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Degree-Constrained Orientation of Maximum Satisfaction: Graph Classes and Parameterized Complexity

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Abstract The problem Max W-Light (Max W-Heavy) for an undirected graph is to assign a direction to each edge so that the number of vertices of outdegree at most W (resp. at least W) is maximized. It is known that these problems are NP-hard even for fixed W. For example, Max 0- Light is equivalent to the problem of finding a maximum independent set. In this paper, we show that for any fixed constant W, Max W-Heavy can be solved in linear time for hereditary graph classes for which treewidth is bounded by a function of degeneracy. We show that such graph classes include chordal graphs, circular-arc graphs, d-trapezoid graphs, chordal bipartite graphs, and graphs of bounded clique-width. To have a polynomial-time algorithm for Max

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W-Light, we need an additional condition of a polynomial upper bound on the number of potential maximal cliques to apply the metatheorem by Fomin et al. (SIAM J Comput 44:54–87, 2015). The aforementioned graph classes, except bounded clique-width graphs, satisfy such a condition. For graphs of bounded clique-width, we present a dynamic programming approach not using the metatheorem to show that it is actually polynomial-time solvable for this graph class too. We also study the parameterized complexity of the problems and show some tractability and intractability results.

Keywords Orientation · Graph class · Width parameter · Parameterized complexity

1 Introduction

Let $G = (V, E)$ be an undirected graph. An *orientation of* G is a function that maps each undirected edge $\{u, v\} \in E$ to one of the two possible directed edges (u, v) and (v, u). For any orientation Λ of G, define $\Lambda(E) = \bigcup_{e \in E}\{\Lambda(e)\}$ and let $\Lambda(G)$ denote the directed graph $(V, \Lambda(E))$. For any vertex $u \in V$, the *outdegree of* u *under* Λ is defined as $d^+_\Lambda(u) = |\{(u, v) : (u, v) \in \Lambda(E)\}|$, i.e., the number of outgoing edges from u in $\Lambda(G)$. For any non-negative integer W, a vertex $u \in V$ is called W*-light in* $\Lambda(G)$ if $d^+_\Lambda(u) \le W$, and W*-heavy in* $\Lambda(G)$ if $d^+_\Lambda(u) \ge W$.

For any fixed integer $W \ge 0$, the following optimization problems (introduced in [3], see also [4]) are defined, where the input is an undirected graph $G = (V, E)$:

- Max W-Light: Output an orientation Λ of G such that $\left|\{u \in V : d^+_\Lambda(u) \le W\}\right|$ is maximized.
- Max W-Heavy: Output an orientation Λ of G such that $\left|\{u \in V : d^+_\Lambda(u) \ge W\}\right|$ is maximized.

Symmetrically, we can consider the following problems:

- Min W-Light: Output an orientation Λ of G such that $\left|\{u \in V : d^+_\Lambda(u) \le W\}\right|$ is minimized.
- Min W-Heavy: Output an orientation Λ of G such that $\left|\{u \in V : d^+_\Lambda(u) \ge W\}\right|$ is minimized.

Observe that Max W-Light (resp., Max W-Heavy) and Min $(W+1)$-Heavy (resp., Min $(W-1)$-Light) are *supplementary problems* in the sense that an exact algorithm for one gives an exact algorithm for the other, though their approximability properties and fixed-parameter tractability may differ. Since this paper mainly focuses on the polynomial-time solvability, we consider only Max W-Light and Max W-Heavy.[1]

It is shown in [3] that Max W-Light is NP-hard for any fixed $W \ge 0$, and Max W-Heavy is NP-hard for any fixed $W \ge 3$. They also show that for $W \le 1$ Max W-Heavy can be solved in polynomial time. Recently Khoshkhah [28] has closed the gap by showing that Max 2- Heavy can be solved in polynomial time.

For these problems, the same authors of [3] investigate the approximability [4]. They got comprehensive results on the approximability of the problems. Some of the results are listed as follows:

[1] We consider parameterized complexity in Sect. 5 where the equivalence does not hold.

- For every fixed $W \geq 1$, Max W-Light cannot be approximated within $(n/W)^{1-\varepsilon}$ in polynomial time, unless P = NP. On the positive side, there exists a polynomial-time $(n/(2W+1))$-approximation algorithm for Max W-Light.
- For every fixed $W \geq 1$, Min W-Heavy cannot be approximated within 1.3606 in polynomial time, unless P = NP. On the positive side, they show that Min W-Heavy can be approximated within a ratio of $\ln(W+1)+1$ in polynomial time for every fixed $W \geq 2$.
- Min W-Light can be approximated within $\ln(W+1)+1$ for any $W \geq 1$. This ratio is almost tight, because it was shown that, for sufficiently large values of W, Min W-Light is NP-hard to approximate within $\ln(W+1) - O(\log\log W)$.
- For sufficiently large values of W, Max W-Heavy is NP-hard to approximate within $(n/W)^{1/2-\varepsilon}$ for any $\varepsilon > 0$. Note that the best known polynomial-time approximation ratio for Max W-Heavy is $W+1$ [3].

Due to the work mentioned above [3], the general (in)approximability of the problems is well understood. In this paper, we thus investigate the problem from another aspect, that is, graph classes. For the two problems Max W-Light and Max W-Heavy, we take similar but slightly different approaches.

The main tool for Max W-Light is the metatheorem of Fomine et al. [19] that can be seen as an extension of Courcelle's theorem [1,13]. It provides a polynomial-time algorithm for finding a maximum induced subgraph of bounded treewidth satisfying a counting monadic second-order logic formula from a given graph with polynomially many potential maximal cliques. We show that if a hereditary graph class has a polynomial upper bound on the number of potential maximal cliques and has a function depending only on degeneracy as an upper bound of treewidth, then the metatheorem of Fomin et al. can be applied to Max W-Light.

Similarly, for Max W-Heavy, we consider hereditary graph classes with treewidth bounded by a function of degeneracy. However, we do not require polynomial upper bounds on the number of potential maximal cliques. We first show that the problem for graphs of bounded treewidth can be solved in linear time. Next we present a linear-time reduction from graphs with a function of degeneracy as an upper bound of treewidth to graphs of bounded treewidth. Combining these results, we obtain a linear-time algorithm for Max W-Heavy on graph classes with the aforementioned property.

We then show that our algorithms can be applied to several well-known graph classes. It is known that chordal graphs, circular-arc graphs, d-trapezoid graphs, and chordal bipartite graphs have polynomial upper bounds on the number of potential maximal cliques (see Sect. 4). We show that these hereditary graph classes have functions of degeneracy as upper bounds on treewidth, and thus our algorithms can be applied. Additionally, we observe that graphs of bounded clique-width admit a function of degeneracy as an upper bounded of treewidth, and thus Max W-Heavy can be solved in linear time. To show that Max W-Light can be solved in polynomial time for graphs of bounded clique-width, we present a dynamic programming based algorithm.

We also consider the parameterized complexity of the problems. We show that for any fixed W, Max W-Light is W[1]-complete, while Max W-Heavy admits a kernel of $O(Wk)$ vertices, where the parameter k is the solution size.

1.1 Related Work

Graph orientations that optimize certain objective functions involving the resulting directed graph or that satisfy some special property such as acyclicity [46] or k-edge connectivity [11,40,44] have many applications to graph theory, combinatorial optimization, scheduling (load balancing), resource allocation, and efficient data structures. For example, an orientation that minimizes the maximum outdegree [5,10,47] can be used to support fast vertex adjacency queries in a sparse graph by storing each edge in exactly one of its two incident vertices' adjacency lists while ensuring that all adjacency lists are short [10]. There are many optimization criteria for graph orientation other than these. See [2] or Chapter 61 in [45] for more details and additional references.

