Effect of the trace anomaly on the cosmological constant

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(Received 2 April 2008; published 8 July 2008)

It has been argued that the quantum (conformal) trace anomaly could potentially provide us with a dynamical explanation of the cosmological constant problem. In this paper, however, we show by means of a semiclassical analysis that the trace anomaly does not affect the cosmological constant. We construct the effective action of the conformal anomaly for flat Friedmann-Lemaître-Robertson-Walker spacetimes consisting of local quadratic geometric curvature invariants. Counterterms are thus expected to influence the numerical value of the coefficients in the trace anomaly and we must therefore allow these parameters to vary. We calculate the evolution of the Hubble parameter in quasi-de Sitter spacetime, where we restrict our Hubble parameter to vary slowly in time, and in Friedmann-Lemaître-Robertson-Walker spacetimes. We show dynamically that a universe consisting of matter with a constant equation of state, a cosmological constant, and the quantum trace anomaly evolves either to the classical de Sitter attractor or to a quantum trace anomaly driven one. When considering the trace anomaly truncated to quasi-de Sitter spacetime, we find a region in parameter space where the quantum attractor destabilizes. When considering the exact expression of the trace anomaly, a stability analysis shows that whenever the trace anomaly driven attractor is stable, the classical de Sitter attractor is unstable, and vice versa. Semiclassically, the trace anomaly does not affect the classical late time de Sitter attractor, and hence it does not solve the cosmological constant problem.

DOI: 10.1103/PhysRevD.78.023508 PACS numbers: 98.80.−k, 04.62.+v, 95.36.+x

I. INTRODUCTION

Recent observations have clearly indicated that the expansion of the Universe is accelerating. According to Einstein's general relativity, this can only be realized if the pressure of the dominant component of the current Universe is negative. These observations have triggered a renewed interest in the cosmological constant problem (for recent reviews, see e.g. [1–3]). What is usually referred to as the ''old'' cosmological constant problem can be phrased as follows: why is the measured (effective) cosmological constant extremely close to zero?

One approach dealing with the cosmological constant problem is concerned with employing the effective field theory of gravity [4,5]. Lacking a full quantum theory of gravity, an effective field theory of gravity adopts the following point of view: in order to describe quantum phenomena at very large and cosmologically relevant distances, the precise physics at the shortest distance scales is irrelevant. In other words, the effective field theory of gravity is the low energy limit of quantum gravity. It combines classical general relativity with knowledge of quantum field theory in curved spacetimes [6].

In order to describe these long distance effects accurately, one supplements the classical Einstein-Hilbert action with certain additional contributions. One of these additions is the trace anomaly or conformal anomaly which quantum field theories are known to exhibit [6–13]. If the classical action is invariant under conformal transformations of the metric, the resulting stress-energy tensor is traceless. As an explicit example, one can easily verify that the trace of a massless, conformally coupled scalar field vanishes. In quantum field theory the stress-tensor is promoted to an operator. A careful renormalization procedure renders its expectation value $\langle T^{\mu}{}_{\nu}\rangle$ finite. However, inevitably, the renormalization procedure results in general in a nonvanishing trace of the renormalized stress-energy tensor. Classical conformal invariance cannot be preserved at the quantum level. Ever since its discovery, the trace anomaly has found many applications in various areas in physics (see e.g. [14]).

An alternative approach to the backreaction problem of quantum fluctuations on the background spacetime deals with quantum fields whose spectrum is nearly flat [15–24]. Consequently, the spectrum in the infrared is not suppressed and is therefore expected to yield a strong backreaction on the background spacetime. Examples of such fields are the minimally coupled massless scalar and the graviton.

A. The connection between the cosmological constant and the trace anomaly

Some authors stated that the trace anomaly could have effects on dark energy and the cosmological constant problem [25,26], whereas it has been argued by other authors that the trace anomaly could potentially provide us with a dynamical explanation of the cosmological constant problem [27–32]. Broadly speaking, the line of rea-

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soning is as follows (for a more in-depth review, we refer to [32]). The new, conformal degree of freedom is usually parametrized by

$$g_{\mu\nu}(x) = e^{2\sigma(x)}\bar{g}_{\mu\nu}(x). \tag{1}$$

According to the authors of e.g. [32], the trace anomaly cannot be generated from a local finite term in the action, but rather stems from a nonlocal effective action that generates the conformal anomaly by variation with respect to the metric [33]. It is this genuine nonlocality of the trace anomaly, revealing a large distance effect of quantum physics, that is at the very foundation of its connection with the effective field theory of gravity. One then argues that the new conformal field should dynamically screen the cosmological constant, thus solving the cosmological constant problem.

