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Stability results for stochastic delayed recurrent neural networks with discrete and distributed delays ☆

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Abstract

We present new conditions for asymptotic stability and exponential stability of a class of stochastic recurrent neural networks with discrete and distributed time varying delays. Our approach is based on the method using fixed point theory, which do not resort to any Liapunov function or Liapunov functional. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In particular, a class of neural networks without stochastic perturbations is also considered. Examples are given to illustrate our main results.

Keywords: Fixed point theory; Asymptotic stability; Exponential stability; Stochastic recurrent neural networks; Variable delays; Doob's inequality

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1. Introduction and main results

Neural networks have received an increasing interest in various areas [3,5]. The stability of neural networks is critical for signal processing, especially in image processing and solving some classes of optimization problems [4,13,32,33]. For the stochastic effects to the dynamical behaviors of neural networks, Liao and Mao [11,12] initiated the study of stability and instability of stochastic neural networks.

Due to the finite switching speed of neurons and amplifiers, time delays which may lead to instability and bad performance in neural processing and signal transmission are commonly encountered in both biological and artificial neural networks. In addition, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [27]. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [34]. In these circumstances the signal propagation is not instantaneous and may not be suitably modeled with discrete delays. Therefore, a more appropriate way which incorporates continuously distributed delays in neural network models has been used. Further, due to random fluctuations and probabilistic causes in the network, noises do exist in a neural network. Thus, it is necessary and rewarding to study stability properties of stochastic delayed neural networks.

Liapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of differential equations. The success of Liapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. However, it may be difficult to look for a good Liapunov functional for some classes of stochastic delay differential equations. Therefore, an alternative may be explored to overcome such difficulties.

It was proposed by Burton [2] and his co-workers to use a fixed point method to study the stability problem for deterministic systems. Luo [16] and Appleby [1] have applied this method to deal with the stability problems for stochastic delay differential equations, and afterwards, a great number of classes of stochastic delay differential equations are investigated by using fixed point methods, see, for example, [3,17,18,21,22]. It turns out that the fixed point method is a powerful technique in dealing with stability problems for deterministic and stochastic differential equations with delays. Moreover, it has an advantage that it can yield the existence, uniqueness and stability criteria of the considered system in one step.

In this paper, we consider a general class of stochastic neural networks with discrete and distributed varying delays which is described by

$$dx_i(t) = \left[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^{n} l_{ij} \int_{t-r(t)}^{t} h_j(x_j(s))\, ds\right] dt + \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, dw_j(t), \tag{1}$$

or

$$dx(t) = \left[-Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + W \int_{t-r(t)}^{t} h(x(s))\, ds\right] dt + \sigma(t, x(t), x(t-\tau(t)))\, dw(t),$$

for $i = 1, 2, 3, \cdots, n$, where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $C = \mathrm{diag}(c_1, c_2, \cdots, c_n) > 0$ where $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $W = (l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \cdots, f_n(x(t)))^T \in \mathbb{R}^n$, $g(x(t)) = (g_1(x(t)), g_2(x(t)), \cdots, g_n(x(t)))^T \in \mathbb{R}^n$, $h(x(t)) = (h_1(x(t)), h_2(x(t)), \cdots, h_n(x(t)))^T \in \mathbb{R}^n$, where $\tau(t)$ and $r(t)$ denote discrete time varying delay and the bound of a distributed time varying delay, respectively, $\tau(t)$ and $r(t)$ are nonnegative continuous functions. Moreover, $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t =$ completion of $\sigma\{\omega(s) : 0 \leq s \leq t\}$) and $\sigma = (\sigma_{ij})_{n\times n} : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the diffusion coefficient matrix. Denote $\vartheta = \inf_{t\geq 0}\{t - \tau(t), t - r(t)\}$.

The initial condition for the system (1) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{2}$$

where $t \mapsto \phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$ with the norm defined by

$$\|\phi\|^p = \sup_{\vartheta\leq s\leq 0} \left\{ \mathbb{E} \sum_{i=1}^{n} |\phi_i(s)|^p \right\},$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and $p \geq 2$.

Many interesting articles [6–8,14,24,28] have considered the special cases of (1). Hu et al. [6] and Wan and Sun [28] studied a class of stochastic neural networks with the delays constant and discrete. The activation functions appearing in [6] are required to be bounded. Liao and Mao [14] investigated exponential stability of a class of stochastic delay interval systems via Razumikihin-type theorems, several exponential stability results were provided. However, the results are not only difficult to verify but also restrict to the case of the interval matrices $\tilde{A} = \tilde{B} = \tilde{C} = 0$. Sun and Cao [24] investigated the pth moment exponential stability of a class of stochastic differential equations with discrete bounded delays by using the method of variation parameter, inequality technique and stochastic analysis. This method was firstly used in [28], which does not require the boundedness, monotonicity and differentiability of the activation functions. However, the stability criteria in [24] requires that the delay functions are bounded, differentiable and their derivatives are simultaneously required to be not greater than 1. This may impose a very strict constraint on model because time delays sometimes vary dramatically with time in real circuits (see [31]). Huang et al. [7,8] investigated the exponential stability of stochastic differential equations with discrete time-varying delays with the help of a Liapunov function and Dini derivative. However, the use of their criteria depends very much on the choice of positive numbers k_{ij} etc. and a positive diagonal matrix M (see Theorem 3.3 in [7] and Theorem 3.3 in [8]).

The aim of this paper is to study the exponential stability and asymptotic stability of the stochastic delayed neural networks (1) with initial condition (2) by using a fixed point method. To obtain our main results, we suppose the following conditions are satisfied:

(A1) The delays $\tau(t)$, $r(t)$ are continuous functions such that $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$.
(A2) The mappings $f_j(\cdot)$, $g_j(\cdot)$ and $h_j(\cdot)$ satisfy $f(0)\equiv 0$, $g(0)\equiv 0$, $h(0)\equiv 0$, $\sigma(t,0,0)\equiv 0$ and are globally Lipschitz functions with Lipschitz constants α_j, β_j and γ_j, respectively, where $j=1,2,3,\cdots n$.
(A3) For each $i,j=1,2,\cdots,n$, there exist nonnegative constants μ_j and ν_j such that

$$(\sigma_{ij}(t,x,y)-\sigma_{ij}(t,u,v))^2\le \mu_j(x_j-u_j)^2+\nu_j(y_j-v_j)^2.$$

Denote by $x(t,0,\phi)$ a solution of the system (1) with the initial condition (2). The local Lipschitz condition and the linear growth condition on the functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and $\sigma(t,\cdot,\cdot)$ guarantee the existence and uniqueness of a global solution for the system (1), we refer to [19] for detailed information. Clearly, system (1) admits a trivial solution $x(t,0)\equiv 0$.

Definition 1.1. The trivial solution of system (1) is said to be stable in pth $(p\ge 2)$ moment if for arbitrary given $\epsilon>0$, there exists a $\delta>0$ such that $\|\phi\|^p<\delta$ yields that

$$\mathbb{E}\|x(t,0,\phi)\|^p<\epsilon,\quad t\ge 0,$$

where $\phi\in C\left([\vartheta,0],L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$. In particular, when $p=2$, the trivial solution is said to be mean square stable.

Definition 1.2. The trivial solution of system (1) is said to be asymptotically stable in pth $(p\ge 2)$ moment if it is stable in pth moment and there exists a scalar $\delta>0$, such that $\|\phi\|^p<\delta$ implies

$$\lim_{t\to\infty}\mathbb{E}\|x(t,0,\phi)\|^p=0,$$

where $\phi\in C\left([\vartheta,0],L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$.

Definition 1.3. The trivial solution of system (1) is said to be pth $(p\ge 2)$ moment exponentially stable if there exists a pair of constants $\lambda, C>0$ such that

$$\mathbb{E}\|x(t,0,\phi)\|^p\le C\mathbb{E}\|\phi\|^p e^{-\lambda t},\quad t\ge 0,$$

holds for $\phi\in C\left([\vartheta,0],L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$. Especially, when $p=2$, we speak of exponentially stable in mean square.

Different choices of norms can be considered on spaces of stochastic processes. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction mapping with respect to the norm. For the system (1) with initial condition (2), we consider the following two different complete spaces which are defined by using two types of norms.

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega):[\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi\in C\left([\vartheta,\infty),L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\to 0$ as $t\to\infty$. If we define a norm by

$$\|\varphi\|^p := \sup_{t\geq\vartheta}\left(\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\right), \tag{3}$$

then $\mathcal{S}_\phi$ is a complete space. Using a contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain our first result which is proved in Section 2.

Theorem 1.4. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r>0$);*
(ii) *and such that*

$$\begin{aligned}&5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\&+5^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+5^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)<1,\end{aligned} \tag{4}$$

where $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$, then the solution of (1)–(2) exists uniquely and is asymptotically stable in pth moment.