On the other hand, degree-constrained graph orientations [20,21,25,35] arise when a degree lower bound $W^l(v)$ and a degree upper bound $W^u(v)$ for each vertex v in the graph are specified in advance or as part of the input, and the outdegree of v in any valid graph orientation is required to lie in the interval $[W^l(v), \ldots, W^u(v)]$. Obviously, a graph does not always have such an orientation, and in this case, one might want to compute an orientation that best fits the outdegree constraints according to some well-defined criteria [2,3]. In case $W^l(v) = 0$ and $W^u(v) = W$ for every vertex v in the input graph, where W is a non-negative integer, and the objective is to maximize (resp., minimize) the number of vertices that satisfy (resp., violate) the outdegree constraints, then we obtain Max W-Light (resp., Min $(W+1)$-Heavy). Similarly, if $W^l(v) = W$ and $W^u(v) = \infty$ for every vertex v in the input graph, then we obtain Max W-Heavy and Min $(W-1)$-Light.

Another related problem is to find a maximum vertex set that induces a subgraph of bounded degeneracy. (See the next section for the definition of degeneracy.) This problem can be seen as a variant of Max W-Light, where we can use acyclic orientations only. This problem is studied in the context of parameterized [37] and exact [43] computation. Concerning the computational complexity on restricted graph classes, we can obtain a result similar to the one for Max W-Light as we observe in the final section of this paper.

2 Preliminaries

We say two vertices $u, v \in V(G)$ are *adjacent in* G if $\{u, v\} \in E(G)$. Let $N_G(u)$ be the set of all vertices that are adjacent to u in G, i.e., $N_G(u) = \{v : \{u, v\} \in E\}$. The *degree of* u *in* G is $d_G(u) = |N_G(u)|$. We define $\delta(G) = \min\{d_G(u) : u \in V(G)\}$. The *degeneracy* of a graph G, denoted by $\hat{\delta}(G)$, is the maximum of the minimum degrees over all induced subgraphs of G. Let $(v_1, \ldots, v_n)$ be an ordering on $V(G)$ such that $d_{G_i}(v_i) = \delta(G_i)$, where $G_i = G[\{v_j : j \geq i\}]$. It is known that such an ordering can be computed in linear time and that $\hat{\delta}(G) = \max_{1 \leq i \leq n} \delta(G_i)$ [38]. For any $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. If $G[U]$ is a complete graph, then U is a *clique* of G. The size of a maximum clique in G is denoted by $\omega(G)$. Let $\omega_{\mathsf{b}}(G)$ be the maximum integer k such that G has a subgraph isomorphic to the complete bipartite graph $K_{k,k}$. From the definition, $\omega(G) - 1$ and $\omega_{\mathsf{b}}(G)$ are lower bounds of $\hat{\delta}(G)$. A class $\mathscr{C}$ of graphs is *hereditary* if $\mathscr{C}$ is closed under taking induced subgraphs. Namely, if $G \in \mathscr{C}$, then all induced subgraphs of G belong to $\mathscr{C}$.

For an integer $W \geq 0$, an orientation of a graph is called a *W-light orientation* if the maximum outdegree is at most W. If a W-light orientation exists, we say that the graph is *W-light orientable*. By replacing "at most" with "at least" in these definitions, we similarly define *W-heavy orientations* and *W-heavy orientable* graphs.

2.1 Minimal Triangulations and Potential Maximal Cliques

A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(\{X_i : i \in I\}, T = (I, F))$ such that each X_i, called a *bag*, is a subset of V, and T is a tree such that

- for each $v \in V$, there is $i \in I$ with $v \in X_i$;
- for each $\{u, v\} \in E$, there is $i \in I$ with $u, v \in X_i$;
- for $i, j, k \in I$, if j is on the i, k-path in T, then $X_i \cap X_k \subseteq X_j$.

The *width* of a tree-decomposition is the size of a maximum bag minus 1. A graph has *treewidth* at most t if and only if it has a tree-decomposition of width at most t. We denote the treewidth of G by $\mathbf{tw}(G)$.

A graph is *chordal* (or *triangulated*) if it has no induced cycle of length 4 or more. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $G' = (V, E')$ such that $E \subseteq E'$. A triangulation G' of G is *minimal* if no proper subgraph of G' is a triangulation of G. It is known that the treewidth of G is the minimum integer t such that there is a (minimal) triangulation H of G with the maximum clique size $t + 1$. A vertex set $P \subseteq V(G)$ is a *potential maximal clique* of G if P is a maximal clique in some minimal triangulation of G. The set of all potential maximal cliques of G is denoted by Π_G. A vertex set $S \subseteq V(G)$ is an *a, b-separator* for $a, b \in V(G)$ if a and b are in different components in $G - S$. An a, b-separator is *minimal* if no proper subset of it is an a, b-separator. A vertex set is a *minimal separator* if it is a minimal a, b-separator for some pair a, b. The set of all minimal separators of G is denoted by Δ_G. By the following proposition, graphs have a polynomial number of minimal separators if and only if they have a polynomial number of potential maximal cliques.

Proposition 2.1 (Bouchitté and Todinca [9]) *For every n-vertex graph G, it holds that* $|\Delta_G|/n \leq |\Pi_G| \leq n|\Delta_G|^2 + n|\Delta_G| + 1$.

3 Metatheorems

In this section we present metatheorems for Max W-Light and Max W-Heavy. We apply them to some well-studied graph classes in the next section.

3.1 MSO Expressibility

We now introduce the *monadic second-order logic* (MSO) of graphs. The syntax of MSO of graphs includes (1) the logical connectives $\vee$, $\wedge$, $\neg$, $\Leftrightarrow$, $\Rightarrow$, (2) variables for vertices, edges, vertex sets, and edge sets, (3) the quantifiers ∀ and ∃ applicable to these variables, and (4) the following binary relations:

- $u \in U$ for a vertex variable u and a vertex set variable U;
- $d \in D$ for an edge variable d and an edge set variable D;
- $\mathsf{inc}(d, u)$ for an edge variable d and a vertex variable u, where the interpretation is that d is incident with u;
- equality of variables.

To express edge orientations in MSO for graphs of bounded treewidth, we use the following observation [7,27] (see also [14]). First observe that a graph of treewidth at most k has a proper $k+1$ coloring. Now an orientation of a graph $G = (V, E)$ with treewidth at most k can be represented by a proper coloring $\gamma : V \to \{1, \dots, k+1\}$ and an edge set $F \subseteq E$ as follows: edge $e = \{u, v\}$ is oriented as (u, v) if and only if either $\gamma(u) < \gamma(v)$ and $e \in F$, or $\gamma(u) > \gamma(v)$ and $e \notin F$.

Lemma 3.1 *For any fixed W,* Max W-Heavy *and* Max W-Light *for graphs of bounded treewidth can be expressed in an optimization version of MSO and thus solved in linear time.*

Proof Let $G = (V, E)$ be a graph of treewidth at most k. For vertex sets $V_1, \dots, V_{k+1} \subseteq V$, one can easily express in MSO that $V_1, \dots, V_{k+1}$ give a proper $k+1$ coloring of G. Let $\mathsf{proper\text{-}coloring}(V_1, \dots, V_{k+1})$ be such an MSO formula.

