

# On Optimal Polyline Simplification Using the Hausdorff and Fréchet Distance

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## Abstract

We revisit the classical polygonal line simplification problem and study it using the Hausdorff distance and Fréchet distance. Interestingly, no previous authors studied line simplification under these measures in its pure form, namely: for a given  $\varepsilon > 0$ , choose a minimum size subsequence of the vertices of the input such that the Hausdorff or Fréchet distance between the input and output polylines is at most  $\varepsilon$ .

We analyze how the well-known Douglas-Peucker and Imai-Iri simplification algorithms perform compared to the optimum possible, also in the situation where the algorithms are given a considerably larger error threshold than  $\varepsilon$ . Furthermore, we show that computing an optimal simplification using the undirected Hausdorff distance is NP-hard. The same holds when using the directed Hausdorff distance from the input to the output polyline, whereas the reverse can be computed in polynomial time. Finally, to compute the optimal simplification from a polygonal line consisting of  $n$  vertices under the Fréchet distance, we give an  $O(kn^5)$  time algorithm that requires  $O(kn^2)$  space, where  $k$  is the output complexity of the simplification.

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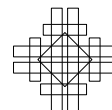
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## 1 Introduction

Line simplification (a.k.a. polygonal approximation) is one of the oldest and best studied applied topics in computational geometry. It was and still is studied, for example, in the context of computer graphics (after image to vector conversion), in Geographic Information Science, and in shape analysis. Among the well-known algorithms, the ones by Douglas and Peucker [11] and by Imai and Iri [18] hold a special place and are frequently implemented and cited. Both algorithms start with a polygonal line (henceforth *polyline*) as the input, specified by a sequence of points  $\langle p_1, \dots, p_n \rangle$ , and compute a subsequence starting with  $p_1$  and ending with  $p_n$ , representing a new, simplified polyline. Both algorithms take a constant  $\varepsilon > 0$  and guarantee that the output is within  $\varepsilon$  from the input.

The Douglas-Peucker algorithm [11] is a simple and effective recursive procedure that keeps on adding vertices from the input polyline until the computed polyline lies within a prespecified distance  $\varepsilon$ . The procedure is a heuristic in several ways: it does not minimize the number of vertices in the output (although it performs well in practice) and it runs in  $O(n^2)$  time in the worst case (although in practice it appears more like  $O(n \log n)$  time). Hershberger and Snoeyink [17] overcame the worst-case running time bound by providing a worst-case  $O(n \log n)$  time algorithm using techniques from computational geometry, in particular a type of dynamic convex hull.

The Imai-Iri algorithm [18] takes a different approach. It computes for every *link*  $\overline{p_i p_j}$  with  $i < j$  whether the sequence of vertices  $\langle p_{i+1}, \dots, p_{j-1} \rangle$  that lie in between in the input lie within distance  $\varepsilon$  to the segment  $\overline{p_i p_j}$ . In this case  $\overline{p_i p_j}$  is a valid link that may be used in the output. The graph  $G$  that has all vertices  $p_1, \dots, p_n$  as nodes and all valid links as edges can then be constructed, and a minimum link path from  $p_1$  to  $p_n$  represents an optimal simplification. Brute-force, this algorithm runs in  $O(n^3)$  time, but with the implementation of Chan and Chin [8] or Melkman and O'Rourke [21] it can be done in  $O(n^2)$  time.

There are many more results in line simplification. Different error measures can be used [6], self-intersections may be avoided [10], line simplification can be studied in the streaming model [1], it can be studied for 3-dimensional polylines [5], angle constraints may be put on consecutive segments [9], there are versions that do not output a subset of the input points but other well-chosen points [16], it can be incorporated in subdivision simplification [12, 13, 16], and so on and so forth. Some optimization versions are NP-hard [12, 16]. It is beyond the scope of this paper to review the very extensive literature on line simplification.

Among the distance measures for two shapes that are used in computational geometry, the *Hausdorff distance* and the *Fréchet distance* are probably the most well-known. They are both *bottleneck measures*, meaning that the distance is typically determined by a small subset of the input like a single pair of points (and the distances are not aggregated over the whole shapes). The Fréchet distance is considered a better distance measure, but it is considerably more difficult to compute because it requires us to optimize over all parametrizations of the two shapes. The Hausdorff distance between two simple polylines with  $n$  and  $m$  vertices can be computed in  $O((n+m) \log(n+m))$  time [3]. Their Fréchet distance can be computed in  $O(nm \log(n+m))$  time [4].

Now, the Imai-Iri algorithm is considered an optimal line simplification algorithm, because it minimizes the number of vertices in the output, given the restriction that the output must be a subsequence of the input. But for what measure? It is not optimal for the Hausdorff distance, because there are simple examples where a simplification with fewer vertices can be given that still have Hausdorff distance at most  $\varepsilon$  between input and output. This comes from the fact that the algorithm uses the Hausdorff distance between a link  $\overline{p_i p_j}$  and the

sub-polyline  $\langle p_i, \dots, p_j \rangle$ . This is more local than the Hausdorff distance requires, and is more a Fréchet-type of criterion. But the line simplification produced by the Imai-Iri algorithm is also not optimal for the Fréchet distance. In particular, the input and output do not necessarily lie within Fréchet distance  $\varepsilon$ , because links are evaluated on their Hausdorff distance only.

The latter issue could easily be remedied: to accept links, we require the Fréchet distance between any link  $\overline{p_i p_j}$  and the sub-polyline  $\langle p_i, \dots, p_j \rangle$  to be at most  $\varepsilon$  [2, 15]. This guarantees that the Fréchet distance between the input and the output is at most  $\varepsilon$ . However, it does not yield the optimal simplification within Fréchet distance  $\varepsilon$ . Because of the nature of the Imai-Iri algorithm, *it requires us to match a vertex  $p_i$  in the input to the vertex  $p_i$  in the output in the parametrizations, if  $p_i$  is used in the output*. This restriction on the parametrizations considered limits the simplification in unnecessary ways. Agarwal et al. [2] refer to a simplification that uses the normal (unrestricted) Fréchet distance with error threshold  $\varepsilon$  as a *weak  $\varepsilon$ -simplification under the Fréchet distance*.<sup>4</sup> They show that the Imai-Iri algorithm using the Fréchet distance gives a simplification with no more vertices than an optimal weak  $(\varepsilon/4)$ -simplification under the Fréchet distance, where the latter need not use the input vertices.

