On the area of constrained polygonal linkages

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Consider a mechanical linkages whose underlying graph is a polygon with a diagonal constraint, or more general, a partial two-tree. We show that (with an appropriate definition) the oriented area is a Bott–Morse function on the configuration space of the linkage. Its critical points are described and Bott–Morse indices are computed. This paper is a generalization of analogous results for polygonal linkages (obtained earlier by G. Khimshiashvili, G. Panina, and A. Zhukova).

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1. Introduction

A *polygonal linkage* is a linkage whose underlying graph is a polygon (or, equivalently, a single-cycle graph). One thinks of it as of a flexible polygon with rigid edges and revolving joints at the vertices whose ambient space is the Euclidean plane. The idea of considering the oriented area as a Morse function on its configuration space has already led to some non-trivial results: the critical points (or, equivalently, critical configurations) are easily describable, and there exists a short formula for the Morse index [6], [7], [8], [12]. In some further generalization [9] the oriented area proves to be an exact Morse function.

In the present paper we extend the class of underlying graphs of linkages in such a way that it is possible to introduce the oriented area with the same nice Morse-theoretical properties.

We start in Section 3 with *three-chain linkages.* It is our first example which is not a polygonal linkage. By definition, a three-chain linkage is a patch of three chains, and therefore the underlying graph has three

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cycles. One of the cycles is distinguished: we consider its area S as the function defined on the configuration space. Generically, it is a Bott–Morse function. We prove that the critical configurations are characterized by a combination of *cyclic* and *aligned* conditions. We also give a formula for Bott–Morse indices of critical points and critical components (Section 4).

As an interesting illustration, we describe a Hessian bifurcation and show that it amounts to a pitchfork bifurcation (Section 5).

Finally, we present the most general result which includes polygonal linkages and three-chains (Section 6). Namely, we show that critical points characterization and Bott–Morse index formula extend to *partial two-trees.* This class of graphs is well-studied and widely used in computer science since many algorithmic problems may be solved much more efficiently for partial k-trees, and in particular for partial two-trees, see [13]. Partial two-trees have a number of graph-theoretical characterizations; we make use of some of them. Also linkages with underlying partial two-tree graph have some specific properties, see [10]; the present paper describes yet another one.

2. Preliminaries and notation

A *linkage* is a graph together with length assignment to its edges. Its *configuration space* is the space of all planar realizations of the linkage. Its *reduced configuration space* is the quotient of the configuration space by orientation-preserving isometries of the plane, that is, by translations and rotations.

A realizability convention. We assume throughout the paper that the edge lengths of a linkage are always realizable, so that the configuration space is non-empty.

A configuration of a linkage is called *aligned* if it fits in a straight line.

Assume that $X \neq Y$ are two vertices of a configuration P of some linkage. If (1) P is a smooth point of the configuration space, and (2) the gradient of the distance function $|XY|$ does not vanish at P, we say that *for the configuration P, the vertex Y moves freely with respect to X.*

This can be reformulated in two ways: (1) the map from the configuration space to the plane which sends a configuration to the vector $\overrightarrow{XY}$ is a submersion, and (2) the vertex Y admits a non-zero infinitesimal motion in any direction in the plane (with X fixed).

Clearly, this notion is symmetric: for a given configuration, X moves freely with respect to Y iff Y moves freely with respect to X. An example follows in Lemma 1.

2.1. Open chains

An *open chain* is a linkage whose underlying graph is a path graph. In robotics it is also called *a robot arm.* We assume that its vertices are numbered, so an open chain has a natural orientation. For a configuration $(p_1, ..., p_{n+1})$ of an open chain consider the distance $D = |p_1 p_{n+1}|$ as a Morse function on the reduced configuration space $(S^1)^{n-1}$. Its critical points are described in [3]:

Theorem 1. *[4]*

(1) *Critical configurations of the distance function D are aligned configurations.*
(2) *The Morse index of a critical configuration equals $f - 1$, where f is the number of forward edges, i.e. the number of (directed) edges $\overrightarrow{p_i p_{i+1}}$ which have the same direction as the vector $\overrightarrow{p_1 p_{n+1}}$.*

Let us reformulate the first statement of the theorem in a way to be used in the paper:

Lemma 1. *For any configuration of an open chain, the terminal point $T = p_{n+1}$ moves freely with respect to the initial point $I = p_1$ unless the configuration is aligned.* $\square$

2.2. Oriented area for polygonal linkages

A *polygonal linkage* is a linkage with a single-cycle underlying graph. We assume that all the vertices are numbered, so the cycle carries a natural orientation.