By a proper coloring $(V_1, \dots, V_{k+1})$ and an edge set $F \subseteq E$, we represent an orientation of G as described above. Now, for an edge $e \in E$ and a vertex $v \in V$, there is an MSO formula that means e is an out-going edge from v. For example, we can express it as follows:

$$\begin{aligned}&\mathsf{out}_{V_1,\dots,V_{k+1},F}(e, v) := \mathsf{inc}(e, v) \wedge \\ &\quad \exists u \left(\left(e \in F \wedge \bigvee_{i<j} (u \in V_i \wedge v \in V_j) \right) \vee \left(e \notin F \wedge \bigvee_{i>j} (u \in V_i \wedge v \in V_j) \right) \right).\end{aligned}$$

Under the orientation represented by $(V_1, \dots, V_{k+1})$ and $F \subseteq E$, W-heaviness and W-lightness of a vertex can be expressed as follows:

$$\begin{aligned}W\text{-}\mathsf{heavy}_{V_1,\dots,V_{k+1},F}(v) &:= \exists e_1, \dots, e_W \, (\mathsf{distinct}(e_1, \dots, e_W) \\ &\qquad \wedge \forall e_i \, (\mathsf{out}_{V_1,\dots,V_{k+1},F}(e_i, v))), \\ W\text{-}\mathsf{light}_{V_1,\dots,V_{k+1},F}(v) &:= \neg \left((W+1)\text{-}\mathsf{heavy}_{V_1,\dots,V_{k+1},F}(v) \right),\end{aligned}$$

where $\mathsf{distinct}(e_1, \dots, e_W)$ means that the edges $e_1, \dots, e_W$ are distinct. Hence the problems are equivalent to finding a vertex set X of the maximum size in the following formulas:

$$\exists V_1, \dots, V_{k+1}, \exists F \left(\mathsf{proper\text{-}coloring}(V_1, \dots, V_k) \wedge \forall v \in X \left(W\text{-}\mathsf{heavy}_{V_1,\dots,V_{k+1},F}(v) \right) \right),$$
$$\exists V_1, \dots, V_{k+1}, \exists F \left(\mathsf{proper\text{-}coloring}(V_1, \dots, V_k) \wedge \forall v \in X \left(W\text{-}\mathsf{light}_{V_1,\dots,V_{k+1},F}(v) \right) \right).$$

It is known that for a fixed MSO formula on a graph of bounded treewidth, one can find in linear time a maximum vertex subset satisfying the formula (see [1,13]). □

Corollary 3.2 *For fixed W and k, the property of being W-light (or W-heavy) orientable can be expressed in MSO for graphs of treewidth at most k.*

3.2 Max W-Light

We can see that the problem of finding a maximum W-light orientable induced subgraph is polynomially equivalent to Max W-Light.

Lemma 3.3 *A graph G has a W-light orientable induced subgraph of at least k vertices if and only if the edges of G can be oriented so that at least k vertices have outdegree at most W. Furthermore, if a maximum W-light orientable induced subgraph of G can be found in $O(f(m,n))$ time, then* Max W-Light *can be solved in $O(f(m,n)+m^{1.5})$ time, where m and n are the numbers of edges and vertices in G, respectively.*

Proof To show the if part, assume that under an orientation Λ, G has at least k vertices of outdegree at most W. Let L be such vertices. The graph $G[L]$ is W-orientable as we can orient the edges of $G[L]$ as in $\Lambda(G)$.

To show the only-if part, let H be a W-light orientable induced subgraph of G with at least k vertices, and let Λ' be a W-light orientation of H. We extend Λ' to an orientation Λ of G by orienting the edges between $V(G)\setminus V(H)$ and $V(H)$ from $V(G)\setminus V(H)$ to $V(H)$, and the other edges completely in $V(G)\setminus V(H)$ arbitrarily. The vertices in $V(H)$ are still W-light under Λ.

For the second part of the lemma, first we find a maximum W-light orientable induced subgraph H of G in $O(f(m,n))$ time. We next compute a W-light orientation Λ' of H in $O(m^{1.5})$ time [6]. Finally, we obtain the orientation Λ of G as described above in linear time. By the characterization above, Λ is an optimal solution of Max W-Light. □

The *counting monadic second-order logic* (CMSO) of graphs is an extension of MSO, with an additional sentence of checking the cardinality of a set modulo some constant.[2] Recently, Fomin et al. [19] have presented the following metatheorem.

Proposition 3.4 (Fomin et al. [19]) *For any fixed t and a CMSO-expressible property $\mathscr{P}$, the following problem can be solved in polynomial time for any class of graphs with a polynomial number of potential maximal cliques: Given a graph G, find a maximum induced subgraph H of treewidth at most t that has the property $\mathscr{P}$.*

This metatheorem is quite powerful and allows us to solve many problems for graphs with polynomially many potential maximal cliques. (See [19] for applications.) However, we cannot apply it to our problem Max W-Light in general because W-light orientable graphs may have large treewidth. For example, grid graphs are 2-light orientable but have unbounded treewidth.

[2] The ordinary MSO is enough for our purpose. We introduce CMSO to precisely state Proposition 3.4.

In the following, we show that with an additional restriction to graph classes, we can apply the metatheorem of Fomin, Todinca, and Villanger to Max W-Light.

Lemma 3.5 *Every W-light orientable graph has degeneracy at most $2W$.*

Proof We prove the contrapositive. Let G be a graph with $\hat{\delta}(G) > 2W$. There is a subgraph H of G such that $\delta(H) > 2W$. Since the average degree $2|E(H)|/|V(H)|$ of H is at least $\delta(H)$, we have $|E(H)|/|V(H)| > W$. Thus for any orientation Λ of G,

$$\max_{u \in V(G)} d^+_\Lambda(u) \geq \max_{u \in V(H)} d^+_\Lambda(u) \geq \sum_{u \in V(H)} d^+_\Lambda(u)/|V(H)| \geq |E(H)|/|V(H)| > W.$$

This implies that G is not W-light orientable. □

Theorem 3.6 *For any fixed W,* Max W-Light *can be solved in polynomial time for a hereditary graph class $\mathscr{C}$ with a polynomial number of potential maximal cliques if the treewidth of each graph in $\mathscr{C}$ is bounded from above by a function of its degeneracy.*

Proof Let f be a function such that $\mathbf{tw}(G) \leq f(\hat{\delta}(G))$ for each $G \in \mathscr{C}$. By Lemma 3.5, a W-light orientable graph in $\mathscr{C}$ has treewidth at most $f(2W)$. Since $\mathscr{C}$ is hereditary, a maximum W-light orientable induced subgraph of a graph in $\mathscr{C}$ can be found in polynomial time by Proposition 3.4 and Corollary 3.2. Now, by Lemma 3.3, the theorem follows. □

3.3 Max W-Heavy

Unlike Max W-Light, the problem Max W-Heavy is not equivalent to the problem of finding a maximum orientable induced subgraph. We here present a way of directly finding an orientation with as many W-heavy vertices as possible for graphs with treewidth bounded by a function of degeneracy.

Proposition 3.7 ([4]) *Every graph of minimum degree at least $2W + 1$ is W-heavy orientable and a W-heavy orientation of it can be found in linear time.*

Theorem 3.8 *For any fixed W,* Max W-Heavy *can be solved in linear time for a hereditary graph class $\mathscr{C}$ if the treewidth of each graph in $\mathscr{C}$ is bounded from above by a function of its degeneracy.*

Proof Let f be a function such that $\mathbf{tw}(G) \leq f(\hat{\delta}(G))$ for each $G \in \mathscr{C}$. Let $G \in \mathscr{C}$ be a graph with n vertices. Let $(v_1, v_2, \ldots, v_n)$ be an ordering of $V(G)$ such that for each i, the vertex v_i has the minimum degree in G_i, where $G_i = G[\{v_j : i \leq j \leq n\}]$. Let h be the first index such that $d_{G_h}(v_h) \geq 2W + 1$. If there is no such index, we set $h = n + 1$.

Let $H = G[\{v_j : 1 \leq j < h\}]$. Since $\mathscr{C}$ is hereditary, we have $H \in \mathscr{C}$, and thus $\mathbf{tw}(H) \leq f(\hat{\delta}(H)) \leq f(2W)$. We obtain H' from H as follows: add a clique C of size $2W + 1$; for each vertex v in H, add edges from v to arbitrarily chosen $d_G(v) - d_H(v)$ vertices in C. It holds that $\mathbf{tw}(H') \leq \mathbf{tw}(H) + |C| \leq f(2W) + 2W + 1$.

By Lemma 3.1, an orientation Λ' of H' with the maximum number of W-heavy vertices can be found in linear time. Note that all vertices in C are W-heavy under Λ' even in $H'[C]$. Otherwise, by Proposition 3.7, we can change the directions of edges in $H'[C]$ so that all vertices in C become W-heavy. Since this modification does not decrease the outdegree of any vertex in $V(H)$, the new orientation has strictly more W-heavy vertices than Λ'. This contradicts the optimality of Λ'.

