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Topology of a compact Lie group

Bachelor Thesis

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Abstract

The main goal of this thesis is to prove the theorem that the fundamental group of a compact Lie group is finite, under the assumption that the Lie group is connected and semisimple. This theorem relies on theory developed in the study of differentiable manifolds and Lie groups as well as constructions from algebraic topology. Roughly the first half of the thesis will be dedicated to analyze the topological part of the approach, whereat the the next part the main focus will be shifted to analysis and differential geometry. This thesis includes various definitions of the structures involved, but prerequisite knowledge on basic theory of manifolds and homology is advised.

Contents

1 The fundamental group

We will begin with the construction of certain notions that are part of the branch of mathematics called algebraic topology. The question whether two spaces X and Y are homeomorphic plays a key role in this discipline. Sometimes it is enough to consider purely topological properties of both spaces to conclude that such a homeomorphism cannot exist. However, it may not be enough to look only at the number of connected components or compactness of spaces to tell if they are different. This is a good motivation for defining new algebraic structures on topological spaces which will allow us to compare them by looking at their corresponding structures, instead of comparing the spaces directly.

Throughout the first two subsections, we will roughly follow Chapters 0 and 1 of Hatcher [6]. Here we will introduce the simplest and most important tool, also known as the fundamental group of a space. In order to define this, we will first look at concept of homotopy. This will allow us to tell if two continuous maps from one topological space to another are 'similar'. Once the notion of fundamental group is established, we will proceed with examples of fundamental groups. Finally, we will apply this theory to the concept of topological groups to prove the main lemma of this section.

1.1 Homotopy

In topology we can bend, compress and curve spaces without changing their topological properties. A homeomorphism is an explicit way to bend one space into another. When looking at general functions between topological spaces, we are interested in similarities between these functions. More specifically, we can ask ourselves the question whether one function can be transformed into another in a 'nice' way.

Example 1.1.1. To make this more concrete, let us consider functions $S^1 \to \mathbb{R}^2 \setminus \{0\}$. The first example that comes to mind is the inclusion $S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$. We can compare this map to the function f where $f(x, y) = \frac{1}{2}(x, y)$ and note that f is actually a different version of the inclusion that is shrunk down. We could transform the inclusion over time into f in a continuous way by linearly collapsing the image of the inclusion until its radius is halved.

Now define the function g by $g(x, y) = (x, y + \frac{1}{2})$. Again, the inclusion and g are similar in the way that g can be obtained from the inclusion. This can be done by moving the image of the inclusion upwards over time until we get g. We call such a transformation of functions an homotopy. △

From this point on I is the unit interval $[0, 1] \subset \mathbb{R}$.

Definition 1.1.2. A homotopy between two maps $f_0, f_1 : X \to Y$ is a continuous function $H : X \times I \to Y$ such that $H|_{X\times\{0\}} = f_0$ and $H|_{X\times\{1\}} = f_1$. If such a function exists, we call f_0 and f_1 homotopic and we write $f_0 \simeq f_1$.

This definition can be directly applied to Example 1.1.1.

Example 1.1.3. The homotopy between the inclusion and f in Example 1.1.1 can be given by

$$H : S^1 \times I \to \mathbb{R}^2 \setminus \{0\} : ((x, y), t) \mapsto (1 - \frac{1}{2}t)(x, y)$$

and the homotopy between the inclusion and g by

$$\tilde{H} : S^1 \times I \to \mathbb{R}^2 \setminus \{0\} : ((x,y),t) \mapsto (x, y + \frac{1}{2}t)$$

Both can be viewed as a family of functions $(H_t)_{t\in I}$, each of them being continuous as a map from S_1 to $\mathbb{R}^2 \setminus \{0\}$. Once we fix a point $(x,y) \in S_1$, the resulting map that sends $t \mapsto H((x,y),t)$ is also continuous.

△

After having seen these examples, it might be reasonable to assume that *all* functions $S^1 \to \mathbb{R}_2 \setminus \{0\}$ are homotopic. It turns out that this is not the case.

Claim 1. Not all maps $S^1 \to \mathbb{R}_2 \setminus \{0\}$ are homotopic.

Observe that the inclusion map, f and g all have one thing in common; the image of the circle is mapped *around* the point that is taken out of the plain. We can define a new function h by $h(x,y) = (x, y+2)$ that maps the circle completely above the missing point. It may seem obvious that there is now way to deform h into the inclusion, since we cannot 'drag' the image of the circle over the singularity in a continuous manner. However, it will turn out to be rather difficult to directly proof this. One way to make this argument into a proof, would be to use the fundamental group of $\pi_1(\mathbb{R}_2 \setminus \{0\})$ and the theory of covering spaces. See Hatcher [6, p. 29].

Proposition 1.1.4. *The homotopy relation $\simeq$ of maps is an equivalence relation.*

Proof. Let $f, g, h : X \to Y$ be maps.
We define a homotopy $K : X \times I \to Y : (x,t) \mapsto f(x)$ to conclude that $f \simeq f$. Since we chose f arbitrarily, the relation is reflexive.

Assume that $f \simeq g$, with corresponding homotopy H. Then $K : X \times I \to Y : (x,t) \mapsto H(x, 1-t)$ defines a homotopy. Note that $K|_{X\times\{0\}} = g$ and $K|_{X\times\{1\}} = f$. Hence we also have $g \simeq f$, which proves the symmetry.

Now assume that $f \simeq g$ and $g \simeq h$ with their respective homotopies H and $\tilde{H}$. Then

$$K(x,t) = \begin{cases} H(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \tilde{H}(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

defines an homotopy. We observe that $K|_{X\times\{0\}} = f$ and $K|_{X\times\{1\}} = h$, verifying that $f \simeq h$. Therefore the relation is transitive. □

1.2 Defining the fundamental group

The fundamental group is defined by looking at loops and deformations of loops. In order to arrive there, we will introduce the notion of a path.

Definition 1.2.1. A path in the space X is a continuous map $\gamma : I \to X$. We call this path γ a loop if its initial point and end point coincide, i.e. if $\gamma(0) = \gamma(1)$. This point is called the base point of γ.

Now that we have defined the notion of path, we are interested in a slightly stronger version of homotopy.

Definition 1.2.2. We say that paths $\gamma_0, \gamma_1 : I \to X$ are path homotopic if there exists an homotopy $H : I \times I \to X$ between γ_0 and γ_1 that does not change the base point. By that we mean for each path $\gamma_t = H(\cdot, t)$ with $t \in I$ we have that $\gamma_t(0) = \gamma_0(0)$ and $\gamma_t(1) = \gamma_0(1)$. Such H is called an homotopy of paths.

If we have two paths $\gamma, \eta : I \to X$ with $\gamma(1) = \eta(0)$, then there is a way to compose these paths to obtain an new path.

Definition 1.2.3. The concatenation of paths $\eta \cdot \gamma$ is defined by

$$\eta \cdot \gamma = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \eta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Finally, before we define the fundamental group there is one lemma that will be useful when dealing with homotopies of paths. It states that traversing a path at different speeds will not change the path with respect to homotopy.

Lemma 1.2.4. *Let $\gamma : I \to X$ be a path and $\phi : I \to I$ a map such that $\phi(0) = 0$ and $\phi(1) = 1$. Then we can conclude that $\gamma \simeq \gamma \circ \phi$.*

Proof. Define the homotopy $H : I \times I \to X$ as

$$H(s,t) = \gamma((1-t)s + t\phi(s))$$

where t acts as the deformation parameter and s is in the domain of a path. Then $H(s,0) = \gamma(s)$ and $H(s,1) = \gamma(\phi(s))$. Moreover, we have $H(0,t) = \gamma(t\phi(0)) = \gamma(0)$ and $H(1,t) = \gamma(1 - t + t\phi(1)) = \gamma(t)$ by the assumption that ϕ does not change the end points of the interval. This shows that H is an homotopy of path and thus proves the lemma. □

Let us now look at the set of all loops $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = x_0$ with $x_0 \in X$. These are the loops in X with base point x_0. Proposition 1.1.4 states that the relation of maps being homotopic is an equivalence relation. If we specifically look at the relation induced by the homotopy of paths, the proof of Proposition 1.1.4 would still apply in this situation. The reflexivity is trivial and the homotopies that are constructed to provide the symmetry and transitivity of the relation, fix the end points throughout the homotopy. Hence it makes sense to view the quotient of the set of all loops at x_0 by the homotopy of paths relation.

Definition 1.2.5. The set $\pi_1(X, x_0)$ is defined as the set of classes $[\gamma]$ of loops $\gamma : I \to X$ with base point $x_0 \in X$ with respect to homotopy of paths.

Proposition 1.2.6. *$\pi_1(X, x_0)$ equipped with the product $[\eta] \cdot [\gamma] = [\eta \cdot \gamma]$ is a group.*

Proof. Let us first check that the product is well defined, because a priori it is not clear if the product depends on the choice of representatives. Assume that $[\eta_0] = [\eta_1]$ and $[\gamma_0] = [\gamma_1]$. Then there are homotopies H_η, connecting η_0 and η_1, and H_γ connecting γ_0 and γ_1. Our goal is to show $[\eta_0 \cdot \gamma_0] = [\eta_1 \cdot \gamma_1]$. Define the homotopy $H : I \times I \to X$ by

$$H(s,t) = \begin{cases} H_\eta(2s,t) & \text{if } 0 \leq s \leq \frac{1}{2} \\ H_\gamma(2s-1,t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Then we can check that $H|_{I\times\{0\}} = \eta_0 \cdot \gamma_0$ and $H|_{I\times\{1\}} = \eta_1 \cdot \gamma_1$. Therefore we indeed have that $\eta_0 \cdot \gamma_0 \simeq \eta_1 \cdot \gamma_1$, proving that the product is well defined.

We proceed by checking that $\pi_1(X, x_0)$ obeys the axioms of a group. Define the identity element to be the class of the constant path $c : I \to X : x \mapsto x_0$. If we apply Lemma 1.2.4 to the function $\phi : I \to I$ where

$$\phi(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2s & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

It follows that $[c \cdot \gamma] = [\gamma \circ \phi] = [\gamma]$. The equation $[\gamma \cdot c] = [\gamma]$ follows in a similar way. Hence c is indeed the identity.

Let $\alpha, \beta, \gamma : I \to X$ be paths. In order to show the associativity of the product, we need to convince ourselves that $[\alpha \cdot (\beta \cdot \gamma)] = [(\alpha \cdot \beta) \cdot \gamma]$. Again these are the same paths up to a difference in speed, so by Lemma 1.2.4 we have the equivalence $\alpha \cdot (\beta \cdot \gamma) \simeq (\alpha \cdot \beta) \cdot \gamma$.

Lastly, every element in the group should have an inverse. Let $\gamma : I \to X$ and define the inverse as $\gamma^{-1}(s) = \gamma(1-s)$. To show that $[\gamma] \cdot [\gamma^{-1}] = [c]$, we can construct another homotopy K where

$$K(s,t) = \begin{cases} \gamma(2st) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma^{-1}(1-2t+2st) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note that $K(s, \frac{1}{2})$ is well defined, $K|_{I\times\{0\}} = \gamma^{-1} \cdot \gamma$ and $K|_{I\times\{1\}} = c$. Hence this is indeed the inverse.

Having checked all group axioms, we can conclude that $\pi_1(X, x_0)$ is a group. □

From now on we will refer to $\pi_1(X, x_0)$ as the fundamental group of X at x_0.

Lemma 1.2.7. *Let X be path connected, then for any $x_0, x_1 \in X$ there is a group isomorphism $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.*

Proof. See Hatcher [6, p. 29]. □

As a consequence, we will denote the fundamental group of any path connected space X as $\pi_1(X)$. Since two different choices of a base points will result in the same fundamental group, this abbreviation leads to no ambiguity.

Definition 1.2.8. A deformation retraction of a space X into A with $A \subset X$ is a homotopy $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$ and H fixes A, i.e. $H(a, t) = a$ for all $a \in A$ and $t \in I$.

1.3 Topological groups

Definition 1.3.1. A topological group is a group G equipped with a topology such that the product function $G \times G \to G : (x, y) \mapsto xy$ and an inversion function $G \to G : x \mapsto x^{-1}$ are continuous. Here $G \times G$ is equipped with the product topology.

Lemma 1.3.2. *Let G be a topological group the identity element e. Then $\pi_1(G, e)$ is abelian.*

Proof. Let $\gamma_1, \gamma_2 \in \pi_1(G, e)$ and $\gamma_1 \cdot \gamma_2$ be the concatenation of these paths. We will take the direct approach by constructing a homotopy to show $\gamma_1 \cdot \gamma_2$ and $\gamma_2 \cdot \gamma_1$ are homotopy equivalent. Define $H : I \times I \to G$ to be

$$H(s,t) = \begin{cases} \gamma_1(2st)^{-1}\gamma_1(2s)\gamma_2(2st) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma_2(t(2s-1))^{-1}\gamma_2(2s-1)\gamma_1(t(2s-1)) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note that H is continuous. Furthermore, evaluating H in $t = 0$, the first and last factor of the product become $\gamma_1(0)$ and $\gamma_2(0)$, which are both equal to e. We are left with the definition of $\gamma_1 \cdot \gamma_2$, so we can conclude that $H(t, 0) = \gamma_1 \cdot \gamma_2(t)$.

