(-1)^{\ell-1}(2\pi)^{w_2-w_3-2}\cos\left(\frac{1}{2}\pi(w_1-w_4)\right)$$
$$\times\int_{(\alpha)}\psi^*(s)\frac{\Gamma\left(\ell-\frac{1}{2}+z_1-s\right)}{\Gamma\left(\ell+\frac{1}{2}-z_1+s\right)}\Gamma(s+1-w_1-w_2)$$
$$\times\Gamma(s+1-w_2-w_4)\,ds$$

with $\nu_V=\ell-\frac{1}{2}$ *and* α *as above.*

Remark. With respect to the process behind the appearance of products of three values of Hecke series, there arises a notable difference between the present work and [19]. Here $H_V(z_1)$ comes from (2.26), $H_V(z_2)$ from (3.7), and $H_V(z_3)$ from the above limiting procedure. In [19] the $H_V(z_1)$ corresponds to the sum over the shift parameter, i.e., the n of (3.9) and thus in a context similar to the present. However, there $H_V(z_2)$ and $H_V(z_3)$ appear combined in the product $H(z_2)H(z_3)$ as a consequence of the multiplicativity of Hecke operators that is irrelevant to our argument (but (2.30) depends on it via the proof). This observation applies also to the product of six values of the zeta-function in the numerator of (6.27) below.

As to the consequence of Lemma 8, it should be enough to state only the end result, which coincides with the sum of (4.5.4) and (4.5.8) of [19]:

Lemma 13. *Let* (6.1) *hold. Then we have*

$$(6.27)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\mathscr{T}\varpi_E^{(1)}\mathscr{P}_0 f(1)$$
$$=\frac{1}{2}\int_{(0)}\frac{\zeta(z_1+\nu)\zeta(z_1-\nu)\zeta(z_2+\nu)\zeta(z_2-\nu)\zeta(z_3+\nu)\zeta(z_3-\nu)}{\zeta(1+2\nu)\zeta(1-2\nu)}$$
$$\times(\Psi^++\Psi^-)(\nu)\,d\nu.$$

It remains to discuss $\mathscr{T}\mathscr{P}_\infty f(1)$ and $\mathscr{T}\varpi_E^{(0)}\mathscr{P}_0 f(1)$. The following assertion coincides with [19], (4.3.16):

Lemma 14. *Let* (6.1) *hold. Then we have*

$$(6.28)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\mathscr{T}\mathscr{P}_\infty f(1)=\psi^*(w_1+w_2-1)$$
$$\times\frac{1}{\zeta(2z_2+1)}\zeta(w_1+w_2-1)\zeta(w_3+w_4)\zeta(w_3-w_2+1)\zeta(w_4-w_1+1),$$

and

$$(6.29)\qquad \frac{1}{2}\lim_{\xi_2\to 0^+}\lim_{\xi_1\to 0^+}\mathscr{T}\varpi_E^{(0)}\mathscr{P}_0 f(1)=\psi^*(w_2+w_4-1)$$
$$\times\frac{1}{\zeta(2z_3+1)}\zeta(w_1+w_3)\zeta(w_2+w_4-1)\zeta(w_3-w_2+1)\zeta(w_1-w_4+1).$$

Proof. To confirm the former, it suffices to observe (3.30), the first equation in (3.41), and that (6.5) gives $\hat{\psi}_\tau(0)=\psi_\tau^*(w_1+w_2-1)$. As to the latter, we apply, on the right side of (5.44), the change of variable $v\mapsto v-z_3$, and shift the contour to (ε). We do not encounter any singularity, because of (6.1). Following the argument for (6.13), we have

$$(6.30)\qquad \lim_{\xi_1\to 0^+}\mathscr{T}\varpi_E^{(0)}\mathscr{P}_0 f(1)=2\frac{Z(-z_3)}{\zeta(2z_3+1)}\left((\hat{\psi}_\tau)^+ + (\hat{\psi}_\tau)^-\right)(w_4-w_1).$$

The inversion of the formula (6.9) gives

$$(6.31)\qquad \left((\hat{\psi}_\tau)^+ + (\hat{\psi}_\tau)^-\right)(w_4-w_1)$$
$$=2(2\pi)^{w_1-w_4}\cos\left(\frac{1}{2}\pi(w_1-w_4)\right)\Gamma(w_4-w_1)\psi_\tau^*(w_2+w_4-1).$$

This and the functional equation for the zeta-function yield (6.29).

Remark. In [19] those two terms corresponding to (6.28) and (6.29) come up as residual terms. Here they are, respectively, understood as the consequence of the automorphic regularization (3.24) and the contribution of the constant terms of Eisenstein series.

