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# On points without universal expansions

by

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*To Robert Tijdeman on his 75th birthday*

**1. Introduction.** Let $1 < \beta < 2$. Given any $x \in [0, (\beta - 1)^{-1}]$, a sequence $(a_n) \in \{0, 1\}^{\mathbb{N}}$ is called a *$\beta$-expansion* of $x$ if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Sidorov [20] proved that for any $1 < \beta < 2$, almost every point in $[0, (\beta-1)^{-1}]$ has uncountably many expansions. If $(a_n)$ is the only $\beta$-expansion of $x$, then we call $x$ a *univoque point* with unique expansion $(a_n)$. Denote by $U_\beta$ the set of univoque points in base $\beta$. There are many results concerning unique expansions: see [9, 12] and references therein.

Let $(a_n)$ be a $\beta$-expansion of $x$. If for any $k \geq 1$ and any $(b_1 \cdots b_k) \in \{0, 1\}^k$ there exists some $k_0$ such that

$$a_{k_0+1} a_{k_0+2} \cdots a_{k_0+k} = b_1 \cdots b_k,$$

then we call $(a_n)$ a *universal $\beta$-expansion* of $x$.

The dynamical approach is a good way to generate $\beta$-expansions effectively. Define $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$ (see Figure 1).

Let $x \in [0, (\beta - 1)^{-1}]$ with an expansion $(a_n)_{n=1}^{\infty}$, and set $T_{a_1 \cdots a_n} = T_{a_n} \circ T_{a_{n-1}} \circ \cdots \circ T_{a_1}$. We call $\{T_{a_1 \cdots a_n}(x)\}_{n=0}^{\infty}$ an *orbit* of $x$ in base $\beta$. For simplicity, we set $T_{a_0}(x) = x$. Clearly, for different expansions, $x$ has distinct orbits. Evidently, for any $n \geq 1$, we have

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \frac{T_{a_1 \cdots a_n}(x)}{\beta^n}.$$

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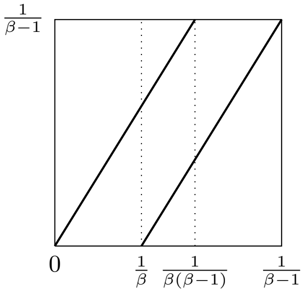


Fig. 1. The dynamical system for $\{T_0, T_1\}$

The digits $(a_n)$ are chosen in the following way: if $T_{a_1\cdots a_{j-1}}(x) \in [0, \beta^{-1})$, then $a_j = 0$, and if $T_{a_1\cdots a_{j-1}}(x) \in ((\beta-1)^{-1}\beta^{-1}, (\beta-1)^{-1}]$, then $a_j = 1$. However, if $T_{a_1\cdots a_{j-1}}(x) \in [\beta^{-1}, (\beta-1)^{-1}\beta^{-1}]$, then we may choose $a_j$ to be 0 or 1. Due to this observation, we call $[\beta^{-1}, (\beta-1)^{-1}\beta^{-1}]$ the *switch region*. All possible $\beta$-expansions can be generated using this idea [8, 3]. If $x$ has exactly $k$ different expansions, then we say $x$ has *multiple expansions* [22, 6, 7].

There exists a criterion that characterizes unique expansions [9, 12]. However, few papers have considered universal expansions and multiple expansions. In this paper, we shall use the dynamical approach to study universal expansions.

Universal expansions have a close connection with the discrete spectra

$$D = \Big\{\sum_{i=0}^{n} a_i\beta^i : a_i \in \{0,1\},\ n \geq 0\Big\}.$$

Let $D = \{y_0 = 0 < y_1 < y_2 < \cdots\}$. Define

$$L^1(\beta) = \limsup_{k\to\infty}(y_{k+1} - y_k).$$

Erdős and Komornik [10] proved that if $L^1(\beta) = 0$, then all the points of $(0, (\beta-1)^{-1})$ have universal expansions. They also proved that if $1 < \beta \leq \sqrt[4]{2} \approx 1.19$, then $L^1(\beta) = 0$. In particular, they proved that $L^1(\sqrt{2}) = 0$. Sidorov and Solomyak [23] also considered some algebraic numbers for which $L^1(\beta) = 0$, and their result has been improved by Akiyama and Komornik [1] who proved that if $1 < \beta \leq \sqrt[3]{2} \approx 1.26$, then $L^1(\beta) = 0$. Feng [15] used the result of [1] and some ideas in fractal geometry to show that for any non-Pisot $\beta \in (1, \sqrt{2}]$, if $\beta^2$ is not a Pisot number, then $L^1(\beta) = 0$.

Concerning generic results, Sidorov [21] showed that for any $1 < \beta < 2$, almost every point in $[0, (\beta-1)^{-1}]$ has at least one universal expansion. Dajani and de Vries [4] used a dynamical approach to show that for any $\beta > 1$, almost every point of $[0, (\beta-1)^{-1}]$ has uncountably many universal expansions.

The results of Sidorov [20] and of Dajani and de Vries [4] imply that the set of points without universal expansions has zero Lebesgue measure. In other words, the Lebesgue measure of $V_\beta$ is zero, where

$$V_\beta = \{x \in [0, (\beta-1)^{-1}] : x \text{ does not have a universal expansion}\}.$$

A natural question is to study the Hausdorff dimension of $V_\beta$. Our main result is Theorem 2.2, where we show that if $\beta$ is the $n$-bonacci number then the Hausdorff dimension of $V_\beta$ equals 1. For $1 < \beta < (1+\sqrt{5})/2$, the Hausdorff dimension of $V_\beta$ has a close connection with an old conjecture of Erdős and Komornik [10]:

Conjecture 1.2. *For any non-Pisot* $\beta \in (1, 1+\sqrt{5}/2)$, $L^1(\beta) = 0$.