Let Λ'' be a W-heavy orientation of $G_h = G[\{v_h, \dots, v_n\}]$. By Proposition 3.7, such an orientation can be found in linear time. We next construct an orientation Λ of G from Λ' and Λ'' as follows: for each edge in $E(H)$ or $E(G_h)$, we use the direction in Λ' or Λ'', respectively; for each edge between $V(H)$ and $V(G_h)$, we use the direction from $V(H)$ to $V(G_h)$. All vertices in $V(G_h)$ are W-heavy in G under Λ. Under Λ, each vertex in $V(H)$ has at least as many out-neighbors as under Λ'. Thus a vertex in $V(H)$ is W-heavy in G under Λ if it is W-heavy in H' under Λ'.

We now show the optimality of Λ. Suppose to the contrary that there is an orientation Λ_{OPT} of G with strictly more W-heavy vertices than Λ. Let F and F_{OPT} be the W-heavy vertices in $V(H)$ under Λ and Λ_{OPT}, respectively. Since the vertices in $V(G_h)$ are W-heavy under Λ, we have $|F| < |F_{\mathsf{OPT}}|$. Now let Λ'_{OPT} be an orientation of H' such that the edges in H are oriented as in Λ_{OPT}, the edges between $V(H)$ and C are oriented from $V(H)$ to C, and the edges in $H[C]$ are oriented so that all the vertices in C become W-heavy. Then, at least $|C| + |F_{\mathsf{OPT}}| > |C| + |F|$ vertices are W-heavy in H' under Λ'_{OPT}. This contradicts the optimality of Λ' since at most $|C| + |F|$ vertices are W-heavy in H' under Λ'. □

4 Graph Classes

In this section, we show that Theorems 3.6 and 3.8 can be applied to several important graph classes. More precisely, we show the following theorems.

Theorem 4.1 *For any fixed W,* Max W-Light *can be solved in polynomial time for the classes of chordal graphs, d-trapezoid graphs, circular-arc graphs, chordal bipartite graphs, and graphs of bounded clique-width.*

Theorem 4.2 *For any fixed W,* Max W-Heavy *can be solved in linear time for the classes of chordal graphs, d-trapezoid graphs, circular-arc graphs, chordal bipartite graphs, and graphs of bounded clique-width.*

Figure 1 is a diagram of graph classes related to our results.

To prove Theorems 4.1 and 4.2, we show for each graph class that it satisfies conditions of Theorems 3.6 and 3.8 in the following subsections. To solve Max W-Light for graphs of bounded clique-width, we present a direct solution as we cannot apply the metatheorem. Note that all graph classes studied in this section are hereditary.

4.1 Chordal Graphs

It is well known that a chordal graph of n vertices has at most n maximal cliques (see [26]). Since a chordal graph is the unique minimal triangulation of itself, the

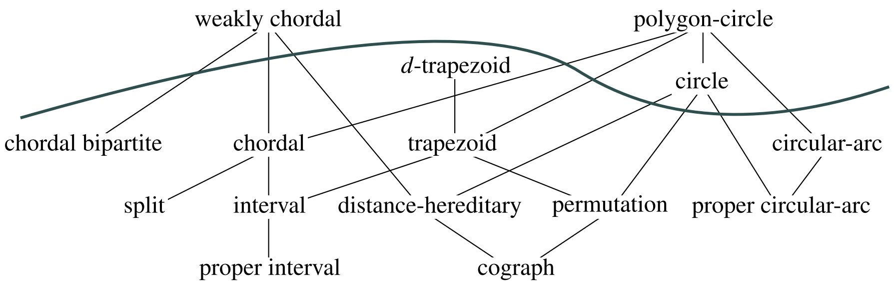

Fig. 1 Graph classes with polynomially many potential maximal cliques

number of potential maximal cliques is at most n for every n-vertex chordal graph. From the definition of chordal graphs, the following equality follows.

Proposition 4.3 (Folklore) *For every chordal graph G, $\mathbf{tw}(G) = \hat{\delta}(G) = \omega(G) - 1$.*

4.2 *d*-Trapezoid Graphs

The *co-comparability graph* of a partial order $(V, \prec)$ is a graph with the vertex set V in which two vertices u and v are adjacent if and only if they are incomparable, that is, $u \not\prec v$ and $v \not\prec u$. A partial order $(V, \prec)$ is an *interval order* if each element $v \in V$ can be represented by an interval $[l_v, r_v]$ such that $u \prec v$ if and only if $r_u < l_v$. A graph is a *d-trapezoid graph* if it is the co-comparability graph of a partial order defined as the intersection of d interval orders [8].

It is known that every d-trapezoid graph of n vertices has at most $(2n - 3)^{d-1}$ minimal separators [34].

Habib and Möhring showed in the proof of Theorem 3.4 in [24] that for every d-trapezoid graph G, $\mathbf{tw}(G) \leq 4d \cdot \omega_{\mathsf{b}}(G) - 1$. This gives the following fact as a direct corollary.

Proposition 4.4 ([24]) *For every d-trapezoid graph G, $\mathbf{tw}(G) \leq 4d \cdot \hat{\delta}(G) - 1$.*

4.3 Circular-Arc Graphs

A graph is a *circular-arc graph* if it is the intersection graph of arcs on a circle. Every n-vertex circular-arc graph has at most $2n^2 - 3n$ minimal separators [31]. A graph is an *interval graph* if it is the intersection graph of intervals on a line. From the definition, every interval graph is a circular-arc graph. Also, every interval graph is a chordal graph [36].

Lemma 4.5 *For every circular-arc graph G, $\mathbf{tw}(G) \leq 2\hat{\delta}(G)$.*

Proof Let p be a point on the circle in a circular-arc representation of G, and S_p be the vertices that correspond to the arcs containing p in the representation. Assume that we chose p so that $|S_p|$ is minimized. The set S_p is a (possibly empty) non-maximal clique, and thus $|S_p| \leq \omega(G) - 1 \leq \hat{\delta}(G)$. Let $G' = G - S_p$. Since

G' has a circular-arc representation in which the arcs do not cover the entire circle (especially they do not cover the point p), G' is an interval graph. By Proposition 4.3, $\mathbf{tw}(G') = \hat{\delta}(G') \leq \hat{\delta}(G)$. Since a removal of a vertex can decrease treewidth by at most 1, we can conclude that $\mathbf{tw}(G) \leq \mathbf{tw}(G') + |S_p| \leq 2\hat{\delta}(G)$. □

4.4 Chordal Bipartite Graphs

A bipartite graph is a *chordal bipartite graph* if it has no induced cycle of length 6 or more. Every chordal bipartite graph has $O(m+n)$ minimal separators [32]. We show in this subsection that for every chordal bipartite graph G, $\mathbf{tw}(G) \leq 2\hat{\delta}(G) - 1$.

Let $G = (X, Y; E)$ be a chordal bipartite graph. We call (A, B) with $A \subseteq X$ and $B \subseteq Y$ a *biclique* if $G[A \cup B]$ is a complete bipartite graph. A biclique (A, B) is *maximal* if there is no other biclique (A', B') satisfying $A \cup B \subsetneq A' \cup B'$. Let $\mathscr{M}_{\mathsf{b}}(G)$ be the set of maximal bicliques (A, B) of G with $\min\{|A|, |B|\} \geq 2$.

Two maximal bicliques (A_1, B_1) and (A_2, B_2) *cross* if either $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, or $A_1 \subsetneq A_2$ and $B_1 \supsetneq B_2$. If $c \colon \mathscr{M}_{\mathsf{b}}(G) \to 2^{V(G)}$ is a mapping such that $c(A, B) \in \{A, B\}$ for each $(A, B) \in \mathscr{M}_{\mathsf{b}}(G)$, then we call the family $\mathscr{C} = \{c(A, B) : (A, B) \in \mathscr{M}_{\mathsf{b}}(G)\}$ a *biclique coloring* of G. A biclique coloring $\mathscr{C}$ is *feasible* if for each pair $(A_1, B_1), (A_2, B_2) \in \mathscr{M}_{\mathsf{b}}(G)$ that cross with $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, not both A_1 and B_2 are in $\mathscr{C}$.

