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Existence and exponential stability of a class of impulsive neutral stochastic partial differential equations with delays and Poisson jumps

Guiling Chen [a,*], Onno van Gaans [b], Sjoerd Verduyn Lunel [c]

[a] School of Mathematics, Southwest Jiaotong University, Chengdu 610031, PR China
[b] Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands
[c] Mathematical Institute, Utrecht University, P.O. Box 80010, 3508 TA, Utrecht, The Netherlands

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ABSTRACT

In this paper, the existence, uniqueness and exponential stability of mild solution for a class of impulsive neutral stochastic partial differential equations with delays and Poisson jumps are studied. The existence and uniqueness of mild solution are studied by means of successive approximations, and the exponential stability in pth moment of mild solution is investigated by employing an appropriate impulsive-integral inequality. An example is given to illustrate our main results.

1. Introduction and preliminaries

Stochastic partial functional differential equations have received an increasing interest recently due to their wide applications in biology, chemistry, physics and economical systems. The existence, uniqueness and stability properties of such equations have been extensively studied by many authors, see, for example, Friedman (1975), Ikeda and Watanabe (1981), Mao (1997), Caraballo and Liu (1999).

On the other hand, impulsive control systems arise naturally in a wide variety of applications involving impulsive control for dosage supply in pharmacokinetics, ecosystems management, stabilization and synchronization in chaotic secure communication systems and other chaos systems. Therefore, an interesting subject is to discuss the existence, uniqueness and stability of mild solutions of stochastic partial functional differential equations.

The techniques dealing with the existence and uniqueness of stochastic partial functional differential equations are mainly by using the successive approximation method (Angura and Vinodkumar, 2010; Jiang and Shen, 2011; Chen, 2010a) and the Banach fixed point method (Sakthivel and Luo, 2009a, b).

In the case of differential equations with delays, in particular when we are concerned with the mild solutions of stochastic partial differential equations, the Lyapunov's second method, although is usually regarded as a powerful tool to study the stability and boundedness, is not suitable to consider such a problem. A difficulty is that mild solutions do not have stochastic differentials, so that one cannot apply the Itö formula to them. Sakthivel and Luo (2009a, b) have investigated the asymptotic

* Corresponding author.
E-mail addresses: guiling@home.swjtu.edu.cn (G. Chen), vangaans@math.leidenuniv.nl (O. van Gaans), S.M.VerduynLunel@uu.nl (S.V. Lunel).

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stability of mild solutions of impulsive stochastic partial differential equations with delays by using fixed point methods. Cui et al. (2011) also applied fixed point methods to deal with exponential stability of mild solutions of stochastic partial differential equations with delays. Recently, Chen (2010b) studied the exponential stability of a class of stochastic partial differential equations with delays by establishing an impulsive-integral inequality, and it turns out that it is an effective way to study exponential stability of mild solutions of impulsive stochastic partial differential equations with delays.

Comparing with stochastic partial differential equations with delays, many dynamical systems not only depend on present and past states but also involve derivatives with delays, and neutral stochastic partial differential equations with delays are often used to describe such systems. However, to the best of our knowledge, there are only a few work, see, for example, Jiang and Shen (2012) and Luo and Taniguchi (2009), discussing the stability of mild solutions to neutral stochastic partial differential equations with delays.

In this paper, we will study existence, uniqueness and exponential stability of a class of impulsive neutral stochastic partial differential equations with delays and Poisson jumps by using a successive approximation method and an appropriate impulsive-integral inequality.

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all P-null sets). Let X, Y be two real separable Hilbert spaces which are both equipped with the norm $\|\cdot\|$, let $\mathcal{L}(Y, X)$ denote the space of all bounded linear operators from Y into X. Suppose $\{p(t), t \geq 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Poisson point process taking values in measurable space $(U, \mathcal{B}(U))$. The random measure N_p defined by $N_p((0,t]\times \Lambda) := \sum_{s\in(0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(U)$ is called the Poisson random measure induced by $p(\cdot)$, thus, we can define the measure $\tilde{N}$ by $\tilde{N}(dt, dy) = N_p(dt, dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure. Let $w = (w_t)_{t\geq 0}$, independent of the Poisson point process, be a Y-valued Wiener process defined on $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ with covariance operator Q, that is

$$\mathbb{E}\langle w(t), x\rangle_Y \langle w(s), y\rangle_K = (t\wedge s)\langle Qx, y\rangle_Y, \quad x, y \in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. Furthermore, $\mathcal{L}_2^0$ denotes the space of all Q-Hilbert–Schmidt operators from Y to X with the norm

$$\|\xi\|^2_{\mathcal{L}_2^0} : tr(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}(K, X).$$

For the construction of a stochastic integral in a Hilbert space, see Da Prato and Zabczyk (1992).