This conjecture is true if $\beta \in (1, \sqrt{2}]$ and $\beta^2$ is not a Pisot number [15]. We briefly discuss the connection between the dimension of $V_\beta$ and this conjecture. If we were able to find some non-Pisot number $1 < \beta < (1+\sqrt{5})/2$ such that the Hausdorff dimension of $V_\beta$ is positive, then $L^1(\beta) > 0$. The reason is that $L^1(\beta) = 0$ implies that all the points of $(0, (\beta-1)^{-1})$ have universal expansions. In other words, we would disprove the Erdős–Komornik conjecture. Therefore, considering the Hausdorff dimension of $V_\beta$ is meaningful to this conjecture.

The dimension of $V_\beta$ has a strong relation to open dynamical systems. Roughly speaking, $V_\beta$ is a union of countable survivor sets generated by some open dynamical systems. These open dynamical systems are smaller than the usual open systems as we consider all the possible orbits, i.e. all the possible orbits should avoid some holes. In this paper, we shall make use of this tool to study the dimension of $V_\beta$.

The paper is arranged as follows. In Section 2, we start with some necessary definitions and notation, and then we state the main results of the paper. In Section 3 we give the proofs, and in Section 4 we make some final remarks.

**2. Preliminaries and main results.** In this section, we give some notation and definitions. Let $\Omega = \{0,1\}^{\mathbb{N}}$, $E = [0, (\beta-1)^{-1}]$, and let $\sigma$ be the left shift. The random $\beta$-transformation $K$ is defined in the following way [8].

Definition 2.1. $K : \Omega \times E \to \Omega \times E$ is defined by

$$K(\omega, x) = \begin{cases} (\omega, \beta x), & x \in [0, \beta^{-1}), \\ (\sigma(\omega), \beta x - \omega_1), & x \in [\beta^{-1}, \beta^{-1}(\beta-1)^{-1}], \\ (\omega, \beta x - 1), & x \in (\beta^{-1}(\beta-1)^{-1}, (\beta-1)^{-1}]. \end{cases}$$

We call $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$ the *switch region* since in this region we can choose the digit to be used and change from 0 to 1 or vica versa.

When the orbits of points hit or enter the switch region, and we always choose the digit 1, then we call this algorithm the *greedy algorithm.* More precisely, the greedy map $G : E \to E$ is defined by

$$G(x) = \begin{cases} \beta x, & x \in [0, \beta^{-1}), \\ \beta x - 1, & x \in [\beta^{-1}, (\beta-1)^{-1}]. \end{cases}$$

Let $(\omega, x) \in \Omega \times E$. For any $n \geq 1$, we denote by $K^n(\omega, x) = K(K^{n-1}(\omega, x))$ the $n$th iteration of $K$, and let $\pi(\omega, x) = x$ be the projection on the second coordinate. We can study $\beta$-expansions via the following iterated function system:

$$f_j(x) = \frac{x + j}{\beta}, \quad j \in \{0, 1\}.$$

The self-similar set [14] for this IFS is the interval $[0, (\beta-1)^{-1}]$. This tool is useful in the proof of Lemma 3.12.

Before we state our main results, we define some sets. Given $1 < \beta < 2$ and any $N \geq 3$, define

$$\begin{aligned} E_{\beta,N} &= \{x \in [0, (\beta-1)^{-1}] : \text{no orbit of } x \text{ hits } [0, \beta^{-N}(\beta-1)^{-1}]\}, \\ F_{\beta,N} &= \{x \in [0, (\beta-1)^{-1}] : \text{the greedy orbit of } x \\ &\qquad\qquad\qquad \text{does not hit } [0, \beta^{-N}(\beta-1)^{-1}]\}. \end{aligned}$$

We can give a simple symbolic explanation of $E_{\beta,N}$: no $\beta$-expansion $(a_n)$ of any point in $E_{\beta,N}$ contains the block $(00\cdots0)$ ($N$ zeros). Let

$$\mathcal{O} = \{\pi(K^n(\omega, 1)) \cup \pi(K^n(\omega, (\beta-1)^{-1} - 1) : n \geq 0,\ \omega \in \Omega\}$$

be the union of all possible orbits of 1 and of $(\beta-1)^{-1} - 1$. An algebraic number $\beta > 1$ is called a *Pisot number* if all of its conjugates lie inside the unit circle.

Now we state our main results.

THEOREM 2.2. *For any $n \geq 2$, let $\beta_n$ be the $n$-bonacci number satisfying*

$$\beta^n = \beta^{n-1} + \beta^{n-2} + \cdots + \beta + 1.$$

*Then* $\dim_H(V_{\beta_n}) = 1$.

The following result gives a sufficient condition under which the Hausdorff dimension of $F_{\beta,N}$ can be calculated.