Evaluating H at $t = 1$ the first and middle factor of the product cancel, because they are each others inverse with respect to the group multiplication on G. We see that $H(s, 1) = \gamma_2 \cdot \gamma_1(s)$.

Finally it is necessary to check that H is a homotopy of paths. Since we have $H(0, t) = H(1, t) = e$ for all $t \in I$, this is indeed the case. Therefore we can conclude that $\gamma_1 \cdot \gamma_2 \simeq \gamma_2 \cdot \gamma_1$, completing the proof. $\square$

2 The first homology group

This section we will establish a connection between the fundamental group of a space X and the first homology group $H_1(X)$ with coefficients in $\mathbb{Z}$. Readers are assumed to be familiar with the concept of singular homology. If this is not the case, then Hatcher [6] provides the constructions we will use in Chapter 2. Whereas the fundamental group only gives information on the way paths can be embedded into the space, the homology measures the way simplices of arbitrary dimension can be embedded. In the 1-dimensional case there is a natural correspondence between the two objects given by $h : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$ where interpret each loop $f : I \to X$ as a 1-simplex. Note that every loop is a cycle because $\partial f = f(1) - f(0) = 0$.

Remark 2.0.1. Before we arrive at the main theorem of this section. It is useful to introduce the notation that we will be using.

- The notation Δ^n stands for an n-simplex. This is a topological space that corresponds to the generalization of a triangle to dimension n.
- A continuous function $\sigma : \Delta^n \to X$ is called a singular n-simplex or n-cycle. Formal sums of the form $\sum_i \sigma_i$ will be referred to as chains.
- The space of all singular n-simplices $\sigma : \Delta^n \to X$ is denoted by $\Delta_n(X)$.
- The group $C_n(X)$ is defined as the direct sum $\bigoplus_{\sigma \in \Delta_n(X)} \mathbb{Z}$. Instead of choosing coefficients in $\mathbb{Z}$, we may also use another abelian group M for this purpose. In that case we denote $C_n(X, M)$.
- The groups $C_n(X)$ together with the maps $\partial : C_n(X) \to C_{n-1}(X)$ form a chain complex (C, ∂). The homology of this complex is called the singular homology and the n-th homology class is denoted by $H_n(X)$. In case we used coefficients in M, we write $H_n(X, M)$.

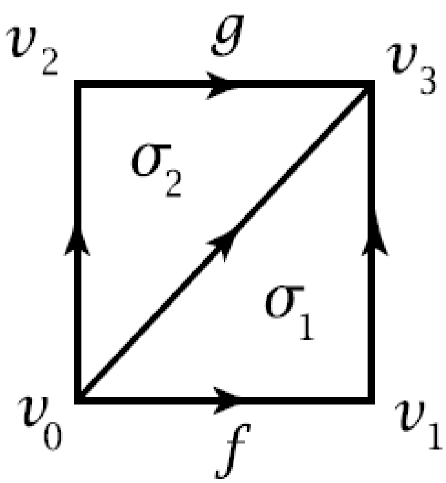
Theorem 2.0.2. *The map $h : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$ is a homomorphism of groups. Furthermore, if X is path connected, then h is surjective and the kernel of h is exactly the commutator subgroup of the fundamental group of X.*

Remark 2.0.3. The proof of Theorem 2.0.2 will be based on the proof that is given in Hatcher [6, p. 166]. We will adopt the convention that throughout this section $f \simeq g$ means that f and g are path homotopic. The notation $f \sim g$ means f is in the same homology class as g, i.e. their difference $f - g$ is the boundary of a 2-cycle.

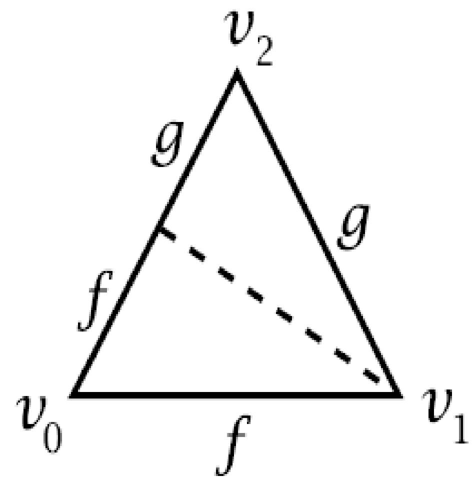
To prove theorem 2.0.2, we will first prove four helpful lemmas.

Lemma 2.0.4. *Let c be the constant path in X such that $c(t) = x_0$ for every $t \in I$. Then we have $c \sim 0$.*

Proof. Define the constant 2-chain $\sigma : \Delta^2 \to X : x \mapsto x_0$. We denote the edges of Δ^2 by v_0, v_1, v_2 and use square brackets for the convex hull of edges, so $\Delta^2 = [v_0, v_1, v_2]$. Then we see that $\partial\sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f$, because σ is also constant on the boundary. Therefore c is in the image of ∂_2 and we conclude that $c \sim 0$. □



(a) Visualization of Lemma 2.0.5



(b) Visualization of Lemma 2.0.6

Lemma 2.0.5. *Let f and g be loops at x_0. If f and g are homotopic, then they are also homologous as cycles. That is, $f \simeq g$ implies $f \sim g$.*

Proof. Let $H : I \times I \to X$ be the homotopy connecting f and g. We can visualize the domain as a square that we divide into two 2-simplices σ_1 and σ_2 (see Figure 1a from Hatcher [6]). Now observe that

$$\partial(\sigma_1 - \sigma_2) = \sigma_1|_{[v_1,v_3]} - \sigma_1|_{[v_0,v_3]} + \sigma_1|_{[v_0,v_1]} - \sigma_2|_{[v_2,v_3]} + \sigma_2|_{[v_0,v_3]} - \sigma_2|_{[v_0,v_2]} = f - g$$

Here we used that σ_1 and σ_2 agree on $[v_0, v_3]$. Hence $f - g$ is a boundary, proving the lemma. □

Lemma 2.0.6. *Let f and g be paths in X with the property that $f(1) = g(0)$, then $f \cdot g \sim f + g$. That is, the concatenation of paths is homologous to the sum of the paths viewed as simplices.*

Proof. Let v' be the middle point of the edge $[v_0, v_2]$ of the simplex Δ^2. Define the map $P : \Delta^2 \to [v_0, v_2]$ as the orthogonal projection of Δ^2 onto $[v_0, v_2]$. By this we mean that v_1 is send to v'. The orthogonality means that for every $x \in \Delta^2$ the line through the points x and $P(x)$ is perpendicular to the edge $[v_0, v_2]$, given that $x \neq P(x)$. Now we define $\sigma : \Delta^2 \to X$ to be a chain that is given by $\sigma = (f \cdot g) \circ P$, where $f \cdot g : [v_0, v_2] \to X$. It now follows that $\sigma|_{[v_0,v_2]} = f \cdot g$, $\sigma|_{[v_0,v_1]} = f$ and $\sigma|_{[v_1,v_2]} = g$ (see Figure 1b from Hatcher [6]). It is easy to see that $\partial\omega = g - f \cdot g + f$. Consequently $g - f \cdot g + f$ is a boundary, so we obtain $f \cdot g \sim f + g$. □

Lemma 2.0.7. *Let f be a loop at x_0 and $\bar{f}$ be the path traversed in the opposite direction. Then $\bar{f} \sim -f$.*

Proof. This lemma is a consequence of the previous three lemmas. Using these relations we derived, we obtain $f + \bar{f} \sim f \cdot \bar{f} \simeq c \sim 0$, where c is the constant path. Therefore we have $f + \bar{f} \sim 0$, which proves the lemma. □

Proof of Theorem 2.0.2. So far we have only defined h on loops. To check that h is well defined on homotopy classes of loops, it is sufficient to make sure that two loops in the same homotopy class are send to the same homology class. Lemma 2.0.5 exactly states that this is the case.

Lemma 2.0.6 applied to the classes of loops in the fundamental group of X, ensures that h respects the group product. In other words, for $[f], [g] \in \pi_1(X)$ we have $h([f] \cdot [g]) = h([f]) + h([g])$. This makes h into a homomorphism, which proves the first statement of the theorem.

From now on we assume X to be path connected. To prove the surjectivity of h, let the finite sum $\sum_i n_i \sigma_i$ be a representative of a homology class in $H_1(X, \mathbb{Z})$. We will proceed by constructing an element of $H_1(X, \mathbb{Z})$ that is homologous to $\sum_i n_i \sigma_i$ and lies in the image of h. That means the element we construct will have to be a loop based at x_0.

First we rewrite

$$\sum_i n_i \sigma_i = \sum_i \sum_{j=1}^{|n_i|} \text{sign}(n_i) \sigma_i = \sum_j \text{sign}(n_j) \sigma_j$$

where on the right hand side we combined the two sums into a single sum over a larger index set. To get rid of the signs, we use lemma 2.0.7 to see that $\sum\limits_j \text{sign}(n_j)\sigma_j \sim \sum\limits_j \tilde{\sigma}_j$ where we define

$$\tilde{\sigma}_j = \begin{cases} \sigma_j & \text{if } \text{sign}(n_j) = 1 \\ \overline{\sigma_j} & \text{if } \text{sign}(n_j) = -1 \end{cases}$$

Since the chain $\sum_j \tilde{\sigma}_j$ is an element of a homology class, we have $\partial(\sum_j \tilde{\sigma}_j) = 0$. This implies that the end points of the $\tilde{\sigma_j}$ cancel. If there is an index j such that $\tilde{\sigma_j}$ is not a loop, then there has to a another index j' such that the composition $\tilde{\sigma_j} \cdot \tilde{\sigma_{j'}}$ is well defined. By iterating this process we obtain a new element of the homology class $\sum_k \eta_k$ that consists only of loops η_k. Lemma 2.0.6 provides that $\sum_k \eta_k \sim \sum_j \tilde{\sigma}_j$.

Since X is path connected, let γ_k be a path from x_0 to the base point of η_k. Then $\gamma_k \cdot \eta_k \cdot \overline{\gamma_k}$ is a loop with base point x_0. Moreover, by using Lemmas 2.0.6 and 2.0.7 we have $\gamma_k \cdot \eta_k \cdot \overline{\gamma_k} \sim \eta_k$.

Returning to the original representative of the homology class we chose, we have now constructed an element $\sum_k \gamma_k \cdot \eta_k \bar{\gamma}_k$ that is homologous to $\sum_i n_i \sigma_i$, and is a sum of loops around x_0. Finally, we use Lemma 2.0.6 to make $\sum_k \gamma_k \cdot \eta_k \bar{\gamma}_k$ into a single loop γ around x_0, which clearly lies in the image of h. This completes the proof that h is surjective.

It rests us to prove that the kernel of h is the commutator subgroup $[\pi_1(X), \pi_1(X)] = \{aba^{-1}b^{-1} | a, b \in \pi_1(X)\}$. We have proven that h is a homomorphism, so the commutator subgroup is mapped to zero due to $H_1(X)$ being abelian. The other inclusion, $\ker(h) \subseteq [\pi_1(X), \pi_1(X)]$, requires a bit more work.

Let $[f] \in \ker(h)$, then there exists a formal sum of simplices $\sum_i n_i \sigma_i \in C_2(X)$ such that its boundary is equal to f. By allowing singular one simplices σ_i to appear multiple times in the sum, we may assume that $n_i = \pm 1$. We write $\partial \sigma_i = \tau_{i0} - \tau_{i1} + \tau_{i2}$ where the τ_{ij} are

1-cycles. Using this expression, we get

$$f = \partial(\sum_i n_i\sigma_i) = \sum_i n_i \partial\sigma_i = \sum_{i,j} (-1)^j n_i \tau_{ij}$$

Since $C_1(X)$ is the direct sum of all singular 1-simplices in X without any relations between elements, we deduce from the equality above that there is one specific 1-cycle such that $f = (-1)^j n_i \tau_{ij}$ and the rest of the cycles on the right hand side cancel pairwise. Define the space K as disjoint union of the domains Δ_i^2 that belong to the 2-cycles σ_i where we identify the previously paired edges τ_{ij}. Using the orientation induced by the edges, K is a Δ-complex. Note that K has a finite number of path components $K^0, \ldots, K^N$. We may number these components such that $I_f \subset K^0$, where I_f is the domain of f. Let $J(K^0)$ be the index set such that $i \in J(K^0)$ if and only if $\Delta_i \subset K^0$ and let $J(K \setminus K^0)$ be the index set consisting of all other indices. Then we may split the sum into two parts:

$$f = \partial(\sum_{i \in J(K^0)} n_i\sigma_i) + \partial(\sum_{i \in J(K \setminus K^0)} n_i\sigma_i).$$

The first term of the right hand side is f and the second term vanishes. Therefore we may as well assume K to be path connected.