For Borel set $z \in \mathcal{B}(U - \{0\})$, we consider a class of impulsive neutral stochastic partial differential equations with delays and Poisson jumps, which is described by

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t)) \\ \qquad\qquad + \displaystyle\int_Z h(t, x(t-\sigma(t)), y)\tilde{N}(dt, dy), \qquad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k x(t_k^-), \quad t = t_k, \quad k = 1, 2, \ldots, \\ x_0(\vartheta) = \phi \in PC, \quad \vartheta \in [-\tau, 0], \quad a.s. \end{cases} \tag{1}$$

where ϕ is $\mathcal{F}_0$-measurable and the functions $\tau(t), \delta(t), \rho(t), \sigma(t) : [0, \infty) \to [0, \tau]$ $(\tau > 0)$ are continuous functions. Let $PC \equiv PC([-\tau, 0]; X)$ be the space of all almost surely bounded $\mathcal{F}_0$ -measurable functions from $[-\tau, 0]$ into X that are continuous everywhere except for a finite number of points s at which $\xi(s)$ and the left limit of $\xi(s)$ exists and $\xi(s^+) = \xi(s)$ as usual, equipped with the supremum norm $\|\phi\|_0 = essup_{\omega\in\Omega} \sup_{t\in[-\tau,0]} \|\phi(t)(\omega)\|$; A is the infinitesimal generator of a strongly continuous semigroup of linear operators $S(t)(t \geq 0)$ in X, refer to Pazy (1983) for detailed information. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time $t = t_k$, $k = 1, 2, \ldots$, respectively. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump in the state x at time t_k with $I_k(\cdot) : X \to X (k = 1, 2, \ldots)$ determining the size of the jump; $u, f : [0, \infty) \times X \to X$, $g : [0, \infty) \times X \to \mathcal{L}_2^0(Y, X)$, $h : [0, \infty) \times X \times U \to X$ are given functions to be specified later.

Definition 1.1. An X-valued stochastic process $x(t)$, $t \in [0, +\infty)$, is called the mild solution of (1) if

(a) $x(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$.
(b) $x(t)$ has càdlàg paths on $[0, +\infty)$ almost surely, and for $t \in [0, +\infty)$, $x(t)$ satisfies the following integral equation

$$\begin{aligned} x(t) = {} & S(t)(\phi(0) + u(0, \phi)) - u(t, x(t-\tau(t))) - \int_0^t AS(t-s)u(s, x(s-\tau(s)))\,ds \\ & + \int_0^t S(t-s)f(s, x(s-\delta(s)))\,ds + \int_0^t S(t-s)g(s, x(s-\rho(s)))\,dw(s) \\ & + \int_0^t \int_Z S(t-s)h(s, x(s-\sigma(s)), y)\tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k)I_k x(t_k^-) \end{aligned} \tag{2}$$

and

$$x_0(\cdot) = \phi \in PC, \quad a.s.$$

To obtain our main results, we impose the following assumptions:

(A1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X such that $0 \in \rho(-A)$, the resolvent set of $-A$, and $S(t)$ is uniformly bounded,

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for some constants $\gamma, M > 0$.

(A2) The mappings $f(t, \cdot)$, $\sigma(t, \cdot)$ and $h(t, \cdot)$ satisfy the following Lipschitz and linear growth conditions: for any $x, y \in H$ and $t \geq 0$,

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leq L_1\|x-y\|, \quad L_1 > 0, \\ \|g(t,x) - g(t,y)\| &\leq L_2\|x-y\|, \quad L_2 > 0, \\ \int_Z \|h(t,x,z) - h(t,y,z)\|^2 \nu(dz) &\leq L_3^2\|x-y\|^2, \quad L_3 > 0, \end{aligned}$$

we further assume that $f(t, 0) = g(t, 0) = h(t, 0, z) = 0$ for all $t \geq 0, z \in Z$. Then Eq. (1) has a trivial solution when $\phi = 0$.

(A3) The mapping $(-A)^\alpha u(t, \cdot)$ satisfies the uniformly Lipschitz condition: there exists a positive constant $K > 0$, such that for any $x, y \in X$

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\| \leq K\|x-y\|, \quad u(t,0) = 0, \quad t \geq 0,$$

for $\alpha \in (1/p, 1](p \geq 2)$ and $u(t, \cdot) \in D((-A)^\alpha)$. Moreover, for $\alpha \in (1/p, 1](p \geq 2)$, $\kappa = \|(-A)^\alpha\|K < 1$.

(A4) $I_k \in C(X, X)$ and there exists a positive constant q_k such that $\|I_k(x) - I_k(y)\| \leq q_k\|x - y\|$ and $I_k(0) = 0$, $k = 1, 2, 3, \ldots,$ for each $x, y \in X$ and $\sum_{k=1}^{+\infty} q_k < +\infty$.

For the definition and properties of fractional powers of $-A$ we refer to Pazy (1983).

Lemma 1.2 (*Theorem 6.13, Pazy, 1983*)**.** *Suppose that the assumption* (A1) *holds, then for any $\beta \in (0, 1]$, we have that*

(i) *for each $x \in \mathcal{D}((-A)^\beta)$,*

$$S(t)(-A)^\beta x = (-A)^\beta S(t)x,$$

(ii) *there exists a positive constant $M_\beta > 0$ such that*

$$\left\|(-A)^\beta S(t)\right\| \leq M_\beta t^{-\beta} e^{-\gamma t}, \quad t > 0.$$

Lemma 1.3 (*Da Prato and Zabczyk, 1992*)**.** *For any $p \geq 2$ and for an arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(\cdot)$,*

$$\sup_{s\in[0,t]} E\left\|\int_0^s \Phi(u)dw(u)\right\|^p \leq c_p\left(\int_0^t \left(E\|\Phi(s)\|^p_{\mathcal{L}_2^0}\right)^{2/p} ds\right)^{p/2}$$

where $c_p = (p(p-1)/2)^{p/2}$.