COROLLARY 2.3. *Let $(1+\sqrt{5})/2 < \beta < 2$. If all the possible orbits of* 1 *hit only finitely many points, then for any $N \geq 3$,* $\dim_H(F_{\beta,N})$ *can be calculated explicitly. In particular, for any Pisot number in* $(1,2)$, $\dim_H(F_{\beta,N})$ *can be calculated.*

This result is indeed a corollary of [5, Theorem 4.2]. Generally, calculating the Hausdorff dimension of $E_{\beta,N}$ is not an easy problem. By definition,

$E_{\beta,N} \subset F_{\beta,N}$ for any $N \geq 3$. Hence, $E_{\beta,N}$ is a smaller survivor set, and it is difficult to calculate its dimension. However, for the sequence $(\beta_n)$, we have the following asymptotic result.

THEOREM 2.4. *For any $n \geq 2$ and $N \geq 3$, let $\beta_n$ be the n-bonacci number. Then* $\dim_H(F_{\beta_n,N-1}) \leq \dim_H(E_{\beta_n,N}) \leq \dim_H(F_{\beta_n,N})$. *Consequently,*

$$\lim_{N\to\infty} \dim_H(E_{\beta_n,N}) = \lim_{N\to\infty} \dim_H(F_{\beta_n,N}) = 1.$$

*Moreover, for any $N > 2n+4$, $\dim_H(F_{\beta_n,N} \setminus E_{\beta_n,N}) > 0$. Furthermore, we can find some set of positive Hausdorff dimension such that every point in this set has uncountably many expansions, but none of them is a universal expansion.*

The last statement strengthens one result of [21, Counterexample].

The following result is about the topological structure of $E_{\beta,N}$.

THEOREM 2.5. *Given any $N \geq 3$, for almost every $\beta \in (1,2)$, $E_{\beta,N}$ is a graph-directed self-similar set.*

**3. Proof of main theorems.** In this section, we give a proof of Theorem 2.2. To begin, we recall some classical results and notation. An expansion $(a_n)$ is called the *quasi-greedy expansion* if it is the largest infinite expansion, in the sense of lexicographical ordering. Denote $\sigma((a_n)_{n=1}^\infty) = (a_n)_{n=2}^\infty$ and $\sigma^k((a_n)_{n=1}^\infty) = (a_n)_{n=k+1}^\infty$. Let $(\alpha_n)$ be the quasi-greedy expansion of 1. The following classical result was proved by Parry [18].

THEOREM 3.1. *Let $(a_n)_{n=1}^\infty$ be an expansion of $x \in [0,(\beta-1)^{-1}]$. Then $(a_n)_{n=1}^\infty$ is the greedy expansion if and only if*

$$\sigma^k((a_n)_{n=1}^\infty) < (\alpha_n)_{n=1}^\infty \quad \textit{whenever } a_k = 0.$$

LEMMA 3.2. *For any $n \geq 2$, let $\beta_n$ be the n-bonacci number. Then for any $N \geq 3$,*

$$F_{\beta_n,N-1} \subset E_{\beta_n,N}.$$

*Proof.* Since $\beta_n$ is the Pisot number satisfying $\beta^n = \beta^{n-1} + \beta^{n-2} + \cdots + \beta + 1$, it follows that the quasi-greedy expansion of 1 is $(1^{n-1}0)^\infty$. Hence the block $1^{n-1}$ can appear in the greedy expansions. In other words, any expansion in base $\beta_n$ can be changed into the greedy expansion using the rule $10^n \sim 01^n$, i.e. the block $10^n$ can be replaced by $01^n$ without changing the value of the corresponding number.

Given any point $x \notin E_{\beta_n,N}$, there exists an expansion of $x$ such that its coding, say $(a_n)$, contains a block $(0\cdots0)$ of length $N$, i.e. there exists some $k_0$ such that $a_{k_0+1}\cdots a_{k_0+N} = 0\cdots0$.

If $(a_n)$ is the greedy expansion of $x$, then clearly $x \notin F_{\beta_n,N-1}$. Assume $(a_n)$ is not the greedy expansion. We can transform $(a_n)$ into the greedy expansion of $x$ by using the rule $10^n \sim 01^n$. Denote by $(b_n)$ that greedy expansion. Notice that the transformation used shrinks a block of zeros in the sequence $(a_n)$ by at most one term. To be more precise, if $a_{k_0+1}\cdots a_{k_0+N}a_{k_0+N+1}a_{k_0+N+n} = 0\cdots 0\,1^n$ ($N$ zeros), then the corresponding block is

$$b_{k_0+1}\cdots b_{k_0+N}b_{k_0+N+1}b_{k_0+N+n} = \underbrace{0\cdots 0}_{N-1}10^n.$$

Thus, $x \notin F_{\beta_n,N-1}$. ∎

Next, we want to prove that

$$\lim_{N\to\infty}\dim_H(F_{\beta_n,N}) = 1.$$

This result can be obtained by applying perturbation theory; it was essentially proved by Ferguson and Pollicott [11, Theorem 1.2].

Lemma 3.3. *For any* $1<\beta<2$, $\lim_{N\to\infty}\dim_H(F_{\beta,N}) = 1$.

Here we give a detailed proof of our desired limit.

Lemma 3.4.

$$\lim_{N\to\infty}\dim_H(F_{\beta_n,N}) = 1.$$

For simplicity, we assume $n=2$; for $n\ge 3$ the proof is similar but the calculation is more complicated. We give an outline of the proof. First, we give a Markov partition for $[0,(\beta-1)^{-1}]$ using the orbit of 1. Hence, we can define an adjacency matrix $S$ and construct an associated subshift $\Sigma$ of finite type. Equivalently, we transform the original space $\{0,1\}^{\mathbb{N}}$ into a subshift of finite type. Next, we define a submatrix $S'$ of $S$, and construct a graph-directed self-similar set with the open set condition [17]. Finally, we identify $F_{\beta_n,N}$ with a graph-directed self-similar set, and prove the desired result.

