

Two models for the homotopy theory of ∞ -operads

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ABSTRACT

We compare two models for ∞ -operads: the complete Segal operads of Barwick and the complete dendroidal Segal spaces of Cisinski and Moerdijk. Combining this with comparison results already in the literature, this implies that all known models for ∞ -operads are equivalent — for instance, it follows that the homotopy theory of Lurie’s ∞ -operads is equivalent to that of dendroidal sets and that of simplicial operads.

1. Introduction

The theory of *operads* is a convenient framework for organizing a variety of algebraic structures, such as associative and commutative algebras, or more interestingly algebras which are associative or commutative up to coherent homotopy. For us, operads will by default be *coloured* operads, that is, we allow them to have many objects — these can be used to describe structures such as enriched categories or a pair of rings together with a bimodule. Roughly speaking, an operad \mathbf{O} consists of a set of objects, for each list of objects (x_1, \dots, x_n, y) a set of multimorphisms $\mathbf{O}(x_1, \dots, x_n; y)$ from (x_1, \dots, x_n) to y , equipped with an action of the symmetric group Σ_n that permutes the inputs x_i , and associative and unital composition operations for the multimorphisms. More generally, we can consider *enriched* operads, where the sets of multimorphisms are replaced by objects of some symmetric monoidal category, such as vector spaces or chain complexes; these can be used to describe algebraic structures such as Lie algebras or Poisson algebras.

In topology, we typically encounter operads enriched in topological spaces (or simplicial sets) such as the E_n -operads of May [10]. There is an evident notion of (weak) homotopy equivalence between such operads and one would like to consider the category of topological operads and weak equivalences as a homotopy theory. Unfortunately, for many purposes it can be difficult to work with this theory, because topological operads are in a sense too rigid — for instance, a weak equivalence between topological operads \mathbf{P} and \mathbf{Q} need not induce an equivalence between the homotopy theories of \mathbf{P} -algebras and \mathbf{Q} -algebras. Moreover, one often encounters structures that are naturally seen as operad algebras in a homotopy-coherent sense, but can be difficult to rigidify to fit in this framework — as a baby example, it is reasonable to think of symmetric monoidal categories as ‘commutative monoids’ in the (2-)category of categories, but actual commutative monoids require the associativity and symmetry conditions to hold strictly, which is essentially never true for interesting examples.

For these reasons, it is desirable to have a usable theory of ‘weak’ or homotopy-coherent operads, where composition of multimorphisms is only associative up to a (specified) coherent choice of higher homotopies, and homotopy-coherent algebras for them. The foundations for

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a theory of such ∞ -operads were set up by Lurie in [9]; his work gives a powerful framework for working with homotopy-coherent algebraic structures, as evidenced by the many results obtained in [9] after building these foundations.

Although Lurie’s model is by far the best developed version of ∞ -operads, a number of other models have been proposed, namely the *dendroidal sets* of Moerdijk and Weiss [11] and the closely related models of complete dendroidal Segal spaces and dendroidal Segal operads of Cisinski and Moerdijk [2], and the *complete Segal operads* of Barwick [1]. Moreover, just as simplicial categories give a model for ∞ -categories, we can consider simplicial operads as a model for ∞ -operads; appropriate model category structures on this category have been constructed by Cisinski and Moerdijk [3] and by Robertson [13].

Some comparisons between these different models are already known.

- Cisinski and Moerdijk compare the three dendroidal models in [2], and also compare dendroidal sets to simplicial operads in [3].
- Barwick compares his model to Lurie’s in [1].
- Heuts, Hinich and Moerdijk obtain a partial comparison between dendroidal sets and Lurie’s model in [5]. However, their result is restricted to operads without units.

In this paper, our goal is to prove one of the missing comparisons: we will show that the homotopy theory of Barwick’s complete Segal operads is equivalent to that of complete dendroidal Segal spaces. To state a more precise version of our result, recall that Barwick’s Segal operads are certain presheaves of spaces on a category $\Delta_{\mathbb{F}}$, forming a full subcategory $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ of the ∞ -category $\mathcal{P}(\Delta_{\mathbb{F}})$ of presheaves — we will refer to them as *Segal presheaves* on $\Delta_{\mathbb{F}}$ to avoid confusion with the Segal operads of Cisinski and Moerdijk [3], which are a dendroidal analogue of Segal categories. Similarly, the dendroidal Segal spaces of Cisinski and Moerdijk are certain presheaves on a category Ω (we will likewise refer to them as *Segal presheaves* on Ω), forming a full subcategory $\mathcal{P}_{\text{Seg}}(\Omega)$ of the ∞ -category $\mathcal{P}(\Omega)$ of all presheaves. We will define a functor $\tau: \Delta_{\mathbb{F}}^{\perp} \rightarrow \Omega$, where $i: \Delta_{\mathbb{F}}^{\perp} \hookrightarrow \Delta_{\mathbb{F}}$ is a certain full subcategory, and prove:

THEOREM 1.1. *Composition with the functors i and τ induces equivalences of ∞ -categories*

$$\mathcal{P}_{\text{Seg}}(\Omega) \xrightarrow{\sim} \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^{\perp}) \xleftarrow{\sim} \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}).$$

These functors restrict further to give equivalences between the full subcategories of complete objects.

Here the *complete* objects are those whose underlying Segal spaces are complete in the sense of Rezk [12]. We will prove that i gives an equivalence in Lemma 2.11, that τ gives an equivalence in Theorem 5.1, and that we get equivalences on complete objects in Corollary 6.3. In fact, these subcategories of complete objects are localizations of the respective ∞ -categories of Segal presheaves. A map of dendroidal Segal spaces is known to become an equivalence after completion if and only if it is *fully faithful* and *essentially surjective* by a result of Cisinski and Moerdijk. In Corollary 6.5 we apply Theorem 1.1 to deduce the analogous statement for Barwick’s Segal operads.

Combining Theorem 1.1 with the above-mentioned comparison results already in the literature, this implies that all known models for ∞ -operads are equivalent. In particular, we obtain the following interesting comparisons as an immediate consequence of our work, answering a question of Lurie [9]:

COROLLARY 1.2. *The homotopy theory of Lurie’s ∞ -operads is equivalent to that of dendroidal sets and to that of simplicial operads.*

Although we have chosen to use the language of ∞ -categories in this paper, as we believe this leads to a cleaner presentation of our work, our result can also be interpreted in the language of model categories: the ∞ -categories $\mathcal{P}_{\text{Seg}}(\Omega)$, $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ and $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ can be obtained from Bousfield localizations of the projective (or Reedy) model structures on the categories $\text{Fun}(\Omega^{\text{op}}, \text{Set}_{\Delta})$, $\text{Fun}(\Delta_{\mathbb{F}}^{1,\text{op}}, \text{Set}_{\Delta})$ and $\text{Fun}(\Delta_{\mathbb{F}}^{\text{op}}, \text{Set}_{\Delta})$ of simplicial presheaves on Ω , $\Delta_{\mathbb{F}}^1$ and $\Delta_{\mathbb{F}}$, respectively. Moreover, it is easy to see that composition with i with τ give right Quillen functors between these localized model structures (with left adjoints given by left Kan extensions). In this language, our result says:

COROLLARY 1.3. *The Quillen adjunctions*

$$\begin{aligned} \tau_! : \text{Fun}(\Delta_{\mathbb{F}}^{1,\text{op}}, \text{Set}_{\Delta}) &\rightleftarrows \text{Fun}(\Omega^{\text{op}}, \text{Set}_{\Delta}) : \tau^*; \\ i_! : \text{Fun}(\Delta_{\mathbb{F}}^{1,\text{op}}, \text{Set}_{\Delta}) &\rightleftarrows \text{Fun}(\Delta_{\mathbb{F}}^{\text{op}}, \text{Set}_{\Delta}) : i^* \end{aligned}$$

are Quillen equivalences, where the categories involved are equipped with the Bousfield localizations of the respective projective model structures at the Segal equivalences. Moreover, they remain Quillen equivalences if we localize further to get the model structures for complete objects.

Since a Quillen adjunction is a Quillen equivalence if and only if it induces an equivalence of homotopy categories, this is an immediate consequence of Theorem 1.1.

1.1. Overview

In §2 we review the definition of Barwick’s Segal operads, which we will call Segal presheaves on $\Delta_{\mathbb{F}}$. We also show that we can equivalently consider Segal presheaves on a full subcategory $\Delta_{\mathbb{F}}^1$ of $\Delta_{\mathbb{F}}$. Next, in §3 we review the dendroidal Segal spaces of Cisinski and Moerdijk, which we will similarly refer to as Segal presheaves on Ω . In §4 we define the functor τ from $\Delta_{\mathbb{F}}^1$ to Ω , and then in §5 we prove our main comparison result, namely that composing with τ gives an equivalence between the two ∞ -categories of Segal presheaves. Finally, in §6 we review the definition of complete Segal presheaves on $\Delta_{\mathbb{F}}$ and Ω , and observe that these agree under our equivalence.