Lemma 1.4 (*Mao, 1997; Luo et al., 2006*)**.** *Let $p \in [1, \infty)$ and $\nu \in (0, 1)$. For any two real positive numbers $a, b > 0$, then*

$$(a+b)^p \leq \nu^{1-p}a^p + (1-\nu)^{1-p}b^p.$$

The rest of this paper is organized as follows. The existence and uniqueness of the mild solution of the system (1) are studied in Section 2. Exponential stability of the mild solution of the system (1) is investigated in Section 3. An example is given to illustrate our main results in Section 4.

2. Existence and uniqueness

In this section, we present the existence and uniqueness of the mild solution of the system (1) by means of a successive approximation method.

Theorem 2.1. *Let $p \geq 2$. Assume that the conditions* (A1)–(A4) *hold. If the following inequality*

$$\begin{aligned} &9^{p-1}(1-\kappa)^{-p}\left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right] \\ &+ 3^{p-1}(1-\kappa)^{-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] + 9^{p-1}\gamma(1-\kappa)^{-p}M^p\left(\sum_{t_k<t} q_k\right)^p < \gamma \end{aligned} \tag{3}$$

is satisfied, where $c_p = (p(p-1)/2)^{p/2}$, then the system (1) has a unique mild solution.

Proof. To obtain the existence of the mild solution of the system (1), let $x^0(t) = S(t)(\phi(0) + u(0, \phi))$, $t \in [0, T]$ and $x_0^n(t) = \phi(t)$, $t \in [-\tau, 0]$, $n = 0, 1, 2, \ldots$. For each integer $n = 1, 2, \ldots$, we define the following successive approximating procedure,

$$\begin{aligned} x^n(t) &= S(t)(\phi(0) + u(0, \phi)) - u(t, x^n(t - \tau(t))) - \int_0^t AS(t-s)u(s, x^n(s - \tau(s)))\,ds \\ &\quad + \int_0^t S(t-s)f(s, x^{n-1}(s - \delta(s)))\,ds + \int_0^t S(t-s)g(s, x^{n-1}(s - \rho(s)))\,dw(s) \\ &\quad + \int_0^t \int_Z S(t-s)h(s, x^{n-1}(s - \sigma(s)), y)\,\tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t - t_k)I_k x^{n-1}(t_k^-). \end{aligned} \tag{4}$$

Step 1. We claim that the sequence $\{x^n(t), n \geq 0\}$ is bounded. For any real numbers a, b, c, d, e and f, we have that

$$\begin{aligned} (a+b+c+d+e+f)^p &\leq 3^{p-1}(a+b+c+d)^p + 3^{p-1}e^p + 3^{p-1}f^p \\ &\leq 3^{p-1}\left[\left(1+\frac{1}{\epsilon}\right)^{p-1} a^p + (1+\epsilon)^{p-1}(b+c+d)^p\right] + 3^{p-1}e^p + 3^{p-1}f^p \\ &\leq 3^{p-1}\left(1+\frac{1}{\epsilon}\right)^{p-1} a^p + 9^{p-1}(1+\epsilon)^{p-1}(b^p + c^p + d^p) + 3^{p-1}e^p + 3^{p-1}f^p. \end{aligned} \tag{5}$$

From Lemma 1.4 and (4), for $0 \leq t \leq T$, we have

$$\begin{aligned} \mathbb{E}\|x^n(t)\|^p &\leq \kappa^{1-p}\mathbb{E}\left\|u(t, x^n(t-\tau(t)))\right\|^p + (1-\kappa)^{1-p}\mathbb{E}\Bigg\|S(t)(\phi(0)+u(0,\phi)) - \int_0^t AS(t-s)u(s, x^n(s-\tau(s)))\,ds \\ &\quad + \int_0^t S(t-s)f(s, x^{n-1}(s-\delta(s)))\,ds + \int_0^t S(t-s)g(s, x^{n-1}(s-\rho(s)))\,dw(s) \\ &\quad + \int_0^t \int_Z S(t-s)h(s, x^{n-1}(s-\sigma(s)), y)\,\tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k)I_k x^{n-1}(t_k^-)\Bigg\|^p \\ &\leq \kappa^{1-p}E\left\|u(t, x^n(t-\tau(t)))\right\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1} E\,\|S(t)(\phi(0)+u(0,\phi))\|^p \\ &\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}\mathbb{E}\left\|\int_0^t AS(t-s)u(s, x^n(s-\tau(s)))\,ds\right\|^p \\ &\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}\mathbb{E}\left\|\int_0^t \int_Z S(t-s)h(s, x^{n-1}(s-\sigma(s)), y)\,\tilde{N}(ds, dy)\right\|^p \\ &\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k x^{n-1}(t_k^-)\right\|^p \\ &\quad + 3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)f(s, x^{n-1}(s-\delta(s)))\,ds\right\|^p \\ &\quad + 3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)g(s, x^{n-1}(s-\rho(s)))\,dw(s)\right\|^p. \end{aligned} \tag{6}$$

Now, we estimate the right-hand side of (6). By assumption (A3), we obtain

$$\mathbb{E}\left\|u(t, x^n(t-\tau(t)))\right\|^p \leq \|(-A)^{-\alpha}\|^p\mathbb{E}\|(-A)^\alpha u(t, x(t-\tau(t)))\|_H^p \leq K^p\|(-A)^{-\alpha}\|^p\mathbb{E}\|x^n(t-\tau(t))\|^p. \tag{7}$$

