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Zero-dimensional Donaldson–Thomas invariants of Calabi–Yau 4-folds

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A B S T R A C T

We study Hilbert schemes of points on a smooth projective Calabi–Yau 4-fold X. We define DT_4 invariants by integrating the Euler class of a tautological vector bundle $L^{[n]}$ against the virtual class. We conjecture a formula for their generating series, which we prove in certain cases when L corresponds to a smooth divisor on X. A parallel equivariant conjecture for toric Calabi–Yau 4-folds is proposed. This conjecture is proved for smooth toric divisors and verified for more general toric divisors in many examples.

Combining the equivariant conjecture with a vertex calculation, we find explicit positive rational weights, which can be assigned to solid partitions. The weighted generating function of solid partitions is given by $\exp(M(q)-1)$, where $M(q)$ denotes the MacMahon function.

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1. Introduction

1.1. Background

Hilbert schemes on a smooth projective variety X are moduli schemes which parametrise subschemes of X with given Hilbert polynomial. From the point of view of coherent sheaves, they can be regarded as moduli schemes of ideal sheaves of subschemes with fixed Chern character. The simplest example is the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points on X, whose ideal sheaves have Chern character $(1, 0, \cdots, 0, -n)$. There are lots of interesting studies on their geometry, topology and representation theory, most of which are concentrated on the cases $\dim_{\mathbb{C}} X \leqslant 2$. The difficulty in extending these studies to higher dimensions comes from the fact that the Hilbert schemes are in general no longer smooth.

One surprising feature about $\dim_{\mathbb{C}} X = 3$ is that, although $\mathrm{Hilb}^n(X)$ can be very singular with different irreducible components of various dimensions, it still carries a degree zero virtual class $[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}$ [17]. The degree of this class is called a degree zero Donaldson–Thomas invariant of X [22]. An expression for the generating series of these invariants was conjectured and verified for local toric surfaces by Maulik–Nekrasov–Okounkov–Pandharipande [17] and confirmed in full generality by Levine–Pandharipande [15] and Li [16]. See also [2] for another proof in the Calabi–Yau case.

Our aim is to go one dimensional higher and restrict to the case of Calabi–Yau manifolds [23]. By the work of Borisov–Joyce [3] and Cao–Leung [7], we have a virtual class construction for Gieseker moduli spaces of stable sheaves on smooth projective Calabi–Yau 4-folds, which is in particular applicable to $\mathrm{Hilb}^n(X)$. A difference from the case of 3-folds is that the virtual class is no longer of degree zero, so we need natural insertions to define invariants.

1.2. The compact case

Let X be a smooth projective Calabi–Yau 4-fold and let $\mathrm{Hilb}^n(X)$ denote the Hilbert scheme of n points on X. Assume the existence of an orientation $o(\mathcal{L})$ on the determinant line bundle $\mathcal{L}$ over $\mathrm{Hilb}^n(X)$. Then the results of [3,7] provide a DT_4 virtual class

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})} \in H_{2n}(\mathrm{Hilb}^n(X), \mathbb{Z}). \tag{1.1}$$

The virtual class (1.1) depends on the choice of orientation $o(\mathcal{L})$. On each connected component of $\mathrm{Hilb}^n(X)$, there are two choices of orientations, which affects the corresponding contribution to the class (1.1) by a sign. We review facts about the DT_4 virtual class in Section 2.1.

In order to define the invariants, we require insertions. Let L be a line bundle on X and denote by $L^{[n]}$ the tautological (rank n) vector bundle over $\mathrm{Hilb}^n(X)$ with fibre $H^0(L|_Z)$ over $Z \in \mathrm{Hilb}^n(X)$. Then it makes sense to define the following:

Definition 1.1. Let X be a smooth projective Calabi–Yau 4-fold and let L be a line bundle on X. Let $\mathcal{L}$ be the determinant line bundle of $\mathrm{Hilb}^n(X)$ with quadratic form Q induced from Serre duality. Suppose $\mathcal{L}$ is given an orientation $o(\mathcal{L})$. We define

$$\mathrm{DT}_4(X, L, n\,; o(\mathcal{L})) := \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})}} e(L^{[n]}) \in \mathbb{Z}, \quad \text{if } n \geqslant 1,$$

where $e(-)$ denotes the Euler class. We also set $\mathrm{DT}_4(X, L, 0\,; o(\mathcal{L})) := 1$.

We make the following conjecture for the generating series of these invariants:

Conjecture 1.2 *(Conjecture 2.2). Let X be a smooth projective Calabi–Yau 4-fold and L be a line bundle on X. There exist choices of orientation such that*

$$\sum_{n=0}^{\infty} \mathrm{DT}_4(X, L, n\,; o(\mathcal{L}))\, q^n = M(-q)^{\int_X c_1(L)\cdot c_3(X)},$$

where

$$M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

denotes the MacMahon function.

We verify Conjecture 1.2 in some good cases based on the following geometric setting, where the line bundle $L = \mathcal{O}_X(D)$ is associated to an effective divisor $D \subseteq X$.

Proposition 1.3 *(Proposition 2.4). Let X be a smooth quasi-projective variety, $D \subseteq X$ any effective divisor, and $L = \mathcal{O}_X(D)$. There exists a tautological section σ and an isomorphism of schemes*

$$\begin{array}{ccc} & & L^{[n]} \\ & & \pi \big\downarrow \big\uparrow \sigma \\ \sigma^{-1}(0) \cong \mathrm{Hilb}^n(D) & \xrightarrow{\iota} & \mathrm{Hilb}^n(X). \end{array}$$

For $n \leqslant 3$ and D, X both smooth, the Hilbert schemes are smooth and we can explicitly compare deformation–obstruction theories on X and D (Proposition 2.8). The latter gives rise to zero-dimensional DT_3 invariants on D, which are known by the work of [15,16].

Theorem 1.4 *(Theorem 2.10). Let X be a smooth projective Calabi–Yau 4-fold, $D \subseteq X$ a smooth divisor, and $L = \mathcal{O}_X(D)$. For each $n \leqslant 3$, there exists a choice of orientation $o(\mathcal{L})$ such that*

$$\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})}} e(L^{[n]}) = \int_{[\mathrm{Hilb}^n(D)]^{\mathrm{vir}}} 1.$$

In particular, Conjecture 1.2 is true in this setting.

The proof for general n will rely on Joyce's theory of D-manifolds or Kuranishi atlases. We hope to return to it in a future paper.

1.3. The toric case

When X is a smooth quasi-projective toric Calabi–Yau 4-fold with action of $(\mathbb{C}^*)^4$, we can study an equivariant version of Conjecture 1.2. Despite the non-compactness of X and $\mathrm{Hilb}^n(X)$, we can still define an equivariant version of the DT_4 virtual class on the torus fixed locus, which consists of a finite number of reduced points.

The definition involves the subtorus $T \subseteq (\mathbb{C}^*)^4$ preserving the Calabi–Yau volume form and hence Serre duality pairing. We note the following equality of fixed loci (Lemma 3.1, 3.6)

$$\mathrm{Hilb}^n(X)^T = \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4}.$$

For any $Z \in \mathrm{Hilb}^n(X)^T$, we consider the equivariant Euler class

$$e_T(\mathrm{Ext}^1(I_Z, I_Z)) \in H^*(BT),$$

and also the half Euler class

$$e_T(\mathrm{Ext}^2(I_Z, I_Z), Q) \in H^*(BT),$$

where Q is the quadratic form induced from the Serre duality pairing on $\mathrm{Ext}^2(I_Z, I_Z)$. We then have

$$e_T(\mathrm{Ext}^2(I_Z, I_Z), Q) = \pm\sqrt{(-1)^{\frac{ext^2(I_Z,I_Z)}{2}}\, e_T\big(\mathrm{Ext}^2(I_Z, I_Z)\big)}, \tag{1.2}$$

where the class $(-)$ in $\sqrt{(-)}$ is a square and the sign depends on the choice of orientation.

Definition 1.5. (Definition 3.8) The T-equivariant virtual class of $\mathrm{Hilb}^n(X)$ is

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{T,o(\mathcal{L})} := \sum_{Z \in \mathrm{Hilb}^n(X)^T} \frac{e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big)}{e_T\big(\mathrm{Ext}^1(I_Z, I_Z)\big)},$$

where $o(\mathcal{L})$ denotes a choice of sign in (1.2) for each $Z \in \mathrm{Hilb}^n(X)^T$.

By fixing a T-equivariant line bundle L on X, we can consider the equivariant Euler class of its tautological bundle $e_T(L^{[n]})$ and define

$$\mathrm{DT}_4(X,T,L,n\,;o(\mathcal{L})) := \sum_{Z\in \mathrm{Hilb}^n(X)^T} \frac{e_T\big(\mathrm{Ext}^2(I_Z,I_Z),Q\big)\cdot e_T(L^{[n]}|_Z)}{e_T\big(\mathrm{Ext}^1(I_Z,I_Z)\big)}.$$

An equivariant version of Conjecture 1.2 can then be posed as follows:

Conjecture 1.6 *(Conjecture 3.12). Let X be a smooth quasi-projective toric Calabi–Yau 4-fold and L be a T-equivariant line bundle on X. Then there exist choices of orientation $o(\mathcal{L})$ such that*

$$\sum_{n=0}^{\infty} \mathrm{DT}_4(X,T,L,n\,;o(\mathcal{L}))\,q^n = M(-q)^{\int_X c_1^T(L)\cdot c_3^T(X)},$$

where $\int_X$ denotes equivariant push-forward to a point.

When $L = \mathcal{O}_X(D)$ corresponds to a smooth toric divisor D, we can prove Conjecture 1.6.

Theorem 1.7 *(Theorem 3.13). Conjecture 1.6 is true for $L = \mathcal{O}_X(D)$, where $D \subseteq X$ is a smooth $(\mathbb{C}^*)^4$-invariant divisor.*

Any smooth quasi-projective toric Calabi–Yau 4-fold X can be covered by open $(\mathbb{C}^*)^4$-invariant subsets (equivariantly) isomorphic to $\mathbb{C}^4$. On each such subset, every $(\mathbb{C}^*)^4$-invariant zero-dimensional subscheme corresponds to a *solid partition* $\pi = \{\pi_{ijk}\}_{i,j,k\geqslant 1}$, i.e. a sequence of non-negative integers $\pi_{ijk} \in \mathbb{Z}_{\geqslant 0}$ satisfying

$$\pi_{ijk} \geqslant \pi_{i+1,j,k}, \quad \pi_{ijk} \geqslant \pi_{i,j+1,k}, \quad \pi_{ijk} \geqslant \pi_{i,j,k+1} \quad \forall\ i,j,k \geqslant 1,$$
$$|\pi| := \sum_{i,j,k\geqslant 1} \pi_{ijk} < \infty,$$

where $|\pi|$ is called the *size* of π.

Using a vertex formalism as in MNOP [17], we reduce Conjecture 1.6 to the case $X = \mathbb{C}^4$ (Proposition 3.20). This leads us to assigning expressions $L_\pi(d_1,d_2,d_3,d_4)$ (coming from $e_T(L^{[n]})$) and w_π (coming from $e_T\big(\mathrm{Ext}^2(I_Z,I_Z),Q\big)/e_T\big(\mathrm{Ext}^1(I_Z,I_Z)\big)$) to any solid partition π. See Definition 3.16. In fact, the equivariant weight w_π is only defined up to sign, reflecting the different signs in (1.2) for different choices of orientation. The case $X = \mathbb{C}^4$ then essentially corresponds to the following conjecture (which now includes a uniqueness assertion).

Conjecture 1.8 *(Conjectures 3.19 and 3.21). There exists a unique way of choosing the signs for the equivariant weights w_π such that*

$$\sum_{\pi} L_\pi(d_1,d_2,d_3,d_4)\,\mathsf{w}_\pi\, q^{|\pi|} = M(-q)^{\frac{(d_1\lambda_1+d_2\lambda_2+d_3\lambda_3+d_4\lambda_4)(-\lambda_1\lambda_2\lambda_3-\lambda_1\lambda_2\lambda_4-\lambda_1\lambda_3\lambda_4-\lambda_2\lambda_3\lambda_4)}{\lambda_1\lambda_2\lambda_3\lambda_4}}$$

holds in $\frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}(d_1,d_2,d_3,d_4)[\![q]\!]$*, where the sum is over all solid partitions and* $M(q)$ *denotes the MacMahon function.*

Besides Theorem 1.7, we verify Conjecture 1.8 in the following setting by using a Maple program, which calculates w_π for a given solid partition π.

Theorem 1.9 *(Theorem 3.22). Conjecture 1.8 is true modulo* q^7*.*

1.4. Application to counting solid partitions

By experimental study of many examples, we find that the specialisation

$$L_\pi(0,0,0,-d)\,\mathsf{w}_\pi\Big|_{\lambda_1+\lambda_2+\lambda_3=0} \tag{1.3}$$

is well-defined. We pose the following conjecture:

Conjecture 1.10 *(Conjecture 4.1). Let* π *be a solid partition and let* w_π *be defined using the unique sign in Conjecture 1.8. Then the following properties hold:*

(a) $L_\pi(0,0,0,-d)\,\mathsf{w}_\pi \in \frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,d)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$ *has no pole at* $\lambda_4 = -(\lambda_1+\lambda_2+\lambda_3)$*.*

(b) *The specialisation* $L_\pi(0,0,0,-d)\,\mathsf{w}_\pi\Big|_{\lambda_1+\lambda_2+\lambda_3=0}$ *is independent of* $\lambda_1,\lambda_2,\lambda_3$*.*

(c) *More precisely, there exists a rational number* $\omega_\pi \in \mathbb{Q}_{>0}$ *(independent of* d*) such that*

$$L_\pi(0,0,0,-d)\,\mathsf{w}_\pi\Big|_{\lambda_1+\lambda_2+\lambda_3=0} = (-1)^{|\pi|}\,\omega_\pi \prod_{l=1}^{\pi_{111}}(d-(l-1)). \tag{1.4}$$

In particular, for $d\in\mathbb{Z}_{>0}$*, the LHS vanishes when* $\pi_{111}>d$ *and otherwise has the same sign as* $(-1)^\pi$*.*

Geometrically, the specialisation (1.3) corresponds to taking $X=\mathbb{C}^4$ and $D=\{x_4^d=0\}\subseteq\mathbb{C}^4$. Then $L=\mathcal{O}(D)\cong\mathcal{O}\otimes t_4^{-d}$. As we have seen in Proposition 1.3, the canonical section of $L^{[n]}$ on $\mathrm{Hilb}^n(\mathbb{C}^4)$ cuts out the sublocus of zero-dimensional subschemes Z contained in D. At the level of torus fixed points, we are therefore considering solid partitions π of height $\pi_{111}\leqslant d$. This is the geometric motivation for the specialisation (1.3).

We have the following evidence for this conjecture:

Proposition 1.11 *(Proposition 4.2).*

- *Conjecture 1.10 is true for any solid partition* π *of size* $|\pi| \leqslant 6$.
- *Properties (a), (b), and the absolute value of equation (1.4) hold for* $d = 1$ *and any solid partition* π *satisfying* $\pi_{111} = 1$ *(in this case* $|\omega_\pi| = 1$*).*
- *Properties (a), (b), and the absolute value of equation (1.4) hold for various individual solid partitions of size* $\leqslant 15$ *listed in Appendix A.*

By combining Conjectures 1.8 and 1.10, we find a formula for enumerating ω_π-weighted solid partitions π.

Theorem 1.12 *(Theorem 4.3). Assume Conjectures 1.8 and 1.10 are true. Then*

$$\sum_{\pi} \omega_\pi \, t^{\,\pi_{111}} \, q^{|\pi|} = e^{t(M(q)-1)}, \tag{1.5}$$

where the sum is over all solid partitions, t *is a formal parameter, and* $M(q)$ *denotes the MacMahon function. In particular, for* $t = 1$

$$\sum_{\pi} \omega_\pi \, q^{|\pi|} = e^{M(q)-1}.$$

This theorem inspired us to define an *explicit* combinatorial weight $\omega_\pi^c \in \mathbb{Q}_{>0}$ associated to each solid partition π (Definition 4.7). Firstly, we *prove* an unconditional version of Theorem 1.12 with ω_π replaced by ω_π^c (Theorem 4.11). Secondly, we conjecture that $\omega_\pi = \omega_\pi^c$ and check this for the cases of Proposition 1.11 (Conjecture 4.13, Proposition 4.14).

The definition of ω_π^c (Definition 4.7) can naturally be extended to d-dimensional partitions for any $d \geqslant 0$, where $d = 3$ corresponds to the case of solid partitions. The proof of Theorem 4.11 immediately gives

$$\log \sum_{d\text{-partitions } \pi} \omega_\pi^c \, q^{|\pi|} = \sum_{(d-1)\text{-partitions } \pi,\, |\pi| \geqslant 1} q^{|\pi|}$$

and we give a similar formula involving the formal parameter t (Remark 4.12). In a future work [5], we relate this formula to equivariant DT type invariants on $\mathbb{C}^{d+1}$.