The discussion begs the following questions: How much worse do the known algorithms and their variations perform in theory, when compared to the optimal Hausdorff and Fréchet simplifications? What if the optimal Hausdorff and Fréchet simplifications use a smaller value than  $\varepsilon$ ? As mentioned, Agarwal et al. [2] give a partial answer. How efficiently can the optimal Hausdorff simplification and the optimal Fréchet simplification be computed (when using the input vertices)?

**Organization and results.** In Section 2 we explain the Douglas-Peucker algorithm and its Fréchet variation; the Imai-Iri algorithm has been explained already. We also show with a small example that the optimal Hausdorff simplification has fewer vertices than the Douglas-Peucker output and the Imai-Iri output, and that the same holds true for the optimal Fréchet simplification with respect to the Fréchet variants.

In Section 3 we will analyze the four algorithms and their performance with respect to an optimal Hausdorff simplification or an optimal Fréchet simplification more extensively. In particular, we address the question how many more vertices the four algorithms need, and whether this remains the case when we use a larger value of  $\varepsilon$  but still compare to the optimization algorithms that use  $\varepsilon$ .

In Section 4 we consider both the directed and undirected Hausdorff distance to compute the optimal simplification. We show that only the simplification under the directed Hausdorff distance from the output to the input polyline can be computed in polynomial time, while the rest is NP-hard to compute. In Section 5 we show that the problem can be solved in polynomial time for the Fréchet distance.

■ **Table 1** Algorithmic results.

|                    | Douglas-Peucker          | Imai-Iri      | Optimal                         |
|--------------------|--------------------------|---------------|---------------------------------|
| Hausdorff distance | $O(n \log n)$ [17]       | $O(n^2)$ [8]  | NP-hard ( <i>this paper</i> )   |
| Fréchet distance   | $O(n^2)$ ( <i>easy</i> ) | $O(n^3)$ [15] | $O(kn^5)$ ( <i>this paper</i> ) |

<sup>4</sup> Weak refers to the situation that the vertices of the simplification can lie anywhere.

## 2 Preliminaries

The line simplification problem takes a maximum allowed error  $\varepsilon$  and a polyline  $P$  defined by a sequence of points  $\langle p_1, \dots, p_n \rangle$ , and computes a polyline  $Q$  defined by  $\langle q_1, \dots, q_k \rangle$  and the error is at most  $\varepsilon$ . Commonly the sequence of points defining  $Q$  is a subsequence of points defining  $P$ , and furthermore,  $q_1 = p_1$  and  $q_k = p_n$ . There are many ways to measure the distance or error of a simplification. The most common measure is a distance, denoted by  $\varepsilon$ , like the Hausdorff distance or the Fréchet distance (we assume these distance measures are known). Note that the Fréchet distance is symmetric, whereas the Hausdorff distance has a symmetric and an asymmetric version (the distance from the input to the simplification).

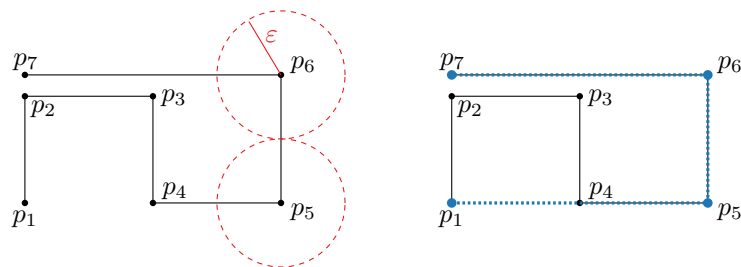
The Douglas-Peucker algorithm for polyline simplification is a simple recursive procedure that works as follows. Let the line segment  $\overline{p_1 p_n}$  be the first simplification. If all points of  $P$  lie within distance  $\varepsilon$  from this line segment, then we have found our simplification. Otherwise, let  $p_f$  be the furthest point from  $\overline{p_1 p_n}$ , add it to the simplification, and recursively simplify the polylines  $\langle p_1, \dots, p_f \rangle$  and  $\langle p_f, \dots, p_n \rangle$ . Then merge their simplifications (remove the duplicate  $p_f$ ). It is easy to see that the algorithm runs in  $O(n^2)$  time, and also that one can expect a much better performance in practice. It is also straightforward to verify that polyline  $P$  has Hausdorff distance (symmetric and asymmetric) at most  $\varepsilon$  to the output. We denote this simplification by  $DP_H(P, \varepsilon)$ , and will leave out the arguments  $P$  and/or  $\varepsilon$  if they are understood.

We can modify the algorithm to guarantee a Fréchet distance between  $P$  and its simplification of at most  $\varepsilon$  by testing whether the Fréchet distance between  $P$  and its simplification is at most  $\varepsilon$ . If not, we still choose the most distant point  $p_f$  to be added to the simplification (other choices are possible). This modification does not change the efficiency of the Douglas-Peucker algorithm asymptotically as the Fréchet distance between a line segment and a polyline can be determined in linear time. We denote this simplification by  $DP_F(P, \varepsilon)$ .