Throughout the paper we assume that no configuration fits a straight line. This is equivalent to smoothness of reduced configuration space of the polygonal linkage, see [3]. Let us be more precise. The parameter space $\mathbb{R}^n = (l_1, ..., l_n)$ is divided into *chambers* by hyperplanes (called "walls") of type $\sum_1^n \pm l_i = 0$ (for all possible combinations of $\pm$'s). If $(l_1, ..., l_n)$ lies on no wall, the reduced configuration space is smooth. For two $(l_1, ..., l_n)$ belonging to one and the same chamber the reduced configuration spaces are diffeomorphic.

Definition 1. The *oriented area* of a polygon with the vertices $p_i = (x_i, y_i)$ is defined by

$$2S(P) = (x_1 y_2 - x_2 y_1) + \ldots + (x_n y_1 - x_1 y_n).$$

Definition 2. A polygon P is *cyclic* if all its vertices p_i lie on a circle.

Cyclic polygons arise in the framework of the paper as critical points of the oriented area. According to [4] it was Banchoff, who considered Morse theory of the area function in relation with isoperimetric problems (unpublished).

Theorem 2. *[6] Generically, S is a Morse function. At smooth points of the reduced configuration space, a polygon P is a critical point of the oriented area S iff P is a cyclic configuration.* $\square$

It is worthy mentioning the paper [5] which served as in impetus and paradigm for the treating oriented area as a Morse function.

We recall a short formula for Morse index of a cyclic configuration from [12], [7]. We fix the following notation for a cyclic configuration.

ω_P is the winding number of P with respect to the center of the circumscribed circle O.

α_i is the half of the angle between the vectors $\overrightarrow{Op_i}$ and $\overrightarrow{Op_{i+1}}$. The angle is defined to be positive, orientation is not involved.

ε_i is the orientation of the edge $p_i p_{i+1}$, that is,

$$\varepsilon_i = \begin{cases} 1, & \text{if the center } O \text{ lies to the left of } p_i p_{i+1}; \\ -1, & \text{if the center } O \text{ lies to the right of } p_i p_{i+1}. \end{cases}$$

$e(P)$ is the number of positive entries in $\varepsilon_1, ..., \varepsilon_n$.

$\mu_P = \mu_P(S)$ is the Morse index of the function S in the point P. That is, $\mu_P(S)$ is the number of negative eigenvalues of the Hessian matrix $Hess_P(S)$.

Theorem 3. *[7], [8] Generically, for a cyclic configuration P of a polygonal linkage,*

$$\mu_P(S) = \begin{cases} e(P) - 1 - 2\omega_P & \text{if } \sum_{i=1}^n \varepsilon_i \tan \alpha_i > 0; \\ e(P) - 2 - 2\omega_P & \text{otherwise.} \end{cases} \quad \square$$

3. Oriented area and its critical points for three-chains

Take three oriented open chains A, B, and Z, and glue together their initial points I, and also glue together their terminal points T. The obtained linkage is called a *three-chain*. If A, B, and Z have p, q, and r edges respectively, we say that the three-chain linkage has the type $[p, q; r]$.

The underlying graph has three simple cycles: AB, AZ, and BZ.

Let us first look at the reduced configuration space of a three-chain.

Theorem 4. *If none of the three cycles AB, AZ, and BZ of a three-chain linkage has an aligned configuration, the reduced configuration space of the three-chain is a smooth manifold.*

Proof. We mimic the proof of the analogous statement for polygonal linkages, see [3]. Let us remove the edge number r from the chain Z. We obtain "a polygon AB with a tail" whose reduced configuration space M is smooth. Next we define the function D on M as the (squared) distance between the end of the tail and the point T. The reduced configuration space of the three-chain is the level set of the function D: it is diffeomorphic to $D^{-1}(l)$, where l is the length of the removed edge. It is a smooth manifold unless l is a critical value of the function D. Lemma 1 implies that this happens only if one of the cycles gets aligned. □

In this way we get a set of linear equations of type:

$$\epsilon_1 l_1 + \cdots + \epsilon_n l_n = 0, \text{where } \epsilon \in \{-1, 0, 1\}$$

The corresponding arrangement of hyperplanes ("*walls*") divides the parameter space into convex *chambers*. In each chamber the reduced configuration spaces are smooth and mutually diffeomorphic.

