β-Stars or On Extending a Drawing of a Connected Subgraph

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Abstract. We consider the problem of extending the drawing of a subgraph of a given plane graph to a drawing of the entire graph using straight-line and polyline edges. We define the notion of star complexity of a polygon and show that a drawing Γ_H of an induced connected subgraph H can be extended with at most $\min\{h/2, \beta + \log_2(h) + 1\}$ bends per edge, where β is the largest star complexity of a face of Γ_H and h is the size of the largest face of H. This result significantly improves the previously known upper bound of $72|V(H)|$ [5] for the case where H is connected.We also show that our bound is worst case optimal up to a small additive constant. Additionally, we provide an indication of complexity of the problem of testing whether a star-shaped inner face can be extended to a straight-line drawing of the graph; this is in contrast to the fact that the same problem is solvable in linear time for the case of star-shaped outer face [9] and convex inner face [12].

1 Introduction

In this paper we study the problem of extending a given partial drawing of a graph. In particular, given a plane graph $G = (V, E)$, i.e. a planar graph with a fixed combinatorial embedding and a fixed outer face, a subgraph H of G and a planar straight-line drawing Γ_H of H, we ask whether Γ_H can be extended to a planar straight-line drawing of G (see Fig. 1). We study both the decision question and the relaxed variation of using bends for the drawing extension.

It is known that a drawing extension always exists even if $H = (V, \emptyset)$, where each edge is represented by a polyline with at most $120n$ bends, here $n = |V|$ [14]. This bound was improved to $3n+2$ by Badent et al. [1]. These upper bounds are asymptotically optimal as there are instances that require $\Omega(n)$ bends on $\Omega(n)$ edges [1]. In terms of the size of the pre-drawn graph H, Chan et al. [5] showed that a drawing extension with $72|V(H)|$ bends per edge is possible for a general subgraph H.

In order to pinpoint the source of multiple necessary bends for the drawing extension we define the notion of a β-star (resp. β-outer-star), a polygon where β bends are necessary and sufficient to reach the kernel of the polygon (resp.

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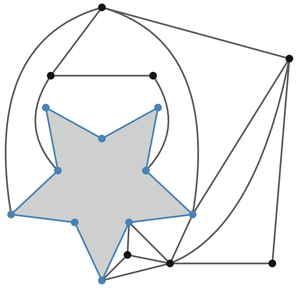

Fig. 1. An embedding of a plane graph alongside a fixed drawing of an inner face (blue) as a star-shaped polygon (gray). (Color figure online)

infinity). We study the upper bounds on the number of bends in a drawing extension as a function of β. We show that a drawing Γ_H of an induced connected subgraph H can be extended with at most $\min\{h/2, \beta + \log_2(h) + 1\}$ bends per edge if each face of H is represented in Γ_H as a β-(outer)-star and h is the size of the largest face of H (Theorem 7). We show that this bound is worst case optimal up to a small additive constant. We observe that in case both G and H are trees a closer to optimal bound of $1 + 2\lceil |V(H)|/2 \rceil$ bends per edge had been provided by Di Giacomo et al. [7].

In case a planar embedding is not provided as a part of the input, it is NP-hard to test whether a straight-line drawing extension exists [15]. The problem is not known to belong to the class NP, as a possible solution may have coordinates which can not be represented with a polynomial number of bits [15]. Very recently, Lubiw et al. have studied a related problem of drawing a graph inside a (not-necessarily simply connected) closed polygon [11]. They showed that this problem can not be shown to lie in NP by the mean of providing vertex coordinates, as these are sometimes irrational numbers. They have also shown that the problem is hard for the existential theory of reals ($\exists\mathbb{R}$-hard) even if a planar embedding of the graph is provided as a part of the input. This problem would be equivalent to partial graph drawing extendability, if the polygon would be open, however this situation has not been investigated. Bekos et al. [2,3] have studied the problem of extending a given partial drawing of bipartite graphs, where one side of the bipartition is pre-drawn. They have shown that this problem lies in NP if each free vertex is required to lie in the convex hull of its pre-drawn neighbors. Regarding drawing extensions with bends, it is NP-hard to test whether a drawing extension with at most k bend per edge exists [2,8].

Despite all the hardness results, it is long known that a straight-line drawing extension always exists if H is the outer face and Γ_H is a convex polygon [4, 16]; and H is a chordless outer face and Γ_H is a star-shaped polygon [9]. An existence of a straight-line drawing extension can be checked by the mean of necessary and sufficient conditions in case where H is an inner face and Γ_H is a convex polygon [12]. As an extension of this work, and with the general goal to better understand the boundary between the easy and the difficult cases, we investigated the question of testing whether a straight-line drawing extension exists for an inner face H drawn as a star-shaped polygon Γ_H. We observe that one can not test whether such an extension exists by just checking each vertex individually, as in the case for a convex inner face, and show that there exists an instance such that the region where a vertex of $V(G) \setminus V(H)$ can lie to allow for a straight-line drawing extension is bounded by a curve of degree $2^{\Omega(|H|)}$(Theorem 8).

