



Stability results for nonlinear functional differential equations using fixed point methods[☆]

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Abstract

We present new conditions for stability of the zero solution for three distinct classes of scalar nonlinear delay differential equations. Our approach is based on fixed point methods and has the advantage that our conditions neither require boundedness of delays nor fixed sign conditions on the coefficient functions. Our work extends and improves a number of recent stability results for nonlinear functional differential equations in a unified framework. A number of examples are given to illustrate our main results.

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1. Introduction and main results

Lyapunov's direct method provides simple geometric theorems for deciding the stability or instability of an equilibrium point of a differential equation. However, in the context of functional differential equations, Lyapunov's direct method is not always as effective, in particular if the delay is unbounded or if the differential equation has unbounded terms. Therefore, it was recently proposed by Burton [5] and co-workers to use fixed point methods as an alternative. While Lyapunov's direct method usually requires pointwise conditions, fixed point methods need conditions of an averaging nature, and, therefore, can handle various delays or unbounded terms more easily.

A typical stability result based on fixed point theory arguments follows a number of standard arguments adapted to the special structure of the equation under consideration. This leads to many different results in the literature for different classes of equations, for example, with time dependent delays, distributed delays, neutral terms, and certain nonlinearities, see [1–11]. The aim of this paper is to study the approach using fixed point theory in a systematic way and to unify recent results in the literature by considering three general classes of equations. For each of these classes of equations, we combine different techniques to prove new stability theorems. In addition, we present a number of examples to illustrate our results.

The first class of functional differential equations that we will study in this paper is of the form

$$x'(t) = - \int_{t-r(t)}^t a(t, s)g(s, x(s)) ds, \quad (1)$$

where $r(t) : [0, \infty) \rightarrow [0, \infty)$, $a(t, s) : [0, \infty) \times [r_0, \infty) \rightarrow \mathbb{R}$ are continuous functions, g is a continuous function that satisfies a Lipschitz condition with respect to x on $[r_0, \infty) \times [-l, l]$, where $r_0 = \inf_{t \geq 0} \{t - r(t)\}$.

A standard fix point argument shows that the differential equation (1) provided with an initial condition

$$x(t) = \phi(t) \quad t \in [r_0, 0], \quad (2)$$

where $\phi(s) \in C([r_0, 0], \mathbb{R})$ defines a well-posed initial-value problem and we denote by $x(t) := x(t, \phi)$ the solution of (1) with initial condition (2).

Definition 1.1. The zero solution of (1) is said to be stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, we have that the corresponding solution satisfies $|x(t)| < \epsilon$ for $t \geq 0$.

Definition 1.2. The zero solution of (1) is said to be asymptotically stable if it is stable and there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, the corresponding solution $x(t)$ tends to zero as $t \rightarrow \infty$.

Theorem 1.3. Consider the functional differential equation (1) and suppose that the following conditions are satisfied

- (i) $g(s, -x) = -g(s, x)$;
- (ii) there exists an $l > 0$ such that g satisfies a Lipschitz condition with respect to x on $[r_0, \infty) \times [-l, l]$, that is, there exists a constant $L > 0$, such that $|g(s, x) - g(s, y)| \leq L|x - y|$ for $s \geq r_0$ and $x, y \in [-l, l]$;

- (iii) there are functions w and W that are continuous, odd and strictly increasing on $[-l, l]$ such that $w(x) \leq g(s, x) \leq W(x)$ for $x \in [0, l]$;
- (iv) $x - w(x)$ is non-decreasing on $[0, l]$;
- (v) $|x - g(s, x)| \leq l - w(l)$ for $x \in [-l, l]$;
- (vi) $v : [r_0, \infty) \rightarrow \mathbb{R}$ is a continuous function, $v(t) \geq 0$ for $t \geq 0$;
- (vii) there exists a continuous function q such that $|\int_t^u a(s, u) ds| \leq q(u)$ for $t - r(t) \leq u \leq t$;
- (viii) a positive number $\alpha < w(l)[W(l)]^{-1}$ exists such that

$$\begin{aligned} & \int_{t-r(t)}^t \left| v(u) + \int_t^u a(s, u) ds \right| du + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \\ & \quad \times \int_{s-r(s)}^s \left| v(u) + \int_s^u a(s, u) ds \right| du ds \\ & \quad + \int_0^t e^{-\int_s^t v(u) du} \left| v(s - r(s)) + \int_s^{s-r(s)} a(u, s - r(s)) du \right| |1 - r'(s)| ds \leq \alpha. \end{aligned}$$

Then there exists a $\delta \in (0, l)$ such that, for each continuous initial function $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, there is a unique solution $x : [0, \infty) \rightarrow \mathbb{R}$ with $x(t) = \phi(t)$ on $[r_0, 0]$ of (1) such that $|x(t)|$ is bounded by l on $[r_0, \infty)$. This implies that the zero solution of (1) is stable.

Remark 1.4. The proof is based on a generalization of some ideas of Jin and Luo [10] who discussed the case when $g(s, x) = g(x)$. We eliminate the condition that $t - r(t)$ is strictly increasing and obtain weaker conditions in Theorem 1.3 than those obtained in Theorem 4.1 of Becker and Burton [1]. See Example 1.5.

Example 1.5. Consider the following integro-differential equation

$$x'(t) = - \int_{0.4635t}^t \frac{0.9}{s^2 + 1} \left(\frac{4}{5} + \frac{1}{10} \sin^2 s \right) x^3(s) ds. \tag{3}$$

We check the condition (vii) of Theorem 4.1 in Becker and Burton [1], $f(t) = \frac{t}{0.4635}$, then

$$G(t, s) = \int_t^{s/0.4635} \frac{0.9}{s^2 + 1} du = \frac{0.9(s/0.4635 - t)}{s^2 + 1},$$

for $t \geq 0$ and $0.4635t \leq s \leq t$. Consequently, $\lim_{t \geq 0} \left\{ 2 \int_{0.4635t}^t |G(t, u)| du \right\} = 0.9 \times 2 \left(-\frac{\ln 0.4635+1}{0.4635} + 1 \right) = 0.9027$. Then there exists some $t_0 > 0$ such that for $t \geq t_0$, we have $2 \int_{0.4635t}^t |G(t, u)| du > 0.9020$. Since $\frac{w(\frac{1}{2})}{w(\frac{1}{2})} = \frac{8}{9} = 0.8889 < 0.9020$. This implies that condition (vii) of Theorem 4.1 in Becker and Burton [1] does not hold. Thus Theorem 4.1 of Becker and Burton [1] cannot be applied to Eq. (3). However, by (11), we obtain $B(t, s) = \int_t^s \frac{0.9}{s^2+1} du = \frac{0.9(s-t)}{s^2+1}$. Choosing $v(t) = \frac{0.9t}{t^2+1}$, we have that

$$\begin{aligned} & \int_{t-r(t)}^t |v(u) + B(t, u)| du = 0.9 \times \int_{0.4635t}^t \left| \frac{2u - t}{u^2 + 1} \right| du = 0.9 \\ & \quad \times \int_{0.4635t}^{0.5t} \frac{t - 2u}{u^2 + 1} du + 0.9 \times \int_{0.5t}^t \frac{2u - t}{u^2 + 1} du \\ & \quad = 0.9 \times [2 \arctan 0.5t - \arctan t - \arctan 0.4635t] + \ln(t^2 + 1) \\ & \quad \quad + \ln(0.4635^2 t^2 + 1) - 2 \ln(0.25t^2 + 1) := w(t). \end{aligned}$$