1.2. Notation

This paper is written in the language of ∞ -categories (or more specifically *quasicategories*), as developed by Joyal [6], Lurie [8, 9] and others. We will use terminology from [8]; here we give a few reminders.

- \mathcal{S} is the ∞ -category of spaces (or ∞ -groupoids).
- If \mathcal{C} is an ∞ -category, we write $\mathcal{P}(\mathcal{C})$ for the ∞ -category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$ of presheaves of spaces on \mathcal{C} .
- Δ is the usual simplicial indexing category. We say a morphism $\phi: [n] \rightarrow [m]$ is *inert* if it is the inclusion of a subinterval in $[m]$, that is, if $\phi(i) = \phi(0) + i$ for all i , and *active* if it preserves the end-points, that is, if $\phi(0) = 0$ and $\phi(n) = m$. The active and inert morphisms form a factorization system on Δ .

2. Segal presheaves on $\Delta_{\mathbb{F}}$ and $\Delta_{\mathbb{F}}^1$

In this section we review the model for ∞ -operads introduced by Barwick in [1], which we will refer to as Segal presheaves on $\Delta_{\mathbb{F}}$. We also show that these are equivalent to Segal presheaves on a full subcategory $\Delta_{\mathbb{F}}^1$, which will be easier to relate to the dendroidal category later on.

DEFINITION 2.1. Write \mathbb{F} for a skeleton of the category of finite sets (possibly empty), that is, the category with objects $\mathbf{k} := \{1, \dots, k\}$, $k = 0, 1, \dots$, and morphisms maps of sets. Let $\Delta_{\mathbb{F}}$ be the category with objects pairs $([n], f: [n] \rightarrow \mathbb{F})$ with a morphism $([n], f) \rightarrow ([m], g)$ given by a morphism $\phi: [n] \rightarrow [m]$ in Δ and a natural transformation $\eta: f \rightarrow g \circ \phi$ such that

- (i) the map $\eta_i: f(i) \rightarrow g(\phi(i))$ is injective for all $i = 0, \dots, m$,
- (ii) the commutative square

$$\begin{array}{ccc} f(i) & \xrightarrow{\eta_i} & g(\phi(i)) \\ \downarrow & & \downarrow \\ f(j) & \xrightarrow{\eta_j} & g(\phi(j)) \end{array}$$

is a pullback square for all $0 \leq i \leq j \leq m$.

We say an object $([n], f) \in \Delta_{\mathbb{F}}$ has *length* n .

NOTATION 2.2. If $([n], f)$ is an object of $\Delta_{\mathbb{F}}$, we will write $f^{ij}: f(i) \rightarrow f(j)$ for the image under f of the map $i \rightarrow j$ in \mathbf{n} ; we abbreviate $f^{i(i+1)}$ to f^{i+1} .

REMARK 2.3. An object of $\Delta_{\mathbb{F}}$ is thus a sequence

$$\mathbf{k}_0 \rightarrow \mathbf{k}_1 \rightarrow \dots \rightarrow \mathbf{k}_n$$

of maps of finite sets. If $\mathbf{k}_n = \mathbf{1}$, we can think of this as a tree with levels: we think of the elements of the sets \mathbf{k}_i as the edges of the tree — in particular, \mathbf{k}_0 is the set of leaves, and the map $\mathbf{k}_i \rightarrow \mathbf{k}_{i+1}$ assigns to an edge e in level i the unique outgoing edge of the vertex that has e as an incoming edge; thus we can also think of the elements of \mathbf{k}_i with $i > 0$ as the vertices of the tree. A general object of $\Delta_{\mathbb{F}}$ can then be thought of as a ‘forest’, that is, a collection of trees indexed by \mathbf{k}_n . To define Segal presheaves we now want to impose relations on presheaves on $\Delta_{\mathbb{F}}$ that force the value on a forest to decompose into the values at the basic corollas, corresponding to the objects $([1], \mathbf{n} \rightarrow \mathbf{1})$, as well as the single edge $([0], \mathbf{1})$.

REMARK 2.4. Since morphisms in $\Delta_{\mathbb{F}}$ are required to induce pullback squares, given an object $([m], f) \in \Delta_{\mathbb{F}}$ and a morphism $q: \mathbf{a} \rightarrow f(m)$, there exists an essentially unique morphism $([m], f_{\mathbf{a}}) \rightarrow ([m], f)$ over $\text{id}_{[m]}$ with value q at m .

REMARK 2.5. The projection $\Delta_{\mathbb{F}} \rightarrow \Delta$ is a Grothendieck fibration: given $([n], f)$ and $\phi: [m] \rightarrow [n]$, the map $\phi^*([n], f) := ([m], f \circ \phi) \rightarrow ([n], f)$ is a Cartesian morphism. In general a morphism $(\phi, \eta): ([m], g) \rightarrow ([n], f)$ is Cartesian if and only if $\eta_i: g(i) \rightarrow f(\phi(i))$ is an isomorphism for all i .

DEFINITION 2.6. We say a map $(\phi, \eta): ([n], f) \rightarrow ([m], g)$ in $\Delta_{\mathbb{F}}$ is

- (1) *injective* if $\phi: [n] \rightarrow [m]$ is injective;
- (2) *surjective* if ϕ is surjective, and $\eta_i: f(i) \rightarrow g(\phi(i))$ is an isomorphism for all i (or equivalently, if ϕ is surjective and (ϕ, η) is Cartesian);
- (3) *inert* if ϕ is inert in Δ ;
- (4) *active* if ϕ is active in Δ , and $\eta_i: f(i) \rightarrow g(\phi(i))$ is an isomorphism for all i (or equivalently, if ϕ is active and (ϕ, η) is Cartesian).

The surjective and injective maps, as well as the active and inert maps, form factorization systems on $\Delta_{\mathbb{F}}$ — this is clear since they are both lifted from factorization systems on Δ via the fibration $\Delta_{\mathbb{F}} \rightarrow \Delta$. We write $\Delta_{\mathbb{F},\text{int}}$ for the subcategory of $\Delta_{\mathbb{F}}$ containing only the inert maps.

A presheaf $\mathcal{F}: \Delta_{\mathbb{F}}^{\text{op}} \rightarrow \mathcal{S}$ is a *Segal presheaf* if it satisfies the following three ‘Segal conditions’:

- (1) for every object $([n], \mathbf{k}_0 \rightarrow \mathbf{k}_1 \rightarrow \cdots \rightarrow \mathbf{k}_n)$ of $\Delta_{\mathbb{F}}$, the natural map

$$\mathcal{F}([n], \mathbf{k}_0 \rightarrow \cdots \rightarrow \mathbf{k}_n) \rightarrow \mathcal{F}([1], \mathbf{k}_0 \rightarrow \mathbf{k}_1) \times_{\mathcal{F}([0], \mathbf{k}_1)} \cdots \times_{\mathcal{F}([0], \mathbf{k}_{n-1})} \mathcal{F}([1], \mathbf{k}_{n-1} \rightarrow \mathbf{k}_n)$$

is an equivalence;

- (2) for every object $([1], \mathbf{k} \rightarrow \mathbf{1})$, the natural map

$$\mathcal{F}([1], \mathbf{k} \rightarrow \mathbf{1}) \rightarrow \prod_{i=1}^l \mathcal{F}([1], \mathbf{k}_i \rightarrow \mathbf{1})$$

(where \mathbf{k}_i is the fibre of \mathbf{k} over $i \in \mathbf{1}$) is an equivalence;

- (3) for every object $([0], \mathbf{k})$, the natural map

$$\mathcal{F}([0], \mathbf{k}) \rightarrow \prod_{i=1}^k \mathcal{F}([0], \mathbf{1})$$

is an equivalence.

For us, a more convenient formulation of this definition will be the following:

DEFINITION 2.7. Let $\Delta_{\mathbb{F}}^{\text{el}}$ denote the full subcategory of $\Delta_{\mathbb{F},\text{int}}$ spanned by the objects $([1], \mathbf{k} \rightarrow \mathbf{1})$, for $k \geq 0$, and $([0], \mathbf{1})$, which we refer to as *elementary objects*. Then we say a presheaf $\mathcal{F}: \Delta_{\mathbb{F}}^{\text{op}} \rightarrow \mathcal{S}$ is a *Segal presheaf* if the restriction $\mathcal{F}|_{\Delta_{\mathbb{F},\text{int}}^{\text{op}}}$ is the right Kan extension of its restriction to $\Delta_{\mathbb{F}}^{\text{el,op}}$. We write $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ for the full subcategory of $\mathcal{P}(\Delta_{\mathbb{F}})$ spanned by the Segal presheaves.

