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Equivalence of formulations of the MKP hierarchy and its polynomial tau-functions

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Abstract. We show that a system of Hirota bilinear equations introduced by Jimbo and Miwa defines tau-functions of the modified KP (MKP) hierarchy of evolution equations introduced by Dickey. Some other equivalent definitions of the MKP hierarchy are established. All polynomial tau-functions of the KP and the MKP hierarchies are found. Similar results are obtained for the reduced KP and MKP hierarchies.

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1. Introduction

The KP hierarchy was introduced by Sato in his seminal paper [15] as the hierarchy of evolution equations of Lax type

$$\frac{dL}{dt_n} = [(L^n)_+, L],\ n = 1, 2, \dots,$$

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on the pseudo-differential operator $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots$, where $\partial = \frac{\partial}{\partial t_1}$ and $+$ stands for the differential part. He also introduced the associated wave functions and the tau-function, and discussed reductions of the KP hierarchy. His ideas have been subsequently developed by his school in a series of papers, which were reviewed in [10] and [5].

In the review [10] Jimbo and Miwa also introduced the modified KP hierarchy (MKP hierarchy), as a set of bilinear equations on the tau-functions τ_ℓ, $\ell \in \mathbb{Z}$, see [10], equation $(2.4)_{l,l'}$, each τ_ℓ being a tau-function of the KP hierarchy. It was subsequently shown in [11] that these equations arise naturally from the fermionic formulation of the MKP hierarchy and the boson-fermion correspondence. This implies that the MKP tau-functions $(\ldots, \tau_{\ell-1}, \tau_\ell, \tau_{\ell+1}, \ldots)$ are naturally parameterized by the infinite-dimensional flag manifold ([11], Corollary 8.1), in analogy with the famous observation of Sato [15] that tau-functions of the KP hierarchy are parametrized by the infinite-dimensional Grassmann manifold. Note that the tau-functions of the discrete KP hierarchy, studied in [2], are precisely those, satisfying the Jimbo–Miwa equations from [10].

On the other hand, Dickey proposed a Lax type formulation of the MKP hierarchy in [6] (see also [7]), which is an extension of the Sato formulation of KP. The first result of the present paper is the equivalence of Jimbo–Miwa's tau-function formulation and Dickey's Lax type formulation of the MKP hierarchy (Theorem 3 in Sect. 4), in analogy with the well developed theory of the KP hierarchy (see e.g. [10], [5]). Similar equivalences are established for the discrete KP hierarchy in [2]. The vertex operator construction of the Lie algebra gl_∞ provides solutions to the tau-function formulation of the MKP hierarchy [11], hence to the Lax type formulation of it. Similar solutions have been constructed in [2] for the discrete KP hierarchy.

In Sect. 5 we give eigenfunction formulations of the MKP hierarchy, closely related to the work [9]. As a byproduct, we find in Sect. 6 an astonishingly simple explicit description of all polynomial tau-functions of the KP and the MKP hierarchies (Theorem 16). Of course, it is a well-known result of Sato [15] that all Schur polynomials are tau-functions of the KP hierarchy. We show that, moreover, all polynomial tau-functions of the KP hierarchy can be obtained from Schur polynomials by certain shifts of arguments.

We discuss in Sect. 7 the reductions of the MKP hierarchy to the modified n-KdV hierarchies for each integer $n \geq 2$, the $n = 2$ case being the classical modified KdV hierarchy (*cf.* [6]). Finally, in Sect. 8 we find all polynomial tau-functions for the n-KdV hierarchy, and (implicitly) for the modified n-KdV hierarchy. This was known only for $n = 2$ [11].

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2. The fermionic formulation of MKP

Recall the semi-infinite wedge representation [11], [13]. Consider the infinite matrix group GL_∞, consisting of all complex matrices $G = (g_{ij})_{i,j\in\mathbb{Z}}$ which are invertible and all but a finite number of $g_{ij} - \delta_{ij}$ are 0. It acts naturally on the vector space $\mathbb{C}^\infty = \bigoplus_{j\in\mathbb{Z}} \mathbb{C}e_j$ (via the usual formula $E_{ij}(e_k) = \delta_{jk}e_i$).

The semi-infinite wedge space $F = \Lambda^{\frac{1}{2}\infty}\mathbb{C}^\infty$ is the vector space with a basis consisting of all semi-infinite monomials of the form $e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots$, where $i_1 > i_2 > i_3 > \cdots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell \gg 0$. One defines the representation R of GL_∞ on F by

$$R(G)(e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge \cdots) = Ge_{i_1} \wedge Ge_{i_2} \wedge Ge_{i_3} \wedge \cdots ,$$

and apply linearity and anticommutativity of the wedge product $\wedge$.

The corresponding representation r of the Lie algebra gl_∞ of GL_∞ can be described in terms of a Clifford algebra. Define the wedging and contracting operators ψ_j^+ and ψ_j^- $(j \in \mathbb{Z} + \frac{1}{2})$ on F by

$$\begin{aligned} &\psi_j^+(e_{i_1} \wedge e_{i_2} \wedge \cdots) = e_{-j+\frac{1}{2}} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots , \\ &\psi_j^-(e_{i_1} \wedge e_{i_2} \wedge \cdots) \\ &\quad = \begin{cases} 0 & \text{if } j - \dfrac{1}{2} \neq i_s \text{ for all } s, \\ (-1)^{s+1} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots & \text{if } j = i_s - \dfrac{1}{2}. \end{cases} \end{aligned}$$

These operators satisfy the relations $(i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)$:

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathscr{C}\ell$. Introduce the following elements of F $(m \in \mathbb{Z})$:

$$|m\rangle = e_m \wedge e_{m-1} \wedge e_{m-2} \wedge \cdots . \tag{1}$$

It is clear that F is an irreducible $\mathscr{C}\ell$-module such that

$$\psi_j^\pm|0\rangle = 0 \quad \text{for } j > 0.$$

The representation r of gl_∞ in F, corresponding to the representation R of GL_∞, is given by the formula $r(E_{ij}) = \psi^+_{-i+\frac{1}{2}}\psi^-_{j-\frac{1}{2}}$. Define the *charge decomposition*

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)}, \qquad \text{where charge}(|m\rangle) = m \text{ and charge}(\psi_j^\pm) = \pm 1.$$

The space $F^{(m)}$ is an irreducible highest weight gl_∞-module, with highest weight vector $|m\rangle$:

$$r(E_{ij})|m\rangle = 0 \quad \text{for } i < j,$$
$$r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \quad \text{if } i > m \text{ (resp. if } i \le m).$$

Let

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$

be the GL_∞-orbit of the highest weight vector $|m\rangle$.

Theorem 1 ([11], Theorem 5.1). *Let I be a non-empty finite subset of $\mathbb{Z}$ and let $f = \bigoplus_{m\in I} f_m \in \bigoplus_{m\in I} F^{(m)}$ be such that all $f_m \neq 0$. Then $f \in \bigoplus_{m\in I} \mathcal{O}_m$ if and only if for all $k, \ell \in I$, such that $k \ge \ell$, one has*

$$\sum_{i\in\mathbb{Z}+\frac{1}{2}} \psi_i^+ f_k \otimes \psi_{-i}^- f_\ell = 0. \tag{2}$$

Equation (2) is called the $(k-\ell)$-th modified KP hierarchy in the fermionic picture. The 0-th modified KP is the KP hierarchy. The collection of all such equations $k, \ell \in \mathbb{Z}$ with $k \ge \ell$ is called the (full) MKP hierarchy in the fermionic picture.

3. The bosonic formulation of MKP

Define the fermionic fields by $\psi^\pm(z) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} \psi_i^\pm z^{-i-\frac{1}{2}}$ and the bosonic field $\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1} =:\psi^+(z)\psi^-(z):$. Then there exists a unique vector space isomorphism, called the boson-fermion correspondence, $\sigma : F \to B = \mathbb{C}[q, q^{-1}] \otimes \mathbb{C}[t_1, t_2, \ldots]$ such that $\sigma(|m\rangle) = q^m$, $\sigma\alpha_n\sigma^{-1} = \frac{\partial}{\partial t_n}$, $\sigma\alpha_{-n}\sigma^{-1} = nt_n$, for $n > 0$ and $\sigma\alpha_0\sigma^{-1} = q\frac{\partial}{\partial q}$. Moreover, one has

$$\sigma\psi^\pm(z)\sigma^{-1} = q^{\pm 1} z^{\pm q\frac{\partial}{\partial q}} \exp\Big(\pm\sum_{k=1}^\infty t_k z^k\Big) \exp\Big(\mp\sum_{k=1}^\infty \frac{\partial}{\partial t_k}\frac{z^{-k}}{k}\Big). \tag{3}$$

For $f_m \in \mathcal{O}_m \cup \{0\}$ we write: $\sigma(f_m) = \tau_m(t)q^m$, where $t = (t_1, t_2, \ldots)$. Such a τ_m is called a tau-function. Under the isomorphism σ we can rewrite (2), using (3), to obtain a Hirota bilinear identity for tau-functions.

The first formulation of the MKP hierarchy: *Let $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots)$, $y = (y_1, y_2, \ldots)$, and* Res $\sum_i f_i z^i \, dz = f_{-1}$, *then*

$$\operatorname{Res} z^{k-\ell}\tau_k(t-[z^{-1}])\tau_\ell(y+[z^{-1}]) \exp\Big(\sum_{i=1}^\infty (t_i - y_i)z^i\Big)\, dz = 0, \quad k \ge \ell. \tag{4}$$

The equations (4) first appeared in [10], $(2.4)_{l,l'}$.

Divide (4) by $\tau_k(t)\tau_\ell(y)$ and introduce the wave functions w_m^+ and adjoint wave function w_m^- ($m \in \mathbb{Z}$) by

$$
\begin{aligned}
w_m^\pm(t,z) =& q^{\mp 1}\frac{\sigma\left(\psi^\pm(z) f_m\right)}{\sigma(f_m)} \\
=& z^{\pm m}\frac{\tau_m(t \mp [z^{-1}])}{\tau_m(t)} e^{\pm t\cdot z}.
\end{aligned}
\tag{5}
$$

Here and thereafter we use the shorthand notation

$$
t \cdot z = \sum_{i=1}^{\infty} t_i z^i .
$$

Then (4) becomes

$$
\operatorname{Res} w_k^+(t,z) w_\ell^-(y,z)\, dz = 0, \quad k \geq \ell. \tag{6}
$$

4. The Lax type formulation of MKP

We now want to express the wave functions in terms of formal pseudo-differential operators in $\partial = \frac{\partial}{\partial t_1}$. A formal pseudo-differential operator is an expression of the form

$$
P(t,\partial) = \sum_{j \leq N} P_j(t)\partial^j ,
$$

where the $P_j(t)$ are functions in t, infinitely differentiable in t_1. The differential part of $P(t,\partial)$ is $P_+(t,\partial) := \sum_{j=0}^{N} P_j(t)\partial^j$, and $P_- := P - P_+$. These operators form an associative algebra with multiplication $\circ$, defined by ($k, \ell \in \mathbb{Z}$)

$$
A(t)\partial^k \circ B(t)\partial^\ell = \sum_{i=0}^{\infty} \binom{k}{i} \frac{\partial^i A(t)}{\partial t_1^i} B(t)\partial^{k+\ell-i}.
$$

The formal adjoint of $P(t,\partial)$ is defined by the following formula:

$$
\Big(\sum_j P_j(t)\partial^j\Big)^* = \sum_j (-\partial)^j \circ P_j(t).
$$

The residue of $P(t,\partial)$ is $\operatorname{Res}_\partial P(t,\partial) := P_{-1}(t)$.

Let

$$
P_m^\pm(t, \pm z) = \frac{\tau_m(t \mp [z^{-1}])}{\tau_m(t)} = 1 \pm p_1^\pm(t) z^{-1} + p_2^\pm(t) z^{-2} \pm \cdots , \tag{7}
$$

so that

$$w_m^{\pm}(t,z) = P_m^{\pm}(t,\pm z)z^{\pm m}e^{\pm t\cdot z} = P_m^{\pm}(t,\partial)\circ(\pm\partial)^{\pm m}(e^{\pm t\cdot z})$$
$$= P_m^{\pm}(t,\partial)\circ(\pm\partial)^{\pm m}\circ\exp\Big(\pm\sum_{i=2}^{\infty}t_i(\pm\partial)^i\Big)(e^{\pm t_1 z}). \tag{8}$$

Then (4) is equivalent to

$$\operatorname{Res} P_k^{+}(t,z)z^k e^{t\cdot z}P_\ell^{-}(y,-z)z^{-\ell}e^{-y\cdot z}\,dz = 0. \tag{9}$$

The following lemma is crucial. It involves only the first variable t_1. When we use it, the variables $t_2, t_3, \dots$ are seen as extra parameters.

Lemma 2 ([13], Lemma 4.1). *Let $P(t_1,\partial)$ and $Q(t_1,\partial)$ be two formal pseudo-differential operators, then*

$$\operatorname{Res} P(t_1,z)e^{t_1 z}Q(y_1,-z)e^{-y_1 z}\,dz = \operatorname{Res}_{\partial} P(t_1,\partial)\circ Q(t_1,\partial)^*\circ e^{u\partial}|_{u=t_1-y_1}\,.$$

Applying the lemma to the bilinear identity (6), while using the expression (8) for the wave functions, one deduces

$$P_k^{-}(t,\partial)^* = P_k^{+}(t,\partial)^{-1},\quad (P_k^{+}(t,\partial)\circ\partial^{(k-\ell)}\circ P_\ell^{+}(t,\partial)^{-1})_- = 0. \tag{10}$$

We obtain the Sato–Wilson equation

$$\frac{\partial P_k^{+}(t,\partial)}{\partial t_j} = (P_k^{+}(t,\partial)\circ\partial^j\circ P_k^{+}(t,\partial)^{-1})_-\circ P_k^{+}(t,\partial)\,, \tag{11}$$

by differentiating (6) by t_j, using the first equation of (10) and then applying Lemma 2 (see e.g. [13], proof of Lemma 4.2) .