Combining the maps σ_i, we obtain a map $\sigma : K \to X$. Let v_j be the vertices of the simplices σ_i and let v_0 be one of end points of I_f. Then there exist paths γ_j in K from v_j to v_0.

Claim 1. The maps γ_j extend to a homotopy $H : K \times I \to X$ that deforms σ such that I_f is left unchanged and the v_j are send to v_0. By this we mean $H|_{K\times\{0\}} = \sigma$ and $H(v_j, 1) = x_0$ for all vertices v_j as well as $H(x,t) = x$ for all $t \in I$ and all $x \in K$ that are mapped to I_f.

Define the new chain $\sum_i \tilde{\sigma}_i$ given by $\tilde{\sigma}_i(x) = H(\sigma_i(x), 1)$. Since the H fixes $\partial(\sum_i n_i\sigma_i) = f$, there exists a $\beta \in C_3(X, \mathbb{Z})$ such that the homotopy H applied to $\sum_i \sigma_i$ gives

$$H_1(\sum_i \sigma_i) - H_0(\sum_i \sigma_i) = \partial\beta.$$

This means that

$$\partial(\sum_i \tilde{\sigma}_i) = \partial \sum_i \sigma_i + \partial\partial\beta = f.$$

Therefore the edges of $\sum_i \tilde{\sigma}_i$ cancel out, leaving only f. This chain has the property that each edge of each simplex in the chain is a loop around x_0.

Finally, we look at the class of f in the abelianization of the fundamental group of X, defined as $\pi_1(X)^{ab} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$. By using additive notation for the group product in $\pi_1(X)^{ab}$, we get

$$[f] = \sum_{i,j} (-1)^j n_i [\tilde{\tau_{ij}}] = \sum_i n_i [\partial\tilde{\sigma}_i]$$

where $\partial\tilde{\sigma}_i = \tilde{\tau_{i0}} - \tilde{\tau_{i1}} + \tilde{\tau_{i2}}$. Observe that the composed loop $\tilde{\tau_{i0}} - \tilde{\tau_{i1}} + \tilde{\tau_{i2}}$ is homotopic to a constant loop, since $\tilde{\tau_{i0}} \cdot (-\tilde{\tau_{i1}} \cdot \tilde{\tau_{i2}})$ is homotopy equivalent to a point by using a contraction of the simplex Δ^2. Therefore $[f] = \sum_i n_i[\partial\tilde{\sigma}_i] = [0]$ in $\pi_1(X)^{ab}$. This shows that the loops at x_0 that are the boundary of a singular 2-chain, are indeed in de commutator subgroup of $\pi_1(X)$. Hence $\ker(h) = [\pi_1(X), \pi_1(X)]$, which proves the theorem. $\square$

To prove Claim 1 we will use the following lemma.

Lemma 2.0.8. *Let $f : X \to Y$ be a continuous map, $A \subset X$ closed and $H : A \times I \to Y$ a homotopy such that $H|_{A\times\{0\}} = f|_A$. If $X \times I$ deformation retracts into $X \times \{0\} \cup A \times I$, then there is an extension of H to a homotopy $X \times I \to Y$ such that $H|_{X\times\{0\}} = f$.*

Proof. Let $U \subseteq Y$ be closed, then $f^{-1}(U) \times \{0\}$ is closed in $X \times I$. Since $H^{-1}(U)$ is closed in $A \times I$ and $A \times I$ is closed in $X \times I$, it follows that $H^{-1}(U)$ is also closed in $X \times I$. We extend H to $\tilde{H} : X \times \{0\} \cup A \times I \to Y$ by $\tilde{H}|_{X\times\{0\}} = f$. To check that $\tilde{H}$ is continuous, it is enough to see $\tilde{H}^{-1}(U) = f^{-1}(U) \cup H^{-1}(U)$ is the union of two closed sets and therefore is closed. First applying the deformation retraction on $X \times I$ and then applying $\tilde{H}$ gives us the desired extension of the homotopy. □

Proof of Claim 1. Let $\sigma : K \to X$ be the continuous map in Lemma 2.0.8. Choose $A = I_f \cup \bigcup_j v_j$, that is, the union of the vertices of σ_i and the edge that is the domain of f. Then $A \subset K$ is closed. We define the homotopy $H : K \times I \to X$ by $H(x, t) = x$ for all $x \in I_f$ and $t \in I$. Furthermore we define $H(v_j, t) = \gamma_j(t)$ for all vertices that do not lie in I_f. Recall that γ_j is a path in K from v_j to v_0. Then we indeed have $H|_{A\times\{0\}} = \sigma|_A$. It is clear by drawing a picture that $X \times I$ deformation retracts into $X \times \{0\} \cup A \times I$. Alternatively, we can also use Proposition 0.16 in Hatcher [6, p. 15] to arrive at the same conclusion.

Hence by Lemma 2.0.8 we have the homotopy extension we required. □

3 Structures on manifolds

In this section we will introduce the notion of forms and vector fields on smooth manifolds. In order to understand these subjects, it is useful to first discuss the underlying linear algebra. Once we have defined forms and vector fields, we will have a closer look at the way they behave on manifolds. In particular, we will formulate a lemma that gives us a criterion to asses whether two forms are equal using vector fields. Our main purpose here is to formulate the theory of manifolds that we can use in the context of compact semisimple Lie groups. Readers that are already familiar with smooth manifolds should feel free to skip this section.

3.1 Covectors and the tensor product

The theory in this subsection is focused on real finite dimensional vector spaces. In general, similar theory can be developed for any vector space over a field with characteristic zero. However, in our case it is sufficient to only concern ourselves with the real case.

Definition 3.1.1. The dual of a vector space V is defined as the set $\{f : V \to \mathbb{R} | f \text{ is linear}\}$. Together with the operations of point wise addition of functions and scalar multiplication V^* is a vector space itself. We denote the dual vector space by V^*. Elements of V^* are called covectors of V.

This rest of this subsection is based on the notes of E.P. van den Ban [1] Chapter 1 and 2.

Definition 3.1.2. Let $e_1, \dots, e_k$ be a basis for the vector space V. We define the dual basis $e^1, \dots, e^k$ of V^* as $e^i(e_j) = \delta^i_j$ for $1 \leq i, j \leq k$. This symbol, also known as the Kronecker delta, is defined by $\delta^i_j = 1$ if $i = j$ and $\delta^i_j = 0$ if $i \neq j$.

Proposition 3.1.3. *The dual basis defined above is a basis for V^*.*

Proof. The elements of the dual basis are defined on the basis of V. By extending them linearly onto V, we see that the dual basis indeed are elements of V^*. Assume $\sum_{i=1}^k \lambda_i e^i = 0$. This implies in particular that $\sum_{i=1}^k \lambda_i e^i(e_j) = 0$ for $1 \leq j \leq k$. Therefore we have $\lambda_i = 0$. This shows the linear independence of $e^1, \dots, e^k$. Every element $\xi \in V^*$ can be written as a linear combination of $e^1, \dots, e^k$ in the following way:

$$\xi = \sum_{i=1}^k \xi(e_i) e^i$$

We can check that both sides are equal by evaluating both sides at the basis elements of V. $\square$

By applying the definition of dual vector space to V^*, we obtain the double dual vector space $V^{**} = (V^*)^*$. There is an canonical embedding $\iota : V \to V^{**}$ defined by $\iota(v)(\xi) = \xi(v)$ for $\xi \in V^*$. This map is clearly linear. Furthermore, if $\iota(v)(\xi) = 0$ for all $\xi \in V^*$, then this also holds for each basis vector of V^*, so $e^i(v) = 0$. This implies that $v = 0$, showing the injectivity of ι.

Lemma 3.1.4. *Let V be a finite dimensional vector space, then the embedding $\iota : V \to V^{**}$ is a linear isomorphism.*

Proof. From Proposition 3.1.3 we get $\dim(V^{**}) = \dim(V^*) = \dim(V)$ and ι is injective linear map. Hence ι is a linear bijection. □

Definition 3.1.5. For vector spaces $V_1, \ldots, V_k$ and W, a W-valued multilinear map f is a map $f : V_1 \times \ldots \times V_k \to W$ that is linear in each argument separately. We denote the space of all W-valued multilinear maps of $V_1, \ldots, V_k$ by $L^k(V_1, \ldots, V_k; W)$.

There is one special class of $\mathbb{R}$-multilinear functions that we are interested in, called k-covectors.

Definition 3.1.6. A k-covector on a real vector space V is a map $f \in L^k(V, \ldots, V; \mathbb{R})$ that is alternating. Let S_k be the group of permutations of length k. Saying that f is alternating is equivalent to the statement that for any permutation $\sigma \in S_k$, the application of σ to the arguments of the function results in the change by a factor equal to the sign of σ. That is $f(v_1, \ldots, v_k) = \mathrm{Sign}(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$ for all $\sigma \in S_k$. We denote the set of all k-vectors of V by $\wedge^k V^*$.

Remark 3.1.7. Note that $L^k(V, \ldots, V; \mathbb{R})$ is closed under the point-wise addition of functions and multiplication by scalars. This makes $L^k(V, \ldots, V; \mathbb{R})$ into a vector space. Moreover, the point-wise addition and scalar multiplication of alternating functions will result in another alternating function. Therefor $\wedge^k V^*$ is a linear subspace of $L^k(V, \ldots, V; \mathbb{R})$.

There is another interesting concept that is obtain by dualizing $L^k(V, \ldots, V; \mathbb{R})$, which is called the tensor product.

Definition 3.1.8. For $V_1, \ldots, V_k$ finite dimensional real vector spaces, we define their tensor product as $V_1 \otimes \cdots \otimes V_k = L^k(V_1^*, \ldots, V_K^*; \mathbb{R})$. For an element $(x_1, \ldots, x_k) \in V_1 \times \ldots \times V_k$ its tensor product is defined as

$$[x_1 \otimes \cdots \otimes x_k](\xi_1, \ldots, \xi_k) = \xi_1(x_1) \cdots \xi_k(x_k)$$

Let $\phi : V_1 \times \ldots \times V_k \to V_1 \otimes \cdots \otimes V_k$ be the map such that $\phi(x_1, \ldots, x_k) = x_1 \otimes \cdots \otimes x_k$. It is easy to check that ϕ is multilinear. Therefore we indeed have $x_1 \otimes \cdots \otimes x_k \in V_1 \otimes \cdots \otimes V_k$.

Lemma 3.1.9. *Let $V_1, \ldots, V_k$ be finite dimensional real vector spaces of dimension d_j, respectively. Let $e_{j,1}, \ldots, e_{j,d_j}$ be a basis for V_j for all $1 \leq j \leq k$. We define the index set I as the set of elements of the form $i = (i(1), \ldots, i(k))$ where $1 \leq i(j) \leq d_j$ for all $1 \leq j \leq k$. Then*

$$\{e_{1,i(1)} \otimes \cdots \otimes e_{k,i(k)} | i \in I\}$$

is a basis for $V_1 \otimes \cdots \otimes V_k$.

Proof. Let $e_j^1, \ldots, e_j^{d_j}$ be the basis of V_j^*. To somewhat shorten the notation, we use multi-index notation and write e^i for $(e_1^{i(d_1)}, \ldots, e_k^{i(d_k)})$. Likewise, we denote $e_i = (e_{1,i(d_1)}, \ldots, e_{k,i(d_k)})$ for $i \in I$. Furthermore, we write

$$(\otimes e)_i = \phi(e_i) = e_{1,i(1)} \otimes \cdots \otimes e_{k,i(k)}.$$

Moreover, will show for $t \in V_1 \otimes \cdots \otimes V_k$ that $t = 0$ if and only if $t(e^i) = 0$ for all $i \in I$ by using its multilinearity. Per definition $t = 0$ if and only if $t(\xi_1, \ldots, \xi_k) = 0$ for all $(\xi_1, \ldots, \xi_k) \in V_1^* \times \ldots \times V_k^*$. We write $\xi_j = \xi_j(e_{j,1})e_j^1 + \ldots + \xi_j(e_{j,d_j})e_j^{d_j}$ for $1 \leq j \leq k$. Now we get

$$t(\xi_1, \ldots, \xi_k) = \sum_{i \in I} \xi^i t(e^i)$$

where the coefficients ξ^i are $\prod_{j=1}^{k} \xi_j(e_{j,i(j)})$. Using the expression on the right, we immediately see that $t(\xi_1, \ldots, \xi_k) = 0$ if and only if $t(e^i) = 0$ for all $i \in I$.
Let $m \in I$, then $(\otimes e)_i(e^m)$ equals 1 if $i = j$ and equals 0 otherwise. To conclude that the vectors $(\otimes e)_i$ are linearly independent, let λ_i be coefficients and assume that we have $\sum_{i \in I} \lambda_i (\otimes e)_i = 0$. Evaluation in the e^j gives us $\lambda_j = 0$ for all $j \in I$. Therefore we have linear independence.
To show that $(\otimes e)_i$ also span $V_1 \otimes \cdots \otimes V_k$, it is sufficient to convince ourselves that we can write $t = \sum_{i \in I} t(e^i)(\otimes e)_i$ for an arbitrary $t \in V_1 \otimes \cdots \otimes V_k$. Again by evaluating both sides at e^j, all terms on the right hand side exempt from one vanish and we see that the equality holds. $\square$

From now on we will use the notation $\otimes^k V$ by which we mean the k-fold product $V \otimes \cdots \otimes V$. Note that $\otimes^k V^*$ is defined as the space $L^k(V^{**}, \ldots, V^{**}; \mathbb{R})$. Lemma 3.1.4 states that we may identify V^{**} with V and therefore have the identification $\otimes^k V^* \cong L^k(V, \ldots, V; \mathbb{R})$ as vector spaces. This identification makes it possible to view the space $\wedge^k(V^*)$ of alternating k-linear maps as a linear subspace of $\otimes^k V^*$. Since we are mainly interested $\wedge^k(V^*)$ it is very useful to construct a projection of $\otimes^k V^*$ onto $\wedge^k(V^*)$.