REMARK 2.8. For $X \in \Delta_{\mathbb{F}}$, write $\Delta_{\mathbb{F}/X}^{\text{el}}$ for the category $\Delta_{\mathbb{F}}^{\text{el}} \times_{\Delta_{\mathbb{F},\text{int}}} (\Delta_{\mathbb{F},\text{int}})_{/X}$; then $\mathcal{F} \in \mathcal{P}(\Delta_{\mathbb{F}})$ is a Segal presheaf if and only if for every $X \in \Delta_{\mathbb{F}}$ the map $\mathcal{F}(X) \rightarrow \lim_{E \in \Delta_{\mathbb{F}/X}^{\text{el}}} \mathcal{F}(E)$ is an equivalence. If we let X_{Seg} denote the presheaf $\text{colim}_{E \in \Delta_{\mathbb{F}/X}^{\text{el}}} E$ (where we regard E as a presheaf via the Yoneda embedding), then this means that \mathcal{F} is a Segal presheaf if and only if it is local with respect to the maps $X_{\text{Seg}} \rightarrow X$ for $X \in \Delta_{\mathbb{F}}$. Thus $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ is the localization of $\mathcal{P}(\Delta_{\mathbb{F}})$ with respect to these maps — in particular, it is an accessible localization of $\mathcal{P}(\Delta_{\mathbb{F}})$; we write $L_{\Delta_{\mathbb{F}}}: \mathcal{P}(\Delta_{\mathbb{F}}) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ for the localization functor. We call the local equivalences for this localization, that is, the maps that are sent to equivalences by $L_{\Delta_{\mathbb{F}}}$, the *Segal equivalences* in $\mathcal{P}(\Delta_{\mathbb{F}})$.

DEFINITION 2.9. Let $\Delta_{\mathbb{F}}^1$ be the full subcategory of $\Delta_{\mathbb{F}}$ spanned by the objects $([n], f)$ such that $f(n) = \mathbf{1}$. The active–inert and surjective–injective factorization systems on $\Delta_{\mathbb{F}}$ clearly restrict to factorization systems on $\Delta_{\mathbb{F}}^1$. We write $\Delta_{\mathbb{F},\text{int}}^1$ for the subcategory of $\Delta_{\mathbb{F}}^1$ containing only the inert maps. Since $\Delta_{\mathbb{F}}^{\text{el}}$ is a full subcategory of $\Delta_{\mathbb{F},\text{int}}^1$, we can again define a presheaf $\mathcal{F}: \Delta_{\mathbb{F}}^{\text{1,op}} \rightarrow \mathcal{S}$ to be a *Segal presheaf* if the restriction $\mathcal{F}|_{\Delta_{\mathbb{F},\text{int}}^{\text{1,op}}}$ is a right Kan extension of its restriction to $\Delta_{\mathbb{F}}^{\text{el,op}}$. Let $i: \Delta_{\mathbb{F}}^1 \hookrightarrow \Delta_{\mathbb{F}}$ denote the inclusion. Then it is clear from the definition that composition with i induces a functor $i^*: \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$.

REMARK 2.10. A presheaf $\mathcal{F} \in \mathcal{P}(\Delta_{\mathbb{F}}^1)$ is again a Segal presheaf if and only if \mathcal{F} is local with respect to the maps $X_{\text{Seg}} \rightarrow X$ for $X \in \Delta_{\mathbb{F}}^1$. Thus $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ is the localization of $\mathcal{P}(\Delta_{\mathbb{F}}^1)$ with respect to these maps — in particular, it is an accessible localization of $\mathcal{P}(\Delta_{\mathbb{F}}^1)$; we write $L_{\Delta_{\mathbb{F}}^1}: \mathcal{P}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ for the localization functor. We call the local equivalences for this localization, that is, the maps that are sent to equivalences by $L_{\Delta_{\mathbb{F}}^1}$, the *Segal equivalences* in $\mathcal{P}(\Delta_{\mathbb{F}}^1)$.

LEMMA 2.11. *The functor $i^*: \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ is an equivalence.*

Proof. We will show that the right Kan extension functor $i_*: \mathcal{P}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}(\Delta_{\mathbb{F}})$, which is right adjoint to $i^*: \mathcal{P}(\Delta_{\mathbb{F}}) \rightarrow \mathcal{P}(\Delta_{\mathbb{F}}^1)$, restricts to an inverse to i^* on Segal presheaves. First of all, as i^* preserves colimits, it is easy to see that it sends Segal equivalences to Segal equivalences; it follows that i_* preserves the property of being a Segal presheaf. To see that i_* indeed gives the desired inverse, we will show that the natural transformations $\text{id}_{\mathcal{P}(\Delta_{\mathbb{F}})} \rightarrow i_*i^*$ and $i^*i_* \rightarrow \text{id}_{\mathcal{P}(\Delta_{\mathbb{F}}^1)}$ are equivalences on Segal presheaves.

Since $i: \Delta_{\mathbb{F}}^1 \hookrightarrow \Delta_{\mathbb{F}}$ is the inclusion of a full subcategory, the functor i_* is fully faithful, and so $i^*i_* \rightarrow \text{id}_{\mathcal{P}(\Delta_{\mathbb{F}}^1)}$ is an equivalence for any presheaf on $\Delta_{\mathbb{F}}^1$.

If \mathcal{F} is a presheaf on $\Delta_{\mathbb{F}}$, then the component of $\mathcal{F} \rightarrow i_*i^*\mathcal{F}$ at $([n], f) \in \Delta_{\mathbb{F}}$ is the natural map $\mathcal{F}([n], f) \rightarrow \lim_{((\Delta_{\mathbb{F}}^1)_{/([n], f)})^{\text{op}}} \mathcal{F}$. If $f(n) = \mathbf{k}$, let $([n], f_i)$ for $i = 1, \dots, k$ denote the subtree

$$f(0)_i \rightarrow f(1)_i \rightarrow \dots \rightarrow f(n-1)_i \rightarrow \{i\}$$

given by the fibres $f(j)_i$ of $f^{j^n}: f(j) \rightarrow f(n) = \mathbf{k}$ at $i \in \mathbf{k}$. To understand the limit $i_*i^*\mathcal{F}([n], f)$ we will prove that the inclusion $\{f_i : i = 1, \dots, k\} \hookrightarrow (\Delta_{\mathbb{F}}^1)_{/([n], f)}$ is cofinal. By Joyal’s version of Quillen’s Theorem A [8, Theorem 4.1.3.1] it suffices to show that for every $(\phi, \eta): ([m], g) \rightarrow ([n], f)$ in $(\Delta_{\mathbb{F}}^1)_{/([n], f)}$, the category $\{f_i : i = 1, \dots, k\}_{(\phi, \eta)_{/}}$ is weakly contractible. But it is clear that this is the one-object set $\{f_j\}$, where $j = f^{\phi(m)^n}(\eta_m(1))$, which is certainly weakly contractible.

It follows that $(i_*i^*\mathcal{F})([n], f) \simeq \prod_{i \in f(n)} \mathcal{F}([n], f_i)$, and the map $\mathcal{F}([n], f) \rightarrow (i_*i^*\mathcal{F})([n], f)$ is the natural map $\mathcal{F}([n], f) \rightarrow \prod_{i \in f(n)} \mathcal{F}([n], f_i)$. But if \mathcal{F} is a Segal presheaf then this map is an equivalence (since $(\Delta_{\mathbb{F}}^{\text{el}})_{/([n], f)}$ is the coproduct of $(\Delta_{\mathbb{F}}^{\text{el}})_{/([n], f_i)}$ over $i \in f(n)$). \square

3. Segal Presheaves on Ω

We now recall the definition of the dendroidal category Ω . It was originally defined by Moerdijk and Weiss [11] as a category of trees, with the morphisms given by the operad maps between the free operads generated by these trees. A combinatorial reformulation of this definition was later given by Kock [7], and it is his definition that we will recall here.

DEFINITION 3.1. A *polynomial endofunctor* is a diagram of sets

$$X_0 \xleftarrow{s} X_2 \xrightarrow{p} X_1 \xrightarrow{t} X_0.$$

A polynomial endofunctor is a *tree* if

- (1) the sets X_i are all finite;
- (2) the function t is injective;
- (3) the function s is injective, with a unique element R (the *root*) in the complement of its image;
- (4) define a successor function $\sigma: X_0 \rightarrow X_0$ as follows. First, set $\sigma(R) = R$. For $e \in s(X_2)$ (which is the complement of R in X_0), take e' in X_2 with $s(e') = e$ and set $\sigma(e) = t(p(e'))$. Then for every e there exists some k such that $\sigma^k(e) = R$.