Contribution and Outline. We start with the necessary definitions in Sect. 2. In Sect. 3, we show that a star-shaped drawing of an inner face can be extended with at most 1 bend per edge. Section 4 is devoted to the study of generalizations of stars. In Sect. 4.1, we start with a generalization of star-shaped polygons to β-star and β-outer-star polygons (β is referred to as star complexity), and show that the number of bends per edge necessary for a drawing extension of an inner face H with a star complexity β is not bounded in terms of β (Theorems 2 and 3). Motivated by the proof of Sect. 3 we define the notion of planar-β-star and planar-β-outer-star (this β is referred to as planar star complexity) and show that the planar star complexity determines the number of bends per edge in a drawing extension (Theorems 4 and 5). In Sect. 4.2, we study the planar star complexity of an arbitrary simple polygon and the relationship between the star complexity and the planar star complexity of a polygon. In particular, we show that every β-star with n vertices is a planar-$\beta + \delta$-star where $\delta \leq \log_2(n)$ (Theorem 6). In Sect. 5, we state the implications of Sect. 4 to the drawing extension of (induced) connected subgraphs. In particular, we prove that a drawing Γ_H of an induced connected subgraph H can be extended with at most $\min\{h/2, \beta + \log_2(h) + 1\}$ bends per edge if the star complexity of Γ_H is β and h is the size of the largest face of H (Theorem 7). Last but not least, in Sect. 6 we provide an indication of complexity of the problem of testing whether a star-shaped inner face H admits a straight-line drawing extension. In particular, we prove that there exists an instance such that the region where a vertex of $V(G) \setminus V(H)$ can lie to allow a straight-line drawing extension is bounded by a curve of degree $2^{\Omega(|H|)}$ (Theorem 8). All omitted proofs can be found in the full version [13].

2 Preliminaries

Basic Geometric Terms. The segment (resp. line) induced by two points a and b is designated by $s(a, b)$ (resp. $l(a, b)$). We denote a curve between a and b by $c(a, b)$. We refer to the ray along $l(a, b)$ starting at a and (not) containing b as $r(a, b)$ ($q(a, b)$). For a polyline c, $\#c$ designates the number of bends on c.

Let P be a polygon. Two points a, b *see* each other if the open segment $s(a, b)$ does not intersect the boundary of P. A simple polygon P is *convex* if each pair of points inside P see each other. A simple polygon P is *star-shaped* or a *star* if there is a non-empty set of points K called the *kernel* inside the polygon such that any point of the kernel can see any vertex of the polygon. By assuming that the vertices of P are in general position, we have that a kernel of P contains an open ball of positive radius.

Graphs and Drawings of Graphs. A *drawing* Γ of a graph is a function that assigns to each vertex a unique point in the plane and to each edge $\{a, b\}$ a curve connecting the points assigned to a and b. A drawing is *straight-line* (resp. *k-bend*) if each edge is drawn as a segment (resp. a polyline with at most k bends). A graph is *planar* if it has a *planar* drawing, i.e. a drawing without edge crossings. A planar drawing Γ subdivides the plane into connected regions called *faces*; the unbounded region is the *outer* and the other regions are the *inner* faces. The cyclic ordering of the edges around each vertex of Γ together with the description of the outer face of Γ characterize a class of drawings with the same combinatorial properties, which is called an *embedding* of G. A planar graph G with a planar embedding is called *plane graph*. A *plane subgraph* H of G is a subgraph of G together with a planar embedding that is the restriction of the embedding of G to H. A plane graph G is (*internally*) *triangulated* if each (inner) face of G is a triangle. For a given cycle, a chord is an edge between two non-consecutive vertices of the cycle.

Let G be a plane graph and let H be a plane subgraph of G. Let Γ_H be a planar straight-line drawing of H. We say that the instance (G, Γ_H) admits a *k-bend* (resp. *straight-line*) *extension* if drawing Γ_H can be completed to a planar k-bend (resp. straight-line) drawing Γ_G of the plane graph G. We refer to k as the *curve complexity* of the drawing Γ_G.

For a given graph $G = (V, E)$, let $N(v) = \{w \in V \mid \{v, w\} \in E\}$ be the *neighbors* of $v \in V$. For a plane graph G and a face F, let $N_F(v) = N(v) \cap F = (w_1, w_2, \ldots, w_\ell)$ be the sequence of neighbors of v that belong to F. For v outside F, let the list $N_F(v)$ be ordered clockwise around F with w_1 chosen such that the area delimited by the cycle C composed of edges $\{v, w_1\}$, $\{v, w_\ell\}$ and the clockwise path H from w_1 to w_ℓ in F does not contain F (see Fig. 2a). A vertex $z \in V \setminus V(F)$ lying in the cycle C is said to be *enclosed* by vertex v.

Let F be a face of G and Γ_F its planar drawing. The *feasibility area* of a vertex $v \in V \setminus V(F)$ is the set of all possible positions of v, such that the implied straight-line drawing of $F \cup \{v\}$ can be extended to a planar straight-line drawing of $V(F) \cup \{v\} \cup Q_v$, where Q_v is the set of all vertices enclosed by v.

3 Star-Shaped Polygons

Let G be a plane graph with n vertices, F be a chordless face of G with h vertices and Γ_F a star-shaped drawing of F. In this section we prove that the instance (G, Γ_F) admits a 1-bend-extension. While the proof itself is rather straightforward, we still present it here as it motivates a specific way to generalize star-shaped polygons by considering planarity issues.

In our construction we place vertices $V \setminus V(F)$ one by one with the property that a vertex is placed only after all vertices enclosed by it have already been placed. This property is achieved by a canonical ordering [10] that lists vertices starting from the face F. The following lemma can be proven along the same lines as the existence of a usual canonical ordering [10]. We say $G \setminus F$ is triangulated if each face of G is triangulated with the exception of the face F.