In the sequel we will concentrate on three-chains with smooth reduced configuration spaces only, but in case of singularities several statements are also valid in the smooth part.

For a three-chain we distinguish the oriented cycle $\Gamma = AB$ assuming that A goes in the positive direction (and therefore, B goes in the negative direction). In figures we indicate the orientation by an arrow, see Fig. 3.

Definition 3. The *oriented area of a three-chain linkage* is the function S defined on the reduced configuration space as the oriented area of the cycle Γ.

A remark on genericity. Throughout the paper we assume that the edge lengths are *generic*. We do not specify the meaning in advance, instead we keep a right to exclude from "generic" any finite number of manifolds of codimension at least one. So the generic case remains an open and everywhere dense subset of the parameter space.

Let us first study the simplest case: three-chains of type $[2, 2; 2]$, or *generalized Peaucellier–Lipkin linkage*. In this case the reduced configuration space is one dimensional, and therefore is a union of circles.

Proposition 1. *For a $[2, 2; 2]$ three-chain, critical points of S fall into two types (see Fig. 1):*

(1) *Circular type: Γ is realized as a cyclic polygon, that is, all the vertices of Γ lie on a circle.*
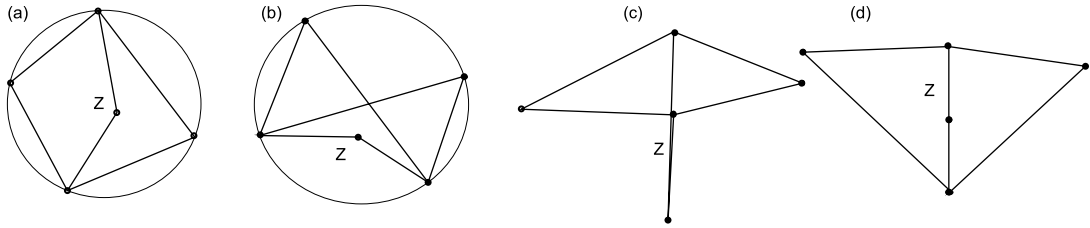(2) *Aligned type: Z is aligned.*

Fig. 1. Critical configurations: (a) and (b) are cyclic, (c) and (d) are aligned. Each of them comes in a four-tuple.
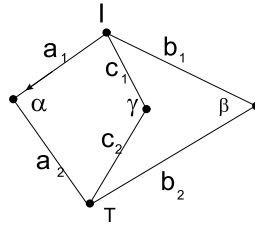

Fig. 2. Notation for Proposition 1.

Proof. In the notation of Fig. 2 we have

$$2S = a_1 a_2 \cdot \sin\alpha + b_1 b_2 \cdot \sin\beta.$$

We use the Lagrange multipliers method with respect to the equations

$$a_1^2 + a_2^2 - 2a_1 a_2 \cdot \cos\alpha = b_1^2 + b_2^2 - 2b_1 b_2 \cdot \cos\beta$$

and

$$a_1^2 + a_2^2 - 2a_1 a_2 \cdot \cos\alpha = c_1^2 + c_2^2 - 2c_1 c_2 \cdot \cos\gamma.$$

The Lagrange multipliers matrix is

$$M = \begin{pmatrix} a_1 a_2 \cdot \cos\alpha & b_1 b_2 \cdot \cos\beta & 0 \\ 2a_1 a_2 \cdot \sin\alpha & -2b_1 b_2 \cdot \sin\beta & 0 \\ 2a_1 a_2 \cdot \sin\alpha & 0 & -2c_1 c_2 \cdot \sin\gamma \end{pmatrix}.$$

At critical points its determinant vanishes. The direct computation gives

$$\det M = 4c_1 c_2 a_1 a_2 b_1 b_2 \cdot \sin\gamma \cdot \sin(\alpha + \beta),$$

which vanishes either $\sin\gamma = 0$ (aligned type) or $\sin(\alpha + \beta) = 0$ (cyclic type). $\quad\square$

Theorem 5. *For a three-chain linkage all critical points of the function S fall into two (generically disjoint) types, see Fig. 3:*

(1) *Circular type: Γ is realized as a cyclic polygon, that is, all the vertices of Γ lie on a circle. For each cyclic configuration of Γ we have a critical manifold which equals the reduced configuration space of a flexible polygon obtained from Z by attaching the corresponding diagonal of the cyclic polygon.*
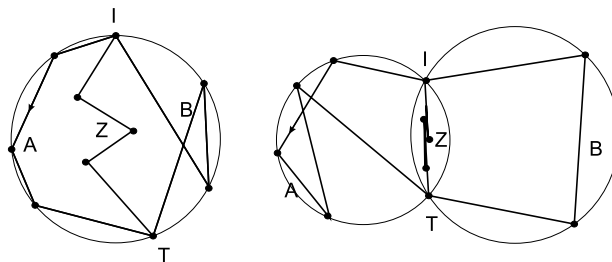
Fig. 3. Critical configurations of a three-chain.