REMARK 3.2. The intuition behind this notion of ‘tree’ is as follows: we think of X_0 as the set of edges of the tree, X_1 as the set of vertices (our trees do not have vertices at their leaves or root), and X_2 as the set of pairs (v, e) where v is a vertex and e is an incoming edge of v . The function s is the projection $s(v, e) = e$, the function p is the projection $p(v, e) = v$, and the function t assigns to each vertex its unique outgoing edge.

REMARK 3.3. The name ‘polynomial endofunctor’ comes from the fact that such a diagram induces an endofunctor of $\text{Set}/_{X_0}$ given by $t_!p_*s^*$. We refer the reader to [7] for more discussion of this.

DEFINITION 3.4. A morphism of polynomial endofunctors $f: X \rightarrow Y$ is a commutative diagram

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow f_0 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ Y_0 & \longleftarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 \end{array}$$

such that the middle square is Cartesian. We write \mathcal{Q}_{int} for the category of trees and morphisms of polynomial endofunctors between them; we will refer to these as the *inert* morphisms between trees, or as *embeddings* of subtrees.

REMARK 3.5. By [7, Proposition 1.1.3] every morphism of polynomial endofunctors between trees is injective, which justifies calling these morphisms embeddings.

The following two definitions fix some terminology which we will need later.

DEFINITION 3.6. Let X be a tree. Then a *leaf* of X is an element of X_0 which is not in the image of $t: X_1 \rightarrow X_0$.

DEFINITION 3.7. We write C_n for the *n-corolla*, namely the tree

$$\{0, 1, \dots, n\} \leftarrow \{1, \dots, n\} \rightarrow \{0\} \hookrightarrow \{0, 1, \dots, n\}.$$

We write η for the *edge*, namely the trivial tree

$$* \leftarrow \emptyset \rightarrow \emptyset \hookrightarrow *.$$

DEFINITION 3.8. If T is a tree, let $\text{sub}(T)$ be the set of subtrees of T , that is, the set of morphisms $T' \rightarrow T$ in \mathcal{Q}_{int} , and let $\text{sub}'(T)$ be the set of subtrees of T with a marked leaf, that is, the set of pairs of morphisms $(\eta \rightarrow T', T' \rightarrow T)$, where the image of the first map is a leaf of T' . We then write \overline{T} for the polynomial endofunctor

$$T_0 \leftarrow \text{sub}'(T) \rightarrow \text{sub}(T) \rightarrow T_0,$$

where the first map sends a marked subtree to its marked edge, the second is the obvious projection, and the third sends a subtree to its root.

DEFINITION 3.9. The category \mathcal{Q} has objects trees, and has as morphisms $T \rightarrow T'$ the morphisms of polynomial endofunctors $\overline{T} \rightarrow \overline{T}'$.

REMARK 3.10. By [7, Corollary 1.2.10], the polynomial endofunctor \overline{T} is in fact the free polynomial monad generated by T , and the category \mathcal{Q} is a full subcategory of the Kleisli category of the monad assigning the free polynomial monad to a polynomial endofunctor. This

means that a morphism $\bar{T} \rightarrow \bar{T}'$ is uniquely determined by the composite $T \rightarrow \bar{T} \rightarrow \bar{T}'$. In fact, more is true.

LEMMA 3.11 [7, Lemma 1.3.5]. *Any morphism $\bar{T} \rightarrow \bar{T}'$ in Ω is uniquely determined by the underlying map $T_0 \rightarrow T'_0$ on sets of edges.*

DEFINITION 3.12. It follows that Ω_{int} is a subcategory of Ω ; we say a morphism in Ω is *inert* if it lies in the image of Ω_{int} . We also say a morphism $\phi: T \rightarrow T'$ in Ω is *active* if it takes the maximal subtree to the maximal subtree, or equivalently if it takes the leaves of T to the leaves of T' (bijectively) and the root of T to the root of T' .

REMARK 3.13. In [7] the inert morphisms are called *free*, and the active ones *boundary-preserving*. Our terminology follows that of Barwick [1] and Lurie [9].

PROPOSITION 3.14 (Kock [7, Proposition 1.3.13]). *The active and inert morphisms form a factorization system on Ω .*

DEFINITION 3.15. Let Ω^{el} be the full subcategory of Ω_{int} spanned by the objects C_n ($n = 0, 1, \dots$) and η . We say a presheaf $\mathcal{F}: \Omega^{\text{op}} \rightarrow \mathcal{S}$ is a *Segal presheaf* if the restriction $\mathcal{F}|_{\Omega_{\text{int}}^{\text{op}}}$ is a right Kan extension of its restriction to $\Omega^{\text{el,op}}$. We write $\mathcal{P}_{\text{Seg}}(\Omega)$ for the full subcategory of $\mathcal{P}(\Omega)$ spanned by the Segal presheaves.

REMARK 3.16. For $T \in \Omega$, write $\Omega_{/T}^{\text{el}}$ for the category $\Omega^{\text{el}} \times_{\Omega_{\text{int}}} (\Omega_{\text{int}})_{/X}$; then a presheaf $\mathcal{F} \in \mathcal{P}(\Omega)$ is a Segal presheaf if and only if the map $\mathcal{F}(T) \rightarrow \lim_{E \in \Omega_{/T}^{\text{el}}} \mathcal{F}(E)$ is an equivalence for every $T \in \Omega$. If we let T_{Seg} denote the presheaf $\text{colim}_{E \in \Omega_{/T}^{\text{el}}} E$ (where we regard E as a presheaf via the Yoneda embedding), then this means that \mathcal{F} is a Segal presheaf if and only if it is local with respect to the maps $T_{\text{Seg}} \rightarrow T$ for $T \in \Omega$. Thus $\mathcal{P}_{\text{Seg}}(\Omega)$ is the localization of $\mathcal{P}(\Omega)$ with respect to these maps — in particular, it is an accessible localization of $\mathcal{P}(\Omega)$; we write $L_{\Omega}: \mathcal{P}(\Omega) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega)$ for the localization functor. We call the local equivalences for this localization the *Segal equivalences* in $\mathcal{P}(\Omega)$.

REMARK 3.17. The ∞ -category $\mathcal{P}_{\text{Seg}}(\Omega)$ corresponds to the model category of *dendroidal Segal spaces* studied by Cisinski and Moerdijk [2].

4. From $\Delta_{\mathbb{F}}^1$ to Ω

In this section we will define a functor $\tau: \Delta_{\mathbb{F}}^1 \rightarrow \Omega$. On objects, the functor τ takes an object $([n], f)$ in $\Delta_{\mathbb{F}}^1$ to the diagram

$$\coprod_{i=0}^n f(i) \xleftarrow{s} \coprod_{i=0}^{n-1} f(i) \xrightarrow{p} \coprod_{i=1}^n f(i) \xrightarrow{t} \coprod_{i=0}^n f(i),$$

where s and t are the obvious inclusions and p takes $x \in f(i)$ to $f^{i+1}(x) \in f(i+1)$.

If $(\phi, \eta): ([n], f) \rightarrow ([m], g)$ is an inert map in $\Delta_{\mathbb{F}}^1$, then we define $\tau(\phi, \eta)$ to be the obvious morphism

$$\begin{array}{ccccccc} \coprod_{i=0}^n f(i) & \longleftarrow & \coprod_{i=0}^{n-1} f(i) & \longrightarrow & \coprod_{i=1}^n f(i) & \longrightarrow & \coprod_{i=0}^n f(i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \coprod_{j=0}^m g(j) & \longleftarrow & \coprod_{j=0}^{m-1} g(j) & \longrightarrow & \coprod_{j=1}^m g(j) & \longrightarrow & \coprod_{j=0}^m g(j). \end{array}$$

Here the middle square is Cartesian, as required, since by definition η is a Cartesian natural transformation.