Since the function $w(t)$ is increasing on $[0, \infty)$ and $\lim_{t \rightarrow \infty} w(t) = 0.9 \times (1/0.4635 - 3 + 2 \ln 2 + 2 \ln 0.927) = 0.3530$, we have $\int_{t-r(t)}^t |v(u) + B(t, u)| du < 0.3530$,

$$\begin{aligned} & \int_0^t e^{-\int_s^t v(u) du} |v(s - r(s)) + B(s, s - r(s))| |1 - r'(s)| ds \\ &= (1/0.4635 - 2) \int_0^t e^{-\int_s^t \frac{0.9u}{u^2+1} du} \frac{0.9s}{s^2 + 1/0.4635^2} ds < 0.1575, \end{aligned}$$

and $\int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r(s)}^s |v(u) + B(s, u)| du ds < 0.3530$. Hence, we have

$$\begin{aligned} & \int_{t-r(t)}^t |v(u) + B(t, u)| du + \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r(s)}^s |v(u) + B(s, u)| du ds \\ &+ \int_0^t e^{-\int_s^t v(u) du} |v(s - r(s)) + B(s, s - r(s))| |1 - r'(s)| ds \\ &< 0.8635 < \frac{w(\frac{1}{2})}{W(\frac{1}{2})} = \frac{8}{9} = 0.8889. \end{aligned}$$

By [Theorem 1.3](#), the zero solution of (3) is stable. Together with [Remark 1.4](#), this shows that our result extends the result of Becker and Burton [1].

The second class consists of scalar neutral integro-differential equations of the form

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + \int_{t-r_3(t)}^t g(t, x(s)) d\mu(t, s), \quad t \geq 0, \quad (4)$$

where for $j = 1, 2, 3$, the delays $r_j(t) : [0, \infty) \rightarrow [0, \infty)$ are continuous functions, the coefficients $a, c : [0, \infty) \rightarrow \mathbb{R}$ are continuous, where $r_0 = \min \{ \inf_{t \geq 0} \{t - r_1(t)\}, \inf_{t \geq 0} \{t - r_2(t)\}, \inf_{t \geq 0} \{t - r_3(t)\} \}$. The kernel $\mu(t, s)$ is of bounded variation for each t and $g : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and for each $t, x g(t, x) > 0$ if $x \neq 0$ is sufficient small. We assume that g satisfies:

(G) $g(t, 0) = 0$, there exists an $l > 0$ such that g satisfies a Lipschitz condition with respect to x on $[0, \infty) \times [-l, l]$, that is, there exists a constant $L > 0$, such that $|g(t, x) - g(t, y)| \leq L|x - y|$, for $t \geq 0$ and $x, y \in [-l, l]$.

In our second result we obtain sufficient and necessary conditions for the asymptotic stability of (4) by introducing two auxiliary continuous functions $h_1(t)$ and $h_2(t)$ which will be used to define an appropriate map defined on a complete metric space so that we can apply a fixed point argument.

Theorem 1.6. *Consider the neutral integro-differential equation (4) an suppose the following conditions are satisfied*

- (i) *the delay $r_2(t)$ is differentiable, the delay $r_1(t)$ is twice differentiable with $r'_1(t) \neq 1$, and $t - r_j(t) \rightarrow \infty$ as $t \rightarrow \infty, j = 1, 2, 3$;*
- (ii) *there exists a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \rightarrow \mathbb{R} (j = 1, 2)$ such that*

$$\begin{aligned} & \left| \frac{c(t)}{1 - r'_1(t)} \right| + \sum_{j=1}^2 \int_{t-r_j(t)}^t |h_j(s)| ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} (V_{[s-r_3(s), s]}(\mu(s, \cdot))) ds \\ &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_2(s - r_2(s))(1 - r'_2(s)) - a(s)| ds \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| ds \\
 &+ \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \left(\int_{s-r_j(s)}^s |h_j(u)| du \right) ds \leq \alpha,
 \end{aligned}$$

where $k(s) = ((1 - r_1'(s))^2)^{-1} [c(s)(h_1(u) + h_2(u)) + c'(s)(1 - r_1'(s)) + c(s)r_1''(s)]$ and $V_{[s-r_3(s),s]}(\mu(s, \cdot))$ denotes the total variation of $\mu(s, \cdot)$ on $[s - r_3(s), s]$;

(iii) and such that $\liminf_{t \rightarrow \infty} \int_0^t (h_1(s) + h_2(s)) ds > -\infty$.

Then the zero solution of (4) is asymptotically stable if and only if

(iv) $\int_0^t (h_1(s) + h_2(s)) ds \rightarrow \infty$ as $t \rightarrow \infty$.

Corollary 1.7. Consider the equation

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + b(t)g(t, x(t - r_3(t))). \tag{5}$$

Assume that $r_2(t)$ is differentiable, $r_1(t)$ is twice differentiable, $r_1'(t) \neq 1$, $t - r_j(t) \rightarrow \infty$ as $t \rightarrow \infty$, $j = 1, 2, 3$, g satisfies condition (G). Suppose that there exist a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \rightarrow \mathbb{R}$, ($j = 1, 2$), such that $\liminf_{t \rightarrow \infty} \int_0^t (h_1(s) + h_2(s)) ds > -\infty$ and

$$\begin{aligned}
 &\left| \frac{c(t)}{1 - r_1'(t)} \right| + \sum_{j=1}^2 \int_{t-r_j(t)}^t |h_j(s)| ds \\
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)| ds \\
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} [|h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| + |b(s)|] ds \\
 &+ \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \int_{s-r_j(s)}^s |h_j(u)| du ds \leq \alpha,
 \end{aligned}$$

where $k(s)$ and $h_j(s)$, ($j = 1, 2$), are defined as in [Theorem 1.6](#). Then the zero solution of (5) is asymptotically stable if and only if $\int_0^t (h_1(s) + h_2(s)) ds \rightarrow \infty$ as $t \rightarrow \infty$.