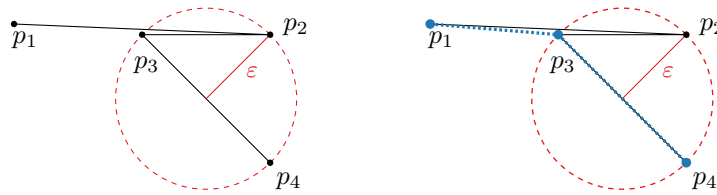
We have already described the Imai-Iri algorithm in the previous section. We refer to the resulting simplification as  $II_H(P, \varepsilon)$ . It has a Hausdorff distance (symmetric and asymmetric) of at most  $\varepsilon$  and never has more vertices than  $DP_H(P, \varepsilon)$ . Similar to the Douglas-Peucker algorithm, the Imai-Iri algorithm can be modified for the Fréchet distance, leading to a simplification denoted by  $II_F(P, \varepsilon)$ .

We will denote the optimal simplification using the Hausdorff distance by  $OPT_H(P, \varepsilon)$ , and the optimal simplification using the Fréchet distance by  $OPT_F(P, \varepsilon)$ . In the case of Hausdorff distance, we require  $P$  to be within  $\varepsilon$  of its simplification, so we use the directed Hausdorff distance.

The example in Figure 1 shows that  $DP_H(P)$  and  $II_H(P)$  – which are both equal to  $P$  itself – may use more vertices than  $OPT_H(P) = \langle p_1, p_5, p_6, p_7 \rangle$ . Similarly, the example in Figure 2 shows that  $DP_F$  and  $II_F$  may use more vertices than  $OPT_F$ .



■ **Figure 1** Simplifications  $II_H$  (same as input, left) and  $OPT_H$  (in blue, right) for an example.



■ **Figure 2** Simplifications  $II_F$  (same as input, left) and  $OPT_F$  (in blue, right) for an example.

**3 Approximation quality of Douglas-Peucker and Imai-Iri simplification**

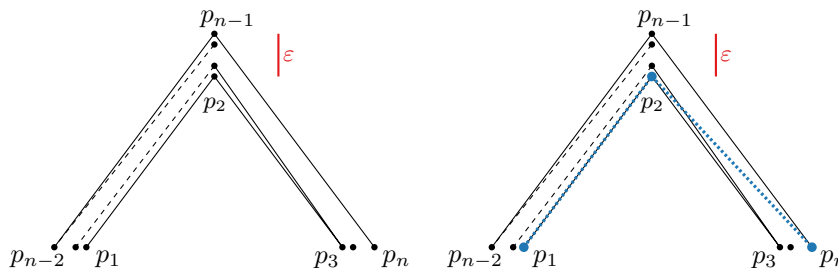
The examples of the previous section not only show that  $II_H$  and  $II_F$  (and  $DP_H$  and  $DP_F$ ) use more vertices than  $OPT_H$  and  $OPT_F$ , respectively, they show that this is still the case if we run  $II$  with a larger value than  $\epsilon$ . To let  $II_H$  use as few vertices as  $OPT_H$ , we must use  $2\epsilon$  instead of  $\epsilon$  when the example is stretched horizontally. For the Fréchet distance, the enlargement factor needed in the example approaches  $\sqrt{2}$  if we put  $p_1$  far to the left. In this section we analyze how the approximation enlargement factor relates to the number of vertices in the Douglas-Peucker and Imai-Iri simplifications and the optimal ones. The interest in such results stems from the fact that the Douglas-Peucker and Imai-Iri algorithms are considerably more efficient than the computation of  $OPT_H$  and  $OPT_F$ .

**3.1 Hausdorff distance**

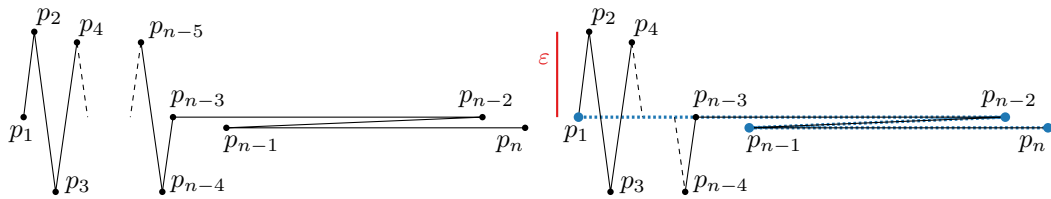
To show that  $II_H$  (and  $DP_H$  by consequence) may use many more vertices than  $OPT_H$ , even if we enlarge  $\epsilon$ , we give a construction where this occurs. Imagine three regions with diameter  $\epsilon$  at the vertices of a sufficiently large equilateral triangle. We construct a polyline  $P$  where  $p_1, p_5, p_9, \dots$  are in one region,  $p_2, p_4, p_6, \dots$  are in the second region, and the remaining vertices are in the third region, see Figure 3. Let  $n$  be such that  $p_n$  is in the third region. An optimal simplification is  $\langle p_1, p_i, p_n \rangle$  where  $i$  is any even number between 1 and  $n$ . Since the only valid links are the ones connecting two consecutive vertices of  $P$ ,  $II_H$  is  $P$  itself. If the triangle is large enough with respect to  $\epsilon$ , this remains true even if we give the Imai-Iri algorithm a much larger error threshold than  $\epsilon$ .

► **Theorem 1.** *For any  $c > 1$ , there exists a polyline  $P$  with  $n$  vertices and an  $\epsilon > 0$  such that  $II_H(P, c\epsilon)$  has  $n$  vertices and  $OPT_H(P, \epsilon)$  has 3 vertices.*

Note that the example applies both to the directed and the undirected Hausdorff distance.



■ **Figure 3** The Douglas-Peucker and Imai-Iri algorithms may not be able to simplify at all, whereas the optimal simplification using the Hausdorff distance has just three vertices (in blue, right).



■ **Figure 4** Left: a polyline on which the Fréchet version of the Douglas-Peucker algorithm performs poorly and the output polyline contains  $n$  vertices. Right: the optimal simplification contains four vertices (in blue).

### 3.2 Fréchet distance

Our results are somewhat different for the Fréchet distance; we need to make a distinction between  $DP_F$  and  $II_F$ .