B. The semiclassical approach to the cosmological constant and the trace anomaly

The proposal advocated in [32] is very interesting and should be investigated further. Before studying the effect of a new conformal degree of freedom (1), we feel that first a proper complete analysis of the dynamics resulting from the effective action of the trace anomaly should be performed. This is what we pursue in this paper.

According to the cosmological principle the Universe is homogeneous and isotropic on the largest and cosmologically relevant scales. The CMB measurements [34] constrain the inhomogeneities at order 10^{-4}–10^{-5}. Moreover, the Universe appears to be spatially flat. Let us make the following observations.

First, the cosmological principle dictates the use of the conformally flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$. Hence, inhomogeneous fluctuations of the metric tensor and, in particular, of the conformal part of the metric tensor (1) are observed and expected to be small at the largest scales, comparable to the Hubble radius (also in the early Universe).

Second, we are led to an essentially semiclassical analysis. The vacuum expectation value of the stress-energy tensor resulting in the trace anomaly has been calculated semiclassically. In a semiclassical analysis quantum fluctuations backreact on the background spacetime. Phase transitions aside, quantum fluctuations naturally affect the homogeneous background *homogenously*. It is a well-known fact that quantum fluctuations can break certain symmetries present in de Sitter [15–18], e.g. time translation invariance. However, we are not aware of quantum fluctuations breaking the homogeneity and isotropy of the background spacetime.

Finally, if quantum fluctuations compensate for or screen the cosmological constant, we must have $T_{\mu\nu} \propto g_{\mu\nu}$. Let us set: $T_{\mu\nu} = \theta(x)g_{\mu\nu}$. Stress-energy conservation and metric compatibility immediately yield: $\nabla_\mu\theta(x) = \partial_\mu\theta(x) = 0$. Hence we conclude that $\theta(x)$ must be a constant: $\theta(x) = \theta_0$. Only homogeneous vacuum fluctuations can compensate the cosmological constant. Moreover, $T_{\mu\nu} = \theta_0 g_{\mu\nu}$ does not break any of the symmetries of a maximally symmetric spacetime.[1] Hence, this form cannot be used to study dynamical backreaction.

The arguments above motivate a semiclassical approach to examining the connection between the cosmological constant and the trace anomaly. Note that we do not consider a new, conformal degree of freedom (1). Hence, we do certainly not exclude any possible effect this (inhomogeneous) conformal degree of freedom might have on the cosmological constant. However, it is plausible that in order to address the link between the cosmological constant and the trace anomaly, a semiclassical analysis suffices.

C. The modified Starobinsky model

Another application of the conformal anomaly can be found in what has become known as trace anomaly induced inflation: in the absence of a cosmological constant, the trace anomaly could provide us with an effective cosmological constant. Originally, Starobinsky [35] realized that quantum one-loop contributions of massless fields can source a de Sitter stage. Subsequently, the theory of trace anomaly induced inflation received significant contributions from [36–39]. If one includes a cosmological constant, the theory of anomaly induced inflation is plagued by instabilities, which we will also come to address. The modified Starobinsky model as advocated by [40–42], takes advantage of these instabilities to account for a graceful exit from inflation. It is argued that supersymmetry breaking changes the degrees of freedom such that it destabilizes the quantum anomaly driven attractor and simultaneously stabilizes the classical de Sitter attractor.

Improving on e.g. [41,43], we incorporate matter with a constant equation of state in the Einstein field equations. We argue that it is simply inconsistent not to include matter. Consider the following analogy: if we examine an empty universe with a cosmological constant only, there is no dynamics and the (00) Einstein equation yields $H^2 = \Lambda/3$. Matter drives the dynamics and $H^2 \neq \Lambda/3$ can only be realized with $\rho_{\rm M} \neq 0$.