Remark 1.8. [Theorem 1.6](#) contains all the stability results for (4) discussed in [1,3–7,9,11]. Note that, in addition, in our result the delays can be unbounded and that the coefficients can change sign. See [Examples 1.10](#) and [1.12](#).

Example 1.9. Consider the neutral differential equation

$$x'(t) = -\frac{1}{t+1}x(t) + \frac{1}{2t+2}x(t - 0.05t) + 0.05x'(t - 0.05t). \tag{6}$$

Define $a(t) = \frac{1}{t+1}$, $b(t) = \frac{1}{2t+2}$, $c(t) = 0.05$, $r(t) = 0.05t$ and $v(t) = \frac{2}{t+1}$. Then $\frac{|c(t)|}{|1-r'(t)|} = \frac{0.05}{1-0.05} = \frac{1}{19} \approx 0.0526$. Since $|v(s) - a(s)| = \frac{1}{s+1}$, $k(s) = \frac{2}{19(s+1)}$, we

have

$$\begin{aligned} & \left| \frac{c(t)}{1-r'(t)} \right| + \int_{t-r(t)}^t |v(s) - a(s)| ds + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \int_{s-r(s)}^s |v(u) - a(u)| du ds \\ & \quad + \int_0^t e^{-\int_s^t v(u) du} \left\{ [v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s) + b(s) \right\} ds \\ & < 0.0513 + 0.0513 + 0.697 = 0.852, \end{aligned}$$

and since $\int_0^t v(s) ds = \int_0^t \frac{2}{s+1} ds = 2 \ln(t+1)$, the conditions of [Corollary 1.7](#) are satisfied. Therefore, the zero solution of (6) is asymptotically stable.

Example 1.10. Consider the following differential equation

$$\begin{aligned} x'(t) = & -\frac{1}{32} \left(\frac{1}{4} - \frac{1}{3} \sin t + \epsilon_1(t) \right) x \left(t - \left(1 - \frac{1}{3} \cos t + \epsilon_2(t) \right) \right) \\ & + \frac{\cos t}{256} g(x(t-r_3(t))), \end{aligned} \tag{7}$$

where $|\epsilon_j(t)| < \epsilon < \frac{2}{51}$, $|\epsilon'_j(t)| < \epsilon < \frac{2}{51}$, $j = 1, 2$, $r_3(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous function which satisfies $t - r_3(t) \rightarrow \infty$ as $t \rightarrow \infty$, g satisfies condition (G).

Define $a(t) = \frac{1}{32} \left(\frac{1}{4} - \frac{1}{3} \sin t + \epsilon_1(t) \right)$, $b(t) = \frac{\cos t}{256}$, $r_2(t) = 1 - \frac{1}{3} \cos t + \epsilon_2(t)$, and $v(t) = \frac{1}{32}$. Hence, we have

$$\begin{aligned} & \int_{t-r_2(t)}^t |v(s)| ds + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \int_{s-r_2(s)}^s |v(u)| du ds \\ & \quad + \int_0^t e^{-\int_s^t v(u) du} (|v(s-r_2(s))(1-r'_2(s)) - a(s)| + |b(s)|) ds \\ & \leq \left(\frac{1}{24} + \frac{\epsilon}{32} \right) + \left(\frac{1}{24} + \frac{\epsilon}{32} \right) + \left(\frac{7}{8} + \epsilon \right) = \frac{23}{24} + \frac{17\epsilon}{16} < 1, \end{aligned}$$

and since $\int_0^t v(s) ds = \int_0^t \frac{1}{32} ds = \frac{1}{32}t$, the conditions of [Corollary 1.7](#) are satisfied. Therefore, the zero solution of (7) is asymptotically stable.

Example 1.11. Consider the following differential equation

$$x'(t) = -a(t)x \left(t - 1 - \frac{1}{3} \cos t \right) + b(t)g(x(t-r_3(t))), \tag{8}$$

where $0 < m_1 \leq a(t)$, $|b(t)| \leq M_2$, $r_3(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous function which satisfies $t - r_3(t) \rightarrow \infty$ as $t \rightarrow \infty$, g satisfies condition (G).

Define $r_2(t) = 1 - \frac{1}{3} \cos t$, if we choose $v(t) = v$ is a constant satisfying $v > \frac{4m_1}{5}$, we have

$$\begin{aligned} & \int_{t-r_2(t)}^t |v(s)| ds + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \int_{s-r_2(s)}^s |v(u)| du ds \\ & \quad + \int_0^t e^{-\int_s^t v(u) du} (|v(s-r_2(s))(1-r'_2(s)) - a(s)| + |b(s)|) ds \\ & \leq \frac{4}{3}v + \frac{4}{3}v + \left(\frac{5}{4} - \frac{m_1 - M_2}{v} \right) = \frac{8}{3}v + \frac{5}{4} - \frac{m_1 - M_2}{v}. \end{aligned}$$

Next, choose v such that $\frac{8}{3}v + \frac{5}{4} - \frac{m_1 - M_2}{v} < 1$, and since $\int_0^t v(s) ds = \int_0^t v ds = vt$, then the conditions of [Corollary 1.7](#) are satisfied. Therefore, the zero solution of (8) is asymptotically stable. For example, if we choose $v = \frac{1}{32}$, $m_1 = \frac{1}{32}$, $M_2 = \frac{1}{64}$, we have $\frac{8}{3}v + \frac{5}{4} - \frac{m_1 - M_2}{v} = \frac{5}{6} < 1$.

Example 1.12. Consider the following differential equation

$$x'(t) = -\frac{1}{128} \left(1 - 2 \sin \frac{5t}{16} \right) x(t - 1). \tag{9}$$

Define $a(t) = \frac{1}{128} (1 - 2 \sin \frac{5t}{16})$, $v(t) = \frac{1}{32}$, we obtain that

$$\begin{aligned} & \int_{t-1}^t |v(s)| ds + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \int_{s-1}^s |v(u)| du ds \\ & + \int_0^t e^{-\int_s^t v(u) du} |v(s - 1) - a(s)| ds < \frac{1}{32} + \frac{1}{32} + 0.104 < 1. \end{aligned}$$

Since $\int_0^t v(s) ds = \frac{t}{32} \rightarrow \infty$ as $t \rightarrow \infty$, the conditions of [Corollary 1.7](#) are satisfied. Therefore, the zero solution of (9) is asymptotically stable.