Lemma 1. *Let $G = (V, E)$ be a plane graph, $|V| = n$, and let F be an inner face with h vertices of G, such that $G \setminus F$ is triangulated. There is an ordering $\mathcal{J} = (v_1, \ldots, v_{n-h})$ of the vertices of $V \setminus V(F)$, such that for each j, $1 \leq j \leq n-h$, the following holds:* (1) *the graph G_j induced by the vertices $\{v_1, \ldots, v_j\} \cup F$ is biconnected,* (2) *$G_j \setminus F$ is internally triangulated,* (3) *v_{j+1} lies in the outer face of G_j,* (4) *vertices $N(v_{j+1}) \cap V(G_j)$ belong to the outer face of G_j.*

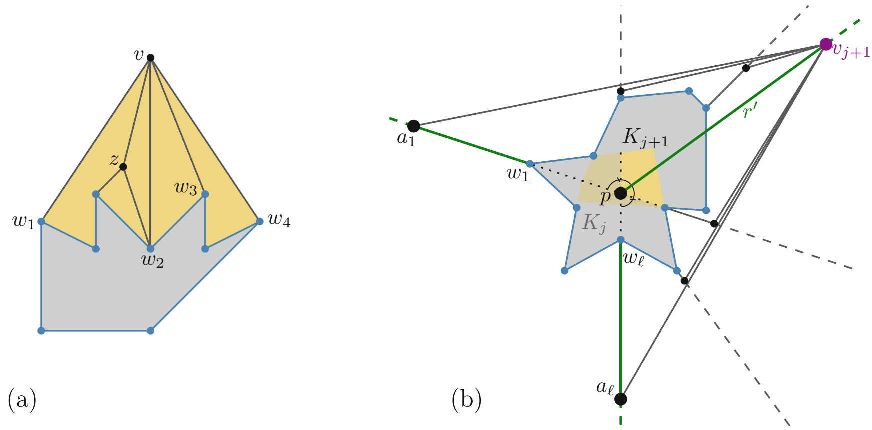

Fig. 2. (a) Yellow area contains vertices enclosed by v. (b) Proof of Theorem 1. (Color figure online)

Theorem 1. *Each instance $(G = (V, E), \Gamma_F)$ where Γ_F is a star-shaped drawing of a chordless inner face F allows a 1-bend-extension.*

Proof. We start with triangulating G by placing a vertex in each non-triangular face and connecting it to the vertices of the face. We delete the added vertices and edges after the triangulated graph has been drawn. We refer to the new graph as G as well. Let $\mathcal{J} = (v_1, \ldots, v_{n-h})$ be an ordering of the vertices $V \setminus V(F)$ as defined by Lemma 1. For $1 \leq j \leq n - h = |V(G)| - |V(F)|$, let G_j be the graph as defined by Lemma 1 and let F_j be the outer face of G_j. Additionally we set $G_0 = F_0 = F$.

We prove the theorem by induction. Assume that for a $0 \leq j \leq n-h$ we have a drawing of G_j, such that F_j forms a star-shaped polygon P_j with kernel K_j. This is true for $j = 0$. Let v_{j+1} be the next vertex according to $\mathcal{J}$ and let p be a point of the kernel of the already drawn star-shaped polygon P_j. For each $w \in N_{F_j}(v_{j+1})$ consider the ray $q(p, w)$. Due to P_j being star-shaped and due to property (4) of Lemma 1, they all lie outside of P_j. Since G_j is biconnected, v_{j+1} has at least two neighbors, i.e. $\ell = |N_{F_j}(v_{j+1})| \geq 2$.

Now we consider the ray r' that is the bisector of the clockwise angle formed by the rays $q(p, w_1)$ and $q(p, w_\ell)$, see Fig. 2b. If we place v_{j+1} sufficiently far away from p on r', v_{j+1} sees $q(p, w_1)$ and $q(p, w_\ell)$, i.e. $\exists a_1 \in q(p, w_1), a_\ell \in q(p, w_\ell)$, with $s(v_{j+1}, a_1) \cap P_j = \emptyset = s(v_{j+1}, a_\ell) \cap P_j$. This is due to the fact that the angles between $q(p, w_1)$ and r' and between $q(p, w_\ell)$ and r' are strictly smaller than π.

Since v_{j+1} is between $q(p, w_1)$ and $q(p, w_\ell)$, v_{j+1} also sees a point a_i on the ray $q(p, w_i)$, $i = 2, \ldots, \ell - 1$. For each $i \in \{1, \ldots, \ell\}$ we draw the edge $\{w_i, v_{j+1}\}$ using the segments $s(w_i, a_i)$ and $s(a_i, v_{j+1})$. Observe that the points $a_1, \ldots, a_\ell$ should be chosen so that they appear around v_{j+1} in a counterclockwise order.

The lines $l(p, w_1)$ and $l(p, w_\ell)$ separate the plane into four quadrants. The new kernel K_{j+1} of the polygon P_{j+1} is the intersection of the old kernel K_j and the quadrant containing v_{j+1}. Since the kernel K_j was an open set, p could not have been on the boundary of K_j, therefore K_{j+1} is a non-empty open set. $\square$

We observe that, according to the proof of Theorem 1, the class of the polygons that allows a 1-bend-extension is wider than stars. In particular, these are the polygons from the vertices of which we can shoot rays to infinity which neither intersect mutually nor intersect the polygon itself. We call such polygons *planar outer-stars*. This gives the following:

Corollary 1. *Each instance* (G, Γ_F) *where* F *is a chordless inner face and* Γ_F *is a planar outer-star, allows a* 1*-bend-extension.*

4 Generalization of Stars

In this section we generalize the notion of stars and planar outer-star polygons and investigate the lower and upper bounds for the number of bends per edge in the drawing extensions.