Applying Lemma 1.2, Hölder inequality and assumption (A3), we have

$$\begin{aligned} &\mathbb{E}\left\|\int_0^t AS(t-s)u(s, x^n(s-\tau(s)))\,ds\right\|^p \\ &\leq M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^n(s-\tau(s))\|^p\,ds. \end{aligned} \tag{8}$$

Similarly, we have

$$\mathbb{E}\left\|\int_0^t S(t-s)f(s, x^{n-1}(s-\delta(s)))\,ds\right\|^p \leq \mathbb{E}\left[\int_0^t \|S(t-s)f(s, x^{n-1}(s-\delta(s)))\|\,ds\right]^p$$

$$
\begin{aligned}
&\le \mathbb{E}\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,x^{n-1}(s-\delta(s)))\|\,ds\right]^p\\
&\le M^pL_1^p\mathbb{E}\left[\int_0^t e^{-\gamma(t-s)}\|x^{n-1}(s-\delta(s))\|\,ds\right]^p\\
&= M^pL_1^p\mathbb{E}\left[\int_0^t e^{-(\gamma(p-1)/p)(t-s)}e^{-(\gamma/p)(t-s)}\|x^{n-1}(s-\delta(s))\|\,ds\right]^p\\
&\le M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\delta(s))\|^p\,ds\\
&\le M^pL_1^p\gamma^{1-p}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\delta(s))\|^p\,ds. \qquad (9)
\end{aligned}
$$

By Lemma 1.3 and Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t S(t-s)g(s,x^{n-1}(s-\rho(s)))\,dw(s)\right\|^p\\
&\le c_pM^p\left\{\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,x^{n-1}(s-\rho(s)))\|^p\right)^{2/p}ds\right\}^{p/2} \qquad (10)\\
&\le c_pM^pL_2^p\left\{\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|x^{n-1}(s-\rho(s))\|^p\right)^{2/p}ds\right\}^{p/2}\\
&= c_pM^pL_2^p\left\{\int_0^t \left(e^{-\gamma(p-1)(t-s)}e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\rho(s))\|^p\right)^{2/p}ds\right\}^{p/2}\\
&\le c_pM^pL_2^p\left\{\int_0^t e^{-\left[\frac{2(p-1)}{p-2}\right]\gamma(t-s)}\,ds\right\}^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\rho(s))\|^p\,ds\\
&\le c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\rho(s))\|^p\,ds \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left\|\int_0^t\int_Z S(t-s)h(s,x^{n-1}(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p\\
&\le c_p\mathbb{E}\left(\int_0^t\int_Z \|S(t-s)h(s,x^{n-1}(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right)^{p/2}\\
&\le c_pM^p\mathbb{E}\left(\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,x^{n-1}(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right)^{p/2}\\
&\le c_pM^p\mathbb{E}\left(\int_0^t e^{-2\gamma(t-s)}\int_Z \|h(s,x^{n-1}(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right)^{p/2}\\
&\le c_pM^pL_3^p\left(\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\sigma(s))\|^2\,ds\right)^{p/2}\\
&= c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\gamma(t-s)}e^{-\frac{2}{p}\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\sigma(s))\|^2\,ds\right)^{p/2}\\
&\le c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}\,ds\right)^{(p-2)/2} e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\sigma(s))\|^p\,ds\\
&\le c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x^{n-1}(s-\sigma(s))\|^p\,ds. \qquad (12)
\end{aligned}
$$

Furthermore, we obtain

$$
\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x^{n-1}(t_k^-))\right\|^p \le M^p\left(\sum_{t_k<t} q_k\right)^{p-1}\sum_{t_k<t} q_ke^{-\gamma p(t-t_k)}\mathbb{E}\|x^{n-1}(t_k^-)\|^p. \qquad (13)
$$

Hence, from (6)–(13), we obtain

$$\begin{aligned}
&\sup_{s\in[0,t]} \mathbb{E}\|x^n(s)\|^p \\
&\le \kappa \sup_{\theta\in[-\tau,t]} \mathbb{E}\|x^n(s)\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1} M^p e^{-\gamma pt}(1+K\|(-A)^{-\alpha}\|)^p \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p_{1-\alpha} K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^n(s)\|^p \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} c_p \gamma^{-1} M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n-1}(s)\|^p \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p \left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n-1}(s)\|^p \\
&+ 3^{p-1}(1-\kappa)^{1-p} M^p L_1^p \gamma^{-p} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n-1}(s)\|^p \\
&+ 3^{p-1}(1-\kappa)^{1-p} c_p \gamma^{-1} M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n-1}(s)\|^p.
\end{aligned} \tag{14}$$