The third class consists of nonlinear delay differential equations of the form

$$x'(t) = -a(t)f(x(t - r_1(t))) + b(t)g(x(t - r_2(t))), \quad t \geq 0, \tag{10}$$

where $r_1, r_2 : [0, \infty) \rightarrow [0, \infty)$ are continuous functions, $r_0 = \min\{\inf_{t \geq 0}\{t - r_1(t)\}, \inf_{t \geq 0}\{t - r_2(t)\}\}$. The coefficients $a, b : [0, \infty) \rightarrow \mathbb{R}$ and $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Suppose, in addition, that $r_1(t)$ is differentiable, $t - r_j(t) \rightarrow \infty$ as $t \rightarrow \infty$, $j = 1, 2$, and that there exists a continuous function $\tilde{a} : [0, \infty) \rightarrow \mathbb{R}$ such that $a(t) = \tilde{a}(t)(1 - r_1'(t))$ and, finally, that the inverse function $h(t)$ of $t - r_1(t)$ exists. We then have the following result.

Theorem 1.13. Consider the nonlinear delay differential equation (10) and suppose that

- (i) $v(t) : [r_0, \infty) \rightarrow \mathbb{R}$ is a continuous function, $v(t) \geq 0$ as $t \geq 0$;
- (ii) there exists a constant $l > 0$ such that $f(x)$, $x - f(x)$, and $g(x)$ satisfy a Lipschitz condition with constant $L > 0$ on the interval $[-l, l]$;
- (iii) the functions f and g are odd, increasing on $[0, l]$, $x - f(x)$ is nondecreasing on $[0, l]$;
- (iv) there exists an $\alpha \in (0, 1)$ with $\alpha g(l) < (1 - \alpha)f(l)$ such that for $t \geq 0$,

$$\begin{aligned} & \int_0^t e^{-\int_s^t v(u) du} |\tilde{a}(h(s))| ds + \int_0^t e^{-\int_s^t v(u) du} |b(s)| ds \\ & + \int_0^t e^{-\int_s^t v(u) du} |v(s - r_1(s))(1 - r_1'(s))| ds \\ & + \int_{t-r_1(t)}^t |\tilde{a}(h(s)) + v(s)| ds + \int_0^t e^{-\int_s^t v(u) du} v(s) \\ & \times \int_{s-r_1(s)}^s |\tilde{a}(h(u)) + v(u)| du ds \leq \alpha. \end{aligned}$$

Then the zero solution of (10) is stable.

Remark 1.14. Burton [5] studied the special case when $b(t) \equiv 0$ and r_1 is a constant. Following the technique developed by Burton [4], Ding and Li [6] studied stability properties of (10) as well. However, condition (iv) in Ding and Li [6] is restrictive. By introducing a continuous

function $v(t)$ for constructing a fixed point mapping argument, the alternative condition (iv) in [Theorem 1.13](#) is obtained. Note that the condition that the functions $t - r_1(t)$ and $t - r_2(t)$ are strictly increasing is not needed in [Theorem 1.13](#).

Remark 1.15. It is an open problem whether the zero solution of (10) is asymptotically stable. Our method of proof cannot be used to solve this problem. The reason is that if we would add the condition to C_ϕ^l that $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, then C_ϕ^l would no longer be complete under the weighted metric.

2. Proof of Theorem 1.3

In this section, we will prove [Theorem 1.3](#). We start with some preparations. First, (1) can be written as

$$x'(t) = B(t, t - r(t))(1 - r'(t))g(t - r(t), x(t - r(t))) + \frac{d}{dt} \int_{t-r(t)}^t B(t, s)g(s, x(s)) ds, \tag{11}$$

where $B(t, s) := \int_t^s a(u, s) du$. If we multiply both sides of (11) by $e^{\int_0^t v(s) ds}$, integrate from 0 to t , and perform an integration by parts, then we obtain

$$x(t) = \left\{ \phi(0) - \int_{-r(0)}^0 [v(s) + B(0, s)]g(s, \phi(s)) ds \right\} e^{-\int_0^t v(s) ds} + \int_{t-r(t)}^t [v(s) + B(t, s)]g(s, \phi(s)) ds - \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r(s)}^s [v(u) + B(s, u)]g(u, x(u)) du ds + \int_0^t e^{-\int_s^t v(u) du} v(s)[x(s) - g(s, x(s))] ds + \int_0^t e^{-\int_s^t v(u) du} [v(s - r(s)) + B(s, s - r(s))] \times (1 - r'(s))g(s - r(s), x(s - r(s))) ds. \tag{12}$$

By (ii), we choose a common Lipschitz constant L for $g(s, x)$ and $x - g(s, x)$ on $[-l, l]$. For $t \in [r_0, \infty)$ and a constant $k > 4$, we define

$$h(t) = kL \int_0^t [v(u) + q(u) + p(u)] du, \tag{13}$$

where q is as defined in (iv) of [Theorem 1.3](#) and $p(u) = [v(u - r(u)) + B(u, u - r(u))](1 - r'(u))$. Let C be the space of all continuous functions $\varphi : [r_0, \infty) \rightarrow \mathbb{R}$ such that $|\varphi|_h := \sup \{ |\varphi(t)|e^{-h(t)} : t \in [r_0, \infty) \} < \infty$, where h is given by (13). Then $(C, |\cdot|_h)$ is a Banach space, which can be verified by Cauchy’s criterion for uniform convergence. Thus (C, d) is a complete metric space, where d denotes the induced metric: $d(\varphi, \eta) = |\varphi - \eta|_h$ for $\varphi, \eta \in C$. Define

$$C_\phi^l = \left\{ \varphi \mid \varphi \in C, \|\varphi\| = \sup_{t \geq r_0} |\varphi(t)| \leq l, \varphi(t) = \phi(t) \text{ for } t \in [r_0, 0] \right\},$$

where $\phi : [r_0, 0] \rightarrow [-l, l]$ is a given continuous initial function. Then C_ϕ^l is a closed subspace of C and hence a complete metric space as well with the metric inherited from C .