(2) *Aligned type: Z is aligned, and each of A and B are cyclic. The two circumscribed circles generically do not coincide.*

The proof comes from using Proposition 1 sufficiently many times. The ruling idea is to freeze some of the joints of the linkage and to get a small linkage treated in Proposition 1. As an example, if A and B contain two bars each, and Z contains three bars, we freeze one of the joints of Z and use Proposition 1 for the first time, then freeze the other joint of Z and use Proposition 1 for the second time. □

A genericity remark. According to our genericity convention, we may assume that cyclic polygon AB and simultaneously aligned Z never occur. We may also assume that each of the cyclic polygons that appear in the above theorem is a non-degenerate Morse point (considered as a single polygon).

Example 1. Three-chain $[2, 2; 2]$.

The reduced configuration space in the smooth case consists of either two or four circles, depending on intervals between minimum and maximal distances between the endpoints of the arms A, B, and Z. The global picture is very sensitive for the mutual position of these intervals and gives rise to many different cases. There exist $[2, 2; 2]$ three-chains with 16 critical points (which is the maximum possible number of critical points). On two components of its reduced configuration space we have only one maximum and one minimum of aligned type. The other components contain four cyclic and two aligned critical points.

Example 2. Three-chains $[2, 2; 2]$ are $[2, 2; 3]$. These are the first cases after $[2, 2; 2]$. Their reduced configuration spaces are one and the same two dimensional manifold. One can meet here the first Bott–Morse singularities:

The three-chain $[2, 2; 3]$ has isolated aligned singularities and Bott–Morse if the 4-cycle is cyclic.

The three-chain $[2, 3; 2]$ has only isolated singularities; both cyclic (5-gon) and aligned.

4. Bott–Morse indices for the three-chain linkage

We start with a reminder about non-isolated critical points, see [1]. A function $f : M^n \to \mathbb{R}$ is called *Bott–Morse* if for every component C of its critical set f is locally of the form $f = f_o - (x_1^2 + ... + x_\lambda^2) + (x_{\lambda+1}^2 + ... + x_k^2)$. $\lambda = \lambda(C)$ is called the Bott–Morse index of the component.

As a consequence, a critical component is a smooth manifold of codimension $n - k$.

The standard formula from Bott–Morse theory relates the critical point theory to the topology of the configuration space:

$$\sum_i (-1)^\lambda \chi(C_i) = \chi(M^n).$$

Here the sum runs over all critical manifolds C_i (including critical points).

Consider first the critical point of cyclic type. For a fixed cyclic configuration of Γ critical points form a manifold which is assumed to be already known: it is the reduced configuration space of a polygon formed by Z with an extra bar $W = IT$.

Theorem 6. *The Bott–Morse index of the critical manifold of the cyclic type equals the Morse index of the cyclic polygon, see Theorem 3.*

Proof. The area function can be described as $S = S(\xi, \eta)$, where ξ are coordinates for Γ, and η are coordinates for Z. Clearly, S does not depend on η. □

Now assume that we have a critical configuration of aligned type. The cycle Γ splits in the homological sum of two cycles AW and BW obtained by cutting Γ along the diagonal $W = IT$. These two polygons are cyclic, and therefore can be treated as critical points of the area function on the reduced configuration space of polygonal linkages defined by their edge lengths.

Before we formulate a theorem let us fix some notation. We introduce the vector $\overrightarrow{W} = \overrightarrow{IT}$. Denote by O_A, and O_B the centers of circumscribed circles. Generically the aligned critical configurations is Morse (as soon as $O_A \neq O_B$ and each of the two cyclic polygons represent non-degenerate Morse points) and we have:

Theorem 7. *The Morse index of an aligned critical configuration is*

$$\mu = \mu_A + \mu_B + \nu,$$

where μ_A and μ_B are the Morse indices of the polygons AW and BW obtained by adding the diagonal W to A and B respectively. The formula for μ_A and μ_B is contained in Theorem 3.

$$\nu = \begin{cases} f - 1, & \text{if the pair of vectors } (\overrightarrow{W}, \overrightarrow{O_A O_B}) \text{ is positively oriented;} \\ r - f, & \text{otherwise,} \end{cases}$$

where r the number of edges in the chain Z, f is the number of forward edges in Z.