4.1 β-Stars

A simple polygon P is a *β-star* if there is an open set of points K called the *kernel* inside P with the following property: for each point $p \in K$ and for each vertex v of P there is a polyline $c(v)$ connecting v and p with at most β bends such that $c(v)$ touches P only at v. The smallest such β is referred to as *star complexity* of the polygon P. This set of curves is referred to as *curve-set* $\mathcal{C}$ of P and p is the *center* of $\mathcal{C}$. In the literature this kernel is also known as the link center of the polygon and it can be calculated in $O(n \log n)$ time [6]. The straight-forward extension of this definition to act "outside" the polygons is as follows: a simple polygon P is a *β-outer-star* if for each vertex v of P there is an infinite polyline $c(v)$ outside of P starting at v with at most β bends. The smallest such β is referred to as *outer star complexity* of the polygon P. Again, $\mathcal{C} = \{c(v) \mid v \in P\}$ is called *curve-set*. The *center* of this set is a point at infinity. One can think about β-outer-star as of β-star with the kernel in infinity.

While β-star and β-outer-star are straight-forward ways to extend the notion of a star inside and outside, and these definitions capture an inherent complexity of the polygon, we can show that restricting the fixed inner face to be a 1-star is not sufficient to ensure a c-bend-extension for any constant c (Theorem 2). Even more, restricting the fixed inner face to a β-outer-star still does not imply the existence of a $c+\beta$-bend-extension for any constant c (Theorem 3).

Theorem 2. *There exist instances* (G, Γ_F) *where* F *is an inner face with* h *vertices and* Γ_F *is a* 1*-star such that any drawing extension of* (G, Γ_F) *contains an edge with at least* $\lfloor \frac{h-3}{2} \rfloor$ *bends.*

Theorem 3. *There exist instances* (G, Γ_F) *where* F *is an inner face with* h *vertices and* Γ_F *is a* β*-outer-star such that any drawing extension of* (G, Γ_F) *has an edge with at least* $\beta + \log_2(\frac{h+5}{6}) + 1$ *bends.*

The above lower bounds and the fact that a planar outer-star admits an extension with one bend per edge guided us to extend definitions of β-star and β-outer star to include planarity. A simple polygon P is a *planar-*β*-star* if there is an open set of points K called the kernel inside P with the following property: for a fixed point $p \in K$ and for each vertex v of P there is an oriented polyline $c(v)$ inside P from v to p with at most β bends such that for any v and v', $c(v)$ and $c(v')$ share the single point p.

A simple polygon P is a *planar-*β*-outer-star* if for each vertex v of P there is an oriented infinite polyline $c(v)$ outside of P starting at v with at most β bends such that for any v and v', $c(v)$ and $c(v')$ neither cross nor touch each other. The smallest such β is referred to as *planar (outer) star complexity* of the polygon P. The set of curves are referred to as *planar curve-set* centered at the fixed point p. Due to these definitions the following two theorems can be proven.

Theorem 4. *Each instance* (G, Γ_F) *where* F *is a chordless outer face and* Γ_F *is a planar-*β*-star allows a* β*-bend-extension.*

Theorem 5. *Each instance* (G, Γ_F) *where* F *is a chordless inner face and* Γ_F *is a planar-*β*-outer-star allows a* $\beta + 1$*-bend-extension.*

4.2 Planar Star Complexity of Polygons

While planar (outer) star complexity nicely bounds the required number of bends per edge in a drawing extension, it does not represent a simple and inherent polygon characteristic. Thus, in the following we first provide an upper bound on the planar (outer) star complexity of a polygon in terms of the size of the polygon (Lemma 2). Then, after preliminary results, we provide an upper bound of a planar (outer) star complexity in terms of (outer) star complexity (Theorem 6).

Lemma 2. *A simple polygon with* h *vertices is a planar* $\frac{h-2}{2}$*-star and a planar* $\frac{h-2}{2}$*-outer-star.*

Proof. For the interior, we set a kernel K to be an intersection of the interior of P with an ε-ball around a vertex u of P. Let p be a point in K. Notice, that by just following the boundary of the polygon it is possible to reach p from any vertex $v \neq u$ with a polyline $c(v)$ with at most $\frac{h-2}{2}$ bends. A set of such curves $\{c(v_i)|v_i \in P\}$, drawn in an appropriate order in order to avoid mutual intersections, represents a planar curve-set of P.

