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Symplectic manifolds with no Kähler structure

Bachelor Thesis

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Abstract
Kähler manifolds are manifolds with three compatible geometric structures. These are called the Riemannian structure, the symplectic structure and the complex structure. The main problem in this thesis is the Thurston-Weinstein problem. The Thurston-Weinstein problem is the problem of constructing symplectic manifolds with no Kähler structure. The first chapter deals with general properties of Kähler manifolds. The main result will be a criterion for dimension of some of its cohomologygroups. The second chapter will deal with differential graded algebra's, which is a generalization of the de Rham-complex. It will show that Kähler manifolds have a property called formality. The third chapter will look into nilmanifolds, which are quotients of Lie groups. It turns out that the only nilmanifolds which admit a Kähler structure are tori. The last chapter will give an outlook on the various ways people have tried to solve the Thurston-Weinstein problem in different situations.

Contents

Chapter 1

Introduction

Most mathematicians study sets equipped with some structure. There are many different kinds of structures. Examples include the structure of a group, the structure of a partial order, the structure of a topological space and so on. Some structures generalize others. A normed vector space generalizes the notion of a vector space with an inner product, and a group is the more general version of an abelian group. This is also the relation between the two main objects of study in this thesis. A Kähler manifold is a special type of symplectic manifold. One natural question which arises is the question whether there even exist symplectic manifolds which do not admit a Kähler structure. This question was first treated by William Thurston in 1976 (see [5]). He attributed the problem to Alan Weinstein, so it has become known as the Thurston-Weinstein problem. Since Thurston's first example, many people have constructed symplectic manifolds which do not admit a Kähler structure. This thesis will develop the main tools used to prove that certain symplectic manifolds do not admit a Kähler structure. It also contains some examples of symplectic manifolds carrying no Kähler structure.

The first chapter will start with defining Kähler manifolds. Käler manifolds are smooth manifolds with a symplectic structure and a complex structure such that the symplectic form and the induced almost complex structure are compatible. The complex structure gives rise to new differential operators ∂ and $\bar{\partial}$. With these new differential operators comes a new cohomologytheory, called the Dolbeault cohomology. The Dolbeault cohomologygroups have two indices as a consequence of the two differential operators. The question of how Dolbeault cohomology relates to de Rhamcohomology is given by Hodge theory, which is the topic of the last subsection of the first chapter. The most important theorem is that for compact Kähler manifolds there is an isomorphism

$$H^k(M;\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$$

which relates de Rhamcohomology to Dolbeaultcohomology. As a consequence, odd de Rhamcohomologygroups have even dimension. This consequence is used in the first example of a symplectic manifold with no Kähler structure.

The second chapter generalizes the notion of the exterior derivative on differential forms to differential graded algebra's. Differential graded algebra's are graded algebra's with a differential which behaves like the exterior derivative. With any *DGA* there is are associated cohomologygroups. Every reasonable *DGA* has a minimal model, which has the same cohomology and satisfies some other conditions. This minimal model is unique and is usually easier to deal with than the *DGA* itself. One property a minimal model can satisfy is called formality. The main result of this section is that compact Kähler manifolds have formal minimal models. Massey products are introduced as a tool to proof non-formality. The setup of this chapter is based on the book *Symplectic Manifolds with no Kähler Structure* by Aleksy Tralle and John Oprea, see [4].

The third chapter looks into a special kind of manifold called a nilmanifold. Nilmanifolds are quotients of nilpotent Liegroups by discrete subgroups. Their minimal model can be described in terms of only the Lie algebra associated to the Lie group. This is used in proving that the only nilmanifolds which admit a Kähler struture are tori. An example is given of an countable family of symplectic nilmanifolds which are not diffeomorphic to tori and do therefore not admit any Kähler structure.

The last chapter gives an outlook in several different ways in which the Thurston-Weinstein problem is solved. This chapter doesn't go in detail as much as the other chapters, but it gives a broad impression on the versatility of the problem.

The reader is assumed to be familiar with smooth manifolds. If this is not the case, the reader is advised to study the basic concepts of smooth manifolds before reading this thesis. A good book for this is *Introduction to Smooth Manifolds* by John Lee ([1]). For understanding some parts of this thesis basic knowledge of algebraic topology is required. This can be obtained from *Algebraic Topology* by Allen Hatcher ([2]).

Chapter 2

Kähler Manifolds

A Kähler manifold is a smooth manifold equipped with three compatible structures. Before we will go into structures on manifolds, we will discuss structures on vector spaces. Recall from linear algebra that an inner product on a real vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying the following conditions:

- $\langle v, w \rangle = \langle w, v \rangle, \forall v, w \in V$ (symmetry)
- $\langle \lambda v + \mu u, w \rangle = \lambda \langle v, w \rangle + \mu \langle u, w \rangle, \forall \lambda, \mu \in \mathbb{R}, \forall u, v, w \in V$ (linearity in the first argument)
- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$ (positive definiteness)

The second linear structure is the linear complex structure.
Let V be a real vector space. A linear complex structure on V is a linear map $J : V \to V$ such that $J^2 = -Id_V$. The basic intuition behind this definition is that J turns real vector spaces into complex vector spaces. We can define the multiplication of a vector v by a complex number $a + b\imath$ to be $(a+b\imath)v = av + bJ(v)$. However, we can also let J act like $-\imath$. This last fact will become important later on. Since complex vector spaces have even real dimensions, V can only admit a linear complex structure if it is even dimensional.

The third and last linear structure is the linear symplectic structure. A linear symplectic form on a vector space V is a map $\omega : V \times V \to \mathbb{R}$ satisfying the following conditions:

- $\omega(v, w) = -\omega(w, v), \forall v, w \in V$ (skew-symmetry)
- $\omega(\lambda v + \mu u, w) = \lambda \omega(v, w) + \mu \omega(u, w), \forall \lambda, \mu \in \mathbb{R}, \forall u, v, w \in V$ (linearity in the first argument)
- $\omega(v, w) = 0 \ \forall w \in V$ if and only if $v = 0$ (nondegeneracy)

Nondegeneracy can also be characterized differently. A linear symplectic form ω is nondegenerate if and only if $\omega^{\wedge n} \neq 0$.
Let V be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$, a linear symplectic form ω and an linear complex structure J. The triple $(\langle \cdot, \cdot \rangle, \omega, J)$ is said to be compatible if $\langle v, w \rangle = \omega(v, J(w))$ for all $v, w \in V$.
Because compatible triples have an equation relating them, only two of the structures are needed to construct the last one. However, not all combinations of two linear structures can be made into a compatible triple.

Let V be a vector space equipped with a linear symplectic form ω and a linear complex structure J. Then ω and J are compatible if $\omega(v, w) = \omega(Jv, Jw)$ for all $v, w \in V$ and if $\omega(v, Jv) > 0$ for all $v \in V \backslash \{0\}$. If ω and J are compatible, then we can define an inner product $\langle \cdot, \cdot \rangle$ by $\langle v, w \rangle = \omega(v, Jw)$.

Then $\langle \cdot, \cdot \rangle$ is obviously bilinear and positive definite. It is also symmetric: $\langle v, w \rangle = \omega(v, Jw) = \omega(Jv, J^2 w) = \omega(Jv, -w) = \omega(w, Jv) = \langle w, v \rangle$.

Let V be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$ and a linear complex structure J. Then $\langle \cdot, \cdot \rangle$ and J are compatible if $\langle v, w \rangle = \langle Jv, Jw \rangle$ for all $v, w \in V$. If $\langle \cdot, \cdot \rangle$ and J are compatible, then we can define a linear symplectic form ω by $\omega(v, w) = \langle Jv, w \rangle$. Then $\omega(v, Jv) = \langle v, v \rangle$, which means that ω is nondegenerate. Since $\omega(v, w) = \langle Jv, w \rangle = \langle J^2 v, Jw \rangle = -\langle Jw, v \rangle = -\omega(w, v)$, ω is also skew-symmetric.

Lemma 2.0.0.1. *Let V be a vector space equipped with a linear symplectic structure ω. Then there exists a linear complex structure compatible with ω.*

Proof. Let g be an arbitrary inner product on V. By nondegeneracy, we get two isomorphisms between V and V^*:

$$\tilde{g} : V \to V^*,\ \tilde{g}(v)(w) = g(v, w)$$

$$\tilde{\omega} : V \to V^*,\ \tilde{\omega}(v)(w) = \omega(v, w)$$

Then define $A = \tilde{g}^{-1} \circ \tilde{\omega}$. Since it is the compostion of isomorphisms, it is an isomorphism. Since $g(Av, w) = \tilde{g}(Av)(w) = \tilde{\omega}(v)(w) = \omega(v, w) = -\omega(w, v) = -g(Aw, v) = -g(v, Aw)$, we get $A^* = -A$. Now AA^* is symmetric ($(AA^*)^* = AA^*$) and positive ($g(AA^*v, v) = g(A^*v, A^*v) > 0$ for $v \neq 0$). From linear algebra we know that this implies that AA^* is diagonalizable. Let

$$AA^* = B\ diag(\lambda_1, \ldots, \lambda_n)\ B^{-1}$$

be the diagonalization. Then define

$$\sqrt{AA^*} = B\ diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})\ B^{-1}$$

and take

$$J = (\sqrt{AA^*})^{-1} A$$

With this definition we get $JJ^* = Id_V$. Since A is skew-adjoint, so is J, so $J^2 = -JJ^* = -Id_V$. This linear complex structure is compatible with ω:

$$\omega(Jv, Jw) = g(AJv, Jw) = g(JAv, Jw) = g(Av, J^*Jw) = g(Av, w) = \omega(v, w)$$

and for $v \neq 0$:

$$\omega(v, Jv) = g(Av, Jv) = g(J^*Av, v) = g(\sqrt{AA^*}v, v) > 0$$

□

This lemma implies that vector spaces which admit linear symplectic structures must be even dimensional.

We want to apply these linear structures to smooth manifolds. We can do this by defining them on each tangent space. However, we want them to respect the smooth structure of the manifold, so these linear structures must vary smoothly. This notion of smoothness can be described in two ways. The first way is in charts.

Let M be a smooth manifold and (U, φ) be a chart of M with local coordinates $(x^1, \ldots, x^n)$. If $g = (g_x)_{x \in M}$ is an inner product on every tangent space $T_xM$of M, it can locally be described by $g = \sum\limits_{1 \leq i,j \leq n} g_{ij} dx^i dx^j$. We call g smooth around a point $x \in M$ if there is a chart (U, φ) with $x \in U$ such that all local functions g_{ij} are smooth. We call g smooth if it is smooth around every $x \in M$.

If $\omega = (\omega_x)_{x \in M}$ is a linear symplectic form on every tangent space T_xM of M, it can locally be described by $\omega = \sum\limits_{i<j} \omega_{ij} dx^i \wedge dx^j$. We call ω smooth around a point $x \in M$ if there is a chart (U, φ)

with $x \in U$ such that all local functions ω_{ij} are smooth. We call ω smooth if it is smooth around every $x \in M$.

The second and more natural way to characterize smoothness is in terms of vectorfields. We call $g = (g_x)_{x \in M}$ smooth if $g(X, Y)$ is a smooth function for all smooth vectorfields X and Y. The same definition holds for linear symplectic forms: $\omega = (\omega_x)_{x \in M}$ is smooth if $\omega(X, Y)$ is a smooth function for all smooth vectorfields X and Y. This definition works for linear complex structures as well. We call $J = (J_x)_{x \in M}$ smooth if $J(X)$ is a smooth vectorfield for every smooth vectorfield X.

Compatibility on vector spaces can be generalized to manifolds. We call two structures on a manifold M compatible if their linear structures are compatible on every tangent space T_xM of M.

Definition 2.0.0.2. A Riemannian manifold (M, g) is a manifold M together with a smoothly varying inner product g_x on the tangent space T_xM. A smoothly varying inner product is called a Riemannian metric.

All smooth manifolds admit Riemannian structures. For our purposes, this will be the least important structure of Kähler manifolds. The next structure is the symplectic structure, which will be our main concern.

Definition 2.0.0.3. An almost symplectic form is a smoothly varying linear symplectic form ω_x on each tangent space T_xM. Since it is skew-symmetric, bilinear and smooth, it is a 2-form. An almost symplectic form is called symplectic if it is closed. The pair (M, ω) is called a symplectic manifold.

Symplectic structures are more rare than Riemannian structures. The first reason for that is that only even dimensional manifolds can admit a symplectic structure.

Example 2.0.0.4. The 2-sphere S^2 admits a symplectic structure. If we see S^2 as an embedded submanifold $S^2 \subset \mathbb{R}^3$, then the symplectic form is given by $\omega_x(v, w) = x \cdot (v \times w)$. Here the $\cdot$ means the dot product in $\mathbb{R}^3$ and $\times$ denotes the cross product in $\mathbb{R}^3$. A useful identity involving the dot product and the cross product is given by $a \cdot (b \times c) = c \cdot (a \times b)$. With this identity we find $\omega_x(v, x \times v) = ||x \times v||^2$. Here $x \times v \in T_xS^2$ and $||x \times v||^2 = 0$ if and only if $v = 0$. Hence ω is nondegenerate. Since ω is a 2-form and S^2 is two dimensional, ω is closed. Hence ω is a symplectic form. $\triangle$

This example does not generalize to higher dimensions. To prove there are no higher dimensional symplectic spheres, we need a lemma.

Lemma 2.0.0.5. *Let (M, ω) be a compact symplectic manifold. Then $H^2(M) \neq 0$.*

Proof. Proof by contradiction. Suppose that $H^2(M) = 0$, then all closed 2-forms are exact. In particular the symplectic form ω is exact, so $\omega = d\alpha$ for some 1-form α. Since ω is nondegenrate, $\omega^{\wedge n}$ is a volume form. By Stokes' theorem, we get

$$0 \neq \int_M \omega^{\wedge n} = \int_M \omega \wedge \omega^{\wedge(n-1)} = \int_M d\alpha \wedge \omega^{\wedge(n-1)} = \int_M d(\alpha \wedge \omega^{\wedge(n-1)}) = \int_{\partial M} \alpha \wedge \omega^{\wedge(n-1)} = \int_{\emptyset} \alpha \wedge \omega^{\wedge(n-1)} = 0$$

From this contradiction we conclude that ω represents a nontrivial cohomologyclass. $\square$

This lemma can easily be generalized to any even dimension up to the dimension of the manifold. Since spheres have only nontrivial cohomology in dimension 0 and in the dimension of the manifold, there are no symplectic spheres of dimension > 2.
The last structure on a manifold is the complex structure.

Definition 2.0.0.6. Let M be a smooth manifold. An almost complex structure on M is a smoothly varying complex structure J_x on the tangent space T_xM.

The name 'almost complex' suggests that there is also something called a complex structure. This is indeed true.

Definition 2.0.0.7. A complex manifold is a topological manifold M together with a maximal atlas of charts (U, φ) between open subsets of M and $\mathbb{C}^n$ which cover M and for which the transition maps are holomorphic.

Any complex manifold can be made into a real smooth manifold by identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. One could expect that every complex manifold has an almost complex structure. This is indeed the case, and it is even canonical.