Definition 3.1.10. For a real vector space V, we define $\mathrm{Alt} : \otimes^k V^* \to \wedge^k(V^*)$ by

$$\mathrm{Alt}(v^1 \otimes \cdots \otimes v^k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Sign}(\sigma) v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(k)}.$$

Strictly speaking, Alt depends on the degree k and it would be reasonable to denote it is by Alt_k. However, the k is commonly omitted in literature and it its absence leads to no large ambiguity.

Proposition 3.1.11. *The map Alt is well-defined and it is a projection. In other words, for each alternating tensor $\omega \in \otimes^k V^*$ we have $\mathrm{Alt}(\omega) = \omega$.*

Proof. Let $\omega \in \otimes^k V^*$ and assume that $\omega = v^1 \otimes \cdots \otimes v^k$. Now define the group action $S_k \times \otimes^k V^* \to \otimes^k V^* : (\sigma, \omega) \mapsto \sigma \cdot \omega$ where $\sigma \cdot \omega$ stand for $v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(k)}$.
Let $\tau \in S_k$, then we have

$$\begin{aligned}\tau(\mathrm{Alt}(\omega)) &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Sign}(\sigma)(\tau\sigma) \cdot \omega = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{Sign}(\sigma) v^{\tau^{-1}\sigma^{-1}(1)} \otimes \cdots \otimes v^{\tau^{-1}\sigma^{-1}(k)} \\ &= \frac{1}{k!} \sum_{\tau\sigma \in S_k} \mathrm{Sign}(\tau\sigma) v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(k)} = \frac{1}{k!} \sum_{\tau\sigma \in S_k} \mathrm{Sign}(\tau\sigma)\sigma \cdot \omega = \mathrm{Sign}(\tau)\,\mathrm{Alt}(\omega)\end{aligned}$$

Therefore we see that $\mathrm{Alt}(\omega)$ is alternating. Furthermore, if $\omega \in \otimes^k V$ is already alternating, we have

$$\begin{aligned}\mathrm{Alt}(\omega) &= \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{Sign}(\sigma) v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(k)} = \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{Sign}(\sigma)\,\mathrm{Sign}(\sigma^{-1}) v^{\sigma\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma\sigma^{-1}(k)} \\ &= \frac{1}{k!}\sum_{\sigma\in S_k} v_1 \otimes \cdots \otimes v_k = v^1 \otimes \cdots \otimes v^k = \omega\end{aligned}$$

In general $\omega \in \otimes^k V^*$ implies that we can write $\omega = \sum_i v_i^1 \otimes \cdots \otimes v_i^k$. The argument above can also be applied to this case. This makes Alt into a projection. □

We can use Alt to define the wedge product of two covectors.

Definition 3.1.12. Define the wedge product $\wedge : \otimes^k V^* \to \wedge^k V^* : (v^1 \otimes \cdots \otimes v^k) \mapsto v^1 \wedge \cdots \wedge v^k$ where $v^1 \wedge \cdots \wedge v^k = k!\,\mathrm{Alt}(v^1 \otimes \cdots \otimes v^k) = \sum\limits_{\sigma\in S_k} \mathrm{Sign}(\sigma) v^{\sigma^{-1}(1)} \otimes \cdots \otimes v^{\sigma^{-1}(k)}$.

Proposition 3.1.13. *The formula for computing the wedge product of two wedges is given by*

$$(v_1 \wedge \ldots \wedge v_k) \wedge (w_1 \wedge \ldots \wedge w_l) = \frac{(k+l)!}{k!l!}\,\mathrm{Alt}[(v_1 \wedge \ldots \wedge v_k) \otimes (w_1 \wedge \ldots \wedge w_l)]$$

Proof. Write $A = v_1 \wedge \ldots \wedge v_k$ and $B = w_1 \wedge \ldots \wedge w_l$. By using the same group action defined in the proof of Proposition 3.1.11, we get $v_1 \wedge \ldots \wedge v_k = \sum_{\tau\in S_k} \mathrm{Sign}(\tau)\tau \cdot A = k!\,\mathrm{Alt}(A)$ and $w_1 \wedge \ldots \wedge w_l = \sum_{\rho\in S_l} \mathrm{Sign}(\rho)\rho \cdot B = l!\,\mathrm{Alt}(B)$. Now observe that

$$\begin{aligned}\frac{(k+l)!}{k!l!}\,\mathrm{Alt}[(v_1 \wedge \ldots \wedge v_k) \otimes (w_1 \wedge \ldots \wedge w_l)] &= \sum_{\sigma\in S_{k+l}} \mathrm{Sign}(\sigma)\sigma \cdot (\mathrm{Alt}\,A \otimes \mathrm{Alt}\,B) \\ &= \sum_{\substack{\sigma\in S_{k+l}\\ \tau\in S_k\\ \rho\in S_l}} \mathrm{Sign}(\sigma\tau\rho)\sigma \cdot (\tau \cdot A \otimes \rho \cdot B) = \sum_{\sigma,\tau,\rho} \mathrm{Sign}(\sigma\tau\rho)(\sigma\tau\rho) \cdot (A \otimes B)\end{aligned}$$

In the last expression we used an embedding $S_k \hookrightarrow S_{k+l} : \tau \mapsto (\tau(1), \ldots, \tau(k), k+1, \ldots, k+l)$ into the first coordinates and $S_k \hookrightarrow S_{k+l} : \rho \mapsto (1, \ldots, k, \rho(k+1), \ldots, \rho(k+l))$ onto the last coordinates. Continuing where we left off, we have

$$\begin{aligned}&= \frac{1}{k!l!}\sum_{\tau,\rho}\sum_{\sigma} \mathrm{Sign}(\sigma\tau\rho)(\sigma\tau\rho) \cdot (A \otimes B) = \frac{1}{k!l!}k!l!\sum_{\sigma} \mathrm{Sign}(\sigma)\sigma \cdot (A \otimes B) \\ &= v_1 \wedge \ldots \wedge v_k \wedge w_1 \wedge \ldots \wedge w_l\end{aligned}$$

□

3.2 Differential forms

Now that we have discussed the necessary linear algebra, we will change our point of view to that of manifolds. From now on manifold will mean C^∞-manifold. If the dimension of a manifold is not specified, we will assume it to be n-dimensional. This is a good moment to clarify the notation that is used for certain basis elements.

Remark 3.2.1. By fixing a chart of M, we obtain a local basis of the tangent space which is denoted by $\partial_1, \ldots, \partial_n$ and the local dual basis of the cotangent space denoted by $dx^1, \ldots, dx^n$. We will denote the induced basis elements of T_pM by $(\partial_i)_p$ and basis elements of T_p^*M by dx_p^i for $1 \leq i \leq n$. Also note that we can evaluate basis elements of the cotangent space in the elements of the tangent space to obtain the Kronecker delta: $\partial^i(dx_j) = \delta^i_j$.

By applying Definition 3.1.6 to the tangent space of a manifold, we arrive at the notion of differential forms.

Definition 3.2.2. A set-theoretical differential k-from on a manifold M is a assignment $M \ni p \mapsto \omega_p \in \wedge^k T_p^*M$, i.e. ω_p is a k-covector of the tangent space at p.

We now want to state what it means for a set-theoretical differential k-from to be smooth. Consider $\chi : U \to \mathbb{R}^n$ a chart around $p \in M$. This chart induces a local dual basis $\partial_p^1, \ldots, \partial_p^n$ on T_p^*M. Using this basis, we can write any set-theoretical k-form as

$$\omega(p) = \sum_{i \in I} \omega^i(p) dx_p^{i_1} \wedge \ldots \wedge dx_p^{i_k}$$

using multi-index notation for $i = (i_1, \ldots i_k)$ where I is the set of all k-tuples of integers such that $1 \leq i_1 < \ldots < i_k \leq n$. The functions $\omega^i : U \to \mathbb{R}$ we obtain, will be called the coordinate functions of ω. A set-theoretical differential k-form is said to be smooth at p if there is a chart χ with domain U around p such that $\omega^i \in C^\infty(U)$ for all $i \in I$. A differential k-form, or simply k-form, is a set-theoretical differential k-form that is smooth at every point $p \in M$. We denote the set of k-forms on M by $\Omega^k(M)$.

Example 3.2.3. The most straight forward example of a 1-form is the differential of a differentiable function $f : M \to \mathbb{R}$. At each point $p \in M$, f gives rise to a tangent map $(df)_p : T_pM \to \mathbb{R}$. The tangent map is linear and it has only one argument, so it is a covector of T_pM. $\triangle$

There is a natural way that allows one to define forms on a manifold M by 'pulling them back' from another manifold N via a smooth map.

Definition 3.2.4. Let $f : M \to N$ be smooth and $\omega \in \Omega^k(N)$. We define the pullback $f^*\omega \in \Omega^k(M)$ of ω by f to be the composition of tangent map of f with ω, i.e. for $V_1, \ldots, V_k \in T_pM$ we have $(f^*\omega)_p(V_1, \ldots, V_k) = \omega_{f(p)}(T_pf(V_1), \ldots, T_pf(V_k))$.

Note that we have $f^*\omega : T_pM \times \ldots \times T_pM \to \mathbb{R}$. Since ω is alternating, so is $f^*\omega$. Furthermore the linearity of the tangent map and the multilinearity of ω imply that $f^*\omega$ is multilinear. Therefore $f^*\omega$ is indeed a k-form on M.

3.3 Vector fields

Definition 3.3.1. A set-theoretical vector field is a assignment $M \ni p \mapsto X_p \in T_pM$. Similar to the way we defined differential forms, we want to characterize the smoothness of vector fields. Let $\chi : U \to \mathbb{R}^n$ be a chart around p. This chart gives induces a basis $(\partial_1)_p, \dots, (\partial_n)_p$ of T_pM. Using this basis, we can write

$$X_p = \sum_{i=1}^{n} X^i(p)(\partial_i)_p$$

where $X^i : M \to \mathbb{R}$ are the component functions of X. We say that X is smooth at p is there is a chart χ with domain U around p such that $X^i \in C^\infty(U)$ for all $1 \leq i \leq n$. A vector field on M is a set-theoretical vector field that is smooth at every $p \in M$. We denote the vector space of vector fields on M by $\mathfrak{X}(M)$.

Now we proceed with the definition of the pullback of vector fields. The difference with the pullback of forms lies in the properties we require of the function we pull back along. Although we could define a pullback of forms along any smooth map, in the case of vector fields we need the function to be a diffeomorphism. That is because we need to be able to use the inverse of the tangent map.

Definition 3.3.2. Let $f : M \to N$ be a diffeomorphism and $X \in \mathfrak{X}(N)$. We define the pullback $f^*X \in \mathfrak{X}$ to be $f^*X(p) = (T_pf)^{-1}X(f(p))$ for all $p \in T_pM$.

Per definition, two k-forms ω_1, ω_2 are equal if they are equal at every point $p \in M$. There is a natural way of evaluating forms in vector fields on M. Vector fields allow us to associate a tangent vector to every point of M. Let $X^{(1)}, \dots, X^{(k)} \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. We define the expression $\omega(X^{(1)}, \dots, X^{(k)})$ in a point $p \in M$ by $\omega(X^{(1)}, \dots, X^{(k)})(p) = \omega_p(X_p^{(1)}, \dots, X_p^{(k)})$. This provides us with the map $\omega(X^{(1)}, \dots, X^{(k)}) : M \to \mathbb{R}$.