Consider the case when both the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau>0$). Let $\phi\in L^p_{\mathcal{F}_0}(\Omega,C([\vartheta,0],\mathbb{R}^n))$, define $\mathcal{C}_\phi$ to be the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega):[-\tau,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi\in L^p(\Omega,C([\vartheta,\infty),\mathbb{R}^n))$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$, $\varphi(t,\cdot)=\phi(\vartheta)$ for $t\in[-\tau,\vartheta]$ (in case $-\tau<\vartheta$), and $\sum_{i=1}^n\mathbb{E}\sup_{t-\tau\leq s\leq t}|\varphi_i(s)|^p\to 0$ for $t\to\infty$. If we define a norm by

$$\|\varphi\|^p=\sup_{t\geq 0}\left[\sum_{i=1}^n\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}|\varphi_i(s)|^p\right)\right], \tag{5}$$

then $\mathcal{C}_\phi$ is a complete space. Using a contraction mapping defined on the space $\mathcal{C}_\phi$ and applying a contraction mapping principle, we obtain our second result which is proved in Section 3.

Theorem 1.5. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau>0$);*

(ii) *and such that*

$$
\begin{aligned}
&5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&+5^{p-1}\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\
&+5^{p-1}n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)<1, \qquad (6)
\end{aligned}
$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$*;*

then the solution of (1)–(2) exists uniquely and is asymptotically stable in pth moment. More than that, for every $\epsilon>0$*, there exists a* $\delta>0$ *such that* $\|\phi\|<\delta$ *implies* $\sum_{i=1}^{n}\mathbb{E}\sup_{t-\tau\le s\le t}|x_i(s)|^p<\epsilon$ *and* $\lim_{t\to\infty}\left\{\mathbb{E}\left(\sup_{t-\tau\le s\le t}\|x(s,0,\phi)\|^p\right)\right\}=0$.

Remark 1.6. In some papers, see, for example, [15,16,29,30], the norm for the space of stochastic process is defined by

$$
\|\varphi\|_{[0,t]}=\left[\mathbb{E}\left(\sup_{s\in[0,t]}|\varphi(s,\omega)|^2\right)\right]^{1/2}.
$$

As in [16], in order to show $P(\mathcal{S})\subseteq\mathcal{S}$, we need to estimate $\mathbb{E}\sup_{s\in[0,t]}|I_5(s)|^2$, where

$$
I_5(s)=\int_0^s e^{-\int_z^s h(u)\,du}\left[c(z)x(z)+e(z)x(z-\delta(z))\right]dw(z).
$$

However, $I_5(s)$ is not a local martingale (see Section 8 for its proof). Hence, Burkholder–Davis–Gundy Inequality can not be applied directly.

Define $\mathcal{B}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega):[\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi\in C\left([\vartheta,\infty),L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $e^{\lambda t}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(t)|^p\right)\to 0$ as $t\to\infty$, where $\lambda<\min\{c_1,c_2,\cdots,c_n\}$. Then $\mathcal{B}_\phi$ is a complete space with respect to the norm (3). Using a contraction mapping defined on the space $\mathcal{B}_\phi$ and applying a contraction mapping principle, we obtain our third result, which is proved in Section 4.

Theorem 1.7. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and* $r(t)$ *in the distributed delay are bounded by a constant* τ $(\tau>0)$*;*
(ii) *and such that*

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \tag{7}$$

$$+5^{p-1}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} + 5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) < 1,$$

where $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$,

then the trivial solution of (1) is exponentially stable in pth moment.

Remark 1.8. Theorem 1.7 can, for example, be applied to establish exponential stability in pth moment of a two dimensional stochastically perturbed Hopfield neural network with time-varying delay, the delay is bounded but not differentiable, see Example 7.3 for details.

Remark 1.9. Many articles, see, for example, [23,24] have studied the case and special case of stochastic neural network (1). However, they impose the following condition on the delays:

(H) the discrete delay $\tau(t)$ is differentiable function and $r(t)$ in the distributed delay is non-negative and bounded, that is, there exist constants τ_M, ζ, r_M such that

$$0 \le \tau(t) \le \tau_M, \quad \tau'(t) \le \zeta, \quad r(t) \le r_M. \tag{8}$$

In our results, condition (H) is replaced by other assumptions, which may be satisfied when (H) is not.

Consider the case when there are no stochastic effects on the system (1), which then comes down to the following neural network described by

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^{n} l_{ij}\int_{t-r(t)}^{t} h_j(x_j(s))\,ds,$$
$$i = 1, 2, 3, \cdots, n, \tag{9}$$

or

$$\frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bg(x-\tau(t)) + D\int_{t-r(t)}^{t} h(x(s))\,ds, \tag{10}$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot))^T$ is the neuron state vector of the transformed system (9).

The initial condition for the system (9) is

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{11}$$

where ϕ is a continuous function with the norm defined by $\|\phi\| = \sup_{\vartheta \le s \le 0}\sum_{i=1}^{n}|\phi_i(s)|$.

Assume that (A1)–(A3) are satisfied, then (9) and (11) admit a trivial solution $x = 0$. Denote by $x(t, 0, \phi) = (x_1(t, 0, \phi_1), \cdots, x_n(t, 0, \phi_n))^T \in \mathbb{R}^n$ the solution of (9) with initial condition (11).

Definition 1.10. For the system (9) with initial condition (11), we have that

(i) the trivial solution of (9) is said to be stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|\phi\| < \delta$ yields that $\|x(t, 0, \phi)\| < \epsilon$ for $t \geq 0$, where $\phi \in C([\vartheta, 0], \mathbb{R}^n)$;
(ii) the trivial solution of (9) is said to be asymptotically stable if it is stable and there exists a $\delta > 0$ such that $\|\phi\| < \delta$ implies that $\lim_{t\to\infty} \|x(t, 0, \phi)\| = 0$, where $\phi \in C([\vartheta, 0], \mathbb{R}^n)$;
(iii) the trivial solution of (9) is said to be globally exponentially stable if there exists a pair of constants $\lambda > 0$ and $C > 0$ such that $\|x(t, 0, \phi)\| \leq Ce^{-\lambda t}\|\phi\|$ for $t \geq 0$, where $\phi \in C([\vartheta, 0], \mathbb{R}^n)$.

Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \times \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $t \mapsto \varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(\theta) = \phi_i(\theta)$ for $\vartheta \leq \theta \leq 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n$. For any $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \mathcal{H}_\phi$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_n) \in \mathcal{H}_\phi$, if we define the metric as

$$d(\varphi, \eta) = \sup_{t\geq\vartheta} \sum_{i=1}^{n} |\varphi_i(t) - \eta_i(t)|, \tag{12}$$

then $\mathcal{H}_\phi$ becomes a complete metric space. Using a contraction mapping defined on the space H_ϕ and applying a contraction mapping principle, we obtain our fourth result. Its proof is given by Section 5.

Theorem 1.11. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r > 0$);*
(ii) *and such that*

$$\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| < 1; \tag{13}$$

then the solution of (9)–(11) *exists uniquely and is asymptotically stable.*

Define $\mathcal{K}_\phi = \mathcal{K}_{1\phi} \times \mathcal{K}_{2\phi} \times \cdots \times \mathcal{K}_{n\phi}$, where $\mathcal{K}_{i\phi}$ is the space consisting of continuous functions $t \mapsto \varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(\theta) = \phi_i(\theta)$ for $\vartheta \leq \theta \leq 0$ and $e^{\lambda t}\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots, n$, where $\lambda < \min\{c_1, c_2, \cdots, c_n\}$. Then $\mathcal{K}_\phi$ is a complete metric space with respect to the metric (12). Using a contraction mapping defined on the space $\mathcal{K}_\phi$ and applying a contraction mapping principle, we obtain our fifth result, which is proved in Section 6.

Theorem 1.12. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau > 0$);*
(ii) *and such that*

$$\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|+\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|+\sum_{i=1}^{n}\frac{\tau}{c_i}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|<1; \quad (14)$$

then the trivial solution of (9) with initial condition (11) is exponentially stable.

Remark 1.13. Several exponential stability results [10,25,26] were provided for the system (9), by constructing an appropriate Liapunov functional and employing linear matrix inequality (LMI) method. However, the delays in those results should satisfy the following condition (H). From our results, we provide other assumptions. The delays in our results are required to be bounded. Furthermore, Theorem 1.11 is an extension and improvement of the result in Lai and Zhang [9].