Theorem 2.0.0.8. *Let M be a complex manifold. Then M has a canonical almost complex structure J.*

Proof. Let (U, φ) be a chart, with $z_1, \ldots, z_n$ as local coordinates, where $z_j = x_j + \imath y_j$. Then define J on U by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j},\ J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

We want to show that this is a well-defined global almost complex structure. Consider two charts (U, φ) and (V, ψ), with local coordinates $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$, where $z_j = x_j + \imath y_j$ and $w_j = u_j + \imath v_j$. The almost complex structure as defined in the chart (U, φ) will be denoted by J_1 and the almost complex structure as defined in the chart (V, ψ) will be called J_2. By the Cauchy-Riemann equations, we see that on the overlap $U \cap V$ they agree:

$$\begin{aligned}
J_2\left(\frac{\partial}{\partial x_j}\right) &= J_2\left(\sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial v_k}\right) \\
&= \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial v_k} - \frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial u_k} \\
&= \sum_{k=1}^{n} \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial v_k} + \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial u_k} \\
&= \frac{\partial}{\partial y_j} \\
&= J_1\left(\frac{\partial}{\partial x_j}\right)
\end{aligned}$$

$$\begin{aligned}
J_2\left(\frac{\partial}{\partial y_j}\right) &= J_2\left(\sum_{k=1}^{n} \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial v_k}\right) \\
&= \sum_{k=1}^{n} \frac{\partial u_k}{\partial y_j}\frac{\partial}{\partial v_k} - \frac{\partial v_k}{\partial y_j}\frac{\partial}{\partial u_k} \\
&= \sum_{k=1}^{n} -\frac{\partial v_k}{\partial x_j}\frac{\partial}{\partial v_k} - \frac{\partial u_k}{\partial x_j}\frac{\partial}{\partial u_k} \\
&= -\frac{\partial}{\partial x_j} \\
&= J_1\left(\frac{\partial}{\partial y_j}\right)
\end{aligned}$$

We conclude that $J_1 = J_2$, so the almost complex structure is well-defined globally. $\square$

This theorem shows that every complex structure gives rise to an almost complex structure. The converse, however, may not be true. To distinguish the complex manifolds from the other almost complex manifolds, we have the following definition:

Definition 2.0.0.9. Let M be a manifold with an almost complex structure J. Then we call J integrable if it is induced by some complex structure on M.

The three geometric structures described in this section combine nicely into a structure called the Kähler structure.

Definition 2.0.0.10. A Kähler manifold is a smooth manifold with a complex structure and a symplectic structure such that the induced almost complex structure and the symplectic form are compatible.

Remark 2.0.0.11. There are several equivalent definitions of a Kähler manifold, each from a different point of view. The definition above emphasizes the symplectic structure and the complex structure since that is the most natural one for the purpose of this thesis.

Definition 2.0.0.12. A symplectic manifold (M, ω) of dimension $2n$ satisfies the hard Lefschetz property if the map

$$[\omega]^j : H^{n-j}(M) \to H^{n+j}(M)$$
$$[\alpha] \mapsto [\omega \wedge \alpha]$$

is an isomorphism for all $1 \leq j \leq n$

The hard Lefschetz theorem states the following:

Theorem 2.0.0.13. *Let (M, ω) be a compact Kähler manifold. Then (M, ω) satisfies the hard Lefschetz property.*

A proof of this theorem can be found in [8].

2.1 Complex forms

In this section, we will work with an almost complex manifold (M, J). The almost complex structure is not necessarily integrable, unless stated otherwise.

Definition 2.1.0.1. The complexified tangent bundle of M is the bundle $TM \otimes \mathbb{C}$. The almost complex structure extends to $TM \otimes \mathbb{C}$ by:

$$J(v \otimes z) = J(v) \otimes z, \forall v \in TM, z \in \mathbb{C}$$

Since $J^2 = -Id_{TM}$, J has eigenvalues $\imath$ and $-\imath$. Hence the complexified tangent bundle is the direct sum of subbundles $T_{1,0}$ and $T_{0,1}$ which are given by

$$\begin{aligned} T_{1,0} =& \{v \in TM \otimes \mathbb{C} \mid J(v) = \imath v\} \\ =& \{v \otimes 1 - Jv \otimes \imath \mid v \in TM\} \\ T_{0,1} =& \{v \in TM \otimes \mathbb{C} \mid J(v) = -\imath v\} \\ =& \{v \otimes 1 + Jv \otimes \imath \mid v \in TM\} \end{aligned}$$

These linear subspaces both come with a natural projection:

$$\pi_{1,0} : TM \otimes \mathbb{C} \to T_{1,0},\ v \otimes z \mapsto \frac{1}{2}(v \otimes z - Jv \otimes \imath z)$$
$$\pi_{0,1} : TM \otimes \mathbb{C} \to T_{0,1},\ v \otimes z \mapsto \frac{1}{2}(v \otimes z + Jv \otimes \imath z)$$

for which $\pi_{1,0}^2 = \pi_{1,0}$ and $\pi_{0,1}^2 = \pi_{0,1}$ and which combine into an isomorphism

$$(\pi_{1,0}, \pi_{0,1}) : TM \otimes \mathbb{C} \xrightarrow{\sim} T_{1,0} \oplus T_{0,1}$$

Just like the tangent space, the cotangent space also admits a complexification.

Definition 2.1.0.2. The complexified cotangent bundle of M is the bundle $T^*M \otimes \mathbb{C}$. The almost complex structure extends to $TM^* \otimes \mathbb{C}$ by

$$J(\alpha \otimes z) = \alpha \circ J \otimes z, \forall \alpha \in T^*M, z \in \mathbb{C}$$

Similarly to the tangent bundle, the complexified cotangent bundle is the direct sum of subbundles $T^{1,0}$ and $T^{0,1}$ which are given by

$$\begin{aligned} T^{1,0} =& T^*_{1,0} = \{\alpha \in T^*M \otimes \mathbb{C} \mid \alpha \circ J = \imath\alpha\} \\ =& \{\alpha \otimes 1 - \alpha \circ J \otimes \imath \mid \alpha \in T^*M\} \\ T_{0,1} =& T^*_{0,1} = \{\alpha \in T^*M \otimes \mathbb{C} \mid \alpha \circ J = -\imath\alpha\} \\ =& \{\alpha \otimes 1 + \alpha \circ J \otimes \imath \mid \alpha \in T^*M\} \end{aligned}$$

These linear subspaces also come with a natural projection:

$$\pi^{1,0} : T^*M \otimes \mathbb{C} \to T^{1,0},\ \alpha \otimes z \mapsto \frac{1}{2}(\alpha \otimes z - \alpha \circ J \otimes \imath z)$$
$$\pi^{0,1} : T^*M \otimes \mathbb{C} \to T^{0,1},\ \alpha \otimes z \mapsto \frac{1}{2}(\alpha \otimes z + \alpha \circ J \otimes \imath z)$$

which combine into an isomorphism

$$(\pi^{1,0}, \pi^{0,1}) : T^*M \otimes \mathbb{C} \xrightarrow{\sim} T^{1,0} \oplus T^{0,1}$$

Elements of $T_{1,0}$ are called J-holomorphic tangent vectors, while elements of $T_{0,1}$ are called J-antiholomorphic tangent vectors. Similarly, elements of $T^{1,0}$ are called J-holomorphic 1-forms and elements of $T^{0,1}$ are called J-antiholomorphic 1-forms. This allows us to define

$$\Lambda^{n,m}(T^*M \otimes \mathbb{C}) = (\Lambda^n T^{1,0}) \wedge (\Lambda^m T^{0,1})$$
$$\Lambda^k(T^*M \otimes \mathbb{C}) = \Lambda^k(T^{1,0} \oplus T^{0,1}) = \bigoplus_{n+m=k} \Lambda^{n,m}(T^*M \otimes \mathbb{C})$$

Now $\Omega^{n,m}(M)$ consists of all sections of $\Lambda^{n,m}(T^*M \otimes \mathbb{C})$. Its elements are called differential forms of type (n, m). Complex valued k-forms are sections of $\Lambda^k(T^*M \otimes \mathbb{C})$. The set of all complex valued k-forms is denoted by $\Omega^k(M; \mathbb{C})$. For $k = n + m$ we have the natural projections

$$\pi^{n,m} : \Lambda^{n,m}(T^*M \otimes \mathbb{C}) \to \Lambda^k(T^*M \otimes \mathbb{C})$$

These projections give rise to new differential operators:

Definition 2.1.0.3. Let (M, J) be an almost complex manifold. Then we define the differential operators ∂ and $\bar\partial$ to be

$$\partial = \pi^{n+1,m} \circ d : \Omega^{n,m}(M) \to \Omega^{n+1,m}(M)$$
$$\bar\partial = \pi^{n,m+1} \circ d : \Omega^{n,m}(M) \to \Omega^{n,m+1}(M)$$

If M is a complex manifold and J is the induced almost complex structure, then these differential operators have a more concrete description in local coordinates. Let (U, φ) be a chart with local coordinates $(z^1, \ldots, z^n)$ and $z^j = x^j + \imath y^j$. Then we define $dz^j = \frac{1}{2}(dx^j + \imath dy^j)$ and $d\bar z^j = \frac{1}{2}(dx^j - \imath dy^j)$. Similarly, we define $\frac{\partial}{\partial z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} - \imath\frac{\partial}{\partial y^j})$ and $\frac{\partial}{\partial \bar z^j} = \frac{1}{2}(\frac{\partial}{\partial x^j} + \imath\frac{\partial}{\partial y^j})$. Then locally for any smooth function f we get

$$\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z^j} dz^j \text{ and } \bar\partial f = \sum_{j=1}^n \frac{\partial f}{\partial \bar z^j} d\bar z^j$$

A function $f \in C^\infty(M,\mathbb{C})$ is J-holomorphic if $df \circ J = \imath df$. Equivalently, it is J-holomorphic if $\bar{\partial} f = 0$. A function $f \in C^\infty(M,\mathbb{C})$ is J-antiholomorphic if $df \circ J = -\imath df$. Equivalently, it is J-antiholomorphic if $\partial f = 0$.
With these definitions it is possible to give some examples of Kähler manifolds.

Example 2.1.0.4.

1. The easiest example of a Kähler manifold is $\mathbb{C}^n$. The identity is a global chart, so it is a complex manifold. We can canonically identify $T_z\mathbb{C}^n \simeq \mathbb{C}^n$. For coordinates $z_j = x_j + \imath y_j$ and $w_j = u_j + \imath v_j$ the Riemannian metric is given by $g(z,w) = \sum\limits_{j=1}^{n} x_j u_j + y_j v_j$, while the symplectoc structure is given by $\omega_0(z,w) = \sum\limits_{j=1}^{n} x_j v_j - y_j u_j$.

2. The standard symplectic structure on $\mathbb{C}^n$ is invariant under the action of $\mathbb{Z}^{2n}$. Hence on the torus $\mathbb{T}^{2n} = {\mathbb{C}^n}/{\mathbb{Z}^{2n}}$ we get a symplectic form. Let $\pi : \mathbb{C}^n \to \mathbb{T}^{2n}$ be the projection, then then symplectic form is uniquely determined by $\pi^*\omega = \omega_0$.

3. A Riemann surface (M,J) is a complex manifold of complex dimension 1, so it has real dimension 2. Since any manifold admits a Riemannian metric, we can always choose a Riemannian metric g. However, it can only be compatible with J if $g(x,y) = g(Jx,Jy)$ always holds. Now define a new Riemannian metric $\tilde{g}$ by $\tilde{g}(x,y) = g(x,y) + g(Jx,Jy)$. This new Riemannian metric is J-invariant. Now $\omega(x,y) = g(Jx,y)$ defines a nondegenerate 2-form. Since the real dimension of M is 2, $d\omega$ vanishes. Hence $(M,J,\tilde{g},\omega)$ is a Kähler manifold.

4. The complex projective space $\mathbb{CP}^n$ has a symplectic form compatible with the induced almost complex structure, which is called the Fubini-Study form. The construction of the Fubini-Study form begins with a 2-form on $\mathbb{C}^{n+1}\backslash\{0\}$. This 2-form is given by $\tilde{\omega} = \frac{\imath}{2}\partial\bar{\partial}log(|\cdot|^2)$. We can construct $\mathbb{CP}^n$ as the space of orbits of the action of $\mathbb{C}\backslash\{0\}$ on $\mathbb{C}^{n+1}\backslash\{0\}$. Since $\tilde{\omega}$ is invariant under this action, it gives rise to a 2-form on $\mathbb{CP}^n$. If π is the projection $\mathbb{C}^{n+1} \to \mathbb{CP}^n$, then the Fubini-Study form ω is uniquely determined by $\pi^*\omega = \tilde{\omega}$.

$\triangle$

2.2 Dolbeault cohomology

Let (M,J) be an almost complex manifold. For all $f \in C^\infty(M,\mathbb{C})$, the relation $df = \partial f + \bar{\partial} f$ holds. However, this relation doesn't need to hold for k-forms with $k > 0$.

Theorem 2.2.0.1. *Let M be a complex manifold with induced almost complex structure J. Then $d = \partial + \bar{\partial}$.*

Proof. Let (U,φ) be a chart of M, with local coordinates $z_1,\ldots,z_n$. Then $dz_1,\ldots,dz_n,d\bar{z}_1,\ldots d\bar{z}_n$ forms a basis for $\Omega^1(M;\mathbb{C})$. Here $dz_j = \frac{1}{2}(dx_j + \imath dy_j)$ and $d\bar{z}_j = \frac{1}{2}(dx_j - \imath dy_j)$. This basis splits the forms of type $(1,0)$ from those of type $(0,1)$, since

$$T^{1,0} = \mathbb{C} - Span\{dz_j \mid 1 \leq j \leq n\} \text{ and } T^{0,1} = \mathbb{C} - Span\{d\bar{z}_j \mid 1 \leq j \leq n\}$$

Now let $\alpha \in \Omega^l(M)$, then we can write locally

$$\alpha = \sum_{k+m=l} \sum_{|I|=m,|J|=k} \alpha_{IJ} dz_I \wedge d\bar{z}_J$$

for some $\alpha_{IJ} \in C^\infty(U;\mathbb{C})$. Here $I = \{i_1,\ldots,i_m\}$ and $J = \{j_1,\ldots,j_k\}$ are multi-indices and $dz_I =$

$dz_{i_1} \wedge \cdots \wedge dz_{i_m}$. Hence we get

$$\begin{aligned}
d\alpha &= \sum_{k+m=l} \sum_{|I|=m,|J|=k} d\alpha_{IJ} dz_I \wedge d\bar{z}_J \\
&= \sum_{k+m=l} \sum_{|I|=m,|J|=k} (\partial + \bar{\partial})\alpha_{IJ} dz_I \wedge d\bar{z}_J \\
&= \sum_{k+m=l} \sum_{|I|=m,|J|=k} \partial\alpha_{IJ} dz_I \wedge d\bar{z}_J + \bar{\partial}\alpha_{IJ} dz_I \wedge d\bar{z}_J \\
&= \partial\Big(\sum_{k+m=l} \sum_{|I|=m,|J|=k} \alpha_{IJ} dz_I \wedge d\bar{z}_J \Big) + \bar{\partial}\Big(\sum_{k+m=l} \sum_{|I|=m,|J|=k} \alpha_{IJ} dz_I \wedge d\bar{z}_J \Big) \\
&= \partial\alpha + \bar{\partial}\alpha
\end{aligned}$$

□

Remark 2.2.0.2. This argument may seem to hold for any almost complex manifold. However, for non-integrable almost complex structures, it is not possible to find local coordinates $z_1, \ldots, z_n$ which induce such a splitting in the space of 1-forms. It turns out that an almost complex structure J is integrable if and only if $d = \partial + \bar{\partial}$. This is part of the Newlander-Nirenberg theorem. For a proof, see [6].

Let $\alpha \in \Omega^{k,l}(M)$, then

$$0 = d^2\alpha = (\partial + \bar{\partial})(\partial + \bar{\partial})\alpha = \partial^2\alpha + \partial\bar{\partial}\alpha + \bar{\partial}\partial\alpha + \bar{\partial}^2\alpha$$

Since $\partial^2\alpha \in \Omega^{k+2,l}(M)$, $\partial\bar{\partial}\alpha + \bar{\partial}\partial\alpha \in \Omega^{k+1,l+1}(M)$ and $\bar{\partial}^2\alpha \in \Omega^{k,l+2}(M)$, we must have $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. This means that these new differential operators square to zero. Hence we can define cohomologygroups associated to these differential operators.

Definition 2.2.0.3. We define the Dolbeault cohomologygroup of type (l, m) to be

$$H^{l,m}(M) = \frac{Ker(\bar{\partial} : \Omega^{l,m}(M) \to \Omega^{l,m+1}(M))}{Im(\bar{\partial} : \Omega^{l,m-1}(M) \to \Omega^{l,m}(M))}$$

With these two cohomology theories one may wonder what the relation is between them. The answer to that will be that they are very strongly related. Before we are able to prove that, we will have to take a look at Hodge theory.