There is a related work due to Nekrasov [19], where he proposes a conjectural formula for a very general equivariant *K-theoretical* partition function on Hilbert schemes of points on $\mathbb{C}^4$. Specialisations of his partition function seem related to our Conjecture 1.8. We briefly discuss a very special instance of his conjecture in Appendix B, where we point out relations to our choices of orientation. As opposed to [19], our study of the $\mathbb{C}^4$ case emerges from first studying the compact case (Conjecture 1.2) and subsequently studying the toric analogues (Conjectures 1.6 and 1.8).

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2. The compact case

Before stating our conjecture for Hilbert schemes of points on smooth projective Calabi–Yau 4-folds, we review the framework of DT_4 invariants.

2.1. Review of DT_4 invariants

Let X be a smooth projective Calabi–Yau 4-fold, i.e. a smooth projective 4-fold X satisfying $K_X \cong \mathcal{O}_X$ and $H^i(\mathcal{O}_X) = 0$ for $i = 1, 2, 3$. Let ω be an ample divisor on X and $v \in H^*(X, \mathbb{Q})$ a cohomology class.

The coarse moduli space $M_\omega(v)$ of ω-Gieseker semistable sheaves E on X with $\mathrm{ch}(E) = v$ exists as a projective scheme. We always assume that $M_\omega(v)$ is a fine moduli space, i.e. any point $[E] \in M_\omega(v)$ is stable and there is a universal family

$$\mathcal{E} \in \mathrm{Coh}(X \times M_\omega(v)).$$

In [3,7], under certain hypotheses, the authors construct a DT_4 virtual class

$$[M_\omega(v)]^{\mathrm{vir}} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}), \tag{2.1}$$

where $\chi(-,-)$ denotes the Euler pairing. This class is not necessarily algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point $[E] \in M_\omega(v)$, a half-dimensional real subspace of the usual obstruction space

$$\mathrm{Ext}^2_+(E,E) \subseteq \mathrm{Ext}^2(E,E)$$

on which the non-degenerate quadratic form Q defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form

$$\kappa_+ = \pi_+ \circ \kappa : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2_+(E,E),$$

where κ is a Kuranishi map of $M_\omega(v)$ at E and π_+ is projection onto the first factor of

$$\mathrm{Ext}^2(E,E) = \mathrm{Ext}^2_+(E,E) \oplus \sqrt{-1} \cdot \mathrm{Ext}^2_+(E,E). \tag{2.2}$$

In [7], local models are glued in three special cases:

1. when $M_\omega(v)$ consists of locally free sheaves only;
2. when $M_\omega(v)$ is smooth;
3. when $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme.

In these cases, the corresponding virtual classes are constructed using either gauge theory or algebro-geometric perfect obstruction theory.

Assuming $M_\omega(v)$ can be given a (-2)-shifted symplectic structure, a general gluing construction was given by Borisov–Joyce [3] based on Pantev–Töen–Vaquié–Vezzosi's theory of shifted symplectic geometry [20] and Joyce's theory of derived C^∞-geometry. The corresponding virtual class is constructed using Joyce's D-manifold theory (a machinery similar to Spivak's theory of derived smooth manifolds or Fukaya–Oh–Ohta–Ono's theory of Kuranishi space structures used in defining Lagrangian Floer theory).

To have a better understanding of what DT_4 virtual classes look like, we briefly review the construction in situations (2) and (3) mentioned above:

- When $M_\omega(v)$ is smooth, the obstruction sheaf

$$\mathrm{Ob} := \mathcal{E}xt^2_{\pi_M}(\mathcal{E},\mathcal{E})$$

is a vector bundle on $M_\omega(v)$ endowed with a non-degenerate quadratic form Q induced by Serre duality, where $\pi_M : X \times M_\omega(v) \to M_\omega(v)$ denotes projection. A family version of (2.2) defines a real subbundle $\mathrm{Ob}^+ \subseteq \mathrm{Ob}$ on which Q is positive definite and $\mathrm{Ob} \cong \mathrm{Ob}^+ \otimes_{\mathbb{R}} \mathbb{C}$ are isomorphic as vector bundles with quadratic forms [9, Lem. 5]. Since $M_\omega(v)$ is smooth, the Zariski tangent space $\mathrm{Ext}^1(E,E)$ at any $[E] \in M_\omega(v)$ has the same dimension as $M_\omega(v)$, which implies that the local Kuranishi maps are zero. The DT_4 virtual class is given by

$$[M_\omega(v)]^{\mathrm{vir}} = \mathrm{PD}\big(e(\mathrm{Ob},Q)\big), \tag{2.3}$$

where $e(\mathrm{Ob},Q)$ denotes the half-Euler class of (Ob,Q), i.e. the Euler class of a real subbundle Ob^+ and $\mathrm{PD}(-)$ denotes the Poincaré dual. Equality (2.3) holds up to a sign on each connected component. This sign is determined by the *choice of orientation*, which we review below. Note that the half-Euler class satisfies

$$\begin{aligned} e(\mathrm{Ob},Q)^2 &= (-1)^{\frac{\mathrm{rk}(\mathrm{Ob})}{2}} e(\mathrm{Ob}), \text{ if } \mathrm{rk}(\mathrm{Ob}) \text{ is even}, \\ e(\mathrm{Ob},Q) &= 0, \text{ if } \mathrm{rk}(\mathrm{Ob}) \text{ is odd}. \end{aligned} \tag{2.4}$$

- Suppose $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme. Roughly speaking, this means that at any closed point $[F] \in M_\omega(v)$, we have a Kuranishi map

$$\kappa\colon\ \mathrm{Ext}^1(F,F)\to \mathrm{Ext}^2(F,F)=V_F\oplus V_F^*,$$

which factors through a maximal isotropic subspace V_F of $(\mathrm{Ext}^2(F,F),Q)$. Then the DT_4 virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_F\}_{F\in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_F\}_{F\in M_\omega(v)}$ over $M_\omega(v)$.

On orientations In order to construct the above virtual class (2.1) with coefficients in $\mathbb{Z}$ (instead of $\mathbb{Z}_2$), we need an orientability result for $M_\omega(v)$, which is stated as follows. Let

$$\mathcal{L}:=\det(\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{E},\mathcal{E}))\in \mathrm{Pic}(M_\omega(v))$$

be the determinant line bundle of $M_\omega(v)$, equipped with the non-degenerate symmetric pairing Q induced by Serre duality. An *orientation* of $(\mathcal{L},Q)$ is a reduction of its structure group from $O(1,\mathbb{C})$ to $SO(1,\mathbb{C})=\{1\}$. In other words, we require a choice of square root of the isomorphism

$$Q:\mathcal{L}\otimes\mathcal{L}\to\mathcal{O}_{M_\omega(v)}$$

in order to construct the virtual class (2.1). The virtual class (2.1) depends on the choice of orientation $o(\mathcal{L})$, so we write $[M_\omega(v)]^{\mathrm{vir}}_{o(\mathcal{L})}$ in order to stress this dependence.

An existence result of orientations is proved in [6, Thm. 2.2] for Calabi–Yau 4-folds X such that $\mathrm{Hol}(X)=SU(4)$ and $H^{\mathrm{odd}}(X,\mathbb{Z})=0$. Notice that, if orientations exist, the different choices form a torsor for $H^0(M_\omega(v),\mathbb{Z}_2)$.

In particular, when $M_\omega(v)$ is smooth, the choice of orientation on $\mathcal{L}$ is equivalent to a choice of orientation of a real subbundle $\mathrm{Ob}^+\subseteq\mathrm{Ob}$. By the homotopy equivalence $O(n,\mathbb{C})\sim O(n,\mathbb{R})$, the real subbundle is unique up to isomorphisms.

2.2. *Conjecture for* DT_4 *invariants of* $\mathrm{Hilb}^n(X)$

Let X be a smooth projective Calabi–Yau 4-fold. For a positive integer n, we consider the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points on X. It can be identified with the Gieseker moduli space of semistable sheaves with Chern character $(1,0,0,0,-n)\in H^{\mathrm{even}}(X)$, which is a fine moduli space whose closed points parametrise ideal sheaves of points.

Given a line bundle L on X, we define its tautological bundle $L^{[n]}$ as follows [13, Sect. 4.1]

$$L^{[n]}:=(\pi_M)_*\big(\mathcal{O}_{\mathcal{Z}_n}\otimes\pi_X^*L\big),$$

where $\mathcal{Z}_n\subseteq\mathrm{Hilb}^n(X)\times X$ denotes the universal subscheme and π_M,π_X are projections from the product $\mathrm{Hilb}^n(X)\times X$ to each factor. Since π_M is a flat finite morphism of degree n, $L^{[n]}$ is a rank n vector bundle on $\mathrm{Hilb}^n(X)$ with fibre $H^0(L|_Z)$ over $Z \in \mathrm{Hilb}^n(X)$. Note that the (real) virtual dimension of $\mathrm{Hilb}^n(X)$ is $2n$ by (2.1). Hence we define:

Definition 2.1. Let X be a smooth projective Calabi–Yau 4-fold and L a line bundle on X. Assume the determinant line bundle $\mathcal{L}$ of $\mathrm{Hilb}^n(X)$, with its non-degenerate quadratic form Q induced from Serre duality, is given an orientation $o(\mathcal{L})$. We define

$$\mathrm{DT}_4(X, L, n\,; o(\mathcal{L})) := \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})}} e(L^{[n]}) \in \mathbb{Z}, \quad \text{if } n \geqslant 1,$$

and $\mathrm{DT}_4(X, L, 0\,; o(\mathcal{L})) := 1$.

We make the following conjecture for the corresponding generating series.

Conjecture 2.2. *Let X be a smooth projective Calabi–Yau 4-fold and L a line bundle on X. Then there exist choices of orientation such that*

$$\sum_{n=0}^{\infty} \mathrm{DT}_4(X, L, n\,; o(\mathcal{L}))\, q^n = M(-q)^{\int_X c_1(L)\cdot c_3(X)},$$

where $M(q)$ denotes the MacMahon function.

Remark 2.3. When $L = \mathcal{O}_X$, Conjecture 2.2 follows from the fact that $\mathcal{O}_X^{[n]}$ has a nowhere vanishing section which sends Z to $1_Z \in H^0(X, \mathcal{O}_Z)$. Then $e(\mathcal{O}_X^{[n]}) = c_1(\mathcal{O}_X) = 0$.

2.3. Geometric motivation of the conjecture

Let us consider the case when $L = \mathcal{O}_X(D)$ corresponds to an effective divisor $D \subseteq X$. The following proposition is similar to [12, Sect. A.2].[1]

Proposition 2.4. *Let $D \subseteq X$ be any effective divisor on a smooth quasi-projective variety X and let $L := \mathcal{O}_X(D)$. The rank n vector bundle $L^{[n]}$ on $\mathrm{Hilb}^n(X)$ has a tautological section σ whose zero locus is isomorphic to the Hilbert scheme $\mathrm{Hilb}^n(D)$ of n points on D.*

[1] We thank the anonymous referee for pointing out a proof which is significantly simpler than our original.

Proof. Consider the universal subscheme

$$\begin{array}{ccc} & \mathcal{Z} \hookrightarrow \mathrm{Hilb}^n(X) \times X & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathrm{Hilb}^n(X) & & X. \end{array}$$

Let $s : D \subseteq X$ be a section defining D. We claim that the tautological section $\sigma := p_* q^* s$ of $L^{[n]} = p_* q^* L$ has the required property, i.e. we have an equality of schemes

$$Z(\sigma) = \mathrm{Hilb}^n(D).$$

In order to see this, it suffices to take any T-flat family

$$\begin{array}{ccc} & \mathcal{Z}_T \hookrightarrow T \times X & \\ {}^{p_T}\swarrow & & \searrow^{q_T} \\ T & & X \end{array}$$

with zero-dimensional length n fibres and prove that

$$\mathcal{Z}_T \subseteq T \times D \subseteq T \times X$$

if and only if the corresponding morphism $f : T \to \mathrm{Hilb}^n(X)$ factors through $Z(\sigma)$.

Now f factors through $Z(\sigma)$ if and only if $f^*\sigma$ is the zero section of $f^* L^{[n]}$. Note that $\mathcal{Z}_T = \mathcal{Z} \times_T \mathrm{Hilb}^n(X)$ and

$$f^*\sigma = f^* p_* q^* s = p_{T*} q_T^* s.$$

Therefore $f^*\sigma$ is the zero section if and only if $\mathcal{Z}_T \subseteq T \times D$ as required. $\square$

Let X be a smooth projective Calabi–Yau 4-fold with *smooth* divisor $D \subseteq X$ and let $L = \mathcal{O}_X(D)$. Ideally, if all moduli spaces are smooth of expected dimensions,[2] i.e. $\dim_{\mathbb{C}} \mathrm{Hilb}^n(D) = 0$ and $\dim_{\mathbb{R}} \mathrm{Hilb}^n(X) = 2n$, then the section σ constructed in Proposition 2.4 is transverse to the zero section and we have

$$\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} e(L^{[n]}) = \int_{[\mathrm{Hilb}^n(D)]^{\mathrm{vir}}} 1,$$

modulo a sign coming from the choice of orientation involved in defining the LHS. Then Conjecture 2.2 would follow from the generating series of zero-dimensional Donaldson–Thomas invariants of a smooth projective 3-fold D [15,16]

[2] Of course, this fantasy situation never occurs.

$$\sum_{n=0}^{\infty}\Big(\int\limits_{[\mathrm{Hilb}^n(D)]^{\mathrm{vir}}} 1\Big)\, q^n = M(-q)^{\int_D c_3(TD\otimes K_D)}$$

and equation (2.5) below.

For later reference, we add the derivation of the equality

$$\int\limits_D c_3(TD\otimes K_D) = \int\limits_X c_1(L)\cdot c_3(TX). \tag{2.5}$$

Indeed, from the short exact sequence

$$0 \to TD \to TX|_D \to N_{D/X} \to 0$$

and the fact that $N_{D/X} \cong \mathcal{O}_D(D) \cong K_D$ (X is Calabi–Yau), we obtain

$$\int\limits_D c(TD\otimes K_D) = \int\limits_X c_1(L)\cdot \frac{c(TX\otimes L)}{c(L\otimes L)},$$

where $c(-)$ denotes total Chern class. The degree 3 part of the fraction is easily calculated:

$$c_3(TX) + c_1(TX)\cdot c_1(L)^2 = c_3(TX),$$

where the last equality again uses the fact that X is Calabi–Yau.

2.4. Preparation on deformation and obstruction theories

We need to compare deformation–obstruction theories of $\mathrm{Hilb}^n(X)$ and $\mathrm{Hilb}^n(D)$ in order to verify our conjecture.

Lemma 2.5. *Let X be a smooth projective variety and $i : D \hookrightarrow X$ be a smooth divisor. For any subscheme $Z \subseteq D$, we have a short exact sequence*

$$0 \to \mathcal{O}_X(-D) \to I_{Z,X} \to i_*I_{Z,D} \to 0 \tag{2.6}$$

of coherent sheaves on X, where $I_{Z,\star}$ is the ideal sheaf of Z in $\star$ ($\star = X$ or D).

Furthermore, if Z is zero-dimensional, we have a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^0_X(i_*I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^0_X(I_{Z,X}, i_*\mathcal{O}_Z) \to H^0(\mathcal{O}_Z(D)) \to \\ &\to \mathrm{Ext}^1_X(i_*I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^1_X(I_{Z,X}, i_*\mathcal{O}_Z) \to H^1(\mathcal{O}_Z(D)) = 0, \end{aligned} \tag{2.7}$$

and canonical isomorphisms

$$\mathrm{Ext}^i_X(i_*I_{Z,D}, i_*\mathcal{O}_Z) \cong \mathrm{Ext}^i_X(I_{Z,X}, i_*\mathcal{O}_Z) \quad \textit{for } i \geqslant 2.$$

Proof. Sequence (2.6) can be easily deduced from the short exact sequences

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$
$$0 \to I_{Z,X} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0,$$
$$0 \to I_{Z,D} \to \mathcal{O}_D \to \mathcal{O}_Z \to 0,$$

and diagram chasing. Applying $\mathbf{R}\mathrm{Hom}_X(-, i_*\mathcal{O}_Z)$ to (2.6), we get a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_X(i_*I_{Z,D}, i_*\mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, i_*\mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X(-D), i_*\mathcal{O}_Z),$$

whose cohomology gives the long exact sequence (2.7) and the desired canonical isomorphisms because Z is zero-dimensional. □

Lemma 2.6. *Let X be a smooth projective variety with $\dim_{\mathbb{C}}(X) \geqslant 3$ and let $L \to X$ be a line bundle on X. For any zero-dimensional subscheme $Z \subseteq X$, we have canonical isomorphisms*

$$\mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X} \otimes L)_0 \cong \mathrm{Hom}_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \cong \mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L),$$
$$\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X} \otimes L)_0 \cong \mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \cong \mathrm{Ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L).$$

Proof. We apply $\mathbf{R}\mathrm{Hom}_X(-, \mathcal{O}_Z \otimes L)$ to $0 \to I_{Z,X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ and get the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L) \to \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_Z \otimes L) \to \mathrm{Hom}_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \\ &\to \mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^1_X(\mathcal{O}_X, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \\ &\to \mathrm{Ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^2_X(\mathcal{O}_X, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^2_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \cdots . \end{aligned}$$

Since $\mathrm{Hom}_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_Z \otimes L)$ and $H^{>0}(X, \mathcal{O}_Z \otimes L) = 0$ for zero-dimensional subschemes $Z \subseteq X$, we obtain isomorphisms

$$\mathrm{Ext}^i_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \cong \mathrm{Ext}^{i+1}_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes L) \quad \text{for } i \geqslant 0. \tag{2.8}$$

In particular, for $\dim_{\mathbb{C}}(X) = 3$, we obtain

$$\dim_{\mathbb{C}} \mathrm{Ext}^2_X(I_{Z,X}, \mathcal{O}_Z \otimes L) = \dim_{\mathbb{C}} \mathrm{Ext}^0_X(\mathcal{O}_Z, \mathcal{O}_Z), \tag{2.9}$$

where we used Serre duality $\mathrm{Ext}^3_X(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^0_X(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X)^*$. We will use this later.