Remark 1.14. From Theorem 1.11 and Theorem 1.12, we find that the terms with f, g, h in equation (10) can be viewed as perturbations of the stable equation $dx(t)/dt = -Cx(t)$. Condition (ii) in Theorem 1.11 and condition (ii) in Theorem 1.12 require the perturbation to be small relative to the stabilizing force of C. Theorem 1.12 can, for example, be applied to establish exponential stability of a two dimensional cellular neural network with time-varying delay, see Example 7.1 for details.

The paper is organized as follows. In Section 2, we present a proof of Theorem 1.4. The proof of Theorem 1.5 is presented in Section 3 and the proof of Theorem 1.7 is given in Section 4. We present the proofs of Theorem 1.11 and Theorem 1.12 in Section 5 and Section 6, respectively. Examples to illustrate our main results are given in Section 7 and an appendix is given in Section 8.

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We start with some preparations.

Lemma 2.1. ([28]) *If $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T$ is a n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then we have the following formula*

$$\mathbb{E}\left(\int_0^t f_i(s)\,dw_i(s)\int_0^t f_j(s)\,dw_j(s)\right) = \mathbb{E}\int_0^t f_i(s)f_j(s)\,d\langle w_i, w_i\rangle_s,$$

where $\langle w_i, w_i\rangle_s = \delta_{ij}s$ are the cross-variations, δ_{ij} is correlation coefficient, f_i is adapted and $f_i \in L^2(\Omega\times[0,t])$, $i, j = 1, 2, \cdots, n$.

If we multiply both sides of (1) by $e^{c_i t}$ and integrate from 0 to t, we obtain that for $t \ge 0$, $i = 1, 2, 3, \cdots, n$,

$$\begin{aligned} x_i(t) = e^{-c_i t}x_i(0) &+ \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(x_j(s))\,ds \\ &+ \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds \end{aligned} \quad (15)$$

$$+\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\,du\,ds$$

$$+\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s).$$

Lemma 2.2. *Define an operator by* $(Q\varphi)(t) = \phi(t)$ *for* $t \in [\vartheta, 0]$, *and for* $t \geq 0$, $i = 1, 2, 3, \cdots, n$,

$$\begin{aligned}(Q\varphi)_i(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\,dw_j(s) \\ &:= J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t).\end{aligned} \tag{16}$$

Suppose that the assumptions (A1)–(A3) *hold. If conditions* (i) *and* (ii) *in Theorem* 1.4 *are satisfied, then* $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* Q *is a contraction mapping.*

Proof. Step 1. From the definition of the metric space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \infty$, for all $t \geq 0$, $\varphi \in \mathcal{S}_\phi$.

Step 2. We prove the continuity in pth moment of Qx on $[0, \infty)$ for $x \in \mathcal{S}_\phi$. Let $x \in \mathcal{S}_\phi$, $t_1 \geq 0$, and let $r \in \mathbb{R}$ with $|r|$ sufficiently small and $r > 0$ if $t_1 = 0$, we have

$$\begin{aligned}\mathbb{E}\sum_{i=1}^n |J_{2i}(t_1+r) - J_{2i}(t_1)|^p &= \mathbb{E}\sum_{i=1}^n \Bigg| \int_0^{t_1} \left(e^{-c_i(t_1+r-s)} - e^{-c_i(t_1-s)}\right) \sum_{j=1}^n a_{ij} f_j(x_j(s))\,ds \\ &\quad + \int_{t_1}^{t_1+r} e^{-c_i(t_1+r-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\,ds \Bigg|^p \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

Similarly, we have that

$$\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t_1+r)-J_{3i}(t_1)|^p\to 0\quad\text{as}\quad r\to 0,$$

$$\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t_1+r)-J_{4i}(t_1)|^p\to 0\quad\text{as}\quad r\to 0.$$

In the following, we check the continuity of $J_{5i}(t)$.

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t_1+r)-J_{5i}(t_1)|^p\\
&=\mathbb{E}\sum_{i=1}^{n}\left|\int_0^{t_1}\left(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\right)\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right.\\
&\quad\left.+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left|\int_0^{t_1}\left(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\right)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right.\\
&\quad\left.+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&\le (2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left|\int_0^{t_1}\left(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\right)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&\quad+(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left|\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&=(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\mathbb{E}\left[\int_0^{t_1}\left(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\right)^2\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\right]^{p/2}\right.\\
&\quad\left.+\mathbb{E}\left[\int_{t_1}^{t_1+r}e^{-2c_i(t_1+r-s)}\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\right]^{p/2}\right\}\to 0\quad\text{as}\quad r\to 0.
\end{aligned}$$

Thus, Q is continuous in pth moment on $[0,\infty)$.

Step 3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

$$\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p=\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{5}J_{ji}(t)\right|^p\le 5^{p-1}\sum_{j=1}^{5}\mathbb{E}\sum_{i=1}^{n}|J_{ji}(t)|^p. \tag{17}$$

Now, we estimate the terms on the right sides of the above inequality.

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &\le \sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}\,ds\right]^{p/q}\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^p ds\right\}\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^p ds\right\}\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^p\right]ds. \qquad (18)
\end{aligned}$$

Since $\varphi(t)\in\mathcal{S}_\phi$, we have that $\lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|=0$. Thus for any $\epsilon>0$, there exists $T_1>0$ such that $t\ge T_1$ implies $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|<\epsilon$, combining with (18), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{T_1} e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^p\right]ds\\
&\quad+\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_{T_1}^{t} e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^p\right]ds\\
&< \sum_{i=1}^{n}c_i^{-p}e^{-c_it}\left(e^{c_iT_1}-1\right)\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\sup_{0\le s\le T_1}\left\{\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^p\right]\right\}\\
&\quad+\epsilon\sum_{i=1}^{n}c_i^{-p}\left[\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right]^{p/q}
\end{aligned}$$

Hence, from the fact that $c_i>0$ $(i=1,2,\cdots,n)$, we obtain that $\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p\to 0$ as $t\to\infty$.

With the similar computation as (18), we obtain that

$$\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p \le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds$$

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p &\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right]ds\\
&\le \sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(u)|^p\right]du\,ds.
\end{aligned}\qquad (19)$$

Using Lemma 2.1 and Hölder inequality, we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^{p}\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,dw_j(s)\right]^2\right\}^{p/2} \qquad (20)\\
&= n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}\right.\\
&\quad\left.+\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\right]\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\right]\\
&\quad+n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-2c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\right.\\
&\quad\left.+\nu^{p/2}\int_0^t e^{-2c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\right.\\
&\quad\left.+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right]ds\right\}.
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$, for each $\epsilon>0$, there exists $T_2>0$ such that $t\ge T_2$ implies $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-\tau(s))|<\epsilon$ and $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-r(t))|<\epsilon$. From (19), we obtain that

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p &< \sum_{i=1}^{n}\left(\frac{1}{c_i}\right)^{p/q} e^{-c_i t}\int_0^{T_2} e^{c_i s}\,ds\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ &\quad\times\sup_{\vartheta\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right)\right]+\epsilon\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p &< \sum_{i=1}^{n} r e^{-c_i t}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ &\quad\times\sup_{\vartheta\le u\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)\right]\int_0^{T_2} e^{c_i s}\,ds\\ &\quad+\sum_{i=1}^{n}\frac{\epsilon r}{c_i}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}.\end{aligned}$$

Further, from (20), we obtain

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p &< n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\sup_{0\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\right.\\ &\quad\left.+\nu^{p/2}\sup_{\vartheta\le s\le T_2}\left[E\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\right\}\int_0^{T_2} e^{-c_i(t-s)}\,ds\\ &\quad+n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\frac{\epsilon(\mu^{p/2}+\nu^{p/2})}{c_i}\right].\end{aligned}$$

Hence, let $t\to\infty$, from the fact that $c_i>0$ $(i=1,2,\cdots,n)$, we obtain that

$$\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p\to 0,\quad \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p\to 0,\quad\text{and}\quad \mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p\to 0.$$

Thus, combining with (17), we obtain that $\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p\to 0$ as $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p\to 0$. Therefore, $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$.

Step 4. We prove that Q is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, from (18)–(20), we obtain

$$
\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(s)-(Q\psi)_i(s)|^p\right\}\\
&\leq 4^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)-\psi(u)|^p\right)du\\
&\quad+4^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad\times\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u-\tau(u)))-\psi_j(u-\tau(u)))|^p\right)du\\
&\quad+4^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\\
&\quad\times\int_{u-r(u)}^{u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(v)-\psi_j(v)|^p\right)dv\,du\\
&\quad+4^{p-1}n^{p-1}\sup_{s\geq\vartheta}\left\{\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)-\psi_j(u)|^p\right)du\right.\right.\\
&\quad\left.\left.+\nu^{p/2}\int_0^s e^{-c_i(s-u)}E\left(\sum_{j=1}^{n}|\varphi_j(u-\tau(u))-\psi_j(u-\tau(u))|^p\right)du\right]\right\}\\
&\leq 4^{p-1}\left\{\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\quad+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\
&\quad\left.+n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\right\}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.
\end{aligned}
$$

From (4), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 1.4.