**Douglas-Peucker.** We construct an example that shows that  $DP_F$  may have many more vertices than  $OPT_F$ , even if we enlarge the error threshold. It is illustrated in Figure 4. Vertex  $p_2$  is placed slightly higher than  $p_4, p_6, \dots$  so that it will be added first by the Fréchet version of the Douglas-Peucker algorithm. Eventually all vertices will be chosen.  $OPT_F$  has only four vertices. Since the zigzag  $p_{n-3}, \dots, p_n$  can be arbitrarily much larger than the height of the vertical zigzag  $p_1, \dots, p_{n-4}$ , the situation remains if we make the error threshold arbitrarily much larger.

► **Theorem 2.** *For any  $c > 1$ , there exists a polyline  $P$  with  $n$  vertices and an  $\varepsilon > 0$  such that  $DP_F(P, c\varepsilon)$  has  $n$  vertices and  $OPT_F(P, \varepsilon)$  has 4 vertices.*

**Remark.** One could argue that the choice of adding the furthest vertex is not suitable when using the Fréchet distance, because we may not be adding the vertex (or vertices) that are to “blame” for the high Fréchet distance. However, finding the vertex that improves the Fréchet distance most is computationally expensive, defeating the purpose of this simple algorithm. Furthermore, we can observe that also in the Hausdorff version, the Douglas-Peucker algorithm does not choose the vertex that improves the Hausdorff distance most (it may even increase when adding an extra vertex).

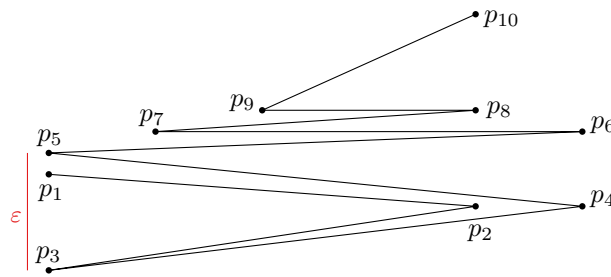
**Imai-Iri.** Finally we compare the Fréchet version of the Imai-Iri algorithm to the optimal Fréchet distance simplification. Our main construction has ten vertices placed in such a way that  $II_F$  has all ten vertices, while  $OPT_F$  has only eight of them, see Figures 5 and 6.

It is easy to see that under the Fréchet distance,  $II_F = OPT_F$  for the previous construction in Figure 4. We give another input polyline  $P$  in Figure 6 to show that  $II_F$  also does not approximate  $OPT_F$  even if  $II_F$  is allowed to use  $\varepsilon$  that is larger by a constant factor.

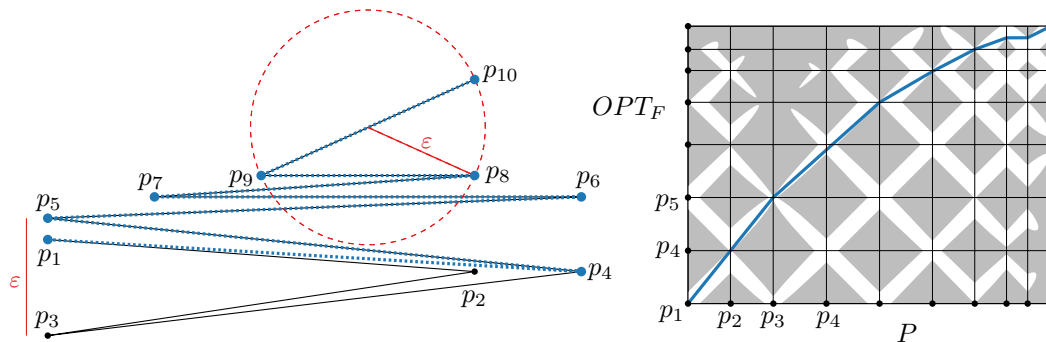
We can append multiple copies of this construction together with a suitable connection in between. This way we obtain:

► **Theorem 3.** *There exist constants  $c_1 > 1$ ,  $c_2 > 1$ , a polyline  $P$  with  $n$  vertices, and an  $\varepsilon > 0$  such that  $|II_F(P, c_1\varepsilon)| > c_2|OPT_F(P, \varepsilon)|$ .*

By the aforementioned result of Agarwal et al. [2], we know that the theorem is not true for  $c_1 \geq 4$ .



■ **Figure 5** The Imai-Iri simplification will have all vertices because the only valid links with a Fréchet distance at most  $\epsilon$  are the ones connecting two consecutive vertices in the polyline.



■ **Figure 6** The optimal simplification can skip  $p_2$  and  $p_3$ ; in the parametrizations witnessing the Fréchet distance,  $OPT_F$  “stays two vertices behind” on the input until the end. Right, the free space diagram of  $P$  and  $OPT_F$ .

## 4 Algorithmic complexity of the Hausdorff distance

The results in the previous section show that both the Douglas-Peucker and the Imai-Iri algorithm do not produce an optimal polyline that minimizes the Hausdorff or Fréchet distance, or even approximate them within any constant factor. Naturally, this leads us to the following question: Is it possible to compute the optimal Hausdorff or Fréchet simplification in polynomial time?

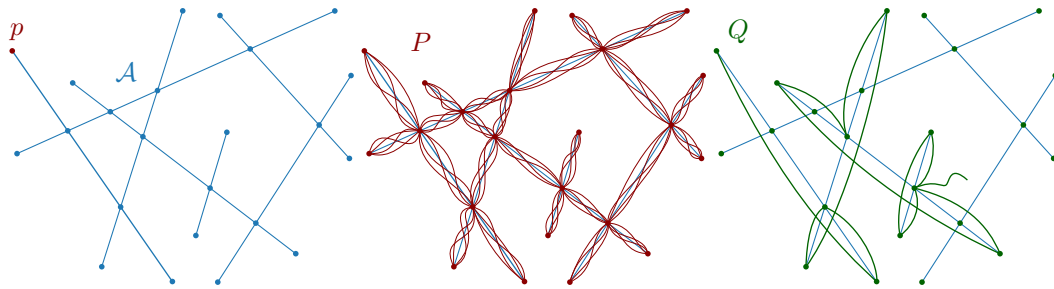
In this section, we present a construction which proves that under the Hausdorff distance, computing the optimal simplified polyline is NP-hard.