Now we transform the symbolic space as follows.

Lemma 3.5. *Let* $\beta=(1+\sqrt{5})/2$ *and* $x\in[0,(\beta-1)^{-1}]$. *Then the greedy expansion of* $x$ *has a coding coming from some subshift of finite type.*

*Proof.* We start by giving a Markov partition for the interval $[0,(\beta-1)^{-1}]$ as follows. Let $N\ge 3$, and set

$$a_1 = 0,\quad a_i = \beta^{-N-2+i}(\beta-1)^{-1},\quad 2\le i\le N-1,$$
$$a_N = \beta^{-1} = \beta^{-2}(\beta-1)^{-1},\quad a_{N+1}=1,\quad a_{N+2}=(\beta-1)^{-1}.$$

Define

$$\begin{aligned} A_1 &= [0,\beta^{-N}(\beta-1)^{-1}],\\ A_i &= [\beta^{-N+i-2}(\beta-1)^{-1},\beta^{-N+i-1}(\beta-1)^{-1}],\quad 2\le i\le N,\\ A_{N+1} &= [1,(\beta-1)^{-1}]. \end{aligned}$$

It is easy to check that

$$T_0(A_1) = A_1 \cup A_2, \quad T_0(A_i) = A_{i+1}, \quad 2 \le i \le N-1,$$

and that $T_1(A_N) = \bigcup_{i=1}^{N-1} A_i$ and $T_1(A_{N+1}) = A_N \cup A_{N+1}$. Hence, we have the following adjacency matrix $S = (s_{ij})_{(N+1)\times(N+1)}$:

$$s_{ij} = \begin{cases} 1, & i = 1,\ j = 1, 2, \\ 1, & 2 \le i \le N-1,\ j = i+1, \\ 1, & i = N,\ j = 1, \dots, N-1, \\ 1, & i = N+1,\ j = N, N+1, \\ 0, & \text{else.} \end{cases}$$

Using $S$, we can construct a subshift $\Sigma$ of finite type. For any

$$(\alpha_i) \in \{1, \dots, N+1\}^{\mathbb{N}},$$

we call $\{A_{\alpha_i}\}_{i=1}^{\infty}$ an *admissible path* if there is some $T_k$, $k = 0$ or $1$, such that

$$T_k(A_{\alpha_i}) \supset A_{\alpha_{i+1}}$$

for any $i \ge 1$. In terms of this definition, we have

$$\Sigma = \big\{(\alpha_i)_{i=1}^{\infty} : \alpha_i \in \{1, \dots, N+1\},\ \{A_{\alpha_i}\}_{i=1}^{\infty} \text{ is an admissible path}\big\}. \ \blacksquare$$

Remark 3.6. Usually, we take the elements $A_i$ of a Markov partition closed on the left and open on the right, i.e. $A_i = [a_i, a_{i+1})$. Under our algorithm, one has a choice at the endpoints. For example, the point $\beta^{-1}$ is the right endpoint of $A_{N-3}$ and the left endpoint of $A_{N-2}$. For this point, we can implement $T_0$ on $A_k = [a_k, \beta^{-1}]$ or $T_1$ on $[\beta^{-1}, a_{k+2}]$. This adjustment is due to the proof of Lemma 3.8. When we construct a graph-directed self-similar set, we need a closed interval—see the graph-directed construction in [17]. This is the reason why we need some compromise here. Although our Markov partition is a little different from the usual definition, this adjustment does not affect our result.

By definition of $E_{\beta,N}$, for any point $x \in E_{\beta,N}$, all possible orbits of $x$ avoid the hole $A_1 = [0, \beta^{-N}(\beta-1)^{-1}]$, which is the first element of the Markov partition. By Lemma 3.5, $x$ also has a coding in the new symbolic space $\Sigma$. For simplicity, we denote by $\{\alpha_{i_n}\}_{n=1}^{\infty}$ this coding of $x$ in $\Sigma$. Since $x \in E_{\beta,N}$, the symbol 1 cannot appear in $\{\alpha_{i_n}\}_{n=1}^{\infty}$.

Motivated by this observation, we construct a new matrix as follows. We delete the first row and first column of $S$, denote the resulting matrix by $S'$, and let $\Sigma'$ be the associated subshift generated by $S'$. Then $S'$ can be represented by a directed graph $(V, E)$. The vertex set consists of the underlying partition $\{A_i\}_{i=2}^{k}$. For any two vertices, if one vertex is one of the components of the image of another vertex, then we can find a similitude, which is the inverse of an expanding map, between these two vertices. For instance, for the vertices $A_2$ and $A_3$, if $T_0(A_2) = A_3$, then we can label a directed edge, from $A_2$ to $A_3$, by a similitude $f(x) = T_0^{-1}(x) = x/\beta$. We denote by $E$ all admissible labels between pairs of vertices. Then by Mauldin and Williams' result [17], we can construct a graph-directed self-similar set $K'_N$ satisfying the open set condition; for the detailed construction, see [17, 5].

Now we have the following lemma.