Proof. From Lemma 2.2, by the contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (1) with $x(t) = \phi(t)$ as $t \in [\vartheta, 0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p \to 0$ as $t \to \infty$.

Now, we prove that the trivial solution of (1) is pth moment stable. Let $\epsilon > 0$ be given and choose $\delta > 0$ ($\delta < \epsilon$) such that $5^{p-1}\delta < (1-\alpha)\epsilon$, where α is the left hand side of (4).

If $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T$ is a solution of (1) with the initial condition (2) satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p < \delta$, then $x(t) = (Qx)(t)$ defined in (16). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p < \epsilon$ for all $t \geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p < \epsilon$ for $t \in [\vartheta, 0]$, we suppose that there exists $t^* > 0$ such that $\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p = \epsilon$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p < \epsilon$ for $\vartheta \leq t < t^*$, then it follows from (4), we obtain that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p &\leq 5^{p-1}\mathbb{E}\sum_{i=1}^{n}e^{-pc_it^*}|x_i(0)|^p \\
&\quad +5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds \\
&\quad +5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \\
&\quad \times\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds \\
&\quad +5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left[\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)} \\
&\quad \times\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds \\
&\quad +5^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\right. \\
&\quad \left.+\nu^{p/2}\int_0^{t^*}e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\right)ds\right] \\
&\leq \left[5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right. \\
&\quad +5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}
\end{aligned}
$$

$$+5^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg]\epsilon+5^{p-1}\delta$$
$$<(1-\alpha)\epsilon+\alpha\epsilon=\epsilon,$$

which is a contradiction. Therefore, the trivial solution of (1) is asymptotically stable in pth moment. □

Corollary 2.3. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r>0$);*
(ii) *and such that*

$$5\sum_{i=1}^{n}c_i^{-2}\left(\sum_{j=1}^{n}a_{ij}^2\alpha_j^2\right)+5\sum_{i=1}^{n}c_i^{-2}\left(\sum_{j=1}^{n}b_{ij}^2\beta_j^2\right)+5\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^2\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)$$
$$+20n\sum_{i=1}^{n}c_i^{-1}(\mu+\nu)<1,$$

where μ, ν are defined as in Theorem 1.4, then the trivial solution of (1) is asymptotically stable in mean square.

Consider the stochastic neural networks without distributed delays

$$dx_i(t)=\left[-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))\right]dt$$
$$+\sum_{j=1}^{n}\sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t) \tag{21}$$

for $i=1,2,3,\cdots,n$.

Corollary 2.4. *Suppose that the assumptions* (A1)–(A3) *hold. The trivial solution of* (21) *is asymptotically stable in pth moment if the following inequality holds,*

$$4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+4^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)<1, \tag{22}$$

where μ, ν are defined as in Theorem 1.4.

Remark 2.5. Condition (A3) can be relaxed. In fact, if $p = 2$, then

$$(\mathrm{A3}') \quad \forall i, \quad \sum_{j=1}^{n} \big(\sigma_{ij}(t,x,y) - \sigma_{ij}(t,u,v)\big)^2 \le \sum_{j=1}^{n} \mu_j (x_j - u_j)^2 + \nu_j (y_j - v_j)^2 \tag{23}$$

is sufficient, as can be easily observed from the proof of Theorem 1.4. If $p \ge 2$, then (A3) can also be replaced by (A3′), but the factor n^{p-1} in front of the last term in (4) has to be replaced by $n^{(3p/2)-2}$. This can be seen from the proof of Theorem 1.4 with the aid of a few more applications of the Hölder inequality.

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. We start with some preparations.

Lemma 3.1. *Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [-\tau, 0]$, and for $t \ge 0$, $(P\varphi)(t)$ is defined by the right hand side of (16), if the conditions* (i) *and* (ii) *in Theorem 1.5 are satisfied, then $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ is a contraction mapping.*

Proof. Observe that all terms at the right hand side of (16) have continuous paths, almost surely. Now, we prove that $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$.

$$\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} |(P\varphi)_i(s)|^p\right] = \sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} \left|\sum_{j=1}^{5} J_{ji}(s)\right|^p\right]$$
$$\le 5^{p-1} \sum_{j=1}^{5} \sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} |J_{ji}(s)|^p\right].$$

We estimate the terms on the right-hand side of the above inequality. Let $c = \min\{c_1, c_2, c_3, \cdots, c_n\}$ and let q be such that $1/p + 1/q = 1$,

$$\mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} |J_{2i}(s)|^p\right]$$
$$\le c^{-p/q} \mathbb{E}\left\{\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j |\varphi_j(u)|\right)^p du\right]\right\}$$
$$\le c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |\varphi_j(u)|^p\right) du\right]\right\}$$
$$\le c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} |\varphi_j(u)|^p\, du\right]\right\}$$

$$
\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\right]\right\}
$$

$$
\le e^{c\tau}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-c(t-u)}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\right]. \tag{24}
$$

Since $\sum_{j=1}^{n}\mathbb{E}\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p\to 0$ as $t\to\infty$, then for any $\epsilon>0$, there exists such that $t\ge T_1$ implies

$$
\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p\right]<\epsilon,
$$

which yields that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t e^{-c(t-u)}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\right] &= \int_0^{T_1} e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du \\
&\quad + \int_{T_1}^{t} e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du \\
&\le \int_0^{T_1} e^{-c(t-u)}\left(\sup_{\vartheta\le v\le T_1}|\varphi_j(v)|^p\right)du+\frac{\epsilon}{c}.
\end{aligned}
$$

Then combining with (24), we obtain that $\mathbb{E}\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|J_{2i}(s)|^p\to 0$ as $t\to\infty$. Similarly, we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|J_{3i}(s)|^p \\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^{n}|\varphi_j(u-\tau(u))|^p\right)du\right]\right\} \\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}|\varphi_j(u-\tau(u))|^p\,du\right]\right\} \\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]\right\} \\
&\le e^{c\tau}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|J_{4i}(s)|^{p}\right]\\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_{j}|^{q}\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_{0}^{s}e^{-c(s-u)}\sum_{j=1}^{n}\left|\int_{u-r(u)}^{u}\varphi_{j}(v)\,dv\right|^{p}du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_{j}|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_{0}^{s}e^{-c(s-u)}\left|\int_{u-r(u)}^{u}\varphi_{j}(v)\,dv\right|^{p}du\right]\right\}\\
&\le \tau^{p}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_{j}|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_{0}^{s}e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_{j}(v)|^{p}\,du\right]\right\}\\
&\le \tau^{p}e^{c\tau}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_{j}|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\int_{0}^{t}e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_{j}(v)|^{p}\,du\right]. \qquad (26)
\end{aligned}
$$

Let $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$. Due to the fact that

$$
\left|\int_{0}^{s}e^{-c_i(s-u)}\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,dw_j(u)\right|^{p}
$$

is a submartingale and the supremum of submartingale is also a submartingale, using Doob's inequality for positive submartingale, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|J_{5i}(s)|^{p}\right]\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}\left|\int_{0}^{s}e^{-c_i(s-u)}\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,dw_j(u)\right|^{p}\right]\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\sup_{t-\tau\le r\le t}\left|\int_{0}^{s}e^{-c(r-u)}\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,dw_j(u)\right|^{p}\right]\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}q^{p}\sup_{t-\tau\le s\le t}\left\{\mathbb{E}\left[\sup_{t-\tau\le r\le t}\left|\int_{0}^{s}e^{-c(r-u)}\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,dw_j(u)\right|^{p}\right]\right\}\\
&\le n^{p-1}e^{pc\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}q^{p}\sup_{t-\tau\le s\le t}\left[\mathbb{E}\left|\int_{0}^{s}e^{-c(t-u)}\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,dw_j(u)\right|^{p}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le n^{p-1}e^{pc\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}q^p \sup_{t-\tau\le s\le t}\left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)}\sigma_{ij}^2(u,\varphi_j(u),\varphi_j(u-\tau(u)))\,du\right)^{p/2}\right]\\
&\le n^{p-1}e^{pc\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}q^p 2^{p/2-1}\sup_{t-\tau\le s\le t}\left\{\mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)}\,du\right)^{p/2-1}\right.\right.\\
&\quad\times\left.\left.\left(\int_0^s e^{-2c(t-u)}\mu_j^{p/2}|\varphi_j(u)|^p\,du+\int_0^s e^{-2c(T-u)}\nu_j^{p/2}|\varphi_j(u-\tau(u))|^p\,du\right)\right]\right\}\\
&\le n^p e^{pc\tau}q^p c^{1-p/2}(\mu^{p/2}+\nu^{p/2})\int_0^t e^{-2c(T-u)}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right].
\end{aligned}
\tag{27}
$$

Using the similar arguments as for the term (24) and combining with (25), (26) and (27), we obtain that

$$
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|(P\varphi)_i(s)|^p\right]\to 0\quad\text{as}\quad t\to\infty.
$$

Thus, $P(\mathcal{C}_\phi)\subseteq\mathcal{C}_\phi$.