Next we consider the following commutative diagram

$$\begin{array}{ccccc} & & \mathbf{R}\Gamma(L)[1] & = & \mathbf{R}\Gamma(L)[1] \\ & & \downarrow & & \downarrow \\ \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, \mathcal{O}_Z \otimes L) & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, I_{Z,X} \otimes L)[1] & \longrightarrow & \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, L)[1] \\ & & \downarrow & & \downarrow \\ & & \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, I_{Z,X} \otimes L)_0[1] & & \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_Z, L)[2], \end{array} \tag{2.10}$$

where the horizontal and vertical rows are distinguished triangles. By taking cones, we obtain a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \mathbf{R}\mathrm{Hom}_X(I_{Z,X}, I_{Z,X} \otimes L)_0[1] \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_Z, L)[2].$$

The long exact sequence of its cohomology gives an isomorphism

$$\mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X} \otimes L)_0 \cong \mathrm{Hom}_X(I_{Z,X}, \mathcal{O}_Z \otimes L),$$

where we used $\mathrm{Ext}^2_X(\mathcal{O}_Z, L) \cong H^{n-2}(X, \mathcal{O}_Z \otimes K_X \otimes L^{-1}) = 0$ because $n = \dim_{\mathbb{C}}(X) \geqslant 3$ and similarly $\mathrm{Ext}^1_X(\mathcal{O}_Z, L) = 0$. Furthermore, we obtain an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X} \otimes L)_0 \to \mathrm{Ext}^3_X(\mathcal{O}_Z, L) \to \\ &\to \mathrm{Ext}^2_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \to \mathrm{Ext}^3_X(I_{Z,X}, I_{Z,X} \otimes L)_0 \to \cdots . \end{aligned} \tag{2.11}$$

When $\dim_{\mathbb{C}}(X) \geqslant 4$, $\mathrm{Ext}^3_X(\mathcal{O}_Z, L) \cong H^{n-3}(X, \mathcal{O}_Z \otimes K_X \otimes L^{-1})^* = 0$ and we are done.

When $\dim_{\mathbb{C}}(X) = 3$, the trace map $\mathrm{Ext}^0_X(I_{Z,X}, I_{Z,X} \otimes L') \cong H^0(X, L')$ is an isomorphism for any line bundle L' because Z has codimension > 1 (cf. [17, I, proof of Lem. 2]). Hence $\mathrm{Ext}^3_X(I_{Z,X}, I_{Z,X} \otimes L)_0 = 0$. Furthermore

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Ext}^3_X(\mathcal{O}_Z, L) &= \dim_{\mathbb{C}} H^0(X, \mathcal{O}_Z) \\ &= \dim_{\mathbb{C}} \mathrm{Ext}^0_X(\mathcal{O}_Z, \mathcal{O}_Z) \\ &= \dim_{\mathbb{C}} \mathrm{Ext}^2_X(I_{Z,X}, \mathcal{O}_Z \otimes L), \end{aligned}$$

where the second equality uses $\mathrm{Hom}_X(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_Z)$ and the third equality uses (2.9). The exact sequence (2.11) yields the desired isomorphism

$$\mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z \otimes L) \cong \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X} \otimes L)_0. \quad \square$$

In the following lemma, we focus attention on $\mathrm{Hilb}^n(X)$, where X is a smooth projective Calabi–Yau 4-fold and $n \leqslant 3$. We recall that for any smooth projective variety Y

and $n \leqslant 3$, the Hilbert scheme $\mathrm{Hilb}^n(Y)$ is smooth of dimension $\dim_{\mathbb{C}}(Y) \cdot n$ (e.g. [14]). In fact, for a subscheme Z of length $n \leqslant 3$, Lemma 2.6 implies

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X})_0 = \dim_{\mathbb{C}} \mathrm{Ext}^0_X(I_{Z,X}, \mathcal{O}_Z) = 4n, \\ \dim_{\mathbb{C}} \mathrm{Ext}^1_D(I_{Z,D}, I_{Z,D})_0 = \dim_{\mathbb{C}} \mathrm{Ext}^0_D(I_{Z,D}, \mathcal{O}_Z) = 3n.\end{aligned}$$

Lemma 2.7. *Let X be a smooth projective Calabi–Yau 4-fold and let $i : D \hookrightarrow X$ be a smooth divisor. For any zero-dimensional subscheme $Z \subseteq D$ of length $\leqslant 3$, the exact sequence (2.7) in Lemma 2.5 breaks into an exact sequence and a canonical isomorphism*

$$\begin{aligned}0 \to \mathrm{Ext}^0_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^0_X(I_{Z,X}, i_*\mathcal{O}_Z) \to H^0(\mathcal{O}_Z(D)) \to 0, \\ \mathrm{Ext}^1_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \cong \mathrm{Ext}^1_X(I_{Z,X}, i_*\mathcal{O}_Z).\end{aligned}$$

Furthermore, using the isomorphism $\mathrm{Ext}^1_X(I_{Z,X}, i_*\mathcal{O}_Z) \cong \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$ of Lemma 2.6, we obtain a canonical inclusion (constructed in the proof)

$$\mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$$

of a half-dimensional subspace which is isotropic with respect to the non-degenerate quadratic form Q on $\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$ defined by Serre duality.

Proof. In the proof, we will use the following dimensions

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{Ext}^0_D(I_{Z,D}, \mathcal{O}_Z) = 3n, \quad \dim_{\mathbb{C}} \mathrm{Ext}^0_X(I_{Z,X}, \mathcal{O}_Z) = 4n, \\ \dim_{\mathbb{C}} \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) = 3n. \quad \dim_{\mathbb{C}} \mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z) = 6n.\end{aligned} \tag{2.12}$$

The first line follows from the fact that $\mathrm{Hilb}^n(X)$ and $\mathrm{Hilb}^n(D)$ are smooth for $n \leqslant 3$ and these are exactly the Zariski tangent spaces at Z. The second line can be seen in several ways. Firstly $\mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) \cong \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0$ and $\mathrm{Ext}^1_X(I_{Z,X}, \mathcal{O}_Z) \cong \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$ by Lemma 2.6, so it suffices to calculate the dimensions of the latter. By Hirzebruch–Riemann–Roch on D we have

$$\begin{aligned}0 = \chi(\mathcal{O}_D) - \chi(I_{Z,D}, I_{Z,D}) &= \dim_{\mathbb{C}} \mathrm{Ext}^1_D(I_{Z,D}, I_{Z,D})_0 - \dim_{\mathbb{C}} \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0 \\ &= 3n - \dim_{\mathbb{C}} \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0.\end{aligned}$$

By Hirzebruch–Riemann–Roch and Serre duality on X we have

$$\begin{aligned}2n = \chi(\mathcal{O}_X) - \chi(I_{Z,X}, I_{Z,X}) &= 2\dim_{\mathbb{C}} \mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X})_0 - \dim_{\mathbb{C}} \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0 \\ &= 8n - \dim_{\mathbb{C}} \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0.\end{aligned}$$

This establishes (2.12).

The spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_D(I_{Z,D}, \mathcal{O}_Z \otimes \wedge^q K_D) \Rightarrow \mathrm{Ext}^{p+q}_X(i_* I_{Z,D}, i_*\mathcal{O}_Z)$$

gives an isomorphism

$$\mathrm{Ext}^0_D(I_{Z,D}, \mathcal{O}_Z) \cong \mathrm{Ext}^0_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \tag{2.13}$$

and an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) \to \mathrm{Ext}^1_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^0_D(I_{Z,D}, \mathcal{O}_Z \otimes K_D) \to \\ &\to \mathrm{Ext}^2_D(I_{Z,D}, \mathcal{O}_Z) \to \mathrm{Ext}^2_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z \otimes K_D) \to 0, \end{aligned} \tag{2.14}$$

where we use $\mathrm{Ext}^3_D(I_{Z,D}, \mathcal{O}_Z) = 0$ (see (2.8)).

Combining (2.12) and (2.13), we know the exact sequence (2.7) in Lemma 2.5 breaks into a short exact sequence and a canonical isomorphism

$$\begin{gathered} 0 \to \mathrm{Ext}^0_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^0_X(I_{Z,X}, i_*\mathcal{O}_Z) \to H^0(\mathcal{O}_Z(D)) \to 0, \\ \mathrm{Ext}^1_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \cong \mathrm{Ext}^1_X(I_{Z,X}, i_*\mathcal{O}_Z). \end{gathered} \tag{2.15}$$

In particular, $\dim_{\mathbb{C}} \mathrm{Ext}^1_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) = 6n$ by (2.12). Therefore (2.12) implies that the six term exact sequence (2.14) splits into two short exact sequences and we obtain

$$0 \to \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) \to \mathrm{Ext}^1_X(i_* I_{Z,D}, i_*\mathcal{O}_Z) \to \mathrm{Ext}^0_D(I_{Z,D}, \mathcal{O}_Z \otimes K_D) \to 0. \tag{2.16}$$

Together (2.15) and (2.16) provide an inclusion

$$\mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z) \hookrightarrow \mathrm{Ext}^1_X(I_{Z,X}, i_*\mathcal{O}_Z) \cong \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0,$$

where the second isomorphism comes from Lemma 2.6. We have obtained a canonical inclusion of a half-dimensional subspace (by (2.12)).

Next, we check $\mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z)$ is an isotropic subspace of $\big(\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0, Q\big)$ under this inclusion. Given $u \in \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z)$, the corresponding element in $\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$ is given by the composition

$$I_{Z,X} \xrightarrow{\alpha} i_* I_{Z,D} \xrightarrow{i_* u} i_*\mathcal{O}_Z[1] \xrightarrow{\beta} I_{Z,X}[2],$$

where α is the morphism constructed in (2.6) and β is the obvious morphism. Given another $u' \in \mathrm{Ext}^1_D(I_{Z,D}, \mathcal{O}_Z)$, it is enough to show the vanishing of the composition

$$I_{Z,X} \xrightarrow{\alpha} i_* I_{Z,D} \xrightarrow{i_* u} i_*\mathcal{O}_Z[1] \xrightarrow{\beta} I_{Z,X}[2] \xrightarrow{\alpha[2]} i_* I_{Z,D}[2] \xrightarrow{i_* u'[2]} i_*\mathcal{O}_Z[3] \xrightarrow{\beta[2]} I_{Z,X}[4].$$

We claim

$$\mathrm{Ext}^1_X(i_*\mathcal{O}_Z, i_*I_{Z,D}) \cong \mathrm{Ext}^1_D(\mathcal{O}_Z, I_{Z,D}). \tag{2.17}$$

This implies that the composition $i_*\mathcal{O}_Z[1] \xrightarrow{\beta} I_{Z,X}[2] \xrightarrow{\alpha[2]} i_*I_{Z,D}[2]$ can be written as $i_*\gamma$, for some $\gamma : \mathcal{O}_Z \to I_{Z,D}[1]$. Therefore the composition

$$i_*I_{Z,D} \xrightarrow{i_*u} i_*\mathcal{O}_Z[1] \xrightarrow{\beta} I_{Z,X}[2] \xrightarrow{\alpha[2]} i_*I_{Z,D}[2] \xrightarrow{i_*u'[2]} i_*\mathcal{O}_Z[3]$$

comes from $\mathrm{Ext}^3_D(I_{Z,D}, \mathcal{O}_Z)$ which is zero by (2.8).

We are left to show (2.17). This follows at once from the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_D(\mathcal{O}_Z, I_{Z,D} \otimes \wedge^q K_D) \Rightarrow \mathrm{Ext}^{p+q}_X(i_*\mathcal{O}_Z, i_*I_{Z,D}),$$

and

$$\mathrm{Ext}^0_D(\mathcal{O}_Z, I_{Z,D} \otimes K_D) \cong \mathrm{Ext}^3_D(I_{Z,D}, \mathcal{O}_Z)^* = 0,$$

where the vanishing is by (2.8). □

Combining Lemma 2.6 and 2.7, we deduce the following:

Proposition 2.8. *Let X be a smooth projective Calabi–Yau 4-fold and let $D \subseteq X$ be a smooth divisor. For any zero-dimensional subscheme $Z \subseteq D$ of length $\leqslant 3$, we have short sequences*

$$0 \to \mathrm{Ext}^1_D(I_{Z,D}, I_{Z,D})_0 \to \mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X})_0 \to H^0(\mathcal{O}_Z(D)) \to 0,$$
$$0 \to \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0 \to \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0 \to \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})^*_0 \to 0,$$

under which $\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0$ is a maximal isotropic subspace of $\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$ with respect to the non-degenerate quadratic form Q defined by Serre duality.

Proof. By Lemma 2.6, we have isomorphisms

$$\mathrm{Ext}^{i+1}_Y(I_{Z,Y}, I_{Z,Y} \otimes L)_0 \cong \mathrm{Ext}^i_Y(I_{Z,Y}, \mathcal{O}_Z \otimes L), \ \text{for } i = 0, 1 \text{ and } Y = X, D.$$

Combining with Lemma 2.7, we obtain the desired short exact sequences and an inclusion

$$\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0 \hookrightarrow \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$$

of a maximal isotropic subspace.

This leads to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0 & \longrightarrow & \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0 & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \exists\, t} & & {\scriptstyle Q}\downarrow{\scriptstyle \cong} & & & & \\ 0 & \longrightarrow & W^* & \longrightarrow & \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0^* & \longrightarrow & \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0^* & \longrightarrow & 0. \end{array}$$

Note that the restriction t of Q is injective, hence also an isomorphism by dimension counting. Thus the quadratic form Q gives an identification $W \cong \mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0^*$. □

A positive real form V_+ on a complex even dimensional vector space V with non-degenerate quadratic form Q is a half-dimensional real subspace on which Q is real and positive definite. When the obstruction space $\mathrm{Ext}^2_X(E, E)_0$ has a maximal isotropic subspace as in Proposition 2.8, we can apply the following useful fact:

Proposition 2.9. *Let V be an even dimensional complex vector space with a non-degenerate quadratic form Q and let V_{iso} be a maximal isotropic subspace of (V, Q). Then for any positive real form V_+ of (V, Q), the composition*

$$c : V_{\mathrm{iso}} \hookrightarrow V \to V_+$$

of the inclusion and projection is an isomorphism of the underlying real vector spaces.

Proof. Since dimensions of V_{iso} and V_+ are the same, we only need to check that the map c is injective. Take $v \in V_{\mathrm{iso}}$ which projects to zero in V_+. By

$$V = V_+ \oplus \sqrt{-1} \cdot V_+,$$

we know $v \in \sqrt{-1} \cdot V_+$. Then $Q(v, v) = 0$, by the isotropic property, which implies that $v = 0$ since Q is negative definite on the subspace $\sqrt{-1} \cdot V_+$. □

2.5. Verification in simple cases: $n \leqslant 3$

When the number n of points satisfies $n \leqslant 3$, the Hilbert schemes $\mathrm{Hilb}^n(X)$ and $\mathrm{Hilb}^n(D)$ are smooth of dimensions $4n$ and $3n$ respectively. Our conjecture can then be verified by direct calculation.

Theorem 2.10. *Let X be a smooth projective Calabi–Yau 4-fold. Let D be a smooth divisor on X and set $L := \mathcal{O}_X(D)$. For each $n \leqslant 3$, there exists a choice of orientation $o(\mathcal{L})$ such that*

$$\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})}} e(L^{[n]}) = \int_{[\mathrm{Hilb}^n(D)]^{\mathrm{vir}}} 1.$$

In particular, Conjecture 2.2 is true modulo q^4 for $L = \mathcal{O}_X(D)$ and $D \subseteq X$ a smooth divisor.

Proof. When $n \leqslant 3$, the Hilbert schemes $\mathrm{Hilb}^n(X)$, $\mathrm{Hilb}^n(D)$ are smooth of dimensions $4n$ and $3n$ respectively. We have also seen that the obstruction sheaf Ob on $\mathrm{Hilb}^n(X)$ is locally free of rank $6n$ ((2.12) and Lemma 2.6).