Lemma 2.1. *Let $x \in C_\phi^l$ and define the operator P by $(Px)(t) = \phi(t)$ for $t \in [r_0, 0]$ and, for $t \geq 0$ we define $(Px)(t)$ by the right hand side of (12). If the conditions (i)–(viii) in Theorem 1.3 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, we have that $P : C_\phi^l \rightarrow C_\phi^l$ and P is a contraction mapping.*

Proof. First of all, given $\varphi \in C_\phi^l$ we show that $P\varphi \in C_\phi^l$. Let $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$ be a continuous function, where $\delta > 0$ satisfies

$$\delta + W(\delta) \int_{-r(0)}^0 |v(u) + B(0, u)| du \leq w(l) - \alpha W(l). \tag{14}$$

Such δ exists, since $W(0) = 0$ and W is continuous on $[0, l]$. Note that $w(l) - \alpha W(l) > 0$ by (iii) and (viii). By (iv), $w(l) \leq l$. For any $\varphi \in C_\phi^l$, by (14), we have $|(P\varphi)(t)| = |\phi(t)| < l$ for $t \in [r_0, 0]$. Now we consider $(P\varphi)(t)$ for $t > 0$. By (i) and (iii), $|g(s, x)| \leq W(l)$ for $x \in [-l, l]$ and $t \geq r_0$, thus using (iii) and (v), we obtain

$$\begin{aligned} |P(\varphi)(t)| &\leq \delta + W(\delta) \int_{-r(0)}^0 |v(u) + B(0, u)| du \\ &\leq w(l) - \alpha W(l) + W(l) \int_{t-r(t)}^t |v(u) + B(t, u)| du \\ &\quad + W(l) \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r(s)}^s |v(u) + B(s, u)| du ds \\ &\quad + W(l) \int_0^t e^{-\int_s^t v(u) du} |v(s - r(s)) + B(s, s - r(s))| |1 - r'(s)| ds \\ &\quad + (l - w(l)) \int_0^t e^{-\int_s^t v(u) du} v(s) ds \\ &\leq w(l) - \alpha W(l) + \alpha W(l) + l - w(l) = l. \end{aligned} \tag{15}$$

So $|P(\varphi)(t)| \leq l$ for $t \in [r_0, \infty)$. Therefore, $P\varphi \in C_\phi^l$.

Next, we show that P is a contraction mapping on C_ϕ^l . Suppose that $\varphi, \eta \in C_\phi^l$,

$$\begin{aligned} |P\varphi(t) - P\eta(t)|e^{-h(t)} &\leq \int_{t-r(t)}^t e^{-kL \int_u^t [v(s)+q(s)] ds} |v(u) + B(t, u)| L |\varphi(u) - \eta(u)| e^{-h(u)} du \\ &\quad + \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r(s)}^s e^{-kL \int_u^s [v(\theta)+q(\theta)] d\theta} |v(u) \\ &\quad + B(s, u)| \cdot L |\varphi(u) - \eta(u)| e^{-h(u)} du ds \\ &\quad + \int_0^t e^{-kL \int_s^t p(u) du} p(s) L |\varphi(s - r(s)) - \eta(s - r(s))| e^{-h(s-r(s))} ds \\ &\quad + \int_0^t e^{-(kL+1) \int_s^t v(u) du} v(s) L |\varphi(s) - \eta(s)| e^{-h(s)} ds, \end{aligned}$$

since $|v(u) + B(t, u)| \leq v(u) + q(u)$, for $t - r(t) \leq u \leq t$, we have $|P\varphi - P\eta|_h \leq (\frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL}) L |\varphi - \eta|_h = \frac{4}{k} |\varphi - \eta|_h < |\varphi - \eta|_h$ for $k > 4$. This shows that P is indeed a contraction mapping. \square

We are now ready to prove [Theorem 1.3](#).

Proof. By the contraction mapping principle, P has a unique fixed point $x \in C^l_\phi$, which is by construction a solution of (1) on $[0, \infty)$ and $|x(t)| \leq l$ for $t \geq r_0$. Hence $x(t)$ is the only continuous function satisfying (1) for $t \geq 0$ with $x(t) = \phi(t)$ on $[r_0, 0]$.

Let $\epsilon > 0$ be given and choose $m > 0$, such that $m < \min\{\epsilon, l\}$, replacing l with m in (15), we see that there is $\delta > 0$ such that $|\phi| \leq m < \epsilon$ for $t \geq r_0$. Hence, the zero solution of (1) is stable. This completes the proof of [Theorem 1.3](#). \square

3. Proof of [Theorem 1.6](#)

In this section, we will prove [Theorem 1.6](#). We start with some preparations. First define

$$S^l_\phi = \left\{ x \mid x \in C([r_0, \infty), \mathbb{R}), \|x\| = \sup_{t \geq r_0} |x(t)| \leq l, x(t) = \phi(t) \text{ for } t \in [r_0, 0], \right. \\ \left. \text{and } x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \right\}.$$

If we define the metric $\rho(x, y) = \|x - y\| = \sup_{t \geq r_0} \{|x(t) - y(t)|\}$, then S^l_ϕ becomes a complete metric space.

If we multiply both sides of Eq. (4) by $e^{\int_0^t (h_1(s)+h_2(s)) ds}$, integrate from 0 to t , and perform an integration by parts, we obtain

$$x(t) = \left\{ \phi(0) - \frac{c(0)}{1 - r'_1(0)} \phi(-r_1(0)) - \sum_{j=1}^2 \int_{-r_j(0)}^0 h_j(s) \phi(s) ds \right\} e^{-\int_0^t (h_1(s)+h_2(s)) ds} \\ + \frac{c(t)}{1 - r'_1(t)} x(t - r_1(t)) \\ + \sum_{j=1}^2 \int_{t-r_j(t)}^t h_j(s) x(s) ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} [h_2(s) - r_2(s)] \\ \times (1 - r'_2(s) - a(s)) x(s - r_2(s)) ds \\ + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} [h_1(s) - r_1(s)] (1 - r'_1(s) - k(s)) x(s - r_1(s)) ds \\ + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} \int_{s-r_3(s)}^s g(s, x(u)) d\mu(s, u) ds \\ - \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} (h_1(s) + h_2(s)) \int_{s-r_j(s)}^s h_j(u) x(u) du ds. \tag{16}$$

Lemma 3.1. Let $x \in S^l_\phi$ and define an operator P by $(Px)(t) = \phi(t)$ for $t \in [r_0, 0]$ and, for $t \geq 0$, $(Px)(t)$ is defined by the right hand side of (16). If conditions (i)–(iv) in [Theorem 1.6](#) are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, we have that $P : S^l_\phi \rightarrow S^l_\phi$ and P is a contraction with respect to the metric defined on S^l_ϕ .