Finally, we prove that P is a contraction mapping. For any $\varphi,\psi\in\mathcal{C}_\phi$, from (24)–(27), we obtain that

$$
\begin{aligned}
&\sup_{t\ge 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|(P\varphi)_i(s)-(P\psi)_i(s)|^p\right]\right\}\\
&\le 4^{p-1}\sup_{t\ge 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(\varphi_j(u))-f_j(\psi_j(u))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\ge 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(\varphi_j(u-\tau(u)))\right.\right.\right.\\
&\quad\left.\left.\left.-g_j(\psi_j(u-\tau(u)))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\ge 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)\,dv\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\ge 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}[\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))\right.\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma_{ij}(u, \psi_j(u), \psi_j(u-\tau(u)))] d\omega_j(u) \Bigg|^p \Bigg] \Bigg\} \\
&\le 4^{p-1} \Bigg\{ e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^q |\alpha_j|^q \right)^{p/q} + e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |b_{ij}|^q |\beta_j|^q \right)^{p/q} \\
&+ \tau^p e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q \right)^{p/q} \\
&+ n^p e^{pc\tau} q^p c^{1-p/2} (2c)^{-1} \left(\mu^{p/2} + \nu^{p/2} \right) \Bigg\} \sup_{t \ge 0} \sum_{j=1}^n \mathbb{E} \left[\sup_{t-\tau \le s \le t} |\varphi_j(s) - \psi_j(s)|^p \right].
\end{aligned}
$$

From (6), we obtain that $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 1.5.

Proof. From Lemma 3.1, by a contraction mapping principle, we obtain that P has a unique fixed point $x(t)$, which is a solution of (1) with $x(t) = \phi(t)$ as $t \in [-\tau, 0]$ and $\sum_{i=1}^n \mathbb{E} \left(\sup_{t-\tau \le s \le t} |x_i(s)|^p \right) \to 0$ as $t \to \infty$.

We prove that the trivial solution of (1) is pth moment stable. Let $\epsilon > 0$ be given, we suppose that there exists $t^* > 0$ such that

$$
\sum_{i=1}^n \mathbb{E} \left(\sup_{t^*-\tau \le s \le t^*} |x_i(s)|^p \right) = \epsilon, \qquad \sum_{i=1}^n \mathbb{E} \left(\sup_{t-\tau \le s \le t} |x_i(s)|^p \right) < \epsilon \quad \text{for} \quad \vartheta \le t < t^*,
$$

choose $0 < \delta < \epsilon$ satisfying $5^{p-1} e^{-pct^*} \delta < (1-\alpha)\epsilon$, where α is the left hand side of (6). If $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T$ is a solution of (1) with the initial condition satisfying $\|\phi\|^p < \delta$, then $x(t) = (Px)(t)$ defined in (16). We claim that $\|x\|^p < \epsilon$ for all $t \ge 0$. It follows from (6), we obtain that

$$
\begin{aligned}
&\sum_{i=1}^n \mathbb{E} \left[\sup_{t^*-\tau \le s \le t^*} |x_i(s)|^p \right] \\
&\le 5^{p-1} e^{-pct^*} \delta + 5^{p-1} \Bigg\{ e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^q |\alpha_j|^q \right)^{p/q} + e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |b_{ij}|^q |\beta_j|^q \right)^{p/q} \\
&+ \tau^p e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q \right)^{p/q} + n^p e^{pc\tau} q^p c^{1-p/2} (2c)^{-1} \left(\mu^{p/2} + \nu^{p/2} \right) \Bigg\} \epsilon \\
&< (1-\alpha)\epsilon + \alpha\epsilon = \epsilon,
\end{aligned}
$$

which is a contradiction. Thus, the proof follows. □

4. Proof of Theorem 1.7

In this section, we prove Theorem 1.7. Define an operator $(Q\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$ and for $t \geq 0$, $(Q\varphi)(t)$ is defined by the right hand side of (16). Following the proof of Theorem 1.4, we find that to show Theorem 1.7, we only need to prove that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p \to 0$ as $t \to \infty$. It follows from (16) that

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p = e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{5}J_{ji}(t)\right|^p \leq 5^{p-1}e^{\lambda t}\sum_{j=1}^{5}\mathbb{E}\left(\sum_{i=1}^{n}|J_{ji}(t)|^p\right). \tag{28}$$

Now, we estimate the right-hand terms of (28). First, by using Hölder inequality,

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p \leq \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{(\lambda-c_i)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds. \tag{29}$$

With the similar computation as (29), we obtain that

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p \leq e^{\lambda\tau}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \times\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds. \tag{30}$$

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p \leq e^{\lambda\tau}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)} \times\int_{s-r(s)}^{s}e^{\lambda u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds. \tag{31}$$

Using Lemma 2.1 and Hölder inequality, we obtain that

$$\begin{aligned} &e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p \\ &\leq n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\right] \\ &\quad + e^{\lambda\tau}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\nu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right]. \end{aligned} \tag{32}$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (28) to (32), we obtain that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p \to 0$ as $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$. Hence, combining the proof of Theorem 1.4, there exists a unique fixed point $\varphi(\cdot)$ of Q in $\mathcal{B}_\phi$, which is a solution of the system (1) such that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$ as $t\to\infty$. This completes the proof.

Corollary 4.1. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau>0$);*
(ii) *and such that*

$$5\sum_{i=1}^{n}c_i^{-2}\sum_{j=1}^{n}a_{ij}^2\alpha_j^2+5\sum_{i=1}^{n}c_i^{-2}\sum_{j=1}^{n}b_{ij}^2\beta_j^2+5\tau^2\sum_{i=1}^{n}c_i^{-2}\sum_{j=1}^{n}l_{ij}^2\gamma_j^2$$
$$+20n\sum_{i=1}^{n}c_i^{-1}(\mu+\nu)<1,$$

where μ, ν are defined as in Theorem 1.4, then the trivial solution of (1) is exponentially stable in mean square.

Corollary 4.2. *Let $p\ge 2$. Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau>0$);*
(ii) *and such that*

$$4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+4^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p}(\mu^{p/2}+\nu^{p/2})<1,$$

where μ, ν are defined as in Theorem 1.4, then the trivial solution of (21) is exponentially stable in pth moment.

5. Proof of Theorem 1.11

In this section, we prove Theorem 1.11. We start with some preparations. Multiply both sides of (9) by $e^{c_i t}$ and integrate from 0 to t, we obtain that for $t\ge 0$, $i=1,2,3,\cdots,n$,

$$x_i(t)=e^{-c_i t}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(x_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds$$
$$+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(x_j(u))\,du\,ds. \tag{33}$$

Lemma 5.1. *Define an operator by* $(Px)(\theta) = \phi(\theta)$*, for* $\vartheta \le \theta \le 0$*, and for* $t \ge 0$*,* $i = 1, 2, 3, \cdots, n$*,*

$$(Px)_i(t) = e^{-c_i t} x_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(x_j(s - \tau(s)))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du\, ds := I_1(t) + I_2(t) + I_3(t) + I_4(t). \tag{34}$$

If the conditions (i) *and* (ii) *in* Theorem 1.11 *are satisfied, then* $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$ *and* P *is a contraction mapping.*

Proof. First, we prove that $P\mathcal{H}_\phi \subseteq \mathcal{H}_\phi$. In view of (34), we have that, for fixed time $t_1 \ge 0$, it is easy to check that $\lim_{r\to 0}[(Px)_i(t_1 + r) - (Px)_i(t_1)] = 0$, $i = 1, 2, 3, \cdots, n$. Thus, P is continuous on $[0, \infty)$. Note that $(Px)(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$, we obtain that P is continuous on $[\vartheta, \infty)$.