Consider the quadric bundle (Ob, Q), where Q is the non-degenerate quadratic form defined by Serre duality. By [9, Lem. 5], we can choose a positive real form Ob^+ of the quadric bundle (Ob, Q), such that $\mathrm{Ob} \cong \mathrm{Ob}^+ \otimes_{\mathbb{R}} \mathbb{C}$ as quadric bundles. Then

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})} = \mathrm{PD}\big(e(\mathrm{Ob}^+)\big) \in H_{2n}(\mathrm{Hilb}^n(X))$$

for an appropriate choice of orientation $o(\mathcal{L})$ in the definition of both sides. Therefore

$$\begin{aligned}\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{o(\mathcal{L})}} e(L^{[n]}) &= \int_{[\mathrm{Hilb}^n(X)]} e(L^{[n]}) \cdot e(\mathrm{Ob}^+) \\ &= \int_{[\mathrm{Hilb}^n(D)]} e(\mathrm{Ob}^+)|_{\mathrm{Hilb}^n(D)},\end{aligned}$$

where the second equality follows from the fact that $\mathrm{Hilb}^n(D) \subseteq \mathrm{Hilb}^n(X)$ represents the Poincaré dual of the Euler class $e(L^{[n]})$ by Proposition 2.4.

Next, we use the fact that the subspaces

$$\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0 \hookrightarrow \mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X})_0$$

determine a maximal isotropic subbundle $V_{\mathrm{iso}} \subseteq \mathrm{Ob}|_{\mathrm{Hilb}^n(D)}$. Note that

$$V_{\mathrm{iso}} \cong \mathrm{Ob}_{\mathrm{Hilb}^n(D)}$$

is precisely the obstruction bundle of the perfect obstruction theory on $\mathrm{Hilb}^n(D)$ studied in [17], whose fibre over $Z \in \mathrm{Hilb}^n(D)$ is $\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D})_0$. By (a family version of) Proposition 2.9, we have

$$e(\mathrm{Ob}^+)|_{\mathrm{Hilb}^n(D)} = e(V_{\mathrm{iso}}) = e(\mathrm{Ob}_{\mathrm{Hilb}^n(D)}).$$

Since $\mathrm{Hilb}^n(D)$ is smooth, we also have

$$[\mathrm{Hilb}^n(D)]^{\mathrm{vir}} = e(\mathrm{Ob}_{\mathrm{Hilb}^n(D)}) \cap [\mathrm{Hilb}^n(D)].$$

Putting everything together, we deduce

$$\int_{[\mathrm{Hilb}^n(D)]} e(\mathrm{Ob}^+)|_{\mathrm{Hilb}^n(D)} = \int_{[\mathrm{Hilb}^n(D)]} e(\mathrm{Ob}_{\mathrm{Hilb}^n(D)}) = \int_{[\mathrm{Hilb}^n(D)]^{\mathrm{vir}}} 1.$$

The final statement of the proposition follows from [15,16] and (2.5). □

For general $\mathrm{Hilb}^n(X)$, we need Joyce's theory of D-manifolds or Kuranishi atlases to prove a similar statement. We hope to return to this in a future work.

3. The toric case

3.1. Definition and conjecture

Following [7, Sect. 8], we can similarly study zero-dimensional DT_4 invariants of toric Calabi–Yau 4-folds (which are never compact).

Let X be a smooth quasi-projective toric Calabi–Yau 4-fold. By this we mean a smooth quasi-projective toric 4-fold X satisfying $K_X \cong \mathcal{O}_X$ and $H^{>0}(\mathcal{O}_X) = 0$. We also assume the fan contains cones of dimension 4. Such cones correspond to $(\mathbb{C}^*)^4$-invariant affine open subsets (equivariantly) isomorphic to $\mathbb{C}^4$. Fix a Calabi–Yau volume form Ω on X and denote by $T \subseteq (\mathbb{C}^*)^4$ the 3-dimensional subtorus which preserves Ω. Let $\bullet$ be $\operatorname{Spec}\mathbb{C}$ with trivial $(\mathbb{C}^*)^4$-action. We denote by $\mathbb{C} \otimes t_i$ the 1-dimensional $(\mathbb{C}^*)^4$-representation with weight t_i and we write $\lambda_i \in H^*_{(\mathbb{C}^*)^4}(\bullet)$ for its $(\mathbb{C}^*)^4$-equivariant first Chern class. Then

$$H^*_{(\mathbb{C}^*)^4}(\bullet) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4],$$
$$H^*_T(\bullet) = \mathbb{C}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \cong \mathbb{C}[\lambda_1, \lambda_2, \lambda_3].$$

The $(\mathbb{C}^*)^4$-action and T-action both canonically lift to the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points on X, where T preserves the Serre duality pairing (for compactly supported sheaves).

Let L be a T-equivariant line bundle on X and let $L^{[n]}$ be its tautological bundle with induced T-equivariant structure. As in Definition 2.1, we would like to evaluate the integral

$$\int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} e(L^{[n]}), \quad \text{for } n \geqslant 1.$$

However, $\mathrm{Hilb}^n(X)$ is non-compact, so the usual virtual class is not well-defined. Nevertheless, $\mathrm{Hilb}^n(X)$ is "equivariantly compact", i.e. the T-fixed locus $\mathrm{Hilb}^n(X)^T$ is compact. In fact, it consists of finitely many points.

Lemma 3.1. *At the level of closed points, we have*

$$\mathrm{Hilb}^n(X)^T = \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4},$$

which consists of finitely many points.

Proof. We cover X by maximal $(\mathbb{C}^*)^4$-invariant open affine subsets $\{U_\alpha\}$ with centres at $(\mathbb{C}^*)^4$-fixed points. There exist coordinates x_1, x_2, x_3, x_4 on $U_\alpha \cong \mathbb{C}^4$, such that the action of $t \in (\mathbb{C}^*)^4$ on U_α is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4.$$

Then the Calabi–Yau torus is given by

$$T = \{t \in (\mathbb{C}^*)^4 \mid t_1 t_2 t_3 t_4 = 1\}$$

and we see that U_α is also T-invariant. Therefore it suffices to prove the lemma for $X = U_\alpha = \mathbb{C}^4$ with the standard torus action.

The $(\mathbb{C}^*)^4$-invariant ideals in $\mathbb{C}[x_1, x_2, x_3, x_4]$ are precisely the monomial ideals. Clearly

$$\mathrm{Hilb}^n(X)^T \supseteq \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4}.$$

By considering the weight of $x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$ under the action of $t \in \mathbb{C}^4$, it is easy to see that any T-invariant ideal $I \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$ is of form

$$I = \langle x_1^{n_{11}} x_2^{n_{12}} x_3^{n_{13}} x_4^{n_{14}} f_1(x_1x_2x_3x_4), \cdots, x_1^{n_{l1}} x_2^{n_{l2}} x_3^{n_{l3}} x_4^{n_{l4}} f_l(x_1x_2x_3x_4) \rangle,$$

where $\{f_i(y)\}$ are polynomials of one variable with constant coefficient 1 and $n_{ij} \in \mathbb{Z}_{\geqslant 0}$. Suppose I is T-invariant and corresponds to a zero-dimensional subscheme Z. Then the underlying reduced subscheme Z_{red} is a zero-dimensional T-invariant subset of $\mathbb{C}^4$, i.e. $Z_{\mathrm{red}} = \{(0, 0, 0, 0)\}$. Therefore I is determined by its restriction to any Zariski open neighbourhood U of $(0, 0, 0, 0)$. Take

$$(0, 0, 0, 0) \in U = \{f_1(x_1x_2x_3x_4) \neq 0\} \cap \cdots \cap \{f_l(x_1x_2x_3x_4) \neq 0\}.$$

The polynomials $f_i(x_1x_2x_3x_3)$ become invertible elements on U and therefore

$$I|_U = \langle x_1^{n_{11}} x_2^{n_{12}} x_3^{n_{13}} x_4^{n_{14}}, \cdots, x_1^{n_{l1}} x_2^{n_{l2}} x_3^{n_{l3}} x_4^{n_{l4}} \rangle.$$

We conclude that

$$I = \langle x_1^{n_{11}} x_2^{n_{12}} x_3^{n_{13}} x_4^{n_{14}}, \cdots, x_1^{n_{l1}} x_2^{n_{l2}} x_3^{n_{l3}} x_4^{n_{l4}} \rangle$$

which shows $\mathrm{Hilb}^n(X)^T \subseteq \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4}$ as sets. $\square$

Example 3.2. Consider $X = \mathbb{C}^4$ with standard torus action. Then

$$I = \langle x_1^3, x_2^3, x_3^3, x_4^3, x_1^2x_2^2x_3^2x_4^2 + x_1x_2x_3x_4 \rangle$$

defines a zero-dimensional T-invariant subscheme. According to the proof of Lemma 3.1, it is equal to $\langle x_1^3, x_2^3, x_3^3, x_4^3, x_1x_2x_3x_4 \rangle$. Indeed, we have

$$x_1x_2x_3x_4 = [x_2^3x_3^3x_4^3]x_1^3 + [1 - x_1x_2x_3x_4](x_1^2x_2^2x_3^2x_4^2 + x_1x_2x_3x_4).$$

Let $U \cong \mathbb{C}^4$ be a maximal $(\mathbb{C}^*)^4$-invariant affine open subset of X. Choose coordinates $x_1, \ldots, x_4$ such that the action is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4.$$

The T-invariant (and therefore $(\mathbb{C}^*)^4$-invariant by Lemma 3.1) zero-dimensional subschemes of U_α can be labelled by solid partitions.

Definition 3.3. A *solid partition* $\pi = \{\pi_{ijk}\}_{i,j,k \geqslant 1}$ consists of a sequence of non-negative integers $\pi_{ijk} \in \mathbb{Z}_{\geqslant 0}$ satisfying

$$\pi_{ijk} \geqslant \pi_{i+1,j,k}, \quad \pi_{ijk} \geqslant \pi_{i,j+1,k}, \quad \pi_{ijk} \geqslant \pi_{i,j,k+1} \quad \forall\, i, j, k \geqslant 1,$$

such that

$$|\pi| := \sum_{i,j,k \geqslant 1} \pi_{ijk} < \infty.$$

Here $|\pi|$ is called the *size* of π.

Specifically, the zero-dimensional subscheme Z_π corresponding to the solid partition $\pi = \{\pi_{ijk}\}_{i,j,k \geqslant 1}$ is defined by the monomial ideal

$$I_{Z_\pi} := \langle x_1^{i-1}x_2^{j-1}x_3^{k-1}x_4^{\pi_{ijk}} \mid i, j, k \geqslant 1 \rangle$$

and $|\pi|$ equals the length of Z_π. The $(\mathbb{C}^*)^4$-equivariant representation of Z_π is given by

$$Z_\pi = \sum_{i,j,k \geqslant 1} \sum_{l=1}^{\pi_{ijk}} t_1^{i-1}t_2^{j-1}t_3^{k-1}t_4^{l-1}, \tag{3.1}$$

where the sum is over all $i, j, k \geqslant 1$ for which $\pi_{ijk} \geqslant 1$.

In order to be able to apply Serre duality for $\mathrm{Ext}^*(I_Z, I_Z)$ on a non-compact toric Calabi–Yau 4-fold X, we will use the following lemma.

Lemma 3.4. *For any* $Z \in \mathrm{Hilb}^n(X)^T$, *we have isomorphisms of* T*-representations*

$$\mathrm{Ext}^i(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^{i+1}(I_Z, I_Z), \;\; i = 0, 1, 2,$$
$$\mathrm{Ext}^i(I_Z, I_Z) \cong \mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z), \;\; i = 1, 2, 3, \quad \mathrm{Ext}^4(I_Z, I_Z) = 0.$$

Proof. All morphisms in this proof are T-equivariant. By applying $\mathbf{R}\mathrm{Hom}(-, \mathcal{O}_Z)$ to the short exact sequence,

$$0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0, \tag{3.2}$$

we obtain isomorphisms

$$\mathrm{Ext}^i(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^{i+1}(\mathcal{O}_Z, \mathcal{O}_Z), \quad i \geqslant 0, \tag{3.3}$$

where we use $H^{i \geqslant 1}(\mathcal{O}_X) = 0$. By applying $\mathbf{R}\mathrm{Hom}(I_Z, -)$ to (3.2) we obtain an exact sequence

$$\cdots \to \mathrm{Ext}^i(I_Z, \mathcal{O}_Z) \to \mathrm{Ext}^{i+1}(I_Z, I_Z) \to \mathrm{Ext}^{i+1}(I_Z, \mathcal{O}_X) \to \cdots . \tag{3.4}$$

By applying $\mathbf{R}\mathrm{Hom}(-, \mathcal{O}_X)$ to (3.2), we find

$$\begin{aligned} &\mathrm{Hom}(I_Z, \mathcal{O}_X) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X), \\ &\mathrm{Ext}^1(I_Z, \mathcal{O}_X) = \mathrm{Ext}^2(I_Z, \mathcal{O}_X) = \mathrm{Ext}^4(I_Z, \mathcal{O}_X) = 0, \\ &\mathrm{Ext}^3(I_Z, \mathcal{O}_X) \cong \mathrm{Ext}^4(\mathcal{O}_Z, \mathcal{O}_X), \end{aligned} \tag{3.5}$$

where we use $H^{i \geqslant 1}(\mathcal{O}_X) = 0$ and $\mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_X) = 0$ for $i \leqslant 3$ (by [11, pp. 78], $\mathcal{E}xt^{i \leqslant 3}(\mathcal{O}_Z, \mathcal{O}_X) = 0$, so the vanishing follows from the local-to-global spectral sequence $H^p(X, \mathcal{E}xt^q(-,-)) \Rightarrow \mathrm{Ext}^{p+q}(-,-)$ [11, pp. 85, (3.16)]). Combining with (3.4), we get the following isomorphisms and exact sequence

$$\begin{aligned} &\mathrm{Ext}^0(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(I_Z, I_Z), \quad \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(I_Z, I_Z), \\ &0 \to \mathrm{Ext}^2(I_Z, \mathcal{O}_Z) \to \mathrm{Ext}^3(I_Z, I_Z) \to \mathrm{Ext}^3(I_Z, \mathcal{O}_X) \xrightarrow{\eta} \mathrm{Ext}^3(I_Z, \mathcal{O}_Z) \to \\ &\to \mathrm{Ext}^4(I_Z, I_Z) \to \mathrm{Ext}^4(I_Z, \mathcal{O}_X) = 0. \end{aligned} \tag{3.6}$$

For the first isomorphism of (3.6), we used $\mathrm{Hom}(I_Z, I_Z) \cong \mathrm{Hom}(I_Z, \mathcal{O}_X)$. This follows from the fact that the isomorphism $H^0(\mathcal{O}_X) \to \mathrm{Hom}(I_Z, \mathcal{O}_X)$ of (3.5) factors through $H^0(\mathcal{O}_X) \to \mathrm{Hom}(I_Z, I_Z)$ (see diagram (2.10)[3]).

[3] Since X is smooth and quasi-projective, any $(\mathbb{C}^*)^4$-equivariant coherent sheaf on X has a finite $(\mathbb{C}^*)^4$-equivariant locally free resolution by [8, Prop. 5.1.28]. Therefore we have T-equivariant trace maps as usual.

We claim that the map η is an isomorphism. In fact, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(I_Z, \mathcal{O}_X[3]) & \xrightarrow{\eta} & \mathrm{Hom}(I_Z, \mathcal{O}_Z[3]) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ \mathrm{Hom}(\mathcal{O}_Z[-1], \mathcal{O}_X[3]) & \xrightarrow{\phi} & \mathrm{Hom}(\mathcal{O}_Z[-1], \mathcal{O}_Z[3]), \end{array}$$

where i_1, i_2 are isomorphisms in (3.5), (3.3) respectively, and ϕ is the map in the exact sequence

$$\to \mathrm{Ext}^4(\mathcal{O}_Z, I_Z) \to \mathrm{Ext}^4(\mathcal{O}_Z, \mathcal{O}_X) \xrightarrow{\phi} \mathrm{Ext}^4(\mathcal{O}_Z, \mathcal{O}_Z) \to 0,$$

obtained by applying $\mathbf{R}\mathrm{Hom}(\mathcal{O}_Z, -)$ to (3.2). By Riemann–Roch and Serre duality, we have[4]

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Ext}^4(\mathcal{O}_Z, \mathcal{O}_X) &= \dim_{\mathbb{C}} H^0(X, \mathcal{E}xt^4(\mathcal{O}_Z, \mathcal{O}_X)) \\ &= \chi(\mathcal{O}_Z, \mathcal{O}_X) = n, \\ \dim_{\mathbb{C}} \mathrm{Ext}^4(\mathcal{O}_Z, \mathcal{O}_Z) &= \dim_{\mathbb{C}} \mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z) = n. \end{aligned}$$

Therefore ϕ is an isomorphism and so is η. We conclude that $\mathrm{Ext}^2(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^3(I_Z, I_Z)$ and $\mathrm{Ext}^4(I_Z, I_Z) = 0$ by (3.6), which finished the proof. □

Remark 3.5. Although a smooth quasi-projective toric Calabi–Yau 4-fold X is non-compact, the sheaf $\mathcal{O}_Z$ has proper support for any $Z \in \mathrm{Hilb}^n(X)^T$. Therefore, we can apply T-equivariant Serre duality to $\mathrm{Ext}^i(\mathcal{O}_Z, \mathcal{O}_Z)$.[5] Consequently, Lemma 3.4 allows us to apply T-equivariant Serre duality to $\mathrm{Ext}^i(I_Z, I_Z)$ for $i = 1, 2, 3$. We will use this throughout the rest of this section.