Introduce the Lax operator L_k by dressing ∂ by (the dressing operator) P_k^{+}:

$$L_k = L_k(t,\partial) = P_k^{+}(t,\partial)\circ\partial\circ P_k^{+}(t,\partial)^{-1}. \tag{12}$$

Differentiate (8) by t_j and apply the Sato–Wilson equation (11). This gives the following linear equation (= linear problem) for the wave function w_k^{+} ($k\in\mathbb{Z}$):

$$L_k w_k^{+}(t,z) = z w_k^{+}(t,z),\quad \frac{\partial w_k^{+}(t,z)}{\partial t_j} = (L_k^j)_+ w_k^{+}(t,z) \tag{13}$$

and the adjoint wave function w_k^{-}:

$$L_k^* w_k^{-}(t,z) = z w_k^{-}(t,z),\quad \frac{\partial w_k^{-}(t,z)}{\partial t_j} = -(L_k^j)_+^* w_k^{-}(t,z)\,. \tag{14}$$

From (11) it is easy to deduce the Lax equations on L_k (see e.g. [13], Lemma 4.3):

$$\frac{\partial L_k}{\partial t_j} = [(L_k^j)_+, L_k], \quad j = 1, 2, \ldots, \tag{15}$$

which are the compatibility conditions of the linear problem (13). From (7) we find that

$$P_k^+(t, \partial) = 1 - \partial(\log \tau_k(t))\partial^{-1} + \cdots,$$

hence the second equation of (10) for $k = \ell + 1$ gives that

$$\begin{aligned} P_{\ell+1}^+(t,\partial) \circ \partial \circ P_\ell^+(t,\partial)^{-1} &= (P_{\ell+1}^+(t,\partial) \circ \partial \circ P_\ell^+(t,\partial)^{-1})_+ \\ &= \partial + \partial(\log(\tau_\ell(t)) - \partial(\log(\tau_{\ell+1}(t)), \end{aligned}$$

and hence

$$P_{\ell+1}^+(t,\partial)\partial = (\partial + v_\ell(t)) \circ P_\ell^+(t,\partial), \quad \text{where } v_\ell(t) = \partial\Big(\log \frac{\tau_\ell(t)}{\tau_{\ell+1}(t)}\Big). \tag{16}$$

This leads to another formulation of MKP, which was suggested by Dickey [6], [7]:

The second formulation of the MKP hierarchy: *Let $U = \mathbb{C}[u_i^{(n)}, v_j^{(n)} \mid i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}]$ be the algebra of differential polynomials in u_i and v_j, where $\partial u_i^{(n)} = u_i^{(n+1)}$, $\partial v_j^{(n)} = v_j^{(n+1)}$. Let $L_0(\partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots \in U((\partial^{-1}))$ be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in U $(j \in \mathbb{Z}_{\geq 1}, i \in \mathbb{Z})$:*

$$\begin{aligned} \frac{\partial L_0(\partial)}{\partial t_j} &= [(L_0(\partial)^j)_+, L_0(\partial)], \\ \frac{\partial v_i}{\partial t_j} &= (L_{i+1}(\partial)^j)_+ \circ (\partial + v_i) - (\partial + v_i) \circ (L_i(\partial)^j)_+, \end{aligned} \tag{17}$$

where $L_i(\partial)$ and $L_{-i}(\partial)$, for $i > 0$, are defined by

$$\begin{aligned} L_i(\partial) &= (\partial + v_{i-1}) \circ L_{i-1}(\partial) \circ (\partial + v_{i-1})^{-1}, \\ L_{-i}(\partial) &= (\partial + v_{-i})^{-1} \circ L_{1-i}(\partial) \circ (\partial + v_{-i}). \end{aligned} \tag{18}$$

Theorem 3. *The first and the second formulation of MKP are equivalent.*

Proof. To prove that the first formulation implies the second, first note that, using the first formula of (16), one indeed gets that for $\ell > 0$:

$$L_\ell = P_\ell^+ \circ \partial \circ (P_\ell^+)^{-1} = (\partial + v_{\ell-1}) \circ P_{\ell-1}^+ \circ \partial (P_{\ell-1}^+)^{-1} \circ (\partial + v_{\ell-1})^{-1}$$
$$= (\partial + v_{\ell-1}) \circ L_{\ell-1} \circ (\partial + v_{\ell-1})^{-1}$$

and

$$L_{-\ell} = P_{-\ell}^+ \circ \partial \circ (P_{-\ell}^+)^{-1} = (\partial + v_{-\ell})^{-1} \circ P_{1-\ell}^+ \circ \partial \circ (P_{1-\ell}^+)^{-1} \circ (\partial + v_{-\ell})$$
$$= (\partial + v_{-\ell})^{-1} \circ L_{1-\ell} \circ (\partial + v_{-\ell}) \,. \tag{19}$$

Secondly, we show that the second equation of (17) holds. This follows from the Sato–Wilson equation (11). Indeed,

$$\frac{\partial P_{\ell+1}^+(t,\partial)}{\partial t_j} = -(L_{\ell+1}(t,\partial)^j)_- \circ (\partial + v_\ell(t)) \circ P_\ell^+(t,\partial)$$
$$= \frac{\partial v_\ell(t)}{\partial t_j} P_\ell^+(t,\partial) - (\partial + v_\ell(t)) \circ (L_\ell(t,\partial)^j)_- \circ P_\ell^+(t,\partial) \,,$$

we deduce that

$$\frac{\partial v_\ell(t)}{\partial t_j} = -\,(L_{\ell+1}(t,\partial)^j)_- \circ (\partial + v_\ell(t)) + (\partial + v_\ell(t)) \circ (L_\ell(t,\partial)^j)_-$$
$$= -\,L_{\ell+1}(t,\partial)^j \circ (\partial + v_\ell(t)) + (L_{\ell+1}(t,\partial)^j)_+ \circ (\partial + v_\ell(t))$$
$$+\,(\partial + v_\ell(t)) \circ L_\ell(t,\partial)^j - (\partial + v_\ell(t)) \circ (L_\ell(t,\partial)^j)_+$$
$$= (L_{\ell+1}(t,\partial)^j)_+ \circ (\partial + v_\ell(t)) - (\partial + v_\ell(t)) \circ (L_\ell(t,\partial)^j)_+ \,.$$

Here we have used that $L_{\ell+1}(t,\partial)^j \circ (\partial + v_\ell(t)) = (\partial + v_\ell(t)) \circ L_\ell(t,\partial)^j$.

To prove the converse, we use a result of Shiota [16], the Claim of Sect. 1.2. He shows that if L_0 satisfies the Lax equation (15), then $w_0^+(t,z)$ is uniquely determined by the linear problem (13), up to multiplication by elements of the form $1 + \sum_{i>0} a_i z^{-i}$, with $a_i \in \mathbb{C}$ or rather $P_0(t,\partial) = 1 + \sum_{i>0} w_i(t)\partial^{-i}$ is a unique solution up to multiplication from the right by elements of the form $1 + \sum_{i>0} a_i \partial^{-i}$, with $a_i \in \mathbb{C}$, of the equations

$$L_0 \circ P_0^+(t,\partial) = P_0^+(t,\partial) \circ \partial, \qquad \frac{\partial P_0^+(t,\partial)}{\partial t_j} = P_0^+(t,\partial) \circ \partial^j - (L_0^j)_+ \circ P_0^+(t,\partial).$$

Hence, $w_0^+(t) = P_0^+(t,\partial)e^{t\cdot z}$ satisfies (13) and thus is a wave function for L_0, so that $w_0^-(t) = (P_0^+(t,\partial))^{*-1}e^{-t\cdot z}$ is the adjoint wave function. For $i > 0$, let

$$P_i^+ = (\partial + v_{i-1}) \circ (\partial + v_{i-2}) \circ \cdots \circ (\partial + v_0) \circ P_0^+,$$
$$P_{-i}^+ = (\partial + v_{-i})^{-1} \circ (\partial + v_{1-i})^{-1} \circ \cdots \circ (\partial + v_{-1})^{-1} \circ P_0^+,$$

and construct all other (adjoint) wave functions via

$$w_i^+ = (\partial + v_{i-1})(w_{i-1}^+), \qquad w_i^- = (\partial + v_{i-1})^{*-1}(w_{i-1}^-),$$
$$w_{-i}^+ = (\partial + v_{-i})^{-1}(w_{1-i}^+), \quad w_{-i}^- = (\partial + v_{-i})^*(w_{1-i}^-). \tag{20}$$

By (17) and (18) these (adjoint) wave functions satisfy the linear problem (13). In order to show that the bilinear identity holds for the wave functions, we first prove that

$$(\partial^j P_k^+(t,\partial) P_\ell^{-*}(t,\partial))_- = 0 \quad \text{for all } k \geq \ell,\ j \geq 0. \tag{21}$$

We show this for $k \geq 0$ and $\ell < 0$ (all other cases are obvious):

$$\begin{aligned} \partial^j \circ P_k^+ P_\ell^{-*} &= \partial^j \circ (\partial + v_{k-1}) \circ \cdots \circ (\partial + v_0) \circ P_0^+ \circ (P_0^+)^{-1} \\ &\quad \circ (\partial + v_{-1}) \circ \cdots \circ (\partial + v_\ell) \\ &= \partial^j \circ (\partial + v_{k-1}) \circ (\partial + v_{k-2}) \circ \cdots \circ (\partial + v_\ell). \end{aligned}$$

Using Lemma 2, we deduce from (21) that

$$\operatorname{Res} \frac{\partial^j w_k^+(s_1, t_2, t_3, \ldots, z)}{\partial s_1^j} w_\ell^-(t_1, t_2, t_3, \ldots, z)\, dz = 0.$$

The second formula of (13) implies that

$$\operatorname{Res} \frac{\partial^{j_1+j_2+\cdots+j_n} w_k^+(s_1, t_2, t_3, \ldots, z)}{\partial s_1^{j_1} \partial t_2^{j_2} \cdots \partial t_n^{j_n}} w_\ell^-(t_1, t_2, t_3, \ldots, z)\, dz = 0.$$

Using Taylor's formula we obtain the bilinear identity (6) for the wave function. The tau-functions τ_i are then obtained up to a scalar factor by the formula (see e.g. [13] equation (111), which is a direct consequence of (7)):

$$\frac{\partial \log \tau_i(t)}{\partial t_j} = \operatorname{Res} z^j \Big(\frac{\partial}{\partial z} - \sum_{k>0} z^{-k-1} \frac{\partial}{\partial t_k} \Big) P_i^+(t,z).$$

Hence, multiplying (6) by $\tau_k(t)\tau_\ell(y)$, we obtain the bilinear identities (4) for the tau-functions, which is the first formulation of MKP. Thus the two formulations are equivalent. □

The v_j are expressed in terms of the tau-functions via the second formula of (16). Using (7), we see that

$$P_0^\pm(t,\partial) = \sum_{i,j=0}^{\infty} \frac{S_i(\mp D)\tau_0}{\tau_0} \partial^{-i}, \quad \text{where } \sum_{i=0}^{\infty} S_i(D) z^i = \exp\Big(\sum_{k=1}^{\infty} \frac{z^k}{k} \frac{\partial}{\partial t_k} \Big).$$

This and the fact that L_0 is given by (12), gives that the u_i can be calculated by the following formula

$$L_0(t,\partial) = \sum_{i,j=0}^{\infty} \frac{S_i(-D)\tau_0}{\tau_0} \partial^{1-i-j} \circ \frac{S_j(D)\tau_0}{\tau_0}.$$

Remark 4. Dickey shows that all flows $\frac{\partial}{\partial t_k}$, defined by (17), commute ([6], Proposition 2.3). Hence (17) is an integrable system of compatible evolution equations in U.

Remark 5. The differential algebra U carries an automorphism S (commuting with ∂), defined by

$$S(v_j) = v_{j+1}, \quad S(L) = (\partial + v) \circ L \circ (\partial + v)^{-1}.$$

The MKP hierarchy can be understood as the following system of partial differential-difference equations ($j = 1, 2, \ldots$):

$$\begin{cases} \dfrac{dL}{dt_j} = [(L^j)_+, L], \\ \dfrac{dv}{dt_j} = (S(L)^j)_+ \circ (\partial + v) - (\partial + v) \circ (L^j)_+ . \end{cases}$$

Here $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots$ and $v = v_0$.

5. Eigenfunction formulation of MKP

There is yet another formulation of MKP. It is given in terms of eigenfunctions and adjoint eigenfunctions of the Lax operators L_k.

Definition 6. *Let $L = L(t, \partial)$ be a pseudo-differential operator with coefficients in $\mathbb{C}(t_1, t_2, \ldots)$, where $\partial = \frac{\partial}{\partial t_1}$. An element $\phi \in \mathbb{C}(t_1, t_2, \ldots)$ is called an eigenfunction (resp. adjoint eigenfunction) for L if*

$$\frac{\partial \phi(t)}{\partial t_n} = (L^n)_+ (\phi(t)) \quad \left(\text{resp. } \frac{\partial \phi(t)}{\partial t_n} = -(L^n)_+^* (\phi(t))\right), \quad n = 1, 2, \ldots . \tag{22}$$

Example 7. Let $L = L(t, \partial)$ be a pseudo-differential operator and $w^+(t, z)$ (resp. $w^-(t, z)$) satisfy

$$\frac{\partial w^+(t,z)}{\partial t_j} = (L^j)_+ w^+(t, z), \quad \left(\text{resp. } \frac{\partial w^-(t,z)}{\partial t_j} = -(L^j)_+^* w^-(t, z)\right),$$

cf. (13) and (14). Then for each $f(z) \in \mathbb{C}((z^{-1}))$ the functions

$$q_f^\pm(t) = \operatorname{Res} f(z) w^\pm(t, z)\, dz, \tag{23}$$

are eigenfunctions (taking $+$) and adjoint eigenfunctions (taking $-$) for L. In particular if $L = \partial$, then

$$q_f^\pm(t) = \operatorname{Res} f(z) e^{\pm t\cdot z}\, dz,$$

are its (adjoint) eigenfunctions.