For the exterior, observe that by following the boundary of the polygon from any vertex u of P it is possible to reach a vertex belonging to the convex hull of P with a polyline $c(u)$ with at most $\frac{h-4}{2}$ bends because the convex hull contains at least three vertices. A set of such curves, drawn in appropriate order in order to avoid mutual crossing, augmented by infinite rays, result in a planar curve-set of P with curve complexity at most $\frac{h-2}{2}$. ☐

Observe that, in general, the planar star complexity of a polygon may be much lower than $\frac{h-2}{2}$. Thus, in the following we aim to bound the planar star complexity in terms of the star complexity. We rely on the following definitions: let $\mathcal{C}$ be a planar curve-set of a planar-β(-outer)-star. For a curve $c(v)$ from $\mathcal{C}$ and a point p on $c(v)$, we denote by $c_v(p)$ the part of the curve split at p, not containing v and by $\#c_v(p)$ the number of bends on $c_v(p)$. Furthermore, $c(v,p)$ designates the part of the curve $c(v)$ between v and p. An intersection between the curves $c(v)$ and $c(w)$ of $\mathcal{C}$ at a point p is called *avoidable* if one of the curves has more bends after the intersection than the other, i.e. if $\#c_v(p) \neq \#c_w(p)$. The term "avoidable" stems from the fact that if $\#c_v(p) > \#c_w(p)$, we can modify $c(v)$ by rerouting it along $c(w)$ starting just before the point p and this way eliminate the intersection without increasing the number of bends per curve. Concerning said avoidable intersections the following holds:

Lemma 3. *For a given β(-outer)-star P there is a curve-set of P with at most β bends each without avoidable intersections.*

In order to resolve all remaining intersections we consider pairs of curves a and b intersecting at a point p, such that p is the first intersection for both a and b. In that case we call p *initial* intersection. However, we first have to show that if there are intersections, then there is always at least one initial intersection. We formalize this in the following definition and Lemma 4. A sequence of vertices $(w_1,\ldots,w_m)$ of P, with respective curves $(c(w_1),\ldots,c(w_m))$ is called *cyclic ordering*, if for each $1 \leq j \leq m$, the first curve that $c(w_j)$ intersects is the curve $c(w_{(j \mod m)+1})$. We can prove the following:

Lemma 4. *For a given polygon P with a curve-set $\{c(v) \mid v \in V(P)\}$ without avoidable intersections there is no cyclic ordering.*

Using Lemmas 3 and 4 we prove a relation between β-stars and planar-β-stars.

Theorem 6. *Every β-star (resp. outer-star) with n vertices is a planar-$(\beta+\delta)$-star (resp. outer-star), where $\delta \leq \log_2(h)$.*

Proof. Let P be a β(-outer)-star with h vertices. By Lemma 3, P has a curve-set with at most β bends per curve without avoidable intersections. Let p be an initial intersection of two curves, which exists by Lemma 4. We resolve the intersection p by adding a bend to one of the curves and rerouting it along and sufficiently close to the other to ensure that they have the same intersections with other curves. We call such curves that follow each other after a resolved intersection a *group*. We then repeat resolving intersections of groups until there are no more intersections. As a final part of the proof we show that during this process for each curve at most $\log_2(h)$ bends have been added.

For a curve c, let $\#^a c$ be the number of bends that were added to c during this algorithm. During the execution of the algorithm we maintain a set of groups $\mathbb{G}$. Each group $\mathcal{G}r_i \in \mathbb{G}$ is a set of curves. For each group $\mathcal{G}r_i$ let $\#^a\mathcal{G}r_i$ be the maximum number of additional bends over all curves in $\mathcal{G}r_i$, i.e. $\#^a\mathcal{G}r_i = \max_{c\in\mathcal{G}r_i}(\#^a c)$. In the beginning each curve is in its own group, that means we start with $\mathbb{G} = \{\{c(v)\} \mid v \in V(P)\}$ and for each $\mathcal{G}r_i \in \mathbb{G}$, $\#^a\mathcal{G}r_i = 0$.

The following step is repeated until there are no more intersections. Let p be an initial intersection of two groups $\mathcal{G}r_i$ and $\mathcal{G}r_j$. We reroute the curves of one of $\mathcal{G}r_i$ and $\mathcal{G}r_j$. If we choose to reroute $\mathcal{G}r_j$, then we add a bend to each curve of $\mathcal{G}r_j$ and then the curves of $\mathcal{G}r_j$ follow along the curves of $\mathcal{G}r_i$, thus increasing $\#^a c$ by one for each $c \in \mathcal{G}r_j$. Resolving the intersection p creates a new group $\mathcal{G}r_k = \mathcal{G}r_i \cup \mathcal{G}r_j$. In order to keep $\#^a\mathcal{G}r_k$ bounded we apply the following strategy: if $\#^a\mathcal{G}r_i \neq \#^a\mathcal{G}r_j$, then we reroute the group with less additional bends and get $\#^a\mathcal{G}r_k = \max\{\#^a\mathcal{G}r_i, \#^a\mathcal{G}r_j\}$. Otherwise, $\#^a\mathcal{G}r_i = \#^a\mathcal{G}r_j$ and we arbitrarily choose one of the groups, so $\#^a\mathcal{G}r_k = \#^a\mathcal{G}r_j + 1$. With each resolved intersection two groups are merged into one, thus the overall number of groups reduces by one. As a result, after at most $h - 1$ resolved crossings between groups this iteration stops.