Proposition 3.3.3. *For any $\omega \in \Omega^k(M)$ and $X^{(1)}, \dots, X^{(k)} \in \mathfrak{X}(M)$ the map $\omega(X^{(1)}, \dots, X^{(k)}) : M \to \mathbb{R}$ is smooth.*

Proof. Let $p \in M$ and fix a chart $\chi : U \to \mathbb{R}^n$. This chart provides us with a basis of T_pM and T_p^*M. By using the notation from Definition 3.2.2 and 3.3.1 for the bases of the tangent and cotangent space, we write $\omega = \sum_{i \in I} \omega^i dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and $X = \sum_{j=1}^{n} X^j \partial_j$.

$$\omega(X^{(1)}, \dots, X^{(k)})(p) = \sum_{i \in I} \omega^i(p) \cdot dx_p^{i_1} \wedge \dots \wedge dx_p^{i_k}(X_p^{(1)}, \dots, X_p^{(k)})$$

□

Corollary 3.3.4. *Let $p \in M$, then the map $\mathfrak{X}(M) \to T_pM : X \mapsto X_p$ is surjective.*

Proof. Let $V \in T_pM$. Choose a chart $\chi : U \to \mathbb{R}^n$ around p. Then the tangent map $T_p\chi : T_pM \to \mathbb{R}^n$ is an isomorphism of vector spaces. Let $K \subset \chi(U)$ be a compact neighborhood of $\chi(p)$. Then there exists a compactly supported function $\varphi \in C_C^\infty(\mathbb{R}^n)$ with the properties that $\varphi = 1$ on a open neighborhood of K and $\mathrm{supp}(\varphi) \subset \chi(U)$. Define the vector field $\tilde{X}$ by $\tilde{X}(y) = \varphi(y) \cdot T_p\chi(X_p)$ for all $y \in \mathbb{R}^n$. Now define the vector field X on M by $X(m) = (\chi^*\tilde{X})(m)$ for $m \in U$ and $X(m) = 0$ otherwise. Note that the extension by zero is smooth, because $\chi^*\tilde{X}$ is strictly supported in the open set U. We can now conclude that X has the property that $X(p) = V$, since $\varphi(\chi(p)) = 1$. □

It turns out the evaluation of forms in vector fields is compatible with the evaluation of forms in points by the lemma below.

Lemma 3.3.5. *Let $\omega, \eta \in \Omega^k(M)$, then the following are equivalent:*

i) $\omega = \eta$

ii) *For all vector fields $X^{(1)}, \dots, X^{(k)} \in \mathfrak{X}(M)$ and points $p \in M$ we have $\omega_p(X_p^{(1)}, \dots, X_p^{(k)}) = \eta_p(X_p^{(1)}, \dots, X_p^{(k)})$.*

Proof. Assume **i)**. Fix vector fields $X^{(1)}, \dots, X^{(k)} \in \mathfrak{X}(M)$ and $p \in M$. Observe that $X_p^{(1)}, \dots, X_p^{(k)} \in T_pM$ per definition of a vector field. By our assumption we can directly conclude that $\omega_p(X_p^{(1)}, \dots, X_p^{(k)}) = \eta_p(X_p^{(1)}, \dots, X_p^{(k)})$.

For the other direction assume **ii)**. Let $p \in M$ and $V_1, \dots, V_k \in T_pM$. Corollary 3.3.4 provides us with vector fields $X^{(i)}$ for all $1 \leq i \leq k$ with the property that $X_p^{(i)} = V_i$. Therefore we have $\omega_p(V_1, \dots, V_k) = \eta_p(V_1, \dots, V_k)$ for all $V_1, \dots, V_k \in T_pM$ and we are done. □

Definition 3.3.6. We define the derivation of a smooth real-valued functions on M by a vector field to be the map $C^\infty(M) \times \mathfrak{X}(M) \to C^\infty(M) : (f, X) \mapsto Xf$, where we define $Xf(p) = (df)_P(X_p)$. This can be interpreted as the directional derivative of f at p along X_p.

4 Lie theory

It is in this section that we introduce Lie groups and Lie algebras. The extra structure that a Lie group has over manifolds, will give rise to so called left invariance of vector fields and forms on the Lie group. In the end of this section we will discuss the proof of the surjectivity of the Lie bracket, under the assumption that G has certain properties.

4.1 Basic properties

Definition 4.1.1. A Lie group is a smooth manifold equipped with a smooth product function $G \times G \to G : (x, y) \mapsto xy$ and a smooth inversion function $G \to G : x \mapsto x^{-1}$ that makes it into a group.

Example 4.1.2.

- $\mathbb{R}^n$ equipped with addition and neutral element 0 is a Lie group. [2]
- The space of invertible $n \times n$-matrices together with matrix multiplication and neutral element I is a Lie group.
- The unit circle $S^1 \subset \mathbb{C}$ together with the multiplication of complex numbers and neutral element 1 again forms a Lie group.

△

Definition 4.1.3. Define the Lie algebra of G to be the tangent space at the identity e, denoted by $\mathfrak{g} = T_eG$.

For any Lie groups there is an identification of the Lie algebra $\mathfrak{g}$ with the space of so called left invariant vector fields on G.

Definition 4.1.4. For $g \in G$ we define the map $l_g : G \to G : x \mapsto xg$ as the left multiplication. A vector field $V \in \mathfrak{X}(G)$ is called left invariant if $(l_g)_*V = V$ for all $g \in G$. The set of left invariant vector fields on G is denoted by $\mathfrak{X}_L(G)$.

Note that the map l_g is smooth, because the multiplication on G is smooth. Its inverse is exactly $l_{g^{-1}}$, which is smooth as well. That makes l_g into a diffeomorphism from G onto itself and therefore the push forward operation by l_g is well defined.

Lemma 4.1.5. *The following are equivalent:*

i) *$V \in \mathfrak{X}(G)$ is left invariant.*

ii) *For all $x, y \in G$ we have $V(xy) = T_y(l_x)V(y)$.*

Proof. Note for $x \in G$ that l_x is a diffeomorphism, which leads to a linear isomorphism on the tangent spaces. In particular, we can write the identity map $Id : G \to G$ as $l_{x^{-1}} \circ l_x$ and apply the chain rule in the point xy to conclude that $T_{xy}(Id) = T_{xy}(l_{x^{-1}} \circ l_x) = T_y(l_x) \circ T_{xy}(l_{x^{-1}})$. This gives us $T_{xy}(l_{x^{-1}})^{-1} = T_y(l_x)$. Assume **i)**, then we use this equality to conclude

$$V(xy) = (l_{x^{-1}})_*V(xy) = T_{xy}(l_{x^{-1}})^{-1}V(l_{x^{-1}}(xy)) = T_y(l_x)V(y)$$

Hence we have **ii)**. The other implication is given by a similar argument. It suffices to see that

$$V(xy) = T_y(l_x)V(y) = T_{xy}(l_{x^{-1}})^{-1}V(l_{x^{-1}}(xy)) = (l_{x^{-1}})_*V(xy)$$

□

This characterization of left invariance shows that a left invariant vector field is completely determined by its value at the identity. By choosing $y = e$ we obtain for an arbitrary left invariant vector field: $V(x) = T_e(l_x)V(e)$. The rest of this subsection is based on Chapter 2,3 and 4 of Lie Groups [2].

Lemma 4.1.6. *The map $v_X : \mathfrak{g} \to \mathfrak{X}_L(G) : X \mapsto v_X$ where we define $v_X(g) = T_e(l_g)X$ is a linear isomorphism.*

Proof. We have to show that v_X is left invariant. By differentiating the map $l_{xy} = l_x \circ l_y$ and applying the chain rule, we get

$$v_X(xy) = T_e(l_{xy})X = T_y(l_x)T_e(l_y)X = Ty(l_x)v_X(y)$$

By lemma 4.1.5 this proves the left invariance of v_X. Note that v is injective, since $v_X = v_Y$ implies $v_X(e) = v_Y(e)$ and thus $X = Y$. The map $V \mapsto V(e)$ that evaluates left invariant vector fields in the identity acts as an inverse of v. That makes v into a bijection. □

We can use these vector fields to construct the exponential map. Recall that an integral curve of a vector field V is any differential map $\gamma : [a, b] \to G$ with $a < 0 < b$ such that $\gamma'(t) = V(\gamma(t))$.

Definition 4.1.7. Let α_X be the maximal integral curve of the left invariant vector field v_X with starting point e. We define the exponential map as $\exp : \mathfrak{g} \to G : X \mapsto \alpha_X(1)$.

It is shown in Lie groups [2, p. 16] that the domain of α_X is $\mathbb{R}$, therefore exp is well defined.

Let us now move on to another important map. First consider the map $C_x : G \to G : y \mapsto xyx^{-1}$ that conjugates elements of G by an $x \in G$. Observe that C_x is a diffeomorphism from G to itself by the same reasoning that was used in the case of the left multiplication l_g. For any $x \in G$ we have $c_x(e) = e$. Hence the tangent map $T_e(C_x)$ is an isomorphism of $\mathfrak{g}$ onto itself. Note that for $x, y \in G$ the composition $T_e(C_x) \circ T_e(C_y)$ is again an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}$. Furthermore, for each $x \in G$ the inverse of $T_e(C_x)$ is given by $T_e(C_x^{-1})$. Therefore the isomorphisms of $\mathfrak{g}$, or in fact any vector space, onto itself is a group which we will denote by $GL(\mathfrak{g})$.

Definition 4.1.8. We define the adjoint representation of a Lie group G as the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ such that $\mathrm{Ad}(x) = T_eC_x$

Let $v_1, \ldots, v_n$ be a basis of T_eG, then there is a unique linear map from $\mathbb{R}^n$ to $\mathfrak{g}$ that sends the standard i-th basis element e_i to v_i. For another basis $w_1, \ldots, w_n$ of $\mathfrak{g}$, there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ that sends w_i to v_i. This is a linear bijection and therefore a linear isomorphism. From this we can conclude that $\mathrm{GL}(\mathfrak{g})$ can be equipped with charts that make it into a smooth manifold, independent of the choice of a basis.

Since $\mathrm{GL}(\mathfrak{g})$ is a group as well as a manifold, it follows that it is a Lie group if the product and inversion map are smooth. This is the case, see page 16 of Lie Groups [2]. Using this new insight, Ad becomes a smooth map between Lie groups.

Definition 4.1.9. We define ad as the tangent map of Ad at the identity e, hence $\mathrm{ad} = T_e \mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$.

Note that $\mathrm{GL}(\mathfrak{g})$ is a linear subspace of $\mathrm{End}(\mathfrak{g})$, the space of all linear transformation from $\mathfrak{g}$ to itself. Now observe that $\mathrm{End}(\mathfrak{g}) \simeq \mathbb{R}^N$ for a certain N, since it is a finite dimensional vector space. Let I stand for the identity element of $\mathrm{GL}(\mathfrak{g})$, then it follows that $T_I \mathrm{GL}(\mathfrak{g}) = T_I\mathrm{End}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$. This is because $\mathrm{GL}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ is open. That explains why ad maps into $\mathrm{End}(\mathfrak{g})$.

From properties of the exponential map and the use of the chain rule, we can derive that $\mathrm{ad}(X) = \frac{d}{dt}|_{t=0} \mathrm{Ad}(\exp tX)$. See [2, p. 19].

Definition 4.1.10. A real Lie algebra is a real vector space V together with a bilinear map $[\cdot,\cdot] : V \times V \to V$ that is anti-symmetric and obeys the Jacobi identity. By that we mean for all $X, Y, Z \in V$ we have $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. This operation is called a Lie bracket.

Definition 4.1.11. For a Lie group, we define the Lie bracket $T_eG \times T_eG \to t_eG : (X, Y) \mapsto [X, Y]$ where $[X, Y] = (\mathrm{ad}\, X)Y$.

Proposition 4.1.12. *The tangent space T_eG equipped with the operation defined above is a Lie algebra.*

Proof. See Lie Groups [2, p. 20-21]. □

Definition 4.1.13. We call a k-form $\omega \in \Omega^k(G)$ left invariant if $l_g^*\omega = \omega$ for all $g \in G$. The space of all left invariant k-forms on G is denoted by $\Omega^k_L(G)$.

Each k-form on G defines a k-covector of T_eG by evaluation in the identity element, so there is a map $Ev : \Omega^k(G) \to \Lambda^k T_e^*G : \omega \mapsto \omega_e$. On the other hand, we will see soon that any k-covector can be made into a left invariant k-form on G.

Definition 4.1.14. We define the map $\phi : \Lambda^k T_e^* G \to \Omega^k(G)$ that makes k-covectors of T_eG into k-forms on G such that for any $\omega_e \in T_eG$ we have

$$\phi(\omega_e)_g = \begin{cases} \omega_e & \text{if } g = e \\ T_e(l_{g^{-1}})^*\omega_e & \text{if } g \neq e \end{cases}$$

Therefore, once we choose a k-covector ω_e, the evaluation of the corresponding form in $g \in G$ becomes $\omega_g(X_1, \dots, X_k) = \omega_e(T_e(l_{g^{-1}})X_1, \dots, T_e(l_{g^{-1}})X_k)$ for $X_1, \dots, X_k \in T_eG$.