Note that $\sup_{s\in[-\tau,t]} E\|x(s)\|^p \le \sup_{s\in[0,t]} E\|x(s)\|^p + \sup_{s\in[-\tau,0]} E\|x(s)\|^p$, from (14), we obtain

$$\begin{aligned}
&\sup_{s\in[0,t]} \mathbb{E}\|x^n(s)\|^p \\
&\le \left\{1-\kappa-9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p_{1-\alpha} K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\right\}^{-1} \\
&\times \left\{\kappa \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1} M^p e^{-\gamma pt}(1+K\|(-A)^{-\alpha}\|)^p \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p\right. \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p_{1-\alpha} K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} c_p \gamma^{-1} M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \left[\sup_{s\in[0,t]} E\|x^{n-1}(s)\|^p + \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p\right] \\
&+ 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p \left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)} \left[\sup_{s\in[0,t]} \mathbb{E}\|x^{n-1}(s)\|^p + \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p\right] \\
&+ 3^{p-1}(1-\kappa)^{1-p} M^p L_1^p \gamma^{-p} \left[\sup_{s\in[0,t]} \mathbb{E}\|x^{n-1}(s)\|^p + \sup_{s\in[-\tau,0]} \mathbb{E}\|\phi(s)\|^p\right] \\
&\left.+ 3^{p-1}(1-\kappa)^{1-p} c_p \gamma^{-1} M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \left[\sup_{s\in[0,t]} E\|x^{n-1}(s)\|^p + \sup_{s\in[-\tau,0]} E\|\phi(s)\|^p\right]\right\}.
\end{aligned} \tag{15}$$

From inequality (3) and (15), we can always find a number $\epsilon > 0$ small enough such that

$$Q := \kappa + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p_{1-\alpha} K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} < 1.$$

Since $\mathbb{E}\|\phi\|^p < \infty$, using mathematical induction to (15), we obtain that the sequence $\{x^n(t), n \ge 0\}$ is bounded.

Step 2. We claim that the sequence $\{x^n(t), n \ge 0\}$ is a Cauchy sequence. For $0 \le t \le T$, from (4), we obtain

$$\begin{aligned}
&\sup_{s\in[0,t]} \mathbb{E}\|x^{n+1}(s) - x^n(s)\|^p \\
&\le \kappa \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n+1}(s) - x^n(s)\|^p \\
&+ 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1} M^p_{1-\alpha} K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^{n+1}(s) - x^n(s)\|^p \\
&+ 6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} c_p \gamma^{-1} M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^n(s) - x^{n-1}(s)\|^p \\
&+ 6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p \left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)} \sup_{s\in[-\tau,t]} \mathbb{E}\|x^n(s) - x^{n-1}(s)\|^p
\end{aligned}$$

$$+3^{p-1}(1-\kappa)^{1-p}M^pL_1^p\gamma^{-p}\sup_{s\in[-\tau,t]}\mathbb{E}\|x^n(s)-x^{n-1}(s)\|^p$$
$$+3^{p-1}(1-\kappa)^{1-p}c_p\gamma^{-1}M^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\sup_{s\in[-\tau,t]}\mathbb{E}\|x^n(s)-x^{n-1}(s)\|^p,$$

which implies that

$$\begin{aligned}
&\sup_{s\in[0,t]}\mathbb{E}\|x^{n+1}(s)-x^n(s)\|^p\\
&\le\left\{1-\kappa-3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1}M_{1-\alpha}^pK^p\gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\right\}^{-1}\\
&\quad\times\left\{6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}c_p\gamma^{-1}M^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right.\\
&\quad+6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}M^p\left(\sum_{t_k<t}q_k\right)^p+3^{p-1}(1-\kappa)^{1-p}M^pL_1^p\gamma^{-p}\\
&\quad\left.+3^{p-1}(1-\kappa)^{1-p}c_p\gamma^{-1}M^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right\}\sup_{s\in[0,t]}\mathbb{E}\|x^n(s)-x^{n-1}(s)\|^p\\
&=\frac{\alpha\sup_{s\in[0,t]}\mathbb{E}\|x^n(s)-x^{n-1}(s)\|^p}{1-Q_1}\le\frac{\alpha^n\sup_{s\in[0,t]}\mathbb{E}\|x^1(s)-x^0(s)\|^p}{(1-Q_1)^n},
\end{aligned}\tag{16}$$

where

$$\begin{aligned}
\alpha&=6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}c_p\gamma^{-1}M^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\\
&\quad+6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}M^p\left(\sum_{t_k<t}q_k\right)^p+3^{p-1}(1-\kappa)^{1-p}M^pL_1^p\gamma^{-p}\\
&\quad+3^{p-1}(1-\kappa)^{1-p}c_p\gamma^{-1}M^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2},\\
Q_1&=\kappa+3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1}M_{1-\alpha}^pK^p\gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}.
\end{aligned}$$

From (4), we obtain that

$$\begin{aligned}
&\sup_{s\in[0,t]}\mathbb{E}\|x^1(s)-x^0(s)\|^p\\
&\le(1-Q_1)^{-1}\left\{\kappa+3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1}M_{1-\alpha}^pK^p\gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\right.\\
&\quad+6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}c_p\gamma^{-1}M^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}+6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}M^p\left(\sum_{t_k<t}q_k\right)^p\\
&\quad\left.+3^{p-1}(1-\kappa)^{1-p}M^pL_1^p\gamma^{-p}+3^{p-1}(1-\kappa)^{1-p}c_p\gamma^{-1}M^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right\}\sup_{s\in[-\tau,0]}E\|\phi(s)\|^p\\
&\quad+\left\{6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}c_p\gamma^{-1}M^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}+6^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}M^p\left(\sum_{t_k<t}q_k\right)^p\right.\\
&\quad\left.+3^{p-1}(1-\kappa)^{1-p}M^pL_1^p\gamma^{-p}+3^{p-1}(1-\kappa)^{1-p}c_p\gamma^{-1}M^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right\}\sup_{s\in[0,t]}E\|x^0(s)\|^p,
\end{aligned}\tag{17}$$

and

$$\sup_{s\in[0,t]}E\|x^0(s)\|^p=\sup_{s\in[0,t]}E\|S(s)\phi(0)\|^p\le M^pE\|\phi(0)\|^p.\tag{18}$$