These (adjoint) eigenfunctions were used by Matveev and Salle [14] to construct new solutions of the KP equation from old ones. In fact we will prove later the following:

Proposition 8. *If $\tau(t)$ is a tau-function, satisfying* (4) *for $k = \ell$, and $L = P^+ \circ \partial \circ (P^+)^{-1}$ is the corresponding Lax operator, where P^+ is given by* (7)*, then $\phi^{\pm}(t)\tau(t)$ is again a tau-function, provided that $\phi^{\pm}(t)$ is an (adjoint) eigenfunction for L.*

We will show (see also [9]) that $\tau(t)$ and $\phi^{\pm}(t)\tau(t)$ satisfy the 1st modified KP hierarchy (4) for $k - \ell = 1$. The converse of this statement also holds, namely we have:

Proposition 9. *Let $\tau_k(t)$ and $\tau_{k+1}(t)$ be KP tau-functions that satisfy* (4) *for $k - \ell = 1$. Then their ratio $\phi_k(t) = \frac{\tau_{k+1}(t)}{\tau_k(t)}$ is an eigenfunction for $L_k = P_k^+ \partial P_k^{+-1}$ and $\frac{1}{\phi_k(t)}$ is an adjoint eigenfunction for $L_{k+1} = P_{k+1}^+ \partial P_{k+1}^{+-1}$, where P_m^+ is given by* (7)*.*

Proof. The tau-function formulation of the 1st MKP hierarchy, i.e., (4) for $k - \ell = 1$ is equivalent to (see e.g. [12], Theorem 2.3 (c), for $l = 1$).

$$\operatorname{Res} z^{-1}\tau_k(t-[z^{-1}])\tau_{k+1}(y+[z^{-1}]) \exp\Big(\sum_{i=1}^{\infty}(t_i - y_i)z^i\Big)\, dz = \tau_{k+1}(t)\tau_k(y). \tag{24}$$

Divide equation (24) by $\tau_{k+1}(t)\tau_k(y)$, to obtain:

$$\operatorname{Res} \phi_k(t)^{-1} w_k^+(t,z)\phi_k(y) w_{k+1}^-(y,z)\, dz = 1. \tag{25}$$

Differentiate this equation by t_n and then multiply by $\phi_k(t)$, to obtain

$$\operatorname{Res}\Big(-\frac{\partial \phi_k(t)}{\partial t_n}\phi_k(t)^{-1} w_k^+(t,z) + \big(L_k^n\big)_+ (w_k^+(t,z))\Big)\phi_k(y) w_{k+1}^-(y,z)\, dz$$
$$= 0.$$

Using Lemma 2, (7), (8) and the fact that

$$w_{k+1}^-(y,z) = \frac{1}{\phi_k(y)}(-\partial)^{-1} \circ \phi_k(y)(P_k^+(y,\partial))^{*-1} e^{-\sum_i y_i z^i},$$

we obtain

$$\Big(-\frac{\partial\phi_k(t)}{\partial t_n}\phi_k(t)^{-1} P_k^+(t) \circ P_k^+(t)^{-1} \circ \phi_k(t)\partial^{-1} + (L_k^n)_+ \circ P_k^+(t) \circ P_k^+(t)^{-1} \circ \phi_k(t)\partial^{-1}\Big)_-$$
$$= 0.$$

Taking the residue of this expression (i.e., the coefficient of ∂^{-1}) gives equation (22). The second formula can be also obtained from (25) in almost the same way, but now one has to differentiate this equation by y_1 and continue in a similar manner. □

One also has:

Proposition 10. *Let $\phi_k(t)$ be as in the previous Proposition and let $w_k^+(t,z) = P_k^+(t,z)z^k e^{t\cdot z}$ and $w_{k+1}^-(t,z) = P_{k+1}^-(t,-z)z^{-k-1}e^{-t\cdot z}$ be the (adjoint) wave function, corresponding to τ_k and τ_{k+1}, i.e., given by* (7) *and* (8) *satisfying* (6) *for $\ell = k+1$. Then*

$$P_{k+1}^+(t,\partial)\circ\partial = \phi_k(t)\partial\circ\frac{1}{\phi_k}(t)P_k^+(t,\partial) \tag{26}$$

and

$$L_{k+1} = \phi_k(t)\partial\circ\frac{1}{\phi_k(t)}L_k\circ\phi_k(t)\partial^{-1}\circ\frac{1}{\phi_k(t)}. \tag{27}$$

Proof. If we divide equation (24) by $\tau_k(t)\tau_{k+1}(y)$, we obtain

$$\operatorname{Res} w_k^+(t,z)\phi_k(y)w_{k+1}^-(y,z)\,dz = \phi_k(t)\frac{1}{\phi_k(y)}. \tag{28}$$

which is equivalent to (6). Using Lemma 2 and (10), we deduce that

$$P_k^+(t,\partial)\circ\partial^{-1}\circ P_{k+1}^+(t,\partial)^{-1} = \phi_k(t)\partial^{-1}\circ\frac{1}{\phi_k(t)},$$

which gives (26). Then (27) follows from (12). □

The converse also holds:

Proposition 11. *Let $\phi^+(t)$ be an eigenfunction and $\phi^-(t)$ be an adjoint eigenfunction for the Lax operator $L = P\partial P^{-1}$, i.e., L satisfies* (15)*, where P is a dressing operator, satisfying the Sato–Wilson equation* (11)*, then*

$$Q = \phi^+(t)\partial\circ\frac{1}{\phi^+(t)}P \quad \text{and} \quad R = \frac{1}{\phi^-(t)}\partial^{-1}\circ\phi^-(t)P$$

also satisfy (11) *and both*

$$Q\circ\partial\circ Q^{-1} \quad \text{and} \quad R\circ\partial\circ R^{-1}$$

are Lax operators.

For a proof of this proposition, see pages 499 and 500 of [9].

Proof of Proposition 8. We will only consider the case of eigenfunctions. The proof for adjoint eigenfunctions is similar. Use the previous Proposition, then

$$\operatorname{Res} Qe^{t\cdot z}(P^*)^{-1}e^{y\cdot z}\,dz = \phi^+(t)\partial_{t_1}\circ\frac{1}{\phi^+(t)}\operatorname{Res} Pe^{t\cdot z}(P^*)^{-1}e^{-y\cdot z}\,dz = 0.$$

Hence the wave function $Qe^{t\cdot z}$ and the adjoint wave function $(P^*)^{-1}e^{-y\cdot z}$ satisfy the 1st modified KP hierarchy, (6) for $k = \ell + 1$. Therefore, $Pe^{t\cdot z}$ and $(Q^*)^{-1}e^{-y\cdot z}$ satisfy (25), i.e.,

$$\operatorname{Res} Pe^{t\cdot z} Q^{*-1} e^{y\cdot z}\, dz = \frac{\phi^+(t)}{\phi^+(y)}\,.$$

Let τ be the tau-function which corresponds to P and τ_1 be the tau-function that corresponds to Q, then

$$\operatorname{Res} z^{-1}\tau(t-[z^{-1}])\tau_1(y+[z^{-1}]) \exp\Big(\sum_{i=1}^{\infty}(t_i-y_i)z^i\Big)\, dz = \tau(t)\phi^+(t)\frac{\tau_1(y)}{\phi^+(y)},$$

which must be equation (24). Thus $\tau_1(t) = \phi^+(t)\tau(t)$. □

Define

$$\phi_k^+(t) = \phi_k(t)\Big(\text{resp. } \phi_k^-(t) = \frac{1}{\phi_{-k-1}}\Big) \quad \text{for } k \geq 0,$$

which are eigenfunctions for L_k (resp. adjoint eigenfunctions for L_{-k}). Then by Proposition 9,

$$\phi_k^+(t) = \frac{1}{\phi_{-k-1}^-(t)} = \frac{\tau_{k+1}(t)}{\tau_k(t)}, \tag{29}$$

and (by (16) and Proposition 9)

$$\partial + v_k(t) = \begin{cases} \partial - \partial(\log \phi_k^+(t)) = \phi_k^+(t)\partial \circ \dfrac{1}{\phi_k^+(t)} & \text{for } k \geq 0, \\ \partial + \partial(\log \phi_{-k-1}^-(t)) = \dfrac{1}{\phi_{-k-1}^-(t)}\partial \circ \phi_{-k-1}^-(t) & \text{for } k < 0, \end{cases} \tag{30}$$

and

$$\begin{aligned} w_{k+1}^{\pm}(t,z) &= \pm(\phi_k^+(t)^{\pm 1}\partial^{\pm 1} \circ \phi_k^+(t)^{\mp 1})w_k^{\pm}(t,z), \\ w_{-k-1}^{\pm}(t,z) &= \pm(\phi_k^-(t)^{\mp 1}\partial^{\mp 1} \circ \phi_k^-(t)^{\pm 1})w_{-k}^{\pm}(t,z). \end{aligned} \tag{31}$$

It is clear that the first and the second formulation of MKP imply the following.

The third formulation of the MKP hierarchy: *Let $W = \mathbb{C}[u_i^{(n)}, \phi_j^{\pm(n)} \mid i \in \mathbb{Z}_{\geq 1},\ j,n \in \mathbb{Z}_{\geq 0}]$ be the algebra of differential polynomials in u_i and $\phi_j^{\pm}$, where $\partial u_i^{(n)} = u_i^{(n+1)}$, $\partial\phi_j^{\pm(n)} = \phi_j^{\pm(n+1)}$. Let $L_0(\partial) = \partial + u_1(t)\partial^{-1} +$*

$u_2(t)\partial^{-2}\cdots \in W((\partial^{-1}))$ be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in W:

$$\begin{aligned}\frac{\partial L_0(\partial)}{\partial t_j} &= [(L_0(\partial)^j)_+, L_0(\partial)], \quad \frac{\partial \phi_i^+}{\partial t_j} = (L_i(\partial)^j)_+(\phi_i^+),\\ \frac{\partial \phi_i^-}{\partial t_j} &= -(L_{-i}(\partial)^j)_+^*(\phi_i^-)\end{aligned} \tag{32}$$

for $j \in \mathbb{Z}_{\geq 1}$ and $i \in \mathbb{Z}_{\geq 0}$, where the L_i and L_{-i}, for $i > 0$, are defined by

$$\begin{aligned}L_i &= \phi_{i-1}^+\partial \circ \frac{1}{\phi_{i-1}^+} L_{i-1} \circ \phi_{i-1}^+\partial^{-1} \circ \frac{1}{\phi_{i-1}^+},\\ L_{-i} &= \frac{1}{\phi_{i-1}^-}\partial^{-1} \circ \phi_{i-1}^- L_{1-i} \circ \frac{1}{\phi_{i-1}^-}\partial \circ \phi_{i-1}^-.\end{aligned}$$

Theorem 12. *All three formulations of the MKP are equivalent.*

Proof. Assume the third formulation of MKP holds. Define for $i \geq 0$ the function $v_i = -\partial \log \phi_i^+$ and $v_{-i-1} = \partial \log \phi_i^-$. Then

$$w_{i+1}^+(t,z) = \phi_i^+(t)\partial \circ \frac{1}{\phi_i^+(t)}(w_i^+(t,z)) = (\partial + v_i(t))(w_i^+(t,z))$$

is a wave function for $L_{i+1} = (\partial + v_i(t))L_i(\partial + v_i(t))^{-1}$. One finds similar wave functions and relations between these wave functions if $i < 0$. Hence, the same proof as the proof of Theorem 3 gives the second equation of (17). Equation (18) is obvious. □

Now, for $i > 0$, the tau-function is equal to (by (29))

$$\tau_{\pm i} = \phi_{i-1}^{\pm}\tau_{\pm(i-1)} = \phi_{i-1}^{\pm}\phi_{i-2}^{\pm}\tau_{\pm(i-2)} = \cdots = \phi_{i-1}^{\pm}\phi_{i-2}^{\pm}\cdots\phi_0^{\pm}\tau_0\,, \tag{33}$$

and the (adjoint) wave function $w_{\pm i}^{\pm}(t,z) = M_{\pm i}(t,\partial)(w_0^{\pm}(t,z))$, where $M_0 = 1$ and by (31) and (30):

$$\begin{aligned}M_{\pm i}(t,\partial) &= (\pm\partial + v_{\pm(i-\frac{1}{2}\mp\frac{1}{2})}) \circ M_{\pm(i-1)}(t,\partial)\\ &= \pm\phi_{i-1}^{\pm}\partial \circ \frac{1}{\phi_{i-1}^{\pm}} M_{\pm(i-1)}(t,\partial)\\ &= \phi_{i-1}^{\pm}\partial \circ \frac{1}{\phi_{i-1}^{\pm}}\phi_{i-2}^{\pm}\partial \circ \frac{1}{\phi_{i-2}^{\pm}} M_{\pm(i-2)}(t,\partial)\\ &= \cdots\\ &= (\pm 1)^i \phi_{i-1}^{\pm}\partial \circ \frac{\phi_{i-2}^{\pm}}{\phi_{i-1}^{\pm}}\partial \circ \frac{\phi_{i-3}^{\pm}}{\phi_{i-2}^{\pm}}\partial \circ \cdots \circ \frac{\phi_0^{\pm}}{\phi_1^{\pm}}\partial \circ \frac{1}{\phi_0^{\pm}}\end{aligned} \tag{34}$$

is an i-th order differential operator. Using the connection between the wave function and adjoint wave function we have, $w^+_{-i}(t,z) = M^{*-1}_{-i}(t,\partial)(w^+_0(t,z))$ and using the relation between the wave function and the Lax operator (12), we find

$$L_i = M_i \circ L_0 \circ M_i^{-1} \quad \text{and} \quad L_{-i} = (M^*_{-i})^{-1} \circ L_0 \circ M^*_{-i}. \tag{35}$$

In the polynomial case, using the boson-fermion correspondence σ, it is not difficult to find these (adjoint) eigenfunctions. We know from the results of [11] that if $\sigma^{-1}(\tau_n q^n) = f_n \in \mathscr{O}_n$, then $\sigma^{-1}(\tau_{n+1}q^{n+1}) = w \wedge f_n$ for some $w = \sum_i a_i e_i \in \mathbb{C}^\infty$. We have

$$\begin{aligned} f_{n+1} &= w \wedge f_n = \Big(\sum_i a_i e_i\Big) \wedge f_n = \sum_i a_i \psi^+_{-i+\frac{1}{2}}(f_n) \\ &= \operatorname{Res} \sum_i a_i z^{-i} \psi^+(z)(f_n)\, dz, \end{aligned}$$

since this holds for $f_n = |n\rangle$ and $f_{n+1} = |n+1\rangle$. Thus if we define $\phi^+_n(t) = \operatorname{Res} \sum_i a_i z^{-i} w^+_n(t,z)\, dz$, then by (5) we find that

$$\begin{aligned} \tau_{n+1}q^{n+1} &= \sigma\Big(\operatorname{Res} \sum_i a_i z^{-i} \psi^+(z)(f_n)\, dz\Big) \\ &= \operatorname{Res} \sum_i a_i z^{-i} \sigma \psi^+(z) \sigma^{-1}\, dz\, \tau_n q^n \\ &= \operatorname{Res} \sum_i a_i z^{-i} w^+_n(t,z)\, dz\, \tau_n q^{n+1} \\ &= \phi^+_n \tau_n q^{n+1}, \end{aligned} \tag{36}$$

hence

$$\tau_{n+1} = \phi^+_n \tau_n, \quad \text{where } \phi^+_n = \operatorname{Res} \sum_i a_i z^{-i} w^+_n(t,z)\, dz. \tag{37}$$

Since $f_{n-1} = \sum_i b_i \psi^-_{i+\frac{1}{2}}(f_n)$, with $b_i \in \mathbb{C}$, we find in a similar way

$$\tau_{-n-1} = \phi^-_n \tau_{-n}, \quad \text{where } \phi^-_n(t) = \operatorname{Res} \sum_i b_i z^i w^-_{-n}(t,z)\, dz\, . \tag{38}$$

Thus we have the following:

Lemma 13. *In the polynomial setting every (adjoint) eigenfunction is of the form* (23).