In the literature, if one solves the trace of the Einstein field equations in a universe with a cosmological constant and trace anomaly, one solves, however, in reality for the dynamics in a universe filled with radiation. For it is only radiation with equation of state $w = 1/3$ that does not contribute to the trace of the Einstein field equation $T_{\rm rad} = 0$. This point has not been included in other papers. We will consider matter with constant but otherwise arbitrary equation of state $w > -1$, and not just (implicitly) radiation with $w = 1/3$.

[1] Maximally symmetric spacetimes are de Sitter, anti-de Sitter, and Minkowski spacetime.

D. Outline

In this paper we show that the cosmological constant problem cannot be solved by taking account of the trace anomaly alone. The outline of this paper is as follows. In Sec. II we recall the basics of the conformal anomaly and discuss how to study its effect on the evolution of the Universe by tracing the Einstein field equations.

In Sec. III, we derive the conformal anomaly from an effective action in flat homogeneous FLRW spacetimes consisting of local quadratic geometric curvature invariants. Since one usually adds infinite counterterms to cancel the radiative one loop divergences, we do not see any reason why we should exclude adding a Gauss-Bonnet counterterm to cancel the anomaly in flat FLRW spacetimes. Even though this term in the effective action is formally divergent, at the level of the equation of motion it yields a finite result. Hence, the coefficients multiplying the curvature invariants in the trace anomaly are not uniquely specified by the anomaly. The physical coefficient, i.e.: the parameter that can be measured, receives contributions both from the trace anomaly and from possible counterterms canceling divergences from the underlying (and yet unknown) fundamental theory. This motivates varying the coupling parameters multiplying the curvature invariants in the anomaly. We can thus study all possible effects of the anomaly on the evolution of our Universe.

In Sec. IV we study the evolution of a quasi-de Sitter universe in the presence of matter with constant equation of state, a cosmological constant and the trace anomaly. In quasi-de Sitter spacetime we assume, loosely speaking, that the Hubble parameter is a slowly varying function of time. Effectively, we truncate the expression of the exact trace anomaly and discard higher order derivative contributions.

In Sec. V we generalize our analysis and study the evolution of a FLRW universe again in the presence of matter with constant equation of state, a cosmological constant and the trace anomaly. We examine the exact trace anomaly and take all higher derivative contributions into account. As the dimensionality of the phase space increases, we must carefully perform a stability analysis of the late time asymptotes.

II. TRACING THE EINSTEIN FIELD EQUATIONS AND THE TRACE ANOMALY

A. The conformal anomaly in four dimensions in FLRW spacetimes

The trace anomaly or the conformal anomaly in four dimensions is in general curved spacetimes given by [6,7,32]:

$$T_{\rm Q} \equiv \langle T^{\mu}{}_{\mu}\rangle = bF + b'(E - \tfrac{2}{3}\Box R) + b''\Box R, \tag{2}$$

where:

$$E \equiv {}^*R_{\mu\nu\kappa\lambda}{}^*R^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{3a}$$

$$F \equiv C_{\mu\nu\kappa\lambda}C^{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2, \tag{3b}$$

where as usual $R_{\mu\nu\kappa\lambda}$ is the Riemann curvature tensor, ${}^*R_{\mu\nu\kappa\lambda} = \varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\kappa\lambda}/2$ its dual, $C_{\mu\nu\kappa\lambda}$ the Weyl tensor and $R_{\mu\nu}$ and R the Ricci tensor and scalar, respectively. Note that E is the Gauss-Bonnet invariant. The general expression for the trace anomaly can also contain additional contributions if the massless conformal field is coupled to other long range gauge fields (see e.g. [6]). Finally, the parameters b, b' and b'' appearing in (2) are dimensionless quantities multiplied by $\hbar$ and are given by

$$b = \frac{1}{120(4\pi)^2}(N_S + 6N_F + 12N_V) \tag{4a}$$

$$b' = -\frac{1}{360(4\pi)^2}\left(N_S + \frac{11}{2}N_F + 62N_V\right), \tag{4b}$$

where N_S, N_F, and N_V denote the number of fields of spin 0, 1/2 and 1, respectively, ($\hbar = 1$). It is important to note that $b > 0$ whereas $b' < 0$ in general. It turns out that the coefficient b'' is regularization dependent and is therefore not considered to be part of the true conformal anomaly. We take this into account and study the effect of b'' on the stability of the solutions we are about to derive. For definiteness, we will assume that these parameters take their standard model values: $N_S = 4$, $N_F = 45/2$, and $N_V = 12$. Note if we were to include right-handed neutrinos, $N_F = 24$. One could also examine the numerical value of the coefficients (4) for the late time Universe. Today's massless particle is just the photon, hence $N_V = 1$, $N_S = 0$ and $N_F = 0$.