2.3 Hodge Theory

Let M be a compact Kähler manifold of dimension 2n with Riemannian metric g, symplectic form ω and (integrable) almost complex structure J. Recall that g induces an isomorphism $\tilde{g} : TM \to T^*M$ given by

$$\tilde{g}(v)(w) = g(v, w),\ \forall v, w \in T_xM,\ x \in M$$

Hence we can define an inner product on the cotangent bundle, which we will also denote by g

$$g(\alpha, \beta) = g(\tilde{g}^{-1}(\alpha), \tilde{g}^{-1}(\beta)),\ \forall \alpha, \beta \in \Omega^1(M)$$

We can extend this inner product to an inner product on $\Omega^k(M)$ for arbitrary k. To do this, first note that the pure wedges span $\Omega^k(M)$. Therefore, we only need to define the inner product on pure wedges and extend linearly. On pure wedges, this inner product is defined to be

$$g(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = det((g(\alpha_i, \beta_j))_{1 \le i,j \le k})$$

Every orientable Riemannian manifold has a canonical volume form on it. If $x^1, \ldots, x^{2n}$ is a positively oriented coordinate frame, then the Riemannian metric can be expressed as $g = \sum_{1 \leq i,j \leq 2n} g_{ij} dx^i dx^j$, where (g_{ij}) is a symmetric positive definite matrix. The volume form is locally given by

$$vol_g = \sqrt{det(g_{ij})} dx^1 \wedge \cdots \wedge x^{2n}$$

Definition 2.3.0.1. The Hodge star operator $\star : \Omega^k(M) \to \Omega^{2n-k}(M)$ is the unique operator which satisfies the equation

$$\alpha \wedge \star\beta = g(\alpha, \beta) vol_g$$

for all $\alpha, \beta \in \Omega^k(M)$. Since we are more interested in complex forms, we want to extend this to complex forms. Hence we extend $\star$ to be complex linear. In the complex case, we define the Hodge star operator $\star : \Omega^{p,q} \to \Omega^{n-q,n-p}$ to be the unique operator which satisfies the equation

$$\alpha \wedge \overline{\star\beta} = g(\alpha, \beta) vol_g$$

To simplify notation, we will write $\bar{\star}\beta = \overline{\star\beta}$. One interesting identity is that for k-forms α we have $\bar{\star}\bar{\star}\alpha = (-1)^k \alpha$. Using this Hodge star operator, we can define an inner product on the space of all complex valued differential forms. This inner product is given by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \bar{\star}\beta$$

With respect to this inner product, we can define an adjoint operator to the exterior derivative, which we will denote by d^*. By Stokes' theorem, we find for $\alpha \in \Omega^k(M; \mathbb{C})$ and $\beta \in \Omega^{k+1}(M; \mathbb{C})$ that

$$0 = \int_M d(\alpha \wedge \bar{\star}\beta) = \int_M d\alpha \wedge \bar{\star}\beta + (-1)^k \int_M \alpha \wedge d\bar{\star}\beta = \langle d\alpha, \beta \rangle + \langle \alpha, (-1)^k \star^{-1} d \star \beta \rangle$$

From this we conclude that $d^* = - \star d \star$. This allows us to define the Laplace-de Rham operator Δ_d, which is given by $\Delta_d = dd^* + d^*d$. If we follow the same procedure for $\bar{\partial}$, we find $\bar{\partial}^* = - \star \partial \star$ and we define $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$.

Definition 2.3.0.2. A k-form α is called harmonic with respect to d if $\Delta_d \alpha = 0$. The space of all harmonic forms of degree k will be denoted by $\mathcal{H}^k(M; \mathbb{C})$. The space of all harmonic forms with respect to $\bar{\partial}$ will be denoted by $\mathcal{H}^{p,q}(M)$.

Both Δ_d and $\Delta_{\bar{\partial}}$ are elliptic differential operators. Such operators have the property that they split $\Omega^k(M; \mathbb{C})$ in a nice way. We will present the following theorem without proof. For more information about elliptic differential operators and the proof of the theorem, see [7] chapters 1 and 4 or [8] chapters 4 and 5.

Theorem 2.3.0.3. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator on a compact manifold. Then*

$$\Gamma(E) = Ker(P) \oplus Im(P^*) \text{ and } \Gamma(F) = Ker(P^*) \oplus Im(P)$$

If we apply this theorem to Δ_d, we see that for compact Kähler manifolds

$$\Omega^k(M; \mathbb{C}) = \mathcal{H}^k(M; \mathbb{C}) \oplus \Delta_d \Omega^k(M; \mathbb{C})$$

And if we apply the theorem to $\Delta_{\bar{\partial}}$, we see that for compact Kähler manifolds

$$\Omega^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \Delta_{\bar{\partial}} \Omega^{p,q}(M)$$

Theorem 2.3.0.4. *Let M be a compact Kähler manifold. Then any element in the de Rham cohomologygroup has a unique harmonic representative. Hence we get the isomorphism*

$$H^k(M;\mathbb{C}) \simeq \mathcal{H}^k(M;\mathbb{C})$$

Proof. Since $\Delta_d = dd^* + d^*d$, we can split

$$\Delta_d \Omega^k(M;\mathbb{C}) = d(\Omega^{k-1}(M;\mathbb{C})) + d^*(\Omega^{k+1}(M;\mathbb{C}))$$

Take $\alpha \in d^*(\Omega^{k+1}(M;\mathbb{C}))$, so $\alpha = d^*\beta$ for some $\beta \in \Omega^{k+1}(M;\mathbb{C})$ and assume that $d\alpha = 0$. Then

$$0 = \langle dd^*\beta, \beta\rangle = \langle d^*\beta, d^*\beta\rangle = \langle \alpha, \alpha\rangle$$

so $\alpha = 0$. This means that we get

$$\Omega^k(M;\mathbb{C}) = \mathcal{H}^k(M;\mathbb{C}) \oplus d^*(\Omega^{k+1}(M;\mathbb{C})) \oplus d(\Omega^{k-1}(M;\mathbb{C}))$$

In particular, the only cohomology comes from $\mathcal{H}^k(M;\mathbb{C})$. What is left is to show that harmonic k-forms are closed but not exact. Take $\alpha \in \mathcal{H}^k(M;\mathbb{C})$. Then

$$0 = \langle \Delta_d\alpha, \alpha\rangle = \langle dd^*\alpha, \alpha\rangle + \langle d^*d\alpha, \alpha\rangle = \langle d^*\alpha, d^*\alpha\rangle + \langle d\alpha, d\alpha\rangle$$

Hence $d\alpha = d^*\alpha = 0$. Suppose $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M;\mathbb{C})$, then

$$\langle \alpha, \alpha\rangle = \langle d\beta, \alpha\rangle = \langle \beta, d^*\alpha\rangle = 0$$

Hence the only exact element of $\mathcal{H}^k(M;\mathbb{C})$ is 0. $\square$

The same argument holds for Dolbeault cohomology. Hence

$$H^{p,q}(M) \simeq \mathcal{H}^{p,q}(M)$$

Definition 2.3.0.5. The conjugate differential operator d^c is defined by conjugating the differential operator d with the (integrable) almost complex structure J

$$d^c = J^{-1} \circ d \circ J$$

Lemma 2.3.0.6. *The conjugate differential operator is given by $d^c = -\imath(\partial - \bar{\partial})$.*

Proof. Let $\alpha \in \Omega^{p,q}(M)$. Then d^c acts on α as follows:

$$\begin{aligned} d^c\alpha =& J^{-1}dJ\alpha \\ =& \imath^{p-q}J^{-1}d\alpha \\ =& \imath^{p-q}J^{-1}(\partial + \bar{\partial})\alpha \\ =& \imath^{p-q}J^{-1}\partial\alpha + \imath^{p-q}J^{-1}\bar{\partial})\alpha \\ =& \imath^{p-q}\imath^{-p-1+q}\partial\alpha + \imath^{p-q}\imath^{-p+q+1}\bar{\partial})\alpha \\ =& -\imath(\partial - \bar{\partial})\alpha \end{aligned}$$

$\square$

Lemma 2.3.0.7. *For the Laplace-de Rham operators the following equality's hold:*

$$\Delta_d = \Delta_{d^c} = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$$

Proof. The proof of this lemma involves the introduction of several new operators and many commutation relations. It can be found in [8]. □

As a consequence of theorem 2.3.0.3, the Laplace-de Rham operator Δ_d is a bijection between $\Delta_d\Omega^k(M;\mathbb{C})$ and itself. The inverse is called the Green operator associated to d and will be denoted by G_d. An immediate consequence of the previous lemma is that

$$G_d = G_{d^c} = \frac{1}{2}G_\partial = \frac{1}{2}G_{\bar\partial}$$

Theorem 2.3.0.8. *The de Rham cohomologygroups and the Dolbeault cohomologygroups of any Kähler manifold are related by*

$$H^k(M;\mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(M)$$

and the Dolbeault cohomologygroups satisfy

$$H^{p,q}(M) \simeq H^{q,p}(M)$$

Proof. Since all elements of $H^k(M;\mathbb{C})$ and $H^{p,q}(M)$ have unique harmonic representatives, the first statement is equivalent to the statement that

$$\mathcal{H}^k(M;\mathbb{C}) \simeq \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M)$$

The isomorphism is given by $(\pi^{0,k},\ldots,\pi^{k,0})$. To see this, let $\alpha \in \mathcal{H}^k(M;\mathbb{C})$. Then by lemma 2.3.0.7 and the fact that $\Delta_{\bar\partial}$ preserves bidegree,

$$\Delta_d\alpha = \Delta_d(\pi^{0,k} + \cdots + \pi^{k,0})\alpha = 2\Delta_{\bar\partial}\pi^{0,k}\alpha + \cdots + 2\Delta_{\bar\partial}\pi^{k,0}\alpha$$

Hence α is harmonic if and only if all of its components are harmonic.
For the second part, it suffices to show that $\mathcal{H}^{p,q}(M) \simeq \mathcal{H}^{q,p}(M)$. Let $\alpha \in \mathcal{H}^{p,q}(M)$, then $\Delta_{\bar\partial}\alpha = 0$. Conjugating this equation yields $\overline{\Delta_{\bar\partial}\alpha} = \Delta_\partial\bar\alpha = \Delta_{\bar\partial}\bar\alpha = 0$ by lemma 2.3.0.7. Hence the map $\alpha \mapsto \bar\alpha$ maps $\mathcal{H}^{p,q}(M)$ to $\mathcal{H}^{q,p}(M)$. It is its own inverse, so it really is an isomorphism. □

Corollary 2.3.0.9. *Let M be a compact Kähler manifold. Then the odd Bettinumbers $b_k = dim(H^k(M))$ are even.*

Proof. Let k be an odd integer. Then

$$\begin{aligned} b_k =& dim(H^k(M)) \\ =& \sum_{p+q=k} dim(H^{p,q}(M)) \\ =& \sum_{p<\frac{k}{2}} dim(H^{p,k-p}(M)) + dim(H^{k-p,p}(M)) \\ =& \sum_{p<\frac{k}{2}} dim(H^{p,k-p}(M)) + dim(H^{p,k-p}(M)) \\ =& 2\sum_{p<\frac{k}{2}} dim(H^{p,q}(M)) \end{aligned}$$

Hence b_k is even. □

This is the first result which distinguishes Kähler manifolds from general symplectic manifolds. If we find a symplectic manifold with an even bettinumber b_k for some odd k, then the manifold cannot admit a Kähler structure.

Example 2.3.0.10. Let $\omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$ be the standard symplectic form on $\mathbb{R}^4$. Consider the manifold $M = {\mathbb{R}^4}/{\Gamma}$, where Γ is the discrete group of affine transformations generated by

$$\begin{aligned} g_1 &: (x^1, x^2, x^3, x^4) \mapsto (x^1 + 1, x^2, x^3, x^4) \\ g_2 &: (x^1, x^2, x^3, x^4) \mapsto (x^1, x^2 + 1, x^3, x^4) \\ g_3 &: (x^1, x^2, x^3, x^4) \mapsto (x^1, x^2, x^3 + 1, x^4) \\ g_4 &: (x^1, x^2, x^3, x^4) \mapsto (x^1 + x^2, x^2, x^3, x^4 + 1) \end{aligned}$$

The fact that M is indeed a manifold follows from the quotient manifold theorem, see [1], theorem 7.10. This theorem says that if the action of a Lie group to a manifold is smooth, free and proper, then the quotient is again a manifold. Smoothness of the action is trivial. For the other two properties, we first extend Γ to a Lie group G by taking linear combinations of g_1, g_2, g_3 and g_4. Now G is diffeomorphic to $\mathbb{R}^4$, so Γ acts on G. Since Γ is a subgroup of G, the action is free. Let $\varphi : G \times G \to G \times G$, $(g, h) \mapsto (gh, h)$ be the map of the action of G onto itself. Then φ is invertible, with smooth and therefore continuous inverse. Since φ^{-1} is continuous, it sends compact subsets to compact subsets, so φ is proper. Hence the restricted action to the action of Γ on G is proper as well.
Clearly $g^*\omega_0 = \omega_0$ for any $g \in \Gamma$. Hence there is a unique symplectic form ω on M for which $\pi^*\omega = \omega_0$. Therefore, we have constructed a symplectic manifold (M, ω). Its fundamental group is given by $\pi_1(M) = \Gamma$. To see why this is the case, first remark that $\mathbb{R}^4$ is the universal cover of M. Then we define $x_g : I \to M$ to be the path $\pi(tg(0))$. From the universal lifting property (see [2]), we see that all paths are homotopic to exactly one of these $x_g's$.
The first homologygroup is given by ${\Gamma}/{[\Gamma, \Gamma]}$, see [2]. The generators g_1, g_2, and g_3 commute among themselves and g_4 commutes with g_1 and g_3. However, $g_4^{-1} g_2^{-1} g_4 g_2 = g_1$, so $[\Gamma, \Gamma]$ is the free group generated by g_1. This means that $H^1(M; \mathbb{Z}) \simeq \mathbb{Z}^3$, so $b_1 = 3$. By corollary 2.3.0.9, M does not admit any Kähler structure. △

Chapter 3

Differential graded algebra's

3.1 Minimal models

Definition 3.1.0.1. A graded algebra is an algebra A with a decomposition $A = \bigoplus_i A^i$ into homogeneous subspaces A^i such that $A^i \cdot A^j \subset A^{i+j}$.

Elements of A^i are called homogeneous elements of degree i. The degree of a homogeneous element $x \in A$ is denoted by $|x| = i$. A graded algebra is commutative if $x \cdot y = (-1)^{|x||y|} y \cdot x$ for all homogeneous elements $x, y \in A$.

Any algebra together with the trivial grading $A = A^0$ is a graded algebra. A more insightful example is a polynomial algebra over an arbitrary field with the usual grading. Such a polynomial is not commutative in the graded sense however. An important example will be the free algebra ΛV of a graded vector space $V = \bigoplus_i V^i$. This is the polynomial algebra on even elements and the exterior algebra on odd elements. Since commutativity of even elements is the same as usual commutativity, the free algebra is commutative in the graded sense.

Definition 3.1.0.2. A commutative differential graded algebra over a field k, or $k - DGA$ for short, is a commutative graded algebra A over a field k together with a differential $d : A^i \to A^{i+1}$ which satisfies the following conditions:

- $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$
- $d^2 = 0$

The exterior derivative d on the space of all differential forms $\Omega^*(M)$ of a manifold M is the main example of a DGA. We will use this later to describe properties of manifolds, which is our main concern. One problem is that this space is usually very big. To capture the essential information about a manifold without dealing with such a big space, we instead study the cohomology associated to it. Elements which vanish under the differential are called closed, and elements in the image of the differential are called exact.