Similarly to [17, I, Lem. 6], we have the following.

Lemma 3.6. *For any $Z \in \mathrm{Hilb}^n(X)^T$, we have an isomorphism of T-representations*

$$\mathrm{Ext}^0(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(I_Z, I_Z).$$

Moreover, $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)^T = 0$. In particular, the scheme $\mathrm{Hilb}^n(X)^T = \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4}$ consists of finitely many reduced *points.*

[4] Although X is non-compact, we can pass to a "toric compactification" $X \subset \overline{X}$, i.e. a smooth projective toric 4-fold containing X as a $(\mathbb{C}^*)^4$-invariant open subset. Since $Z \subset X$ has proper support, we get $(\mathbb{C}^*)^4$-equivariant isomorphisms $H^0(X, \mathcal{E}xt^4_X(\mathcal{O}_Z, \mathcal{O}_X)) \cong H^0(\overline{X}, \mathcal{E}xt^4_{\overline{X}}(\mathcal{O}_Z, \mathcal{O}_{\overline{X}}))$ and $\mathrm{Ext}^*_X(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^*_{\overline{X}}(\mathcal{O}_Z, \mathcal{O}_Z)$.

[5] See footnote 4.

Proof. The isomorphism $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(I_Z, I_Z)$ was proved in Lemma 3.4.

Next we show $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)^T = 0$. In fact it suffices to prove this when $X = \mathbb{C}^4$. Then there exists a convenient basis for $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)$ of $(\mathbb{C}^*)^4$-equivariant homomorphisms. This basis is described by combinatorial objects, which we call *Haiman arrows*. See [18] (and also [4]) for details. These are arrows α in the character lattice $\mathbb{Z}^4$ such that:

- the tail $t(\alpha) \in \mathbb{Z}^4$ satisfies $(I_Z)_{t(\alpha)} \neq 0$, i.e. it lies on a nonzero weight space of I_Z,
- the head $h(\alpha) \in \mathbb{Z}^4$ satisfies $(\mathcal{O}_Z)_{h(\alpha)+(n_1,n_2,n_3,n_4)} \neq 0$ for some $n_1, n_2, n_3, n_4 \geqslant 0$.

Denote the standard basis of $\mathbb{Z}^4$ by

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (0,0,1,0), \quad e_4 = (0,0,0,1).$$

Suppose α is a Haiman arrow such that the arrow defined by $t(\alpha) \pm e_i$, $h(\alpha) \pm e_i$, for some choice of $\pm$ and some basis vector e_i, is also a Haiman arrow. I.e. the Haiman arrow α can be translated to another neighbouring Haiman arrow β. Then we call these Haiman arrows *equivalent*. This induces an equivalence relation on the collection of all Haiman arrows. Next, we consider the collection $\mathcal{C}$ of equivalence classes c of Haiman arrows such that all representatives $\alpha \in c$ satisfy $h(\alpha) \in (\mathcal{O}_Z)_{h(\alpha)} \neq 0$. Then the elements of $\mathcal{C}$ are in 1-1 correspondence with a basis of $(\mathbb{C}^*)^4$-equivariant homomorphisms of $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)$ as follows. To each class $c \in \mathcal{C}$ we assign a module morphism $\phi_c : I_Z \to \mathcal{O}_Z$, which is determined as follows. For each $\alpha \in c$ such that $t(\alpha)$ corresponds to a minimal homogeneous generator of I_Z, we define

$$\phi_c(x^{t(\alpha)}) = x^{h(\alpha)}$$

and all other minimal homogeneous generators are mapped to zero. Here we use multi-index notation $x^w := x_1^{w_1} x_2^{w_2} x_3^{w_3} x_4^{w_4}$. It is part of Haiman's theory that this is well-defined and defines a basis $\{\phi_c\}_{c \in \mathcal{C}}$ of $\mathrm{Hom}(I_Z, \mathcal{O}_Z)$. Clearly the weight of ϕ_c equals

$$h(\alpha) - t(\alpha),$$

which is independent of the choice $\alpha \in c$. The statement we are after follows from the fact that any Haiman arrow β with the property that $h(\beta) - t(\beta) = (n,n,n,n)$, for some n, is equivalent to a Haiman arrow γ satisfying $(\mathcal{O}_Z)_{h(\gamma)} = 0$, i.e. $[\beta] \notin \mathcal{C}$. We conclude $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)^T = 0$. $\square$

Example 3.7. Suppose $I_Z := (x_1, x_2, x_3, x_4)^2$. Then $\mathcal{C}$ consists of 40 elements (implying that $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)$ is 40-dimensional and $\mathrm{Hilb}^5(\mathbb{C}^4)$ is singular at Z). Explicitly, the basis ϕ_c described in the proof of the previous lemma consists of the following 40 homomorphisms:

$$\phi_{ij} : x_i^2 \mapsto x_j, \quad \text{any other minimal homogeneous generator} \mapsto 0$$

$$\phi_{abc} : x_a x_b \mapsto x_c, \quad \text{any other minimal homogeneous generator} \mapsto 0$$

for all i, j and a, b, c with $a < b$. Observe that none of these homomorphisms has weight of the form (n, n, n, n). Therefore $\mathrm{Ext}^0(I_Z, \mathcal{O}_Z)^T = 0$.

We continue with the definition of equivariant DT_4 invariants. For $Z \in \mathrm{Hilb}^n(X)^T$, one can form complex vector bundle

$$\begin{array}{c} ET \times_T \mathrm{Ext}^i(I_Z, I_Z) \\ \downarrow \\ ET \times_T \{I_Z\} = BT \end{array} \quad \text{for } i = 1, 2,$$

whose Euler class is the T-equivariant Euler class $e_T\big(\mathrm{Ext}^i(I_Z, I_Z)\big)$.

When $i = 2$, the Serre duality pairing on $\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ defines a non-degenerate quadratic form Q on $\mathrm{Ext}^2(I_Z, I_Z)$ (via Lemma 3.4) and also on $ET \times_T \mathrm{Ext}^2(I_Z, I_Z)$ as T preserves the Calabi–Yau volume form. We define

$$e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big) \in \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3] \tag{3.7}$$

as the *half Euler class* of $(ET \times_T \mathrm{Ext}^2(I_Z, I_Z), Q)$. By definition, this is the Euler class of its positive real form,[6] which exists because the classifying space BT is simply connected. The half Euler class (3.7) depends on a choice of orientation on a positive real form.

Following [7, Sect. 8], we can define the equivariant virtual class as follows:

Definition 3.8. Let X be a smooth quasi-projective toric Calabi–Yau 4-fold. Denote by $T \subseteq (\mathbb{C}^*)^4$ the three-dimensional subtorus which preserves the Calabi–Yau volume form. The T-equivariant virtual class of $\mathrm{Hilb}^n(X)$ is

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}_{T,o(\mathcal{L})} := \sum_{Z \in \mathrm{Hilb}^n(X)^T} \frac{e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big)}{e_T\big(\mathrm{Ext}^1(I_Z, I_Z)\big)} \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),$$

where $o(\mathcal{L})$ denotes a choice of orientation of a positive real form of $(ET \times_T \mathrm{Ext}^2(I_Z, I_Z), Q)$ for each $Z \in \mathrm{Hilb}^n(X)^T$.

Note that we have $\mathrm{Ext}^i(I_Z, I_Z) = \mathrm{Ext}^i(I_Z, I_Z)_0$ for $i = 1, 2$, because $H^{>0}(\mathcal{O}_X) = 0$.[7]

Remark 3.9. For each $Z \in \mathrm{Hilb}^n(X)^T$, $o(\mathcal{L})$ is equivalent to the choice of sign in the square root (1.2). If the number of fixed points $\mathrm{Hilb}^n(X)^T$ is N, the number of choices of $o(\mathcal{L})$ is 2^N.

[6] I.e. a half rank real subbundle on which Q is real and positive definite.
[7] See footnote 3 on the existence of T-equivariant trace maps.

The T-equivariant version of Definition 2.1 is given as follows.

Definition 3.10. In the setup of Definition 3.8, let L be a T-equivariant line bundle on X with corresponding tautological bundle $L^{[n]}$ on $\mathrm{Hilb}^n(X)$. Then

$$\begin{aligned}
&\mathrm{DT}_4(X, T, L, n\,; o(\mathcal{L})) \\
&\quad := \sum_{Z \in \mathrm{Hilb}^n(X)^T} \frac{e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big) \cdot e_T(L^{[n]}|_Z)}{e_T\big(\mathrm{Ext}^1(I_Z, I_Z)\big)} \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3), \text{ if } n \geqslant 1, \\
&\mathrm{DT}_4(X, T, L, 0\,; o(\mathcal{L})) := 1.
\end{aligned}$$

We recall the notion of equivariant push-forward for (not necessarily compact) manifolds with torus action (e.g. toric Calabi–Yau 4-folds). In the compact case, this coincides with the usual proper push-forward in the Atiyah–Bott localisation formula.

Definition 3.11. Let X be a smooth manifold with $T \cong (\mathbb{C}^*)^k$-action such that the torus fixed locus X^T consists of finite number of (necessarily reduced) points. The equivariant push-forward of $\pi : X \to pt$ is

$$\int_X : H_T^*(X) \to H_T^*(pt)_{\mathrm{loc}}, \quad \text{s.t.} \int_X \alpha = \sum_{x \in X^T} \frac{\iota_x^* \alpha}{e_T(T_x X)},$$

where $H_T^*(pt)_{\mathrm{loc}}$ is the ring of fractions of $H_T^*(pt)$, which is isomorphic to $\mathbb{C}(\lambda_1, \cdots, \lambda_k)$ if we identify $H_T^*(pt) \cong \mathbb{C}[\lambda_1, \cdots, \lambda_k]$, and $\iota_x : \{x\} \times_T ET \to X \times_T ET$ is the natural inclusion.

We propose the following T-equivariant version of Conjecture 2.2.

Conjecture 3.12. *Let X be a smooth quasi-projective toric Calabi–Yau 4-fold. Denote by $T \subseteq (\mathbb{C}^*)^4$ the three-dimensional subtorus which preserves the Calabi–Yau volume form. Let L be a T-equivariant line bundle on X. Then there exist choices of orientation such that*

$$\sum_{n=0}^{\infty} \mathrm{DT}_4(X, T, L, n\,; o(\mathcal{L}))\, q^n = M(-q)^{\int_X c_1^T(L) \cdot c_3^T(X)},$$

where $M(q)$ denotes the MacMahon function.

3.2. Proof for smooth toric divisors

Let $L = \mathcal{O}_X(D)$ for a T-invariant divisor $D \subseteq X$. Note that if D is not $(\mathbb{C}^*)^4$-invariant, by the proof of Lemma 3.1, D can locally be written as the sum of a $(\mathbb{C}^*)^4$-invariant divisor and a T-invariant divisor which is not $(\mathbb{C}^*)^4$-invariant. Hence, locally near each fixed point, L is T-equivariantly isomorphic to a $(\mathbb{C}^*)^4$-equivariant line bundle. Therefore it suffices to consider Conjecture 3.12 for $(\mathbb{C}^*)^4$-equivariant divisors only.

We prove Conjecture 3.12 when $D \subseteq X$ is a *smooth* $(\mathbb{C}^*)^4$-equivariant divisor.

Theorem 3.13. *Let X be a smooth quasi-projective toric Calabi–Yau 4-fold. Denote by $T \subseteq (\mathbb{C}^*)^4$ the three-dimensional subtorus which preserves the Calabi–Yau volume form. Let $L = \mathcal{O}_X(D)$, where $D \subseteq X$ is a smooth $(\mathbb{C}^*)^4$-invariant divisor. Then Conjecture 3.12 is true.*

Proof. For $Z \in \mathrm{Hilb}^n(X)^T$ such that $Z \nsubseteq D$, i.e. Z does not lie scheme theoretically in D, we claim that

$$e_T(L^{[n]}|_Z) = 0. \tag{3.8}$$

Let $U \cong \mathbb{C}^4$ be any $(\mathbb{C}^*)^4$-invariant affine open subset of X. As D is smooth and $(\mathbb{C}^*)^4$-invariant, we can choose coordinates x_1, x_2, x_3, x_4 on U such that the action is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4,$$

and $D \cap U$ is defined by $x_4 = 0$. Equation (3.8) then follows from Lemma 3.14 below.

Now we only need to calculate

$$\sum_{Z \in \mathrm{Hilb}^n(X)^T,\, Z \subseteq D} \frac{e_T(\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X}), Q) \cdot e_T(L^{[n]}|_Z)}{e_T(\mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X}))}. \tag{3.9}$$

For $Z \in \mathrm{Hilb}^n(X)^T$ and $Z \subseteq D \subseteq X$, Lemma 3.4 gives T-equivariant isomorphisms

$$\mathrm{Ext}^i_X(I_{Z,X}, I_{Z,X}) \cong \mathrm{Ext}^i_X(\mathcal{O}_Z, \mathcal{O}_Z), \text{ for } i = 1, 2, 3,$$
$$\mathrm{Ext}^i_D(I_{Z,D}, I_{Z,D}) \cong \mathrm{Ext}^i_D(\mathcal{O}_Z, \mathcal{O}_Z), \text{ for } i = 1, 2,$$

where the isomorphisms on D can be deduced similarly as for X.

From the T-equivariant distinguished triangle (e.g. [11, Cor. 11.4, pp. 248–249])

$$\mathbf{R}\mathrm{Hom}_D(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathbf{R}\mathrm{Hom}_D(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_D)[-1],$$

we obtain a T-equivariant exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_D(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}_D(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_D) \to \\ &\to \mathrm{Ext}^2_D(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1_D(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_D) \to \\ &\to \mathrm{Ext}^3_D(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^3_X(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2_D(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_D) \to 0. \end{aligned}$$

By T-equivariant Serre duality, this gives

$$\begin{aligned}\mathrm{Ext}^1_X - \mathrm{Ext}^2_X + \mathrm{Ext}^3_X = \mathrm{Ext}^1_D + (\mathrm{Ext}^1_D)^* - (\mathrm{Ext}^2_D + (\mathrm{Ext}^2_D)^*) \\ + H^0(D, \mathcal{O}_Z \otimes K_D) + H^0(D, \mathcal{O}_Z \otimes K_D)^* \in K_T(\bullet)\end{aligned}$$

in the T-equivariant K-theory of a point, where we abbreviate $\mathrm{Ext}^i_A := \mathrm{Ext}^i_A(\mathcal{O}_Z, \mathcal{O}_Z)$. For the corresponding Euler classes, we deduce

$$\frac{e_T(\mathrm{Ext}^1_X) \cdot e_T(\mathrm{Ext}^3_X)}{e_T(\mathrm{Ext}^2_X)} = (-1)^n \cdot \left(\frac{e_T(\mathrm{Ext}^1_D) \cdot e_T(H^0(D, \mathcal{O}_Z \otimes K_D))}{e_T(\mathrm{Ext}^2_D)} \right)^2 .$$

Therefore we have

$$\begin{aligned} &\frac{e_T(\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X}), Q) \cdot e_T(L^{[n]}|_Z)}{e_T(\mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X}))} \\ &= \frac{e_T(\mathrm{Ext}^2_X(I_{Z,X}, I_{Z,X}), Q) \cdot e_T(H^0(X, \mathcal{O}_Z \otimes \mathcal{O}_X(D)))}{e_T(\mathrm{Ext}^1_X(I_{Z,X}, I_{Z,X}))} \\ &= \frac{e_T(\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D}))}{e_T(\mathrm{Ext}^1_D(I_{Z,D}, I_{Z,D}))}, \end{aligned}$$

where we used (1.2) and $L|_D = K_D$ (X is Calabi–Yau). Moreover, the second equality is up to sign corresponding to the choice of orientation in defining the half Euler class.