Lemma 3.7. *Let $\beta = (1+\sqrt{5})/2$. Then $F_{\beta,N} = K'_N$ up to a countable set, i.e. there exists a countable set $C_1$ such that $F_{\beta,N} \subset K'_N \subset C_1 \cup F_{\beta,N}$.*

*Proof.* Evidently, $F_{\beta,N} \subset K'_N$. Take $x \in K'_N$. Then by the definition of $K'_N$, the greedy orbit of $x$ does not hit $[0, \beta^{-N}(\beta-1)^{-1})$. If the greedy orbit of $x$ does not hit the closed interval $[0, \beta^{-N}(\beta-1)^{-1}]$, then $x \in F_{\beta,N}$. If there exists some $(i_1 \cdots i_{n_0})$ such that

$$T_{i_1 \cdots i_{n_0}}(x) = \beta^{-N}(\beta-1)^{-1},$$

then

$$x \in \bigcup_{n=1}^{\infty} \bigcup_{(i_1\cdots i_n)\in\{0,1\}^n} f_{i_1\cdots i_n}(\beta^{-N}(\beta-1)^{-1}),$$

where $f_0(x) = \beta^{-1}x$ and $f_1(x) = \beta^{-1}x + \beta^{-1}$. Therefore,

$$K'_N \subset E_{\beta,N} \cup \bigcup_{n=1}^{\infty} \bigcup_{(i_1\cdots i_n)\in\{0,1\}^n} f_{i_1\cdots i_n}(\beta^{-N}(\beta-1)^{-1}). \ \blacksquare$$

Lemma 3.8. *Let $\beta = (1+\sqrt{5})/2$. Then*

$$\dim_H(F_{\beta,N}) = \frac{\log \lambda_N}{\log \beta},$$

*where $\lambda_N$ is the largest positive root of the equation*

$$x^{N-1} = \sum_{i=0}^{N-3} x^i.$$

*Moreover,*

$$\lim_{N\to\infty} \lambda_N = \frac{1+\sqrt{5}}{2} = \beta.$$

*Proof.* By Lemma 3.7, $\dim_H(F_{\beta,N}) = \dim_H(K'_N)$. Since $K'_N$ is a graph-directed self-similar set with the open set condition, we can explicitly calculate its Hausdorff dimension: $\dim_H(F_{\beta,N}) = \log \lambda_N/\log \beta$, where $\lambda_N$ is the spectral radius of $S'$ (for the detailed method, see [17]). The second statement is a simple exercise.

This finishes the proof of Lemma 3.4 for the case $n = 2$. For $n \geq 3$, the proof is similar. $\blacksquare$

A similar result is available for the doubling map with hole [13]. Let $D(x) = 2x \bmod 1$ be the doubling map defined on $[0,1)$. Given any $\epsilon > 0$, set

$$D_\epsilon = \{x \in [0,1) : D^n(x) \notin [0,\epsilon] \text{ for any } n \geq 0\}.$$

Clearly, $\lim_{\epsilon\to 0} \dim_H(D_\epsilon)$ exists. Hence, to find this limit it suffices to consider the set

$$D_{2^{-N}} = \{x \in [0,1) : D^n(x) \notin [0,2^{-N}] \text{ for any } n \geq 0\}.$$

We have the following result.

EXAMPLE 3.9.

$$\dim_H(D_{2^{-N}}) = \frac{\log \gamma_N}{\log 2},$$

where $\gamma_N$ is the $N$-bonacci number satisfying the equation

$$x^N = x^{N-1} + x^{N-2} + \cdots + x + 1.$$

It is easy to see that $\lim_{N\to\infty} \gamma_N = 2$. Therefore,

$$\lim_{\epsilon\to 0} \dim_H(D_\epsilon) = \lim_{N\to\infty} \dim_H(D_{2^{-N}}) = 1.$$

*Proof of Theorem 2.2.* Let $\beta_n$ be an $n$-bonacci number. By Lemmas 3.4 and 3.7, we have

$$E_{\beta_n,N} \subset F_{\beta_n,N} \subset K'_N \subset F_{\beta_n,N} \cup C_1.$$

By Lemma 3.4,

$$\lim_{N\to\infty} \dim_H(F_{\beta_n,N}) = 1.$$

Therefore,

$$\dim_H(F_{\beta_n,N}) = \dim_H(E_{\beta_n,N}) \leq \dim_H(V_{\beta_n}) \leq 1,$$

which implies that $\dim_H(V_{\beta_n}) = 1$. ∎

It is easy to show that when $\beta$ is a Pisot number, then all the possible orbits of $x \in \mathbb{Q}([\beta]) \cap [0,(\beta-1)^{-1}]$ hit finitely many points only. The following lemma is standard. However for the sake of convenience, we give a detailed proof.