Definition 3.1.0.3. The nth cohomologygroup $H^n(A, d)$ of a $k - DGA$ (A, d) is defined to be:

$$H^n(A, d) = \frac{Ker(d : A^n \to A^{n+1})}{Im(d : A^{n+1} \to A^n)}$$

Since $d^2 = 0$ we know that $Im(d : A^{n+1} \to A^n) \subset Ker(d : A^n \to A^{N+1})$, so this is well defined. In all cases we will be interested in, this group will have the structure of a finite dimensional vector space. Elements in the cohomologygroup will be denoted by $[x]$, where x is a representative.

Definition 3.1.0.4. Let $(\mathcal{M}_A, d)$ and (A, d_A) be $k - DGA$'s. Then $(\mathcal{M}_A, d)$ is a model for (A, d_A) if there exists a algebra homomorphism $\varphi : \mathcal{M}_A \to A$ which intertwines the differentials and for which $\varphi_* : H^n(\mathcal{M}_A, d) \to H^n(A, d_A), [x] \mapsto [\varphi(x)]$ defines an isomorphism.

Such a homomorphism is called a DGA homomorphism.
A generating set of an algebra A is a set X such that A is spanned by all elements of the type $x_1 \cdot \dots \cdot x_n$ for $n \in \mathbb{N}$ and $x_i \in X$ for $1 \leq i \leq n$.

Definition 3.1.0.5. Let $(\mathcal{M}_A, d)$ be a model of (A, d_A). Then it is called a minimal model if $(\mathcal{M}_A, d)$ is a free algebra of some graded vector space $V = \bigoplus\limits_i V^i$ and if there exists an ordering on a generating set $X = \{x_\alpha \mid \alpha \in I\}$ such that $x_\alpha < x_\beta$ implies $|x_\alpha| < |x_\beta|$ and such that dx_β contains only generators $x_\alpha < x_\beta$. Since $|dx_\beta| = |x_\beta| + 1$, there is no term in dx_β which is linear in the generators x_α.

Theorem 3.1.0.6. *Let (A, d_A) be a $k - DGA$ for which $H^0(A, d_A) = k$, $H^1(A, d_A) = 0$ then (A, d_A) has a minimal model.*

Proof. To prove the theorem, we construct a minimal model step-by-step. First, we define $\mathcal{M}(0) = \mathcal{M}(1) = \mathbb{Q}$. Now let Z^0 be a space of closed elements in A^0. Then define $\varphi_0 : \mathcal{M}(0) \xrightarrow{\sim} Z^0$ to be a vector space splitting of the natural projection.

Next, we define $\mathcal{M}(2) = \mathcal{M}(1) \otimes \Lambda H^2(A, d_A)$. Let $Z^2(A)$ be the space of closed elements in A^2. We set $d(H^2(A, d_A)) = 0$ and extend φ_1 to φ_2 by a vector space splitting $\varphi_2 : H^2(A, d_A) \xrightarrow{\sim} Z^2$ and extending freely. We will continue in this manner to construct the minimal model. Note that $\mathcal{M}(2)$ is minimal and that φ_2 induces an isomorphism in cohomology up to degree 2 and is injective in degree 3. This last statement is trivial, since $\mathcal{M}(2)$ doesn't contain any elements of degree 3. For the induction step, we assume the following:

1. $\mathcal{M}(n)$ is minimal
2. φ_n induces isomorphisms on cohomology up to degree n.
3. φ_n induces an injection on cohomology in degree $n + 1$.

For any natural number k, we will denote the space of closed elements of A^k by $Z^k(A)$, and the space of closed elements of $M(n)^k$ by $Z^k(M(n))$. There is a natural projection from $Z^k(A)$ to $H^k(A, d_A)$ by taking the cohomologyclass. Let $\alpha : H^{n+1}(A, d_A) \to Z^{n+1}$ and $\beta : H^{n+2}(A, d_A) \to Z^{n+2}(\mathcal{M}(n))$ be two vector space splittings of this natural projection. Denote by γ a vector space splitting of $d : A^{n+1} \to A^{n+2}$.
We define $V^{n+1} = Coker((\varphi_n)_* : H^{n+1}(\mathcal{M}(n), d) \to H^{n+1}(A, d_A))$ and $W^{n+1} = Ker((\varphi_n)_* : H^{n+2}(\mathcal{M}(n), d) \to H^{n+2}(A, d_A))$. Now we can go on with our construction and define $\mathcal{M}(n + 1) = \mathcal{M}(n) \otimes \Lambda(V^{n+1} \oplus W^{n+1})$. We also define the differential

$$d(v) = \begin{cases} d_{\mathcal{M}(n)}(v) & \text{if } v \in \mathcal{M}(n) \\ 0 & \text{if } v \in V^{n+1} \\ \beta(v) & \text{if } v \in W^{n+1} \end{cases}$$

as well as the homomorphism φ_{n+1}:

$$\varphi_{n+1}(v) = \begin{cases} \varphi_n(v) & \text{if } v \in \mathcal{M}(n) \\ \alpha(v) & \text{if } v \in V^{n+1} \\ \gamma(\varphi_n(\beta(v))) & \text{if } v \in W^{n+1} \end{cases}$$

With these definitions, it is easy to check that $\mathcal{M}(n+1)$ is minimal and that φ_{n+1} commutes with the differential. It also induces an isomorphism in cohomology up to degree $n+1$ and an injection in degree $n+2$. Hence, by induction $\mathcal{M}(n)$ satisfies these conditions for all natural numbers n. We conclude that $(\mathcal{M}_A = \cup_{n\in\mathbb{N}}\mathcal{M}(n), d)$ is a minimal model of (A, d_A). □

Remark 3.1.0.7. The theorem above also holds in the case that $H^1(A, d_A) \neq 0$. However, to prove the more general case is way more complicated while it doesn't give much more insight. For the whole proof, I refer the reader to [9].

This theorem shows the existence of minimal models of spaces satisfying some reasonable conditions. However, with existence always comes the question of uniqueness: Given a DGA, is its minimal model unique up to isomorphism? To answer this question, we first need a lemma:

Lemma 3.1.0.8. *Let $(\mathcal{M} = \Lambda V, d)$ and $(\mathcal{M}(n) = \Lambda \bigoplus_{i\leq n} V^i), d)$ be two minimal $DGA's$ with the natural inclusion $\imath_n : \mathcal{M}(n) \hookrightarrow \mathcal{M}$. Then $H^{n+2}(C(\imath_n)) \simeq V^{n+1}$. Here $C(\imath_n)$ denotes the the algebraic mapping cone of $\imath_n$. (see Appendix)*

Proof. Let (a, b) be an element of $C(\imath_n)$. Then (a, b) is closed if and only if $da = 0$ and $db = -a$. Hence for any $b \in \mathcal{M}$, there can be only one $a \in \mathcal{M}(n)$ for which (a, b) is closed. Because of this, the second coordinate captures all the necessary information about the pair. If $b \in \mathcal{M}^{n+1}$, we can uniquely write it as $b = b_1 + b_2$, where $b_1 \in V^{n+1}$ and $b_2 \in \mathcal{M}(n)^{n+1}$. Define

$$\theta : Z^{n+2}(C(\imath_n)) \to V^{n+1}, (a, b) \mapsto b_1$$

Let $v \in V^{n+1}$. Then $\theta(-dv, v) = v$, so θ is surjective. Now we want to find the kernel of θ. To do this, we note that $\theta(d(a, b)) = \theta(-da, a + db) = 0$ for any $(a, b) \in (C(\imath_n))^{n+1}$, since $a \in \mathcal{M}(n)^{n+2}$ trivially and $db \in \mathcal{M}(n)^{n+2}$ since $d(V^{n+1}) \subset \mathcal{M}(n)^{n+2}$. Hence $Im(d : (C(\imath_n))^{n+1} \to (C(\imath_n))^{n+2}) \subset Ker(\theta)$. Now let $(a, b) \in Ker(\theta)$, then $b \in \mathcal{M}(n)^{n+1}$, so $(-b, 0) \in \mathcal{M}(n)^{n+2}$ and $d(-b, 0) = (a, b)$. Hence $Ker(\theta) = Im(d)$, which means that $\theta_* : H^{n+2}(C(\imath_n)) \to V^{n+1}$ is an isomorphism. □

This lemma allows us to prove uniqueness of minimal models.

Theorem 3.1.0.9. *Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a map between minimal $DGA's$ which induces an isomorphism φ_* in cohomology. Then φ is an isomorphism.*

Proof. Since both $DGA's$ are minimal, $\mathcal{M}(0) = \mathcal{N}(0)$ and φ induces an isomorphism. Assume inductively that φ induces an isomorphism $\mathcal{M}(n) \to \mathcal{N}(n)$. Consider the following diagram:

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H^{n+1}(\mathcal{M}(n)) & \longrightarrow & H^{n+1}(\mathcal{M}) & \longrightarrow & H^{n+2}(C(\imath_{\mathcal{M}}) & \longrightarrow & H^{n+2}(\mathcal{M}(n)) & \longrightarrow & H^{n+2}(\mathcal{M}) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H^{n+1}(\mathcal{N}(n)) & \longrightarrow & H^{n+1}(\mathcal{N}) & \longrightarrow & H^{n+2}(C(\imath_{\mathcal{N}}) & \longrightarrow & H^{n+2}(\mathcal{N}(n)) & \longrightarrow & H^{n+2}(\mathcal{N}) & \longrightarrow & \ldots \end{array}$$

By induction hypothesis the first and the fourth vertical arrows are isomorphisms. Since φ induces isomorphisms in cohomology, the second and fifth vertical arrows are isomorphisms as well. The five lemma (see Appendix) states that the third vertical arrow is also an isomorphism. By the previous lemma, if $\mathcal{M} = \Lambda V$ and $\mathcal{N} = \Lambda W$, φ induces an isomorphism $V^{n+1} \to W^{n+1}$. Since $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda V^{n+1}$ and $\mathcal{N}(n+1) = \mathcal{N}(n) \otimes \Lambda W$, we get an induced isomorphism $\mathcal{M}(n+1) \to \mathcal{N}(n+1)$. By induction, we conclude that φ is an isomorphism. □

Corollary 3.1.0.10. *The minimal model of a DGA is unique up to isomorphism.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two minimal models of A, with homomorphisms $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{N}}$. Define $\psi : \mathcal{M} \to \mathcal{N}$ such that $\varphi_{\mathcal{M}} = \varphi_{\mathcal{N}} \circ \psi$. Since $\varphi_{\mathcal{M}}$ and $\varphi_{\mathcal{N}}$ induce isomorphisms in cohomology, ψ also induces an isomorphism in cohomology. By the previous theorem, we conclude that ψ is an isomorphism. □

3.2 Homotopy invariance

Let X, Y be two topological spaces and $f, g : X \to Y$ two maps between them. We always assume maps between topological spaces to be continuous. Then we say that f is homotopic to g, denoted $f \simeq g$, if there exists a map $H : X \times I \to Y$ such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

holds for all $x \in X$. Here I is the unit interval $[0, 1]$. A pointed topological space is a pair (X, x), where X is a topological space and $x \in X$ a point. A basepoint preserving homotopy between two pointed topological spaces (X, x) and (Y, y) is a homotopy H between X and Y for which $H(x, t) = y$ for all $t \in I$. We want to find a similar notion for $DGA's$. To do this, we note that there is an equivalent way of defining homotopy. We define two maps $\lambda_0, \lambda_1 : Y^I \to Y$ by

$$\lambda_0(\tau) = \tau(0) \text{ and } \lambda_1(\tau) = \tau(1)$$

Where Y^I denots the space of maps $I \to Y$. Then $f \simeq g$ if and only if there exists a map $K : X \to Y^I$ such that

$$\lambda_0 \circ K = f \text{ and } \lambda_1 \circ K = g$$

This formulation allows us to define homotopy for $DGA's$. We will define homotopy for minimal $DGA's$. Let $\mathcal{M} = (\Lambda X, \tilde{d})$ be a minimal model. Then we define $\mathcal{M}^I = (\Lambda X \otimes \Lambda \tilde{X} \otimes \Lambda d\tilde{X}, d)$, where the differential d is extended to generators of $\Lambda X \otimes \Lambda \tilde{X} \otimes \Lambda d\tilde{X}$ by

$$dx_i = \tilde{d}x_i,\ d(\tilde{x}_i) = d\tilde{x}_i \text{ and } d(d\tilde{x}_i) = 0$$

The degrees are given by $|\tilde{x}_i| = |x_i| - 1$ and $|d\tilde{x}_i| = |\tilde{x}_i| + 1 = |x_i|$.
A derivation of degree k on a graded differential algebra X is a map $\rho : X^n \to X^{n+k}$ for which

$$\rho(a \cdot b) = \rho(a) \cdot b + (-1)^{k|a|} a \cdot \rho(b)$$

The differential is an example of a derivation. It has degree 1. On $\mathcal{M}^I$, we can define another derivation of degree -1. We denote it by i and on generators it is given by

$$i(x_j) = \tilde{x}_j,\ i(\tilde{x}_j) = i(d\tilde{x}_j) = 0$$

This new derivation i also satisfies $i^2 = 0$. We define a new map $\theta = id + di$. Then

$$\begin{aligned}
\theta(a \cdot b) &= (id + di)(a \cdot b) \\
&= i(d(a) \cdot b) + (-1)^{|a|} i(a \cdot db) + d(i(a) \cdot b) + (-1)^{|a|} d \\
&= id(a) \cdot b + (-1)^{|a|+1}(d(a) \cdot ib) + (-1)^{|a|}(i(a) \cdot db) + a \cdot idb \\
&+ di(a) \cdot b + (-1)^{|a|+1}(i(a) \cdot db) + (-1)^{|a|}(d(a) \cdot ib) + a \cdot dib \\
&= id(a) \cdot b + a \cdot idb + di(a) \cdot b + a \cdot dib \\
&= \theta(a) \cdot b + a \cdot \theta(b)
\end{aligned}$$

Hence θ is a derivation of degree 0. For every degree of x there is a N such that $\theta^N = 0$ on that degree. Hence we can define

$$e^\theta = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}$$

without any concerns about convergence. Its inverse is given by

$$e^{-\theta} = \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!}$$

Hence e^θ is an automorphism of $\mathcal{M}^I$. Since θ commutes with d, so does e^θ. We define the maps $\lambda_0, \lambda_1 : \mathcal{M} \to \mathcal{M}^I$ by

$$\lambda_0(x_i) = x_i \text{ and } \lambda_1(x_i) = e^\theta x_i$$

With these maps we can define homotopy for $DGA's$.

Definition 3.2.0.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two minimal $DGA's$ and $f, g : \mathcal{M} \to \mathcal{N}$ two maps between them. Then f is DGA-homotopic to g if there exists a DGA-homomorphism $H : \mathcal{M}^I \to \mathcal{N}$ such that

$$H \circ \lambda_0 = f \text{ and } H \circ \lambda_1 = g$$

Remark 3.2.0.2. There is also a notion of based DGA-homotopy's. That involves augmentations, which we will not deal with here. For more information, see [4].

If x is closed, then e^θ reduces to $e^\theta = 1 + di$. Hence, we get

$$\begin{aligned} H(\lambda_1(x) &= H(\lambda_0(x) + H(d\tilde{x}) \\ g(x) &= f(x) + dH(\tilde{x}) \\ g(x) - f(x) &= dH(\tilde{x}) \end{aligned}$$

Since the difference $g(x) - f(x)$ is exact, DGA-homotopic maps induce the same homomorphism on cohomology. This reminds us of homotopy's between general topological spaces, where the same statement applies. This is no coincidence. The next theorem shows the connection between the two types of homotopy. For a proof, see [10].

Theorem 3.2.0.3. *Homotopic maps between smooth manifolds induce homotopic DGA-homomorphisms of their minimal models.*

To simplify notation, we will write $x_i' = \lambda_1(x_i)$ and we introduce

$$\Omega = \sum_{n=1}^{\infty} \frac{(id)^n}{n!}$$

such that $x_i' = x_i + d\tilde{x}_i + \Omega(x_i)$.

Lemma 3.2.0.4. *With these notations, we get the equality* $(\Lambda(x_i, \tilde{x}_i, d\tilde{x}_i), d) = (\Lambda(x_i, x_i', \tilde{x}_i), d)$.