Hence, by employing (16)–(18), we obtain that for any $m > n \geq 1$,

$$\sup_{s\in[0,t]} \mathbb{E}\|x^m(s) - x^n(s)\|^p \leq \sum_{n}^{+\infty} \sup_{s\in[0,t]} \mathbb{E}\|x^{n+1}(s) - x^n(s)\|^p \leq \sum_{n}^{+\infty} \frac{\alpha^n}{(1-Q_1)^n} \sup_{s\in[0,t]} \mathbb{E}\|x^1(s) - x^0(s)\|^p \to 0 \quad \text{as} \quad n \to \infty,$$

since $\alpha + Q_1 < 1$. This shows that $\{x^n(t), n \geq 0\}$ is a Cauchy sequence.

Step 3. We prove the existence and uniqueness of the mild solution of the system (1). Borel–Cantelli lemma shows that, as $n \to \infty, x^n(t) \to x(t)$ holds uniformly for $0 \leq t \leq T$. Hence, taking limits on both sides of (4), we obtain that $x(t)$ is a solution of the system (1). This shows the existence. The uniqueness of the solutions could be obtained by the same procedure as Step 2. □

3. Exponential stability

In this section, we study exponential stability of mild solutions of the system (1) by using an appropriate impulsive-integral inequality. We start with some preparations.

Definition 3.1. Let $p \geq 1$ be an integer. Eq. (1) is said to be exponentially stable in *pth* moment, if for any initial value ϕ, there exists a pair of positive constants $\lambda > 0$ and C such that

$$E\|x(t)\|^p \leq C\|\phi\|_0^p e^{-\lambda t}, \quad t \geq 0.$$

To obtain sufficient conditions for the exponential stability in *pth* moment of mild solutions of system (1), we establish the following impulsive-integral inequality, which is essential to the proof of the main result.

Lemma 3.2. *Consider a $\gamma > 0$, positive constants: $\lambda_0, \lambda, \lambda^*, \lambda_k (k = 1, 2, \dots)$ and a function $y : [-\tau, \infty) \to [0, \infty)$. If $\frac{\lambda^*}{\gamma} + \lambda + \sum_{k=1}^{\infty} \lambda_k < 1$, and the following inequality*

$$y(t) \leq \begin{cases} \lambda_0 e^{-p\gamma t} + \lambda \sup_{\theta\in[-\tau,0]} y(t+\theta) + \lambda^* \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} y(s+\theta)\,ds + \sum_{t_k<t} \lambda_k e^{-p\gamma(t-t_k)} y(t_k^-), & t \geq 0, \\ \lambda_0 e^{-p\gamma t}, \quad t \in [-\tau, 0], \end{cases} \tag{19}$$

holds, then we have $y(t) \leq M^* e^{-p\mu t} (t \geq -\tau)$, where μ is a positive root of the algebraic equation $\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k = 1$ and $M^* = \max\left\{\frac{\lambda_0(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\} > 0$.

Proof. Let $F(\mu) = \left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k - 1$. We have $F(0)F(\gamma^-) < 0$, that is, there exists a positive constant $\mu \in (0, \gamma)$ such that $F(\mu) = 0$. For any $\epsilon > 0$, let

$$C_\epsilon = \max\left\{\frac{(\epsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \epsilon + \lambda_0\right\}.$$

To prove the lemma, we claim that (22) implies

$$y(t) \leq C_\epsilon e^{-p\mu t}, \quad t \geq -\tau. \tag{20}$$

It is easily shown that (20) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\epsilon e^{-p\mu t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\epsilon e^{-p\mu t_1^*}. \tag{21}$$

Combining with (22), we have

$$\begin{aligned} y(t_1^*) &\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\epsilon \sup_{\theta\in[-\tau,0]} e^{-p\mu(t_1^*+\theta)} + \lambda^* C_\epsilon \int_0^{t_1^*} e^{-\gamma(t_1^*-s)} \sup_{\theta\in[-\tau,0]} e^{-p\mu(s+\theta)}\,ds + C_\epsilon \sum_{t_k<t_1^*} \lambda_k e^{-p\gamma(t_1^*-t_k)} e^{-p\mu t_k} \\ &\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\epsilon e^{-p\mu t_1^*} e^{p\mu\tau} - \frac{\lambda^* C_\epsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} C_\epsilon e^{-p\mu t_1^*} + \left(\sum_{k=1}^{\infty} \lambda_k\right) C_\epsilon e^{-p\mu t_1^*} \\ &= \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\epsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \left[\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k\right] C_\epsilon e^{-p\mu t_1^*}. \end{aligned} \tag{22}$$

From the definition of C_ϵ, we have

$$\lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\epsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \le \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \cdot \frac{(\epsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}} < 0.$$