For a family of vertex sets $\mathscr{S} \subseteq 2^{V(G)}$, we denote by $G_{\mathscr{S}}$ the graph obtained from G by making each $S \in \mathscr{S}$ a clique.

Proposition 4.6 (Kloks and Kratsch [30]) *If $\mathscr{C}$ is a feasible biclique coloring of a chordal bipartite graph G, then $G_{\mathscr{C}}$ is chordal.*

Let $\mathscr{C}_{\min}(G)$ be a family that contains a smaller one of A and B for each $(A, B) \in \mathscr{M}_{\mathsf{b}}(G)$. That is, $\mathscr{C}_{\min}(G) = \{\arg\min_{C \in \{A,B\}} |C| : (A, B) \in \mathscr{M}_{\mathsf{b}}(G)\}$, where the ties in arg min are broken arbitrarily.

Note that if (A, B) and (A, B') are bicliques, then $(A, B \cup B')$ is also a biclique. Hence, if (A, B) and (A', B') are distinct elements of $\mathscr{M}_{\mathsf{b}}(G)$, then $A \neq A'$ and $B \neq B'$. Thus, for each $(A, B) \in \mathscr{M}_{\mathsf{b}}(G)$, $A \in \mathscr{C}_{\min}(G)$ implies $|A| \leq |B|$.

Lemma 4.7 *$\mathscr{C}_{\min}(G)$ is a feasible biclique coloring for every chordal bipartite graph G.*

Proof Suppose to the contrary that

1. $(A_1, B_1), (A_2, B_2) \in \mathscr{M}_{\mathsf{b}}(G)$ cross with $A_1 \supsetneq A_2$ and $B_1 \subsetneq B_2$, and
2. $A_1, B_2 \in \mathscr{C}_{\min}(G)$.

The first assumption gives $|A_1| > |A_2|$ and $|B_1| < |B_2|$, and the second gives $|A_1| \leq |B_1|$ and $|B_2| \leq |A_2|$. They cause a contradiction $|A_1| \leq |B_1| < |B_2| \leq |A_2| < |A_1|$. □

Proposition 4.8 (Kloks and Kratsch [30]) *Let $\mathscr{C}$ be a feasible biclique coloring of a chordal bipartite graph $G = (X, Y; E)$. Let K be a maximal clique in $G_{\mathscr{C}}$ with $|K| > 2$. Let $K_X = K \cap X$ and $K_Y = K \cap Y$. If $|K_X| \geq 2$, then one of the following two cases holds:*

1. $|K_Y| = 1$ *and there exists* $(A, B) \in \mathscr{M}_\mathsf{b}(G)$ *such that* $K_X = A$ *and* $A \in \mathscr{C}$.
2. $|K_Y| > 1$ *and there exist* $(A_1, B_1), (A_2, B_2) \in \mathscr{M}_\mathsf{b}(G)$ *with* $A_1, B_2 \in \mathscr{C}$ *such that* $K_X \subseteq A_1$ *and* $K_Y \subseteq B_2$.

Lemma 4.9 *Let* $G = (X, Y; E)$ *be a chordal bipartite graph and* $\mathscr{C} = \mathscr{C}_{\min}(G)$. *Then* $\omega(G_\mathscr{C}) \leq 2 \cdot \omega_\mathsf{b}(G)$.

Proof By Lemma 4.7, $\mathscr{C}$ is a feasible biclique coloring of G. Let K be a maximal clique in $G_\mathscr{C}$ with $|K| > 2$. Let $K_X = K \cap X$ and $K_Y = K \cap Y$. Assume without loss of generality that $|K_X| \geq 2$. Now by Proposition 4.8, one of the following two cases holds:

1. $|K_Y| = 1$ and there exists $(A, B) \in \mathscr{M}_\mathsf{b}(G)$ such that $K_X = A$ and $A \in \mathscr{C}$.
2. $|K_Y| > 1$ and there exist $(A_1, B_1), (A_2, B_2) \in \mathscr{M}_\mathsf{b}(G)$ with $A_1, B_2 \in \mathscr{C}$ such that $K_X \subseteq A_1$ and $K_Y \subseteq B_2$.

In the first case, $|A| \leq |B|$ holds, and thus $|K| = |A| + 1 \leq \omega_\mathsf{b}(G) + 1$. In the second case, $|A_1| \leq |B_1|$ and $|B_2| \leq |A_2|$ together imply that $|K| \leq |A_1| + |B_2| \leq 2 \cdot \omega_\mathsf{b}(G)$. □

By Proposition 4.6, $G_\mathscr{C}$ is chordal if $\mathscr{C} = \mathscr{C}_{\min}(G)$. Proposition 4.3 and Lemma 4.9 imply that $\mathbf{tw}(G) \leq 2 \cdot \omega_\mathsf{b}(G) - 1$ holds for every chordal bipartite graph G. Since $\omega_\mathsf{b}(G) \leq \hat{\delta}(G)$, we have the following corollary.

Corollary 4.10 *For every chordal bipartite graph* G, $\mathbf{tw}(G) \leq 2 \cdot \hat{\delta}(G) - 1$.

4.5 Graphs of Bounded Clique-Width

A *k-expression* is a rooted binary tree such that

- each leaf has label $\circ_i$ for some $i \in \{1, \ldots, k\}$,
- each node with one child has a label $\rho_{i,j}$ or $\eta_{i,j}$ ($i, j \in \{1, \ldots, k\}, i \neq j$), and
- each node with two children has a label $\cup$.

Each node in a k-expression represents a vertex-labeled graph as follows:

- a $\circ_i$-node represents a graph with one vertex of label i;
- a $\cup$-node represents the disjoint union of the labeled graphs represented by its children;
- a $\rho_{i,j}$-node represents the labeled graph obtained from the one represented by its child by relabeling the label-i vertices with label j;
- an $\eta_{i,j}$-node represents the labeled graph obtained from the one represented by its child by adding all edges between the label-i vertices and the label-j vertices.

A k-expression represents the graph represented by its root. The *clique-width* of a graph G, denoted by $\mathbf{cw}(G)$, is the minimum integer k such that there is a k-expression representing a graph isomorphic to G.

It is known that graphs of bounded treewidth have bounded clique-width [12]. The converse is not true in general. For example, the complete graph K_n ($n \geq 2$) has clique-width 2 and treewidth $n - 1$. On the other hand, the following bound is known for graphs with no large complete bipartite subgraphs.

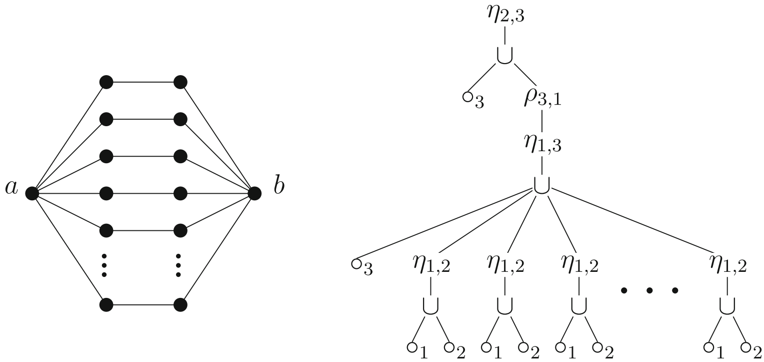

Fig. 2 A graph with n vertices, $2^{(n-2)/2}$ minimal a, b-separators, and a 3-expression

Proposition 4.11 (Gurski and Wanke [23]) *For every graph G of clique-width at most k,* $\mathbf{tw}(G) \leq 3k \cdot \omega_{\mathsf{b}}(G) - 1$.