### 4.1 Undirected Hausdorff distance

We first consider the undirected (or bidirectional) Hausdorff distance; that is, we require both the maximum distance from the initial polyline  $P$  to the simplified polyline  $Q$  and the maximum distance from  $Q$  to  $P$  to be at most  $\epsilon$ .

► **Theorem 4.** *Given a polyline  $P = \langle p_1, p_2, \dots, p_n \rangle$  and a value  $\epsilon$ , the problem of computing a minimum length subsequence  $Q$  of  $P$  such that the undirected Hausdorff distance between  $P$  and  $Q$  is at most  $\epsilon$  is NP-hard.*

We prove the theorem with a reduction from Hamiltonian cycle in segment intersection graphs. It is well-known that Hamiltonian cycle is NP-complete in planar graphs [14], and by Chalopin and Gonçalves’ proof [7] of Scheinerman’s conjecture [22] that the planar graphs



■ **Figure 7** The construction:  $\mathcal{A}$  is the arrangement of a set of segments  $S$ . We build an input path  $P$  that “paints” over  $S$  completely, and we are looking for an output path  $Q$  that corresponds to a Hamiltonian cycle. In this case, there is no Hamiltonian cycle, and the path gets stuck.

are included in the segment intersections graphs it follows that Hamiltonian cycle in segment intersections graphs is NP-complete.

Let  $S$  be a set of  $n$  line segments in the plane, and assume all intersections are proper (if not, extend the segments slightly). Let  $G$  be its intersection graph (i.e.  $G$  has a vertex for every segment in  $S$ , and two vertices in  $G$  are connected by an edge when their corresponding segments intersect). We assume that  $G$  is connected; otherwise, clearly there is no Hamiltonian cycle in  $G$ .

We first construct an initial polyline  $P$  as follows. (Figure 7 illustrates the construction.) Let  $\mathcal{A}$  be the arrangement of  $S$ , let  $p$  be some endpoint of a segment in  $S$ , and let  $\pi$  be any path on  $\mathcal{A}$  that starts and finishes at  $p$  and visits all vertices and edges of  $\mathcal{A}$  (clearly,  $\pi$  may reuse vertices and edges). Then  $P$  is simply  $3n + 1$  copies of  $\pi$  appended to each other. Consequently, the order of vertices in  $Q$  now must follow the order of these copies. We now set  $\varepsilon$  to a sufficiently small value.

Now, an output polyline  $Q$  with Hausdorff distance at most  $\varepsilon$  to  $P$  must also visit all vertices and edges of  $\mathcal{A}$ , and stay close to  $\mathcal{A}$ . If  $\varepsilon$  is sufficiently small, there will be no benefit for  $Q$  to ever leave  $\mathcal{A}$ .

► **Lemma 5.** *A solution  $Q$  of length  $3n + 1$  exists if and only if  $G$  admits a Hamiltonian cycle.*

**Proof.** Clearly, any simplification  $Q$  will need to visit the  $2n$  endpoints of the segments in  $S$ , and since it starts and ends at the same point  $p$ , will need to have length at least  $2n + 1$ . Furthermore,  $Q$  will need to have at least two internal vertices on every segment  $s \in S$ : once to enter the segment and once to leave it (note that we cannot enter or leave a segment at an endpoint since all intersections are proper intersections). This means the minimum number of vertices possible for  $Q$  is  $3n + 1$ .

Now, if  $G$  admits a Hamiltonian cycle, it is easy to construct a simplification with  $3n + 1$  vertices as follows. We start at  $p$  and collect the other endpoint of the segment  $s_1$  of which  $p$  is an endpoint. Then we follow the Hamiltonian cycle to segment  $s_2$ ; by definition  $s_1 s_2$  is an edge in  $G$  so their corresponding segments intersect, and we use the intersection point to leave  $s_1$  and enter  $s_2$ . We proceed in this fashion until we reach  $s_n$ , which intersects  $s_1$ , and finally return to  $p$ .

On the other hand, any solution with  $3n + 1$  vertices must necessarily be of this form and therefore imply a Hamiltonian cycle: in order to have only 3 vertices per segment the vertex at which we leave  $s_1$  must coincide with the vertex at which we enter some other segment, which we call  $s_2$ , and we must continue until we visited all segments and return to  $p$ . ◀



## 4.2 Directed Hausdorff distance: $P \rightarrow Q$

We now shift our attention to the directed Hausdorff distance from  $P$  to  $Q$ : we require the maximum distance from  $P$  to  $Q$  to be at most  $\varepsilon$ , but  $Q$  may have a larger distance to  $P$ . The previous reduction does not seem to work because there is always a Hamiltonian Cycle of length  $2n$  for this measure. Therefore, we prove the NP-hardness differently.

The idea is to reduce from COVERING POINTS BY LINES, which is known to be both NP-hard [20] and APX-hard [19]: given a set  $S$  of points in  $\mathbb{R}^2$ , find the minimum number of lines needed to cover the points. The complete proof is explained in full detail in [23]; here we give the main part of the construction.