Next, we prove that $\lim_{t\to\infty}(Px)_i(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$, $i = 1, 2, 3, \cdots, n$. Since $x_i(t) \in \mathcal{H}_{i\phi}$, we have that $\lim_{t\to\infty} x_i(t) = 0$. Then for any $\epsilon > 0$, there exists $T_i > 0$ such that $s \ge T_i$ implies $|x_i(s)| < \epsilon$. Choose $T = \max_{i=1,2,\cdots,n}\{T_i\}$, combining with condition (A2), we obtain that

$$|I_2(t)| \le \int_0^T e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}k_j||x_j(s)|\, ds + \int_T^t e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}\alpha_j||x_j(s)|\, ds$$
$$\le e^{-c_i t} \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\le s\le T} |x_j(s)| \int_0^T e^{-c_i s}\, ds + \frac{\epsilon}{c_i} \sum_{j=1}^n |a_{ij}\alpha_j|. \tag{35}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the estimate (35), we have that $I_2(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists $T_i' > 0$ such that $s \ge T_i'$ implies $|x_i(s - \tau(s))| < \epsilon$ for $i = 1, 2, \cdots, n$. Choose $T' = \max_{i=1,2,\cdots n}\{T_i'\}$, we obtain

$$|I_3(t)| \le \int_0^{T'} e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}\beta_j||x_j(s - \tau(s))|\, ds + \int_{T'}^t e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}k_j||x_j(s - \tau(s))|\, ds$$
$$\le e^{-c_i t} \sum_{j=1}^n |b_{ij}\beta_j| \sup_{\vartheta\le s\le T'} |x_j(s)| \int_0^{T'} e^{c_i s}\, ds + \frac{\epsilon}{c_i} \sum_{j=1}^n |b_{ij}\beta_j|. \tag{36}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the estimate (36), we have that $I_3(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - r(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists $T_i^* > 0$ such that $s \geq T_i^*$ implies $|x_i(s - r(s))| < \epsilon$ for $i = 1, 2, \cdots, n$. Choose $T^* = \max_{i=1,2,\cdots n}\{T_i^*\}$, we obtain

$$
\begin{aligned}
|I_4(t)| &\leq \int_0^{T^*} e^{-c_i(t-s)} \sum_{j=1}^n |l_{ij}\gamma_j| \int_{s-r(s)}^{s} |x_j(u)|\,du\,ds + \epsilon r \int_{T^*}^{t} e^{-c_i(t-s)} \sum_{j=1}^n |l_{ij}\gamma_j|\,ds \\
&\leq r \sum_{j=1}^n |l_{ij}\gamma_j| \sup_{\vartheta \leq u \leq T^*} |x_j(u)| \int_0^{T^*} e^{-c_i(t-s)}\,ds + \frac{\epsilon r}{c_i} \sum_{j=1}^n |l_{ij}\gamma_j|. \qquad (37)
\end{aligned}
$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the estimate (37), we have that $I_4(t) \to 0$ as $t \to \infty$. From the above estimate, we conclude that $\lim_{t\to\infty}(Px)_i(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$, $i = 1, 2, 3, \cdots, n$. Therefore, $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$.

Now, we prove that P is a contraction mapping. For any $x, y \in \mathcal{H}_\phi$, from (35)–(37), we obtain that

$$
\begin{aligned}
&\sum_{i=1}^n |(Px)_i(t) - (Py)_i(t)| \\
&\leq \sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s) - y_j(s)|\,ds \\
&\quad + \sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s-\tau(s)) - y_j(s-\tau(s))|\,ds \\
&\quad + \sum_{i=1}^n \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \int_{s-r(s)}^{s} |x_j(u) - y_j(u)|\,du\,ds \\
&\leq \sum_{i=1}^n \left\{ \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \right\} \\
&\quad \times \sup_{\vartheta \leq s \leq t} \sum_{j=1}^n |x_j(s) - y_j(s)|.
\end{aligned}
$$

From (13), we obtain that P is a contraction mapping. $\square$

We are now ready to prove Theorem 1.11.

Proof. Let P be defined as in Lemma 5.1, by a contraction mapping principle, P has a unique fixed point $x \in \mathcal{H}_\phi$ with $x(\theta) = \phi(\theta)$ on $\vartheta \leq \theta \leq 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotically stable, we need to prove that the trivial equilibrium $x = 0$ of (9) is stable. For any $\epsilon > 0$, choose $0 < \delta < \epsilon$ satisfying the condition $\delta + \epsilon\alpha < \epsilon$, where α is the left hand side of the inequality (13). If $x(t, 0, \phi) = (x_1(t, 0, \phi), x_2(t, 0, \phi), \cdots, x_n(t, 0, \phi))$ is the solution of (9) with the initial condition $\|\phi\| < \delta$, then we claim that $\|x(t, 0, \phi)\| < \epsilon$ for all $t \geq 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$\sum_{i=1}^{n} |x_i(t^*, 0, \phi)| = \epsilon, \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i(t, 0, \phi)| < \epsilon \quad \text{for} \quad 0 \leq t < t^*. \tag{38}$$

From (13) and (33), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} |x_i(t^*, 0, \phi)| \leq \sum_{i=1}^{n} \Bigg[& |e^{-c_i t^*} x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |a_{ij} f_j(x_j(s))| \, ds \\
& + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |b_{ij} g_j(x_j(s - \tau(s)))| \, ds \\
& + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))| \, du \, ds \Bigg] \\
< \delta + \epsilon \Bigg(& \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \\
& + \sum_{i=1}^{n} \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \Bigg) \leq \delta + \epsilon\alpha < \epsilon,
\end{aligned}$$

which contradicts (38). Therefore, $\|x(t, 0, \phi)\| < \epsilon$ for all $t \geq 0$. This completes the proof. □

Let $l_{ij} \equiv 0$ for $i, j = 1, 2, \cdots, n$, the system is reduced to

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau(t))), \tag{39}$$

which is the description of cellular neural network with time-varying delays. Following the result of Theorem 1.11, we have the following corollary.

Corollary 5.2. *Suppose that the assumptions* (A1)–(A3) *hold. If the following condition is satisfied,*

$$\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| < 1, \tag{40}$$

then the trivial solution of (39) *is asymptotically stable.*

Remark 5.3. Note that the delay in Corollary 5.2 can be unbounded. Lai and Zhang [9] studied the asymptotic stability (39) as well. However, the additional condition

$$\max_{i=1,2,\cdots,n}\left[\frac{1}{c_i}\sum_{j=1}^{n}|a_{ij}k_j|+\frac{1}{c_i}\sum_{j=1}^{n}|b_{ij}k_j|\right]<\frac{1}{\sqrt{n}} \tag{41}$$

is needed in Theorem 4.1 of [9]. It is clear that Corollary 5.2 is an improvement of the result in [9].

6. Proof of Theorem 1.12

Proof. In this section, we prove Theorem 1.12. Define an operator $(P\varphi)(t)=\phi(t)$ for $t\in[\vartheta,0]$ and for $t\geq 0$, $(P\varphi)(t)$ is defined by the right hand side of (34). Following the proof of Theorem 1.11, we only need to show that $e^{\lambda t}(P\varphi)_i(t)\to 0$ as $t\to\infty$, $i=1,2,\cdots,n$. We estimate the right-hand terms of (34),

$$e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|$$

$$\leq \max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\sum_{j=1}^{n}|\varphi_j(s)|\,ds$$

$$e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\right|$$

$$\leq e^{\lambda\tau}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|\,ds$$

$$e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(\varphi_j(u))\,du\,ds\right|$$

$$\leq e^{\lambda\tau}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^{s}e^{\lambda u}\sum_{j=1}^{n}|\varphi_j(u)|\,du\,ds.$$

From the fact that $\lambda<\min\{c_1,c_2,\cdots,c_n\}$, $c_i>0$ $(i=1,2,\cdots,n)$ and the above estimate, we obtain that $e^{\lambda t}(P\varphi)_i(t)\to 0$ as $t\to\infty$. □

For the cellular neural network (39), we have the following result.

Corollary 6.1. *Suppose that the assumptions* (A1)–(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ is bounded by a constant τ ($\tau > 0$);*
(ii) *and such that*

$$\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}k_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}k_j| < 1,$$

then the trivial solution of (39) with initial condition (11) is exponentially stable.

7. Examples

Example 7.1. Consider the following two-dimensional cellular neural network

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x - \tau(t)),$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 3/7 \\ -1/7 & -1/7 \end{pmatrix},$$
$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 2/7 \\ 3/7 & 1/7 \end{pmatrix}.$$

The activation function is described by $g_i(x) = tanh(x)$, for $i = 1, 2$. The time-varying delay $\tau(t)$ is continuous and $|\tau(t)| \leq \tau$, where τ is a constant.

It is clear that $\alpha_i = \beta_i = 1$ for $i = 1, 2$. We check the condition (40) in Corollary 5.2,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| \leq \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7}\right) = \frac{16}{21} < 1.$$

Hence, by Corollary 5.2, the trivial equilibrium $x = 0$ of this cellular neural network is asymptotically stable. See Fig. 1.

However, the condition (41) becomes

$$\max_{i=1,2} \left\{ \frac{1}{c_i} \sum_{j=1}^{2} |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^{2} |b_{ij}\beta_j| \right\} = \frac{17}{21} > \frac{1}{\sqrt{2}}.$$

Hence, Theorem 4.1 of [9] is not applicable.