The proposition above with Theorem 3.8 imply that Max W-Heavy can be solved in linear time for graphs of bounded clique-width. However, we cannot apply Theorem 3.6 since graphs of bounded clique-width may have a super-polynomial number of potential maximal cliques. The n-vertex graph in Fig. 2 has at least $2^{(n-2)/2}$ minimal a, b-separators. Hence the graph has at least $\frac{1}{n} \cdot 2^{(n-2)/2}$ potential maximal cliques. On the other hand, the 3-expression in Fig. 2 represents the graph, and thus it has clique-width at most 3.

In the rest of this section, we directly show that Max W-Light is polynomial-time solvable for graphs of bounded clique-width. A k-expression of a graph is *irredundant* if for each edge $\{u, v\}$, there is exactly one node $\eta_{i,j}$ that adds the edge between u and v. We will show that:

Theorem 4.12 *Given a graph with an irredundant k-expression,* Max W-Light *can be solved in time* $O(n^{2k(W+2)+4} \log n)$.

For a graph of clique-width k, one can compute a $(2^{3k} - 1)$-expression of it in $O(n^3)$ time [41] (see also [42]), while exact computation of the clique-width and a corresponding k-expression is NP-hard [18]. A k-expression of a graph can be transformed into an irredundant one with $O(n)$ nodes in linear time [16]. Now the following is a corollary to Theorem 4.12.

Corollary 4.13 *For graphs of clique-width at most k,* Max W-Light *can be solved in time* $O(n^{2(2^{3k}-1)(W+2)+4} \log n)$.

We now prove Theorem 4.12. Let G be an n-vertex graph and T be an irredundant k-expression of G with $O(n)$ nodes. We denote by r the root of T. For each node t in T, let G_t be the graph represented by t with $V_t := V(G_t)$. For each $i \in \{1, \ldots, k\}$, let V_t^i be the set of label-i vertices in G_t.

For a node t in T, a $k \times (W + 2)$ integer matrix $A = (A_{i,j})_{i \in \{1,\ldots,k\},\ j \in \{0,\ldots,W+1\}}$ is an *outdegree signature* of G_t if there is an orientation Λ of G_t such that for each $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, W\}$, $A_{i,j}$ is the number of label-i vertices with outdegree

j in G_t under Λ, and for each $i \in \{1, \ldots, k\}$, $A_{i,W+1}$ is the number of label-i vertices with outdegree at least $W+1$ in G_t under Λ. The *weight* $w(A)$ of an outdegree signature A of G_t is $|V(G_t)| - \sum_{i \in \{1,\ldots,k\}} A_{i,W+1}$. Note that there are at most $n^{k(W+2)}$ outdegree signatures for each node in T.

Observation 4.14 The optimal value of Max W-Light for G is $\max_A w(A)$, where the maximum is taken over all outdegree signatures A of $G_r = G$.

By Observation 4.14, if we have all possible outdegree signatures for all nodes in T, then we can obtain the optimal value of Max W-Light. We compute the outdegree signatures by a bottom-up dynamic programming over the k-expression T. In a standard way, we can modify the dynamic programming to compute an optimal solution as well.

Computing outdegree signatures for the leaf, $\cup$-, and $\rho_{p,q}$-nodes is fairly straightforward.

Lemma 4.15 *For a leaf node, its outdegree signature can be computed in $O(1)$ time.*

Proof Let t be a leaf node with label $\circ_i$. The graph G_t has only one outdegree signature A such that $A_{i,0} = 1$ and all other entries are 0. □

Lemma 4.16 *For a $\cup$-node, its outdegree signatures can be computed in $O(n^{2k(W+2)})$ time from the outdegree signatures of its children.*

Proof Let t be a $\cup$-node with the children t' and t''. An orientation of $G_{t'}$ with outdegree signature A' and an orientation of $G_{t''}$ with outdegree signature A'' can be combined into an orientation of G_t with outdegree signature A such that $A_{i,j} = A'_{i,j} + A''_{i,j}$ for any i and j. As there is no other way to construct an orientation of G_t, we can construct all outdegree signatures of it by merging all combinations of outdegree signatures of $G_{t'}$ and $G_{t''}$ in $O(1)$ time for each. □

Lemma 4.17 *For a $\rho_{p,q}$-node, its outdegree signatures can be computed in $O(n^{k(W+2)})$ time from the outdegree signatures of its child.*

Proof Let t be a $\rho_{p,q}$-node with the child t'. An orientation of G_t is also an orientation of $G_{t'}$ and vice versa. The corresponding outdegree signatures A and A' satisfy that $A_{p,j} = 0$ and $A_{q,j} = A'_{p,j} + A'_{q,j}$ for all j, and $A_{i,j} = A'_{i,j}$ for all $i \notin \{p, q\}$ and for all j. Because each outdegree signature of $G_{t'}$ corresponds to one outdegree signature of G_t and the corresponding signature can be computed in $O(1)$ time, the lemma holds. □

To compute outdegree signatures for $\eta_{p,q}$-nodes, we need the following result.[3]

Proposition 4.18 (Asahiro et al. [2]) *Given an undirected n-vertex m-edge graph $G = (V, E)$ with lower and upper bounds $(\mathtt{l}(v), \mathtt{u}(v)) \in \{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ for each $v \in V$, it can be decided in $O(m^{1.5} \log n)$ time whether there is an orientation Λ such that $\mathtt{l}(v) \le d^+_\Lambda(v) \le \mathtt{u}(v)$ for each $v \in V$.*

[3] The result is presented in a more general way in the original paper (see [2, Theorem 1]).

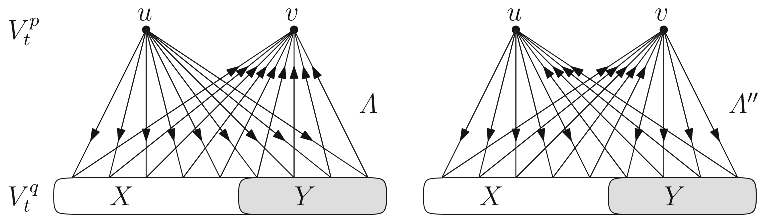

Fig. 3 The new orientation Λ'' swaps the outdegrees of u and v

Lemma 4.19 *For an $\eta_{p,q}$-node, its outdegree signatures can be computed in time $O(n^{2k(W+2)+3}\log n)$ from the outdegree signatures of its child.*

Proof Let t be an $\eta_{p,q}$-node with the child t'. By the definition of k-expression, $V_t^i = V_{t'}^i$ for all i. Recall that T is irredundant. Hence there is no edge between V_t^p and V_t^q in $G_{t'}$, while G_t has all possible edges between V_t^p and V_t^q.

Let Λ' be an orientation of $G_{t'}$ and A' the corresponding outdegree signature. We say that A' can be *extended* to an outdegree signature A of G_t if there is an orientation Λ of G_t that corresponds to A such that $\Lambda(e) = \Lambda'(e)$ for every $e \in E(G_{t'})$.

Claim 4.20 If A' can be extended to A, then there is an orientation Λ of G_t that corresponds to A such that $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$ implies $d^+_{\Lambda}(u) \le d^+_{\Lambda}(v)$ for $u, v \in V_t^i$ and $i \in \{p, q\}$.

Proof (Claim 4.20) Let Λ be an orientation of G_t that corresponds to A. A pair of vertices u and v is *reversed* in Λ if $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$, $d^+_{\Lambda}(u) > d^+_{\Lambda}(v)$, and either $u, v \in V_t^p$ or $u, v \in V_t^q$. We assume that Λ is chosen so that the number of reversed pairs is minimized. Let u, v be a reversed pair with $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$. We assume without loss of generality that $u, v \in V_t^p$.

Let X be the set of vertices in V_t^q that have arcs from u but not from v under Λ. Since $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$, we have $|X| > d^+_{\Lambda}(u) - d^+_{\Lambda}(v)$. Let Y be a subset of X such that $|Y| = d^+_{\Lambda}(u) - d^+_{\Lambda}(v)$. We denote by Λ'' the orientation obtained from Λ by reversing the directions of the edges between u, v and Y (see Fig. 3). Since $d^+_{\Lambda''}(u) = d^+_{\Lambda}(v)$, $d^+_{\Lambda''}(v) = d^+_{\Lambda}(u)$, and $d^+_{\Lambda''}(w) = d^+_{\Lambda}(w)$ for any other $w \notin \{u, v\}$, the new orientation Λ'' also gives the outdegree signature A. Note that the pair u, v is not reversed in Λ''.