After the above procedure no two curves intersect, thus P is a planar-$\beta+\delta$-star (resp. outer-star) with $\delta = \max_{\mathcal{G}r\in\mathbb{G}}(\#^a\mathcal{G}r)$. In the following we prove by induction over the group size that for each group $\mathcal{G}r$ it holds that $\#^a\mathcal{G}r \leq \log_2(|\mathcal{G}r|)$. For the induction base we observe that if $|\mathcal{G}r| = 1$ we have $\#^a\mathcal{G}r = 0 = \log_2(|\mathcal{G}r|)$. As an induction hypothesis, assume that for a $k \geq 1$ and each group $\mathcal{G}r$ with $|\mathcal{G}r| \leq k$, it holds that $\#^a\mathcal{G}r \leq \log_2(|\mathcal{G}r|)$. Let $\mathcal{G}r_l$ be a group with $|\mathcal{G}r_l| = k+1$, which is the result of merging two groups $\mathcal{G}r_i$ and $\mathcal{G}r_j$. Since $|\mathcal{G}r_i|, |\mathcal{G}r_j| < |\mathcal{G}r_l|$, the induction hypothesis holds for both $\mathcal{G}r_i$ and $\mathcal{G}r_j$. If $\#^a\mathcal{G}r_i \neq \#^a\mathcal{G}r_j$, we have $\#^a\mathcal{G}r_l = \max\{\#^a\mathcal{G}r_i, \#^a\mathcal{G}r_j\} \leq \log_2(\max\{|\mathcal{G}r_i|, |\mathcal{G}r_j|\}) < \log_2(|\mathcal{G}r_l|)$. Otherwise, if $\#^a\mathcal{G}r_i = \#^a\mathcal{G}r_j$, lets assume w.l.o.g. $|\mathcal{G}r_i| \geq |\mathcal{G}r_j|$, and therefore $|\mathcal{G}r_l| \geq 2|\mathcal{G}r_j|$. We have $\#^a\mathcal{G}r_l = \#^a\mathcal{G}r_j + 1 \leq \log_2(|\mathcal{G}r_j|) + 1 \leq \log_2(|\mathcal{G}r_l|/2) + \log_2(2) = \log_2(|\mathcal{G}r_l|)$.

Since for each v of P the curve $c(v)$ appears in exactly one group, we have that the maximum size of a group is h. It follows that P is a planar-$\beta+\delta$-star (resp. outer-star) with $\delta = \max_{\mathcal{G}r\in\mathbb{G}}(\#^a\mathcal{G}r) \leq \log_2(h)$. □

5 Drawing Extensions of Connected Subgraphs

In this section we apply the results from the previous section to provide a tight upper bound on the number of bends in a drawing extension of a connected subgraph.

Theorem 7. *Each instance* (G, Γ_H) *where* H *is an induced connected subgraph of* G *allows a* $\min\{h/2, \beta + \log_2(h) + 1\}$*-bend-extension, where* h *is the maximum face size of* H *and* β *is the maximum (outer) star complexity of a face in* Γ_H*. This bound is tight up to an additive constant.*

Above theorem implies an upper bound on the number of bends in case of a non-induced subgraph by simply subdividing the induced edges by dummy vertices and removing them after construction. The tightness of the bound follows from the fact that the lower bound proofs (Theorems 2 and 3) can easily be adapted to work for chords.

Corollary 2. *Each instance* (G, Γ_H) *where* H *is a connected subgraph of* G*, allows a* $\min\{h + 1, 2\beta + 2\log_2(h) + 3\}$*-bend-extension, where* h *is the maximum face size of* H *and* β *is the maximum star complexity of a face in* Γ_H*. This bound is tight up to an additive constant.*

6 Extending Stars with Straight Lines

Let $G = (V, E)$ be a plane graph and F a chordless face, fixed on the plane as a star-shaped polygon Γ_F. In this section we study the question whether (G, Γ_F) admits a straight-line extension. Note that for F being the outer face of G, Hong and Nagamochi [9] showed that (G, Γ_F) always admits a straight-line extension. In the following F is an inner face.

If F is an inner face fixed as a convex polygon Γ_F, Mchedlidze *et al.* [12] showed that it can easily be tested if an instance (G, Γ_F) admits a straight-line extension. In their case a necessary and sufficient condition for an extension to exist is that for each vertex individually there is a valid position outside Γ_F. For stars a comparable result is not possible. Even if each vertex could be drawn individually this does not mean that the whole instance admits a straight-line extension. Even more, testing whether pairs of vertices can be drawn together would not be sufficient as the construction in Fig. 3 suggests.

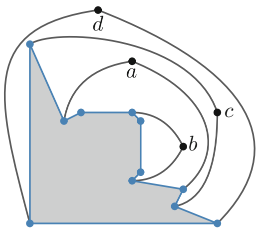

Fig. 3. The drawing cannot be extended to a straight-line drawing of the entire graph, even though this is not revealed when testing individual parts. (Color figure online)

In case of Γ_F being a convex inner face [12], the feasibility area of a vertex adjacent to the fixed face is just a wedge, formed by the intersection of two half planes induced by two edges of Γ_F. In this section we show that the situation for the star shaped inner face is dramatically different, thus there exists an instance for which the feasibility area of a vertex is partially bounded by a curve of exponential complexity.

Theorem 8. *There is an instance* (G, Γ_F) *where* Γ_F *is a star-shaped inner face, such that the feasibility area of some vertex* $v \in G$ *is partially bounded by a curve whose implicit representation is a polynomial of degree* $2^{\Omega(|V|)}$.

Sketch of Proof. A curve is *i-exponentially-complex* if it has a parametric representation of the form $\left\{\left(\frac{r(t)}{u(t)}, \frac{s(t)}{u(t)}\right) \mid t \in \mathcal{I}\right\}$ where r, s and u are polynomials of degree 2^i and $\mathcal{I}$ is an interval. In the following we describe an instance (G, Γ_F), for which a feasibility area of a vertex v is bounded by an $2^{\Omega(|V|)}$-exponentially-complex curve. By slightly pertubing the positions of the vertices of Γ_F to achieve points in general position we have that the implicit representation of this curve is a polynomial of degree at least $2^{\Omega(|V|)}$.