Let  $S = \{s_1, \dots, s_n\}$  be an instance of the COVERING POINTS BY LINES problem. We fix  $\varepsilon$  based on  $S$  and present the construction of a polyline connecting a sequence of  $m = \text{poly}(n)$  points:  $P = \langle p_1, p_2, \dots, p_m \rangle$  such that for every  $1 \leq i \leq n$ , we have  $s_i = p_j$  for some  $1 \leq j \leq m$ . The idea is to force the simplification  $Q$  to cover all points in  $P$  except those in  $S$ , such that in order for the final simplification to cover all points, we only need to collect the points in  $S$  using as few line segments as possible. To this end, we will place a number of *forced points*  $F \subset P$ , where a point  $f$  is *forced* whenever its distance to any line through any pair of points in  $P$  is larger than  $\varepsilon$ . Since  $Q$  must be defined by a subset of points in  $P$ , we will never cover  $f$  unless we choose  $f$  to be a vertex of  $Q$ . Figure 8 shows this idea. On the other hand, we need to place points that allow us to freely draw every line through two or more points in  $S$ . We create two point sets  $L$  and  $R$  to the left and right of  $S$ , such that for every line through two of more points in  $S$ , there are a point in  $L$  and a point in  $R$  on that line. Finally, we need to build additional scaffolding around the construction to connect and cover the points in  $L$  and  $R$ . Figure 9 shows the idea.

The construction has three parts with different purposes:

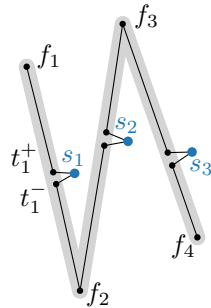
1. a sub-polyline that contains  $S$ ;
2. a sub-polyline that contains  $L$  and  $R$ ; and
3. two disconnected sub-polylines which share the same purpose: to guarantee that all vertices in the previous sub-polyline are themselves covered by  $Q$ .

First, we assume that every point in  $S$  has a unique  $x$ -coordinate; if this is not the case, we rotate  $S$  until it is.<sup>5</sup> We also assume that every line through at least two points of  $S$  has a slope between  $-1$  and  $+1$ ; if this is not the case, we vertically scale  $S$  until it is. Now, we fix  $\varepsilon$  to be smaller than half the minimum difference between any two  $x$ -coordinates of points in  $S$ , and smaller than the distance from any line through two points in  $S$  to any other point in  $S$  not on the line.

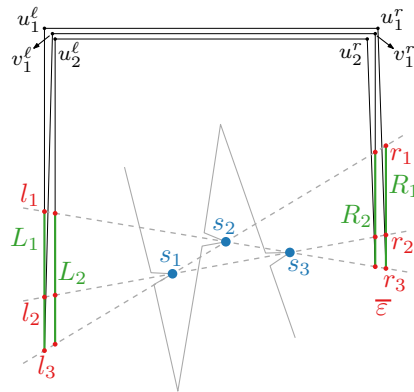
We place  $n + 1$  forced points  $f_1, f_2, \dots, f_n, f_{n+1}$  such that the  $x$ -coordinate of  $f_i$  lies between the  $x$ -coordinates of  $s_{i-1}$  and  $s_i$  and the points lie alternately above and below  $S$ ; we place them such that the distance of the line segment  $\overline{f_i f_{i+1}}$  to  $s_i$  is  $\frac{3}{2}\varepsilon$  and the distance of  $\overline{f_i f_{i+1}}$  to  $s_{i-1}$  is larger than  $\varepsilon$ . Next, we place two auxiliary points  $t_i^+$  and  $t_i^-$  on  $\overline{f_i f_{i+1}}$  such that the distance of each point to  $s_i$  is  $2\varepsilon$ ; refer to Figure 8. Then let  $\tau_1 = \langle f_1, t_1^+, s_1, t_1^-, f_2, t_2^-, s_2, t_2^+, f_3, \dots, f_{n+1} \rangle$  be a polyline connecting all points in the construction;  $\tau_1$  will be part of the input segment  $P$ .

The idea here is that all forced points must appear on  $Q$ , and if only the forced points appear on  $Q$ , everything in the construction will be covered *except* the points in  $S$  (and some arbitrarily short stubs of edges connecting them to the auxiliary points). Of course, we could

<sup>5</sup> Note that, by nature of the COVERING POINTS BY LINES problem, we cannot assume  $S$  is in general position; however, a rotation for which all  $x$ -coordinates are unique always exists.



■ **Figure 8** Example of  $\tau_1$  where  $n = 3$ . For a given  $\varepsilon$ , the (simplified) polyline  $f_1, f_2, f_3, f_4$  covers the gray area but not the blue points  $s_1, s_2, s_3$ .



■ **Figure 9** Construction to allow the lines that can be used to cover the points of  $S$ . To ensure the order of vertices in  $Q$ , we create copies of  $L$  and  $R$ . Then,  $Q$  can use them alternately.

choose to include more points in  $\tau_1$  in  $Q$  to collect some points of  $S$  already. However, this would cost an additional three vertices per collected point (note that using fewer than three, we would miss an auxiliary point instead), and in the remainder of the construction we will make sure that it is cheaper to collect the points in  $S$  separately later.

The second part of the construction serves to allow shortcuts that have the role of the lines in the COVERING POINTS BY LINES problem. Since we only need the  $O(n^2)$  lines that cover at least two points, this part of the construction has  $O(n^2)$  vertices in  $P$ . An indication of this construction is given in Figure 9; further details are in [23].

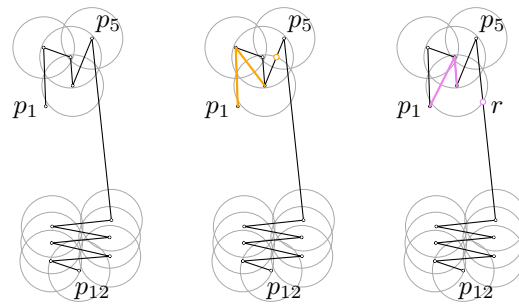
► **Theorem 6.** *Given a polyline  $P = \langle p_1, p_2, \dots, p_n \rangle$  and a value  $\varepsilon$ , the problem of computing a minimum length subsequence  $Q$  of  $P$  such that the directed Hausdorff distance from  $P$  to  $Q$  is at most  $\varepsilon$  is NP-hard.*

### 4.3 Directed Hausdorff distance: $Q \rightarrow P$

Finally, we finish this section with a note on the reverse problem: we want to only bound the directed Hausdorff distance from  $Q$  to  $P$  (we want the output segment to stay close to the input segment, but we do not need to be close to all parts of the input). This problem seems more esoteric but we include it for completeness. In this case, a polynomial time algorithm (reminiscent of Imai-Iri) optimally solves the problem.