To define τ for a general map in $\Delta_{\mathbb{F}}^1$, it is convenient to first introduce an intermediate object $\tilde{\tau}([n], f)$ between $\tau([n], f)$ and its free monad $\bar{\tau}([n], f)$:

DEFINITION 4.1. For $([n], f) \in \Delta_{\mathbb{F}}^1$, let $\text{sub}_{\Delta_{\mathbb{F}}^1}([n], f)$ denote the set of subtrees of $([n], f)$ given by maps in $\Delta_{\mathbb{F}}^1$, that is, the set of inert maps $([m], g) \hookrightarrow ([n], f)$ in $\Delta_{\mathbb{F}}^1$, or equivalently the set of pairs $(x \in f(i), 0 \leq j \leq i)$, corresponding to the subtree

$$f(j)_x \rightarrow f(j+1)_x \rightarrow \cdots \rightarrow f(i-1)_x \rightarrow \{x\},$$

where $f(k)_x$ is the fibre of $f^{ki}: f(k) \rightarrow f(i)$ at x . Similarly, let $\text{sub}'_{\Delta_{\mathbb{F}}^1}([n], f)$ be the set of subtrees in $\text{sub}_{\Delta_{\mathbb{F}}^1}([n], f)$ with a marked leaf, or equivalently the set of triples $(x \in f(i), 0 \leq j \leq i, y \in f(j)_x)$. We then let $\tilde{\tau}([n], f)$ denote the polynomial endofunctor

$$\coprod_{i=0}^n f(i) \leftarrow \text{sub}'_{\Delta_{\mathbb{F}}^1}([n], f) \rightarrow \text{sub}_{\Delta_{\mathbb{F}}^1}([n], f) \rightarrow \coprod_{i=0}^n f(i),$$

where the first map takes $(x \in f(i), j, y \in f(j)_x)$ to the marked leaf y and the second projects it to $(x \in f(i), j)$, and the third takes the subtree $(x \in f(i), j)$ to its root x . The definition of τ on inert maps clearly gives an injective map $\tilde{\tau}([n], f) \hookrightarrow \bar{\tau}([n], f)$ of polynomial endofunctors, and the canonical map $\tau([n], f) \rightarrow \bar{\tau}([n], f)$ factors through this.

For a general map $(\phi, \eta): ([n], f) \rightarrow ([m], g)$, we then define a map of polynomial endofunctors $\tau([n], f) \rightarrow \tilde{\tau}([m], g)$, that is,

$$\begin{array}{ccccccc} \coprod_{i=0}^n f(i) & \longleftarrow & \coprod_{i=0}^{n-1} f(i) & \longrightarrow & \coprod_{i=1}^n f(i) & \longrightarrow & \coprod_{i=0}^n f(i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \coprod_{j=0}^m g(j) & \longleftarrow & \text{sub}'_{\Delta_{\mathbb{F}}^1}([m], g) & \longrightarrow & \text{sub}_{\Delta_{\mathbb{F}}^1}([m], g) & \longrightarrow & \coprod_{j=0}^m g(j), \end{array}$$

as follows.

- The component $\coprod_{i=0}^n f(i) \rightarrow \coprod_{j=0}^m g(j)$ is the obvious map, given on $f(i)$ by $\eta_i: f(i) \rightarrow g(\phi(i))$.
- The component $\coprod_{i=1}^n f(i) \rightarrow \text{sub}_{\Delta_{\mathbb{F}}^1}([m], g)$ is given by

$$(x \in f(i)) \mapsto (\eta_i(x) \in g(\phi(i)), \phi(i-1)).$$

- The component $\coprod_{i=0}^{n-1} f(i) \rightarrow \text{sub}'_{\Delta_{\mathbb{F}}^1}([m], g)$ is defined by

$$(x \in f(i)) \mapsto (\eta_{i+1}(f^{i+1}(x)) \in g(\phi(i+1)), \phi(i), \eta_i(x) \in g(\phi(i))_{\eta_{i+1}(f^{i+1}(x))})$$

We see that the middle square in the diagram above is then Cartesian, since η is a Cartesian natural transformation, so this does indeed define a map of polynomial endofunctors. We then define $\tau(\phi, \eta)$ to be the map $\tilde{\tau}([n], f) \rightarrow \tilde{\tau}([m], g)$ induced by the composite $\tau([n], f) \rightarrow \tilde{\tau}([m], g) \hookrightarrow \bar{\tau}([m], g)$.

LEMMA 4.2. τ is a functor $\Delta_{\mathbb{F}}^1 \rightarrow \Omega$.

Proof. Since τ clearly preserves identities, it remains to check that it respects composition, that is, that for

$$([n], f) \xrightarrow{(\phi, \eta)} ([m], g) \xrightarrow{(\psi, \lambda)} ([k], h)$$

in $\Delta_{\mathbb{F}}^1$ the maps $\tau((\psi, \lambda) \circ (\phi, \eta))$ and $\tau(\psi, \lambda) \circ \tau(\phi, \eta)$ agree. But by Lemma 3.11 it suffices to show that they are given by the same map on the set of edges. By definition, for $\tau(\phi, \eta)$ this is the map $\coprod_{i=0}^n f(i) \rightarrow \coprod_{j=0}^m g(j)$ given on $f(i)$ by $\eta_i: f(i) \rightarrow g(\phi(i))$, so it is evident that the two maps agree on the edge sets. \square

The definition of τ immediately implies the following lemma.

LEMMA 4.3. *The functor τ preserves the surjective–injective and active–inert factorization systems.*

LEMMA 4.4. *The functor τ restricts to an equivalence $\Delta_{\mathbb{F}}^{\text{el}} \rightarrow \Omega^{\text{el}}$. Moreover, for any $X \in \Delta_{\mathbb{F}}^1$, it induces an equivalence of categories $\Delta_{\mathbb{F}/X}^{\text{el}} \rightarrow \Omega_{/\tau(X)}^{\text{el}}$.*

Proof. The first claim follows immediately from the definition of τ . Since every object $E \rightarrow \tau(X)$ in $\Omega_{\text{el}/\tau(X)}$ lies in the image of τ , the induced functor $\Delta_{\mathbb{F}, \text{el}/X} \rightarrow \Omega_{\text{el}/\tau(X)}$ is essentially surjective. We now observe that a morphism in $\Omega_{\text{el}/\tau(X)}$ is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad} & E' \\ & \searrow & \swarrow \\ & \tau(X), & \end{array}$$

where the horizontal map is either the identity map or the image of a unique map of the form $([0], \mathbf{1}) \rightarrow ([1], f)$ under τ . This shows that the functor $\Delta_{\mathbb{F}, \text{el}/X} \rightarrow \Omega_{\text{el}/\tau(X)}$ is also fully faithful. \square

- LEMMA 4.5. (i) *The functor $\tau_1: \mathcal{P}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}(\Omega)$ preserves Segal equivalences.*
 (ii) *Composition with τ restricts to a functor $\tau^*: \mathcal{P}_{\text{Seg}}(\Omega) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$.*
 (iii) *This functor has a left adjoint $L_{\Omega} \circ \tau_1: \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega)$.*

Proof. To prove (i) it suffices to show that the images under τ_1 of the generating Segal equivalences $X_{\text{Seg}} \rightarrow X$ in $\mathcal{P}(\Delta_{\mathbb{F}}^1)$ are Segal equivalences in $\mathcal{P}(\Omega)$. But since τ_1 preserves colimits, Lemma 4.4 implies that $\tau_1(X_{\text{Seg}}) \simeq (\tau X)_{\text{Seg}}$, and so these maps are among the generating Segal equivalences for $\mathcal{P}(\Omega)$. Then the claims (ii) and (iii) are immediate consequences of (i). \square

LEMMA 4.6. *For $E \in \Delta_{\mathbb{F}}^{\text{el}}$, the map $E \rightarrow \tau^*(\tau E)$ in $\mathcal{P}(\Delta_{\mathbb{F}}^1)$ is an equivalence.*

Proof. First consider the case where $E = ([0], \mathbf{1})$, so that $\tau E = \eta$. For any map of trees $\varphi: T \rightarrow \eta$, the tree T must be linear, that is, have only unary vertices. But then $T = \tau([n], \mathbf{1} = \cdots = \mathbf{1})$ for some n and $\varphi = \tau(\psi)$ for the unique map $\psi: ([n], \mathbf{1} = \cdots = \mathbf{1}) \rightarrow ([0], \mathbf{1})$ in $\Delta_{\mathbb{F}}^1$. It follows that $\tau^*\eta = ([0], \mathbf{1})$. The argument for $E = ([1], \mathbf{k} \rightarrow \mathbf{1})$ is similar. Note that τE is the corolla C_k . Consider an $X \in \Delta_{\mathbb{F}}^1$ with a map $\tau X \rightarrow \tau E$. If it is not surjective, then it factors as $\tau X \rightarrow \eta \rightarrow \tau E$, where $\eta \rightarrow \tau E$ is the inclusion of some edge of C_k , and one reduces to the previous case to see that $\tau X \rightarrow \tau E$ is the image of the unique map $X \rightarrow E$ in $\Delta_{\mathbb{F}}^1$. If