LEMMA 3.10. *Suppose $\beta$ is a Pisot number and $x \in \mathbb{Q}([\beta]) \cap [0,(\beta-1)^{-1}]$. Then the set*

$$\{\pi(K^n(\omega,x)) : n \geq 0,\, \omega \in \Omega\}$$

*is a finite set.*

*Proof.* Let $p(X) = X^d - q_1 X^{d-1} - \cdots - q_d$ be the minimal polynomial of $\beta$ with $q_i \in \mathbb{Z}$, $1 \leq i \leq d$. Since $\mathbb{Q}(\beta)$ is generated by $\{\beta^{-1},\ldots,\beta^{-d}\}$, there exist $a_1, \ldots, a_d \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that

$$x = b^{-1} \sum_{i=1}^{d} a_i \beta^{-i}.$$

We assume that $b$ is smallest possible to ensure uniqueness. Let $\beta_1 = \beta$, let $\beta_2, \ldots, \beta_d$ be the Galois conjugates of $\beta$, and set $B = (b_{ij})_{1\le i,j\le d} = (\beta_j^{i-1})_{1\le i,j\le d}$. Define, for $n \ge 0$ and $\omega \in \Omega$,

$$r_n^{(1)}(\omega) = \beta^n \Big( x - \sum_{k=1}^{n} b_k(\omega, x)\beta^{-k} \Big),$$

$$r_n^{(j)}(\omega) = \beta_j^n \Big( b^{-1} \sum_{i=1}^{d} a_i \beta_j^{-i} - \sum_{k=1}^{n} b_k(\omega, x)\beta_j^{-k} \Big) \quad \text{for } j = 2, \ldots, d.$$

Consider the vector $R_n(\omega) = (r_n^{(1)}(\omega), \ldots, r_n^{(d)}(\omega))$. We first show that the set $\{R_n(\omega) : n \ge 0,\, \omega \in \Omega\}$ is uniformly bounded (in $n$ and $\omega$). First note that $r_n^{(1)}(\omega) = \pi(K^n(\omega, x))$, hence $r_n^{(1)}(\omega) \le 1/(\beta - 1)$ for all $n$ and all $\omega$. Let $\eta = \max_{2\le j\le d} |\beta_j|$. Then $\eta < 1$. For $j = 2, \ldots, d$,

$$\begin{aligned} |r_n^{(j)}| &= \Big| b^{-1} \sum_{i=1}^{d} a_i \beta_j^{n-i} - \sum_{k=1}^{n} b_k(\omega, x)\beta_j^{n-k} \Big| \\ &\le b^{-1} \sum_{i=1}^{d} |a_i| \eta^{n-i} + \sum_{k=1}^{n} b_k(\omega, x)\eta^{n-k} \le \frac{b^{-1}\max_{1\le i\le d}|a_i| + 1}{1 - \eta}. \end{aligned}$$

Let

$$C = \max\Big\{ \frac{1}{\beta - 1}, \frac{b^{-1}\max_{1\le i\le d}|a_i| + 1}{1 - \eta} \Big\}.$$

Then $r_n^{(j)} < C$ for all $1 \le j \le d$, $n \ge 0$ and $\omega \in \Omega$. Thus the set $\{R_n(\omega) : n \ge 0,\ \omega \in \Omega\}$ is uniformly bounded.

Next we show that for each $\omega \in \Omega$ and $n \ge 0$, there exists $Z_n(\omega) \in \mathbb{Z}^d$ such that $R_n(\omega) = b^{-1} Z_n(\omega) B$. If $\beta$ is a root of some polynomial $p'(X) \in \mathbb{Z}[X]$, then the elements $\beta_2, \ldots, \beta_d$ are also roots of $p'(X)$. Hence it is sufficient to show that

$$r_n^{(1)} = b^{-1} \sum_{k=1}^{d} z_n^{(k)}(\omega)\beta^{-k} \tag{1}$$

for some $z_n^{(k)}(\omega) \in \mathbb{Z}$. The proof is by induction. Let $n = 1$ and note that $1 = q_1\beta^{-1} + \cdots + q_d\beta^{-d}$. Now

$$
\begin{aligned}
r_1^{(1)}(\omega) &= \beta x - b_1(\omega,x) = \beta b^{-1}\sum_{k=1}^{d} a_k\beta^{-k} - b_1(\omega,x)\sum_{k=1}^{d} q_k\beta^{-k} \\
&= b^{-1}\Big(\sum_{k=1}^{d-1}(a_1q_k - b_1(\omega,x)bq_k + a_{k+1})\beta^{-k} + (a_1 - b_1(\omega,x)b)q_d\beta^{-d}\Big) \\
&= b^{-1}\sum_{k=1}^{d} z_1^{(k)}(\omega)\beta^{-k}
\end{aligned}
$$

with

$$
z_1^{(k)}(\omega) = \begin{cases} (a_1 - b_1(\omega,x)b)q_k + a_{k+1} & \text{if } k \neq d, \\ (a_1 - b_1(\omega,x)b)q_d & \text{if } k = d. \end{cases}
$$

Suppose now that $r_i^{(1)} = b^{-1}\sum_{k=1}^{d} z_i^{(k)}(\omega)\beta^{-k}$ for $z_i^{(k)} \in \mathbb{Z}$. Since $r_n^{(1)} = \pi(K^n(\omega,x))$ for all $n \geq 0$, we have

$$
\begin{aligned}
r_{i+1}^{(1)} &= \beta r_i^{(1)} - b_{i+1}(\omega,x) = \beta b^{-1}\sum_{k=1}^{d} z_i^{(k)}(\omega)\beta^{-k} - b_{i+1}(\omega,x)\sum_{k=1}^{n} q_k\beta^{-k} \\
&= b^{-1}\Big(\sum_{k=1}^{d-1}\big(z_i^{(1)}(\omega)q_k - b_{i+1}(\omega,x)bq_k + z_i^{(k+1)}\big)\beta^{-k} \\
&\qquad\qquad + \big(z_i^{(1)}(\omega) - b_{i+1}(\omega,x)b\big)q_d\beta^{-d}\Big) \\
&= b^{-1}\sum_{k=1}^{d} z_{i+1}^{(k+1)}(\omega)\beta^{-k}
\end{aligned}
$$

with

$$
z_{i+1}^{(k)}(\omega) = \begin{cases} (z_i^{(1)}(\omega) - b_{i+1}(\omega,x)b)q_k + z_i^{(k+1)}(\omega) & \text{if } k \neq d, \\ (z_i^{(1)}(\omega) - b_{i+1}(\omega,x)b)q_d & \text{if } k = d. \end{cases}
$$