Observe that since $\phi_1^\pm = \operatorname{Res} f(z)w^\pm_{\pm 1}(z)\,dz$ for some $f(z)$, we find that if we define $q_0^\pm = \phi_0^\pm$ and $q_1^\pm = \operatorname{Res} f(z)w_0^\pm(z)\,dz$, which are both (adjoint) eigenfunctions of L_0, then using (31) we deduce that

$$\begin{aligned}
\phi_1^\pm &= \operatorname{Res} f(z)w^\pm_{\pm 1}(z)\,dz\\
&= \pm \operatorname{Res} f(z)\phi_0^\pm \partial\Big(\frac{w_0^\pm(z)}{\phi_0^+}\Big)\,dz\\
&= \pm\, q_0^\pm \partial\Big(\frac{q_1^\pm}{q_0^\pm}\Big)\\
&= \pm\,\Big(\partial(q_1^\pm) - \frac{q_1^\pm}{q_0^\pm}\partial(q_0^\pm)\Big).
\end{aligned}$$

Thus

$$\tau_{\pm 2} = \phi_0^\pm\phi_1^\pm\tau_0 = \pm q_0^\pm\Big(\partial(q_1^\pm) - \frac{q_1^\pm}{q_0^\pm}\partial(q_0^\pm)\Big)\tau_0 = \pm\det\begin{pmatrix} q_0^\pm & q_1^\pm\\ \partial(q_0^\pm) & \partial(q_1^\pm)\end{pmatrix}\tau_0.$$

Note that we can remove the possible minus sign in front of the determinant. If τ_2 is a tau-function, then a multiple of τ_2 is also a tau-function. From now on we will always do so, i.e., forget about the sign of the tau-function.

Using formula (34), we deduce that

$$M_{\pm 1} = \pm\phi_0^\pm\partial\circ\frac{1}{\phi_0^\pm}$$

and

$$\begin{aligned}
M_{\pm 2} &= \phi_1^\pm\partial\circ\frac{\phi_0^\pm}{\phi_1^\pm}\partial\circ\frac{1}{\phi_0^\pm}\\
&= \frac{1}{q_0^\pm}\left(q_0^\pm\partial(q_1^\pm) - q_1^\pm\partial(q_0^\pm)\right)\partial\circ\frac{(q_0^\pm)^2}{q_0^\pm\partial(q_1^\pm) - q_1^+\partial(q_0^\pm)}\partial\circ\frac{1}{q_0^\pm}\\
&= \left(\det\begin{pmatrix} q_0^\pm & q_1^\pm\\ \partial(q_0^\pm) & \partial(q_1^\pm)\end{pmatrix}\right)^{-1}\det\begin{pmatrix} q_0^\pm & q_1^\pm & 1\\ \partial(q_0^\pm) & \partial(q_1^\pm) & \partial\\ \partial^2(q_0^\pm) & \partial^2(q_1^\pm) & \partial^2\end{pmatrix}.
\end{aligned}$$

Continuing in this way, see e.g. Theorem 5.1 of [9] for more details, it is possible to express $M_{\pm i}$ in terms of certain (adjoint) eigenfunctions $q_k^\pm(t)$ of the operator L_0, i.e., if

$$\phi_k^\pm = \operatorname{Res} f_k(z)w^\pm_{\pm k}\,dz,$$

for some $f_k(z)\in\mathbb{C}[z,z^{-1}]$, then we define

$$q_k^\pm = \operatorname{Res} f_k(z)w_0^\pm\,dz.$$

These $q_k^{\pm}(t)$ are (adjoint) eigenfunctions for $L_0(\partial)$ by (23). We have the following formulas:

$$\tau_{\pm i} = W_{\pm i}\tau_0 \quad \text{and} \quad w_{\pm i}^{\pm} = M_{\pm i}(w_0^{\pm}) \quad \text{and} \quad w_{-i}^{+} = (M_{-i}^{*})^{-1}(w_0^{+}), \tag{39}$$

where $M_{\pm i} = (\pm 1)^i W_{\pm i}(\partial)/W_{\pm i}$, and

$$\begin{aligned} W_{\pm i}(\partial) &= \det \begin{pmatrix} q_0^{\pm} & \cdots & q_{i-1}^{\pm} & 1 \\ \partial(q_0^{\pm}) & \cdots & \partial(q_{i-1}^{\pm}) & \partial \\ \vdots & \ddots & \vdots & \vdots \\ \partial^i(q_0^{\pm}) & \cdots & \partial^i(q_{i-1}^{\pm}) & \partial^i \end{pmatrix} \quad \text{and} \\ W_{\pm i} &= \det \begin{pmatrix} q_0^{\pm} & \cdots & q_{i-1}^{\pm} \\ \partial(q_0^{\pm}) & \cdots & \partial(q_{i-1}^{\pm}) \\ \vdots & \ddots & \vdots \\ \partial^{i-1}(q_0^{\pm}) & \cdots & \partial^{i-1}(q_{i-1}^{\pm}) \end{pmatrix} \end{aligned} \tag{40}$$

are Wronskian determinants. The determinants $W_{\pm i}(\partial)$ are computed by expanding along the last column, putting the cofactors to the left of the ∂^j's.

Let us prove the formulas of (39). If $\tau_{\pm i} = W_{\pm i}\tau_0$, then

$$\begin{aligned} \tau_{\pm i \pm 1} &= \phi_i^{\pm}\tau_{\pm i} \\ &= \operatorname{Res} f_i(z) w_{\pm i}^{\pm}\, dz\, W_{\pm i}\tau_0 \\ &= \operatorname{Res} f_i(z) M_{\pm i}(w_0^{\pm})\, dz\, W_{\pm i}\tau_0 \\ &= \operatorname{Res} f_i(z) W_{\pm i}(\partial)(w_0^{\pm})\, dz\, \tau_0 \\ &= W_{\pm i}(\partial)(\operatorname{Res} f_i(z) w_0^{\pm}\, dz)\tau_0 \\ &= W_{\pm i}(\partial)(q_i^{\pm})\tau_0 \\ &= W_{\pm(i+1)}\tau_0. \end{aligned}$$

Thus

$$\phi_i^{\pm} = \frac{W_{\pm(i+1)}}{W_{\pm i}},$$

and using this, we find that

$$\begin{aligned} w_{\pm(i+1)}^{\pm} &= \pm\phi_i^{\pm}\partial \circ \frac{1}{\phi_i^{\pm}}(w_{\pm i}^{\pm}) \\ &= (\pm 1)^{i+1}\frac{W_{\pm(i+1)}}{W_{\pm i}}\partial \circ \frac{W_{\pm i}}{W_{\pm(i+1)}} \circ M_{\pm i}(w_0^{\pm}) \\ &= (\pm 1)^{i+1}\frac{W_{\pm(i+1)}}{W_{\pm i}}\partial \circ \frac{W_{\pm i}}{W_{\pm(i+1)}}\Big(\frac{W_{\pm i}(\partial)(w_0^{\pm})}{W_{\pm i}}\Big) \end{aligned}$$

$$
\begin{aligned}
&= (\pm 1)^{i+1} \frac{W_{\pm(i+1)}(\partial)(w_0^{\pm})}{W_{\pm(i+1)}} \\
&= M_{\pm(i+1)}(w_0^{\pm}).
\end{aligned}
$$

The next to the last equality follows from Crum's Identity for Wronskian determinants (which is in fact the Desnanot–Jacobi identity for Wronskians, see [4], Sect. 3):

$$
W_{\pm(i+1)}\partial \circ W_{\pm i}(\partial) - \partial(W_{\pm(i+1)})W_{\pm i}(\partial) = W_{\pm i}W_{\pm(i+1)}(\partial). \tag{41}
$$

Thus $w_{-i}^+ = (M_{-i}^*)^{-1}(w_0^+)$. Now by (35) we find that

$$
\begin{aligned}
L_i &= M_i \circ L_0 \circ M_i^{-1} &&= W_i(\partial)/W_i \circ L_0 \circ (W_i(\partial)/W_i)^{-1}\,, \\
L_{-i} &= M_{-i}^{*-1} \circ L_0 \circ M_{-i}^* &&= (W_{-i}(\partial)/W_{-i})^{*-1} \circ L_0 \circ (W_{-i}(\partial)/W_{-i})^*\,.
\end{aligned} \tag{42}
$$

Remark 14. Let $i \geq 0$ and let $f_i = \sigma^{-1}(\tau_i(t)q^i)$. Then $f_i \in \mathscr{O}_i$, which means that

$$
f_i = v_i \wedge v_{i-1} \wedge \cdots \wedge v_2 \wedge v_1 \wedge f_0, \quad \text{where } v_j = \sum_s a_{sj}e_s,\ f_0 \in \mathscr{O}_0, \tag{43}
$$

and the eigenfunctions of L_j are of the form

$$
\phi_j^+(t) = \operatorname{Res} w_j^+(t,z) \sum_i a_{i,j+1} z^{-i}\, dz.
$$

Hence, this eigenfunction is determined by $w_j^+(t,z)$ and by v_{j+1}. Define

$$
q_j^+(t) = \operatorname{Res} w_0^+(t,z) \sum_i a_{i,j+1} z^{-i}\, dz.
$$

Since M_i is of the form (34), $\phi_0^+(t) = q_0^+(t)$ is in the kernel of M_i. However, if we reorder the v_j's in (43) we get the same element up to a sign. This gives different eigenfunctions ϕ_j^+ and different L_j for $j = 1, 2, \ldots, i-1$, but M_i is the same and L_i is the same. Hence we can put every v_j in (43) just before f_0, which means that the new $f_1 = v_j \wedge f_0$, thus we get a new eigenfunction ϕ_0^+ which is now equal to $q_j^+(t)$. Moreover, if $f_i \neq 0$, then $q_j^+(t) \neq 0$. Thus $q_0^+(t), q_1^+(t), \ldots, q_{i-1}^+(t)$ are non-zero eigenfunctions for L_0 which are all in the kernel of M_i, and clearly must be linearly independent otherwise the element f_i would be 0. Similarly

$$
f_{-i} = v_{-i}(v_{1-i}(\cdots(v_{-2}(v_{-1}(f_0)\cdots)))),
$$

where $v_j = \sum_i b_{ij} \psi^-_{i+\frac{1}{2}}$. Then

$$\phi^-_{j-1}(t) = \operatorname{Res} w^-_{1-j}(t,z) \sum_i b_{i,-j} z^i \, dz,$$

and

$$q^-_{j-1}(t) = \operatorname{Res} w^-_0(t,z) \sum_i b_{i,-j} z^i \, dz,$$

and all $q^-_j(t)$ for $0 \le j < i$ are in the kernel of M_{-i}.

The fourth formulation of the MKP hierarchy: *Let $V = \mathbb{C}[u_i^{(n)}, q_j^{\pm(n)} \mid i \in \mathbb{Z}_{\ge 1},\ j,n \in \mathbb{Z}_{\ge 0}]$ be the algebra of differential polynomials in u_i and $q_j^{\pm}$. Let $L_0 = \partial + u_1(t)\partial^{-1} + \cdots \in V((\partial^{-1}))$ be a pseudo-differential operator. Then the MKP hierarchy is the following system of evolution equations in V:*

$$\begin{aligned} &\frac{\partial L_0(\partial)}{\partial t_j} = [(L_0(\partial)^j)_+, L_0(\partial)], \quad \frac{\partial q_i^+}{\partial t_j} = (L_0(\partial)^j)_+(q_i^+), \\ &\frac{\partial q_i^-}{\partial t_j} = -(L_0(\partial)^j)^*_+(q_i^-). \end{aligned} \tag{44}$$

Now we are able to prove the following:

Theorem 15. *In the polynomial setting, all four formulations of MKP are equivalent.*

Proof. It suffices to establish the equivalence between the third and fourth formulation. To obtain the fourth formulation from the third, we use the fact that if $\phi_i^{\pm}(t)$ is given, then by Lemma 13 this (adjoint) eigenfunction for $L_{\pm i}$ is equal to

$$\phi_i^{\pm}(t) = \operatorname{Res} f^{\pm}(z) w^{\pm}_{\pm i}(z) \, dz \quad \text{for some } f(z) \in \mathbb{C}((z^{-1})).$$

Then we define the $q_i^{\pm}(t)$ of the fourth formulation by

$$q_i^{\pm}(t) = \operatorname{Res} f^{\pm}(z) w_0^{\pm}(z) \, dz \quad \text{for the same } f(z) \in \mathbb{C}((z^{-1})),$$

which now is an (adjoint) eigenfunction for L_0. This $q_i^{\pm}(t)$ for $i \ge 0$ is (by Remark 14) in the kernel of $M_{\pm j}$ (defined in (34)) for $j \ge i$, and since it is an (adjoint) eigenfunction for L_0, it satisfies the second (third) formula of (44). Hence this establishes the fourth formulation of MKP.