$\tau X \rightarrow \tau E$ is surjective, then clearly X must be of the form $([n], \mathbf{k} \simeq \cdots \simeq \mathbf{k} \rightarrow \mathbf{1} = \cdots = \mathbf{1})$ for some $n \geq 1$. Again one observes that there is a unique map $X \rightarrow E$ whose image is $\tau X \rightarrow \tau E$, which implies the lemma. \square

5. Proof of the comparison result

Our goal in this section is to prove that the ∞ -categories $\mathcal{P}_{\text{Seg}}(\Omega)$ and $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ are equivalent. More precisely, we saw in the previous section that the map $\tau: \Delta_{\mathbb{F}}^1 \rightarrow \Omega$ induces a functor between the ∞ -categories of Segal presheaves, and we will show that this gives the desired equivalence:

THEOREM 5.1. *The functor $\tau^*: \mathcal{P}_{\text{Seg}}(\Omega) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ is an equivalence of ∞ -categories.*

Since τ preserves inert-active factorizations, it restricts to a functor $\tau_{\text{int}}: \Delta_{\mathbb{F}, \text{int}}^1 \rightarrow \Omega_{\text{int}}$, and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{\text{Seg}}(\Omega) & \xrightarrow{\tau^*} & \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1) \\ j_{\Omega}^* \downarrow & & \downarrow j_{\Delta_{\mathbb{F}}^1}^* \\ \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}}) & \xrightarrow{\tau_{\text{int}}^*} & \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1), \end{array}$$

where j_{Ω} and $j_{\Delta_{\mathbb{F}}^1}$ denote the inclusions $\Omega_{\text{int}} \rightarrow \Omega$ and $\Delta_{\mathbb{F}, \text{int}}^1 \rightarrow \Delta_{\mathbb{F}}^1$, and $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1)$ and $\mathcal{P}_{\text{Seg}}(\Omega_{\text{int}})$ denote the full subcategories of $\mathcal{P}(\Delta_{\mathbb{F}, \text{int}}^1)$ and $\mathcal{P}(\Omega_{\text{int}})$ spanned by the presheaves that are right Kan extensions of their restrictions to $\Delta_{\mathbb{F}}^{\text{el}}$ and Ω^{el} , respectively.

LEMMA 5.2. *The functor $\tau_{\text{int}}^*: \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}}) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1)$ is an equivalence.*

Proof. Consider the commutative square

$$\begin{array}{ccc} \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}}) & \xrightarrow{\tau_{\text{int}}^*} & \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1) \\ \downarrow & & \downarrow \\ \mathcal{P}(\Omega^{\text{el}}) & \xrightarrow{\tau_{\Delta_{\mathbb{F}}^{\text{el}}}^*} & \mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}}). \end{array}$$

The map τ restricts to an equivalence $\Delta_{\mathbb{F}}^{\text{el}} \rightarrow \Omega^{\text{el}}$ by Lemma 4.4, so the bottom horizontal map here is an equivalence. Moreover, the vertical maps are equivalences by [8, Proposition 4.3.2.15], since $\mathcal{P}_{\text{Seg}}(\Omega_{\text{int}})$ and $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1)$ are by definition the ∞ -categories of presheaves that are right Kan extensions of presheaves on $\Omega^{\text{el}} \simeq \Delta_{\mathbb{F}}^{\text{el}}$. By the 2-out-of-3 property, it follows that the top horizontal map τ_{int}^* is also an equivalence. \square

LEMMA 5.3. (i) *The functor $j_{\Omega}^*: \mathcal{P}_{\text{Seg}}(\Omega) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}})$ has a left adjoint $F_{\Omega} := L_{\Omega} j_{\Omega, !}$, and the adjunction $L_{\Omega} j_{\Omega, !} \dashv j_{\Omega}^*$ is monadic.*

(ii) *The functor $j_{\Delta_{\mathbb{F}}^1}^*: \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1)$ has a left adjoint $F_{\Delta_{\mathbb{F}}^1} := L_{\Delta_{\mathbb{F}}^1} j_{\Delta_{\mathbb{F}}^1, !}$, and the adjunction $L_{\Delta_{\mathbb{F}}^1} j_{\Delta_{\mathbb{F}}^1, !} \dashv j_{\Delta_{\mathbb{F}}^1}^*$ is monadic.*

Proof. We will prove (i); the proof of (ii) is the same. The existence of the left adjoint $L_{\Omega}j_{\Omega,!}$ is obvious, so by [9, Theorem 4.7.4.5] it remains to show that j_{Ω}^* detects equivalences and that j_{Ω}^* -split simplicial objects in $\mathcal{P}_{\text{Seg}}(\Omega)$ have colimits and these are preserved by j_{Ω}^* . Since Ω_{int} contains all the objects of Ω it is clear that j_{Ω}^* detects equivalences, and we also know that $\mathcal{P}_{\text{Seg}}(\Omega)$ has small colimits. Suppose then that we have a j_{Ω}^* -split simplicial object X_{\bullet} in $\mathcal{P}_{\text{Seg}}(\Omega)$, that is, $j_{\Omega}^*X_{\bullet}$ extends to a split simplicial object $X'_{\bullet}: \Delta_{-\infty}^{\text{op}} \rightarrow \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}})$. If we consider X_{\bullet} as a diagram in $\mathcal{P}(\Omega)$ with colimit X , then the colimit of X_{\bullet} in $\mathcal{P}_{\text{Seg}}(\Omega)$ is $L_{\Omega}X$. On the other hand, the colimit X is preserved by $j_{\Omega}^*: \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega_{\text{int}})$ (since this functor is a left adjoint). But by [9, Remark 4.7.3.3], the diagram X'_{\bullet} is a colimit diagram also when viewed as a diagram in $\mathcal{P}(\Omega_{\text{int}})$, so $j_{\Omega_{\text{int}}}^*X \simeq X'_{-\infty}$. This means that the presheaf X satisfies the Segal condition, and so $X \simeq L_{\Omega}X$, that is, X is also the colimit of X_{\bullet} in $\mathcal{P}_{\text{Seg}}(\Omega)$. Since its image in $\mathcal{P}_{\text{Seg}}(\Omega_{\text{int}})$ is $X'_{-\infty}$, this colimit is indeed preserved. \square

The two preceding lemmas imply that $\mathcal{P}_{\text{Seg}}(\Omega)$ and $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ are both the ∞ -categories of algebras for monads on $\mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}}) \simeq \mathcal{P}(\Omega^{\text{el}})$. To show that these ∞ -categories are the same, it will therefore be sufficient to prove that these two monads are equivalent. Our proof of this makes use of the existence of a *right* adjoint to τ^* :

PROPOSITION 5.4. *The functor τ_* given by right Kan extension along τ restricts to a functor*

$$\tau_*: \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1) \rightarrow \mathcal{P}_{\text{Seg}}(\Omega),$$

right adjoint to τ^ .*

Let us show how to deduce Theorem 5.1 from this; the remainder of this section is then devoted to proving Proposition 5.4.

LEMMA 5.5. *The canonical map $\tau_{\text{int}}^*j_{\Omega}^*\tau_* \simeq j_{\Delta_{\mathbb{F}}^1}^*\tau^*\tau_* \rightarrow j_{\Delta_{\mathbb{F}}^1}^*$ is a natural equivalence.*

Proof. Recall the commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{\text{Seg}}(\Omega) & \xrightarrow{\tau^*} & \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1) \\ \downarrow j_{\Omega}^* & & \downarrow j_{\Delta_{\mathbb{F}}^1}^* \\ \mathcal{P}_{\text{Seg}}(\Omega_{\text{int}}) & \xrightarrow{\tau_{\text{int}}^*} & \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}, \text{int}}^1) \\ \downarrow & & \downarrow \\ \mathcal{P}(\Omega^{\text{el}}) & \longrightarrow & \mathcal{P}(\Delta_{\mathbb{F}}^{\text{el}}). \end{array}$$

We saw in the proof of Lemma 5.2 that the lower two vertical arrows are equivalences. Therefore it suffices to check that for $\mathcal{F} \in \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$ and $E \in \Delta_{\mathbb{F}}^{\text{el}}$ the natural map $\tau^*\tau_*\mathcal{F}(E) \rightarrow \mathcal{F}(E)$ is an equivalence. We may identify the domain of this map as

$$(\tau_*\mathcal{F})(\tau E) \simeq \lim_{X \in (\Delta_{\mathbb{F}}^{1, \text{op}})_{\tau E/}} \mathcal{F}(X),$$

where $(\Delta_{\mathbb{F}}^{1, \text{op}})_{\tau E/} \simeq ((\Delta_{\mathbb{F}}^1)_{/\tau E})^{\text{op}}$ and $(\Delta_{\mathbb{F}}^1)_{/\tau E} := \Delta_{\mathbb{F}}^1 \times_{\Omega} \Omega_{/\tau E}$. But the unit morphism $E \rightarrow \tau^*\tau_1 E \simeq \tau^*(\tau E)$ is an equivalence for $E \in \Delta_{\mathbb{F}}^{\text{el}}$ by Lemma 4.6, hence $(E, \tau E = \tau E)$ is a terminal object in $(\Delta_{\mathbb{F}}^1)_{/\tau E}$. Therefore it is initial in $(\Delta_{\mathbb{F}}^{1, \text{op}})_{\tau E/}$ and this implies the map is an equivalence. \square