Example 7.2. Consider the two-dimensional stochastic recurrent neural network with time-varying delays

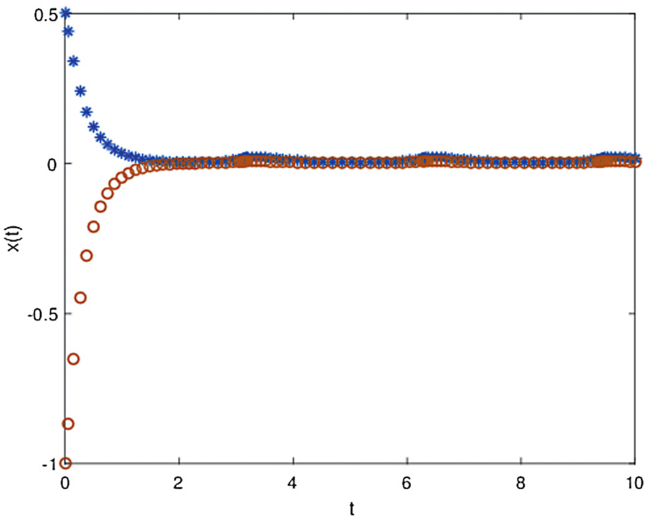

Fig. 1. The solution of Example 7.1.

$$
\begin{aligned}
dx(t) = & -\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\
& + \begin{pmatrix} -0.8 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt \\
& + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \int_{t-r(t)}^{t} 0.2\tanh(x_1(s))\,ds \\ \int_{t-r(t)}^{t} 0.2\tanh(x_2(s))\,ds) \end{pmatrix} dt \\
& + \sigma(t, x(t), x(t-\tau(t)))\,dw(t),
\end{aligned} \tag{42}
$$

where $\tau(t)$, $r(t)$ are continuous functions such that $t - \tau(t) \to \infty$ as $t \to \infty$ and $|r(t)| \leq 1$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ satisfies

$$
trace\left[\sigma^T(t, x, y)\sigma(t, x, y)\right] \leq 0.003(x_1^2 + x_2^2 + y_1^2 + y_2^2).
$$

We suppose $p = 2$, and take $\mu_i = \nu_i = 0.003$ for $i = 1, 2$, by simple computation, we have $\alpha_i = 0.2$, for $i = 1, 2$, $c = \min\{c_1, c_2\} = 5$, $\mu = \nu = 0.003$. By Corollary 2.3, we have that

$$
\begin{aligned}
& 5\sum_{i=1}^{2} c_i^{-2}\left[\sum_{j=1}^{2} a_{ij}^2\alpha_j^2\right] + 5\sum_{i=1}^{2} c_i^{-2}\left[\sum_{j=1}^{2} b_{ij}^2\alpha_j^2\right] + 5\sum_{i=1}^{2}\left(\frac{\tau}{c_i}\right)^2\left[\sum_{j=1}^{2} l_{ij}^2\alpha_j^2\right] \\
& + 20 \times 2 \times \sum_{i=1}^{2} c_i^{-1}(\mu + \nu) < 0.256 < 1.
\end{aligned}
$$

Then the trivial solution of (42) is mean square asymptotically stable. See Fig. 2.

If $\tau(t)$ is bounded, by Corollary 4.1, we obtain that the trivial solution of (42) is mean square exponentially stable.

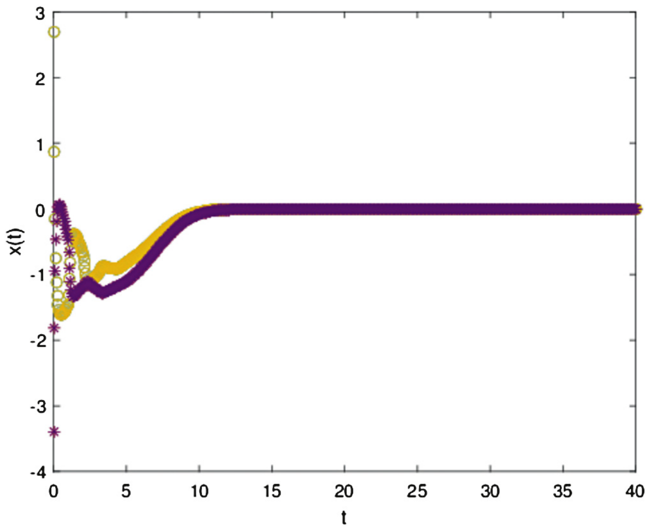

Fig. 2. The solution of Example 7.2.

Example 7.3. Consider a two-dimensional stochastically perturbed HNN with time-varying delays,

$$dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]dt + \sigma(t, x(t), x_\tau(t))d\omega(t), \tag{43}$$

where $f(x) = \frac{1}{5}\arctan x$, $g(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^x + e^{-x})$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, take $\alpha_j = 0.2$, $\beta_j = 0.2$, $j = 1, 2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ satisfies

$$\sigma_{i1}(t, x, y)^2 \le 0.01(x_1^2 + y_1^2) \quad \text{and} \quad \sigma_{i2}(t, x, y)^2 \le 0.01(x_2^2 + y_2^2), \quad i = 1, 2,$$

and $w(t)$ is a two dimensional Brownian motion.

Note that the exponential stability of (43) has been studied in Sun and Cao [24] by employing the method of variation parameter, inequality technique and stochastic analysis.

Now, we check the condition in Corollary 4.2,

$$4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left[\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right]^{p/q} + 4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+ 4^{p-1}2^p c^{-p/2}(\mu^{p/2} + \nu^{p/2}) < 0.18 < 1.$$

By Corollary 4.2, the trivial solution of (43) is exponentially stable. See Fig. 3.

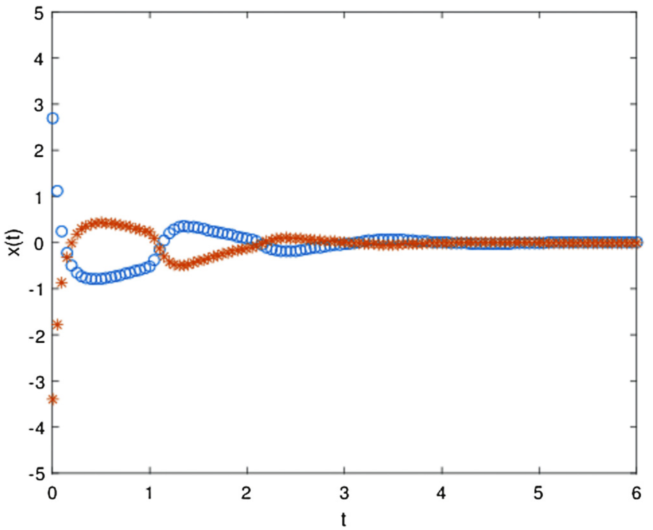

Fig. 3. The solution of Example 7.3.

8. Appendix

In this section, we first show that $I_5(s)$ in [16] is not a local martingale and then we present some examples about Banach spaces.

Definition 8.1. A real valued $\mathcal{F}_t$-adapted process $M = \{M(t) : t \geq 0\}$ is a martingale if $\mathbb{E}|M(t)| < \infty$ for all $t \geq 0$ and

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s), \qquad \text{a.s.} \quad \text{for} \quad \text{all} \quad 0 \leq s < t < \infty.$$

Lemma 8.2. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a martingale.*

Proof. In fact, for $0 \leq u \leq t$,

$$\begin{aligned}\mathbb{E}\left[\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] &= \mathbb{E}\left[\int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] \\ &\quad + \mathbb{E}\left[\int_u^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] \\ &= \int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \neq \int_0^u e^{-c(u-s)}\sigma(s)\,dw(s). \qquad \square\end{aligned} \tag{44}$$

Lemma 8.3. ([20]) *If $M(t)$ is a local martingale and for every t, $\mathbb{E}\sup_{s\in[0,t]}|M(s)| < \infty$, then $M(t)$ is a martingale.*

Lemma 8.4. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a local martingale.*

Proof. We suppose that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is a local martingale. For every t, we have that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{-c(s-u)}\sigma(u)\,dw(u)\right| &\le \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{cu}\sigma(u)\,dw(u)\right| \\ &\le K_1\mathbb{E}\left(\int_0^t e^{2cu}\sigma^2(u)\,du\right)^{1/2} \\ &\le K_1\left(\int_0^t e^{2cu}\mathbb{E}\sigma^2(u)\,du\right)^{1/2} < \infty.\end{aligned}$$

From Lemma 8.3, we obtain that M is a martingale. However, from Lemma 8.2, we know that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a martingale, which is a contradiction. □

Example 8.5. The space $C([a,b])$ of continuous, real-valued (or complex-valued) functions on $[a,b]$ with the sup-normed is a Banach space. More generally, we have the following examples.