Gathering (3.40), (6.4) and Lemmas 11–14 together, we see that we have now proved (4.3.15)–(4.3.16) and Lemmas 4.5 and 4.6 of [19], with a new argument. One should note that the domain (6.1) is not disjoint with (4.3.10) of [19]. Those assertions of [19] give rise to the spectral decomposition of $J(w;g)$ defined at (1.8). Hence, we have achieved the same. What remains is to continue analytically the spectral decomposition of $J(w;g)$ to a neighbourhood of $\mathfrak{p}_{\frac{1}{2}}$. This is naturally the same as Sections 4.6–4.7 of [19], and we skip it.

Thus, for instance, the sum (6.21) converges absolutely uniformly in a neighbourhood of $\mathfrak{p}_{\frac{1}{2}}$. The value at $\mathfrak{p}_{\frac{1}{2}}$ is equal to

$$(6.32)\qquad \frac{1}{2}\sum_V\left(|\varrho_V(1)|^2+|\varrho_V(-1)|^2\right)H_V\left(\frac{1}{2}\right)^3(\psi_+ + \psi_-)(\nu_V),$$

where V are in the unitary principal series, and $\psi_\delta(\nu)$ is the value of $2\pi i\Psi^\delta(\nu)$ with $w = \mathrm{p}_{\frac{1}{2}}$. Here we have used the fact that $H_V\left(\frac{1}{2}\right) = 0$ if $\epsilon_V = -1$, as implied by (2.32). The formula (6.22) gives, for $\nu \in i\mathbb{R}$,

$$(6.33)\qquad \psi_\delta(\nu) = \frac{1}{\pi^2 i}\int_{(\frac{1}{4})} \psi^*(s)\cos\left(\frac{1}{2}\pi\left((1+\delta)\left(\frac{1}{2}-s\right)+(1-\delta)\nu\right)\right)$$
$$\times\cos\left(\frac{1}{2}\pi(1-\delta)s\right)\Gamma\left(\frac{1}{2}-s+\nu\right)\Gamma\left(\frac{1}{2}-s-\nu\right)\Gamma(s)^2\,ds.$$

Inserting (6.23) with $w_2 = 1/2$, we get an absolutely convergent double integral. Exchanging the order of integration and invoking (2.41)–(2.42), we find that

$$(6.34)\qquad \psi_\delta(\nu) = \int_0^\infty \frac{1}{x(x+1)^{\frac{1}{2}}}\psi\left(\frac{1}{x+1}\right)\int_0^\infty j_0(-\delta u)j_\nu(\delta xu)\frac{du}{u^{\frac{3}{2}}}\,dx,$$

which implies (2.34) with an obvious specialization of ψ. The real reason for the appearance of j_ν here should of course be traced back to (4.9). This concerns the contribution of unitary principal series representations only, but other parts are dealt with similarly. Therefore we have fully proved Theorem B, with a method based solely upon the spectral theory of $L^2(\Gamma\backslash G)$.

Concluding remark. Theorem A is not an isolated fact in the theory of zeta-functions. It has been extended to $\mathcal{M}_4(\zeta_F; g)$, where ζ_F are the Dedekind zeta-functions of certain quadratic number fields F with class number one. See [5] and [7] (also [6], Part X). The relevant Lie groups are $\mathrm{PSL}_2(\mathbb{C})$ and the product of two copies of $\mathrm{PSL}_2(\mathbb{R})$, respectively, for the imaginary and the real quadratic cases. What is interesting is that these extensions of $\mathcal{M}_4(\zeta; g)$ admit formulations highly analogous to Theorem B, despite the difference of the representation theories of the two Lie groups from that of $\mathrm{PSL}_2(\mathbb{R})$. The construction (2.37)–(2.38) extends with cubic powers of central values of analogous Hecke series. Besides, (2.34)–(2.35) extends with Bessel functions of representations of corresponding Lie groups.

Also, an analogue of Theorem A is known for $\mathcal{M}_2(H; g)$ with Hecke series H attached to any holomorphic cusp form. Again a formulation similar to Theorem B is possible. See [18] and [20]. Hence, Theorem B could be a typical instance of a certain general structure among mean values of automorphic L-functions. However, the case of real analytic cusp forms is yet to be included in this general picture. Moreover, it is not known if the argument of the present work, i.e., the Poincaré series approach, extends to $\mathcal{M}_2(H; g)$. To these topics we shall return elsewhere.

Finally, it should be stressed that M. Jutila has developed a method with which one can treat L-functions of all types equally as far as the underlying group is $\mathrm{PSL}_2(\mathbb{R})$, although it does not produce results as explicit as Theorem A. See his important work [14].

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Department of Mathematics, Utrecht University, P.O. Box 80.010, TA 3508 Utrecht, the Netherlands
e-mail: bruggeman@math.uu.nl

Department of Mathematics, College of Science and Technology, Nihon University, Surugadai,
Tokyo 101-8308, Japan
e-mail: ymoto@math.cst.nihon-u.ac.jp
e-mail: am8y-mths@asahi-net.or.jp

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