Assume that for a vertex $w \in V_t^p$, the pair u, w is reversed in Λ'' but not in Λ. Since $d^+_{\Lambda''}(u) < d^+_{\Lambda}(u)$ and $d^+_{\Lambda''}(w) = d^+_{\Lambda}(w)$, we have $d^+_{\Lambda'}(w) < d^+_{\Lambda'}(u)$, $d^+_{\Lambda}(w) \le d^+_{\Lambda}(u)$, and $d^+_{\Lambda''}(u) < d^+_{\Lambda''}(w)$. Observe that $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$ implies that $d^+_{\Lambda'}(w) < d^+_{\Lambda'}(v)$. Next, $d^+_{\Lambda}(v) = d^+_{\Lambda''}(u)$ and $d^+_{\Lambda''}(w) = d^+_{\Lambda}(w)$ together imply that $d^+_{\Lambda}(v) < d^+_{\Lambda}(w)$. Now $d^+_{\Lambda''}(w) = d^+_{\Lambda}(w)$ and $d^+_{\Lambda}(u) = d^+_{\Lambda''}(v)$ give that $d^+_{\Lambda''}(w) \le d^+_{\Lambda''}(v)$. Thus the pair v, w is reversed in Λ but not in Λ''. In an almost the same way, we can show that if pair v, w is reversed in Λ'' but not in Λ, then pair u, w is reversed in Λ but not in Λ''.

Therefore, Λ'' has strictly less reversed pairs than Λ. This contradicts the assumption that Λ has the minimum number of reversed pairs. □

Let A be a candidate of an outdegree signature of G_t. That is, A is a $k \times (W + 2)$ integer matrix $A = (A_{i,j})_{i\in\{1,...,k\},\ j\in\{0,...,W+1\}}$. For $i \in \{p, q\}$, let $(d_{i,1}, \ldots, d_{i,|V_t^i|})$

be the nondecreasing sequence such that for each $j \in \{0, \dots, W+1\}$, the value j appears exactly $A_{i,j}$ times. From A', we define $(d'_{i,1}, \dots, d'_{i,|V_t^i|})$ in the same way. For $i \in \{p, q\}$ and $h \in \{1, \dots, |V_t^i|\}$, we define the lower bound $\mathtt{l}_{i,h}$ and the upper bound $\mathtt{u}_{i,h}$ as follows:

$$\mathtt{l}_{i,h} = d_{i,h} - d'_{i,h},$$
$$\mathtt{u}_{i,h} = \begin{cases} d_{i,h} - d'_{i,h} & \text{if } d_{i,h} \le W, \\ n-1 & \text{if } d_{i,h} = W+1. \end{cases}$$

Now let $B = (W_p, W_q; E_B)$ be the complete bipartite graph, where $W_i = \{w_{i,h} : i \in \{p,q\},\ h \in \{1, \dots, |V_i^t|\}\}$ for $i \in \{p, q\}$.

Claim 4.21 A' can be extended to A if and only if there is an orientation Λ_B of B such that for each vertex $w_{i,h}$, it holds that $\mathtt{l}_{i,h} \le d^+_{\Lambda_B}(w_{i,h}) \le \mathtt{u}_{i,h}$.

Proof (Claim 4.21) ($\Longrightarrow$) To show the only-if part, assume that A' can be extended to A. By Claim 4.20, we can assume that there is an orientation Λ of G_t that corresponds to A such that $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$ implies $d^+_{\Lambda}(u) \le d^+_{\Lambda}(v)$ for $u, v \in V_t^p$ and $u, v \in V_t^q$.

For $i \in \{p, q\}$, let $\prec_i$ be the partial ordering on V_t^i such that $u \prec_i v$ if and only if either $d^+_{\Lambda'}(u) < d^+_{\Lambda'}(v)$ or $d^+_{\Lambda'}(u) = d^+_{\Lambda'}(v)$ and $d^+_{\Lambda}(u) < d^+_{\Lambda}(v)$. Let $(v_{i,1}, v_{i,2}, \dots, v_{i,|V_t^i|})$ be a linear extension of $\prec_i$ for each $i \in \{p, q\}$.

Let $i' \in \{p, q\} \setminus \{i\}$. By the assumption of Λ, a vertex $v_{i,h}$ has exactly $d_{i,h} - d'_{i,h}$ neighbors in $V_t^{i'}$ if $d_{i,h} \le W$, and at least $d_{i,h} - d'_{i,h}$ neighbors in $V_t^{i'}$ if $d_{i,h} = W+1$. Now we can obtain Λ_B by orienting each edge $\{w_{p,h}, w_{q,h'}\}$ in the same way as $\{v_{p,h}, v_{q,h'}\}$ in Λ.

($\Longleftarrow$) To show the if part, assume that Λ_B exists. We construct a bijection $f_i \colon V_t^i \to W_i$ for each $i \in \{p, q\}$ in such a way that $w_{i,h}$ becomes the image of a vertex v that satisfies $d^+_{\Lambda'}(v) = d'_{i,h}$ if $d'_{i,h} \le W$, and $d^+_{\Lambda'}(v) \ge W+1$ otherwise. Such a bijection exists for each $i \in \{p, q\}$ from the definition of B.

Now we extend Λ' by orienting the edges between V_t^p and V_t^q using Λ_B. For all $u \in V_t^p$ and $v \in V_t^q$, we orient the edge $\{u, v\}$ using the direction of the edge $\{f_p(u), f_q(v)\}$ in Λ_B. We denote the obtained orientation of G_t by Λ.

Let $f_i(v) = w_{i,h}$ for $i \in \{p, q\}$. If $d_{i,h} \le W$, then $d^+_{\Lambda}(v) = d^+_{\Lambda'}(v) + d^+_{\Lambda_B}(w_{i,h}) = d'_{i,h} + (d_{i,h} - d'_{i,h}) = d_{i,h}$. If $d_{i,h} = W+1$, then $d^+_{\Lambda}(v) = d^+_{\Lambda'}(v) + d^+_{\Lambda_B}(w_{i,h}) \ge d'_{i,h} + (d_{i,h} - d'_{i,h}) = d_{i,h} = W+1$. This implies that Λ corresponds to the outdegree signature A. ☐

For each candidate A, we construct B from A and A'. We also compute the lower and upper bounds of outdegree as described above. Then we check orientability under these bounds. By Proposition 4.18, it can be done in time $O(|E_B|^{1.5} \log |W_p \cup W_q|)$. We can bound this by $O(n^3 \log n)$, and thus the lemma holds. ☐

We have proved that for each node in T, we can compute its outdegree signatures in $O(n^{2k(W+2)+3} \log n)$ time. This completes the proof of Theorem 4.12.

5 Parameterized Complexity

In this section, we study the parameterized complexity of the problems. See the recent textbook [17] for standard concepts in the field of parameterized complexity. The parameter is the number of vertices of outdegree at most (at least) W in MAX W-LIGHT (resp. MAX W-HEAVY). We call it the *solution size*. We show that when parameterized by the solution size, MAX W-LIGHT is W[1]-complete, while MAX W-HEAVY admits a kernel of size $O(Wk)$.

5.1 MAX W-LIGHT is W[1]-Complete

A *graph property* is a collection of graphs. A graph property is *nontrivial* if it is nonempty and does not include all graphs. A nontrivial graph property Π is *hereditary* if $G \in \Pi$ implies that every induced subgraph of G is also in Π.

For a graph property Π, the problem $P(G, k, \Pi)$ is defined as follows: given a graph G and a parameter $k \in \mathbb{Z}$, find a subset $S \subseteq V(G)$ with $|S| = k$ such that $G[S] \in \Pi$.