Assume the fourth formulation holds. Define $\phi^{\pm}_{\pm n} = (-1)^n \frac{W_{\pm n \pm 1}}{W_{\pm n}}$; together with L_0 they form the data of the third formulation. Since $q_i^{\pm}$ is an (adjoint) eigenfunction of L_0, then by Lemma 13 there exist functions $f_i^{\pm}(z) \in \mathbb{C}((z^{-1}))$, such that

$$q_i^{\pm}(t) = \operatorname{Res} f_i^{\pm}(z) w_0^{\pm}(t,z) \, dz. \tag{45}$$

Let τ_0 be the tau-function for L_0. Since $q_0^{\pm} = \phi_0^{\pm}$ is an (adjoint) eigenfunction of L_0, by Proposition 8, the tau-functions for $L_{\pm}$ are

$$\tau_{\pm 1} = W_{\pm 1}\tau_0 = \phi_0^{\pm}\tau_0.$$

The corresponding (adjoint) wave functions are (by Propositions 10 and 11)

$$\begin{aligned} w_1^+(t,z) &= M_1(w_0^+(t,z)) &&= \phi_0^+(t)\partial \circ \frac{1}{\phi_0^+(t)}(w_0^+(t,z)),\\ w_{-1}^-(t,z) &= M_{-1}(w_0^-(t,z)) &&= -\phi_0^-(t)\partial \circ \frac{1}{\phi_0^+(t)}(w_0^-(t,z)), \end{aligned} \tag{46}$$

where $M_{\pm 1}$ is given by (39). The corresponding Lax operator $L_{\pm 1}$ is defined by (42), which is the same as $L_{\pm 1}$ in the third formulation, because of (46). Let

$$\begin{aligned} \phi_1^{\pm}(t) &= \operatorname{Res} f_1^{\pm}(z) w_{\pm 1}^{\pm}(t,z)\, dz\\ &= \pm \operatorname{Res} f_1^{\pm}(z)\frac{W_{\pm 1}(\partial)(w_0^{\pm}(t,z))}{W_{\pm 1}}\\ &= \pm\frac{W_{\pm 1}(\partial)(q_1^{\pm}(t))}{W_{\pm 1}}\\ &= \pm\frac{W_{\pm 2}}{W_{\pm 1}}, \end{aligned}$$

where $f_1^{\pm}(z)$ is given by (45), which are non-zero by Remark 14. Now, $\phi_1^{\pm}(t)$ is an (adjoint) eigenfunction for $L_{\pm 1}$, hence the second equation of (32) holds for $L_{\pm 1}$ and $\phi_1^{\pm}$. Thus (by (39)) we obtain that the tau-functions for $L_{\pm 2}$ are equal to

$$\tau_{\pm 2} = W_{\pm 2}\tau_0 = \frac{W_{\pm 2}}{W_{\pm 1}} W_{\pm 1}\tau_0 = \pm\phi_1^{\pm}\tau_{\pm 1}.$$

The corresponding (adjoint) wave functions are given by (39) and (40), and we have

$$w_2^+(t,z) = M_2(w_0^+(t,z)) = \frac{W_2(\partial)(w_0^+(t,z))}{W_2}.$$

By Crum's identity (41) we find that

$$\begin{aligned} w_2^+(t,z) &= \frac{W_2}{W_1}\partial \circ \frac{W_1}{W_2}\Big(\frac{W_1(\partial)(w_0^+(t,z))}{W_1}\Big)\\ &= \phi_1^+(t)\partial \circ \frac{1}{\phi_1^+(t)} \circ M_1(w_0^+(t,z))\\ &= \phi_1^+(t)\partial \circ \frac{1}{\phi_1^+(t)}(w_1^+(t,z)),\\ w_{-2}^-(t,z) &= M_{-2}(w_0^-(t,z)) = -\phi_1^-(t)\partial \circ \frac{1}{\phi_1^-(t)}(w_{-1}^-(t,z)). \end{aligned} \tag{47}$$

The corresponding Lax operator $L_{\pm 2}$ is defined by (42), which is the same as the one in the third formulation, because of (47). Let

$$\phi_2^{\pm}(t) = \operatorname{Res} f_2^{\pm}(z) w_{\pm 2}^{\pm}(t,z)\, dz = \operatorname{Res} f_2^{\pm}(z) \frac{W_{\pm 2}(\partial)(w_0^{\pm}(t,z))}{W_{\pm 2}}\, dz = \frac{W_{\pm 3}}{W_{\pm 2}},$$

where again $f_2^{\pm}(z)$ is given by (45). This is again an (adjoint) eigenfunction for $L_{\pm 2}$ and hence the second equation of (32) holds for $L_{\pm 2}$ and $\phi_2^{\pm}$. Continuing along these lines gives the third formulation and hence we have proved that all four formulations are equivalent. □

6. Polynomial solutions of MKP

We are now going to construct polynomial tau-functions for MKP. We assume that $f_0 = |0\rangle$ which means that $\tau_0(t) = 1$, $w^{\pm}(t,z) = e^{\pm t\cdot z}$ and $L_0 = \partial$. We construct a $L_0 = \partial$ eigenfunction by the procedure described in Example 7 at the beginning of Sect. 5. Since $f_1 = w \wedge f_0 = w \wedge |0\rangle$ and the vacuum is given by (1), such a w can be chosen of the form $w = \sum_{j=0}^{\infty} a_j e_{j+1}$, thus the corresponding eigenfunction $q^+(t) = \tau_1(t)$ is of the form (see Example 7)

$$q^+(t) = \operatorname{Res} \sum_{j=0}^{\infty} a_j z^{-j-1} e^{t\cdot z}\, dz.$$

A similar construction is possible for the adjoint eigenfunction, in fact we have that all (adjoint) eigenfunctions are of the form

$$q_i^{\pm}(t) = \operatorname{Res} f_i^{\pm}(z) e^{\pm t\cdot z}\, dz \quad \text{for some } f_i^{\pm}(z) = \sum_{j=0}^{\infty} a_{ji}^{\pm} z^{-j-1}. \tag{48}$$

Since $\tau_0 = 1$ and $\tau_n = W_n \tau_0$ (see (39)), the corresponding tau-function is equal to $\tau_n = W_n$, for $n \in \mathbb{Z}$, the Wronskian determinant of the (adjoint) eigenfunctions. Now using the elementary Schur polynomials, which are defined by

$$e^{t\cdot z} = \sum_{j=0}^{\infty} s_j(t) z^j, \tag{49}$$

we find (see (48)) that

$$q_i^{\pm}(t) = \operatorname{Res} f_i^{\pm}(z) e^{\pm t\cdot z}\, dz = \sum_{j=0}^{\infty} a_{ji}^{\pm} s_j(\pm t).$$

One obtains polynomial tau-functions by taking $f_i^{\pm}(z) = \sum_{j=0}^{M_i^{\pm}} a_{ji}^{\pm} z^{-j-1}$. To simplify notation we shall sometimes drop the superscripts $\pm$. Without loss of generality we may assume that $a_{M_i,i} = 1$, then

$$q_i^{\pm}(t) = s_{M_i}(\pm t) + \sum_{j=0}^{M_i-1} a_{ji} s_j(\pm t).$$

One can find recursively constants $c_i = (c_{1i}, c_{2i}, \ldots, c_{M_i i})$, such that

$$q_i^{\pm}(t) = s_{M_i}(\pm t) + \sum_{j=0}^{M_i-1} a_{ji} s_j(\pm t) = s_{M_i}(\pm(t + c_i)). \tag{50}$$

Indeed, since, $s_{M_i}(t + c_i) = \sum_{j=0}^{M_i} s_j(c_i) s_{M_i - j}(t)$, which follows immediately from (49), one has to solve equations of the form $s_j(c_i) = a_{M_i-j,i}$ and this can be done recursively since $s_j(c_i) = c_{ji} + p_j(c_{1i}, \ldots, c_{j-1,i})$, where p_j is some polynomial. First, determine $c_{1,i}$, which is determined by $a_{M_i-1,i}$, then $c_{2,i}$, which is determined by $a_{M_i-2,i}$ and c_{1i}, then c_{3i}, which is determined by $a_{M_i-3,i}$, c_{1i} and c_{2i}, etc. In fact there is an explicit formula for these constants. Since

$$1 + \sum_{j=1}^{M_i} a_{M_i-j,i} z^j = \sum_{j=0}^{M} s_j(c_i) z^j,$$

which is equal to the first $M_i + 1$ terms of $\exp(\sum_{j=1}^{M_i} c_{ji} z^j)$, the logarithm of this gives that

$$\sum_{\ell=1}^{M_i} c_{\ell i} z^{\ell} i + \text{ higher order terms} = \log\Big(1 + \sum_{k=1}^{M_i} a_{M_i-k,i} z^k\Big).$$

Hence

$$c_{ki} = - \sum_{\substack{m_1+2m_2+\cdots+km_k=k \\ m_1\ge 0, m_2\ge 0,\ldots,m_k\ge 0}} \prod_{j=1}^{k} \frac{(-a_{M_i-j,i})^{m_j}}{m_j}.$$

Since $\tau_0 = 1$ and $\tau_{\pm n} = W_{\pm n}\tau_0$, we have (see (40) and (50)) that

$$\begin{aligned}\tau_{\pm n}(t) &= W(q_0^{\pm}(t), q_1^{\pm}(t), \ldots, q_{n-1}^{\pm}(t)) \\ &= W(s_{M_0^{\pm}}(\pm t + c_0^{\pm}), s_{M_1^{\pm}}(\pm t + c_1^{\pm}), \ldots, s_{M_{n-1}^{\pm}}(\pm t + c_{n-1}^{\pm})),\end{aligned} \tag{51}$$

where $W(\cdot)$ stands for the Wronskian determinant of those (adjoint) eigenfunctions, satisfies the KP hierarchy. This shows that every function of the form (51) is a polynomial tau-function. Moreover, one has the following remarkable theorem:

Theorem 16. *(a) All polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form*

$$\begin{aligned}&\tau_{\lambda_1,\lambda_2,\ldots,\lambda_k}(t;c_1,c_2,\ldots,c_k)\\&=\det\left(s_{\lambda_i+j-i}(t_1+c_{1i},t_2+c_{2i},t_3+c_{3i},\ldots)\right)_{1\le i,j\le k},\end{aligned}\tag{52}$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k)$ *is a partition and* $c_i=(c_{1i},c_{2i},\ldots)\in\mathbb{C}^{\lambda_i+k-i}$ *are arbitrary.*

(b) All polynomial tau-functions of the MKP hierarchy are the sequences $(\ldots,\tau_n,\tau_{n+1},\ldots)$*, where each* τ_n *is, up to a constant factor, of the form* (52)*, and* τ_{n+1} *is obtained from* τ_n*, up to a constant factor, in one of the following three possible ways:*

- $\tau_{\mu,\lambda_1,\lambda_2,\ldots,\lambda_k}(t;d,c_1,c_2,\ldots,c_k)$*, with* $\mu\ge\lambda_1$*;*
- $\tau_{\lambda_1-1,\lambda_2-1,\ldots,\lambda_i-1,\mu,\lambda_{i+1},\ldots,\lambda_k}(t;c_1,c_2,\ldots,c_i,d,c_{i+1},\ldots,c_k)$*, for* $i=1,2,\ldots,k$*, with* $\lambda_i>\mu\ge\lambda_{i+1}$*;*
- $\tau_{\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1}(t;c_1,c_2,\ldots,c_k)$.

Here $d=(d_1,d_2,\ldots)$ *is a set of constants connected to the part* μ *of the partition, that appears in* τ_{n+1}*, in the first two cases. In the third case one has to delete* λ_j-1*'s and the corresponding* c_j*'s, whenever* λ_j-1 *is equal to* 0.

Proof. (a) First reorder the functions in (51) such that $M_0>M_1>M_2>\cdots>M_{k-1}$, which leaves the tau-function unchanged up to a sign. If one writes out (51), (*cf.* (40)), where q_i^+ is an elementary Schur function s_{M_i}, using that $\frac{\partial^\ell s_{M_i}}{\partial t_1^\ell}=s_{M_i-\ell}$, it is immediate to check that the the Wronskian matrix of (51) is the transposed of the matrix in:

$$\tau_k(t)=\det(s_{M_{i-1}+j-k}(t_1+c_{1i},t_2+c_{2i},t_3+c_{3i},\ldots))_{ij}.\tag{53}$$

Now, $\tau_n(t)$ is the image under the map σ in B of the following element of $F^{(0)}$ (*cf.* (50), where we remove the upper index +, to simplify notation):

$$\begin{aligned}&\left(e_{M_0+1-k}+\sum_{j=1}^{M_0}a_{j-1,0}e_{j-k}\right)\wedge\cdots\\&\qquad\wedge\left(e_{M_{k-1}+1-k}+\sum_{j=1}^{M_{k-1}}a_{j-1,k-1}e_{j-k}\right)\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots\\&=R\left(I+\sum_{\ell=0}^{k-1}\sum_{j=1}^{M_\ell}a_{j-1,\ell}E_{j-k,M_\ell+1-k}\right)\\&\qquad\cdot(e_{M_0+1-k}\wedge\cdots\wedge e_{M_{k-1}+1-k}\wedge e_{-k}\wedge e_{-k-1}\wedge\cdots).\end{aligned}$$

Recall that (see [11])

$$\sigma(e_{M_0+1-k}\wedge e_{M_1+1-k}\wedge\cdots\wedge e_{M_{k-1}+1-k}\wedge e_{-k}\wedge e_{-k-1}\wedge e_{-k-2}\wedge\cdots) = s_\lambda(t),$$

where

$$s_\lambda(t) = \det(s_{\lambda_i+j-i}(t))_{1\le i,j\le k}$$

is the Schur polynomial, corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, with $\lambda_i = M_{i-1} + i - k$. Thus (53) lies in $\sigma R(U)\sigma^{-1} \cdot s_\lambda(t)$, where R is the representation of GL_∞ in F (see Sect. 2), so that $\sigma R\sigma^{-1}$ is the corresponding representation in B, and U is the subgroup of GL_∞, consisting of upper triangular matrices with 1's on the diagonal.