Proof. Let $J = \sup_{t \geq 0} \left\{ e^{-\int_0^t (h_1(s)+h_2(s)) ds} \right\}$, by (iv), J is well defined. Suppose that (iv) holds. Let $x \in S^l_\phi$, it is clear that $Px \in C([r_0, \infty), \mathbb{R})$. Hence, by (ii) and condition (G), we

have

$$\begin{aligned}
 |(Px)(t)| &\leq \|\phi\| \left(1 + \left| \frac{c(0)}{1-r'_1(0)} \right| + \sum_{j=1}^2 \int_{-r_j(0)}^0 |h_j(s)| ds \right) e^{-\int_0^t (h_1(s)+h_2(s)) ds} \\
 &+ l \left\{ \left| \frac{c(t)}{1-r'_1(t)} \right| + \sum_{j=1}^2 \int_{t-r_j(t)}^t |h_j(s)| ds \right. \\
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} (V_{[s-r_3(s),s]}(\mu(s, \cdot))) ds \\
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_2(s-r_2(s))(1-r'_2(s)) - a(s)| ds \\
 &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s-r_1(s))(1-r'_1(s)) - k(s)| ds \\
 &\left. + \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \int_{s-r_j(s)}^t |h_j(u)| du ds \right\} \\
 &\leq \|\phi\| \left(1 + \left| \frac{c(0)}{1-r'_1(0)} \right| + \sum_{j=1}^2 \int_{-r_j(0)}^0 |h_j(s)| ds \right) J + l\alpha.
 \end{aligned}$$

From this estimate, it follows that if $\delta := \frac{(1-\alpha)l}{\left(1 + \frac{|c(0)|}{|1-r'_1(0)|} + \sum_{j=1}^2 \int_{-r_j(0)}^0 |h_j(s)| ds\right)J}$, then $\|\phi\| \leq \delta$ implies that $|(Px)(t)| \leq l$.

Next, we show that $(Px)(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $x(t) \rightarrow 0$ and $t - r_j(t) \rightarrow \infty$ as $t \rightarrow \infty$, for each $\epsilon > 0$, there exists a $T_1 > 0$ such that $t > T_1$ implies $|x(t - r_j(t))| < \epsilon$, for $j = 1, 2$. Thus for $t \geq T_1$,

$$\begin{aligned}
 |I_2| &:= \left| \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u)) du} (h_1(s) + h_2(s)) \left(\int_{s-r_j(s)}^s h_j(u)x(u) du \right) ds \right| \\
 &\leq l \sum_{j=1}^2 \int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \left(\int_{s-r_j(s)}^s |h_j(u)| du \right) ds \\
 &+ \epsilon \sum_{j=1}^2 \int_{T_1}^t e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \left(\int_{s-r_j(s)}^s |h_j(u)| du \right) ds. \tag{17}
 \end{aligned}$$

By condition (iv), there exists $T_2 > T_1$ such that $t > T_2$ implies

$$l \sum_{j=1}^2 \int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u)) du} |h_1(s) + h_2(s)| \left(\int_{s-r_j(s)}^s |h_j(u)| du \right) ds < \epsilon.$$

Applying (ii), we have $|I_2| \rightarrow 0$ as $t \rightarrow \infty$. Since $x(t) \rightarrow 0$ and $t - r_3(t) \rightarrow \infty$ as $t \rightarrow \infty$, for each $\epsilon > 0$, there exists a $T_3 > 0$ such that $t > T_3$ implies $|x(t - r_3(t))| < \epsilon$. Thus for $t \geq T_3$,

$$\begin{aligned}
 |I_3| &:= \left| \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))du} \int_{s-r_3(s)}^s g(s, x(u)) d\mu(s, u) ds \right| \\
 &\leq l \int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))du} (V_{[s-r_3(s),s]}(\mu(s, \cdot))) ds \\
 &\quad + \epsilon \int_{T_3}^t e^{-\int_s^t (h_1(u)+h_2(u))du} (V_{[s-r_3(s),s]}(\mu(s, \cdot))) ds.
 \end{aligned} \tag{18}$$

By condition (iv), there exists $T_4 > T_3$ such that $t > T_4$ implies $l \int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))du} (V_{[s-r_3(s),s]}(\mu(s, \cdot))) ds < \epsilon$. Applying (ii), we have $|I_3| \rightarrow 0$ as $t \rightarrow \infty$. Similarly, we can show that the rest terms approach zero as $t \rightarrow \infty$, which yields $(Px)(t) \rightarrow 0$ as $t \rightarrow \infty$.

Finally, we show that P is a contraction mapping with contraction constant α . In fact, for $\varphi, \eta \in S_\phi^l$,

$$\begin{aligned}
 |(P\varphi)(t) - (P\eta)(t)| &\leq \left\{ \left| \frac{c(t)}{1 - r_1'(t)} \right| + \sum_{j=1}^2 \int_{t-r_j(t)}^t |h_j(s)| ds \right. \\
 &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))du} (V_{[s-r_3(s),s]}(\mu(s, \cdot))) ds \\
 &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))du} |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)| ds \\
 &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))du} |h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| ds \\
 &\quad + \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))du} |h_1(s) + h_2(s)| \\
 &\quad \left. \times \int_{s-r_j(s)}^t |h_j(u)| du ds \right\} \|\varphi - \eta\| \leq \alpha \|\varphi - \eta\|.
 \end{aligned}$$

Thus, $P : S_\phi^l \rightarrow S_\phi^l$ and P is a contraction mapping. \square

We are now ready to prove [Theorem 1.6](#).

Proof. Let P be defined as in [Lemma 3.1](#). By the contraction mapping principle, P has a unique fixed point x in S_ϕ^l which is by construction a solution of (4) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let $\epsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \epsilon\}$. By considering S_ϕ^m , we obtain that there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4) with $x(t) = \phi(t)$ on $[r_0, 0]$ satisfies $|x(t)| \leq m < \epsilon$ for all $t \geq r_0$. This shows that the zero solution of (4) is asymptotically stable if (iv) holds.

Conversely, we suppose that condition (iv) fails. Then by (iii), there exists a sequence $\{t_n\}$ with $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \int_0^{t_n} (h_1(s) + h_2(s)) ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \leq \int_0^{t_n} (h_1(s) + h_2(s)) ds \leq M, \quad \text{for all } n \geq 1. \tag{19}$$

To simplify our expressions, we define

$$w(s) := |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)| + |h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| + V_{[s-r_3(s),s]}(\mu(s, \cdot)) + |(h_1(s) + h_2(s))| \sum_{j=1}^2 \int_{s-r_j(s)}^s |h_j(u)| du, \quad s \geq 0.$$

By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} (h_1(u)+h_2(u)) du} w(s) ds \leq \alpha, \quad \text{for all } n \geq 1. \tag{20}$$