Proof. We only need to show that $d\tilde{x}_i$ can be obtained from the generators $\{x_i, x_i', \tilde{x}_i\}$. The minimal $DGA's$ are built up in stages, so if x_i is in the first stage, then $\Omega(x_i) = 0$, so $d\tilde{x}_i = x_i' - x_i$. Now assume inductively that $d\tilde{x}_j$ can be obtained through stage k, and take x_i from stage $k+1$. Then $d\tilde{x}_i = x_i' - x_i - \Omega(x_i)$. But by induction $\Omega(x_i)$ can be obtained. Hence we can obtain $d\tilde{x}_i$ as well. $\square$

Lemma 3.2.0.5. *Let $f : \mathcal{M} \to A$ be a DGA homomorphism. Now extend $\mathcal{M}$ to $\tilde{\mathcal{M}} = \mathcal{M} \otimes V$ such that $d(V) \subset \mathcal{M}$, where $V = \langle x_1, \ldots, x_k \rangle$ for some generators $\{x_1, \ldots, x_k\}$ of degree n. Then we can extend f to a DGA homomorphism $\tilde{f} : \tilde{\mathcal{M}} \to A$ if and only if $[f(dx_i)] = 0 \in H^{n+1}(A)$ for all $1 \leq i \leq n$.*

Proof. First assume that such an extension $\tilde{f}$ exists. Then $f(dx_i) = df(x_i)$, so $[f(dx_i)] = 0$. Now assume that $[f(dx_i)] = 0$, then $f(dx_i) = da$ for some $a \in A$. Hence we can define $\tilde{f}(x_i) = a$. This construction extends f and intertwines the differentials. □

The extension which is used in the lemma above is called an elementary extension or an Hirsch extension. Now consider the following homotopy commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, "i"] & A \arrow[d, "\varphi"] \\ \tilde{\mathcal{M}} \arrow[r, "g"] & B \end{tikzcd}$$

Being homotopy commutative means that $f \circ \varphi \simeq i \circ g$. Here $\tilde{\mathcal{M}} = \mathcal{M} \otimes V$ as before.

Theorem 3.2.0.6. *There exist extensions $\tilde{f} : \tilde{\mathcal{M}} \to A$ and $\tilde{H} : \tilde{\mathcal{M}}^I \to B$ if and only if all cohomologyclasses of the form*

$$[(-f(dx_i), g(x_i) - H(\Omega(x_i)))] = 0 \in H^{n+1}(C(\varphi))$$

are zero. Here $C(\varphi)$ denotes the algebraic mapping cone of φ (see Appendix).

Proof. Since $\lambda_1(x_i) = \lambda_0(x_i) + d\tilde{x}_i + \Omega(x_i)$, we get $d\Omega(x_i) = d\lambda_1(x_i) - d\lambda_0(x_i) = (\lambda_1 - \lambda_0)(dx_i)$. By definition of DGA homotopy, we get $H\lambda_0 = \varphi f$ and $H\lambda_1 = gi$. Hence it follows that

$$\begin{aligned} d(-f(dx_i), g(x_i) - H(\Omega(x_i))) &= (df(dx_i), dg(x_i) - dH(\Omega(x_i)) - \varphi f(dx_i)) \\ &= (f(d^2x_i), g(dx_i) - H(d\Omega x_i) - \varphi(f(dx_i))) \\ &= (0, g(dx_i) - H(\lambda_1 - \lambda_0)dx_i - \varphi(f(dx_i))) \\ &= (0, g(dx_i) - g(dx_i) + \varphi(f(dx_i)) - \varphi(f(dx_i))) \\ &= (0, 0) \end{aligned}$$

Now assume that the extensions $\tilde{f}$ and $\tilde{H}$ exist. Then

$$\begin{aligned} d(\tilde{f}(x_i), \tilde{H}(\tilde{x}_i)) &= (d\tilde{f}(x_i), d\tilde{H}(\tilde{x}_i) + \varphi\tilde{(}f(x_i))) \\ &= (-\tilde{f}(dx_i), \tilde{H}(d\tilde{x}_i) + \varphi(\tilde{f}(x_i))) \\ &= (-f(dx_i), g(i(x_i)) - \varphi(\tilde{f}(x_i)) - H(\Omega(x_i)) + \varphi(\tilde{f}(x_i))) \\ &= (-f(dx_i), g(i(x_i)) - H(\Omega(x_i))) \end{aligned}$$

Hence if both extensions exist, then $[(-f(dx_i), g(x_i) - H(\Omega(x_i)))] = 0 \in H^{n+1}(C(\varphi))$.
Now assume that $[(-f(dx_i), g(x_i) - H(\Omega(x_i)))] = 0 \in H^{n+1}(C(\varphi))$. Then there exists $(a, b) \in C(\varphi)$ for which $d(a, b) = (-da, db + \varphi(a)) = (-f(dx_i), g(x_i) - H(\Omega(x_i)))$. This means that $da = f(dx_i)$ and $db + \varphi(a) = g(x_i) - H(\Omega(x_i))$. Hence we define $\tilde{f}(x_i) = a$. By lemma 3.2.0.4, define

$$\begin{cases} \tilde{H} \mid_{\mathcal{M}} = H \\ \tilde{H}(x_i) = \varphi(\tilde{x}_i) \\ \tilde{H}(x_i') = g(x_i) \\ \tilde{H}(\tilde{x}_i) = b \end{cases}$$

Then $\tilde{H}$ intertwines the differentials and extends H, so we have constructed both extensions. □

Corollary 3.2.0.7. *If φ induces an isomorphism on cohomology, then the extensions $\tilde{f}$ and $\tilde{H}$ exist. If φ is also surjective and $\varphi \circ f = g$, then $\tilde{f}$ can be constructed in such a way that $\tilde{f} \circ i = f$ and $\varphi \circ \tilde{f} = g$.*

Proof. Note that there is a long exact sequence of cohomologygroups:

$$\ldots \longrightarrow H^n(A) \xrightarrow{\varphi^*} H^n(B) \xrightarrow{j^*} H^{n+1}(C(\varphi)) \xrightarrow{p^*} H^{n+1}(A) \longrightarrow \ldots$$

where j and p are the natural inclusion and natural projection. Since φ^* is an isomorphism, $H^n(C(\varphi)) = 0$. Hence by the theorem both extensions exist.
Now suppose that φ is surjective and that $\varphi \circ f = g$. Then the original homotopy H can be chosen to satisfy $H(y) = \varphi(f(y))$, $H(\lambda_1(y)) = g(y)$ and $H(\tilde{y}) = 0$ for all generators $y \in \mathcal{M}$. Since $\lambda_1(y) = y + d\tilde{y} + \Omega(y)$, these equations imply that $H(\Omega(y)) = 0$. Therefore $(-f(dx_i), g(x_i) - H(\Omega(x_i)))$ reduces to $(-f(dx_i), g(x_i))$. Hence $[(-f(dx_i), g(x_i))] = 0$, so there exists (a, b) such that $d(a, b) = (-da, db + \varphi(a)) = (-f(dx_i), g(x_i)$. This means that $da = f(dx_i)$ and $db + \varphi(a) = g(x_i)$. Since φ is surjective, there exists an $a' \in A$ for which $\varphi(a') = b$. Then $g(x_i) = \varphi(a + da')$ and we define $\tilde{f}(x_i) = a + da'$, which then satisfies $\tilde{f} \circ i = f$ and $\varphi \circ \tilde{f} = g$. □

Theorem 3.2.0.8. *Let (A, d_A) and (B, d_B) be two $DGA's$ with minimal models $(\mathcal{M}_A, d_1)$ and $\mathcal{M}_B, d_2)$. Then any DGA homomorphism $f : (A, d_A) \to (B, d_B)$ can be lifted to a DGA homomorphism $F : (\mathcal{M}_A, d_1) \to (\mathcal{M}_B, d_2)$ completing the following homotopy commutative diagram:*

$$\begin{tikzcd}
(\mathcal{M}_A, d_1) \arrow[r, "F"] \arrow[d, "\varphi_A"] & (\mathcal{M}_B, d_2) \arrow[d, "\varphi_B"] \\
(A, d_A) \arrow[r, "f"] & (B, d_B)
\end{tikzcd}$$

Proof. The minimal model is built up out of Hirsch extensions. Hence each extension has the same structure as $\tilde{\mathcal{M}}$ in the previous corollary. That means that we can apply the corollary to each extension, with $\varphi = \varphi_B$. Then we take F to be the limit of all those extended maps. This limit is understood as follows. All elements of $\mathcal{M}_A$ are obtained after a finite amount of Hirsch extensions. Hence how F acts on an element is determined and fixed after a finite amount of Hirsch extensions. Since the diagram is homotopy commutative after all finite extensions, the resulting diagram is homotopy commutative. □

The map F is called a model for f.

3.3 Formality

Definition 3.3.0.1. A minimal DGA $(\Lambda V, d)$ is called formal if there is a DGA homomorphism $\varphi : (\Lambda V, d) \to H^*(\Lambda V, d)$ which induces an isomorphism in cohomology.

This definition seems quite abstract and vague. We therefore would like to have some other equivalent definition. This definition is provided by the following theorem:

Theorem 3.3.0.2. *A minimal DGA $(\Lambda V, d)$ is formal if and only if V decomposes into a direct sum $V = C \oplus N$ such that d vanishes on C, d is injective on N and every closed element in the ideal generated by N is exact.*

Proof. Assume that we can decompose $V = C \oplus N$ such that the conditions in the theorem are met. Then the ideal of N doesn't add any cohomology, since all of it's closed elements are exact. On the other hand, all elements of C are closed. Hence we can define the DGA map

$$\varphi : V^i \to H^i(\Lambda V, d), \begin{cases} \varphi(c) = [c], & \text{for } c \in C^i \\ \varphi(n) = 0, & \text{for } n \in N^i \end{cases}$$

which we extend freely to ΛV. This map clearly induces an isomorphism in cohomology, so we only have to check that it intertwines the differentials. On C, $\varphi \circ d = d \circ \varphi = 0$ since d vanishes on C as well as on $H^*(\Lambda V, d)$. Now let $n \in N^i$. Since φ vanishes on N^i, $d(\varphi(n)) = 0$. We can decompose $dn = x + y$, where $x \in C$ and $y \in Ideal(N)$. Since d vanishes on C, we get $0 = d^2 n = dx + dy = dy$. Since y is a closed element in $Ideal(N)$, it is exact, say $y = dz$. This means that $x = d(n - y)$, so $\varphi(d(n)) = [x] = 0$. We see that φ is a DGA homomorphism between $(\Lambda V, d)$ and $H^*(\Lambda V, d)$, so $(\Lambda V, d)$ is formal.
For the other implication, assume that $(\Lambda V, d)$ is formal. Let $\varphi : (\Lambda V, d) \to H^*(\Lambda V, d)$ be the DGA homomorphism which induces an isomorphism in cohomology. Recall from the construction of minimal models in theorem 3.1.0.6 that we defined $\mathcal{M}(n+1) = \mathcal{M}(n) \otimes \Lambda(V^{n+1} \oplus W^{n+1})$. By theorem 3.1.0.9, $(\Lambda V, d)$ is isomorphic to the constucted minimal model. Hence we will identify the two. The differential vanishes on V^{n+1} and is injective on W^{n+1}. To induce an isomorphism in cohomology, φ_{n+1} has to be injective on V^{n+1} and zero on W^{n+1}. Since the differential is zero in $H^*(\Lambda V, d)$, φ_n is surjective. We define $C = \bigoplus_i V^i$ and $N = \bigoplus_i W^i$. Now d is zero on C and injective on N. If we take a closed element n in $Ideal(N)$, then $\varphi(n) = 0$. Therefore $\varphi_*([n]) = 0$, but since φ_* is an isomorphism, $[n] = 0$. Hence all closed elements in the ideal generated by N are exact. □

It may seem like all $DGA's$ are formal, but this is not the case. To prove that there are non-formal $DGA's$, we take a look at Massey products. The first kind of Massey product is the triple Massey product, which will be constructed below.

Let (A, d) be a DGA and let $[a] \in H^p(A, d)$, $[b] \in H^q(A, d)$ and $[c] \in H^r(a, d)$, with representatives a, b and c. Suppose that

$$[a] \cdot [b] = 0 \in H^{p+q}(A, d) \text{ and that } [b] \cdot [c] = 0 \in H^{q+r}(A, d).$$

That means that there are $x^{12} \in H^{p+q-1}(A, d)$ and $x^{23} \in H^{q+r-1}(A, d)$ such that

$$dx^{12} = (-1)^p a \cdot b \text{ and } dx^{23} = (-1)^q b \cdot c.$$

Then we can take $y = (-1)^p a \cdot x^{23} + (-1)^{p+q-1} x^{12} \cdot c$. This element will be closed by construction. Hence it defines a cohomologyclass $[y] \in H^{p+q+r-1}$. Unfortunately, this cohomologyclass is dependent on choice of representatives. To see this, we take three different representatives $\tilde{a} = a + da' \in [a]$, $\tilde{b} = b + db' \in [b]$ and $\tilde{c} = c + dc' \in [c]$. The corresponding change will be:

$$\begin{aligned} d\tilde{x}^{12} &= (-1)^p a \cdot b + (-1)^p a \cdot db' - (-1)^{p-1} da' \cdot b - (-1)^{p-1} da' \cdot db' \\ &= (-1)^p a \cdot b + d(a \cdot b' - (-1)^{p-1} a' \cdot b - (-1)^{p-1} a' \cdot db') \end{aligned}$$

and

$$\begin{aligned} d\tilde{x}^{23} &= (-1)^q b \cdot c + (-1)^q b \cdot dc' - (-1)^{q-1} db' \cdot c - (-1)^{q-1} db' \cdot dc' \\ &= (-1)^q b \cdot c + d(b \cdot c' - (-1)^{q-1} b' \cdot c - (-1)^{q-1} b' \cdot dc') \end{aligned}$$

Hence we can make the choices

$$\begin{aligned} \tilde{x}^{12} &= x^{12} + a \cdot b' - (-1)^{p-1} a' \cdot b - (-1)^{p-1} a' \cdot db' \\ \tilde{x}^{23} &= x^{23} + b \cdot c' - (-1)^{q-1} b' \cdot c - (-1)^{q-1} b' \cdot dc' \end{aligned}$$

With these new choices we get

$$\begin{aligned} \tilde{y} =& (-1)^p (a + da') \cdot (x^{23} + b \cdot c' - (-1)^{q-1} b' \cdot c - (-1)^{q-1} b' \cdot dc') \\ &+ (-1)^{p+q-1} (x^{12} + a \cdot b' - (-1)^{p-1} a' \cdot b - (-1)^{p-1} a' \cdot db') \cdot (c + dc') \\ =& y + a \cdot X_1 + c \cdot X_2 + dX_3 \end{aligned}$$

for some X_1, X_2 and X_3. Hence the indeterminancy in $[y]$ lies in the ideal generated by $[a]$ and $[c]$. Let $\pi : H^*(A,d) \to H^*(A,d)/([a],[c])$ denote the projection into the quotient of the cohomologygroup by the ideal generated by $[a]$ and $[c]$. Then we can define the triple Massey product as follows:

Definition 3.3.0.3. Let (A,d) be a DGA, and let $[a] \in H^p(A,d)$, $[b] \in H^q(A,d)$ and $[c] \in H^r(A,d)$ satisfying the conditions

$$[a] \cdot [b] = 0, \ [b] \cdot [c] = 0$$

the we define the triple Massey product by

$$\langle [a],[b],[c] \rangle = \pi([y])$$

As mentioned before, triple Massey products are not the only kind of Massey products. There are quadruple Massey products, quintuple Massey products and so on. They are all defined similarly as the triple Massey product, but they require one more condition to be defined. For the definition of the quadruple Massey product, the triple Massey products not only have to vanish but they have to vanish simultaneously.