Then, from the definition of μ and (22) , we obtain that $y(t_1^*) < C_\epsilon e^{-p\mu t_1^*}$, which contradicts (21), so (20) holds. As $\epsilon > 0$ is arbitrarily small, in view of (20), it follows that $y(t) \le M^* e^{-p\mu t}$, for $t \ge -\tau$, where $M^* = \max\left\{\frac{\lambda_0(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\}$. □

Theorem 3.3. *Let $p \ge 2$. If the conditions in Theorem 2.1 are satisfied, then the mild solution of (1) is exponentially stable in pth moment.*

Proof. Using the similar the estimation as (6)–(13), we obtain that

$$\begin{aligned}
&\mathbb{E}\|x(t)\|^p \\
&\le \kappa \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(t+\theta)\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\epsilon}\right)^{p-1} M^p e^{-\gamma p t}(1 + K\|(-A)^{-\alpha}\|)^p \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(\theta)\|^p \\
&\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p ds \\
&\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p ds \\
&\quad + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p \left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p \\
&\quad + 3^{p-1}(1-\kappa)^{1-p} M^p L_1^p \gamma^{1-p} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p ds \\
&\quad + 3^{p-1}(1-\kappa)^{1-p} c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p ds.
\end{aligned} \tag{23}$$

From inequality (3), (14) and the condition (A3), we can always find a number $\epsilon > 0$ small enough such that

$$\begin{aligned}
&\kappa + \gamma^{-1}\Bigg\{9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1}\left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right] \\
&\quad + 3^{p-1}(1-\kappa)^{1-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right]\Bigg\} + 9^{p-1}(1-\kappa)^{1-p}(1+\epsilon)^{p-1} M^p \left(\sum_{t_k<t} q_k\right)^p < 1.
\end{aligned}$$

By Lemma 3.2, we obtain that $E\|x(t)\|^p \le \tilde{M} e^{-p\mu t}(\tilde{M} > 0, \mu \in (0, \gamma))$, that is, the mild solution of system (1) is exponentially stable in *pth* moment. □

If $h \equiv 0$ or $\nu \equiv 0$, system (1) becomes

$$\begin{cases}
d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t)) & t \ge 0, \quad t \ne t_k, \\
\Delta x(t_k) = I_k x(t_k^-), \quad t = t_k, \quad k = 1, 2, \ldots, \\
x_0(\cdot) = x_0 \in PC,
\end{cases} \tag{24}$$

Corollary 3.4. *Assume that the conditions* (A1)–(A4) *hold, then the mild solution of* (24) *is exponentially stable in pth moment, if the following inequality*

$$\begin{aligned}
&6^{p-1}(1-\kappa)^{-p} M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\quad + 3^{p-1}(1-\kappa)^{-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] + 6^{p-1}\gamma(1-\kappa)^{-p} M^p \left(\sum_{t_k<t} q_k\right)^p < \gamma
\end{aligned}$$

holds, where $c_p = (p(p-1)/2)^{p/2}$.

Remark 3.5. If $I_k(\cdot) \equiv 0$ $(k = 1, 2, \ldots)$ in (24), Corollary 3.4 is consistent with Theorem 3.2 in Chen (2009).

If the impulsive effects $I_k(\cdot) \equiv 0$ $(k = 1, 2, \ldots)$, system (1) reduces to the following neutral stochastic partial differential equations with delays

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t)) \\ \qquad\qquad + \int_Z h(t, x(t-\sigma(t)), y)\tilde{N}(dt, dy), \qquad t \geq 0, \\ x_0(\cdot) = \phi \in PC. \end{cases} \tag{25}$$

Corollary 3.6. *Assume that the conditions* (A1)–(A4) *hold, then the mild solution of* (25) *is exponentially stable in pth moment, if the following inequality*

$$6^{p-1}(1-\kappa)^{-p}\left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + c_p M^p L_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right]$$
$$+ 3^{p-1}(1-\kappa)^{-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] < \gamma \tag{26}$$

holds, where $c_p = (p(p-1)/2)^{p/2}$.

Remark 3.7. Cui et al. (2011) have studied the existence and exponential stability in mean square of the system (25) by means of fixed point method. The assumption (A1) in Cui et al. (2011) is restricted to

(A1′) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X such that the following inequality holds:

$$\|S(t)\| \leq e^{-\gamma t}, \quad t \geq 0,$$

for some constant γ. It is clear that Corollary 3.6 is an improvement and generalization of the result in Cui et al. (2011).

If $h \equiv 0$ and the impulsive effects $I_k(\cdot) \equiv 0$ $(k = 1, 2, \ldots)$, system (1) becomes

$$\begin{cases} dx(t) = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t)) \qquad t \geq 0, \\ x_0(\cdot) = x_0 \in PC. \end{cases} \tag{27}$$

Corollary 3.8. *Assume that the conditions* (A1)–(A4) *hold, then the mild solution of* (27) *is exponential stability in pth moment, if the following inequality*

$$3^{p-1}M^p\left[L_1^p\gamma^{1-p} + c_p L_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] < \gamma \tag{28}$$

holds, where $c_p = (p(p-1)/2)^{p/2}$.

Remark 3.9. Luo (2008) has studied the mild solution of (27) by using fixed point theory, the result in Luo (2008) is consistent with Corollary 3.8.