(i) If X is a Banach space, the space $C([a,b];X)$ of continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.
(ii) If X is a Banach space, the space $BC([a,b];X) := \{\varphi \mid \varphi\in C([a,b];X),\ \|\varphi\|<\infty\}$ of bounded continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.
(iii) If X is a Banach space, the space $\{\varphi \mid \varphi\in C([a,b];X),\ \lim_{t\to\infty}\varphi(t)=0\}$ and the space

$$\left\{\varphi \mid \varphi\in C([a,b];X), \|\varphi\| = \sup_{s\in[a,b]}|\varphi(s)| \text{ is bounded and } \lim_{t\to\infty}\varphi(t)=0\right\}$$

are Banach spaces with respect to the sup-norm. Clearly, the space

$$C_0([a,b];L^p(\Omega,\mathbb{R}^n)) := \left\{\varphi \mid \varphi\in C([a,b];L^p(\Omega,\mathbb{R}^n)),\ \lim_{t\to\infty}\mathbb{E}|\varphi(t)|^p=0\right\}$$

is a Banach space with respect to the norm defined by $\|\varphi\|^p := \sup_s\left[\mathbb{E}|\varphi(s)|^p\right]$.

The following lemma presents a Banach space that is used in this paper.

Lemma 8.6. *Suppose that $\mathcal{F}_t$ is complete, that is, contains all null sets. Then the space*

$$D := \left\{\varphi\in C_0([a,b];L^p(\Omega,\mathbb{R}^n)),\ \varphi(t) \text{ is } \mathcal{F}_t - \text{measurable for all } t\right\}$$

is a closed subspace of $C_0([a,b];L^p(\Omega,\mathbb{R}^n))$.

Proof. Let $\varphi, \psi \in D$, then $\varphi(t)$ and $\psi(t)$ are $\mathcal{F}_t$-measurable for all t, so $\varphi(t)+\psi(t)$ and $\alpha\varphi(t)$ ($\alpha \in \mathbb{C}$) are $\mathcal{F}_t$-measurable for all t.

Suppose that the sequence $\varphi_1, \varphi_2, \cdots, \varphi_n, \cdots \in D$, $\varphi \in C_0([a,b]; L^p(\Omega, \mathbb{R}^n))$ and $\varphi_n \to \varphi$ as $n \to \infty$ for all t, we claim that $\varphi(t)$ is $\mathcal{F}_t$-measurable. In fact, since $\varphi_n \to \varphi$ as $n \to \infty$, then

$$\sup_{s\in\Omega}\left[\mathbb{E}|\varphi_n(s)-\varphi(s)|^p\right] \to 0 \quad \text{as} \quad n \to \infty.$$

So, for every t, we obtain that $\mathbb{E}|\varphi_n(t)-\varphi(t)|^p \to 0$ as $n \to \infty$, which implies that there exists a subsequence $(\varphi_{n_k}(t))_k$ such that $\varphi_{n_k}(t) \to \varphi(t)$ a.e. on Ω. On the other hand, $\mathcal{F}_t$ is complete. Hence, we obtain that $\varphi(t)$ is $\mathcal{F}_t$-measurable, which implies that D is a closed subspace of the space $C_0([a,b]; L^p(\Omega, \mathbb{R}^n))$. $\square$

References

[1] J.A.D. Appleby, Fixed points, stability and harmless stochastic perturbations, Preprint, 2008.
[2] T.A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover Publication, New York, 2006.
[3] A. Djoudi, R. Khemis, Fixed points techniques and stability for neutral nonlinear differential equations with unbounded delays, Georgian Math. J. 13 (1) (2006) 25–34.
[4] M. Forti, A. Tesi, New conditions for global stability of neural networks with application to linear and quadratic programming problems, IEEE Trans. Circuits Syst. I 42 (1995) 354–366.
[5] S. Haykin, Neural Networks, Prentice-Hall, Englewood Cliffs, 1994.
[6] J. Hu, S. Zhong, L. Liang, Exponential stability analysis of stochastic delayed cellular neural network, Chaos Solitons Fractals 27 (2006) 1006–1010.
[7] C.X. Huang, Y.G. He, H.N. Wang, Mean square exponential stability of stochastic recurrent neural networks with time-varying delays, Comput. Math. Appl. 56 (2008) 1773–1778.
[8] C.X. Huang, Y.G. He, L.H. Huang, W.J. Zhu, pth moment stability analysis of stochastic recurrent neural networks with time-varying delays, Inform. Sci. 178 (2008) 2194–2203.
[9] X.H. Lai, Y.T. Zhang, Fixed point and asymptotic analysis of cellular neural networks, J. Appl. Math. 2012 (2012) 689845, https://doi.org/10.1155/2012/689845.
[10] T. Li, Q. Luo, C.Y. Sun, B.Y. Zhang, Exponential stability of recurrent neural networks with time-varying discrete and distributed delays, Nonlinear Anal. Real World Appl. 10 (2009) 2581–2589.
[11] X. Liao, X. Mao, Exponential stability and instability of stochastic neural networks, Stoch. Anal. Appl. 14 (2) (1996) 165–185.
[12] X. Liao, X. Mao, Stability of stochastic neural networks, Neural Parallel Sci. Comput. 4 (2) (1996) 205–224.
[13] X.X. Liao, J. Wang, Algebraic criteria for global exponential stability of cellular neural networks with multiple time delays, IEEE Trans. Circuits Syst. I 50 (2003) 268–275.
[14] X. Liao, X. Mao, Exponential stability of stochastic delay interval systems, Systems Control Lett. 40 (2000) 171–181.
[15] D.Z. Liu, G.Y. Yang, W. Zhang, The stability of neutral stochastic delay differential equations with Poisson jumps by fixed points, J. Comput. Appl. Math. 235 (2011) 3115–3120.
[16] J.W. Luo, Fixed points and stability of neutral stochastic delay differential equations, J. Math. Anal. Appl. 334 (2007) 431–440.
[17] J.W. Luo, T. Taniguchi, Fixed points and stability of stochastic neutral partial differential equations with infinite delays, Stoch. Anal. Appl. 27 (2009) 1163–1173.
[18] J.W. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, J. Math. Anal. Appl. 342 (2008) 753–760.
[19] X.R. Mao, Stochastic Differential Equations and Applications, Horwood, New York, 1997.
[20] P.E. Protter, Stochastic Integration and Differential Equations, 2nd edition, 2004.
[21] R. Sakthivel, J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, Statist. Probab. Lett. 79 (2009) 1219–1223.
[22] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays, J. Math. Anal. Appl. 356 (2009) 1–6.

[23] R. Sakthivel, R. Samidurai, S.M. Anthoni, Asymptotic stability of stochastic delayed recurrent neural networks with impulsive effects, J. Optim. Theory Appl. 147 (2010) 583–596.
[24] Y.H. Sun, J.D. Cao, *p*th moment exponential stability of stochastic recurrent neural networks with time-varying delays, Nonlinear Anal. Real World Appl. 8 (2007) 1171–1185.
[25] J.K. Tian, S.M. Zhong, Y. Wang, Improved exponential stability criteria for neural networks with time-varying delays, Neurocomputing 97 (2012) 164–173.
[26] J.K. Tian, S.M. Zhong, New delay-dependent exponential stability criteria for neural networks with discrete and distributed time-varying delays, Neurocomputing 74 (2011) 3365–3375.
[27] B.D. Vries, J.C. Principle, The gamma model—a new neural model for temporal processing, Neural Netw. 5 (1992) 565–576.
[28] L. Wan, J. Sun, Mean square exponential stability of stochastic delayed Hopfield neural networks, Phys. Lett. A 343 (2005) 306–318.
[29] M. Wu, N.J. Huang, C.W. Zhao, Fixed points and stability in neutral stochastic differential equations with variable delays, Fixed Point Theory Appl. 2008 (2008) 407352.
[30] M. Wu, N.J. Huang, C.W. Zhao, Stability of half-linear neutral stochastic differential equations with delays, Bull. Aust. Math. Soc. 80 (2009) 369–383.
[31] Z. Yuan, L. Yuan, L. Huang, Dynamics of periodic Cohen–Grossberg neural networks with varying delays, Neurocomputing 70 (2006) 164–172.
[32] Z.G. Zeng, D.S. Wang, Global stability of a general class of discrete-time recurrent neural networks, Neural Comput. 18 (2006) 848–870.
[33] Z.G. Zeng, J. Wang, Complete stability of cellular neural networks with time-varying delays, IEEE Trans. Circuits Syst. I 53 (2006) 944–955.
[34] H.Y. Zhao, Global asymptotic stability of Hopfield neural network involving distributed delays, Neural Netw. 17 (2004) 47–53.