Definition 3.3.0.4. Let (A,d) be a DGA and let $[a] \in H^p(A,d)$, $[b] \in H^q(A,d)$, $[c] \in H^r(A,d)$ and $[d] \in H^s(A,d)$ such that the Massey products

$$\langle [a],[b],[c] \rangle \text{ and } \langle [b],[c],[d] \rangle$$

are defined. Then the Massey products vanish simultaneously if there are choices of representatives such that both representatives of their triple Massey products y and z are exact.

Definition 3.3.0.5. Let (A,d) be a DGA and let $[a] \in H^p(A,d)$, $[b] \in H^q(A,d)$, $[c] \in H^r(A,d)$ and $[d] \in H^s(A,d)$ such that the Massey products

$$\langle [a],[b],[c] \rangle \text{ and } \langle [b],[c],[d] \rangle$$

are defined and vanish simultaneously. Take x^{13} and x^{24} such that

$$dx^{13} = y \text{ and } dx^{24} = z$$

Then we can define $w \in A^{p+q+r+s-2}$ as follows:

$$w = (-1)^p a \cdot x^{24} + (-1)^{p+q-1} x^{12} \cdot x^{34} + (-1)^{p+q+r-1} x^{13} \cdot d$$

This element will be closed by construction. Hence we define the quadruple Massey product as the set of all cohomologyclasses $[w]$ that we can obtain by this construction.

We will now go back to the main topic of this section, which is formality. There is a really nice connection between Massey products and formal spaces, which will be clear with the following theorem:

Theorem 3.3.0.6. *Let $(\Lambda V, d)$ be a formal minimal DGA. Then it is possible to make uniform choices such that all Massey products are exact.*

Proof. By theorem 3.3.0.2 we can decompose V as $V = C \oplus N$ such that d vanishes on C and all closed elements in the ideal generated by N are exact. Hence, if the equation $dx = y$ has a solution for $x \in V$, then we can write $x = c + n$, where $c \in C$ and $n \in (N)$. But since $dc = 0$, we find a solution of the equation for $n \in (N)$. Consider the triple Massey product $\langle [a],[b],[c] \rangle$ represented by

$$y = (-1)^p a \cdot x^{23} + (-1)^{p+q-1} x^{12} \cdot c$$

Then we can take $x^{12}, x^{23} \in (N)$, so $y \in (N)$. But since y is a closed element in the ideal generated by N, is is exact.
This argument also holds for higher order Massey products, since we can choose x^{13} and x^{24} in (N) as well, so $w = (-1)^p a \cdot x^{24} + (-1)^{p+q-1} x^{12} \cdot x^{34} + (-1)^{p+q+r-1} x^{13} \cdot d$ will also be in (N) and therefore exact. We can do this for all higher order Massey products. Hence with these choices of representatives all Massey products are exact. $\square$

This theorem allows us to prove that a DGA is not formal. If we can find a Massey product for which all representatives are not exact, then the DGA can not be formal.

Example 3.3.0.7. Let V be the free algebra generated by a, b, x_1 and x_2. Then $(\Lambda V, d)$ is a DGA with

$$|a| = |b| = 2,\, |x_1| = |x_2| = 3, da = db = 0,\, dx_1 = a^2,\, dx_2 = a \cdot b.$$

The Massey product $\langle [a], [a], [b] \rangle$ is given by

$$\langle [a], [a], [b] \rangle = [a \cdot x_2 - x_1 \cdot b]$$

Since this Massey product has degree 5 and all elements of degree 4 are closed, it can not be exact. Hence this is an example of a minimal DGA which is not formal. $\triangle$

Lemma 3.3.0.8. *Let M be a compact Kähler manifold and let $\alpha \in \Omega^k(M)$ such that $d\alpha = d^c\alpha = 0$. Assume that $\alpha = d\beta$ or $\alpha = d^c\beta$ for some $\beta \in \Omega^{k-1}(M)$. Then there exists a $\gamma \in \Omega^{k-2}(M)$ for which $\alpha = dd^c\gamma = -d^c d\gamma$.*

Proof. Consider the case that $\alpha = d\beta$ for some $\beta \in \Omega^k(M)$. By theorem 2.3.0.4 α has no harmonic part. Hence $\alpha = G_d \Delta_d \alpha = G_d(dd^* + d^*d)\alpha = G_d dd^*\alpha$. By lemma 2.3.0.7 α has no d^c-harmonic part either, so $\alpha = G_{d^c} d^c (d^c)^*\alpha$. Since d^c and d^* commute (see [8]), we get $\alpha = G_d dd^* G_{d^c} d^c (d^c)^*\alpha = dd^c d^* G_d G_{d^c} (d^c)^*\alpha = dd^c(d^* G_d G_{d^c} (d^c)^*\alpha) = dd^c\gamma$. In the case $\alpha = d^c\beta$ the same argument can be applied. $\square$

Theorem 3.3.0.9. *Compact Kähler manifolds are formal.*

Proof. Let M be a compact Kähler manifold and define $\Omega^c(M) = Ker(d^c)$ and $H_{d^c}(M) = H^*(\Omega^*(M), d^c)$. If $[\alpha] \in H_{d^c}(M)$, then $d^c d\alpha = -dd^c\alpha = 0$, so $d\alpha \in H_{d^c}(M)$. Hence $(H_{d^c}(M), d)$ is a DGA. By lemma 3.3.0.8 we see that $d\alpha = d^c(-d\gamma)$, so $d = 0$ on $H_{d^c}(M)$. Let $p : \Omega^c(M) \to (H_{d^c}(M), d)$ be the projection. Then we want to show that p induces an isomorphism p^* on cohomology. Let $[\alpha] \in H_{d^c}(M)$. Then by lemma 3.3.0.8 we get $d\alpha = dd^c\gamma$, so $d(\alpha - d^c\gamma) = 0$. This means that $[\alpha - d^c\gamma] \in H^*(\Omega^c, d)$, and $p^*[\alpha - d^c\gamma] = [\alpha]$. Hence p^* is surjective.
Suppose that $p^*[\alpha] = 0$, then $\alpha = d^c\beta$. By lemma 3.3.0.8, $\alpha = dd^c\gamma$, so $[\alpha] = 0$. Hence p^* is injective, so it is an isomorphism.
Let $i : (\Omega^c(M), d) \to (\Omega^*(M), d)$ be the inclusion. Then we want to show that i induces an isomorphism i^* on cohomology. Let $\alpha \in H^*(\Omega^*(M), d)$, then by lemma 3.3.0.8 we get $d^c\alpha = dd^c\gamma$. Now define $\beta = \alpha + d\gamma$, then $d^c\beta = d^c\alpha + d^c d\gamma = dd^c\gamma - dd^c\gamma = 0$. hence $[\beta] \in \Omega^c(M)$ and $i^*[\beta] = [\alpha]$, so i^* is surjective.
Suppose that $i^*[\alpha] = 0$, then $\alpha = d\beta$. By lemma 3.3.0.8, $\alpha = dd^c\gamma$, so $[\alpha] = 0$. Hence i^* is injective, so it is an isomorphism. Since these maps both induce an isomorphism on cohomology, and since $d = 0$ on $H_{d^c}(M)$, we get

$$H^*(\Omega^*, d) \simeq H^*(\Omega^c(M), d) \simeq H^*(H_{d^c}(M), d) \simeq H_{d^c}(M)$$

By theorem 3.2.0.8 we can lift these DGA homomorphisms completing the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_1 \arrow[d, "\varphi_1"] & \mathcal{M}_2 \arrow[l, "F"'] \arrow[r, "G"] \arrow[d, "\varphi_2"] & \mathcal{M}_3 \arrow[d, "\varphi_3"] \\
H_{d^c}(M) & (\Omega^c(M), d) \arrow[l, "p"'] \arrow[r, "i"] & (\Omega^*(M), d)
\end{tikzcd}$$

By theorem 3.1.0.9, F and G are actually isomorphisms. Hence there is a map from the minimal model $\mathcal{M}_3$ of $(\Omega^*(M), d)$ to $H^*(\mathcal{M}_3)$ inducing an isomorphism on cohomology. □

Chapter 4

Nilmanifolds

Nilmanifolds are a certain type of quotient of Lie groups. Before we go into nilmanifolds, some general properties of Lie groups will be discussed. The Lie algebra will turn out to be of vital importance for the minimal model of nilmanifolds. Therefore the Lie algebra and the interplay between Lie groups and Lie algebra's will also be addressed.

4.1 Lie groups and Lie algebra's

Recall the definition of a Lie group.

Definition 4.1.0.1. A Lie group is a set with a group structure and the structure of a smooth manifold such that all group operations are smooth.

Lie groups need not be connected. Sometimes it is useful to assume that a Lie group is connected. The next theorem deals with this.

Theorem 4.1.0.2. *Let G be a Lie group and denote the connected component of the identity by G^0. Then G^0 is a normal subgroup of G*

Proof. The image of connected components under continuous maps is again connected. Hence the map

$$i : G \to G,\, g \mapsto g^{-1}$$

sends connected components to connected components. Since it sends e tot e, it must send G^0 to G^0. For the same reason, the map

$$m : G \times G \to G,\, (g, h) \mapsto gh$$

sends $G^0 \times G^0$ to G^0. Because G^0 is closed under inversion and multiplication, it is a subgroup.
To see that G^0 is a normal subgroup, take an arbitrary $h \in G$. Then the map

$$c_h : G^0 \to G,\, g \mapsto hgh^{-1}$$

is continuous and sends e to e. Hence it maps G^0 to G^0, so G^0 is a normal subgroup of G. □

A Lie group action is an action of a Lie group which is smooth. Important examples of Lie group actions are actions of G on itself. They are given by

$$L_g : G \to G,\, h \mapsto gh$$

$$R_g : G \to G,\, h \mapsto hg^{-1}$$

$$Ad_g : G \to G,\, h \mapsto ghg^{-1}$$

These actions are called the left action, the right action and the adjoint action. These actions extend to TG by their push forward. In other words, the map $v \mapsto (L_g)_*v$ defines an action on TG, and the same holds for R_g and Ad_g. Since Ad_g preserves the identity, we get an action on the finite dimensional vector space T_eG given by $(Ad_g)_*$. We will write gv for $(L_g)_*v$ and vg for $(R_{g^{-1}})_*v$.

Definition 4.1.0.3. A Lie algebra is a vector space $\mathfrak{g}$ over a field k together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following conditions:

- $[v, w] = -[w, v], \forall v, w \in \mathfrak{g}$ (skew-symmetry)
- $[\lambda u + \mu v, w] = \lambda[u, w] + \mu[v, w],), \forall \lambda, \mu \in k, \forall u, v, w \in \mathfrak{g}$ (linearity in the first argument)
- $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0,), \forall u, v, w \in \mathfrak{g}$ (Jacobi identity)

The binary operation is called the Lie bracket.

Every element of a Lie algebra induces a linear map, called the adjoint map. It is given by

$$ad_v : \mathfrak{g} \to \mathfrak{g},\ ad_v(w) = [v, w]$$

Given a Lie algebra $\mathfrak{g}$, we can find a basis $\{X_1, \ldots, X_n\}$. With respect to this basis, we can always express the Lie bracket explicitly by $[X_i, X_j] = \sum\limits_k c_{ij}^k X_k$. The symbols c_{ij}^k are called the structure constants of the Lie algebra with respect to the basis $\{X_1, \ldots, X_n\}$. The concepts of Lie groups and Lie algebra's are strongly related. A vector field X on G is invariant under left translations or simply left invariant if

$$(dL_g)_h(X_h) = X_{gh}$$

The set of all left invariant vector fields is denoted by $Lie(G)$. It is closed under the usual Lie bracket on vector fields, so the pair $(Lie(G), [\cdot,\cdot])$ is a Lie algebra. The map

$$Lie(G) \to T_eG,\ X \mapsto X_e$$

is an isomorphism of vector spaces. Because of this, $Lie(G)$ is usually identified with T_eG.
The relation between Lie groups and Lie algebra's also goes the other way.

Lemma 4.1.0.4. *Let G be a Lie group, $\mathfrak{g} = T_eG$ its Lie algebra and let $x \in \mathfrak{g}$. Then there exists a unique Lie group homomorphism $\gamma_x : \mathbb{R} \to G$ such that*

$$\dot{\gamma}_x(0) = x$$

Proof. We first prove uniqueness. Since γ_x is a Lie group homomorphism, we get $\dot{\gamma}_x(t) = \gamma_x(t)\dot{\gamma}_x(0) = \dot{\gamma}_x(0)\gamma_x(t)$. Let v_x be the left invariant vector field such that $v_x(e) = x$. Then γ_x is the integral curve of v_x. Hence γ_x is uniquely defined and exists locally.
Now we prove global existence. Let φ_x^t be the flow of v_x. Since v_x is left invariant, so is φ_x^t, which means that $g\varphi_x^t(h) = \varphi_x^t(gh)$. Now $\gamma_x(t) = \varphi_x^t(e)$, so $\gamma_x(t+s) = \varphi_x^{t+s}(e) = \varphi_x^s(\varphi_x^t(e)) = \varphi_x^s(\gamma_x(t)) = \gamma_x(t)\varphi_x^s(e) = \gamma_x(t)\gamma_x(s)$. Hence it can be extended to any $t \in \mathbb{R}$. □

The map γ_x is called the one parameter subgroup of G corresponding to x.

Definition 4.1.0.5. Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. Then the exponential map $exp : \mathfrak{g} \to G$ is given by

$$exp(x) = \gamma_x(1)$$

From this definition it follows that $d_0 exp : \mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity. Hence from the inverse function theorem we conclude that exp is a local diffeomorphism between neighborhoods of $0 \in \mathfrak{g}$ and $e \in G$.
With these concepts and properties of Lie groups and Lie algebra's settled, we can go on to define nilmanifolds.

Definition 4.1.0.6. Let G be a group. Then the lower central series of G is a series

$$G = G_0 \triangleright G_1 \triangleright \dots$$

where $G_{n+1} = [G_n, G]$ is the subgroup generated by all commutators $xyx^{-1}y^{-1}$ with $x \in G_n$ and $y \in G$. If the lower central series ends in $\{e\}$ after finitely many steps, then G is called nilpotent.

Definition 4.1.0.7. A compact nilmanifold is a compact manifold of the type N/Γ, where N is a simply connected nilpotent Lie group and Γ is a discrete subgroup.

A discrete subgroup Γ of G for which G/Γ is compact is called a lattice.

4.2 Chevalley-Eilenberg complex

Let $\mathfrak{g}$ be a Lie algebra with basis $\{X_1, \dots, X_n\}$. Denote the dual of $\mathfrak{g}$ by $\mathfrak{g}^*$ and the dual basis by $\{x_1, \dots, x_n\}$. Let $\Lambda\mathfrak{g}^*$ be the free graded algebra generated by $\mathfrak{g}^*$, such that all elements of $\mathfrak{g}^*$ have degree 1. Then we can define a map δ on $\mathfrak{g}^*$ by

$$\delta x_k(X_i, X_j) = -x_k([X_i, X_j]) = -c_{ij}^k$$

and extend it on $\Lambda\mathfrak{g}^*$ as a graded derivation. That means that for wedge products the relation

$$\delta(x \wedge y) = \delta(x) \wedge y + (-1)^{|x|} x \wedge \delta y$$

is satisfied. More explicitly, the differential is given by

$$\delta x_k = -\sum_{i,j} c_{ij}^k dx_i \wedge dx_j$$

The requirement that $\delta^2 = 0$ is equivalent to the Jacobi identity. This equivalence follows from a simple calculation.

Definition 4.2.0.1. Let $\mathfrak{g}$ be a Lie algebra. Then $(\Lambda\mathfrak{g}^*, \delta)$ is called the Chevalley-Eilenberg complex associated to $\mathfrak{g}$.