Being a toric prime divisor, $D \subseteq X$ is itself a smooth toric 3-fold [10, Sect. 3.1]. As above, on any $(\mathbb{C}^*)^4$-invariant open $U \cong \mathbb{C}^4$ we can choose coordinates such that $t \cdot x_i = t_i x_i$, for all $i = 1, 2, 3, 4$, and $D \cap U = \{x_4 = 0\}$. In these coordinates, the torus of D is obtained from $T = \{t_1 t_2 t_3 t_4 = 1\}$ by setting $t_4 = 1$, i.e. at the level of equivariant parameters we have $\lambda_1 + \lambda_2 + \lambda_3 = \lambda_4 = 0$. We conclude that (3.9) becomes the T-equivariant Donaldson–Thomas invariants of n points on D which, by [17, II, Thm. 2], are equal to

$$\sum_{Z \in \mathrm{Hilb}^n(D)^T} \frac{e_T(\mathrm{Ext}^2_D(I_{Z,D}, I_{Z,D}))}{e_T(\mathrm{Ext}^1_D(I_{Z,D}, I_{Z,D}))} q^n = M(-q)^{\int_D c_3^T(TD \otimes K_D)}.$$

By the definition of equivariant push-forward (Definition 3.11), we have

$$\begin{aligned} \int_X c_3^T(X) \cdot c_1^T(L) &:= \sum_{x \in X^T} \frac{\iota_x^*(c_3^T(X) \cdot c_1^T(L))}{c_4^T(T_x X)} \\ &= \sum_{x \in X^T} \frac{c_3^T(T_x X) \cdot c_1^T(L|_x)}{c_4^T(T_x X)} \\ &= \sum_{x \in D^T} \frac{c_3^T(T_x X) \cdot c_1^T(L|_x)}{c_4^T(T_x X)}, \end{aligned}$$

where $\iota_x : \{x\} \times_T ET \to X \times_T ET$ is the natural inclusion and the last equality follows from Lemma 3.14 below. Similarly, we have

$$\int_D c_3^T(TD \otimes K_D) := \sum_{x \in D^T} \frac{\iota_x^*(c_3^T(TD \otimes K_D))}{c_3^T(T_x D)}$$
$$= \sum_{x \in D^T} \frac{c_3^T(T_x D \otimes K_D|_x)}{c_3^T(T_x D)}.$$

From the T-equivariant short exact sequence

$$0 \to TD \to TX|_D \to K_D \to 0,$$

we obtain

$$c_3^T(T_x X) = c_3^T(T_x D) + c_2^T(T_x D) \cdot c_1^T(K_D|_x), \quad c_4^T(T_x X) = c_3^T(T_x D) \cdot c_1^T(K_D|_x),$$
$$c_3^T(T_x D \otimes K_D|_x)$$
$$= c_3^T(T_x D) + c_2^T(T_x D) \cdot c_1^T(K_D|_x) + c_1^T(T_x D) \cdot c_1^T(K_D|_x)^2 + c_1^T(K_D|_x)^3.$$

Since $K_D|_x = \wedge^3 T_x^* D$, we have

$$c_1^T(T_x D) \cdot c_1^T(K_D|_x)^2 + c_1^T(K_D|_x)^3 = (c_1^T(T_x D) + c_1^T(K_D|_x)) \cdot c_1^T(K_D|_x)^2 = 0$$

and therefore $\int_X c_3^T(X) \cdot c_1^T(L) = \int_D c_3^T(TD \otimes K_D)$ for $L = \mathcal{O}_X(D)$. □

In order to prove (3.8), let $X = \mathbb{C}^4$ with coordinates x_1, x_2, x_3, x_4 such that the action of $t \in (\mathbb{C}^*)^4$ satisfies

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4,$$

and the $(\mathbb{C}^*)^4$-equivariant line bundle L is given by

$$D := \{x_4 = 0\} \subseteq \mathbb{C}^4 \text{ and } L := \mathcal{O}(D).$$

Lemma 3.14. *We have a $(\mathbb{C}^*)^4$-equivariant isomorphism $L^{[n]} \cong \mathcal{O}^{[n]} \otimes t_4^{-1}$. Moreover, for any $Z \in \mathrm{Hilb}^n(\mathbb{C}^4)^T$ such that Z does not lie scheme theoretically in D, we have*

$$e_T(L^{[n]}|_Z) = 0.$$

Proof. Consider the ideal sheaf $\mathcal{O}(-D) \subseteq \mathcal{O}$. This corresponds to the inclusion

$$(x_4) \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$$

and therefore $\mathcal{O}(-D) \cong \mathcal{O} \otimes t_4$ and $L \cong \mathcal{O} \otimes t_4^{-1}$. The fibres of $L^{[n]}$ are given by

$$L^{[n]}|_Z \cong H^0(L|_Z) \cong H^0(\mathcal{O}_Z) \otimes t_4^{-1},$$

where all isomorphisms are $(\mathbb{C}^*)^4$-equivariant isomorphisms. Hence, we have a $(\mathbb{C}^*)^4$-equivariant isomorphism

$$L^{[n]} \cong \mathcal{O}^{[n]} \otimes t_4^{-1}.$$

Now suppose $Z \in \mathrm{Hilb}^n(\mathbb{C}^4)$ is a T-fixed (and therefore $(\mathbb{C}^*)^4$-fixed) element. Then Z corresponds to a solid partitions $\pi = \{\pi_{ijk}\}_{i,j,k \geqslant 1}$. Suppose $Z \not\subseteq D$, i.e. Z is not scheme theoretically contained in D, then $(x_4) \not\subseteq I_Z$. Therefore, $\pi_{111} > 1$ and the class of Z in the $(\mathbb{C}^*)^4$-equivariant K-group $K_{(\mathbb{C}^*)^4}(\bullet)$ contains the term t_4. Hence

$$\begin{aligned} e_{(\mathbb{C}^*)^4}(L^{[n]}|_Z) &= e_{(\mathbb{C}^*)^4}(Z \otimes t_4^{-1}) = e_{(\mathbb{C}^*)^4}(1 + \text{other terms}) \\ &= e_{(\mathbb{C}^*)^4}(1)\, e_{(\mathbb{C}^*)^4}(\text{other terms}) = 0. \end{aligned}$$

This equality holds for T-equivariant Euler classes as well, which corresponds to setting $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$. □

3.3. Vertex formalism

In order to prove Conjecture 3.12, it is in fact enough to prove it for affine space $\mathbb{C}^4$. In this section, we develop the necessary vertex formalism from which this follows. We follow the original arguments developed in the 3-dimensional case by MNOP [17] very closely.

Let X be a smooth quasi-projective toric Calabi–Yau 4-fold and let $\{U_\alpha\}$ be the cover by maximal $(\mathbb{C}^*)^4$-invariant affine open subsets. Let $Z \subseteq X$ be a T-invariant zero-dimensional subscheme (hence also $(\mathbb{C}^*)^4$-invariant by Lemma 3.1). For each α, the restriction $Z_\alpha := Z|_{U_\alpha}$ corresponds to a solid partitions $\pi^{(\alpha)}$, as described previously, and we write

$$I_\alpha := I_{Z_{\pi^{(\alpha)}}}.$$

By footnote 3, we have T-equivariant trace maps and we can take the trace-free part

$$-\mathbf{R}\mathrm{Hom}_X(I_Z, I_Z)_0 \in K_T(\bullet).$$

Denote the global section functor by $\Gamma(-)$. The local-to-global spectral sequence and calculation of sheaf cohomology with respect to the Čech cover $\{U_\alpha\}$ yields

$$-\mathbf{R}\mathrm{Hom}_X(I_Z, I_Z)_0 = \sum_{\alpha,i} (-1)^i \Big(\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}) - \Gamma(U_\alpha, \mathcal{E}xt^i(I_\alpha, I_\alpha)) \Big).$$

Here we use $H^{>0}(U_\alpha, -) = 0$, because U_α is affine. We also use that intersections $U_\alpha \cap U_\beta \cap \cdots$, with $\alpha \neq \beta$, do not contribute because Z is zero-dimensional and therefore

$$I_Z|_{U_\alpha \cap U_\beta \cap \cdots} = \mathcal{O}_{U_\alpha \cap U_\beta \cap \cdots}.$$

This reduced the calculation to

$$-\mathbf{R}\mathrm{Hom}_{U_\alpha}(I_\alpha, I_\alpha)_0 = \sum_i (-1)^i \Big(\Gamma(U_\alpha, \mathcal{O}_{U_\alpha}) - \Gamma(U_\alpha, \mathcal{E}xt^i(I_\alpha, I_\alpha)) \Big).$$

On $U_\alpha \cong \mathbb{C}^4$, we use coordinates x_1, x_2, x_3, x_4 such that the $(\mathbb{C}^*)^4$-action is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4.$$

Let $U := U_\alpha$, $Z := Z_\alpha$, $I := I_\alpha$, $\pi := \pi^{(\alpha)}$, and $R := \Gamma(\mathcal{O}_{U_\alpha}) \cong \mathbb{C}[x_1, x_2, x_3, x_4]$. Consider class $[I]$ in the equivariant K-group $K_{(\mathbb{C}^*)^4}(U)$. By identifying $[R]$ with 1, we obtain a ring isomorphism

$$K_{(\mathbb{C}^*)^4}(U) \cong \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm, t_4^\pm].$$

The Laurent polynomial $\mathsf{P}(I)$ corresponding to $[I]$ under this isomorphism is called the Poincaré polynomial of I. For any $w = (w_1, w_2, w_3, w_4) \in \mathbb{Z}^4$, we use multi-index notation

$$t^w := t_1^{w_1} t_2^{w_2} t_3^{w_3} t_4^{w_4}.$$

Then $[R \otimes t^w] \in K_{(\mathbb{C}^*)^4}(U)$ corresponds to $t^w \in \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm, t_4^\pm]$.

Define an involution $\overline{(\cdot)}$ on $K_{(\mathbb{C}^*)^4}(U)$ by $\mathbb{Z}$-linear extension of

$$\overline{t^w} := t^{-w}.$$

By definition, the trace map

$$\mathrm{tr} : K_{(\mathbb{C}^*)^4}(U) \to \mathbb{Z}(\!(t_1, t_2, t_3, t_4)\!)$$

corresponds to $(\mathbb{C}^*)^4$-equivariant restriction to the fixed point of U.

Take a $(\mathbb{C}^*)^4$-equivariant graded free resolution

$$0 \to F_s \to \cdots \to F_0 \to I \to 0,$$

as in [17], where

$$F_i = \bigoplus_j R \otimes t^{d_{ij}},$$

for certain $d_{ij} \in \mathbb{Z}^4$. Then

$$\mathsf{P}(I) = \sum_{i,j} (-1)^i t^{d_{ij}}. \tag{3.10}$$

The $(\mathbb{C}^*)^4$-character of $\mathcal{O}_Z$ is given by (3.1) and can be expressed in terms of the Poincaré polynomial of I as follows

$$Z = \sum_{i,j,k \geqslant 1} \sum_{l=1}^{\pi_{ijk}} t_1^{i-1} t_2^{j-1} t_3^{k-1} t_4^{l-1} = \mathrm{tr}(\mathcal{O}_U - I) = \frac{1 - \mathsf{P}(I)}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)}. \tag{3.11}$$

We deduce

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_U(I, I) &= \sum_{i,j,k,l} (-1)^{i+k} \mathrm{Hom}(R \otimes t^{d_{ij}}, R \otimes t^{d_{kl}}) \\
&= \sum_{i,j,k,l} (-1)^{i+k} R \otimes t^{d_{kl} - d_{ij}} \\
&= \mathsf{P}(I)\overline{\mathsf{P}(I)} \\
\mathrm{tr}_{\mathbf{R}\mathrm{Hom}_U(I,I)} &= \frac{\mathsf{P}(I)\overline{\mathsf{P}(I)}}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)},
\end{aligned}$$

where we used (3.10) for the third equality. Eliminating $\mathsf{P}(I)$ by using (3.11), the trace of $-\mathbf{R}\mathrm{Hom}_{U_\alpha}(I_\alpha, I_\alpha)_0$ is then given by

$$\mathsf{V}_\alpha := Z_\alpha + \frac{\overline{Z}_\alpha}{t_1 t_2 t_3 t_4} - \frac{Z_\alpha \overline{Z}_\alpha (1-t_1)(1-t_2)(1-t_3)(1-t_4)}{t_1 t_2 t_3 t_4}, \tag{3.12}$$

where we re-introduced the index α. Summing up, we have proved the following lemma:

Lemma 3.15. *Let $Z \subseteq X$ be a T-fixed zero-dimensional subscheme. Then*

$$\mathrm{tr}_{-\mathbf{R}\mathrm{Hom}_X(I_Z, I_Z)_0} = \sum_\alpha \mathrm{tr}_{-\mathbf{R}\mathrm{Hom}_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})_0} = \sum_\alpha \mathsf{V}_\alpha,$$

where the equivariant vertex V_α is defined by (3.12).

For a fixed α, after specialisation $t_1 t_2 t_3 t_4 = 1$, we have

$$\begin{aligned}
\mathsf{V}_\alpha &= \mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}) + \mathrm{Ext}^3_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}) - \mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}) \\
&= \mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}) + \mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})^* - \mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}),
\end{aligned}$$

where each $\mathrm{Ext}^i_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})$, with $i \neq 0$, is a finite-dimensional T-representation by Lemma 3.4 and $\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})$ is self-dual. Consequently

$$e_T(-\mathsf{V}_\alpha) = (-1)^{\dim_{\mathbb{C}} \mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})} \cdot \frac{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))}{e_T(\mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))^2}.$$

Since the Serre duality pairing on $\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})$ is T-invariant, there exists a half Euler class $e_T\big(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q\big)$ as in (3.7). By its property (2.4), we know

$$e_T\big(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q\big)^2 = (-1)^{\frac{1}{2}\dim_{\mathbb{C}} \mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})} \cdot e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha})).$$

Denoting the length of the zero-dimensional subscheme Z_α by n_α and using $\chi(\mathcal{O}_{U_\alpha}) - \chi(I_\alpha, I_\alpha) = 2n_\alpha$, we obtain

$$e_T(-\mathsf{V}_\alpha) = (-1)^{n_\alpha} \cdot \left(\frac{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q)}{e_T(\mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))} \right)^2. \tag{3.13}$$

Definition 3.16. Let π be a solid partition of size $|\pi|$ and let V_π be the expression defined by (3.12), where Z is the T-invariant zero-dimensional subscheme determined by (3.11). We define

$$\mathsf{w}_\pi := \pm\sqrt{(-1)^{|\pi|} \cdot e_T(-\mathsf{V}_\pi)} \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4),$$

i.e. the square root of $(-1)^{|\pi|}$ times (3.13). We only define w_π up to a sign $\pm$.

From Lemma 3.15 and Definition 3.16, we conclude:

Proposition 3.17. *Let $Z \subseteq X$ be a T-fixed zero-dimensional subscheme. Suppose the restriction $Z|_{U_\alpha} \subseteq U_\alpha$ corresponds to a solid partition $\pi^{(\alpha)}$. Then*

$$\frac{e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big)}{e_T\big(\mathrm{Ext}^1(I_Z, I_Z)\big)} = \pm \prod_\alpha \mathsf{w}_{\pi^{(\alpha)}}.$$

Insertions Let L be a $(\mathbb{C}^*)^4$-equivariant line bundle on X. For each α, there exists a character $d^{(\alpha)} = (d_1^{(\alpha)}, d_2^{(\alpha)}, d_3^{(\alpha)}, d_4^{(\alpha)}) \in \mathbb{Z}^4$ such that

$$L|_{U_\alpha} = \mathcal{O}_{U_\alpha} \otimes t^{d^{(\alpha)}}.$$

As above, write $U := U_\alpha$, $d := d^{(\alpha)}$, and suppose we have the standard torus action $t \cdot x_i = t_i x_i$ for all $i = 1, 2, 3, 4$. Let $Z \subseteq U$ be a 0-dimensional T-fixed subscheme corresponding to a solid partition π. Then we define

$$L_\pi(d_1, d_2, d_3, d_4) := e_T\big(H^0(U, \mathcal{O}_Z \otimes L|_U)\big) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4),$$

where

$$H^0(U, \mathcal{O}_Z \otimes L|_U) = \sum_{i,j,k \geqslant 1} \sum_{l=1}^{\pi_{ijk}} t_1^{d_1+i-1} t_2^{d_2+j-1} t_3^{d_3+k-1} t_4^{d_4+l-1}.$$

Then for any $Z \subseteq X$ we have

$$e_T(L^{[n]})|_Z = \prod_{\alpha} L_{\pi^{(\alpha)}}(d_1^{(\alpha)}, d_2^{(\alpha)}, d_3^{(\alpha)}, d_4^{(\alpha)}).$$

Example 3.18. Let $Z_\pi = 1 + t_1 + t_4$. The corresponding solid partition π satisfies

$$\pi_{111} = 2, \quad \pi_{211} = 1, \quad \pi_{ijk} = 0, \text{ otherwise}.$$

Hence $I_{Z_\pi} = \langle x_1^2, x_1x_4, x_4^2, x_2, x_3\rangle$. After specialisation $t_1t_2t_3t_4 = 1$, we get

$$\begin{aligned}\mathsf{V}_\pi =& \Big(t_1^3t_2^2t_3^2 - t_1^3t_2^2t_3 - t_1^3t_2t_3^2 + t_1^3t_2t_3 - t_1t_2^2t_3^2 + t_1t_2^2t_3 + t_1t_2t_3^2 \\ &+ 2t_1t_2t_3 - 2t_1t_2 + 2t_1 + t_1t_3^{-1} + t_1t_2^{-1} - 2t_1t_3 - t_1t_2^{-1}t_3^{-1} + t_2 + t_3 - t_2t_3\Big) + \\ &\Big(t_1^{-3}t_2^{-2}t_3^{-2} - t_1^{-3}t_2^{-2}t_3^{-1} - t_1^{-3}t_2^{-1}t_3^{-2} + t_1^{-3}t_2^{-1}t_3^{-1} - t_1^{-1}t_2^{-2}t_3^{-2} + t_1^{-1}t_2^{-2}t_3^{-1} \\ &+ t_1^{-1}t_2^{-1}t_3^{-2} + 2t_1^{-1}t_2^{-1}t_3^{-1} - 2t_1^{-1}t_2^{-1} + 2t_1^{-1} + t_1^{-1}t_3 + t_1^{-1}t_2 - 2t_1^{-1}t_3^{-1} \\ &- t_1^{-1}t_2t_3 + t_2^{-1} + t_3^{-1} - t_2^{-1}t_3^{-1}\Big),\end{aligned}$$

where all terms come in Serre dual pairs. One readily calculates

$$\begin{aligned}\mathsf{w}_\pi =& \pm\big((\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2(\lambda_2+\lambda_3)(\lambda_1-\lambda_2-\lambda_3)(\lambda_1+2\lambda_2+2\lambda_3) \\ &\cdot(3\lambda_1+2\lambda_2+\lambda_3)(3\lambda_1+\lambda_2+2\lambda_3)\big) \\ &\times\big(\lambda_1^2\lambda_2\lambda_3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_1+2\lambda_2+\lambda_3) \\ &\cdot(\lambda_1+\lambda_2+2\lambda_3)(3\lambda_1+\lambda_2+\lambda_3)(3\lambda_1+2\lambda_2+2\lambda_3)\big)^{-1}, \\ L_\pi(d_1,d_2,d_3,d_4) =& \big((d_1-d_4)\lambda_1 + (d_2-d_4)\lambda_2 + (d_3-d_4)\lambda_3\big) \\ &\cdot\big((d_1-d_4+1)\lambda_1 + (d_2-d_4)\lambda_2 + (d_3-d_4)\lambda_3\big) \\ &\cdot\big((d_1-d_4-1)\lambda_1 + (d_2-d_4-1)\lambda_2 + (d_3-d_4-1)\lambda_3\big),\end{aligned}$$

where we used $\lambda_4 = -\lambda_1 - \lambda_2 - \lambda_3$.