Let us specialize to flat Friedmann-Lemaître-Robertson-Walker or FLRW spacetimes in which the metric is given by $g_{\alpha\beta} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t))$ where $a(t)$ is the scale factor of the Universe in cosmic time t. Recall that a conformal transformation leaves the Weyl tensor invariant. Hence, in FLRW spacetimes $F = 0$. Given the FLRW metric one can easily verify that

$$R^2 = 36[\dot{H}^2 + 4(\dot{H}H^2 + H^4)] \tag{5a}$$

$$R_{\mu\nu}R^{\mu\nu} = 12[\dot{H}^2 + 3(\dot{H}H^2 + H^4)] \tag{5b}$$

$$R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda} = 12[\dot{H}^2 + 2(\dot{H}H^2 + H^4)] \tag{5c}$$

$$\Box R = -6[\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2]. \tag{5d}$$

Hence, the exact expression for the trace anomaly in FLRW spacetimes in four dimensions reads

$$T_{\rm Q} = 4b'\{\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 18\dot{H}H^2 + 6H^4\} - 6b''\{\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\}. \tag{6}$$

To capture the leading order dynamics we work in quasi-de Sitter spacetime and allow for a mildly time dependent Hubble parameter:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \text{constant} \ll 1, \tag{7}$$

i.e.: we assume that ϵ is both small and time independent. This would truncate the trace anomaly up to terms linear in $\dot{H}$ yielding:

$$T_{\rm Q} = 24b'\{3\dot{H}H^2 + H^4\} - 72b''\dot{H}H^2. \tag{8}$$

We will examine both the exact form of the trace anomaly (6) and its truncated form (8).

Truncating the expression for the trace anomaly is motivated by the following realization. In general backgrounds we need a nonlocal effective action to generate the trace anomaly in the equation of motion. The nonlocality at the level of the effective action corresponds to an expansion in derivatives at the level of the equation of motion. Generally, higher derivative contributions in an equation of motion have the tendency to destabilize a system unless the initial conditions are highly fine-tuned. Formally, this is known as the theorem of Ostrogradsky and its relevance to cosmology is outlined, for example, in [44].

Note that when $b'' = 2b'/3$ the truncated version is exact. We discuss this further in Sec. IV B. Finally, note that although we have truncated Eq. (6) to obtain (8), Eq. (8) is still covariant.

B. The dynamics driven by the trace anomaly

From the Einstein-Hilbert action:

$$S = S_{\rm EH} + S_{\rm M} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) + \int d^4x\sqrt{-g}\mathcal{L}_{\rm M}, \tag{9}$$

where:

$$\mathcal{L}_{\rm M} = -\tfrac{1}{2}\partial_\alpha\phi(x)\partial_\beta\phi(x)g^{\alpha\beta} - \tfrac{1}{2}m^2\phi^2(x) - V(\phi(x)), \tag{10}$$

the Einstein field equations follow as usual as

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{11}$$

of which the trace can easily be verified to be

$$R - 4\Lambda = -8\pi G T, \tag{12}$$

where $T = T^\mu{}_\mu$. If one considers an empty universe with a cosmological constant, the (00) Einstein field equation acts as a constraint equation for the Hubble parameter and one simply finds $H^2 = \Lambda/3$ as usual. However, for a nonempty universe, the (00) Einstein field equation becomes a dynamical constraint. The Bianchi identity for the left-hand side of Eq. (11) straightforwardly results in stress-energy conservation for the right-hand side:

$$\nabla^\mu T_{\mu\nu} = 0. \tag{13}$$

Because of stress-energy conservation, the (00) and the (ij) components of the Einstein field equations are not independent.[2] Therefore, any linear combination of the (00) and (ij) components of the Einstein field equations combined with stress-energy conservation suffice to describe the time evolution of the Hubble parameter. In particular, the trace equation (12) and stress-energy conservation (13) contain all relevant dynamics for H.