We will next show that the dimension of the space of all polynomials of the form (53) is $-\frac{1}{2}k(k-1) + \sum_{i=0}^{k-1} M_i$, or in terms of the corresponding partition λ, it is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. To show this, we first calculate the degrees of freedom of such a solution. Since it is difficult to determine this in terms of the degrees of freedom of the constants c_{ij}, we calculate this for the constants $a_{j\ell}$ which appear in (50), or rather in $f_i(z) = z^{-M_i-1} + \sum_{j=0}^{M_i-1} a_{ji} z^{-j-1}$. Note that, the corresponding tau-function does not change if we use Gauss elimination, i.e., if we add a multiple of the function $f_i(z)$ to the function $f_j(z)$. With this we can eliminate with $f_i(z)$ the constant $a_{M_i,j}$ in $f_j(z)$ for all $j < i$. This eliminates all dependence in the constants $a_{j\ell}$ and no more constants can be set to zero. Hence, the degrees of freedom that remain are $\lambda_k = M_{k-1}$ for $f_{k-1}(z)$, $\lambda_{k-1} = M_{k-2} - 1$ for $f_{k-1}(z)$, …, $\lambda_1 = M_0 - k + 1$ for f_0. If we add this all up, we obtain $|\lambda| = -\frac{1}{2}k(k-1) + \sum_{i=0}^{k-1} M_i$, the desired result.

Now recall that the set of all polynomial tau-functions of the KP hierarchy is the orbit $\mathcal{O}_0$ of $\mathbb{C}1 \in B$ under the representation $\sigma R\sigma^{-1}$ of the group GL_∞. Let P be the stabilizer of the line $\mathbb{C}1$, let W be the subgroup of permutations of basis vectors of $\mathbb{C}^\infty$ and let W_0 be its subgroup, consisting of permutations, permuting vectors with non-positive indices between themselves. Then one has the Bruhat decomposition:

$$GL_\infty = \bigcup_{w\in W/W_0} UwP \quad \text{(disjoint union)}.$$

Applying this to $\mathbb{C}1$, we obtain that the projectivised orbit $\mathbb{P}\mathcal{O}_0$ is a disjoint union of Schubert cells $C_w = Uw \cdot 1$, $w \in W/W_0$. It is well known (see, e.g. [11]) that each $w \cdot 1$ is a Schur polynomial s_λ for some partition $\lambda = \lambda(w)$, and the corresponding Schubert cell $C_\lambda = U \cdot s_{\lambda(w)}$ is an affine algebraic variety isomorphic to $\mathbb{C}^{|\lambda|}$.

On the other hand, by the previous discussion, we have constructed an injective polynomial map from the space $\mathbb{C}^{|\lambda|}$ to the Schubert cell C_λ. But, by Nagata's lemma, if an affine variety X is embedded in an irreducible affine variety Y of the same dimension, then either $X = Y$, or the complement Z of X

in Y is a closed subvariety of Y of codimension 1. Since in our situation Y is an affine space, there exists a polynomial F on Y, whose set of zeros is Z. But then the restriction of F to X is a non-constant invertible polynomial function on X, which in our situation is an affine space as well. This is a contradiction.

(b) By part (a), every τ_n must be of the form (52). Since we can shift the index n of τ_n, we may assume, without loss of generality, that $n = k$ and that $\tau_k(t) = \tau_{\lambda_1,\lambda_2,\dots,\lambda_k}(t; c_1, c_2, \dots, c_k)$. Since (51) and (52) give the same tau-function, we find that

$$\tau_k(t) = W(s_{\lambda_1+k-1}(t + c_1), s_{\lambda_2+k-2}(t + c_2), \dots, s_{\lambda_k}(t + c_k)).$$

Using the relation between MKP tau-functions and the infinite flag manifold, as used in [11] and [9], see also Remark 14, we have

$$\sigma^{-1}(\tau_k) = w_k \wedge w_{k-1} \wedge \cdots \wedge w_1 \wedge |0\rangle$$

and

$$\sigma^{-1}(\tau_{k+1}) = w_{k+1} \wedge w_k \wedge w_{k-1} \wedge \cdots \wedge w_1 \wedge |0\rangle,$$

hence the non-zero polynomial tau-function $\tau_{k+1}(t)$ must be the Wronskian determinant of the same functions, but now with one eigenfunction of $L = \partial$ added. Such an eigenfunction is of the form (50), thus

$$\tau_{k+1}(t) = W(s_M(t+d), s_{\lambda_1+k-1}(t+c_1), s_{\lambda_2+k-2}(t+c_2), \dots, s_{\lambda_k}(t+c_k)).$$

Moreover, we may assume that $M \neq \lambda_i + k - i$, otherwise we can use Gauss elimination to get a smaller M. Now reorder $M,\ \lambda_1+k-1,\ \lambda_2+k-2, \dots,\ \lambda_k$ to a decreasing order. If $M > \lambda_1 + k - 1$, then the Wronskian determinant is equal to the first possibility, where $\mu = M - k$. If $\lambda_i + k - i > M > \lambda_{i+1} + k - i - 1$ or $\lambda_k > M \neq 0$, we get the second possibility with $\mu = M + i - k$. And finally, when $M = 0$, we obtain the last possibility. □

7. Reduction of MKP to n-MKdV

Let n be an integer, $n \geq 2$. The n-th Gelfand–Dickey hierarchy, or n-KdV, describes the group orbit in a projective representation of the loop group of SL_n. This is not a subgroup of Gl_∞, one has to take a bigger group, containing it, as, e.g. in [11]. Then the representation R of GL_∞ extends to a projective representation, denoted by $\hat{R}$, of this bigger group. An element of the loop group of SL_n commutes with the operator q^n (in the space B), which means that $\tau_{k+n}(t) = \tau_k(t)$ and hence $v_{k+n}(t) = v_k(t)$ and $P^{\pm}_{n+k}(t,\partial) = P^{\pm}_k(t,\partial)$. This gives that $L_{k+n} = L_k$ and that

$$(L_k^n)_- = (P_k^+ \circ \partial^n \circ P_k^{+-1})_- = (P_{n+k}^+ \circ \partial^n \circ P_k^{+-1})_- = 0\,,$$

which means that L_k^n is a differential operator. Using the Sato–Wilson equations (11), we deduce that $\frac{\partial P_k^+}{\partial t_{jn}} = 0$, for $j = 1, 2, \ldots$, and hence, since $L_k = P_k^+ \circ \partial^n \circ P_k^{+-1}$, that also $\frac{\partial L_k}{\partial t_{jn}} = 0$. The corresponding tau-function then satisfies $\frac{\partial \tau_k}{\partial t_{jn}} = a_j \tau_k$ for some constants a_j, and hence is of the form

$$\tau_k(t) = T_k(t) \exp\Big(\sum_{j=1}^{\infty} a_j t_{jn}\Big), \quad \text{where } \frac{\partial T_k(t)}{\partial t_{jn}} = 0 \text{ for } j = 1, 2, \ldots. \tag{54}$$

Differentiating (6) by t_{jn} and using that $\frac{\partial P_k^+}{\partial t_{jn}} = 0$, we obtain the following.

The first formulation of the n-MKdV:

$$\operatorname{Res} z^{jn+k-\ell} \tau_k(t - [z^{-1}]) \tau_\ell(y + [z^{-1}]) \exp\Big(\sum_{i=1}^{\infty} (t_i - y_i) z^i\Big)\, dz = 0, \tag{55}$$

for all $0 \le k, \ell \le n-1$ *and* $j \ge 0$*, provided that* $jn + k - \ell \ge 0$.

Let $\epsilon = \exp \frac{2\pi i}{n}$. One can reformulate (55) to one identity for each pair k and ℓ as in [8], equation (8):

$$z^{-1} \sum_{a=1}^{n} (\epsilon^a z)^{k-\ell+1+\delta n} \tau_k(t - [(\epsilon^a z)^{-1}]) \tau_\ell(y + [(\epsilon^a z)^{-1}]) \exp\Big(\sum_{i=1}^{\infty} (t_i - y_i)(\epsilon^a z)^i\Big)$$

has no negative powers of z, for $0 \le k, \ell \le n-1$, and $\delta = 0$ if $k - \ell \ge 0$ and $= 1$ if $k - \ell < 0$.

The fact $P_n^+ = P_0^+$ and that L_0^n is a differential operator, gives that L_0 is the n-th root of a differential operator [6], [7]

$$\begin{aligned}
\mathscr{L}_0 &= \partial^n + w_{n-2}(t)\partial^{n-2} + \cdots + w_1(t)\partial + w_0(t) \\
&= L_0^n = P_n^+(t) \circ \partial^n \circ P_0^+(t)^{-1} \\
&= (\partial + v_{n-1}(t)) \circ (\partial + v_{n-2}(t)) \circ \cdots \circ (\partial + v_0(t)) P_0^+(t) P_0^+(t)^{-1} \\
&= (\partial + v_{n-1}(t)) \circ (\partial + v_{n-2}(t)) \circ \cdots \circ (\partial + v_0(t)).
\end{aligned}$$

The explicit form (16) of the $v_j(t)$ expressed in terms of the tau-functions (54), gives that

$$v_0(t) + v_1(t) + \cdots + v_{n-1}(t) = 0, \quad \text{and that } \frac{\partial v_k(t)}{\partial t_{jn}} = 0, \text{ for all } j = 1, 2, \ldots.$$

Note that by (18):

$$\mathscr{L}_j := L_j^n$$

$$= (\partial + v_{j-1}(t)) \circ (\partial + v_{j-2}(t)) \circ \cdots$$
$$\circ (\partial + v_0(t)) \circ (\partial + v_{n-1}(t)) \circ (\partial + v_{n-2}(t)) \circ \cdots \circ (\partial + v_j(t)),$$

which is a Darboux transformation of $\mathscr{L}_0$, i.e., a cyclic permutation of the factors $\partial + v_j$ of $\mathscr{L}_0$.

Since now $L_i = \mathscr{L}_i^{\frac{1}{n}}$ is only expressed in the v_j, the second set of equations of (17), which now have the form

$$\frac{\partial v_i}{\partial t_j} = (\mathscr{L}_{i+1}^{\frac{j}{n}})_+ \circ (\partial + v_i) - (\partial + v_i) \circ (\mathscr{L}_i^{\frac{j}{n}})_+, \quad \text{where } \mathscr{L}_{n+i} = \mathscr{L}_i, \tag{56}$$

imply the first ones, the Lax equations, of (17).

We can reformulate the equations (56) by one compact formula (see e.g. [17]). Let

$$\mathscr{L} = \operatorname{diag}\,(\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_{n-1}) \tag{57}$$

and

$$M = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \partial + v_{n-1}(t) \\ \partial + v_0(t) & 0 & & & 0 \\ 0 & \partial + v_1(t) & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \partial + v_{n-2}(t) & 0 \end{pmatrix}. \tag{58}$$

Then $\mathscr{L} = M^n$, and the equation (56) is exactly the $(i+2) \mod n$-th row of the equation

$$\frac{\partial M}{\partial t_j} = [(\mathscr{L}^{\frac{j}{n}})_+, M], \quad j = 1, 2, \ldots. \tag{59}$$

Hence we obtain:

The second formulation of the n-MKdV: *Let $U_n = \mathbb{C}[v_i^{(m)} \mid i = 0, 1, 2, \ldots, n-1,\ m \in \mathbb{Z}_{\geq 0}]/(v_0 + v_1 + \cdots + v_{n-1})$ be the quotient of the algebra of differential polynomials in v_j by the differential ideal, generated by $v_0 + v_1 + \cdots + v_{n-1}$. Then the n-MKdV hierarchy is the system of evolution equations* (59) *in U_n, where $\mathscr{L}$ and M are given by* (57) *and* (58)*.*

Example. For $n = 2$, we get the modified KdV equation in $v = v_0 = -v_1$. Indeed:

$$\mathscr{L}_0 = \partial^2 + u_0 = (\partial - v) \circ (\partial + v) = \partial^2 + \frac{\partial v}{\partial t_1} - v^2,$$
$$\mathscr{L}_1 = \partial^2 + u_1 = (\partial + v) \circ (\partial - v) = \partial^2 - \frac{\partial v}{\partial t_1} - v^2,$$

and

$$\frac{\partial v}{\partial t_j} = (\mathscr{L}_1^{\frac{j}{2}})_+ \circ (\partial + v) - (\partial + v) \circ (\mathscr{L}_0^{\frac{j}{2}})_+, \quad j = 1, 3, 5, \ldots .$$

For $j = 3$ this gives the classical modified KdV equation:

$$\frac{\partial v}{\partial t_3} = -\frac{3}{2} v^2 \frac{\partial v}{\partial t_1} + \frac{\partial^3 v}{\partial t_1^3}.$$

8. Polynomial solutions of n-KdV and n-MKdV

We can use the ideas of Sect. 6 to obtain polynomial tau-functions of n-MKdV. We will first construct a polynomial tau-function for the n-KdV hierarchy. Let π be a permutation of $1, 2, \ldots, n$, such that $\pi(i) = j_i$, and choose n formal power series

$$f_i(z) = z^{j_i - 1} + \sum_{k=j_i}^{\infty} a_{ki} z^k, \quad i = 1, 2, \ldots, n.$$

Choose non-negative integers $m_1, m_2, \ldots, m_n$, such that at least one $m_i = 0$ and one m_i non-zero (all $m_i = 0$ would lead to the trivial solution $\tau_0 = 1$). We construct $L_0 = \partial$ eigenfunctions from these data. For $\ell = 1, 2, \ldots, m_i$, define

$$\begin{aligned} q_{\ell,i}(t) &= \mathrm{Res} z^{-\ell n} f_i(z) e^{t \cdot z}\, dz = s_{\ell n - j_i}(t) + \sum_{k \geq j_i} a_{ki} s_{\ell n - k - 1}(t) \\ &= s_{\ell n - j_i}(t + c_i), \end{aligned} \tag{60}$$

for certain constants $c_i = (c_{1i}, c_{2i}, \ldots)$. Then $\tau_0(t)$ is the Wronskian determinant of all functions

$$s_{\ell n - j_i}(t + c_i), \quad \text{for } 1 \leq i \leq n,\ 1 \leq \ell \leq m_i \text{ and } \ell n - j_i \geq 0\,.$$

This determinant clearly becomes zero after differentiating by t_{pn} since differentiating the function $s_{\ell n - j_i}(t + c_i)$ by t_{pn} gives $s_{(\ell - p)n - j_i}(t + c_i)$, which is either zero if $(\ell - p)n - j_i < 0$ or it already appears as an eigenfunction in the Wronskian determinant. Hence $\tau_0(t)$ is an n-KdV tau-function.