Theorem 4.2.0.2. *Let N/Γ be a nilmanifold. Then the Chevalley-Eilenberg complex associated to the Lie algebra of N is isomorphic to the minimal model of N/Γ*

Proof. The proof of this theorem can be found in [4]. □

Let N/Γ be a nilmanifold of dimension 2n and let $\mathfrak{n}$ be the Lie algebra of N. Suppose that $\mathfrak{n}$ is $r+1$-step nilpotent, so $\mathfrak{n} = \mathfrak{n}_0 \supset \dots \supset \mathfrak{n}_{r+1} = 0$, where $\mathfrak{n}_{i+1} = [\mathfrak{n}_i, \mathfrak{n}]$. We assume $r > 0$, which holds if and only if $\mathfrak{n}$ is not commutative. Choose a vector space complement $\mathfrak{a}_i$ such that $\mathfrak{n}_i = \mathfrak{n}_{i+1} \oplus \mathfrak{a}_i$. Then $\mathfrak{n} = \mathfrak{a}_0 \oplus \dots \oplus \mathfrak{a}_r$. Hence we can write

$$\Lambda^k \mathfrak{n}^* = \sum \Lambda^{i_0, \dots, i_r}$$

where

$$\Lambda^{i_0, \dots, i_r} = \Lambda^{i_0} \mathfrak{a}_0^* \dots \Lambda^{i_r} \mathfrak{a}_r$$

Of course, i_j must be smaller or equal to $n_j = dim(\mathfrak{a}_j)$.

Lemma 4.2.0.3. *In this decomposition, $H^1(\Lambda\mathfrak{n}^*, \delta)$ is equal to $\Lambda^{1,\dots,0}$.*

Proof. Obviously, no element of degree 1 is exact. Hence only closedness needs to be proven. Note that

$$\delta\alpha(X_i, X_j) = -\alpha([X_i, X_j])$$

This means that $\delta\alpha = 0$ if and only if α vanishes on $\mathfrak{n}_1$, which only occurs when $\alpha \in \mathfrak{a}_0$. □

Lemma 4.2.0.4. *Any closed 2-form σ can be decomposed as $\sigma = \sigma_1 + \sigma_2$, where $\sigma_1 \in \Lambda^{1,\dots,1}$ and $\sigma_2 \in \sum \Lambda^{i_0,\dots,i_{r-1},0}$.*

Proof. Let σ be a closed 2-form and decompose it as $\sigma = \sigma_1 + \sigma_2$ such that σ_2 is the part in $\sum \Lambda^{i_o,\dots,i_{r-1},0}$. Let $X, Y \in \mathfrak{n}$ and $Z \in \mathfrak{n}_r$. Then Z will be in the center of n, so $[X, Z] = [Y, Z] = 0$. Hence

$$\delta\sigma(X, Y, Z) = -\sigma([X, Y], Z) = -\sigma_1([X, Y], Z)$$

Since $[X, Y] \in \mathfrak{n}_1$, this is zero for all $X, Y \in \mathfrak{n}$ if and only if $\sigma_1 \in \Lambda^{1,\dots,1}$. □

By theorem 4.2.0.2, $H^*(\Lambda\mathfrak{n}^*, \delta) \simeq H^*({}^{N}\!/_{\Gamma})$. Denote the isomorphism by φ. Suppose now that ${}^{N}\!/_{\Gamma}$ is a Kähler manifold. By theorem 2.0.0.13, $[\omega]^j$ is an isomorphism between $H^{n-j}({}^{N}\!/_{\Gamma})$ and $H^{n+j}({}^{N}\!/_{\Gamma})$. Hence the same must hold for $\varphi([\omega]^j) = \Omega^j$. Choose a basis $\lambda_1, \dots, \lambda_{n_r}$ of $\Lambda^{0,\dots,0,1}$. Then we can write

$$\Omega = \sum_i \beta_i \wedge \lambda_i + \Omega_2$$

for some $\beta_i \in \Lambda^{1,0,\dots,0}$ and $\Omega_2 \in \sum \Lambda^{i_0,\dots,i_{r-1},0}$. By nondegeneracy the β_i's must be linearly independent, so we can extend it to a basis $(\beta_j)_{1\le j\le n_0}$ of $\Lambda^{1,0,\dots,0}$.

Lemma 4.2.0.5. *Every $\sigma \in \Lambda^{2n-1}\mathfrak{n}$ is closed. If σ is exact, then is is divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_0}$.*

Proof. Let $\mathfrak{b}_i = \mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_i$. For $\alpha \in \Lambda^{i_0,\dots,i_r}$, each term of $\delta\alpha$ belongs to $\Lambda^{j_0,\dots,j_r}$, where for some $k > 0$ we have $j_k = i_k - 1$ and $j_0 + \cdots + j_{k-1} = i_0 + \cdots + i_{k-1} + 2$. Hence, is $i_0 + \cdots + i_r = 2n - 1$, then $i_0 + \cdots + i_{k-1} \ge dim(\mathfrak{b}_{k-1}) - 1$, so $j_0 + \cdots + j_{k-1} \ge dim(\mathfrak{b}_{k-1})$ and $\delta\sigma = 0$. Hence σ is closed. If $i_0 + \cdots + i_r = 2n - 2$, then any term must satisfy $j_0 + \cdots + j_{k-1} = dim(\mathfrak{b}_{k-1})$, so in particular $j_0 = n_0$. Therefore $\sigma \in \sum \Lambda^{n_0,\dots,i_r}$, so it is divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_0}$. □

Theorem 4.2.0.6. *Let M be a compact nilmanifold. If it admits a Kähler structure, then it is diffeomorphic to a torus.*

Proof. Assume that M is not diffeomorphic to a torus. Then its Lie algebra is noncommutative, hence the assumptions on top of this page hold. Using the previous lemmata it suffices to show that $[\Omega]^{n-1}$ is not an isomorphism. Suppose $\sigma \in \Lambda^{2n-1}\mathfrak{n}^*$ is divisible by $\lambda_1 \wedge \cdots \wedge \lambda_{n_r}$ but not by $\beta \wedge \cdots \wedge \beta_{n_r}$, then $[\sigma]$ is not in the image of ${}^{[}\Omega^{n-1}]$. To prove this, we have to show that every $\alpha \in \Lambda^{1,0,\dots,0}$ does not differ from σ by a form divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_r}$.
Decompose Ω^{n-1} as $\Omega^{n-1} = \gamma_1 + \gamma_2$, where $\gamma_1 \in \Lambda^{n_0-2,n_1,\dots,n_r}$ and $\gamma_2 \in \sum \Lambda^{n_0-1,i_1,\dots,i_r} + \sum \Lambda^{n_0,i_1,\dots,i_r}$. Each term in γ_1 is divisible by $\lambda_1 \wedge \cdots \wedge \lambda_{n_r}$, so it is also divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_r}$. Since $\alpha \wedge \gamma_2$ is divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_r}$, $\Omega^{n-1} \wedge \alpha$ is divisible by $\beta_1 \wedge \cdots \wedge \beta_{n_r}$, so $[\sigma]$ is not in the image of $[\Omega]^{n-1}$. □

This theorem gives us another instrument to show that a given symplectic manifold does not admit a Kähler structure.

Example 4.2.0.7. Let $H(1, p)$ be the group of $(p + 2) \times (p + 2)$ matrices of the form

$$\begin{pmatrix} I_p & A & C \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where I_p is the $p \times p$ identity matrix, A and C are $p \times 1$ matrices and b is a real number. This group is called the general Heisenberg group. It is nilpotent and the subgroup of matrices with integral coefficients $\Gamma(p)$ is a lattice. Hence $M_p = {}^{H(1,p)}\!\diagup_{\Gamma}$ defines a nilmanifold. We use the global coordinates

$$\begin{pmatrix} I_p & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

If L_a denotes left multiplication by $a \in H(1,p)$, then $L_a^*(dX) = dX$, $L_a^*(dy) = dy$ and $L_a^*(dZ - Xdy) = dZ - Xdy$. Therefore $\{dX, dy, dZ - Xdy\}$ forms a basis for the Lie algebra of $H(1,p)$. With this, we can define a new manifold $G(p,q) = H(1,p) \times H(1,q)$. Since this is the product of two nilpotent groups, it is nilpotent as well. Again the integral matrices $\Gamma(p,q)$ form a lattice in $G(p,q)$. Therefore $M(p,q) = {}^{G(p,q)}\!\diagup_{\Gamma(p,q)}$ is a nilmanifold. The fact that it is indeed a manifold follows from the quotient manifold theorem. We denote the projection $G(p,q) \to M(p,q)$ by π. We define the 2-form

$$\omega = dX^1 \wedge (dZ^1 - X_1 dy^1) + dX^2 \wedge (dZ^2 - X^2 dy^2) + dy^1 \wedge dy^2$$

where $dX^i \wedge (dZ^i - X^i dy^i) = \sum_j dX_j^i \wedge (dZ_j^i - X_j^i dy^i)$. It is easy to check that ω is closed and nondegenerate, hence symplectic. It is also invariant under left multiplication, so there exists a symplectic form $\tilde{\omega}$ on $M(p,q)$ such that $\pi^*\tilde{\omega} = \omega$. The fundamental group of $M(p,q)$ is given by $\Gamma(p,q)$ for the same reason as in example 2.3.0.10. This is not abelian for $p > 0$ or $q > 0$. Hence for $p > 0$ or $q > 0$ $M(p,q)$ is not diffeomorphic to a torus, so this construction yields a countably infinite amount of symplectic manifolds with no Kähler structure. $\triangle$

Chapter 5

Outlook

The Thurston-Weinstein problem is still studied by many mathematicians. This chapter will briefly cover some of the methods used in constructing symplectic manifolds without Kähler structure. It also covers some conjectures that mathematicians are still working on. Since this is an outlook, all sections can be read separately.

5.1 Solvmanifolds

Definition 5.1.0.1. A solvable group is a group G which has a finite series of subgroups

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_N = G$$

for which ${}^{G_i}\!/_{G_{i-1}}$ is abelian for $1 \leq i \leq N$.

For all nilpotent groups the lower central series satisfies the property, so all nilpotent groups are solvable.

Definition 5.1.0.2. A compact solvmanifold is a compact manifold of the form ${}^{G}\!/_{\Gamma}$, where G is a solvable simply connected Lie group and Γ is a lattice.

Since any nilpotent group is sovable, all nilmanifolds are solvmanifolds. Before we can go on about the structure of solvmanifolds, we have to introduce the concept of a fiber bundle.

Definition 5.1.0.3. A fiber bundle is a structure (E, F, B, π), where E, F and B are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying the following property:
For all $x \in E$ there exists a neighborhood $U \subset B$ of $\pi(x)$ such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ for which the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

where $p_1 : U \times F \to U$ is the canonical projection. Here E is called the total space, F is called the fiber, B is called the base space and π is called the projection map. The fiber bundle is often denoted by

$$F \to E \to B$$

A smooth fiber bundle is a fiber bundle where E, F and B are smooth manifolds and all maps involved are smooth.
The Mostow structure theorem shows that any solvmanifold has the structure of a fiber bundle. For the proof of the theorem, see [13].

Theorem 5.1.0.4. *Let G/Γ be a solvmanifold and let N be the maximal connected nilpotent subgroup of G. Then:*

1. *$N\Gamma$ is a closed subgroup of G.*
2. *$N \cap \Gamma$ is a lattice in N.*
3. *$G/N\Gamma$ is a torus*
4. *G/Γ is the total space of the fiber bundle*

$$N/N \cap \Gamma = N\Gamma/\Gamma \to G/\Gamma \to G/N\Gamma$$

This bundle is called the Mostow bundle.

Definition 5.1.0.5. A solvmanifold G/Γ is called completely solvable if the Lie algebra $\mathfrak{g}$ associated to G satisfies the condition that all adjoint maps ad_v have only real eigenvalues.

Just as in the case of nilmanifolds, solvmanifolds have an associated Chevalley-Eilenberg complex $(\Lambda\mathfrak{g}^*, \delta)$. However, for solvmanifolds the Chevalley-Eilenberg complex does not have to be minimal. Hence we can not generalize theorem 4.2.0.2 for arbitrary solvmanifolds. For solvmanifold that are completely solvable, Hattori's theorem is a weakened version of theorem 4.2.0.2. The proof of this theorem can be found in [12].

Theorem 5.1.0.6. *Let G/Γ be a completely solvable compact solvmanifold, then there exists a DGA homomorphism $\varphi : (\Lambda\mathfrak{g}^*, \delta) \to \Omega^*(G/\Gamma)$ which induces an isomorphism on cohomology.*

This theorem shows that some properties of nilmanifolds can be generalized to solvmanifolds under certain conditions. This raises the question whether this is also the case for theorem 4.2.0.6. In four dimensions, the theorem indeed generalizes:

Theorem 5.1.0.7. *Let M be a compact 4-dimensional solvmanifold. If it admits a Kähler structure, then it is diffeomorphic to a torus.*

The proof of this theorem can be found in [4]. This seems promising for the case of manifolds of arbitrary dimension. However, it is still not known if this holds for all dimensions. This is called the Benson-Gordon conjecture.

Conjecture Any compact solvmanifold which carries a Kähler structure is diffeomorphic to a torus.

5.2 Symplectic Hodge structure

Let (M, ω) be a symplectic manifold of dimension $2n$. Recall that ω induces an isomorphism $\tilde{\omega} : TM \to T^*M$ given by

$$\tilde{\omega}(v)(w) = \omega(v, w),\ \forall v, w \in T_xM,\ x \in M$$

Hence we can define a linear symplectic form on the cotangent bundle, which we will also denote by ω

$$\omega(\alpha, \beta) = \omega(\tilde{\omega}^{-1}(\alpha), \tilde{\omega}^{-1}(\beta)),\ \forall \alpha, \beta \in \Omega^1(M)$$

We can extend this linear symplectic form to a linear symplectic form $\Omega^k(M)$ for all k. To do this, note that the pure wedges span $\Omega^k(M)$. Therefore, we only need to define the linear symplectic form on pure wedges and extend linearly. On pure wedges, this linear symplectic form is defined to be

$$\omega(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = det((\omega(\alpha_i, \beta_j))_{1\leq i,j\leq k})$$

Definition 5.2.0.1. The symplectic Hodge star operator $\star_\omega : \Omega^k(M) \to \Omega^{2n-k}(M)$ is the unique operator which satisfies the equation

$$\alpha \wedge \star_\omega \beta = \omega(\alpha, \beta)\frac{\omega^n}{n!}$$

for all $\alpha, \beta \in \Omega^k(M)$. Similar to the normal Hodge star operator, we define a map

$$d^* : \Omega^k(M) \to \Omega^{k-1}(M),\ d^* = (-1)^k \star_\omega d\star_\omega$$

We call $\alpha \in \Omega^k(M)$ symplectic harmonic if $d\alpha = d^*\alpha = 0$. Let $H^k_{hr}(M)$ be the set of cohomologyclasses which can be represented by symplectic harmonic k-forms. This is a linear subspace of $H^k(M)$, so we define

$$h_k(M, \omega) = dim(H^k_{hr}(M))$$

The next theorem relates symplectic harmonic forms to the hard Lefschetz property.

Theorem 5.2.0.2. *Let (M, ω) be a compact symplectic manifold. Then $h_k(M, \omega) = b_k(M)$ for all $1 \leq k \leq n$ if and only if ω satisfies the hard Lefschetz property.*

The proof can be found in [14]. Since all compact Kähler manifolds satisfy the hard Lefschetz property, we get $h_k(M, \omega) = b_k(M)$ for all compact Kähler manifolds. Hence if the equality does not hold, then M can not admit a Kähler structure. In particular, if $h_k(M, \omega)$ is odd for some odd k, then M can not admit a Kähler structure, regardless of its Bettinumbers. Either $b_k(M) \neq h_k(M, \omega)$ or $b_k(M) = h_k(M, \omega)$. In the first case, the theorem shows that there can be no Kähler structure. In the second case $b_k(M)$ is odd, so again there can be no Kähler structure.