Proposition 4.1.15. *The vector space of left invariant k-forms on G is isomorphic to the space of k-covectors of T_eG. More precisely, the linear isomorphism is given by $Ev : \Omega^k_L(G) \to \Lambda^k T_e^* G : \omega \mapsto \omega_e$.*

Proof. Let $\omega \in \Omega^k_L(G)$ be a left invariant form. Since ω is left invariant, evaluating ω at $x \in G$ is the same as evaluating $(l_{x^{-1}})^*\omega$ at x. Therefore, we have

$$\omega_x(X_1, \dots, X_k) = \omega_e(T_e(l_{x^{-1}})X_1, \dots, T_e(l_{x^{-1}})X_k)$$

This implies that ω is completely determined by its evaluation in the identity. Hence Ev is injective.

For surjectivity we will check that the map ϕ from Definition 4.1.14 actually provides *left invariant* forms. Let $\omega_e \in \Lambda^k T_e^* G$ and, for simplicity, we denote $\omega = \phi(\omega_e)$. We want to prove that $l_x^*\omega = \omega$. Let $g \in G$, then for $X_1, \dots, X_k \in T_g G$ we have

$$\begin{aligned}(l_x^*\omega)_g(X_1, \dots, X_k) &= \omega_{xg}(T_g(l_x)X_1, \dots, T_g(l_x)X_k)\\ &= \omega_e(T_{xg}(l_{xg}^{-1})T_g(l_x)X_1, \dots)\\ &= \omega_e(T_{xg}(l_g^{-1}l_x^{-1})T_g(l_x)X_1, \dots)\\ &= \omega_e(T_g(l_g^{-1}l_x^{-1}l_x)X_1, \dots)\\ &= \omega_e(T_g(l_g^{-1})X_1, \dots) = \omega_g(X_1, \dots, X_k)\end{aligned}$$

Therefore ω is indeed left invariant. Now note that $Ev \circ \phi$ is the identity map on $\Lambda^k T_e^* G$, hence Ev is surjective. □

4.2 Surjectivity of the Lie bracket

In this subsection we assume that G is a connected semisimple compact Lie group. That is, G is compact, connected and $\ker(\text{ad}) = 0$. Note that

$$\ker(\text{ad}) = \{X \in \mathfrak{g} | [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$$

is precisely the center of $\mathfrak{g}$, because $[X, Y] = 0$ if and only if $[X, Y] = -[X, Y]$, which is equivalent to $[X, Y] = [Y, X]$ by the anti-symmetry of the Lie bracket.

Our goal is to prove the following lemma:

Lemma 4.2.1. *If G is a semisimple compact Lie group, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. That is, for every element $Z \in \mathfrak{g}$ we have $\sum_i [X_i, Y_i] = Z$ for certain $X_i, Y_i \in \mathfrak{g}$.*

Lemma 4.2.2. *There exists a positive definite inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that for $X, Y \in \mathfrak{g}$ we have $\langle \text{Ad}(g)X, \text{Ad}(g)Y \rangle = \langle X, Y \rangle$ for all $g \in \mathfrak{g}$.*

Proof. Let $(\cdot, \cdot)$ be any positive definite inner product defined on $\mathfrak{g}$. Similar to the way we constructed a left invariant form that is nowhere vanishing on G from an element $\omega_e \in \Lambda^k T_e^* G$, we can construct a right invariant form on G. Let dg be a right invariant top form, i.e. it has order n equal to the dimension of G. We will use the orientation on G induced by the form dg. With this orientation, dg is per definition positively oriented. We use the integration by top forms to define a new inner product

$$\langle X, Y \rangle = \int_G (\text{Ad}(g)X, \text{Ad}(g)Y) dg.$$

Note that per definiteness of $(\cdot,\cdot)$, $(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y)$ is continuous in g. Moreover, $(\mathrm{Ad}(g)X, \mathrm{Ad}(g)X)$ is non-negative, because we assumed $(\cdot,\cdot)$ to be positive definite. Since dg is a positive form, we also have $\langle X, X\rangle \geq 0$. In the case that $\langle X, X\rangle = 0$, we integrate a non-negative function, so the map $g \mapsto (\mathrm{Ad}(g)X, \mathrm{Ad}(g)X)$ must be identically zero. By choosing $g = e$ we obtain $(\mathrm{Ad}(e)X, \mathrm{Ad}(e)X) = (X, X) = 0$. This implies $X = 0$, because $(\cdot,\cdot)$ is positive definite. This shows $\langle\cdot,\cdot\rangle$ is positive definite.

We claim that the inner product is invariant. Let $g_0 \in G$, then

$$\begin{aligned}\langle \mathrm{Ad}(g_0)X, \mathrm{Ad}(g_0)Y\rangle &= \int_G (\mathrm{Ad}(g)\,\mathrm{Ad}(g_0)X, \mathrm{Ad}(g)\,\mathrm{Ad}(g_0)Y)dg \\ &= \int_G (\mathrm{Ad}(gg_0)X, \mathrm{Ad}(gg_0)Y)dg \\ &= \int_G \varphi(gg_0)dg = \int_G \varphi(g)dg = \langle X, Y\rangle\end{aligned}$$

Here we viewed $(\mathrm{Ad}(\cdot)X, \mathrm{Ad}(\cdot)Y)$ as a function $\varphi : G \to \mathbb{R}$. Therefore we can use the right invariance of dg to conclude that integrating over $\varphi(gg_0)$ is the same as integrating over $\varphi(g)$. This verifies our claim. □

Lemma 4.2.3. *The inner product $\langle\cdot,\cdot\rangle$ defined in Lemma 4.2.2 has the property that $\langle [U, X], Y\rangle = -\langle X, [U, Y]\rangle$ for all $U, X, Y \in \mathfrak{g}$.*

Proof. Let $U, X, Y \in \mathfrak{g}$. From the properties of the inner product, ad and the exponential map, it follows that

$$\begin{aligned}\langle [U, X], Y\rangle &= \langle \mathrm{ad}(U)X, Y\rangle \\ &= \langle \tfrac{d}{dt}|_{t=0}\,\mathrm{Ad}(\exp(tU))X, Y\rangle \\ &= \tfrac{d}{dt}|_{t=0}\langle \mathrm{Ad}(\exp(tU))X, Y\rangle \\ &= \tfrac{d}{dt}|_{t=0}\langle \mathrm{Ad}(\exp(-tU))\,\mathrm{Ad}(\exp(tU))X, \mathrm{Ad}(\exp(-tU))Y\rangle \\ &= \tfrac{d}{dt}|_{t=0}\langle \mathrm{Ad}(\exp(-tU + tU))X, \mathrm{Ad}(\exp(-tU))Y\rangle \\ &= \langle X, \tfrac{d}{dt}|_{t=0}\,\mathrm{Ad}(\exp(-tU))Y\rangle \\ &= \langle X, -\,\mathrm{ad}(U)Y\rangle = -\langle X, [U, Y]\rangle\end{aligned}$$

This is proves the lemma. □

Definition 4.2.4. We define an ideal $\mathfrak{a}$ in $\mathfrak{g}$ to be a linear subspace $\mathfrak{a} < \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$, or equivalently for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$ we have $[X, Y] \subset \mathfrak{a}$. We write $\mathfrak{a} \lhd \mathfrak{g}$.

We denote the orthogonal complement of $\mathfrak{a}$ as $\mathfrak{a}^\perp = \{X \in \mathfrak{g} | \langle X, Y\rangle = 0 \quad \forall Y \in \mathfrak{a}\}$.

Corollary 4.2.5. *If $\mathfrak{a} \lhd \mathfrak{g}$, then $\mathfrak{a}^\perp \lhd \mathfrak{g}$.*

Proof. Note that $\mathfrak{a}^\perp$ is a linear subspace of $\mathfrak{g}$. Therefore, we only have to check that $[\mathfrak{g}, \mathfrak{a}^\perp] \subset \mathfrak{a}^\perp$. Let $X \in \mathfrak{a}^\perp$ and $Y \in \mathfrak{g}$, then for any $U \in \mathfrak{a}$ we use Lemma 4.2.3 to conclude $\langle [Y, X], U\rangle = \langle X, -[Y, U]\rangle$. Since $\mathfrak{a}$ is an ideal, we have $[Y, U] \in \mathfrak{a}$. Finally, recall that $X \in \mathfrak{a}^\perp$, therefore $\langle X, -[Y, U]\rangle = 0$.

The equality $\langle [Y, X], U\rangle = 0$ shows that $\mathrm{ad}(Y)\mathfrak{a}^\perp \subset \mathfrak{a}^\perp$, which proves that $\mathfrak{a}^\perp$ is indeed an ideal. □

Theorem 4.2.6. *Let $\mathfrak{g}$ admit a $\mathrm{ad}(\mathfrak{g})$-invariant positive definite inner product. Then $\mathfrak{g}$ decomposes as the direct sum of minimal ideals that are pairwise disjoint.*

Proof. We will proceed by a proof by induction on the dimension of $\mathfrak{g}$. In the 0-dimensional case, the statement of this theorem is trivial. Assume that the statement is true for all $\mathfrak{g}$ with $\dim(\mathfrak{g}) \leq k$. Now assume that $\dim(\mathfrak{g}) = k+1$. If $\mathfrak{g}$ is a minimal ideal, we are done. If this is not the case, then there exists a strictly smaller ideal $\mathfrak{a}_1$ of $\mathfrak{g}$ that is unequal to $\{0\}$. By choosing smaller and smaller ideals in $\mathfrak{g}$, we obtain the following sequence.

$$\mathfrak{g} \rhd \mathfrak{a}_1 \rhd \ldots \rhd \mathfrak{a}$$

Since $\mathfrak{g}$ has finite dimension and each consecutive ideal will have a strictly smaller dimension, the sequence is finite and has a smallest element $\mathfrak{a}$ unequal to $\{0\}$. Hence we write $\mathfrak{g}$ as the orthogonal direct sum $\mathfrak{a} \oplus \mathfrak{a}^{\perp}$. Note that $\langle \cdot, \cdot \rangle|_{\mathfrak{a}}$ still is positive definite and ad-invariant. So by the induction hypothesis $\mathfrak{a}^{\perp} = \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_k$.

We now want to prove that $\mathfrak{b}_i$ is an ideal of $\mathfrak{g}$. Observe that $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$ are both ideals, so $[\mathfrak{a}, \mathfrak{a}^{\perp}] \subset \mathfrak{a} \cap \mathfrak{a}^{\perp}$. For every $0 \neq X \in \mathfrak{a}$ we have $\langle X, X \rangle \neq 0$, hence $X \notin \mathfrak{a}^{\perp}$, so $\mathfrak{a} \cap \mathfrak{a}^{\perp} = 0$. This implies that $[\mathfrak{a}, \mathfrak{a}^{\perp}] = 0$. With this information we conclude for all $1 \leq i \leq k$ that

$$\begin{aligned}
[\mathfrak{g}, \mathfrak{b}_i] &= [\mathfrak{a}, \mathfrak{b}_i] \oplus [\mathfrak{a}^{\perp}, \mathfrak{b}_i] \\
&= [\mathfrak{a} \cap \mathfrak{b}_i \oplus \mathfrak{b}_i \cap \mathfrak{b}_i^{\perp}, \mathfrak{b}_i] \oplus [\mathfrak{a}^{\perp}, \mathfrak{b}_i] \\
&= [\mathfrak{a} \cap \mathfrak{b}_i, \mathfrak{b}_i] \oplus [\mathfrak{b}_i \cap \mathfrak{b}_i^{\perp}, \mathfrak{b}_i] \oplus [\mathfrak{a}^{\perp}, \mathfrak{b}_i]
\end{aligned}$$

The last expression is the direct sum of three Lie brackets. Note that in all three cases the left argument is a linear subspace of $\mathfrak{a}^{\perp}$. Since $\mathfrak{b}_i \lhd \mathfrak{a}^{\perp}$, we have $[\mathfrak{g}, \mathfrak{b}_i] \subset \mathfrak{b}_i$. Therefore $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}_1 \oplus \ldots \oplus \mathfrak{b}_k$ is a direct sum of pairwise orthogonal ideals, all of which are minimal. That completes the induction step. □

Corollary 4.2.7. *For each minimal ideal $\mathfrak{a}_i$ from Theorem 4.2.6, we have $[\mathfrak{a}_i, \mathfrak{a}_i] = \mathfrak{a}_i$.*

Proof. Let $X \in \mathfrak{g}$ and $U, V \in \mathfrak{a}_i$, then by the Jacobi identity and anti-symmetry of the Lie bracket, we have $[X, [U, V]] = [[X, U], V] + [U, [X, V]]$. Note that $U, V \in \mathfrak{a}_i$, which implies that $[X, U], [X, V] \in \mathfrak{a}_i$. Therefore $[[X, U], V], [U, [X, V]] \in \mathfrak{a}_i$. Finally, we use the fact that $\mathfrak{a}_i$ is a linear subspace to conclude that $[X, [U, V]] \in \mathfrak{a}_i$. Hence, $[\mathfrak{a}_i, \mathfrak{a}_i] \lhd \mathfrak{g}$.