We obtain τ_1 by adding the eigenfunction $s_{(m_1+1)n - j_1}(t + c_1)$ to the Wronskian determinant. We obtain τ_2 by adding this function and also $s_{(m_2+1)n - j_2}(t + c_2)$. For τ_3 we add besides these two also $s_{(m_3+1)n - j_3}(t + c_3)$, etc. For τ_n we add the functions

$$s_{(m_1+1)n - j_1}(t + c_1), s_{(m_2+1)n - j_2}(t + c_2), \ldots, s_{(m_n+1)n - j_n}(t + c_n).$$

This however gives no new tau-function: it is straightforward to check, but rather tedious, that τ_n is a scalar multiple of τ_0. In fact the theorem, that we shall prove later on in this section, then implies that this construction gives all possible polynomial tau-functions for n-MKdV.

Example 17. Let us inspect the case $n = 2$. In this case either $m_1 = 0$ or $m_2 = 0$ and π is the identity or the transposition (12). This gives two possible solutions, viz

$$\tau_0(t) = s_{k,k-1,\ldots,2,1}(t+c) \quad \text{and} \quad \tau_1(t) = s_{k+1,k,k-1,\ldots,2,1}(t+c),$$

or

$$\tau_0(t) = s_{k,k-1,\ldots,2,1}(t+c) \quad \text{and} \quad \tau_1(t) = s_{k-1,k-2,\ldots,2,1}(t+c),$$

where $c = (c_1, c_2, \ldots)$, which are all polynomial tau-functions of the KdV and the modified KdV hierarchies. This is a result of [11], Theorem 9.1 (b). Note that these tau-functions are independent of the even times t_{2k}.

For general n to describe all tau-functions that satisfy the n-MKdV hierarchy in terms of a formula like (52) is rather complicated. Not only are there special partitions λ connected to the case of n-KdV. But also instead of arbitrary constants $c_i = (c_{1i}, c_{2i}, \ldots)$ connected to part λ_i of the partition λ, there are certain restrictions. This time there are series of constants that depend on the shifted parts $\lambda_i - i + 1$, but then calculated modulo n. Hence, there are n of such series $c_{\bar{i}} = (c_{1\bar{i}}, c_{2\bar{i}}, \ldots)$ of which at most $n-1$ appear in the tau-function. Here and thereafter $\bar{s}$ stands for remainder of the division of s by n.

We claim that the Wronskian determinant

$$W(s_{\lambda_1+k-1}(t + c_{\overline{\lambda_1}}), s_{\lambda_2+k-2}(t + c_{\overline{\lambda_2-1}}), \ldots, s_{\lambda_k}(t + c_{\overline{\lambda_k-k+1}})), \tag{61}$$

is a polynomial tau-function of the n-KdV if and only if the set of shifted parts

$$V_\lambda = \{\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \ldots, \lambda_k - k + 1, -k, -k-1, -k-2, \ldots\}$$

satisfies the condition that

$$\text{if } j \in V_\lambda, \text{ then also } j - n \in V_\lambda.$$

This condition reflects the condition that if the eigenfunction $q_{\ell,i}(t)$, defined in (60) appears in the Wronskian determinant, then also $q_{\ell-n,i}(t)$, if it is nonzero, must appear in this determinant as well. Or stated differently, if $s_{\lambda_i+k-i}(t + c_{\overline{\lambda_i-i+1}})$ appears in the Wronskian determinant of (61), then either $\frac{\partial s_{\lambda_i+k-i}(t+c_{\overline{\lambda_i-i+1}})}{\partial t_n} = 0$ or $s_{\lambda_i+k-i-n}(t + c_{\overline{\lambda_i-i+1}})$ also appears in this determinant as well. This leads us to the following notion.

Definition 18. *A partition λ is called n-periodic if the corresponding infinite sequence V_λ is mapped to itself when subtracting n from each term.*

Theorem 19. *All polynomial tau-functions of the n-KdV hierarchy are, up to a constant factor, of the form*

$$\begin{aligned} &\tau^n_{\lambda_1,\lambda_2,\ldots,\lambda_k}(t; c_{\overline{\lambda_1}}, c_{\overline{\lambda_2-1}}, \ldots, c_{\overline{\lambda_k-k+1}}) \\ &= \det(s_{\lambda_i+j-i}(t_1 + c_{1,\overline{\lambda_i-i+1}}, t_2 + c_{2,\overline{\lambda_i-i+1}}, \ldots))_{1\le i,j\le k}, \end{aligned} \tag{62}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is an n-periodic partition. Here the $c_{\bar{i}} = (c_{1\bar{i}}, c_{2\bar{i}}, \ldots)$ for $i = 1, 2, \ldots, n$ (where at most $n-1$ of such $\bar{i}$'s appear) are arbitrary constants.

Before we give the proof, let us make calculations in an explicit example. Let $n = 4$ and $\lambda = (6, 3, 2, 1)$. Then

$$V_\lambda = \{6, 2, 0, -2, -4, -5, -6, \ldots\},$$

hence λ is 4-periodic, and the corresponding tau-function is

$$\begin{aligned}\tau^4_{(6,3,2,1)}(t; c_{\bar{2}}, c_{\bar{2}}, c_{\bar{4}}, c_{\bar{2}}) &= W(s_9(t+c_{\bar{2}}), s_5(t+c_{\bar{2}}), s_3(t+c_{\bar{2}}), s_1(t+c_{\bar{2}})) \\ &= \begin{vmatrix} s_6(t+c_{\bar{2}}) & s_7(t+c_{\bar{2}}) & s_8(t+c_{\bar{2}}) & s_9(t+c_{\bar{2}}) \\ s_2(t+c_{\bar{2}}) & s_3(t+c_{\bar{2}}) & s_4(t+c_{\bar{2}}) & s_5(t+c_{\bar{2}}) \\ s_0(t+c_{\bar{4}}) & s_1(t+c_{\bar{4}}) & s_2(t+c_{\bar{4}}) & s_3(t+c_{\bar{4}}) \\ 0 & 0 & s_0(t+c_{\bar{2}}) & s_1(t+c_{\bar{2}}) \end{vmatrix},\end{aligned} \tag{63}$$

which depends on two series of constants, viz. $c_{\bar{6}} = c_{\bar{2}}$ and $c_{\bar{0}} = c_{\bar{4}}$. The $\bar{6}$ and $\bar{0}$ are the elements of the following set

$$U_\lambda^{(4)} = \{6, 2, 0, -2\} \backslash \{2, -2, -4, -6\} = \{6, 0\},$$

which are all the elements j of V_λ where one removes all elements $j - 4$.

Now

$$\begin{aligned}
s_9(t+c_{\bar{2}}) &= s_9(t) + \sum_{j=0}^{8} a_{9-j,\bar{2}} s_j(t) \quad \text{and} & f_6(z) &= z^{-10} + \sum_{j=0}^{8} a_{9-j,\bar{2}} z^{-j-1}, \\
s_5(t+c_{\bar{2}}) &= s_5(t) + \sum_{j=0}^{4} a_{5-j,\bar{2}} s_j(t), & f_2(z) &= z^{-6} + \sum_{j=0}^{4} a_{5-j,\bar{2}} z^{-j-1}, \\
s_3(t+c_{\bar{4}}) &= s_3(t) + \sum_{j=0}^{2} a_{3-j,\bar{4}} s_j(t), & f_0(z) &= z^{-4} + \sum_{j=0}^{2} a_{3-j,\bar{4}} z^{-j-1}, \\
s_1(t+c_{\bar{2}}) &= s_1(t) + a_{1,\bar{2}} s_0(t), & f_{-2}(z) &= z^{-2} + a_{1,\bar{2}} z^{-1},
\end{aligned}$$

where $a_{k,\bar{j}} = s_k(c_{\bar{j}})$. And as in the proof of Theorem 16 we can eliminate the coefficients of z^{-2}, in $f_0(z)$, $f_2(z)$ and $f_6(z)$, and the coefficient of z^{-4} in $f_2(z)$ and $f_6(z)$ and the coefficient of z^{-6} in $f_6(z)$, leaving a freedom of $9 - 3 = 6 = \lambda_1$ in $f_6(z)$ and similarly a freedom of $3 - 1 = 2 = \lambda_3$ in $f_0(z)$. Hence the dimension of the space of polynomials (63) is

$$8 = 6 + 2 = \lambda_1 + \lambda_3 = \sum_{\lambda_i \in \Lambda^{(4)}(\lambda)} \lambda_i,$$

where

$$\Lambda^{(4)}(6,3,2,1) = \{\lambda_1, \lambda_3\} = \{6,2\}.$$

Let us next investigate the element $s_{(6,3,2,1)}(t)$ the corresponding element under σ^{-1} is

$$\begin{aligned}
&\sigma^{-1}(s_{(6,3,2,1)}(t)) \\
&= e_6 \wedge e_2 \wedge e_0 \wedge e_{-2} \wedge e_{-4} \wedge e_{-5} \wedge \cdots \\
&= t^{-1}u_2 \wedge u_2 \wedge tu_4 \wedge tu_2 \wedge t^2u_4 \wedge t^2u_3 \wedge t^2u_2 \wedge t^2u_1 \wedge t^3u_4 \wedge \cdots .
\end{aligned}$$

Here we make the identification $t^{-k}u_j = e_{4k+j}$ and $t^k e_{ij} = \sum_{s\in\mathbb{Z}} E_{4(s-k)+i,4s+j}$ as in [11], equation (9.1)–(9.2). And this is up to some infinite reordering "equal to"

$$\begin{aligned}
&\hat{R}(t^1e_{11} + t^{-2}e_{22} + t^1e_{33} + e_{44})(tu_4 \wedge tu_3 \wedge tu_2 \wedge tu_1 \wedge t^2u_4 \wedge \cdots) \\
&= \hat{R}(t^1e_{11} + t^{-2}e_{22} + t^1e_{33} + e_{44})|0\rangle.
\end{aligned}$$

We now reconstruct our λ from the element $t^1e_{11} + t^{-2}e_{22} + t^1e_{33} + e_{44}$. For this we invert the process above. We first calculate the corresponding infinite wedge product and need to find the place of $e_6 = t^{-1}u_2 = t^{-2}e_{22}tu_2$ and $e_0 = tu_4 = e_{44}tu_4$ in this product. It is the place 0 and the place -2, which gives the elements $\lambda_1 = 6 - 0$ and $\lambda_3 = 0 - (-2) = 2$ of λ .

We now want to use some of the above features of the example in the following proof:

Proof of Theorem 19. First observe that (61) is equal to (62).

As in the proof of Theorem 16, we can calculate the degrees of freedom of the constants in a similar way. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. As before,

$$\begin{aligned}
s_{\lambda_i+k-i}(t + c_{\overline{\lambda_i-i+1}}) &= s_{\lambda_i+k-i}(t) + \sum_{j=0}^{\lambda_i+k-i-1} a_{\lambda_i+k-i-j,\overline{\lambda_i-i+1}} s_j(t) \\
&= \operatorname{Res} f_{\lambda_i-i+1}(z)e^{t\cdot z}\,dz,
\end{aligned}$$

for $a_{j,\overline{\lambda_i-i+1}} = s_j(c_{\overline{\lambda_i-i+1}})$, and

$$f_{\lambda_i-i+1}(z) = z^{-(\lambda_i-i+1)-k} + \sum_{j=0}^{\lambda_i+k-i-1} a_{j,\overline{\lambda_i-i+1}} z^{-(\lambda_i+k-i)+j-1}.$$

Note that $f_{\lambda_i-i+1-n}(z)$ also appears as some $z^n f_{\lambda_j-j+1}(z)$, for some $j > i$ and it has the form

$$\begin{aligned}
&f_{\lambda_i-i+1-n}(z) \\
&= (z^n f_{\lambda_i-i+1}(z))_-
\end{aligned}$$

$$= z^{-(\lambda_i-i+1)-k-n} + \sum_{j=0}^{\lambda_i+k-i-1-n} a_{j,\overline{\lambda_i-i+1}} z^{-(\lambda_i+k-i)+j+n-1}\,.$$

Hence, proceeding in a similar way as in the proof of Theorem 16, we can use $f_{\lambda_i-i+1}(z)$ to eliminate the constant $a_{\lambda_\ell-\lambda_i+i-\ell-1,\overline{\lambda_\ell-\ell+1}}$, in front of $z^{-(\lambda_i-i+1)-k}$ in $f_{\lambda_\ell-\ell+1}(z)$ for all $\ell < i$. Note that we cannot eliminate more constants. Hence we have λ_k degrees of freedom for $f_{\lambda_k-k+1}(z)$, λ_{k-1} for $f_{\lambda_{k-1}-k+2}(z)$, λ_{k-2} for $f_{\lambda_{k-2}-k+3}(z), \ldots, \lambda_1$ for $f_{\lambda_1}(z)$. This is similar to the KP case, except that some of the $f_{\lambda_i-i+1}(z)$ are related, as described above. Hence we have to find those $f_{\lambda_i-i+1}(z)$ with the highest possible index that are not related to the one with a higher index. These are all the $f_j(z)$'s, with j from the following set:

$$U_\lambda^{(n)} = \{\lambda_1, \lambda_2 - 1, \ldots, \lambda_k - k + 1\}\backslash\{\lambda_1 - n, \lambda_2 - n + 1, \ldots, \lambda_k - n - k + 1\}.$$