Proposition 5.1 (Khot and Raman [29]) *Let Π be a hereditary property that includes all edgeless graphs but not all complete graphs. Then the problem $P(G, k, \Pi)$ is* W[1]-*complete.*

Corollary 5.2 *For any fixed integer $W \geq 0$,* MAX W-LIGHT *is W[1]-complete when parameterized by the solution size.*

Proof Let Π be the graph property that includes all W-light orientable graphs. An edgeless graph is 0-light orientable and thus in Π, while a complete graph of $2W+2$ or more vertices is not in Π by Lemma 3.5. The property Π is hereditary because every induced subgraph of a W-light orientable graph is W-light orientable as one can just use the same (induced) orientation of the induced subgraph. Therefore, $P(G, k, \Pi)$ is W[1]-complete by Proposition 5.1. By Lemma 3.3, $P(G, k, \Pi)$ is equivalent to MAX W-LIGHT parameterized by the solution size. Hence the corollary follows. □

5.2 A Kernel for MAX W-HEAVY

Let (G, k) be an instance of the parameterized version of MAX W-HEAVY, where the parameter k is the solution size. We show the following theorem.

Theorem 5.3 MAX W-HEAVY *parameterized by the solution size k admits a kernel with at most $(2W + 4)k + W - 2$ vertices.*

In the following, we assume that $W \geq 3$ since otherwise the problem can be solved in polynomial time [3,28].

Let $A \subseteq V(G)$ be the set of vertices of degree at least W, and let $B = V(G) \setminus A$. We first bound the number of vertices in A.

Lemma 5.4 *If $|A| \geq k \cdot (W + 1)$, then (G, k) is a yes-instance.*

Proof We prove the lemma by constructing an orientation. We first mark all vertices as 'unused.' The mark 'unused' means that none of the incident edges is oriented. Now we repeat the following process until all vertices in A are marked as 'used':

1. pick an unused vertex v in A;
2. select W neighbors of v, and orient the edges from v to the neighbors;
3. mark v and the selected neighbors as 'used.'

In each iteration, one vertex becomes W-heavy in the partial orientation, and at most $W + 1$ vertices are newly marked as 'used.' Since $|A| \geq k \cdot (W + 1)$, we can repeat the process above at least k times and have k or more W-heavy vertices in the partial orientation. We then complete the partial orientation arbitrarily. This final process does not decrease the number of W-heavy vertices. Thus the lemma holds. $\square$

By the lemma above, we can assume that $|A| < k \cdot (W + 1)$. On the other hand, the size of B is not bounded in general. To further reduce the instance size, we use the following lemma.

Lemma 5.5 *Let G' be a graph with $A = V(G') \cap V(G)$ and $B' := V(G') \setminus V(G)$ such that $G'[A] = G[A]$, and $|N_{G'}(v) \cap B'| = \min\{|N_G(v) \cap B|, W\}$ for all $v \in A$. If $\deg_{G'}(v) < W$ for all $v \in B'$, then (G, k) is a yes-instance if and only if so is (G', k).*

Proof Observe that the vertices in B and B' cannot be W-heavy in any orientation of G and of G', respectively, since they have degree less than W in both graphs. Hence we may assume without loss of generality that in any orientation, the edges between A and B (A and B') are oriented from A to B (resp. A to B').

To show the only-if part, let Λ be an orientation of G. We orient each edge of G' with both endpoints in A as in Λ. We call this orientation Λ'. Since a vertex $v \in A$ has the same out-neighbors in A under both Λ and Λ', it suffices to compare the number of neighbors in B and B'. If $|N_G(v) \cap B| \geq W$, then v is W-heavy under both Λ and Λ'. Otherwise, v has the same number of neighbors in B and B'. Therefore, v is W-heavy under Λ if and only if it is W-heavy under Λ'.

To show the if part, let Λ' be an orientation of G'. We orient each edge of G with both endpoints in A as in Λ'. We then arbitrarily orient each edge of G with both endpoints in B. The rest is almost the same as the one for the only-if part. $\square$

From G, we obtain G' satisfying the assumptions in Lemma 5.5 as follows:

1. remove all vertices of B from G;
2. add an independent set B' of size $\lceil |A| \cdot W/(W - 1) \rceil + W - 2$;
3. for each $v \in A$, repeat the following process:
   (a) find $\min\{|N_G(v) \cap B|, W\}$ vertices in B' that have degree at most $W - 2$ in the current graph;
   (b) add the edges between v and the vertices chosen.

Since $W \geq 3$, it holds that $(W + 1)W/(W - 1) \leq W + 3$, and thus $|B'| \leq k(W + 3) + W - 2$. This implies that $|V(G')| = |A| + |B'| \leq k(2W + 4) + W - 2$.

Note that the step 3a is always possible. To see this, observe that before an execution of the step 3a, at most $W(|A| - 1)$ edges between A and B' are added. On the other hand, if there are at most $W-1$ vertices of degree at most $W-2$ in B', then there are at least $(W-1)(|B'|-(W-1)) \geq (W-1)(|A| \cdot W/(W-1)+W-2-(W-1)) = W(|A|-1)+1$ edges between A and B'.

By Lemma 5.5, G' is a kernel we claimed in the statement of Theorem 5.3.

6 Concluding Remarks

We have presented metatheorems to show linear-time and polynomial-time solvability of Max W-Heavy and Max W-Light, respectively. The metatheorems are applied to several important classes of graphs.

We believe our metatheorems can be applied to many other graph classes. It is known that the classes of weakly chordal graphs, circle graphs, and polygon circle graphs have polynomial upper bounds on the number of potential maximal cliques (see [19]). It is known that some of them have functions on the maximum degree as upper bounds of treewidth [22]. It would be interesting to investigate whether we can strengthen these upper bounds to functions on degeneracy.

A graph is k-chordal if it does not have an induced cycle of length more than k. For example, the chordal graphs are exactly the 3-chordal graphs, and d-trapezoid graphs and weakly chordal graphs are 4-chordal graphs. It can be easily seen that k-chordal graphs for $k \geq 4$ have a superpolynomial number of potential maximal cliques. On the other hand, it is known that their treewidth is upper bounded roughly by $k \cdot \Delta$ [33], where Δ is the maximum degree. It would be interesting to study our problems for k-chordal graphs with $k \geq 4$. Even for 4-chordal graphs, it is not known whether Maximum Independent Set (= Max 0- Light) can be solved in polynomial time [39].

As the final remark, we present a result similar to Theorems 4.1 and 4.2 for the problem of finding a maximum induced subgraph with bounded degeneracy.

Theorem 6.1 *For any fixed W, the problem of finding a maximum set of vertices that induces a subgraph of degeneracy at most W can be solved in polynomial time for the classes of chordal graphs, d-trapezoid graphs, circular-arc graphs, and chordal bipartite graphs, and in cubic time for graphs of bounded clique-width.*

Proof We first show that the property of having degeneracy at most W can be expressed in MSO without edge or edge set variables. In such a subclass of MSO, $\mathsf{inc}(d, u)$ is replaced by $\mathsf{adj}(u, v)$ that means vertices u and v are adjacent. It is known that a problem that can be expressed in this restricted MSO is cubic-time solvable for graphs of bounded clique-width [15,41]. The following formula expresses that a vertex u has degree more than W in the subgraph induced by Y:

$$\mathsf{degree}_{>W}(Y, u) := \exists v_1, \ldots, v_{W+1} \in Y\, (\mathsf{distinct}(v_1, \ldots, v_{W+1}) \wedge \forall v_i (\mathsf{adj}(u, v_i)))\,.$$

Now we can express the property of $G[X]$ having degeneracy at most W as follows:

$$W\text{-}\mathsf{degenerated}(X) := \forall Y \subseteq X, \exists u \in Y, \neg \mathsf{degree}_{>W}(Y, u).$$

As we showed in Sect. 4, bounded degeneracy means bounded treewidth for chordal graphs, d-trapezoid graphs, circular-arc graphs, and chordal bipartite graphs. Thus we can apply Proposition 3.4 and have the theorem. □

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