Let us first prove the following lemma.

Lemma 2. *Assume that in a triangle we fix side lengths a and b and let c vary. Let $S = S(c)$ be the area of the triangle.[1]*
    Then

$$\frac{\partial S(c)}{\partial c} = \begin{cases} |OM|, & \text{if } \gamma < \frac{\pi}{2}; \\ -|OM|, & \text{otherwise.} \end{cases}$$

Here γ is the angle opposite to c, M is the center of the edge c, and O is the center of the circumscribed circle (with radius R).

Proof of the lemma. First note, that in terms of γ the area is strictly increasing between 0 and $\frac{\pi}{2}$, maximal at $\frac{\pi}{2}$ and decreasing between $\frac{\pi}{2}$ and π. According to the cosine rule:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma \quad \text{so} \quad 2c = 2ab \cdot \sin\gamma \cdot \frac{d\gamma}{dc}.$$

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[1] We mean here the "usual" area, which is positive.

Moreover $2S = ab \cdot \sin \gamma$; therefore

$$\frac{\partial(2S)}{\partial c} = ab \cdot \cos \gamma \cdot \frac{d\gamma}{dc} = \frac{c \cos \gamma}{\sin \gamma} = \frac{2R \cdot |OM|}{R} = \pm 2|OM|. \quad \square$$

Now we are ready to prove the theorem. In the neighborhood of an aligned critical point we choose the following coordinate system. It consists of three families of coordinates. Namely, we first choose coordinates $\alpha = (\alpha_1, ..., \alpha_{p-2})$ for the polygon AW and $\beta = (\beta_1, ..., \beta_{q-2})$ for the polygon BW assuming that $w = |IT|$ is constant. We specify these coordinates in more detail, when necessary (they could be angles or lengths of diagonals, etc). Next we choose coordinates $\gamma = (\gamma_1, ..., \gamma_{r-1})$ for the chain Z. Note that α, β, w are coordinates for the polygon AB. Consider a part of the Taylor expansion of $S = S(\alpha, \beta, w)$:

$$S = S_0 + \left(\frac{\partial S}{\partial w}\right)(w - w_0) + \frac{1}{2}\sum\left(\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j}\right)\alpha_i \alpha_j + \frac{1}{2}\sum\left(\frac{\partial^2 S}{\partial \beta_i \partial \beta_j}\right)\beta_i \beta_j$$

Note that $\frac{\partial^2 S}{\partial \alpha_i \partial \beta_j} = 0$. By Morse lemma and Theorem 7, at the critical point we can suppose that: $w - w_0 = -(\gamma_1^2 + \cdots \gamma_N^2) + (\gamma_{N+1}^2 + \cdots \gamma_{r-1}^2)$. After substituting in the formula this gives the 2-jet of S as function of α, β, γ. So the Hessian matrix has a block structure:

$$Hess(S) = \begin{pmatrix} H_A & 0 & 0 \\ 0 & H_B & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix}$$

The first block H_A (respectively, H_B) equals the Hessian matrix of the area of the polygon AW (respectively, BW). The third block is a diagonal matrix $\mathcal{A}$, which up to the constant $(\frac{\partial S}{\partial w})$ equals the Hessian matrix of the distance function D of the open chain Z, see Theorem 1.

To determine the sign of $(\frac{\partial S}{\partial w})$, we specialize the choice of coordinates α and β as follows: triangulate the polygon AW (respectively, BW), and take the lengths of the diagonals $d_1^A, ..., d_k^A$ (respectively, $d_1^B, ..., d_m^B$) as coordinates.