If $j \in U_\lambda$, then $j = \lambda_i - i + 1$ for some i and $f_j(z) = f_{\lambda_i-i+1}(z)$ has λ_i degrees of freedom. Hence, defining

$$\Lambda^{(n)}(\lambda) = \{\lambda_i \mid \lambda_i - i + 1 \in U_\lambda^{(n)}\},$$

the freedom of choosing constants (or the dimension of this subspace of polynomials) is equal to

$$\sum_{\lambda_i \in \Lambda^{(n)}(\lambda)} \lambda_i\,.$$

As before, the tau-function (62) is the image under σ in B of the following element of $F^{(0)}$:

$$\Big(e_{\lambda_1} + \sum_{j=1}^{\lambda_1+k-1} a_{j-1,\overline{\lambda_1}} e_{\lambda_1-j}\Big) \wedge \Big(e_{\lambda_2-1} + \sum_{j=1}^{\lambda_2+k-2} a_{j-1,\overline{\lambda_2-1}} e_{\lambda_2-1-j}\Big) \wedge \cdots$$
$$\wedge \Big(e_{\lambda_k-k+1} + \sum_{j=1}^{\lambda_k} a_{j-1,\overline{\lambda_k-k+1}} e_{\lambda_k-k+1-j}\Big) \wedge e_{-k} \wedge e_{-k-1} \wedge \cdots\,, \tag{64}$$

which is equal to

$$R\Big(I + \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i+k-i} a_{j-1,\overline{\lambda_i-i+1}} E_{\lambda_i-i+1-j,\lambda_i-i+1}\Big)$$
$$\cdot\, (e_{\lambda_1} \wedge e_{\lambda_2-1} \wedge \cdots \wedge e_{\lambda_k-k+1} \wedge e_{-k} \wedge e_{-k-1} \wedge \cdots),$$

where

$$\sigma(e_{\lambda_1} \wedge e_{\lambda_2-1} \wedge \cdots \wedge e_{\lambda_k-k+1} \wedge e_{-k} \wedge e_{-k-1} \wedge \cdots) = s_\lambda(t).$$

We can rewrite (64) as follows:

$$R\Big(I + \sum_{p\in U_\lambda^{(n)}} \sum_{0\le s<\frac{k+p}{n}} \sum_{j=1}^{p+k-sn-1} a_{j-1,\overline{p}} E_{p-j-sn,p-sn}\Big) \cdot (e_{\lambda_1} \wedge e_{\lambda_2-1} \wedge \cdots \wedge e_{\lambda_k-k+1} \wedge e_{-k} \wedge e_{-k-1} \wedge \cdots).$$

Note that replacing the upper bound $p+k-sn-1$ of j by $p+k-1$ does not change the element. We can also drop the lower bound of s because this will give a matrix element that acts as zero on every vector of the wedge product $\sigma^{-1}(s_\lambda(t))$. We can also drop the upper bound of s. Indeed, if we do that, the new element transforms the element e_ℓ for $\ell \le -k$ into an element of the form $v_\ell = e_\ell + \sum_{-\infty\ll i<\ell} b_i e_i$. We can then use the v_j for $j < \ell$ to eliminate all the coefficients of b_i (we have to do this procedure infinitely many times). In this way we get that (64) is equal to

$$\hat{R}\Big(I + \sum_{p\in U_\lambda^{(n)}} \sum_{j=1}^{p+k-1} a_{j-1,\overline{p}} \sum_{s\in\mathbb{Z}} E_{p-j+sn,p+sn}\Big) \cdot (e_{\lambda_1} \wedge e_{\lambda_2-1} \wedge \cdots \wedge e_{\lambda_k-k+1} \wedge e_{-k} \wedge e_{-k-1} \wedge \cdots).$$

Now we relate the above element of the completed GL_∞ to an element of the loop group $SL_n(\mathbb{C}[t,t^{-1}])$ by making the identification $t^{-k}u_j = e_{kn+j}$ and $t^k e_{ij} = \sum_{s\in\mathbb{Z}} E_{(s-k)n+i,sn+j}$ as in [11], equation (9.1)–(9.2). Let

$$U = \{A(t) \in SL_n(\mathbb{C}[t]) \mid A(0) \text{ is upper triangular with 1's on the diagonal}\}.$$

Then, under the above identification we have

$$I + \sum_{p\in U_\lambda^{(n)}} \sum_{j=1}^{p+k-1} a_{j-1,\overline{p}} \sum_{s\in\mathbb{Z}} E_{p-j+sn,p+sn} \in U.$$

Let $T = \{\sum_{i=1}^n t^{k_i} e_{ii} \mid k_i \in \mathbb{Z}, \sum_{i=1}^n k_i = 0\} \subset SL_n(\mathbb{C}[t,t^{-1}])\}$. Fix $w = \sum_{i=1}^n t^{k_i} e_{ii} \in T$. We want to find the partition that corresponds to $\hat{R}(w)|0\rangle$, i.e., to find λ such that $\sigma(\hat{R}(w)|0\rangle) = s_\lambda(t)$. In fact, if $\lambda = (\lambda_1, \lambda_2, \ldots)$, we want to find its parts λ_i that are in $\Lambda^{(n)}(\lambda)$. We will denote these elements by $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_p$. Now, $\hat{R}(w)|0\rangle$ is a semi-infinite wedge product of the elements $t^{k_i+j}u_i = e_{-(k_i+j)n+i}$, for $j > 0$ and all $1 \le i \le n$. We have to order these e_ℓ in a decreasing order in this wedge product, from which we then can determine the corresponding partition λ. For this, first reorder the elements k_i to the decreasing order without interchanging k_i's, if they are the same. Then p is the same as the number of k_i's which are smaller than the maximum of this set. Let π be the permutation that assigns to i the number

j if k_j is in the i-th place in the decreasing order. The corresponding $\Lambda^{(n)}(\lambda)$ has p elements $\hat\lambda_i$, which we put in decreasing order: $\hat\lambda_1 \geq \hat\lambda_2 \geq \cdots \geq \hat\lambda_p$. The part λ_1, which is always an element of $\Lambda^n(\lambda)$, corresponds to the place of $t^{k_{\pi(n)}+1}u_{\pi(n)} = e_{-k_{\pi(n)}n+\pi(n)-n}$ in the semi-infinite wedge product, which is always on the 0-th place. Hence

$$\hat\lambda_1 = \lambda_1 = -k_{\pi(n)}n + \pi(n) - n$$

and $\lambda_2 = -k_{\pi(n)}n + \pi(n) - 2n + 1$, since it corresponds to $t^{k_{\pi(n)}+2}u_{\pi(n)} = e_{-k_{\pi(n)}n+\pi(n)-2n}$, then $\lambda_3 = -k_{\pi(n)}n + \pi(n) - 3n + 2$ and we continue as long as $k_{\pi(n)} + 1, k_{\pi(n)} + 2, \dots$ is smaller than $k_{\pi(n-1)}$. To determine $\hat\lambda_2$ of $\Lambda^{(n)}(\lambda)$, is already a bit more complicated. One has to consider two cases. It is $\lambda_{k_{\pi(n-1)}-k_{\pi(n)}+2}$, if $k_{\pi(n-1)} = k_{\pi(n)}$ or if $k_{\pi(n-1)} > k_{\pi(n)}$ and $\pi(n-1) < \pi(n)$. Then the element $t^{k_{\pi(n-1)}+1}u_{\pi(n-1)}$, which is equal to $e_{-k_{\pi(n-1)}n+\pi(n-1)-n}$ is in the $(-k_{\pi(n-1)} + k_{\pi(n)} - 1)$-th place in the semi-infinite wedge product. Hence $\hat\lambda_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1)$. However, if $k_{\pi(n-1)} > k_{\pi(n)}$ and $\pi(n-1) > \pi(n)$, then $\hat\lambda_2 = \lambda_{k_{\pi(n-1)}-k_{\pi(n)}+1}$ and this corresponds to the same element $e_{-k_{\pi(n-1)}n+\pi(n-1)-n}$, hence $\hat\lambda_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1) - 1$. The extra -1 at the end comes from the inversion of π between the elements $n-1$ and n, viz. in this case $\pi(n-1) > \pi(n)$. The number of inversions will turn out to be important, so let us introduce some notation. Let

$$J_j = |\{i > j \mid \pi(i) < \pi(j)\}|,$$

then

$$\hat\lambda_2 = -k_{\pi(n-1)}(n-1) - k_{\pi(n)} + \pi(n-1) - (n-1) - J_{n-1}.$$

For the next one we have $\hat\lambda_3 = \lambda_{2k_{\pi(n-2)}-k_{\pi(n-1)}-k_{\pi(n)}} + 2 - J_{n-2}$ and the corresponding element is $t^{k_{\pi(n-2)}+1}u_{\pi(n-2)} = e_{-k_{\pi(n-2)}n+\pi(n-2)-n}$, which gives

$$\hat\lambda_3 = -k_{\pi(n-2)}(n-2) - k_{\pi(n-1)} - k_{\pi(n)} + \pi(n-2) - (n-2) - J_{n-2}.$$

Continuing in this way we find

$$\begin{aligned}\hat\lambda_j = &- k_{\pi(n-j+1)}(n-j+1) - k_{\pi(n-j+2)} - \cdots - k_{\pi(n-1)} - k_{\pi(n)} \\ &+ \pi(n-j+1) - (n-j+i) - J_{n-j+1},\end{aligned}$$

where the last one is $\hat{\lambda}_p$. The dimension of this space is $\hat{\lambda}_1 + \hat{\lambda}_2 + \cdots + \hat{\lambda}_p$, which is equal to

$$\sum_{j=n-p+1}^{n} (n-p-2j+1)k_{\pi(j)} + \pi(j) - j - J_j. \tag{65}$$

Since $\sum_i k_i = 0$, we can add a multiple of this sum, thus equation (65) is equal to

$$p\sum_{i=1}^{n-p} k_{\pi(i)} + \sum_{j=n-p+1}^{n} (n-2j+1)k_{\pi(j)} + \pi(j) - j - J_j. \tag{66}$$

Now, $k_{\pi(1)} = k_{\pi(2)} = \cdots = k_{\pi(n-p)}$, hence

$$p\sum_{i=1}^{n-p} k_{\pi(i)} = p(n-p)k_{\pi(1)} = \sum_{i=1}^{n-p}(n-2i+1)k_{\pi(i)}.$$

Thus (66) is equal to

$$\sum_{i=1}^{n}(n-2i+1)k_{\pi(i)} - \sum_{j=n-p+1}^{n} j - \pi(j) + J_j. \tag{67}$$

Note that $\pi(1) < \pi(2) < \cdots < \pi(n-p)$ and $j - \pi(j) + J_j$ are the number of inversions between j and all elements i with $i < j$, thus

$$\sum_{j=n-p+1}^{n} j - \pi(j) + J_j = \text{ number of inversions of } \pi,$$

hence, the dimension of the space which corresponds to w is

$$\sum_{\lambda_i \in \Lambda^{(n)}(\lambda)} \lambda_i = \sum_{i=1}^{n}(n-2i+1)k_{\pi(i)} - (\text{number of inversions of } \pi). \tag{68}$$

We now have to prove that this is indeed the right dimension to obtain all possible polynomial tau-functions. Recall that the set of all polynomial tau-functions of the n-KdV hierarchy is the orbit $\mathscr{O}_0^n$ of $\mathbb{C}1 \in B$ under the projective representation $\hat{R}$ of the group $SL_n(\mathbb{C}[t,t^{-1}])$. Let $P = SL_n(\mathbb{C}[t])$. Then one has the Bruhat decomposition:

$$SL_n(\mathbb{C}[t,t^{-1}]) = \bigcup_{w\in T} UwP \quad \text{(disjoint union)}.$$

Applying this to $\mathbb{C}1$, we obtain that the projectivization of the orbit $\mathscr{O}_0^n$ is a disjoint union of Schubert cells $C_w = Uw \cdot 1$, for all possible $w = \operatorname{diag}(t^{k_1}, \ldots, t^{k_n})$

$\in T$. Now, $UwP = ww^{-1}UwP$, hence elements of U that w conjugates to elements in P get absorbed in P , and the elements $t^c e_{ij} \in U$ that get mapped under conjugation by w to elements $t^d e_{ij}$ with $d < 0$ give the cell. Hence we have to count the possible values of c such that $c - k_i + k_j < 0$. This is straightforward, for $i < j$ it is $|k_i - k_j|$ if $k_i > k_j$ and 0 otherwise. For $j < i$ we find $|k_i - k_j| - 1$ if $k_i > k_j$ and 0 otherwise. Hence, we obtain as dimension the sum of all values $|k_i - k_j|$ for $1 \le i < j \le n$, where we have to subtract 1 if $k_i > k_j$. We find that the dimension of this Schubert cell is

$$\sum_{1 \le i < j \le n} \left(|k_i - k_j| - \begin{cases} 1 & \text{if } k_i > k_j, \\ 0, & \text{otherwise.} \end{cases} \right)$$

Now ordering the k_i's in decreasing order (where π is the permutation as before), we can remove the absolute value and obtain that the dimension is equal to

$$\sum_{1 \le i < j \le n} \left(k_{\pi(i)} - k_{\pi(j)} - \begin{cases} 1 & \text{if } \pi(i) > \pi(j), \\ 0, & \text{otherwise.} \end{cases} \right)$$

In this sum $k_{\pi(i)}$ appears $n-1$ times, with $n-i$ plus signs and $i-1$ minus signs, hence we obtain that the dimension of the Schubert cell C_w is equal to

$$\sum_{i=1}^{n} (n - 2i + 1) k_{\pi(i)} - (\text{number of inversions of } \pi) = \sum_{\lambda_i \in \Lambda^{(n)}(\lambda)} \lambda_i ,$$

which is the dimension of the space of polynomials of the form (62). The same algebro-geometric argument as in the KP case completes the proof of the theorem. $\square$

Example 20. For $n = 3$ we have the following possible polynomial tau-functions of the 3-KdV hierarchy. Let $k, \ell = 0, 1, 2, \ldots$, then we find two series (see (62)):

$$\tau^3_{k+2\ell, k+2\ell-2, \ldots, \ell+2, \ell, \underline{\ell}, \ell-1, \underline{\ell-1}, \ldots, 1, \underline{1}}(t; c, c, \ldots, c, c, \underline{c}, c, \underline{c}, \ldots, c, \underline{c})$$

and

$$\tau^3_{k+2\ell+1, k+2\ell-1, \ldots, \ell+3, \ell+1, \underline{\ell}, \ell, \underline{\ell-1}, \ell-1, \ldots, \underline{1}, 1}(t; c, c, \ldots, c, c, \underline{c}, c, \underline{c}, c, \ldots, \underline{c}, c).$$

We have at most two series of constants that appear, viz. $c = (c_1, c_2, c_3, \ldots)$ and $\underline{c} = (\underline{c}_1, \underline{c}_2, \underline{c}_3, \ldots)$, and c is coupled to the parts of the partition which are not underlined and $\underline{c}$ to all underlined parts of the partition. In both cases the tau-functions are independent of all times t_{3k}.

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