Combining (19) and (20), we have $\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds \leq \alpha e^{\int_0^{t_n} (h_1(u)+h_2(u)) du} \leq \alpha e^M$, for all $n \geq 1$, which yields that the sequence $\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds$ is bounded. Therefore, there exists a convergent subsequence and without loss of generality, we can assume that $\lim_{k \rightarrow \infty} \int_0^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds = \gamma$, for some $\gamma \in \mathbb{R}^+$. Choose a positive integer \bar{k} so large that $\lim_{k \rightarrow \infty} \int_0^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds \leq \frac{\delta_0}{4J}$ for all $n_k > n_{\bar{k}}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$. Now, we consider the solution $x(t) = x(t, t_{n_{\bar{k}}}, \phi)$ of (4) with $x(t_{n_{\bar{k}}}) = \delta_0$ and $x(s) \leq \delta_0$ for $t_{n_{\bar{k}}} - r_0 \leq s \leq t_{n_{\bar{k}}}$, and we may choose ϕ such that $|x(t)| \leq 1$ for $t \geq t_{n_{\bar{k}}}$ and

$$x(t_{n_{\bar{k}}}) - \frac{c(t_{n_{\bar{k}}})}{1 - r_1'(t_{n_{\bar{k}}})} x(t_{n_{\bar{k}}} - r_1(t_{n_{\bar{k}}})) - \sum_{j=1}^2 \int_{t_{n_{\bar{k}}} - r_j(t_{n_{\bar{k}}})}^{t_{n_{\bar{k}}}} h_j(s)x(s) ds \geq \frac{1}{2} \delta_0. \tag{21}$$

So, it follows from (21) with $x(t) = (Px)(t)$ that for $k \geq \bar{k}$,

$$\begin{aligned} & \left| x(t_{n_k}) - \frac{c(t_{n_k})}{1 - r_1'(t_{n_k})} x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^2 \int_{t_{n_k} - r_j(t_{n_k})}^{t_{n_k}} h_j(s)x(s) ds \right| \\ & \geq \frac{1}{2} \delta_0 e^{-\int_{t_{n_{\bar{k}}}}^{t_{n_k}} (h_1(u)+h_2(u)) du} - \int_{t_{n_{\bar{k}}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}} (h_1(u)+h_2(u)) du} w(s) ds \\ & = e^{-\int_{t_{n_{\bar{k}}}}^{t_{n_k}} (h_1(u)+h_2(u)) du} \left(\frac{1}{2} \delta_0 - e^{-\int_0^{t_{n_{\bar{k}}}} (h_1(u)+h_2(u)) du} \int_{t_{n_{\bar{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds \right) \\ & \geq e^{-\int_{t_{n_{\bar{k}}}}^{t_{n_k}} (h_1(u)+h_2(u)) du} \left(\frac{1}{2} \delta_0 - J \int_{t_{n_{\bar{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u)) du} w(s) ds \right) \geq \frac{1}{4} \delta_0 e^{-2M} > 0. \tag{22} \end{aligned}$$

On the other hand, suppose that $x(t) = x(t, t_{n_{\bar{k}}}, \phi) \rightarrow 0$ as $t \rightarrow \infty$. Since $t_{n_k} - r_j(t_{n_k}) \rightarrow \infty$ as $k \rightarrow \infty$, $j = 1, 2$ and (ii) holds, this would imply that $x(t_{n_k}) - \frac{c(t_{n_k})}{1 - r_1'(t_{n_k})} x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^2 \int_{t_{n_k} - r_j(t_{n_k})}^{t_{n_k}} h_j(s)x(s) ds \rightarrow 0$ as $k \rightarrow \infty$, which contradicts the estimate. Hence condition (iv) is necessary for the asymptotic stability of the zero solution of (4). This completes the proof of Theorem 1.6. \square

4. Proof of Theorem 1.13

In this section, we will prove Theorem 1.13. We start with some preparations. Eq. (10) can be written in the following equivalent form

$$x'(t) = -\tilde{a}(h(t))f(x(t)) + \frac{d}{dt} \int_{t-r_1(t)}^t \tilde{a}(h(s))f(x(s)) ds + b(t)g(x(t - r_2(t))). \tag{23}$$

If we multiply both sides of (23) by $e^{\int_0^t v(s) ds}$, integrate from 0 to t , and perform an integration by parts, then we obtain

$$\begin{aligned}
 x(t) = & \left\{ \phi(0) - \int_{-r_1(0)}^0 [\tilde{a}(h(s)) + v(s)] f(\phi(s)) ds \right\} e^{-\int_0^t v(s) ds} \\
 & + \int_0^t e^{-\int_s^t v(u) du} v(s) [x(s) - f(x(s))] ds \\
 & - \int_0^t e^{-\int_s^t v(u) du} \tilde{a}(h(s)) f(x(s)) ds \\
 & - \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r_1(s)}^s [\tilde{a}(h(u)) + v(u)] f(x(u)) du ds \\
 & + \int_0^t e^{-\int_s^t v(u) du} v(s - r_1(s)) (1 - r_1'(s)) f(x(s - r_1(s))) ds \\
 & + \int_{t-r_1(t)}^t [\tilde{a}(h(s)) + v(s)] f(x(s)) ds \\
 & + \int_0^t e^{-\int_s^t v(u) du} b(s) g(x(s - r_2(s))) ds.
 \end{aligned} \tag{24}$$

Let C be the weighted space of all continuous functions $\varphi : [r_0, \infty) \rightarrow \mathbb{R}$ with $|\varphi|_q := \sup\{|\varphi(t)|e^{-q(t)} : t \in [r_0, \infty)\} < \infty$, where the weight function $q : [r_0, \infty) \rightarrow \mathbb{R}$ is defined as follows

$$q(t) = \begin{cases} 1 & \text{for } t \in [r_0, 0], \\ d \int_0^t [v(s) + |\tilde{a}(h(s))| + |b(s)| + w(s)] ds & \text{for } t \in [0, \infty), \end{cases}$$

where $d > 5$ is a constant, and

$$w(s) = \begin{cases} 0 & \text{for } s \in [r_0, 0], \\ |v(s - r_1(s))(1 - r_1'(s))| & \text{for } s \in [0, \infty). \end{cases}$$

The space $(C, |\cdot|_q)$ becomes a Banach space, which can be verified by Cauchy’s criterion for uniform convergence. Define

$$C_\phi^l = \left\{ \varphi : [r_0, \infty) \rightarrow \mathbb{R} \mid \varphi \in C, \|\varphi\| = \sup_{t \geq r_0} |\varphi(t)| \leq l, \varphi(t) = \phi(t) \text{ for } t \in [r_0, 0] \right\}$$

where $\phi : [r_0, 0] \rightarrow [-l, l]$ is a given continuous initial function. Then C_ϕ^l is a closed subset of $(C, |\cdot|_q)$ and hence a complete metric space with the metric inherited from C .