The following conjecture is a *combinatorial version* of Conjecture 3.12 when $X = \mathbb{C}^4$.

Conjecture 3.19. *There exists a way of choosing the signs for the equivariant weights* w_π *in Definition 3.16 such that the following identity holds in* $\frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}(d_1,d_2,d_3,d_4)[\![q]\!]$

$$\sum_\pi L_\pi(d_1,d_2,d_3,d_4)\,\mathsf{w}_\pi\, q^{|\pi|} = M(-q)^{\frac{(d_1\lambda_1+d_2\lambda_2+d_3\lambda_3+d_4\lambda_4)(-\lambda_1\lambda_2\lambda_3-\lambda_1\lambda_2\lambda_4-\lambda_1\lambda_3\lambda_4-\lambda_2\lambda_3\lambda_4)}{\lambda_1\lambda_2\lambda_3\lambda_4}},$$

where the sum is over all solid partitions and $M(q)$ *denotes the MacMahon function.*

Combining Conjecture 3.19 with the vertex formalism, we can deduce Conjecture 3.12.

Proposition 3.20. *Conjecture 3.19 is equivalent to Conjecture 3.12.*

Proof. Conjecture 3.19 is a special case of Conjecture 3.12 when $X = \mathbb{C}^4$. Conversely, assuming Conjecture 3.19 is true, we want to prove Conjecture 3.12.

Let X be a smooth quasi-projective toric Calabi–Yau 4-fold with $(\mathbb{C}^*)^4$-equivariant line bundle L. Let $\{U_\alpha\}_{\alpha=1,\dots e}$ be the cover by maximal open affine $(\mathbb{C}^*)^4$-invariant subsets. Suppose $(\mathbb{C}^*)^4$ acts on the coordinates of $U_\alpha \cong \operatorname{Spec} \mathbb{C}[x_1^{(\alpha)}, x_2^{(\alpha)}, x_3^{(\alpha)}, x_4^{(\alpha)}]$ by

$$t \cdot x_i^{(\alpha)} = \chi_i^{(\alpha)}(t)\, x_i^{(\alpha)}, \quad \text{for all } i = 1,2,3,4,$$

for certain characters $\chi_i^{(\alpha)} : (\mathbb{C}^*)^4 \to \mathbb{C}^*$. If $\chi_i^{(\alpha)}(t) = t_i$ is the standard torus action, then

$$\begin{aligned} &\frac{c_1^T(L|_{p_\alpha})\, c_3^T(TU_\alpha|_{p_\alpha})}{c_4^T(TU_\alpha|_{p_\alpha})} \\ &\quad = \frac{(d_1\lambda_1 + d_2\lambda_2 + d_3\lambda_3 + d_4\lambda_4)(-\lambda_1\lambda_2\lambda_3 - \lambda_1\lambda_2\lambda_4 - \lambda_1\lambda_3\lambda_4 - \lambda_2\lambda_3\lambda_4)}{\lambda_1\lambda_2\lambda_3\lambda_4}, \end{aligned}$$

where $p_\alpha = (0,0,0,0) \in U_\alpha$ is the unique $(\mathbb{C}^*)^4$-fixed point. For other characters, the RHS gets adapted accordingly. We deduce

$$\begin{aligned} &\sum_{n=0}^{\infty} \mathrm{DT}_4(X, T, L, n\,; o(\mathcal{L}))\, q^n \\ &= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Hilb}^n(X)^T} \frac{e_T(\mathrm{Ext}^2_X(I_Z, I_Z), Q) \cdot e_T(L^{[n]}|_Z)}{e_T(\mathrm{Ext}^1_X(I_Z, I_Z))} \\ &= \sum_{n=0}^{\infty} q^n \sum_{Z \in \mathrm{Hilb}^n(X)^{(\mathbb{C}^*)^4}} \frac{e_T(\mathrm{Ext}^2_X(I_Z, I_Z), Q) \cdot e_T(L^{[n]}|_Z)}{e_T(\mathrm{Ext}^1_X(I_Z, I_Z))} \\ &= \sum_{n_1=0}^{\infty} \sum_{Z_1 \in \mathrm{Hilb}^{n_1}(U_1)^{(\mathbb{C}^*)^4}} \cdots \\ &\quad \sum_{n_e=0}^{\infty} \sum_{Z_e \in \mathrm{Hilb}^{n_e}(U_e)^{(\mathbb{C}^*)^4}} \prod_{\alpha=1}^{e} q^{n_\alpha} \frac{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q) \cdot e_T(L^{[n_\alpha]}|_{Z_\alpha})}{e_T(\mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))} \\ &= \prod_\alpha \sum_{n_\alpha=0}^{\infty} q^{n_\alpha} \sum_{Z_\alpha \in \mathrm{Hilb}^{n_\alpha}(U_\alpha)^{(\mathbb{C}^*)^4}} \frac{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q) \cdot e_T(L^{[n]}|_{Z_\alpha})}{e_T(\mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))} \\ &= \prod_\alpha \sum_{\text{solid partitions } \pi^{(\alpha)}} L_{\pi^{(\alpha)}}(d_1^{(\alpha)}, d_2^{(\alpha)}, d_3^{(\alpha)}, d_4^{(\alpha)})\, \mathsf{w}_{\pi^{(\alpha)}}\, q^{|\pi^{(\alpha)}|} \end{aligned}$$

$$= \prod_\alpha M(-q)^{\frac{c_1^T(L|_{p_\alpha})c_3^T(TU_\alpha|_{p_\alpha})}{c_4^T(TU_\alpha|_{p_\alpha})}} = M(-q)^{\sum_\alpha \frac{c_1^T(L|_{p_\alpha})c_3^T(TU_\alpha|_{p_\alpha})}{c_4^T(TU_\alpha|_{p_\alpha})}} = M(-q)^{\int_X c_1^T(L)c_3^T(T_X)}.$$

Here for each $Z \in \mathrm{Hilb}^n(X)^T$, the signs of $e_T(\mathrm{Ext}^2_X(I_Z, I_Z), Q)$ are induced from the choice of signs of $\{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q)\}_\alpha$ when taking the square root of the following equation

$$(-1)^{\frac{\chi(I_Z,I_Z)_0}{2}} \frac{e_T(\mathrm{Ext}^2_X(I_Z, I_Z))}{e_T(\mathrm{Ext}^1_X(I_Z, I_Z))\, e_T(\mathrm{Ext}^3_X(I_Z, I_Z))}$$
$$= \prod_\alpha (-1)^{\frac{\chi(I_{Z_\alpha},I_{Z_\alpha})_0}{2}} \frac{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))}{e_T(\mathrm{Ext}^1_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))\, e_T(\mathrm{Ext}^3_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}))}.$$

In turn, the signs of $\{e_T(\mathrm{Ext}^2_{U_\alpha}(I_{Z_\alpha}, I_{Z_\alpha}), Q)\}_\alpha$ are determined by the signs of $\{\mathsf{w}_{\pi(\alpha)}\}_\alpha$ provided by Conjecture 3.19 (via Definition 3.16 and Proposition 3.17). □

We implemented the calculation of w_π in Definition 3.16 into a Maple program. Using this in the context of Conjecture 3.19 leads us to conjecture the following:

Conjecture 3.21. *There exists a* unique *way of choosing the signs for the equivariant weights* w_π *such that Conjecture 3.19 holds.*

Using our Maple program, we checked the following:

Theorem 3.22. *Conjectures 3.19 and 3.21 are true modulo* q^7*.*

Remark 3.23. A priori there are many possible choices of orientation, i.e. signs for w_π, in Conjecture 3.19. E.g. there are 140 solid partitions of size 6, so in this case there are $2^{140} \approx 10^{42}$ choices! However, we have a (conjectural) very quick way of finding orientations which work. In fact, Conjecture 4.1 of the next section asserts that the specialisation $L_\pi(0, 0, 0, -d)\,\mathsf{w}_\pi$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$ is well-defined (and we check this in many cases). This specialisation is conjecturally equal to $(-1)^{|\pi|} \prod_{l=1}^{\pi_{111}} (d - (l - 1))$ times a non-zero rational number. By choosing the sign of w_π in such a way that this rational number is *positive*, we end up with *existence* of a collection of signs for which Conjecture 3.19 holds in the cases that we checked, i.e. modulo q^7. For order q^6, the calculation can be efficiently organised by comparing the coefficients of each monomial $d_1^{i_1} d_2^{i_2} d_3^{i_3} d_4^{i_4}$ separately.

Remark 3.24. For orders $q^{\leqslant 3}$ we check brute force that the choices of orientation, i.e. signs for w_π, in Conjecture 3.19 are *unique*. For orders q^4, q^5, q^6, we first specialise to $d_1 = d_2 = d_3 = 0$, $d_4 = -d$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (after observing that this specialisation is well-defined) in which case LHS and RHS of Conjecture 3.19 become polynomials of degree $\delta = 4, 5, 6$ respectively. We then compare the coefficients of the terms of the polynomials starting with the leading term: $d^\delta, d^{\delta-1}, \cdots, d$. It turns out that each comparison uniquely determines some of the signs. E.g. for q^6, comparing the coefficients of d^6 fixes 1 sign, comparing the coefficients of d^5 fixes 3 further signs, comparing the coefficients of d^4 fixes 9 further signs, comparing the coefficients of d^3 fixes 25 further signs, comparing the coefficients of d^2 fixes 54 further signs, and comparing the coefficients of d fixes the last 48 signs.

4. Application to counting solid partitions

4.1. Weighted count of solid partitions

In this section, we study Conjecture 3.19 for a special choice of insertions

$$(d_1, d_2, d_3, d_4) = (0, 0, 0, -d), \quad d \geqslant 1.$$

This has applications to enumerating solid partitions.

For a solid partition $\pi = \{\pi_{ijk}\}_{i,j,k\geqslant 1}$, we refer to π_{111} as its *height*. By experimental study of many examples (i.e. Proposition 4.2), we pose the following conjecture:

Conjecture 4.1. *Let π be a solid partition and let w_π be defined using the unique sign in Conjecture 3.21. Then the following properties hold:*

(a) $L_\pi(0,0,0,-d)\,\mathsf{w}_\pi \in \frac{\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,d)}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)}$ *has no pole at* $\lambda_4 = -(\lambda_1+\lambda_2+\lambda_3)$.

(b) *The specialisation* $L_\pi(0,0,0,-d)\,\mathsf{w}_\pi\Big|_{\lambda_1+\lambda_2+\lambda_3=0}$ *is independent of* $\lambda_1, \lambda_2, \lambda_3$.

(c) *More precisely, there exists a rational number $\omega_\pi \in \mathbb{Q}_{>0}$ (independent of d) such that*

$$L_\pi(0,0,0,-d)\,\mathsf{w}_\pi\Big|_{\lambda_1+\lambda_2+\lambda_3=0} = (-1)^{|\pi|}\,\omega_\pi \prod_{l=1}^{\pi_{111}} (d-(l-1)). \tag{4.1}$$

In particular, for $d \in \mathbb{Z}_{>0}$, the LHS vanishes when $\pi_{111} > d$ and otherwise has the same sign as $(-1)^\pi$.

Geometrically, this specialisation corresponds to taking $X = \mathbb{C}^4$ and $D = \{x_4^d = 0\} \subseteq \mathbb{C}^4$. Then $L = \mathcal{O}(D) \cong \mathcal{O} \otimes t_4^{-d}$. As we have seen in Proposition 2.4, the canonical section of $L^{[n]}$ on $\mathrm{Hilb}^n(\mathbb{C}^4)$ cuts out the sublocus of zero-dimensional subschemes Z contained in D. At the level of T-fixed (and therefore $(\mathbb{C}^*)^4$-fixed) points, this means we are considering solid partitions π of height $\pi_{111} \leqslant d$. This is the geometric motivation for the specialisation of Conjecture 4.1.

We give the following evidence for Conjecture 4.1:

Proposition 4.2.

- *Conjecture 4.1 is true for any solid partition π of size $|\pi| \leqslant 6$.*
- *Properties (a), (b), and the absolute value of equation (4.1) hold for $d = 1$ and any solid partition π satisfying $\pi_{111} = 1$ (in this case $|\omega_\pi| = 1$).*
- *Properties (a), (b), and the absolute value of equation (4.1) hold for various individual solid partitions of size $\leqslant 15$ listed in Appendix A.*

Proof. The second statement follows from Theorem 3.13 and [17, I, Sect. 4]. For the other cases, we use our Maple program, which calculates w_π for any given solid partition π. For the first statement, we use the unique choice of signs that we found when verifying Conjecture 3.21 (Theorem 3.22). □

Combining Conjectures 4.1 and 3.19, we obtain a generating function counting *weighted* solid partitions:

Theorem 4.3. *Assume Conjectures 3.19 and 4.1 are true. Then*

$$\sum_{\pi} \omega_\pi \, t^{\pi_{111}} \, q^{|\pi|} = e^{t(M(q)-1)}, \tag{4.2}$$

where the sum is over all solid partitions, t is a formal variable, and $M(q)$ denotes the MacMahon function. In particular, when $t = 1$, we have

$$\sum_{\pi} \omega_\pi \, q^{|\pi|} = e^{M(q)-1}.$$

Proof. Consider Conjecture 3.19 for $d_1 = d_2 = d_3 = 0$, $d_4 = -d$, and the specialisation

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Then the power of $M(-q)$ in Conjecture 3.19 becomes d. According to Conjecture 4.1, this specialisation is well-defined and we get

$$\sum_{\pi} \omega_\pi \cdot \left(\prod_{l=1}^{\pi_{111}} (d - (l-1)) \right) q^{|\pi|} = M(q)^d,$$

for any $d \geqslant 1$. Then it is easy to see that

$$1 + \sum_{\pi_{111}=1} \omega_\pi \, q^{|\pi|} = M(q),$$

$$1 + 2 \sum_{\pi_{111}=1} \omega_\pi \, q^{|\pi|} + 2! \sum_{\pi_{111}=2} \omega_\pi \, q^{|\pi|} = M(q)^2,$$

$$1 + 3 \sum_{\pi_{111}=1} \omega_\pi \, q^{|\pi|} + 3 \times 2 \sum_{\pi_{111}=2} \omega_\pi \, q^{|\pi|} + 3! \sum_{\pi_{111}=3} \omega_\pi \, q^{|\pi|} = M(q)^3,$$

$$\cdots$$

$$1 + \sum_{i=1}^{k} \frac{k!}{(k-i)!} \sum_{\pi_{111}=i} \omega_\pi \, q^{|\pi|} = M(q)^k, \quad k \geqslant 1.$$

Rearranging gives

$$t \sum_{\pi_{111}=1} \omega_\pi \, q^{|\pi|} = t(M(q) - 1),$$

$$t^2 \sum_{\pi_{111}=2} \omega_\pi \, q^{|\pi|} = \frac{t^2}{2}\big(M(q)^2 - 2M(q) + 1\big),$$

$$t^3 \sum_{\pi_{111}=3} \omega_\pi \, q^{|\pi|} = \frac{t^3}{3!}\big(M(q)^3 - 3M(q)^2 + 3M(q) - 1\big),$$

$$\cdots$$

$$t^k \sum_{\pi_{111}=k} \omega_\pi \, q^{|\pi|} = \frac{t^k}{k!}\big(M(q) - 1\big)^k, \quad k \geqslant 1,$$

whose summation gives the equality we want. □

Remark 4.4. Counting solid partitions is a very difficult question. In fact, MacMahon initially proposed an *incorrect* formula for its generating function [1]

$$\sum_{\pi} q^{|\pi|} \overset{?}{=} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{\frac{1}{2}n(n+1)}}.$$

Exact enumeration using computers also does not go very far. As Stanley wrote in his PhD thesis [21][8]

> "The case $r = 2$ has a well-developed theory — here 2-dimensional partitions are known as plane partitions. (...) For $r \geqslant 3$, almost nothing is known and (...) casts only a faint glimmer of light on a vast darkness."