Let $S_1^A, ..., S_{k-1}^A$ and $S_1^B, ..., S_{m-1}^B$ be the (oriented) areas of the triangles of the triangulations. We may assume that S_1^A and S_1^B correspond to triangles that are incident to the edge W. Then $S = S_1^A + ... + S_{k-1}^A + S_1^B + ... + S_{m-1}^B$, however, only two of the summands depend on w. Eventually, Lemma 2 implies:

$$\left(\frac{\partial S}{\partial w}\right) = \left(\frac{\partial S_1^A}{\partial w}\right) + \left(\frac{\partial S_1^B}{\partial w}\right) = \begin{cases} |O_A O_B|, & \text{if the pair of vectors } (\overrightarrow{W}, \overrightarrow{O_A O_B}) \\ & \quad \text{is positively oriented;} \\ -|O_A O_B|, & \text{otherwise.} \end{cases} \quad (*)$$

For generic parameter values, S will be Morse and the Morse index (which equals the number of negative eigenvalues) sums up. $\square$

5. Hessian bifurcation

We discuss a bifurcation occurring in generic one-parameter families at an aligned singularity.

Hessian for $[2, 2; 2]$. We discover a pitchfork bifurcation when the Hessian vanishes. We assume an aligned critical point and use the notations from Fig. 2. Let $w = |IT|$ be the length of the diagonal. In the aligned situation we can assume that $w = w_0 \pm \epsilon^2$ (where ϵ is a coordinate for the chain Z and the sign $\pm$ depends on the type of alignment.
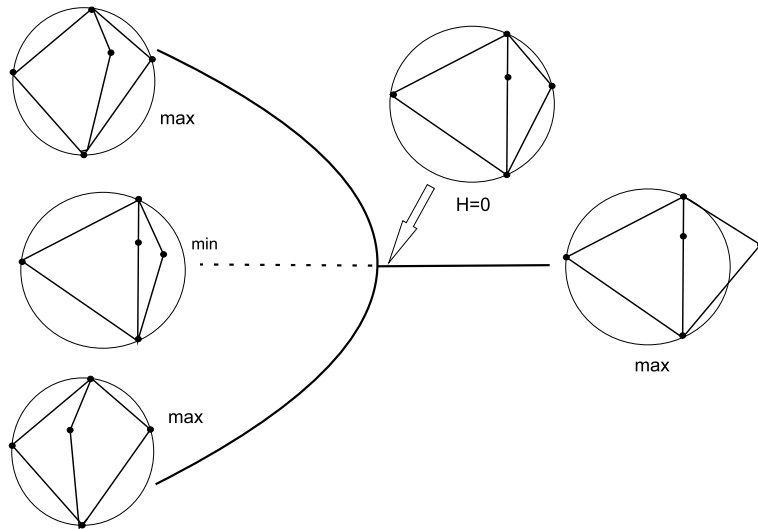
Fig. 4. Pitchfork bifurcation in the [2, 2; 2] case.

Next consider the 2-jet of S at w_0 in terms of w:

$$S = S_0 + \left(\frac{dS}{dw}\right)(w - w_0) + \frac{1}{2}\left(\frac{d^2 S}{dw^2}\right)(w - w_0)^2$$

So the 4-jet of S at w_0 in terms of ϵ is:

$$S = S_0 + \pm\left(\frac{dS}{dw}\right)\epsilon^2 + \frac{1}{2}\left(\frac{d^2 S}{dw^2}\right)\epsilon^4$$

The Hessian is already computed in Lemma 2 for the two triangles. The conditions $(*)$ in the proof of Theorem 7 implies that the Hessian is equal to zero if all the four points of the cycle are on one circle. So we have now a combination of the cyclic and the aligned condition. The formula for S starts in this case with degree four terms. This means that locally we have a singularity of type A_3. By changing the side length slightly it is possible to unfold the singularity such that the sign of the Hessian changes exactly at the initial value of the length vector. The unfolding is of type $S = S_0 + \epsilon^4 \pm \lambda\epsilon^2$ and this describes a pitchfork bifurcation: one aligned maximum splits into one aligned minimum and two maxima of cyclic type. By symmetry, we could also have a similar bifurcation where maximum and minimum are interchanged (see Fig. 4).

Note that this bifurcation takes place in the interior of a chamber: during the deformation one still has diffeomorphic reduced configuration spaces.

This pitchfork bifurcation also shows that in several cases our Morse function is not exact.

Hessian bifurcation for $[p, q; r]$. Assume that $\left(\frac{\partial S}{\partial w}\right) = 0$. Then the Hessian becomes zero. We take the next term to the Taylor expansion of S and get

$$S = S_0 + \left(\frac{\partial S}{\partial w}\right)(w - w_0) + \frac{1}{2}\left(\frac{\partial^2 S}{\partial w^2}\right)(w - w_0)^2 + Q(\alpha, \beta),$$

where $Q(\alpha, \beta)$ is supposed to be a non-degenerate quadratic form. Remind that $w - w_0 = -(\gamma_1^2 + \cdots \gamma_s^2) + (\gamma_{s+1}^2 + \cdots \gamma_{r-1}^2)$, so we work essentially with degree 4 in γ-coordinates.