Let $k \geq 1$ be a fixed integer. Figure 4a displays the plane graph $G_k = (V, E)$ and the drawing of its inner face as a star-shaped polygon. The vertices v_i and w_i still need to be drawn. For $0 \leq i \leq k$, the feasibility area of v_i is denoted by A_i and the boundary of A_i is referred to as B_i. We show that B_k contains a 2^k-exponentially-complex curve. The proof is by induction on $0 \leq i \leq k$.

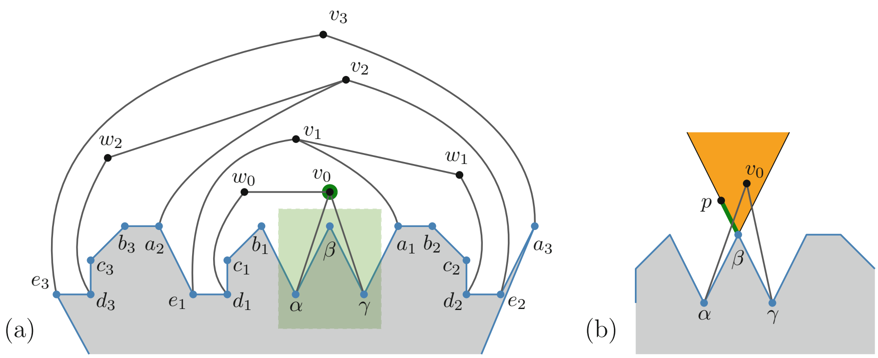

Fig. 4. (a) Graph G_3, the fixed face is drawn in gray. The vertices in the green area are part of the base case. (b) The base case. The green curve $\mathcal{C}_0$ is on the boundary B_0 of the feasibility area A_0 of v_0. (Color figure online)

As the base of the induction we consider the boundary B_0 of vertex v_0 as shown in Fig. 4b. The feasibility area of v_0 is the upper quadrant formed by the lines $l(\alpha, \beta)$ and $l(\gamma, \beta)$. Let $p = (-1/2, 1)$ be a point on the left boundary of A_0. Let $\mathcal{C}_0$ be the segment $s(\beta, p)$ not containing the point p. It holds that $\mathcal{C}_0 = \{(\frac{-t}{t+1}, \frac{2t}{t+1}) \mid t \in \mathcal{I} = [0, 1)\}$. The curve $\mathcal{C}_0$ is 0-exponentially-complex. An implicit equation of $l(\beta, p)$ is $y + 2x = 0$.

In the following we assume that the feasibility area A_{i-1} of v_{i-1} is partially bounded by an $i-1$-exponentially-complex curve satisfying additional invariants and prove that the feasibility area A_i of v_i is partially bounded by an i-exponentially-complex curve that also satisfies these invariants. The invariants are given in three groups, the *universal* invariants, holding after each inductive step, the *even* and the *odd* invariants holding after each even and odd step $i \geq 0$, respectively. Below are the universal and even invariants, with the odd invariants being symmetric.

Universal Invariants

$\mathcal{UI}$.1: A_i is partially bounded by an i-exponentially-complex curve $\mathcal{C}_i = \{v_i(t) = (v_i^x(t), v_i^y(t)) \mid t \in \mathcal{I}\}$, where $\mathcal{I} = [0, \mathcal{I}_{\max})$ and $\mathcal{I}_{\max} > 0$,
$\mathcal{UI}$.2: $v_i^y(t)$ is strictly increasing for $t \in [0, \mathcal{I}_{\max})$.

Even Invariants

$\mathcal{EI}$.1: $b_{i+1}^x < v_i^x(0) < a_{i+1}^x$ and $v_i^y(0) = 0$,
$\mathcal{EI}$.2: A_i is on the right of $\mathcal{C}_i$,
$\mathcal{EI}$.3: Ray $q(a_{i+1}, v_i(0))$ intersects no point of A_i to the left of $v_i(0)$.

We observe that universal and even invariants hold for the base case $i = 0$.

Let $\mathcal{C}_{i-1} = \left\{v_{i-1}(t) = \left(\frac{r(t)}{u(t)}, \frac{s(t)}{u(t)}\right) \mid t \in \mathcal{I}\right\}$. The position $w_{i-1}(t)$ of vertex w_{i-1} is described as the intersection of the rays $q(v_{i-1}(t), b_i)$ and $q(d_i, c_i)$. The position $v_i(t)$ of v_i is described as $q(a_i, v_{i-1}(t)) \cap q(e_i, w_{i-1}(t))$.