Lemma 4.1. *Let $x \in C_\phi^l$. Define the operator P by $(Px)(t) = \phi(t), t \in [r_0, 0]$, and, for $t \geq 0$, define $(Px)(t)$ by the right hand side of (24). If the conditions (i)–(iv) in Theorem 1.13 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$, we have that $P : C_\phi^l \rightarrow C_\phi^l$ and P is a contraction.*

Proof. Since f is odd and satisfies a Lipschitz condition on $[-l, l]$, and $f(0) = 0$, we can choose a $\delta < l$ that satisfies

$$\delta + f(\delta) \int_{-r_1(0)}^0 |\tilde{a}(h(s)) + v(s)| ds \leq (1 - \alpha)f(l) - \alpha g(l).$$

Let $\phi : [r_0, 0] \rightarrow (-\delta, \delta)$ be a continuous function. Thus $|\phi(t)| \leq l$ for $t \in [r_0, 0]$. Now we show for such ϕ , $P : C_\phi^l \rightarrow C_\phi^l$. In fact, for arbitrary $x \in C_\phi^l$, it follows from the conditions in [Theorem 1.13](#) that we have for $t > 0$,

$$\begin{aligned} |(Px)(t)| &\leq \delta + f(\delta) \int_{-r_1(0)}^0 |\tilde{a}(h(s)) + v(s)| ds + f(l) \left\{ \int_0^t e^{-\int_s^t v(u) du} |\tilde{a}(h(s))| ds \right. \\ &\quad + \int_{t-r_1(t)}^t |\tilde{a}(h(s)) + v(s)| ds \\ &\quad + \int_0^t e^{-\int_s^t v(u) du} |v(s)| \int_{s-r_1(s)}^s |\tilde{a}(h(u)) + v(u)| du ds \\ &\quad \left. + \int_0^t e^{-\int_s^t v(u) du} |v(s - r_1(s))(1 - r_1'(s))| ds \right\} \\ &\quad + g(l) \int_0^t e^{-\int_s^t v(u) du} |b(s)| ds + (l - f(l)) \int_0^t e^{-\int_s^t v(u) du} v(s) ds \\ &\leq (1 - \alpha)f(l) - \alpha g(l) + \alpha f(l) + \alpha g(l) + l - f(l) = l. \end{aligned}$$

Hence, $|(Px)(t)| \leq l$ for $t \in [r_0, \infty)$. Therefore $Px \in C_\phi^l$.

Next, we will show that P is a contraction mapping in C_ϕ^l . For $\varphi, \eta \in C_\phi^l$,

$$\begin{aligned} |(P\varphi)(t) - (P\eta)(t)| e^{-q(t)} &\leq \int_0^t e^{-dL \int_s^t [v(u) + |\tilde{a}(h(u))|]} du [v(s) + |\tilde{a}(h(s))|] L|\varphi(s) - \eta(s)| e^{-q(s)} ds \\ &\quad + \int_0^t e^{-\int_s^t v(u) du} v(s) \int_{s-r_1(s)}^s e^{-dL \int_s^t [v(s) + |\tilde{a}(h(s))|]} ds [|\tilde{a}(h(u))| + v(u)] \\ &\quad \times L|\varphi(u) - \eta(u)| e^{-q(u)} du ds \\ &\quad + \int_0^t e^{-dL \int_s^t w(u) du} w(s) L|\varphi(s - r_1(s)) - \eta(s - r_1(s))| e^{-q(s-r_1(s))} ds \\ &\quad + \int_{t-r_1(t)}^t e^{-dL \int_s^t [v(u) + |\tilde{a}(h(u))|]} du [|\tilde{a}(h(s))| + v(s)] L|\varphi(s) - \eta(s)| e^{-q(s)} ds \\ &\quad + \int_0^t e^{-dL \int_s^t |b(u)| du} |b(s)| L|\varphi(s - r_2(s)) - \eta(s - r_2(s))| e^{-q(s-r_2(s))} ds. \end{aligned}$$

Thus $|P\varphi - P\eta|_q \leq \frac{5}{d} |\varphi - \eta|_q$. Since $d > 5$, we conclude that P is a contraction on $(C_\phi^l, |\cdot|_q)$. \square

We are now ready to prove [Theorem 1.13](#).

Proof. By the contraction mapping principle, P has a unique fixed point x in C_ϕ^l , which is by construction a solution of (10) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $|x(t)| \leq l$.

Let $\epsilon > 0$ be given. Then, we choose $m > 0$ so that $m < \min\{l, \epsilon\}$. By considering C_ϕ^m , we obtain existence of a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (10) with $x(t) = \phi(t)$ on $[r_0, 0]$ satisfies $|x(t)| \leq m < \epsilon$ for all $t \geq r_0$. This shows that the zero solution of (10) is stable. This completes the proof of [Theorem 1.13](#). \square

References

[1] L.C. Becker, T.A. Burton, Stability, fixed points and inverse of delays, *Proc. Roy. Soc. Edinburgh* 136A (2006) 245–275.

- [2] T.A. Burton, Furumochi, Tetsuo, Fixed points and problems in stability theory for ordinary and functional differential equations, *Dynam. Systems Appl.* 10 (2001) 89–116.
- [3] T.A. Burton, Stability by fixed point theory or Liapunov's theory: a comparison, *Fixed Point Theory* 4 (2003) 15–32.
- [4] T.A. Burton, Stability by fixed point methods for highly nonlinear delay equations, *Fixed Point Theory* 5 (2004) 3–20.
- [5] T.A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publication, New York, 2006.
- [6] L.M. Ding, X. Li, Z.X. Li, Fixed points and stability in nonlinear equations with variable delays, *Fixed Point Theory Appl.* (2010) 14 Art. ID 195916.
- [7] A. Djoudi, R. Khemis, Fixed points techniques and stability for neutral nonlinear differential equations with unbounded delays, *Georgian Math. J.* 13 (2006) 25–34.
- [8] C.H. Jin, J.W. Luo, Fixed points and stability in neutral differential equations with variable delays, *Proc. Amer. Math. Soc.* 136 (2008) 909–918.
- [9] C.H. Jin, J.W. Luo, Stability in functional differential equations established using fixed point theory, *Nonlinear Anal.* 68 (2008) 3307–3315.
- [10] C.H. Jin, J.W. Luo, Stability of an integro-differential equation, *Comput. Math. Appl.* 57 (2009) 1080–1088.
- [11] Y.N. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed-point theory, *Math. Comput. Modelling* 40 (2004) 691–700.