Note that passing through Hessian zero changes the sign in the matrix $\mathcal{A}$ and therefore the change in index can be greater than 1.
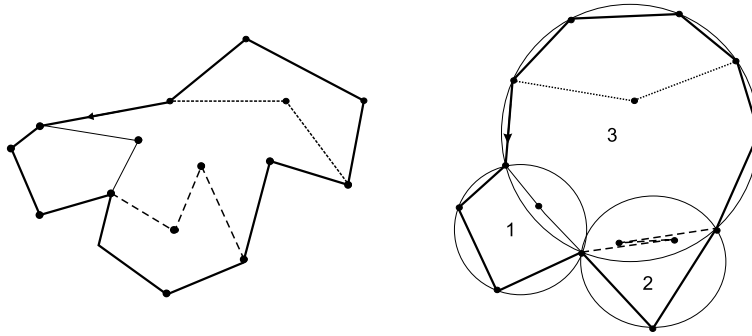
Fig. 5. A polygon with non-crossing diagonals and one of its critical configurations. Its orientation is indicated by an arrow.

Let us consider the $[2, 2; 3]$ case: if $w - w_0 = -\gamma_1^2 + \gamma_2^2$ then there occur non-isolated singularities; the singular set $-\gamma_1^2 + \gamma_2^2 = 0$ is even not smooth. Especially interesting is the case where $w - w_0 = -(\gamma_1^2 + \gamma_2^2)$. If the Hessian is zero there is an aligned isolated maximum and this bifurcates from an isolated maximum into an isolated minimum together with a maximum of circle type. This is a generalized pitchfork bifurcation.

6. Area function for partial two-trees

Let us start this section with an introductory example of a constrained linkage which lies in between three-chains and partial two-trees.

Definition 4. A *polygon with non-crossing diagonals* is a graph G with a distinguished cycle Γ such that

(1) G can be embedded in the plane in such a way that Γ is the outer cycle, and
(2) G is obtained by attaching to Γ a number of path graphs. Each path graph is attached by its initial and terminal points I and T.

In other words, G is polygon with a number of non-crossing diagonals added. Each diagonal is a path graph.

The above definition is combinatorial: we have defined a class of graphs, no edge lengths are involved so far. So the words "non-crossing diagonals" have the combinatorial meaning only. For a configuration of an associated linkage diagonals might cross.

Next, we consider linkages based on this type of graphs. It is easy to see that the approach of Sections 3 and 4 can be repeated literally (we give the statements in the full generality below in the section). Fig. 5 depicts a polygon with non-crossing diagonals and one of the critical configurations.

A natural question is: how far can we extend the class of graphs keeping a similar cyclic-aligned characterization of critical points and easily computable Bott–Morse indices. To our opinion, a proper generalization are linkages whose underlying graphs are partial two-trees (PTTs). It has been observed during AIM meeting "Configuration spaces of linkages" (October 2014) that for a PTT linkage, the parameter space (that is, the space of the edge lengths of the linkage) is divided by some hyperplanes (called "walls") into chambers. In each of the chamers the configuration space is smooth. The diffeomorphic type of the reduced configuration space depends on the chamber only. Two proofs of this claim were discussed during the meeting, both are not published yet. One is by Panina, Sitharam, Wang, and Willoughby, and the other by Streinu and Borcea. However, we do not use it here: Theorem 8 applies to all smooth points of the configuration space.

A *two-tree* is a graph obtained by starting with K_3 (complete graphs on three vertices) and then repeatedly taking the connected sum via edges with a number of copies of K_3.

A *partial two-tree* (PTT, for short) is a graph that can be characterized by the following equivalent conditions:

(1) A partial two-tree becomes a two-tree after adding some extra edges.
(2) A partial two-tree has no K_4 minor, see [11].
(3) PTTs are *two-terminal series-parallel* graphs, see [2].

Let us explain this in details. A *two-terminal graph* (TTG) is a graph with two distinguished vertices, I and T. The *parallel composition* of two TTGs X and Y is a graph obtained by patching I_X to I_Y, and also patching T_X to T_Y. The *series composition* of two TTGs X and Y is a graph obtained by patching I_Y to T_X. Finally, a *two-terminal series-parallel graph* is a two-terminal graph that may be constructed by a sequence of series and parallel compositions starting from a set of copies of a single-edge graph K_2.