Proof of Theorem 5.1. By [9, Corollary 4.7.4.16] it suffices to show that the canonical natural transformation $F_{\Delta_{\mathbb{F}}^1} \circ \tau_{\text{int}}^* \rightarrow \tau^* F_{\Omega}$ is an equivalence. But by Proposition 5.6 these functors are both left adjoints, and so we have an equivalence of left adjoints if and only if the corresponding transformation of right adjoints $j_{\Omega}^* \tau_* \rightarrow (\tau_{\text{int}}^*)^{-1} j_{\Delta_{\mathbb{F}}^1}^*$ is an equivalence. This now follows from Lemma 5.5. \square

Proposition 5.4 is an immediate consequence of the following result, to which we now turn:

PROPOSITION 5.6. *The functor $\tau^* : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Delta_{\mathbb{F}}^1)$ preserves Segal equivalences.*

Our proof of Proposition 5.6 is based on the proof of [5, Proposition 5.5.9]. Before we give it, we must introduce some notation and prove two technical lemmas:

DEFINITION 5.7. For $T \in \Omega$ a tree with at least two vertices, let $\partial^{\text{ext}}T$ denote the *external boundary* of T , namely the presheaf on Ω constructed as the union of all the *external faces* of T . To be precise, let $\text{sub}(T)$ be the full subcategory of $(\Omega_{\text{int}})_{/T}$ on the proper subtrees of T and define $\partial^{\text{ext}}T$ to be the colimit of the composition $\text{sub}(T) \rightarrow \Omega \rightarrow \mathcal{P}(\Omega)$.

LEMMA 5.8. *For T in Ω with at least two vertices, let $(\partial^{\text{ext}}T)_{\text{Seg}}$ denote the colimit of the functor*

$$\text{sub}(T) \rightarrow \mathcal{P}(\Omega), \quad S \mapsto S_{\text{Seg}}$$

(this is well-defined since the maps in Ω involved are all inert). Then the natural map $(\partial^{\text{ext}}T)_{\text{Seg}} \rightarrow T_{\text{Seg}}$ is an equivalence.

Proof. Let $\mathcal{I} \rightarrow \text{sub}(T)$ denote the Grothendieck opfibration associated to the functor sending S to $\Omega_{/S}^{\text{el}}$. By [4, Corollary 5.7] we can regard $(\partial^{\text{ext}}T)_{\text{Seg}}$ as the colimit of the functor $\mathcal{I} \rightarrow \mathcal{P}(\Omega)$ sending $(S, (E \rightarrow S) \in \Omega_{/S}^{\text{el}})$ to E , and the map $(\partial^{\text{ext}}T)_{\text{Seg}} \rightarrow T_{\text{Seg}}$ is the map on colimits induced by the functor $\Phi : \mathcal{I} \rightarrow \Omega_{/T}^{\text{el}}$ that takes $(S, E \rightarrow S)$ to $E \rightarrow S \rightarrow T$. It therefore suffices to prove that Φ is cofinal. But Φ admits a left adjoint, given by the functor sending $E \rightarrow T$ to $(E, E = E)$. \square

For the following definition and lemma it will be clearer to work with (Segal presheaves on) $\Delta_{\mathbb{F}}$ rather than $\Delta_{\mathbb{F}}^1$; this makes no difference due to Lemma 2.11.

DEFINITION 5.9. Let $\text{sd}(\Delta^n)$ denote the partially ordered set of faces of Δ^n (meaning injective maps $[m] \hookrightarrow [n]$ in Δ) or equivalently the partially ordered set of non-empty subsets of $\{0, \dots, n\}$; in other words, $\text{sd}(\Delta^n)$ is the *barycentric subdivision* of Δ^n . We will denote the subset $\{i_1, \dots, i_k\}$ where $i_1 < i_2 < \dots < i_k$ by (i_1, \dots, i_k) . Given a full subcategory (that is, partially ordered subset) $\mathcal{G} \subseteq \text{sd}(\Delta^n)$ and $X \in \Delta_{\mathbb{F}}$ of length n , let $X(\mathcal{G})$ denote the colimit in $\mathcal{P}(\Delta_{\mathbb{F}})$ over $\varphi \in \mathcal{G}$ of $\varphi^* X$. For $\text{sd}(\Lambda_i^n)$ the subcategory containing all objects except $(0, \dots, n)$ and $(0, \dots, i - 1, i + 1, \dots, n)$, we write $\Lambda_i^n X$ for $X(\text{sd}(\Lambda_i^n))$.

LEMMA 5.10. *For any $X \in \Delta_{\mathbb{F}}$ of length n , the map $\Lambda_{n-1}^n X \rightarrow X$ is a Segal equivalence.*

Proof. By the 2-out-of-3 property, it suffices to show that the map $X_{\text{Seg}} \rightarrow \Lambda_{n-1}^n X$ is a Segal equivalence. To prove this, we consider the following filtration of $\text{sd}(\Lambda_{n-1}^n)$: we let $\mathcal{G}_d \subseteq \text{sd}(\Lambda_{n-1}^n)$ contain all subsets of length at most d together with those of length $d + 1$ that are of the form $(i_0, \dots, i_{d-1}, i_d - 1, i_d)$. Then $\mathcal{G}_{n-2} = \text{sd}(\Lambda_{n-1}^n)$ and we have a filtration

$$X_{\text{Seg}} \rightarrow X(\mathcal{G}_0) \rightarrow X(\mathcal{G}_1) \rightarrow \dots \rightarrow X(\mathcal{G}_{n-2}) \simeq \Lambda_{n-1}^n X.$$

It thus suffices to show that the maps $X_{\text{Seg}} \rightarrow X(\mathcal{G}_0)$ and $X(\mathcal{G}_{d-1}) \rightarrow X(\mathcal{G}_d)$ ($d = 1, \dots, n - 2$) are Segal equivalences.

Note that the map $X_{\text{Seg}} \rightarrow X(\mathcal{G}_0)$ is an equivalence by construction; indeed, \mathcal{G}_0 consists of the subsets with one element, together with the subsets of the form $(i, i + 1)$. To see that $X(\mathcal{G}_{d-1}) \rightarrow X(\mathcal{G}_d)$ is a Segal equivalence, we consider a filtration

$$\mathcal{G}_{d-1} = \mathcal{H}_d^d \subseteq \mathcal{H}_d^{d+1} \subseteq \dots \subseteq \mathcal{H}_d^n = \mathcal{G}_d,$$

where \mathcal{H}_d^j is the full subcategory containing \mathcal{G}_{d-1} together with those objects (i_0, \dots, i_k) of \mathcal{G}_d with $i_k \leq j$. Let T_d^j denote the objects of length $d + 1$ in \mathcal{H}_d^j that do not lie in \mathcal{H}_d^{j-1} . Note that for every $\sigma \in T_d^j$ the d -dimensional face $d_i\sigma$ lies in \mathcal{H}_d^{j-1} for $i \neq d$, while $d_d\sigma$ does not lie in \mathcal{H}_d^{j-1} . Using [8, Corollary 4.2.3.10] we therefore have pushout squares

$$\begin{array}{ccc} \coprod_{\sigma \in T_d^j} \Lambda_d^{d+1} \sigma^* X & \longrightarrow & \coprod_{\sigma \in T_d^j} \sigma^* X \\ \downarrow & & \downarrow \\ X(\mathcal{H}_d^{j-1}) & \longrightarrow & X(\mathcal{H}_d^j). \end{array}$$

Since pushouts of Segal equivalences are again Segal equivalences, by inducting on n this completes the proof. \square

Proof of Proposition 5.6. It suffices to show that the images under τ^* of the generating Segal equivalences $T_{\text{Seg}} \rightarrow T$ for $T \in \Omega$ are Segal equivalences in $\mathcal{P}(\Delta_{\mathbb{F}}^1)$. We will prove this by induction on the number of vertices in T , noting that if T is η or T has one vertex, that is, $T \in \Omega^{\text{el}}$, then the statement is vacuous. Given $T \in \Omega$ with two or more vertices, we have a commutative square

$$\begin{array}{ccc} (\partial^{\text{ext}} T)_{\text{Seg}} & \longrightarrow & \partial^{\text{ext}} T \\ \downarrow & & \downarrow \\ T_{\text{Seg}} & \longrightarrow & T. \end{array}$$

Here the left vertical map is an equivalence by Lemma 5.8, and the top horizontal map is the colimit over $S \in \text{sub}(T)$ of the maps $S_{\text{Seg}} \rightarrow S$. Since τ^* preserves colimits and S has fewer vertices than T for all $S \in \text{sub}(T)$, we know by the inductive hypothesis that τ^* of this map is a Segal equivalence. By the 2-out-of-3 property, to show that $\tau^* T_{\text{Seg}} \rightarrow \tau^* T$ is a Segal equivalence it therefore suffices to show that $\tau^*(\partial^{\text{ext}} T) \rightarrow \tau^* T$ is a Segal equivalence.