We find that a specialisation of the weights $L(d_1, d_2, d_3, d_4)_\pi \, \mathsf{w}_\pi$, coming naturally from DT_4 theory, gives a weighted count of solid partitions with a nice closed formula (4.2). Of course, one can always find ω_π such that (4.2) holds (e.g. simply by expanding the RHS of (4.2) and giving all solid partitions of the same size and height an equal weight).

[8] This quote is taken from slides of a talk by S. Govindarajan, Aspects of Mathematics, IMSc, Chennai (2014).

Below we will find an explicit (conjectural) formula of ω_π for any solid partition π (see Conjecture 4.13 and Proposition 4.14). In terms of this explicit formula, it actually becomes rather elementary to prove the counterpart of Theorem 4.3 (i.e. Proposition 4.11). Nevertheless, we find it interesting that such weights ω_π naturally arise from DT_4 theory, even though they may have limited combinatorial interest.

4.2. Combinatorial approach to ω_π

In this section, we assign an explicit weight ω_π^c to any solid partition (Definition 4.7). Firstly, we unconditionally prove the analogue of Theorem 4.3 with ω_π replaced by ω_π^c (Proposition 4.11). Secondly, an obvious generalisation of Proposition 4.11 turns out to hold for partitions of any dimension d (Remark 4.12). Thirdly, we conjecture $\omega_\pi = \omega_\pi^c$, for any solid partition π, and we verify this in many examples (Conjecture 4.13 and Proposition 4.14).

Definition 4.5. Let $\xi = \{\xi_{ij}\}_{i,j \geqslant 1}$ be a plane partition, i.e. a sequence of non-negative integers satisfying

$$\xi_{ij} \geqslant \xi_{i+1,j}, \quad \xi_{ij} \geqslant \xi_{i,j+1}, \quad \forall\, i,j \geqslant 1,$$
$$|\xi| := \sum_{i,j} \xi_{ij} < \infty.$$

We define the *binary representation* of ξ to be the sequence of integers $\{\xi(i,j,k)\}_{i,j,k \geqslant 1}$ given by

$$\xi(i,j,k) := \begin{cases} 1 & \text{if } k \leqslant \xi_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.6. Suppose ξ is given by $\xi_{11} = 2$, $\xi_{21} = 1$, $\xi_{12} = 1$. Then $\xi(1,1,1) = \xi(1,1,2) = \xi(2,1,1) = \xi(1,2,1) = 1$ and $\xi(i,j,k) = 0$ for all other $i,j,k \geqslant 1$.

Definition 4.7. Let $\pi = \{\pi_{ijk}\}_{i,j,k \geqslant 1}$ be a (non-empty) solid partition and consider all possible sequences of integers $\{m_\xi\}_\xi$, where the index ξ runs over all (non-empty) plane partitions and $m_\xi \in \mathbb{Z}_{\geqslant 0}$. Define the following collection

$$\mathcal{C}_\pi := \left\{ \{m_\xi\}_\xi \;\middle|\; \pi_{ijk} = \sum_\xi m_\xi \cdot \xi(i,j,k) \text{ for all } i,j,k \right\}. \tag{4.3}$$

We define

$$\omega_\pi^c := \sum_{\{m_\xi\}_\xi \in \mathcal{C}_\pi} \prod_\xi \frac{1}{(m_\xi)!}. \tag{4.4}$$

For the empty solid partition $\pi = \varnothing$ we define $\omega^c_\pi := 1$.

Remark 4.8. For each $\{m_\xi\}_\xi \in \mathcal{C}_\pi$, we have

$$|\pi| = \sum_\xi m_\xi \cdot |\xi|.$$

Hence, $m_\xi = 0$ if $|\xi|$ is large. Therefore, the collection $\mathcal{C}_\pi$ is a finite set and, for each $\{m_\xi\}_\xi \in \mathcal{C}_\pi$, there are only finitely many nonzero m_ξ.

Example 4.9. Suppose $\pi = \{\pi_{ijk}\}_{i,j,k \geqslant 1}$ satisfies $\pi_{ijk} = 0$ unless $i = j = 1$. Then

$$\omega^c_\pi = \prod_{k=1}^\infty \frac{1}{(\pi_{11k} - \pi_{11,k+1})!}.$$

This is due to the fact that the only plane partitions $\xi = \{\xi_{ijk}\}_{i,j,k \geqslant 1}$ contributing to the defining equation in (4.3) satisfy $\xi(i,j,k) = 0$ unless $i = j = 1$. Define $\xi^{(n)}$ to be the plane partition with binary representation satisfying $\xi(1,1,k) = 1$ for all $1 \leqslant k \leqslant n$ and $\xi(i,j,k) = 0$ otherwise. Then $\mathcal{C}_\pi$ only consists of one element $\{m_\xi\}_\xi$:

$$m_\xi = \begin{cases} \pi_{11k} - \pi_{11,k+1} & \text{if } \xi = \xi^{(k)} \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.10. Consider the solid partition π of Example 3.7, i.e. $\pi_{111} = 2$, $\pi_{211} = \pi_{121} = \pi_{112} = 1$, and $\pi_{ijk} = 0$ otherwise. Then $\omega^c_\pi = 4$. Indeed $\mathcal{C}_\pi$ contains the following four sequences, each contributing 1 to the sum in (4.4):

- Consider the plane partitions $\xi^{(1)}$ and $\xi^{(2)}$ defined by the following binary representations: $\xi^{(1)}(1,1,1) = \xi^{(1)}(2,1,1) = \xi^{(1)}(1,2,1) = \xi^{(1)}(1,1,2) = 1$ and $\xi^{(1)}(i,j,k) = 0$ otherwise; $\xi^{(2)}(1,1,1) = 1$ and $\xi^{(2)}(i,j,k) = 0$ otherwise. Define $\{m_\xi\}_\xi$ by $m_\xi = 1$ if $\xi = \xi^{(1)}$ or $\xi^{(2)}$ and $m_\xi = 0$ otherwise.
- Consider the plane partitions $\xi^{(1)}$ and $\xi^{(2)}$ defined by the following binary representations: $\xi^{(1)}(1,1,1) = \xi^{(1)}(2,1,1) = \xi^{(1)}(1,2,1) = 1$ and $\xi^{(1)}(i,j,k) = 0$ otherwise; $\xi^{(2)}(1,1,1) = \xi^{(2)}(1,1,2) = 1$ and $\xi^{(2)}(i,j,k) = 0$ otherwise. Define $\{m_\xi\}_\xi$ by $m_\xi = 1$ if $\xi = \xi^{(1)}$ or $\xi^{(2)}$ and $m_\xi = 0$ otherwise.
- Consider the plane partitions $\xi^{(1)}$ and $\xi^{(2)}$ defined by the following binary representations: $\xi^{(1)}(1,1,1) = \xi^{(1)}(2,1,1) = \xi^{(1)}(1,1,2) = 1$ and $\xi^{(1)}(i,j,k) = 0$ otherwise; $\xi^{(2)}(1,1,1) = \xi^{(2)}(1,2,1) = 1$ and $\xi^{(2)}(i,j,k) = 0$ otherwise. Define $\{m_\xi\}_\xi$ by $m_\xi = 1$ if $\xi = \xi^{(1)}$ or $\xi^{(2)}$ and $m_\xi = 0$ otherwise.
- Consider the plane partitions $\xi^{(1)}$ and $\xi^{(2)}$ defined by the following binary representations: $\xi^{(1)}(1,1,1) = \xi^{(1)}(1,2,1) = \xi^{(1)}(1,1,2) = 1$ and $\xi^{(1)}(i,j,k) = 0$ otherwise; $\xi^{(2)}(1,1,1) = \xi^{(2)}(2,1,1) = 1$ and $\xi^{(2)}(i,j,k) = 0$ otherwise. Define $\{m_\xi\}_\xi$ by $m_\xi = 1$ if $\xi = \xi^{(1)}$ or $\xi^{(2)}$ and $m_\xi = 0$ otherwise.

The combinatorial weights ω_π^c lead to the following generating series:

Proposition 4.11. *The following identity holds*

$$\sum_\pi \omega_\pi^c \, t^{\pi_{111}} \, q^{|\pi|} = e^{t(M(q)-1)},$$

where the sum is over all solid partitions, t is a formal variable, and $M(q)$ denotes the MacMahon function. In particular, when $t = 1$, we have

$$\sum_\pi \omega_\pi^c \, q^{|\pi|} = e^{M(q)-1}.$$

Proof. The RHS can be rewritten as

$$\left(\prod_{\xi \vdash 1} e^{tq}\right)\left(\prod_{\xi \vdash 2} e^{tq^2}\right)\left(\prod_{\xi \vdash 3} e^{tq^3}\right)\cdots \tag{4.5}$$

where $\prod_{\xi \vdash n}$ denotes the finite product over all plane partitions ξ of size n.

Choose a sequence of multiplicities $\{m_\xi \in \mathbb{Z}_{\geqslant 0}\}_\xi$ with only finitely many $m_\xi > 0$. This choice gives rise to a solid partition π defined as follows

$$\pi_{ijk} := \sum_\xi m_\xi \cdot \xi(i,j,k), \quad \text{for all } i,j,k \geqslant 1,$$

which we call the *solid partition associated to* $\{m_\xi\}_\xi$. Conversely, for a fixed solid partition π, we can consider the collection of all sequences $\{m_\xi \in \mathbb{Z}_{\geqslant 0}\}_\xi$ with only finitely many $m_\xi > 0$ whose associated solid partition is π. This collection is precisely $\mathcal{C}_\pi$.

Each term arising from multiplying out the infinite product (4.5) corresponds to a sequence $\{m_\xi \in \mathbb{Z}_{\geqslant 0}\}_\xi$ with only finitely many $m_\xi > 0$. Such a term contributes

$$\prod_\xi \frac{t^{m_\xi}}{(m_\xi)!} \, q^{m_\xi |\xi|}. \tag{4.6}$$

Now collect all terms of the form (4.6) such that $\{m_\xi\}_\xi$ has associated solid partition π. This gives

$$\sum_{\{m_\xi\}_\xi \in \mathcal{C}_\pi} \prod_\xi \frac{t^{m_\xi}}{(m_\xi)!} q^{m_\xi|\xi|} = \left(\sum_{\{m_\xi\}_\xi \in \mathcal{C}_\pi} \prod_\xi \frac{1}{(m_\xi)!}\right) t^{\pi_{111}} q^{|\pi|} = \omega_\pi^c \, t^{\pi_{111}} q^{|\pi|},$$

where we use $\sum_\xi m_\xi = \pi_{111}$ in the first equality. Summing over all distinct solid partitions gives the formula of the proposition. □

Remark 4.12. We may also start with d-partitions[9] π for any $d \geqslant 1$ and define ω_π^c completely analogously using $(d-1)$-partitions ξ and their binary representations. The same proof yields

$$\log \sum_{d\text{-partitions } \pi} \omega_\pi^c \, t^{\pi_{111}} \, q^{|\pi|} = t \sum_{(d-1)\text{-partitions } \pi,\, |\pi| \geqslant 1} q^{|\pi|},$$

where we use the convention that there exists a single zero-dimensional partition of each size.

We end this section with the observation that a specialisation of DT_4 theory precisely seems to recover the combinatorics that we just described (and this is how we found the weights ω_π^c in the first place).

Conjecture 4.13. *For any solid partition π, we have $\omega_\pi = \omega_\pi^c$, where ω_π is defined using* DT_4 *theory in Conjecture 4.1 and ω_π^c is the explicit combinatorial weight of Definition 4.7.*

Using our Maple program, which calculates w_π for a given π, we verified the following:

Proposition 4.14.

- *Conjecture 4.13 is true for any solid partition π of size $|\pi| \leqslant 6$.*
- *$|\omega_\pi| = \omega_\pi^c$ for any solid partition π satisfying $\pi_{111} = 1$.*
- *$|\omega_\pi| = \omega_\pi^c$ for the explicit list of solid partitions of size $\leqslant 15$ given in Appendix A.*

Appendix A. Explicit calculations of $|\omega_\pi|$

Using our Maple program, which calculates w_π for a given solid partition π, we checked that

$$\begin{aligned} L_\pi(0,0,0,-d)\, \mathsf{w}_\pi \Big|_{\lambda_1+\lambda_2+\lambda_3=0} &= (-1)^{|\pi|}\, \omega_\pi \prod_{l=1}^{\pi_{111}} (d-(l-1)), \\ \omega_\pi &= \omega_\pi^c, \end{aligned} \tag{A.1}$$

hold for all solid partitions π with $|\pi| \leqslant 6$. Here the signs of w_π are the ones induced from Conjecture 3.21. We also checked that the *absolute value* of equations (A.1) hold for:

- **(Height 1 and $d = 1$)** Let π be a solid partition with $\pi_{111} = 1$. Then $|\omega_\pi| = \omega_\pi^c = 1$.

[9] E.g. 1-partitions are partitions, 2-partitions are plane partitions, 3-partitions are solid partitions.

- **(1-Partitions of size $\leqslant 10$)** All solid partitions $\pi = \{\pi_{ijk}\}_{i,j,k\geqslant 1}$ with $\pi_{ijk} = 0$ unless $i = j = 1$ and $|\pi| \leqslant 10$. Then

$$|\omega_\pi| = \omega_\pi^c = \prod_{k=1}^{\infty} \frac{1}{(\pi_{11k} - \pi_{11,k+1})!}.$$

- **(Size 7)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_2 + t_1t_2 + t_3 + t_4 + t_4^2.$$

Then $|\omega_\pi| = \omega_\pi^c = \frac{3}{2}$.
- **(Size 8)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_2 + t_1t_2 + t_3 + t_4 + t_1t_4 + t_4^2.$$

Then $|\omega_\pi| = \omega_\pi^c = 3$.
- **(Size 9)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_3 + t_4 + t_1t_4 + t_4^2.$$

Then $|\omega_\pi| = \omega_\pi^c = 6$.
- **(Size 10)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_4^2.$$

Then $|\omega_\pi| = \omega_\pi^c = 2$.
- **(Size 11)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_2t_4 + t_4^2.$$

Then $|\omega_\pi| = \omega_\pi^c = 8$.
- **(Size 12)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_2t_4 + t_4^2 + t_4^3.$$

Then $|\omega_\pi| = \omega_\pi^c = 6$.
- **(Size 13)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_2t_4 + t_4^2 + t_4^3 + t_4^4.$$

Then $|\omega_\pi| = \omega_\pi^c = \frac{8}{3}$.

- **(Size 14)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_2t_4 + t_4^2 + t_4^3 + t_4^4 + t_4^5.$$

  Then $|\omega_\pi| = \omega_\pi^c = \frac{5}{6}$.
- **(Size 15)** Consider the solid partition π corresponding to

$$Z_\pi = 1 + t_1 + t_1^2 + t_2 + t_2^2 + t_1t_2 + t_2t_3 + t_3 + t_4 + t_1t_4 + t_2t_4 + t_4^2 + t_4^3 + t_4^4 + t_4^5.$$

  Then $|\omega_\pi| = \omega_\pi^c = \frac{5}{3}$.

Appendix B. Nekrasov's conjecture

The first author heard the following related conjecture (written below in terms of equivariant DT_4 theory) from Professor Nikita Nekrasov during a visit to the Simons Center for Geometry and Physics in October 2016. For a recent much more general K-theoretical version, see [19].

Let $X = \mathbb{C}^4$ and let $T = \{t \in (\mathbb{C}^*)^4 \,|\, t_1t_2t_3t_4 = 1\}$ be the Calabi–Yau torus. Denote the equivariant parameters of $(\mathbb{C}^*)^4$ by λ_i $(i = 1, 2, 3, 4)$. We define

$$[\mathrm{Hilb}^n(\mathbb{C}^4)]^{\mathrm{vir}}_{T,o(\mathcal{L})} := \sum_{Z \in \mathrm{Hilb}^n(\mathbb{C}^4)^T} \frac{e_T\big(\mathrm{Ext}^2(I_Z, I_Z), Q\big)}{e_T\big(\mathrm{Ext}^1(I_Z, I_Z)\big)}.$$

As in Definition 3.8, this depends on a choice of orientation $o(\mathcal{L})$ as in Definition 3.8 which is used to define the half Euler classes. Consider the generating function

$$Z_{\mathbb{C}^4} := \sum_{n=0}^{\infty} \Bigg(\int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]^{\mathrm{vir}}_{T,o(\mathcal{L})}} 1 \Bigg) \cdot q^n \in \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} [\![q]\!].$$

Conjecture B.1. *There exist choices of orientation such that*

$$Z_{\mathbb{C}^4} = e^{\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)} q}.$$

Using the signs discussed in Remark 3.23, we checked the following with our Maple program:

Proposition B.2. *Conjecture B.1 is true modulo q^7.*

In fact, the signs of Nekrasov's conjecture seem to be unique as well:

Proposition B.3. *Modulo q^5, there are unique choices of signs for which Conjecture B.1 holds.*

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