We already have $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_i$. Because $\mathfrak{a}_i$ is a minimal ideal, we are left with two possibilities; either $[\mathfrak{a}_i, \mathfrak{a}_i] = 0$ or $[\mathfrak{a}_i, \mathfrak{a}_i] = \mathfrak{a}_i$. If $[\mathfrak{a}_i, \mathfrak{a}_i] = 0$, we also know that $[\mathfrak{a}_i, \mathfrak{a}_j] = 0$ for $i \neq j$. This leads to $[\mathfrak{a}_i, \oplus_{j=1}^{N} \mathfrak{a}_j] = [\mathfrak{a}_i, \mathfrak{g}] = 0$. Hence $\mathfrak{a}_i \subset \ker(\mathrm{ad})$. At the beginning of this subsection we stated that $\ker \mathrm{ad} = 0$, so $\mathfrak{a}_i = 0$. However, that cannot be the case. Therefore we have $[\mathfrak{a}_i, \mathfrak{a}_i] = \mathfrak{a}_i$. □

Proof of Lemma 4.2.1. Using the decomposition of Theorem 4.2.6 and the result of Corollary 4.2.7, we get

$$[\mathfrak{g}, \mathfrak{g}] = \sum_{i,j} [\mathfrak{a}_i, \mathfrak{a}_j] = \sum_i [\mathfrak{a}_i, \mathfrak{a}_i] = \sum_i \mathfrak{a}_i = \mathfrak{g}$$

□

5 The relation between cohomology and homology

In this section we have establish the last ingredients we need to be able to prove that $\pi_1(G)$ is finite. For that we will define the de Rham cohomology by using differential forms. Further on we will look at modules to extend the notion of tensor product. At the end we will arrive at the universal coefficient theorem for homology.

5.1 The Rham cohomology

Definition 5.1.1. We define the exterior differential for all $k \geq 1$ on a manifold M to be the unique map $d : \Omega^{k-1}(M) \to \Omega^k(M)$ such that for $\omega \in \Omega^{k-1}(M)$ and $X_1, \dots, X_k \in \mathfrak{X}(M)$ the following holds:

$$\begin{aligned} d\omega(X_1, \dots, X_k) = & \sum_{1 \leq i \leq k} (-1)^{i-1} X_i(\omega(X_1, \dots, \widehat{X_i}, \dots, X_k)) \\ & + \sum_{1 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X_1}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned}$$

The notation $\widehat{X_i}$ indicates that this argument is left out and $[X_i, X_j]$ is the Lie bracket of vector fields defined in Lee [7, p. 186].

Proposition 5.1.2. *The exterior differential d has the following properties:*

a) *d is $\mathbb{R}$-linear.*

b) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

c) *$d \circ d = 0$.*

d) *d is the usual differential of smooth functions on $\Omega^0(M)$.*

e) *d commutes with pullbacks. Let $F : M \to N$ be a smooth map between manifolds and $F^* : \Omega^k(N) \to \Omega^k(M)$ the associated pull back map. Then we have*

$$F^*(d\omega) = d(F^*\omega).$$

for all $\omega \in \Omega^k(M)$.

Proof. See Lee [7] page 365. □

We call $\omega \in \Omega^k(M)$ exact if there exists an $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$. Moreover, $\omega \in \Omega^k(M)$ is said to be closed if $d\omega = 0$. Note that $d \circ d = 0$ implies that every exact form is closed. The vector spaces of k-forms on M together with the exterior differential give rise to a complex called the de Rham-complex.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

Definition 5.1.3. Let $Z^k(M) = \{\omega \in \Omega^k(M) | \omega \text{ is closed}\}$ and $B^k(M) = \{\omega \in \Omega^k(M) | \omega \text{ is exact}\}$. We define the de Rham cohomology $H^k_{dR}(M, \mathbb{R})$ as the homology of this complex, so the k-th de Rham cohomology is the quotient $Z^k(M)/B^k(M)$.

Now we return to the case that we have a connected, compact semisimple Lie group G. In this case we have the notion of left invariance of forms.

For every $k \geq 0$, we have $\Omega^k_L(G) \subset \Omega^k(G)$. Also note that taking the differential of a left invariant form, will result in another left invariant form, i.e. $d(\Omega^k_L(G)) \subset \Omega^{k+1}_L(G)$ by the following corollary.

Corollary 5.1.4. *Let $\omega \in \Omega^k_L(G)$ be a left invariant form, then $d\omega$ is also left invariant.*

Proof. This follows directly from property **e)** of Proposition 5.1.2. Let $g \in G$, then we have

$$l_g^*(d\omega) = d(l_g^*\omega) = d\omega.$$

by the left invariance of ω. $\square$

Therefore we obtain a chain complex consisting of left invariant forms as a subcomplex of the de Rham-complex. We will now define a new complex, which will turn out to be isomorphic to this subcomplex.

Definition 5.1.5. Proposition 4.1.15 provided an linear isomorphism $\phi : \Omega^k_L(G) \to \mathfrak{g}^*$. We define the long sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{d_{\text{kos, 0}}} \mathfrak{g}^* \xrightarrow{d_{\text{kos, 1}}} \bigwedge\nolimits^2 \mathfrak{g}^* \xrightarrow{d_{\text{kos, 2}}} \dots.$$

The differential d_{kos} is defined by $d_{\text{kos, i}} = \phi \circ d_i \circ \phi^{-1}$. Hence we have

$$d_{\text{kos}} \circ d_{\text{kos}} = (\phi \circ d \circ \phi^{-1}) \circ (\phi \circ d \circ \phi^{-1}) = \phi \circ d \circ \ d \circ \phi^{-1} = 0.$$

Therefore the long sequence is a complex, which we will call the Koszul-complex.

Observe that it follows immediately from the construction of the Koszul-complex that $\phi : (\Omega^k_L(G), d) \to (\wedge^k \mathfrak{g}^*, d_{\text{kos}})$ is an isomorphism of chain complexes. That is, $\phi : \Omega^k_L(G) \to \mathfrak{g}^*$ is an isomorphism for all $k \geq 0$ and $d_{\text{kos}} \circ \phi = \phi \circ d$. We obtain the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "d_{\text{kos}}"] \arrow[d, "\phi^{-1}"] & \mathfrak{g}^* \arrow[r, "d_{\text{kos}}"] \arrow[d, "\phi^{-1}"] & \bigwedge^2 \mathfrak{g}^* \arrow[r, "d_{\text{kos}}"] \arrow[d, "\phi^{-1}"] & \dots \\
0 \arrow[r] & \Omega^0_L(G) \arrow[r, "d"] \arrow[d, hook, "i"] & \Omega^1_L(G) \arrow[r, "d"] \arrow[d, hook, "i"] & \Omega^2_L(G) \arrow[r, "d"] \arrow[d, hook, "i"] & \dots \\
0 \arrow[r] & \Omega^0(G) \arrow[r, "d"] & \Omega^1(G) \arrow[r, "d"] & \Omega^2(G) \arrow[r, "d"] & \dots
\end{tikzcd}$$

Lemma 5.1.6. *Let $\omega \in \Omega_L^k(G)$ and $X^1, \ldots, X^k \in \mathfrak{X}_L(G)$. Then $\omega(X^1, \ldots, X^k) \in C^\infty(G)$ is constant.*

Proof. Let $g \in G$. Since ω as well as the vector fields are left invariant, we derive by using the chain rule that

$$\begin{aligned}\omega_g(X_g^1, \ldots, X_g^k) &= (l_{g^{-1}}^* \omega)_g(X_g^1, \ldots, X_g^k) \\ &= \omega_e(T_g(l_{g^{-1}})X_g^1, \ldots, T_g(l_{g^{-1}})X_g^k) \\ &= \omega_e(T_g(l_{g^{-1}})T_e(l_g)X_e^1, \ldots, T_g(l_{g^{-1}})T_e(l_g)X_e^k) \\ &= \omega_e(T_e(l_e)X_e^1, \ldots, T_e(l_e)X_e^k) = \omega_e(X_e^1, \ldots, X_e^k).\end{aligned}$$

Therefore the value of $\omega(X^1, \ldots, X^k)$ is independent of $g \in G$. □

Lemma 5.1.7. *The cohomology class $H^1(\bigwedge^\bullet \mathfrak{g}^*, d_{kos})$ is trivial.*

Proof. Let $\omega_e \in \mathfrak{g}^*$. By using $d_{\text{kos, }1} = \phi \circ d \circ \phi^{-1}$, we get for all $X_1, X_2 \in \mathfrak{g}$ that

$$d_{\text{kos}}\omega(X_1, X_2) = d\omega(X^1, X^2).$$

Here $\omega = \phi(\omega_e)$ and $X^1, X^2 \in \mathfrak{X}_L(G)$ are the unique left invariant vector fields such that $X_e^1 = X_1$ and $X_e^2 = X_2$. Now we can use the formula from Definition 5.1.1 to see that

$$d\omega(X^1, X^2) = X^1(\omega(X^2)) - X^2(\omega(X^1)) - \omega([X^1, X^2]) = -\omega([X^1, X^2]).$$

The last equality follows from Lemma 5.1.6, which tells us that $\omega(X^1)$ and $\omega(X^2)$ are constant. Therefore we have that $\omega_e \in \ker(d_{\text{kos},1})$ if and only if $-\omega([X^1, X^2]) = 0$ for all $X^1, X^2 \in \mathfrak{X}_L(G)$. By the surjectivity of the Lie bracket this is equivalent to $\omega(X) = 0$ for all $X \in \mathfrak{X}_L(G)$, so $\omega = 0$. Hence $\ker(d_{\text{kos},1}) = 0$, which proves that $H^1(\bigwedge^\bullet \mathfrak{g}^*, d_{\text{kos}})$ is trivial. □

Lemma 5.1.8. *For $k \geq 0$ we have $H^k(\bigwedge^\bullet \mathfrak{g}^*, d_{kos}) = H_{dR}^k(G, \mathbb{R})$.*

Proof. See [8]. □

Theorem 5.1.9. *For $k \geq 0$ we have $H_{dR}^k(G, \mathbb{R}) = H^k(G, \mathbb{R})$.*

Proof. See Lee [7, p. 484]. □

Remark 5.1.10. By combining the previous three results, we can conclude that $H^1(G, \mathbb{R}) = 0$. We will use the characterization of singular homology used in Lee [7] on page 472, which states that $H^1(G, \mathbb{R}) \simeq \text{Hom}(H_1(G, \mathbb{R}); \mathbb{R})$. By $\text{Hom}(H_1(G, \mathbb{R}); \mathbb{R})$ we mean the space all linear maps from $H_1(G, \mathbb{R})$ to $\mathbb{R}$. Since $H^1(G, \mathbb{R})$ is trivial and $H_1(G, \mathbb{R})$ is a real vector space, we also have $H_1(G, \mathbb{R}) = 0$.

5.2 Modules

We will now introduce the concept of modules. After the definition is given, we will extend the the tensor product to modules. Then certain properties of this tensor product are discussed. Finally, we will apply these constructions to $H_1(G, \mathbb{R})$ to analyze $H_1(G, \mathbb{Z})$.

The following definition comes from Modules and Rings [5].

Definition 5.2.1. Let R be a ring. An abelian group $(M, +)$ is a right R-module is there exists a function

$$M \times R \to M : (x, a) \mapsto xa$$

such that for all $x, y \in M$ and $a, b \in R$ the following conditions are fulfilled:

1. $(x + y)a = xa + ya$
2. $x(a + b) = xa + xb$
3. $x(ab) = (xa)b$

A left R-module is defined likewise.

From now on we will focus on the case $R = \mathbb{Z}$.

Definition 5.2.2. A $\mathbb{Z}$-module morphism is a map $f : M \to N$ between $\mathbb{Z}$-modules M and N such that

- $f(x + y) = f(x) + f(y)$
- $f(kx) = kf(x)$

for all $x, y \in M$ and $kin\mathbb{Z}$.