► **Theorem 7.** *Given a polyline  $P = \langle p_1, p_2, \dots, p_n \rangle$  and a value  $\varepsilon$ , the problem of computing a minimum length subsequence  $Q$  of  $P$  such that the directed Hausdorff distance from  $Q$  to  $P$  is at most  $\varepsilon$  can be solved in polynomial time.*

**Proof.** We compute the region with distance  $\varepsilon$  from  $P$  explicitly. For every link we compute if it lies within that region, and if so, add it as an edge to a graph. Then we find a minimum link path in this graph. For a possibly self-intersecting polyline as the input a simple algorithm takes  $O(n^4)$  time (faster is possible). ◀



■ **Figure 10** An example where the farthest-reaching simplification up to  $p_4$  using 2 links is not part of any solution that uses  $p_4$ . Left: the input curve  $P$  in black, with circles of radius  $\varepsilon$  around all vertices in light gray. Middle: A 2-link simplification of  $\langle p_1, p_2, p_3, p_4 \rangle$  that reaches up to a point on  $\overline{p_4 p_5}$  (in yellow) which can be extended to a 4-link simplification of  $P$ . Right: A 2-link simplification of  $\langle p_1, p_2, p_3, p_4 \rangle$  that reaches point  $r$  on  $\overline{p_5 p_6}$  (in pink) which does not allow simplification.

## 5 Algorithmic complexity of the Fréchet distance

In this section, we show that for a given polyline  $P = \langle p_1, p_2, \dots, p_n \rangle$  and an error  $\varepsilon$ , the optimal simplification  $Q = OPT_F(P, \varepsilon)$  can be computed in polynomial time using a dynamic programming approach.

### 5.1 Observations

Note that a link  $\overline{p_i p_j}$  in  $Q$  is not necessarily within Fréchet distance  $\varepsilon$  to the sub-polyline  $\langle p_i, p_{i+1}, \dots, p_j \rangle$  (for example,  $\overline{p_1 p_3}$  in Figure 2). Furthermore, a (sequence of) link(s) in  $Q$  could be mapped to an arbitrary subcurve of  $P$ , not necessarily starting or ending at a vertex of  $P$ . For example, in Figure 6, the sub-polyline  $\langle p_1, p_4, p_5, p_6 \rangle$  has Fréchet distance  $\varepsilon$  to a sub-polyline of  $P$  that starts at  $p_1$  but ends somewhere between  $p_4$  and  $p_5$ . At this point, one might imagine a dynamic programming algorithm which stores, for each vertex  $p_i$  and value  $k$ , the point  $p(i, k)$  on  $P$  which is the farthest along  $P$  such that there exists a simplification of the part of  $P$  up to  $p_i$  using  $k$  links that has Fréchet distance at most  $\varepsilon$  to the part of  $P$  up to  $p(i, k)$ . However, the following lemma shows that even this does not yield optimality; its proof is the example in Figure 10.

► **Lemma 8.** *There exists a polyline  $P = \langle p_1, \dots, p_{12} \rangle$  and an optimal  $\varepsilon$ -Fréchet-simplification that has to use  $p_4$ ,  $Q = \langle p_1, p_2, p_4, p_5, p_{12} \rangle$  using 4 links, with the following properties:*

- *There exists a partial simplification  $R = \langle p_1, p_3, p_4 \rangle$  of  $\langle p_1, \dots, p_4 \rangle$  and a point  $r$  on  $\overline{p_5 p_6}$  such that the Fréchet distance between  $R$  and the subcurve of  $P$  up to  $r$  is  $\leq \varepsilon$ , but*
- *there exists no partial simplification  $S$  of  $\langle p_4, \dots, p_{12} \rangle$  that is within Fréchet distance  $\varepsilon$  to the subcurve of  $P$  starting at  $r$  that uses fewer than 7 links.*

### 5.2 A dynamic programming algorithm

Lemma 8 shows that storing a single data point for each vertex and value of  $k$  is not sufficient to ensure that we find an optimal solution. Instead, we argue that if we maintain the set of *all* points at  $P$  that can be “reached” by a simplification up to each vertex, then we can make dynamic programming work. We now make this precise and argue that the complexity of these sets of reachable points is never worse than linear.

First, we define  $\pi$ , a parameterization of  $P$  as a continuous mapping:  $\pi : [0, 1] \rightarrow \mathbb{R}^2$  where  $\pi(0) = p_1$  and  $\pi(1) = p_n$ . We also write  $P[s, t]$  for  $0 \leq s \leq t \leq 1$  to be the subcurve of  $P$  starting at  $\pi(s)$  and ending at  $\pi(t)$ , also writing  $P[t] = P[0, t]$  for short.

We say that a point  $\pi(t)$  can be *reached* by a  $(k, i)$ -simplification for  $0 \leq k < i \leq n$  if there exists a simplification of  $\langle p_1, \dots, p_i \rangle$  using  $k$  links which has Fréchet distance at most  $\varepsilon$  to  $P[t]$ . We let  $\rho(k, i, t) = \mathbf{true}$  in this case, and  $\mathbf{false}$  otherwise. With slight abuse of notation we also say that  $t$  itself is reachable, and that an interval  $I$  is reachable if all  $t \in I$  are reachable (by a  $(k, i)$ -simplification).