Given a PTT one can turn it to a two-terminal series-parallel graph by an appropriate choice of I and T. Not any two distinct vertices can serve as I and T, so we shall need the following sufficiency condition[2]:

Lemma 3. *If two vertices of a PTT are connected by an edge, they can serve as I and T.*

The proof goes by induction. We can assume that our PTT is a two-tree. For two vertices Q and R sharing an edge, two cases are possible: (1) QR is incident to more that one triangle of the two-tree. Then G is a connected sum of several smaller PTT over the edge QR, so we use the inductive assumption. (2) QR is incident to exactly one triangle. Then removal of the edge QR leaves a connected sum of two smaller PTT over a single vertex incident to both R and Q, and again inductive assumption is applicable. □

Assume now that we have a linkage whose underlying graph is a partial two-tree. An analog of Lemma 1 is valid:

Lemma 4. *For a configuration of a PTT linkage, the vertex T moves freely with respect to the vertex I unless there exists an aligned path connecting T and I.*

The proof comes by induction based on the above characterization (3) combined with Lemma 1. □

Given a PTT linkage with underlying graph G, choose a cycle Γ with no repeating vertices and take its area S as the function on the reduced configuration space of the linkage.
We need the following combinatorial lemma.

Lemma 5. *In the above notation G equals the connected sum of Γ and a number of disjoint graphs G_i.*

(1) *Each of G_i is a PTT, and therefore, a two-terminal series-parallel graph.*
(2) *The connected sum with each of G_i is taken at most two vertices, and*
(3) *these two vertices I_i and T_i can serve as I and T for G_i.*

Proof. (1) is clear. (2) is true since G has no K_4 minors. (3) follows from Lemma 3. □

We give the characterization of critical points of S and the formula for the Bott–Morse indices in an algorithmic way:

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[2] Probably this lemma has already appeared in the literature.

Theorem 8. *Critical points and Bott–Morse indices for partial two-trees.* *Given a configuration P in a smooth part of the configuration space of a PTT linkage,*

(1) *take all aligned paths with endpoints on Γ and replace them by single segments which we call straight line diagonals. Since there is no K_4 minor, they never cross (in the combinatorial sense).*
(2) *Take the new configuration composed of Γ and straight line diagonals (the underlying graph changes!).*
(3) *The straight line diagonals decompose Γ into the homological sum of elementary cycles*

$$\Gamma = \Gamma_1 + ... + \Gamma_k.$$

Then we have:

(1) *P is a critical point of the area S if and only if each of Γ_k has a circumscribed circle.*
(2) *All critical manifolds are products of reduced configuration spaces of partial two-trees with a smaller number of edges.*
(3) *Generically the critical configuration is Bott–Morse and the Bott–Morse index of a critical point (or critical manifold) is:*

$$\mu = \sum_i \mu_i + \sum_j \nu_j,$$

where μ_i is the Morse index of Γ_i; ν_j refer to the j-th aligned path.

Proof. (1) By Lemma 5, the vertices of Γ move freely with respect to each other unless they are connected by an edge of Γ or by an aligned path. Since on this step it suffices to consider only first-order terms, we may think that each aligned diagonal of Γ fixes the distance between its endpoints. The area of Γ is critical if and only if all the Γ_i are critical, and therefore all of them are cyclic. (2) is clear. The proof of (3) follows the pattern of the proof of Theorem 7: in a neighborhood of an aligned critical point we choose the following coordinate system: we first choose coordinates $\alpha^i = (\alpha_1^i, ..., \alpha_{p_i-2}^i)$ for each of the polygons Γ_i. Next we choose coordinates $\gamma^j = (\gamma_1^j, ..., \gamma_{r-1}^j)$ for each of the aligned chains Z_j. In these coordinates the Hessian matrix becomes a block matrix, with blocks corresponding to each of α^i and γ^j. These are blocks that correspond to Γ_i and to the aligned diagonals, so the Morse indices sum up. $\square$

Example. For the Morse index of the critical configuration depicted in Fig. 5, we have:

$$\mu = \mu_1 + \mu_2 + \mu_3 + \nu_{1,3} + \nu_{2,3} =$$
$$1 + 0 + 5 + (2 - 1) + (3 - 2) = 8.$$

Acknowledgements

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