Thus, $z_{i+1}^{(k)}(\omega) \in \mathbb{Z}$. Setting $Z_n(\omega) = (z_n^{(1)},\ldots,z_n^{(d)})$, we have $Z_n(\omega) \in \mathbb{Z}^d$ and $R_n(\omega) = b^{-1}Z_n(\omega)B$.

Since $B$ is invertible, and $R_n(\omega)$ is uniformly bounded in $n$ and $\omega$, we see that $Z_n(\omega)$ is uniformly bounded, and hence takes only finitely many values. It follows that $(R_n(\omega))$ and thus $r_n^{(1)}$ takes only finitely many values. Therefore, the set $\{\pi(K^n(\omega,x)) : n \geq 0,\ \omega \in \Omega\}$ is finite. ∎

Corollary 3.11. *Let $\beta \in (1,2)$ be a Pisot number. For any $a_{i_1}\cdots a_{i_n} \in \{0,1\}^n$, the orbits of the endpoints of the interval $f_{a_{i_1}\cdots a_{i_n}}([0,(\beta-1)^{-1}])$ hit only finitely many points.*

*Proof.* By symmetry, we only need to prove that for the left endpoint $\sum_{j=1}^{n} a_{i_j}\beta^{-j}$, all of its orbits hit finitely many points. This is a direct consequence of Lemma 3.10. ∎

*Proof of Corollary 2.3 and Theorem 2.4.* By Lemma 3.10, Corollary 3.11 and the main result of Mauldin and Williams [17], we can calculate the Hausdorff dimension of $\dim_H(F_{\beta,N})$. By Lemma 3.2 and Corollary 2.3, we have the asymptotic result of Theorem 2.4.

For the "moreover" statement of Theorem 2.4, we define

$$D = \{10^{i_1}10^{i_2}10^{i_3}\cdots : n+1 \le i_k \le N-1 \text{ and there are infinitely many } i_k = N-1\}.$$

By Theorem 3.1, all the codings in $D$ are greedy in base $\beta_n$. Clearly, $D$ has uncountably many elements. Moreover,

$$p(D) = \Big\{\sum_{j=1}^{\infty} a_j \beta^{-j} : (a_j) \in D\Big\} \subset F_{\beta_n,N}.$$

Now we want to show that $p(D) \cap E_{\beta_n,N} = \emptyset$ and $\dim_H(F_{\beta_n,N} \setminus E_{\beta_n,N}) > 0$. By the definition of $D$, for any $10^{i_1}10^{i_2}10^{i_3}\cdots$ there are infinitely many $i_k = N-1$. Without loss of generality, we assume that $i_1 = N-1$, i.e. let

$$(a_k) = 10^{N-1}10^{i_2}10^{i_3}\cdots.$$

Using the rule $10^n \sim 01^n$, we have

$$x = (10^{N-1}10^{i_2}10^{i_3}\cdots)_\beta = (10^N 1^n 0^{i_2-n}10^{i_3}\cdots)_\beta \notin E_{\beta,N},$$

where $(b_k)_\beta = \sum_{k=1}^{\infty} b_k \beta^{-k}$. Hence, $p(D) \cap E_{\beta_n,N} = \emptyset$.

In order to prove $\dim_H(F_{\beta,N} \setminus E_{\beta,N}) > 0$, it suffices to show that $\dim_H(p(D)) > 0$. Here, the set $(D,\sigma)$ is indeed a subset of some $S$-gap shift [16] $D \subset D'$, where

$$D' = \{10^{i_1}10^{i_2}10^{i_3}\cdots : n+1 \le i_k \le N-1\}.$$

The entropy of $D'$ can be calculated: $h(D') = \log\lambda$, where $\lambda$ is the largest positive root of the equation

$$1 = \sum_{k \in \{n+1,\ldots,N-1\}} x^{-k-1}.$$

Now we construct a subset of $p(D)$ as follows: Let $J$ be the self-similar set with the IFS

$$\Big\{g_1(x) = \frac{x}{\beta^{n+2}} + \frac{1}{\beta},\ g_2(x) = \frac{x}{\beta^N} + \frac{1}{\beta}\Big\},$$

i.e.

$$J = g_1(J) \cup g_2(J).$$

By the definitions of $p(D)$ and $J$, we have $J \subset p(D)$. Let

$$E := \big(\beta^{N-1}(\beta^N-1)^{-1}, \beta^{n+1}(\beta^{n+2}-1)^{-1}\big).$$

It is easy to check that $g_1(E) \cap g_2(E) = \emptyset$ and $g_i(E) \subset E$. In other words, the IFS satisfies the open set condition [14]. Hence, $\dim_H(J) = s > 0$, where

$s$ is the unique solution of the equation $\beta^{(-n-2)s} + \beta^{-Ns} = 1$. Consequently,

$$0 < \dim_H(J) = s \le \dim_H(p(D)).$$

For the last statement of Theorem 2.4, it suffices to consider the set $p(D)$. ∎

Now we prove Theorem 2.5. We partition the proof into several lemmas. The following result is essentially proved in [2]. For convenience, we give a detailed proof.