Definition 5.2.3. Let M be a right $\mathbb{Z}$-module, N a left R-module and G be an abelian group. The tensor product $M \otimes_{\mathbb{Z}} N$ of M and N over $\mathbb{Z}$ is defined as a $\mathbb{Z}$-module together with a $\mathbb{Z}$-linear map $M \times N \to M \otimes_{\mathbb{Z}} N : (m, n) \mapsto m \otimes n$ that is defined uniquely up to isomorphisms by the following universal property: Let $b : M \times N \to L$ be a $\mathbb{Z}$-linear map to the $\mathbb{Z}$-module L. Then there exists a unique $\mathbb{Z}$-module morphism $\tilde{b} : M \otimes_{\mathbb{Z}} N \to L$ such that the following diagram commutes.

$$\begin{tikzcd}
M \times N \arrow[r, "b"] \arrow[d] & L \\
M \otimes_{\mathbb{Z}} N \arrow[ur, "\tilde{b}"] &
\end{tikzcd}$$

Lemma 5.2.4. *The tensor product $M \otimes_{\mathbb{Z}} N$ exists and is uniquely determined up to group isomorphisms.*

Proof. See Tensor Products [4]. □

Proposition 5.2.5. *The group $M \otimes_{\mathbb{Z}} N$ is generated by elements of the form $m \otimes n$ with $m \in M$ and $n \in N$.*

Proof. This proof is based on the proof given on Wikipedia [9]. Let Q be the subgroup of $M \otimes_{\mathbb{Z}} N$ generated by the elements in the statement of the proposition. Since $M \otimes_{\mathbb{Z}} N$ is abelian, Q is a normal subgroup. Let $\pi : M \otimes_{\mathbb{Z}} N \to (M \otimes_R N)/Q$ be the corresponding quotient map, then $M \otimes_R N/Q$ is abelian. We obtain the following diagram:

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "\pi\circ\otimes"] & M \otimes_{\mathbb{Z}} N \arrow[d, "\pi"] \\ & (M \otimes_R N)/Q \end{tikzcd}$$

Now observe that the map $\pi \circ \otimes$ sends every $(m, n) \in M \times N$ to zero. If we were to replace π in the diagram above with the map that is identically 0, the diagram would still commute. By the universal property of the tensor product, we know that the quotient $\pi = 0$. Hence $Q = M \otimes_{\mathbb{Z}} N$. $\square$

Remark 5.2.6. Let H be an abelian group, then there exists a natural way to give H the structure of an left $\mathbb{Z}$-module. Namely, define the map $H \times \mathbb{Z} \to H : (h, k) \mapsto h^k$. It is easy to check that this map satisfies the conditions of an $\mathbb{Z}$-module. There is an analogous map that makes H into a right $\mathbb{Z}$-module.

Proposition 5.2.7. *The groups $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{R}$ are isomorphic.*

Proof. Define the $\mathbb{Z}$-module morphism $\psi : \mathbb{R} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$ by $\psi(r) = 1 \otimes_{\mathbb{Z}} r$. Assume that ψ is not injective. Then there exists an $r \in \mathbb{R}$ with $r \neq 0$ such that $1 \otimes r = 0$. Therefore $\mathbb{Z} \times r = 0$. However, this implies the map $Pr : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} : (k, r) \mapsto kr$ cannot be lifted to a map $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$, since $Pr(1, r) = r$ and $\tilde{Pr}(1 \otimes r) = \tilde{Pr}(0) = 0$ for every choice of $\tilde{Pr}$. This contradicts with the assumption that every $\mathbb{Z}$-bilinear map can be lifted. Therefore ψ is injective.

For the surjectivity, let $x \in \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$. By Proposition 5.2.5 we can write $x = \sum_{i=1}^{n} k_i \otimes_{\mathbb{Z}} r_i$. It follows from elementary properties of the tensor product that

$$\sum_{i=1}^{n} k_i \otimes_{\mathbb{Z}} r_i = \sum_{i=1}^{n} 1 \otimes_{\mathbb{Z}} k_i r_i = 1 \otimes_{\mathbb{Z}} \sum_{i=1}^{n} k_i r_i$$

Hence $x \in \operatorname{im}(\psi)$. Therefore ψ is an isomorphism. $\square$

5.3 Coefficients for homology

From now on we will de following the approach of Hatcher [6, p. 261-265] in order to understand the universal coefficient theorem for homology.

Definition 5.3.1. Let $A_0, A_1, \ldots$ be groups. A long sequence has the form

$$\ldots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \to 0$$

where the $f_i : A_i \to A_{i-1}$ are group homomorphisms and 0 is the trivial group. We will use additive notation for the group product. A sequence is called exact at A_i if $\ker(f_i) = \operatorname{im}(f_{i+1})$. The sequence is exact if it is exact at every group.

Lemma 5.3.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of right $\mathbb{Z}$-modules A, B, C and let M be a left $\mathbb{Z}$-module. Then the sequence given by*

$$A \otimes_R M \xrightarrow{f \otimes Id} B \otimes_R M \xrightarrow{g \otimes Id} C \otimes_R M \to 0$$

is also exact.

Proof. This proof is based on Hatcher [6, p. 262]. Note that exactness of the short sequence at C precisely states that g is surjective. Therefore $g \otimes Id$ is also surjective. This proves the exactness at $C \otimes_{\mathbb{Z}} M$.

We now proceed by showing that the sequence is exact at $B \otimes_{\mathbb{Z}} M$. It is enough to prove that the map $B \otimes_{\mathbb{Z}} M/\mathrm{im}(f \otimes Id) \to C \otimes_R M$ induced by $g \otimes_{\mathbb{Z}} Id$ is an isomorphism. We will achieve this by constructing the inverse. Define the map $\psi : C \times M \to B \otimes_{\mathbb{Z}} M/\mathrm{im}(f \otimes Id)$ by $\psi(c, m) = b \otimes m$ where $b \in B$ such that $g(b) = c$. Note that ψ is well defined, because if $g(b) = g(b') = c$, then $b - b' \in \ker(g) = \mathrm{im}(f)$. Therefore there exists an $a \in A$ such that $f(a) = b - b'$. This leads to $b \otimes_{\mathbb{Z}} m - b' \otimes m = (b - b') \otimes_{\mathbb{Z}} m \in \mathrm{im}(f \otimes Id)$. Since ψ is $\mathbb{Z}$-bilinear, by the universal property we obtain a $\mathbb{Z}$-module morphism $C \otimes_R M \to B \otimes_R M/\mathrm{im}(f \otimes Id)$. $\square$

Definition 5.3.3. Let H be an abelian group, then a free resolution F of H is a long exact sequence

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

such that each F_i is a free group.

Let M be a abelian group and F be a free resolution of H. By viewing all groups involved as $\mathbb{Z}$-modules, we tensor the free resolution F of H by M over $\mathbb{Z}$ to obtain a long sequence

$$\ldots \to F_2 \otimes_{\mathbb{Z}} M \xrightarrow{f_2 \otimes Id} F_1 \otimes_{\mathbb{Z}} M \xrightarrow{f_1 \otimes Id} F_0 \otimes_{\mathbb{Z}} M \xrightarrow{f_0 \otimes Id} H \otimes_{\mathbb{Z}} M \to 0.$$

Observe that Lemma 5.3.2 states that this sequence is exact at $F_0 \otimes_{\mathbb{Z}} M$ and $H \otimes_{\mathbb{Z}} M$. We write $H_n(F \otimes_{\mathbb{Z}} M)$ for the homology group $\ker(f_n \otimes Id)/\mathrm{im}(f_{n+1} \otimes Id)$.

Lemma 5.3.4. *For any two free resolutions F and F' of H we have $H_n(F \otimes_{\mathbb{Z}} M) \simeq H_n(F' \otimes_{\mathbb{Z}} M)$.*

Proof. See Hatcher [6] page 263. $\square$

Therefore $H_n(F \otimes_{\mathbb{Z}} M)$ is only dependent on the choice of M and H.

Remark 5.3.5. Let $\mathrm{Tor}(H, M)$ denote $H_1(F \otimes_{\mathbb{Z}} M)$. Furthermore, by $H_n(C; M)$ we mean the n-th homology class of the chain complex C with coefficients in the abelian group M.

Theorem 5.3.6 (The universal coefficient theorem for homology)**.** *Let C be a chain complex of free abelian groups and M an abelian group. Then there are natural short exact sequences*

$$0 \to H_n(C) \otimes_{\mathbb{Z}} M \to H_n(C; M) \to \mathrm{Tor}(H_{n-1}(C), M) \to 0$$

for all $n \geq 1$.

Proof. See Hatcher [6, p. 264]. $\square$

6 Finiteness of the fundamental group

Throughout this section, let G be a connected, compact semi simple Lie group.

Lemma 6.0.1. *The natural map $H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} \to H_1(G;\mathbb{R}) : (\sum_i r_1\sigma_i, \lambda) \mapsto \lambda \sum_i r_1\sigma_i$ is a group isomorphism.*

Proof. We will make use of Theorem 5.3.6. Let C be the singular complex of G

$$\ldots \to C^2(G) \xrightarrow{\partial_2} C^1(G) \xrightarrow{\partial_1} C^0(G) \to 0 \tag{1}$$

Note that C_i are all free abelian groups. Choose $M = \mathbb{R}$. Then we have in particular $H_{n-1}(C) = H_0(G) = \mathbb{Z}$, because G is connected.

Now we want to calculate $\mathrm{Tor}(\mathbb{Z}, \mathbb{R})$. We choose the free resolution F by $F_0 = \mathbb{Z}$ and $F_i = 0$ for $i > 0$. This gives us the exact sequence

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{Id} \mathbb{Z} \to 0.$$

After tensoring with $\mathbb{R}$ we obtain the sequence $F \otimes_{\mathbb{Z}} \mathbb{R}$:

$$\ldots \to 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{Id} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} \to 0$$

By Proposition 5.2.7 this sequence is equal to

$$\ldots \to 0 \to \mathbb{R} \xrightarrow{Id} \mathbb{R} \to 0$$

Hence $\mathrm{Tor}(\mathbb{Z}, \mathbb{R}) = H_1(F \otimes_{\mathbb{Z}} \mathbb{R}) = 0$. By Theorem 5.3.6 we now have the following short exact sequence:

$$0 \to H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} \to H_1(G;\mathbb{R}) \to 0$$

Therefore the natural inclusion $i : H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} \to H_1(G;\mathbb{R})$ given by $i(\sum_j k_j\sigma_j \otimes r_j) = \sum_j r_j k_j \sigma_j$ is an isomorphism. □

Definition 6.0.2. Let H be a group, then an element $h \in H$ is called torsion, if there exists an $n \in \mathbb{N}$ such that $h^n = e$.

Example 6.0.3. The only torsion elements in the additive groups $\mathbb{Z}$ and $\mathbb{R}$ are 0. Any finite group contains only torsion elements. △

Lemma 6.0.4. *If $H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} = 0$, then every element of $H_1(G)$ is torsion.*

Proof. Define the map $\phi : H_1(G) \to H_1(G, \mathbb{R})$ such that $\phi(c) = c \otimes_{\mathbb{Z}} 1$. Fix $c \in H_1(G)$ and define $\mathbb{Z}c = \{kc \in H_1(G) | k \in \mathbb{Z}\}$. Then we have $\mathbb{Z}c \subset H_1(G)$, so $\phi(\mathbb{Z}c) = 0$. Define $\varphi : \mathbb{Z} \to \mathbb{Z}c$ that maps n onto nc. We obtain

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{Z}c \\ \mathbb{Z}/\ker(\varphi) \arrow[ur, "f"] & \end{tikzcd}$$

Note that φ is a surjective homomorphism of $\mathbb{Z}$-modules. Therefore f is an isomorphism of $\mathbb{Z}$-modules and $\ker(\varphi)$ is an ideal of $\mathbb{Z}$. From the fact that $\mathbb{Z}$ is prime ideal domain, we derive $\ker(\varphi) = (n)$ for a certain $n \in \mathbb{N}$ or $\ker(\varphi) = 0$. Assume the latter, then $\mathbb{Z}c$ is isomorphic to $\mathbb{Z}$ as $\mathbb{Z}$-module. Hence $\mathbb{Z}c \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$ by 5.2.7. This contradicts with $H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} = 0$. Therefore $\ker(\varphi) = (n)$, so c is torsion. $\square$

Remark 6.0.5. From Remark 5.2.6 it follows that $H_1(G, \mathbb{R}) = 0$. By Lemma 6.0.1 we get that $H_1(G) \otimes_{\mathbb{Z}} \mathbb{R} = 0$. Using Lemma 6.0.4, we now have that $H_1(G)$ only contains torsion elements. Finally, we can conclude that $\pi_1(G)$ contains only torsion elements, since $\pi_1(G) \simeq \pi_1(G)^{ab} \simeq H_1(G)$.

Theorem 6.0.6. *Let G be a compact manifold, then $H_1(G)$ is finitely generated.*

Proof. See Bott and Tu [3] page 42 Theorem 5.1 and Proposition 5.3.1 $\square$

By using $H_1(G) \simeq \pi_1(G)$, we get that $\pi_1(G)$ is finitely generated.

Theorem 6.0.7. *The fundamental group of a connected, compact semisimple Lie group G is finite.*

Proof. Since $\pi_1(G)$ is finitely generated, let $x_1, \ldots, x_k \in \pi_1(G)$ be generators of the fundamental group. Since every element is torsion, for all $1 \leq i \leq k$ there exists an $n_i \in \mathbb{N}$ such that $x_i^{n_i} = e$. Let $y \in \pi_1(G)$ be arbitrary. Then we write $y = x_1^{j_1} \ldots x_n^{j_n}$, because $\pi_1(G)$ is abelian. Furthermore, we may assume that $0 \leq j_i < n_i$. Therefore the number of distinct elements of $\pi_1(G)$ is at most $\prod_{i=1}^{k} n_i$. This proves the theorem. $\square$

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