4. An example

Example 4.1. Consider the following neutral stochastic partial differential equation with delays and Poisson jumps of the form:

$$\begin{cases} d\left[x(t,\xi) + \frac{\alpha_3}{M_{1-\alpha}\|(-A)^\alpha\|}x(t-\tau(t),\xi)\right] = \left[\frac{\partial^2}{\partial\xi^2}x(t,\xi)dt + \alpha_1 x(t-\delta(t),\xi)\right]dt + \alpha_2 x(t-\rho(t),\xi)dw(t) \\ \qquad\qquad + \int_Z \alpha_4 y x(t-\sigma(t),\xi)\tilde{N}(dt,dy), \qquad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k,\xi) = b_k x(t_k,\xi), \quad t = t_k, \quad k = 1, 2, \ldots, m, \quad b_k \geq 0, \sum_{k=1}^m b_k < \infty, \\ x(t,0) = x(t,\pi) = 0, \quad \alpha_i > 0, i = 1, 2, 3, \quad 0 < \tau(t), \delta(t), \rho(t), \sigma(t) < \tau, \\ x(s,\xi) = \phi(s,\xi), \quad \phi(\cdot,\xi) \in C, \quad \phi(s,\cdot) \in \mathcal{L}^2[0,\pi], \quad -\tau \leq s \leq 0, \quad 0 \leq \xi \leq \pi, \quad \tau \geq 0, \quad t \geq 0, \end{cases} \tag{29}$$

where $w(t)$ is a standard one-dimensional Wiener process and $\|\phi\|_C < +\infty$ a.s., and $M_{1-\alpha} \geq 1 (\alpha \in (1/2, 1])$.

Take $X = L^2[0,\pi]$, $Y = R^1$, define $A : X \to X$ by $-A = \frac{\partial^2}{\partial\xi^2}$ with domain

$$D(-A) = \left\{\omega \in X : \quad \omega, \quad \frac{\partial\omega}{\partial\xi} \quad \text{are absolutely continuous}, \quad \frac{\partial^2\omega}{\partial\xi^2} \in X, \quad \omega(0) = \omega(\pi) = 0\right\}.$$

Then

$$(-A)\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(-A),$$

where $\omega_n(\xi) = \sqrt{2/\pi}\sin n\xi$, $n = 1, 2, 3, \ldots$, is an orthonormal set of eigenvectors of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)(t \geq 0)$ in X and is given (see Pazy, 1983, Page 70) by

$$S(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \quad \omega \in X,$$

it satisfies $\|S(t)\| \leq \exp(-\pi^2 t)$, $t \geq 0$, and hence is a contraction semigroup. Let

$$u(t, x(t-\tau(t), \xi)) = \frac{\alpha_3}{M_{1-\alpha}\|(-A)^{\alpha}\|}x(t-\tau(t), \xi), \quad f(t, x(t-\delta(t), \xi)) = \alpha_1 x(t-\delta(t), \xi),$$
$$g(t, x(t-\rho(t), \xi)) = \alpha_2 x(t-\rho(t), \xi), \quad h(t, x(t-\sigma(t), \xi), z) = \alpha_4 z x(t-\sigma(t), \xi).$$

It is not difficult to check that

$$\|f(t, x(t-\delta(t), \xi)) - f(t, y(t-\delta(t), \xi))\| \leq \alpha_1\|x(t-\delta(t), \xi) - y(t-\delta(t), \xi)\|,$$
$$\|g(t, x(t-\rho(t), \xi)) - g(t, y(t-\rho(t), \xi))\| \leq \alpha_2\|x(t-\delta(t), \xi) - y(t-\delta(t), \xi)\|,$$
$$\int_Z \|h(t, x(t-\sigma(t), \xi), z) - h(t, y(t-\sigma(t), \xi), z)\|^2\,\nu(dz) \leq \alpha_4^2\int_Z z^2\,\nu(dz)\|x(t-\sigma(t), \xi) - y(t-\sigma(t), \xi)\|^2,$$
$$\|(-A)^{\alpha}u(t, x(t-\tau(t), \xi)) - (-A)^{\alpha}u(t, y(t-\tau(t), \xi))\| \leq \frac{\alpha_3}{M_{1-\alpha}}\|(x(t-\tau(t)), \xi) - (y(t-\tau(t)), \xi)\|,$$
$$f(t, 0) = 0, \quad g(t, 0) = 0, \quad h(t, 0, z) = 0, \quad (-A)^{\alpha}u(t, 0, \xi) = 0.$$

From the definition of $(-A)^{-\alpha}$, we have

$$\|(-A)^{-\alpha}\| \leq \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}\|S(t)\|\,dt \leq \frac{1}{\pi^{2\alpha}}.$$

Thus, when $\alpha_3 < M_{1-\alpha}\pi^{2\alpha}(\alpha \in (1/2, 1])$, by Theorem 3.3, the mild solution of (1) is exponentially stable in mean square provided that

$$9\alpha_3^2\pi^{2-4\alpha}\Gamma(2\alpha-1) + 9\alpha_4^2\pi^{-2}\int_Z z^2\,\nu(dz) + 3\left(\alpha_1^2\pi^{-2} + \alpha_2^2\right) + 9\pi^2\left(\sum_{k=1}^{m} q_k\right)^2$$
$$< \left(\pi - \frac{\alpha_3}{M_{1-\alpha}\pi^{2\alpha-1}}\right)^2, \quad \alpha \in \left(\frac{1}{2}, 1\right].$$

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