Using this we calculate the curve $\mathcal{C}_i$, i.e. we calculate the position of v_i as a function of t. This can be done by calculating the equation of the line $l(v_{i-1}(t)), b_i)$, the position of the vertex $w_{i-1}(t)$ and then the equations if the lines $l(a_i, v_{i-1}(t))$ and $l(e_i, w_{i-1}(t))$. The intersection of the latter lines is $v_i(t)$ and we obtain $\mathcal{C}_i = \{(\frac{r_i(t)}{u_i(t)}, \frac{s_i(t)}{u_i(t)}) \mid t \in \mathcal{I}\}$, where each of $r_i(t)$, $s_i(t)$, $u_i(t)$ is quadratic in $r(t)$, $s(t)$ and $u(t)$. By induction hypothesis, C_{i-1} is an $i-1$-exponentially complex curve, i.e. $u(t)$, $s(t)$, $r(t)$ contain terms $t^{2^{i-1}}$. So the curve $\mathcal{C}_i$ is i-exponentially complex, provided that the coefficients of highest degree do no cancel themselves out, which can be avoided by slightly perturbing the position of vertex e_i. This proves Invariant $\mathcal{UI}$.1. A proof that the remaining invariants hold after the induction step concludes the proof of the theorem. □

7 Conclusion

We have shown that a drawing Γ_H of an induced connected subgraph H can be extended with at most $\min\{h/2, \beta + \log_2(h) + 1\}$ bends per edge if the star complexity of Γ_H is β and h is the size of the largest face of H and that this bound is tight up to a small additive constant. In the event of a disconnected subgraph H the known upper bound is $72|V(H)|$. It is tempting to investigate whether the constant 72 can be lowered and to provide a matching lower bound.

We have proven that there is an instance (G, Γ_F) where Γ_F is a star-shaped inner face, such that the feasibility area of some vertex $v \in G$ is partially bounded by an exponential degree curve. This is an indication that for a given instance (G, Γ_F) it is difficult to test whether (G, Γ) admits a straight-line extension. It would be interesting to establish the computational complexity of this problem. We were not able to show the NP-hardness of the problem. Due to its similarity with visibility and stretchability problems we conjecture that the problem is as hard as the existential theory of reals.

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References

1. Badent, M., Giacomo, E.D., Liotta, G.: Drawing colored graphs on colored points. In: Amato, N.M., Lee, D.T., Pietracaprina, A., Tamassia, R. (eds.) Theoretical Computer Science, vol. 408, pp. 129–142 (2008). https://doi.org/10.1016/j.tcs.2008.08.004. http://www.sciencedirect.com/science/article/pii/S0304397508005562
2. Bekos, M.A., et al.: Planar drawings of fixed-mobile bigraphs. In: Frati, F., Ma, K.-L. (eds.) GD 2017. LNCS, vol. 10692, pp. 426–439. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-73915-1_33
3. Bekos, M.A., et al.: Planar drawings of fixed-mobile bigraphs (2017). http://arxiv.org/abs/1708.09238
4. Chambers, E.W., Eppstein, D., Goodrich, M.T., Löffler, M.: Drawing graphs in the plane with a prescribed outer face and polynomial area. **16**, 243–259 (2012). https://doi.org/10.7155/jgaa.00257
5. Chan, T.M., Frati, F., Gutwenger, C., Lubiw, A., Mutzel, P., Schaefer, M.: Drawing partially embedded and simultaneously planar graphs. In: Duncan, C., Symvonis, A. (eds.) J. Graph Algorithms Appl. **19**, 681–706 (2015). https://doi.org/10.1007/978-3-662-45803-7_3
6. Djidjev, H.N., Lingas, A., Sack, J.R.: An O(n log n) algorithm for computing the link center of a simple polygon. Discrete Comput. Geom. **8**(2), 131–152 (1992). https://doi.org/10.1007/BF02293040
7. Giacomo, E.D., Didimo, W., Liotta, G., Meijer, H., Wismath, S.K.: Point-set embeddings of trees with given partial drawings. Comput. Geom. **42**(6), 664–676 (2009). https://doi.org/10.1016/j.comgeo.2009.01.001. http://www.sciencedirect.com/science/article/pii/S0925772109000108
8. Goaoc, X., Kratochvíl, J., Okamoto, Y., Shin, C.S., Spillner, A., Wolff, A.: Untangling a planar graph. Discrete Comput. Geom. **42**(4), 542–569 (2009). https://doi.org/10.1007/s00454-008-9130-6
9. Hong, S.H., Nagamochi, H.: Convex drawings of graphs with non-convex boundary constraints. Discrete Appl. Math. **156**(12), 2368–2380 (2008). https://doi.org/10.1016/j.dam.2007.10.012
10. Kant, G.: Drawing planar graphs using the canonical ordering. Algorithmica **16**(1), 4–32 (1996). https://doi.org/10.1007/BF02086606

11. Lubiw, A., Miltzow, T., Mondal, D.: The complexity of drawing a graph in a polygonal region. In: Biedl, T., Kerren, A. (eds.) Graph Drawing and Network Visualization (2018, to appear)
12. Mchedlidze, T., Nöllenburg, M., Rutter, I.: Extending convex partial drawings of graphs. Algorithmica **76**(1), 47–67 (2016). https://doi.org/10.1007/s00453-015-0018-6
13. Mchedlidze, T., Urhausen, J.: β-stars or on extending a drawing of a connected subgraph (2018). https://arxiv.org/abs/1808.10366
14. Pach, J., Wenger, R.: Embedding planar graphs at fixed vertex locations. Graphs Comb. **17**(4), 717–728 (2001). https://doi.org/10.1007/PL00007258
15. Patrignani, M.: On extending a partial straight-line drawing. Int. J. Found. Comput. Sci. **17**(05), 1061–1069 (2006)
16. Tutte, W.T.: How to draw a graph. Proc. Lond. Math. Soc. **s3-13**(1), 743–767 (1963). https://doi.org/10.1112/plms/s3-13.1.743. http://plms.oxfordjournals.org/cgi/reprint/s3-13/1/743