To prove this, we will consider a filtration on $\tau^* T$. In order to define this we must first introduce some terminology; let us say that a map $\varphi: X \rightarrow \tau^* T$ is *non-degenerate* if it does not factor through any non-trivial surjections in $\Delta_{\mathbb{F}}^1$ — more precisely, we require that for every factorization $X \xrightarrow{\psi} Y \rightarrow \tau^* T$ with ψ a surjective map in $\Delta_{\mathbb{F}}^1$, the map ψ must be an isomorphism. We then say that φ is *admissible* if it is non-degenerate and preserves the root vertex — more precisely, recall that if $X = ([n], f)$, then the adjunct map $\tau(X) \rightarrow T$ is a map of polynomial endofunctors

$$\begin{array}{ccccccc} \coprod_{i=0}^n f(i) & \longleftarrow & \coprod_{i=0}^{n-1} f(i) & \longrightarrow & \coprod_{i=1}^n f(i) & \longrightarrow & \coprod_{i=0}^n f(i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_0 & \longleftarrow & \text{sub}'(T) & \longrightarrow & \text{sub}(T) & \longrightarrow & T_0; \end{array}$$

we say that φ is admissible if it is non-degenerate and the map $\coprod_{i=1}^n f(i) \rightarrow \text{sub}(T)$ takes the root vertex of X , that is, $f(n) = \mathbf{1}$, to the root corolla of T viewed as a subtree of T .

We now define a subpresheaf F_n of the (discrete) presheaf τ^*T as follows: $F_n(X)$ is the union of the image of $\tau_*(\partial^{\text{ext}}T)(X)$ with the maps $X \rightarrow \tau^*T$ that factor through an admissible morphism $Y \rightarrow \tau^*T$ with Y of length $\leq n$.

Every map $\tau(Y) \rightarrow T$ with $Y \in \Delta_{\mathbb{F}}^1$ factors through an admissible map, so $\tau^*T \simeq \text{colim}_{n \rightarrow \infty} F_n$; it hence suffices to show that the inclusions $F_{n-1} \hookrightarrow F_n$ are all Segal equivalences. Let S_n denote the set of isomorphism classes of admissible maps $\varphi: \tau(X) \rightarrow T$, where $X \in \Delta_{\mathbb{F}}^1$ is of length n . For such a φ , we have that

- by the assumption that φ is non-degenerate, the faces $d_i^*X \rightarrow X \rightarrow \tau^*T$ with $i = 0, n$ factor through $\tau_*(\partial^{\text{ext}}T)$, and so in particular through F_{n-1} ;
- the faces $d_i^*X \rightarrow X \rightarrow \tau^*T$ with $0 < i < n - 1$ are admissible of length $n - 1$ and so factor through F_{n-1} ;
- the face $d_{n-1}^*X \rightarrow X \rightarrow \tau^*T$ is *not* admissible — if it were, then it is straightforward to see that φ must be degenerate, which is not the case by assumption.

Note also that if $\varphi: X \rightarrow \tau^*T$ is non-degenerate and does not factor through $\partial^{\text{ext}}T$, but is not admissible, with X of length $n - 1$, then there exists (up to isomorphism) a *unique* admissible map $\varphi': X' \rightarrow \tau^*T$ with X' of length n such that $d_{n-1}^*X' \rightarrow X' \rightarrow \tau^*T$ equals φ . Choosing representatives for the elements of S_n therefore gives a pushout diagram

$$\begin{array}{ccc} \coprod_{S_n} \Lambda_{n-1}^n X & \longrightarrow & F_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{S_n} X & \longrightarrow & F_n. \end{array}$$

Here the left vertical morphism is a Segal equivalence by Lemma 5.10, hence so is the right vertical morphism. □

6. Completion

DEFINITION 6.1. Let $u: \Delta \hookrightarrow \Delta_{\mathbb{F}}^1$ denote the fully faithful inclusion given by sending $[n]$ to $([n], \mathbf{1} = \mathbf{1} = \dots = \mathbf{1})$. If $\mathcal{F}: \Delta_{\mathbb{F}}^{1, \text{op}} \rightarrow \mathcal{S}$ is a Segal presheaf, then $u^*\mathcal{F}$ is a Segal space in the sense of Rezk [12]. We say that \mathcal{F} is *complete* if the Segal space $u^*\mathcal{F}$ is complete. Similarly, we say a Segal presheaf $\mathcal{F}: \Delta_{\mathbb{F}}^{\text{op}} \rightarrow \mathcal{S}$ is *complete* if $u^*i^*\mathcal{F}$ is a complete Segal space, and that a Segal presheaf $\mathcal{F}: \Omega^{\text{op}} \rightarrow \mathcal{S}$ is complete if and only if $u^*\tau^*\mathcal{F}$ is a complete Segal space. We write $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}^1)$, $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}})$ and $\mathcal{P}_{\text{CS}}(\Omega)$ for the full subcategories of $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$, $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ and $\mathcal{P}_{\text{Seg}}(\Omega)$, respectively, spanned by the complete Segal presheaves.

REMARK 6.2. The ∞ -categories $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}^1)$, $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}})$ and $\mathcal{P}_{\text{CS}}(\Omega)$ are accessible localizations of $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$, $\mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ and $\mathcal{P}_{\text{Seg}}(\Omega)$, respectively. In particular, the inclusions $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}^1) \hookrightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}}^1)$, $\mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}) \hookrightarrow \mathcal{P}_{\text{Seg}}(\Delta_{\mathbb{F}})$ and $\mathcal{P}_{\text{CS}}(\Omega) \hookrightarrow \mathcal{P}_{\text{Seg}}(\Omega)$ all have left adjoints.

Putting together our results from the previous sections, we get

COROLLARY 6.3. *Composition with the functors i and τ give equivalences of ∞ -categories*

$$\mathcal{P}_{\text{CS}}(\Omega) \xrightarrow{\sim} \mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}^1) \xleftarrow{\sim} \mathcal{P}_{\text{CS}}(\Delta_{\mathbb{F}}).$$

Proof. Immediate from Theorem 5.1, Lemma 2.11 and the definition of complete Segal presheaves. \square

Using results of Cisinski and Moerdijk in the context of dendroidal Segal spaces, this allows us to characterize the morphisms that are local equivalences with respect to the complete objects as the fully faithful and essentially surjective morphisms, in the following sense:

DEFINITION 6.4. A morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ of Segal presheaves on Ω is *fully faithful* if for every n the commutative square

$$\begin{array}{ccc} \mathcal{F}(C_n) & \longrightarrow & \mathcal{G}(C_n) \\ \downarrow & & \downarrow \\ \mathcal{F}(\eta)^{\times(n+1)} & \longrightarrow & \mathcal{G}(\eta)^{\times(n+1)} \end{array}$$

is a pullback square. We say φ is *essentially surjective* if the morphism $u^* \tau^* \mathcal{F} \rightarrow u^* \tau^* \mathcal{G}$ of Segal spaces is essentially surjective. Obvious variants of this definition also give notions of fully faithful and essentially surjective morphisms between Segal presheaves on $\Delta_{\mathbb{F}}$ and $\Delta_{\mathbb{F}}^1$.

COROLLARY 6.5. A morphism of Segal presheaves (on $\Delta_{\mathbb{F}}$, $\Delta_{\mathbb{F}}^1$ or Ω) maps to an equivalence of complete Segal presheaves if and only if it is fully faithful and essentially surjective. In other words, the localization functors from Segal presheaves to complete Segal presheaves exhibit the latter as the localization of the Segal presheaves at the fully faithful and essentially surjective functors.

Proof. For Ω , this holds by [2, Theorem 8.11]. The other two cases then follow from Theorem 5.1, Lemma 2.11 and the definitions of complete objects and fully faithful and essentially surjective morphisms. \square

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