► **Observation 9.** *A point  $\pi(t)$  can be reached by a  $(k, i)$ -simplification if and only if there exist a  $0 < h < i$  and a  $0 \leq s \leq t$  such that  $\pi(s)$  can be reached by a  $(k-1, h)$ -simplification and the segment  $\overline{p_h p_i}$  has Fréchet distance at most  $\varepsilon$  to  $P[s, t]$ .*

**Proof.** Follows directly from the definition of the Fréchet distance. ◀

Observation 9 immediately suggests a dynamic programming algorithm: for every  $k$  and  $i$  we store a subdivision of  $[0, 1]$  into intervals where  $\rho$  is true and intervals where  $\rho$  is false, and we calculate the subdivisions for increasing values of  $k$ . We simply iterate over all possible values of  $h$ , calculate which intervals can be reached using a simplification via  $h$ , and then take the union over all those intervals. For this, the only unclear part is how to calculate these intervals.

We argue that, for any given  $k$  and  $i$ , there are at most  $n-1$  reachable intervals on  $[0, 1]$ , each contained in an edge of  $P$ . Indeed, every  $(k, i)$ -reachable point  $\pi(t)$  must have distance at most  $\varepsilon$  to  $p_i$ , and since the edge  $e$  of  $P$  that  $\pi(t)$  lies on intersects the disk of radius  $\varepsilon$  centered at  $p_i$  in a line segment, every point on this segment is also  $(k, i)$ -reachable. We denote the farthest point on  $e$  which is  $(k, i)$ -reachable by  $\hat{t}$ .

Furthermore, we argue that for each edge of  $P$ , we only need to take the farthest reachable point into account during our dynamic programming algorithm.

► **Lemma 10.** *If  $k, h, i, s$ , and  $t$  exist such that  $\rho(k-1, h, s) = \rho(k, i, t) = \mathbf{true}$ , and  $\overline{p_h p_i}$  has Fréchet distance  $\leq \varepsilon$  to  $P[s, t]$ , then  $\overline{p_h p_i}$  also has Fréchet distance  $\leq \varepsilon$  to  $P[\hat{s}, \hat{t}]$ .*

**Proof.** By the above argument,  $P[s, \hat{s}]$  is a line segment that lies completely within distance  $\varepsilon$  from  $p_h$ , and  $P[\hat{t}, t]$  is a line segment that lies completely within distance  $\varepsilon$  from  $p_i$ .

We are given that the Fréchet distance between  $\overline{p_h p_i}$  and  $P[s, t]$  is at most  $\varepsilon$ ; this means a mapping  $f : [s, t] \rightarrow \overline{p_h p_i}$  exists such that  $|\pi(x) - f(x)| \leq \varepsilon$ . Let  $q = f(s')$ . Then  $|p_h - \pi(\hat{s})| \leq \varepsilon$  and  $|q - \pi(\hat{s})| \leq \varepsilon$ , so the line segment  $\overline{p_h q}$  lies fully within distance  $\varepsilon$  from  $\hat{s}$ .

Therefore, we can define a new  $\varepsilon$ -Fréchet mapping between  $P[\hat{s}, \hat{t}]$  and  $\overline{p_h p_i}$  which maps  $\hat{s}$  to the segment  $\overline{p_h q}$ , the curve  $P[\hat{s}, t]$  to the segment  $\overline{qp_i}$  (following the mapping given by  $f$ ), and the segment  $\overline{\pi(t)\pi(\hat{t})}$  to the point  $p_i$ . ◀

Now, we can compute the optimal simplification by maintaining a  $k \times n \times n$  table storing  $\rho(k, i, \hat{t})$ , and calculate each value by looking up  $n^2$  values for the previous value of  $k$ , and testing in linear time for each combination whether the Fréchet distance between the new link and  $P[\hat{s}, \hat{t}]$  is within  $\varepsilon$  or not.

► **Theorem 11.** *Given a polyline  $P = \langle p_1, \dots, p_n \rangle$  and a value  $\varepsilon$ , we can compute the optimal polyline simplification of  $P$  that has Fréchet distance at most  $\varepsilon$  to  $P$  in  $O(kn^5)$  time and  $O(kn^2)$  space, where  $k$  is the output complexity of the optimal simplification.*

## 6 Conclusions

In this paper, we analyzed well-known polygonal line simplification algorithms, the Douglas-Peucker and the Imai-Iri algorithm, under both the Hausdorff and the Fréchet distance. Both algorithms are not optimal when considering these measures. We studied the relation between the number of vertices in the resulting simplified polyline from both algorithms and the enlargement factor needed to approximate the optimal solution. For the Hausdorff distance, we presented a polyline where the optimal simplification uses only a constant number of vertices while the solution from both algorithms is the same as the input polyline, even if we enlarge  $\varepsilon$  by any constant factor. We obtain the same result for the Douglas-Peucker algorithm under the Fréchet distance. For the Imai-Iri algorithm, such a result does not exist but we have shown that we will need a constant factor more vertices if we enlarge the error threshold by some small constant, for certain polylines.

Next, we investigated the algorithmic problem of computing the optimal simplification using the Hausdorff and the Fréchet distance. For the directed and undirected Hausdorff distance, we gave NP hardness proofs. Interestingly, the optimal simplification in the other direction (from output to input) is solvable in polynomial time. Finally, we showed how to compute the optimal simplification under the Fréchet distance in polynomial time. Our algorithm is based on the dynamic programming method and runs in  $O(kn^5)$  time and requires  $O(kn^2)$  space.

A number of challenging open problems remain. First, we would like to show NP-hardness of computing an optimal simplification using the Hausdorff distance when the simplification may not have self-intersections. Second, we are interested in the computational status of the optimal simplification under the Hausdorff distance and the Fréchet distance when the simplification need not use the vertices of the input. Third, it is possible that the efficiency of our algorithm for computing an optimal simplification with Fréchet distance at most  $\varepsilon$  can be improved. Fourth, we may consider optimal polyline simplifications using the weak Fréchet distance.

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