LEMMA 3.12. *Let $1 < \beta < 2$ and $N \ge 3$. If there exists some $(\eta_1 \cdots \eta_p) \in \{0,1\}^p$ such that $T_{\eta_1 \cdots \eta_p}(\beta^{-N}(\beta-1)^{-1}) \in (0, \beta^{-N}(\beta-1)^{-1})$, then $E_{\beta,N}$ is a graph-directed self-similar set.*

*Proof.* By assumption and the continuity of the $T_j$'s, there exists $\delta > 0$ such that

$$T_{\eta_1 \dots \eta_p}\big(\beta^{-N}(\beta-1)^{-1}, \beta^{-N}(\beta-1)^{-1} + \delta\big) \subset (0, \beta^{-N}(\beta-1)^{-1}).$$

Set $H = [0, \beta^{-N}(\beta-1)^{-1} + \delta]$. We partition $[0, (\beta-1)^{-1}]$ in terms of the iterated function system

$$f_j(x) = \frac{x+j}{\beta}, \quad j \in \{0,1\}.$$

For any $L$ we have

$$[0, (\beta-1)^{-1}] = \bigcup_{(i_1,\dots,i_L)\in\{1,\dots,m\}^L} f_{i_1} \circ \dots \circ f_{i_L}([0,(\beta-1)^{-1}]).$$

We assume without loss of generality that $L$ is so large that

$$|f_{i_1} \circ \dots \circ f_{i_L}([0,(\beta-1)^{-1}])| < \delta$$

for all $(i_1, \dots, i_L) \in \{0,1\}^L$. Correspondingly, we partition the symbolic space $\{0,1\}^{\mathbb{N}}$ into cylinders of length $L$. For every $(i_1, \dots, i_L) \in \{0,1\}^L$ let

$$C_{i_1 \dots i_L} = \big\{(x_n) \in \{0,1\}^{\mathbb{N}} : x_n = i_n \text{ for } 1 \le n \le L\big\}.$$

Then $\{C_{i_1 \dots i_L}\}_{(i_1,\dots,i_L)\in\{0,1\}^L}$ is a partition of $\{0,1\}^{\mathbb{N}}$, and

$$f_{i_1} \circ \dots \circ f_{i_L}([0,(\beta-1)^{-1}]) = \pi(C_{i_1 \dots i_L}).$$

Let

$$\mathbb{F} = \big\{(i_1, \dots, i_L) \in \{1, \dots, m\}^L : \\ f_{i_1} \circ \dots \circ f_{i_L}([0,(\beta-1)^{-1}]) \cap [0, \beta^{-N}(\beta-1)^{-1}] \ne \emptyset\big\}$$

and

$$\mathbb{F}' = \bigcup_{(i_1,\dots,i_L)\in\mathbb{F}} \pi(C_{i_1 \dots i_L}).$$

By our assumptions on the size of the cylinders, we have

$$[0, \beta^{-N}(\beta-1)^{-1}] \subset \mathbb{F}' \subset H.$$

Using these inclusions we can show that $x \notin E_{\beta,N}$ if and only if there exists $(\theta_1, \dots, \theta_{n_1}) \in \{1, \dots, m\}^{n_1}$ such that $T_{\theta_1 \dots \theta_{n_1}}(x) \in \mathbb{F}'$. If $x \notin E_{\beta,N}$, then by the above observation there exists $(\theta_1, \dots, \theta_{n_1}) \in \{1, \dots, m\}^{n_1}$ such that $T_{\theta_1 \dots \theta_{n_1}}(x) \in \mathbb{F}'$. Therefore, $x$ has a coding containing a block from $\mathbb{F}$. Conversely, if there exists $(\theta_1, \dots, \theta_{n_1}) \in \{1, \dots, m\}^{n_1}$ such that $T_{\theta_1 \dots \theta_{n_1}}(x) \in \mathbb{F}'$, then the condition

$$T_{\eta_1 \dots \eta_p}\big(\beta^{-N}(\beta-1)^{-1}, \beta^{-N}(\beta-1)^{-1}+\delta\big) \subset (0, \beta^{-N}(\beta-1)^{-1})$$

yields $x \notin E_{\beta,N}$. Taking $\mathbb{F}$ to be the set of forbidden words defining a subshift of finite type, we see that $E_{\beta,N}$ is a graph-directed self-similar set (see [5, 17]). ∎

Schmeling [19] proved the following result.

Lemma 3.13. *For almost every $\beta \in (1,2)$, the greedy orbits of $1$ and the lazy orbit of $\bar{1} = (\beta-1)^{-1} - 1$ are dense.*

*Proof of Theorem 2.5.* Theorem 2.5 follows immediately from Lemmas 3.12 and 3.13. ∎

**4. Final remarks.** Similar results are available if we consider $\beta$-expansions with more than two digits. For some Pisot numbers, we may implement similar ideas which are utilized in Lemmas 3.4 and 3.2. Finally, we pose a problem.

Problem 4.1. *Does there exist $\delta > 0$ such that $\dim_H(V_\beta) = 1$ for any $\beta \in (2-\delta, 2)$?*

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