5.3 Blow ups

Until now, all examples of symplectic manifolds with no Kähler structure were non-simply connected. It is harder to come up with simply connected examples, but it has been done. The first examples of simply connected symplectic manifolds with no Kähler structure were constructed by Dusa McDuff. For the original article, see [15]. This construction made use of symplectic blows ups.
We first take a look at classical blow ups. Consider the submanifold $\gamma_n \subset \mathbb{C}^{n+1} \times \mathbb{CP}^n$ given by

$$\gamma_n = \{(z, l) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n \mid z \in l\}$$

This manifold has a natural structure of a line bundle. The projection $p : \gamma_n \to \mathbb{CP}^n$ provides this structure. In what follows we will be more interested in the projection to the first coordinate.
We first construct the new manifold

$$D(\gamma_n) = \{(z, l) \in \gamma_n \mid |z| < 1\}$$

Let $\pi : D(\gamma_n) \to D(0, 1)$ denote the projection of $D(\gamma_n)$ to the open unit disc in $\mathbb{C}^{n+1}$ given by $\pi((z, l)) = z$. All lines $l \in \mathbb{CP}^n$ are uniquely determined by a single point $z \in \mathbb{C}^{n+1}\backslash\{0\}$. Hence for all $z \in D(0, 1)\backslash\{0\}$ we get $\pi^{-1}(z) = (z, l)$, which is a single point. On the other hand, $\pi^{-1}(0) = \{(0, l) \in \gamma_n \mid 0 \in l\} = \mathbb{CP}^n$. This means that $\pi|_{D(\gamma_n)\backslash\{0\}}$ is a diffeomorphism. The new manifold $D(\gamma_n)$ is

thus the unit disc in $\mathbb{C}^{n+1}$ where the origin is replaced by $\mathbb{CP}^n$. In terminology, the origin has been blown up to $\mathbb{CP}^n$. One can check that $D(\gamma_n)$ is simply connected by considering long exact homotopy sequences.
Let X be a complex manifold of complex dimension $n+1$, and let $x \in X$. Take a complex chart (U, φ) with $\varphi : (U, x) \to (D(0,1), 0)$. Then we can define $\tilde{\pi} = \pi|_{D(\gamma_n)\setminus\{0\}} \circ \varphi|_{U\setminus\{x\}}$.

Definition 5.3.0.1. Let X be a complex manifold of complex dimension $n+1$ and $x \in X$ a point. The blow up of X at the point x is given by

$$Bl(X, x) = (X\setminus\{x\}) \cup_{\tilde{\pi}} D(\gamma_n)$$

This is again a complex manifold. The preimage $\pi^{-1}(0)$ is called the exceptional divisor $\mathcal{E}$ of the blow up.

This procedure gives us a way to blow up points. This can be generalized to blow up submanifolds instead of only points.
First note that the condition $z \in l$ is equivalent to the condition $z_i l_j = z_j l_i$ for all $1 \leq i, j \leq n+1$. To generalize blow ups to submanifolds, consider the disc $D = D(0,1) \subset \mathbb{C}^n$ and define

$$\tilde{D} = \{(z, l) \in D \times \mathbb{CP}^{n-k-1} \mid z_i l_j = z_j l_i \text{ for } k+1 \leq i, j \leq n\}$$

The projection onto the first coordinate is again denoted by π. The exceptional divisor is now given by $\mathbb{CP}^{n-k-1}$.
Let X be a complex manifold of (complex) dimension n and M a complex submanifold of (complex) dimension k. Take a collection of charts $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ such that in local coordinates M is given by $z_\alpha^{k+1} = \cdots = z_\alpha^n = 0$, where $z_\alpha^1, \ldots, z_\alpha^n$ denote coordinates in the chart $(U_\alpha, \varphi_\alpha)$. By identifying U_α with D, we get the blow up $\tilde{U}_\alpha$. On intersections, we define the transition maps

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \pi_\alpha^{-1}(U_\alpha \cap U_\beta)$$

Now we define

$$\tilde{U} = \cup_{\pi_{\alpha\beta}} \tilde{U}_\alpha$$

with projection $\pi : \tilde{U} \to \cup_\alpha U_\alpha$.

Definition 5.3.0.2. Let X be a complex manifold of complex dimension $n+1$ and $M \subset X$ a complex submanifold of complex dimension k. The blow up of X along M is given by

$$Bl(X, M) = (X\setminus M) \cup_\pi \tilde{U}$$

together with a projection $\tilde{\pi} : Bl(X, M) \to X$ extending π.

This complex blow up has a symplectic analog, which is described and applied in the article of McDuff (see [15]). Using the symplectic blow up, McDuff constructed symplectic simply connected manifolds with no Kähler structure.

5.4 Gompf's construction

In 1995 Robert Gompf used surgery to construct new symplectic manifolds out of old ones. He used this new construction to find a lot of new examples of symplectic manifolds which do not admit a Kähler structure. For the original article, see [16].
This result is precisely formulated in the following theorem.

Theorem 5.4.0.1. *For any even number $n \geq 6$, finitely generated group G and natural number b there is a closed symplectic manifold M with $\pi_1(M) \simeq G$ and $b_i \geq b$ for $2 \leq i \leq n-2$ such that M is not homotopy equivalent to any Kähler manifold. For $n \geq 8$ we may choose either b_2 or b_3 to be any preassigned sufficiently large number.*

For the proof, the reader is again referred to the original paper ([16]). The construction used in the proof will be discussed below. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds of dimension $2n$. Suppose we have a symplectic manifold (N, σ) of dimension $2n-2$, together with embeddings $i_j : (N, \sigma) \to (M_j, \omega_j)$. The normal bundle of a submanifold $N \subset M_j$ is defined to be $\nu N = {}^{TM_j}\!/_{TN}$. Suppose that both embeddings have a trivial normal bundle. Then the symplectic neighborhood theorem (see [3]) gives us symplectic embeddings $\varphi_j : N \times D_\varepsilon \to M_j$ such that $\varphi_j|_{N\times\{0\}} = i_j$, where D_ε is the open disc of radius ε with standard symplectic form $dx \wedge dy$. Remove the origin from D_ε to get the annulus $A_\varepsilon = D_\varepsilon \backslash \{0\}$. This annulus admits a symplectic automorphism which interchanges the boundary components. In polar coordinates, it is given by

$$\psi : A_\varepsilon \to A_\varepsilon,\ (r, \theta) \mapsto (\sqrt{\varepsilon^2 - r^2}, -\theta)$$

The fiber connected sum is now constructed as

$$M_1 \#_N M_2 = M_1 \backslash i_1(N) \cup_{Id\times\psi} M_2 \backslash i_2(N)$$

where $Id \times \psi : \varphi_1(N \times D_\varepsilon) \to \varphi_2(N \times D_\varepsilon)$. This new manifold carries a natural symplectic form which agrees with ω_1 on M_1 and with ω_2 on M_2 outside a neigborhood of $\varphi_1(N)$ and $\varphi_2(N)$.

Appendix A

Algebraic Topology prerequisites

A.1 Splitting lemma

Definition A.1.0.1. An exact sequence is a sequence $(A^i)_{i\in I}$ of groups together with maps $d^n : A^n \to A^{n+1}$ such that $Im(d^n) = Ker(d^{n+1})$.

A special case of an exact sequence is a short exact sequence, where $I = \{0, 1, 2, 3, 4\}$ and $A^0 = A^4 = 0$. The counterpart is a long exact sequence, in which the index set I is infinite.
The splitting lemma states that under certain conditions it is possible to split a group using a short exact sequence.

Theorem A.1.0.2. *Suppose we have a short exact sequence of abelian groups*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Then the following three statements are equivalent:

1. *There exists a map $h_1 : B \to A$ such that $h_1 \circ f = Id_A$.*
2. *There exists a map $h_2 : C \to B$ such that $g \circ h_2 = Id_C$.*
3. *B is isomorphic to $A \oplus C$ with f as natural inclusion and g as natural projection.*

Proof. First assume that statement 3 holds. Then we can simply take h_1 to be the natural projection and h_2 to be the natural inclusion.
Now we go on with the interesting part of the proof. Assume that statement 1 holds. Then we can write every $b \in B$ as $b = (b - f(h_1(b))) + f(h_1(b))$, where $f(h_1(b)) \in Im(f)$ and $b - f(h_1(b)) \in Ker(h_1)$. Now let $b \in Ker(h_1) \cap Im(f)$, then $b = f(a)$ for some $a \in A$ and $0 = h_1(b) = h_1(f(a)) = a$, so $a = b = 0$. We conclude that $B = Im(f) \oplus Ker(h_1)$. Since the sequence is exact, $Im(f) = Ker(g)$ and g is surjective. Hence any $c \in C$ can be written as $g(b)$ for some unique $b \in Ker(h_1)$. This means that $g : Ker(h_1) \to C$ is an isomorphism. By exactness again, f is injective, which means that $B \simeq A \oplus C$.
For the last part, assume that statement 2 holds. Then we can write every $b \in B$ as $b = (b - h_2(g(b))) + h_2(g(b))$, where $b - h_2(g(b)) \in Ker(g)$ and $h_2(g(b)) \in Im(h_2)$. Let $b \in Ker(g) \cap Im(h_2)$, then $b = h_2(c)$ for some $c \in C$ and $0 = g(b) = g(h_2(c) = c$, so $c = b = 0$. Hence $B = Ker(g) \oplus Im(h_2)$. Since the sequence is exact, $Ker(g) = Im(f)$ and f is injective. Hence $A \simeq Im(f) = Ker(g)$. Since $g \circ h_2$ is a bijection, h_2 is an injection, so $Im(h_2) \simeq C$. We conclude that $B \simeq A \oplus C$. □

A.2 Five Lemma

The Five lemma is a useful tool which combines commutative diagrams with exact sequences. The proof of it heavily depends on two other lemmata called the first and second four lemma.

A.2.1 First Four Lemma

Theorem A.2.1.1. *Consider the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[r, "g"] \arrow[d, "k"] & C \arrow[r, "h"] \arrow[d, "l"] & D \arrow[d, "m"] \\
F \arrow[r, "o"] & G \arrow[r, "p"] & H \arrow[r, "q"] & I
\end{tikzcd}$$

and suppose that the rows are exact, k and m are injective and j is surjective. Then l is injective.

Proof. Let $\tilde{c} \in Ker(l)$, then $l(\tilde{c}) = 0$, so $q(l(\tilde{c})) = 0$. By commutativity of the right diagram, $m(h(\tilde{c})) = 0$. Since m is injective, $h(\tilde{c}) = 0$, so $\tilde{c} \in Ker(h) = Im(g)$. This means that $\tilde{c} = g(\tilde{b})$ for some $\tilde{b} \in B$. By commutativity of the middle diagram, $p(k(\tilde{b})) = l(g(\tilde{b}) = l(\tilde{c}) = 0$. Hence $k(\tilde{b}) \in Ker(p) = Im(o)$, which means that $k(\tilde{b}) = o(\tilde{f})$ for some $\tilde{f} \in F$. Since j is surjective, $\tilde{f} = j(\tilde{a})$ for some $\tilde{a} \in A$. By commutativity of the left diagram, $k(\tilde{b}) = o(\tilde{f}) = o(j(\tilde{a})) = k(f(\tilde{a}))$. Since k is injective, this means that $\tilde{b} = f(\tilde{a})$, so $\tilde{b} \in Im(f) = Ker(g)$. Hence $\tilde{c} = g(\tilde{b}) = 0$, so l is injective. □

A.2.2 Second Four Lemma

Theorem A.2.2.1. *Consider the following diagram:*

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "k"] & C \arrow[r, "h"] \arrow[d, "l"] & D \arrow[r, "i"] \arrow[d, "m"] & E \arrow[d, "n"] \\
G \arrow[r, "p"] & H \arrow[r, "q"] & I \arrow[r, "r"] & J
\end{tikzcd}$$

and suppose that the rows are exact, k and m are surjective and n is injective. Then l is surjective.

Proof. Let $\tilde{h} \in H$, then by surjectivity of m there exists a $\tilde{d} \in D$ for which $q(\tilde{h}) = m(\tilde{d})$. By commutativity of the right diagram and exactness, $0 = r(q(\tilde{h})) = r(m(\tilde{d})) = n(i(\tilde{d}))$. Since n is injective, this means that $i(\tilde{d}) = 0$, so $\tilde{d} \in Ker(i) = Im(h)$. Therefore, $\tilde{d} = h(\tilde{c})$ for some $\tilde{c} \in C$. By commutativity of the middle diagram, $q(\tilde{h}) = m(h(\tilde{c})) = q(l(\tilde{c}))$, so $\tilde{h} - l(\tilde{c}) \in Ker(q) = Im(p)$. Therefore we can write $\tilde{h} - l(\tilde{c}) = p(\tilde{g})$ for some $\tilde{g} \in G$. Since k is surjective, $\tilde{g} = k(\tilde{a})$ for some $\tilde{a} \in A$, so by commutativity of the left diagram $\tilde{h} - l(\tilde{c}) = p(k(\tilde{a})) = l(g(\tilde{a}))$. We conclude that $\tilde{h} = l(\tilde{c} + g(\tilde{a}))$, so l is surjective. □

Remark A.2.2.2. The method of both of these proofs is called diagram chasing. It is widely used in algebraic topology and related areas of mathematical research.

With both of these lemmata in mind it is easy to prove the Five Lemma:

Theorem A.2.2.3. *Consider the following diagram:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[r, "g"] \arrow[d, "k"] & C \arrow[r, "h"] \arrow[d, "l"] & D \arrow[r, "i"] \arrow[d, "m"] & E \arrow[d, "n"] \\
F \arrow[r, "o"] & G \arrow[r, "p"] & H \arrow[r, "q"] & I \arrow[r, "r"] & J
\end{tikzcd}$$

and suppose that the rows are exact, k and m are isomorphisms, n is injective and j is surjective. Then l is also an isomorphism.

Proof. By the first four lemma, l is injective. By the second four lemma l is surjective. Hence l is an isomorphism. □

A.3 Algebraic Mapping Cone

Definition A.3.0.1. A cochain complex is a sequence $A = (A^i)_{i \in I}$ of abelian groups together with differentials $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$

Any exact sequence is a cochain complex. The reverse does not hold in general however. The number i is called the degree. A map between cochain complexes is called a cochain homomorphism if it conserves the degree and intertwines the differentials.

Definition A.3.0.2. Let $f : A \to B$ be a cochain homomorphism. Then we construct the algebraic mapping cone $C(f)$ as the following cochain complex: The sequence of abelian groups is $(A^{i+1} \oplus B^i)_{i \in I}$ and the differential is given by $d^n : (a, b) \mapsto (-d_A^{n+1} a, f(a) + d_B^n b)$.

Bibliography

[1] John M. Lee *Introduction to Smooth Manifolds*, version 3.0, 2000.

[2] Allen Hatcher *Algebraic Topology*, Cambridge University Press, 2002.

[3] Dusa Mc Duff and Dietmar Salomon *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 1998.

[4] Aleksy Tralle and John Oprea *Symplectic manifolds with no Kähler structure*, Springer-Verlag, 1997.

[5] William Thurston *Some simple examples of compact symplectic manifolds*, Proceedings of the American Mathematical Society, 1976.

[6] August Newlander and Louis Nirenberg *Complex analytic coordinates in almost complex manifolds*, Annals of Mathematics 65, 1957.

[7] Olivier Biquard and Andreas Höring *Kähler Geometry and Hodge Theory*, 2008.

[8] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia Prada.

[9] Phillip Griffiths and John Morgan *Rational Homotopy Theory and Differential Forms*, Birkh-iuser, Basel, 1981.

[10] Stephen Halpering *Lectures on Minimal Models*, Hermann, Paris, 1982.

[11] Chal Benson and Carolyn S. Gordon *Kähler and Symplectic Structures on Nilmanifolds*, 1986.

[12] Akio Hattori *Spectral sequence in the de Rham cohomology of fibre bundles*, Journal of the Faculty of Science of the University of Tokyo 8, 1960.

[13] George Mostow *Factor spaces of solvable Lie groups*, Annals of Mathematics 60, 1954.

[14] Dong Yan *Hodge structure on symplectic manifolds*, Advances in Mathematiccs 120, 1996.

[15] Dusa McDuff *Examples of symplectic simply connected manifolds with no Kähler struture*, Journal of Differential Geometry 20, 1984.

[16] Robert Gompf *A New Construction of Symplectic Manifolds*, Annals of Mathematics 142, 1995.