Let us set $\phi(x) = \phi_{\rm cl}(x) + \varphi(x)$ for the quantum field in $S_{\rm M}$ and require that the classical field obeys the equation of motion. Note that the quantum perturbation $\varphi(x)$ does not obey this equation of motion. We expand in terms of the quantum field and construct the effective action as usual:

$$\begin{aligned}\exp[i\Gamma[\phi_{\rm cl}]] &= \exp[iS_{\rm M}[\phi_{\rm cl}]] \\ &\quad\times\int\mathcal{D}\varphi\exp\bigg[i\bigg(\int_x\frac{\delta S_{\rm M}}{\delta\phi_{\rm cl}(x)}\varphi(x) \\ &\quad+\frac{1}{2}\int_{x,y}\frac{\delta^2 S_{\rm M}}{\delta\phi_{\rm cl}(x)\phi_{\rm cl}(y)}\varphi(x)\varphi(y) + \mathcal{O}(\varphi^3)\bigg)\bigg] \\ &= \exp[iS_{\rm M}[\phi_{\rm cl}] + i\Gamma_{\rm Q}[\phi_{\rm cl}]].\end{aligned} \tag{14}$$

The first contribution to the effective action corresponds to the classical part of the action and $\Gamma_{\rm Q}[\phi_{\rm cl}]$ is the contribution to the effective action taking account of the vacuum fluctuations. The stress-energy tensor now follows as

$$\begin{aligned}T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\Gamma[\phi_{\rm cl}] \\ &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(S_{\rm M}[\phi_{\rm cl}] + \Gamma_{\rm Q}[\phi_{\rm cl}]) \equiv T^{\rm C}_{\mu\nu} + T^{\rm Q}_{\mu\nu}.\end{aligned} \tag{15}$$

Hence, there are both classical and quantum contributions to the full stress-energy tensor. Classically, from the equation of motion the scalar field obeys, we have

$$\nabla^\mu T^{\rm C}_{\mu\nu} = 0. \tag{16}$$

Hence, from (13) we derive:

$$\nabla^\mu T^{\rm Q}_{\mu\nu} = 0. \tag{17}$$

Concluding, due to stress-energy conservation at the classical level and for the full stress-energy tensor, we have derived stress-energy conservation for the quantum contributions as well.

Analogously to the classical stress-energy tensor, we can symbolically write $T^\mu{}_{\nu,{\rm Q}} = (-\rho_{\rm Q}, p_{\rm Q}, p_{\rm Q}, p_{\rm Q})$. Combining

$$T_{\rm Q}(t) = -\rho_{\rm Q}(t) + 3p_{\rm Q}(t) \tag{18a}$$

$$\dot{\rho}_{\rm Q}(t) = -3H\{\rho_{\rm Q}(t) + p_{\rm Q}(t)\}, \tag{18b}$$

yields

[2]For example, stress-energy conservation combined with the (00) Einstein field equation straightforwardly yield the (ij) component of the Einstein field equations.

$$\frac{d}{dt}[a^4(t)\rho_Q(t)] = -a^4(t)H(t)T_Q(t). \tag{19}$$

We thus find (identical to [36])

$$\rho_Q(t) = -\frac{1}{a^4(t)}\int^t d\tau a^4(\tau)H(\tau)T_Q(\tau) \tag{20a}$$

$$p_Q(t) = \tfrac{1}{3}(T_Q(t) + \rho_Q(t)). \tag{20b}$$

Although in general spacetimes it is not possible to perform this integral, in cosmologically relevant FLRW spacetimes we can [45]. If we consider the exact form of the trace anomaly (6) in flat FLRW spacetimes, we can easily see that $\rho_Q(t)$ should be of the following form:

$$\rho_Q(t) = b'[c_1\ddot{H}H + c_2\dot{H}^2 + c_3\dot{H}H^2 + c_4H^4] + b''[c_5\ddot{H}H + c_6\dot{H}^2 + c_7\dot{H}H^2]. \tag{21}$$

Upon inserting this ansatz into Eq. (19) and equating the contributions at each order, we immediately find

$$\rho_Q = 2b'[-2\ddot{H}H + \dot{H}^2 - 6\dot{H}H^2 - 3H^4] + 3b''[2\ddot{H}H - \dot{H}^2 + 6\dot{H}H^2]. \tag{22}$$

Because ρ_Q can be expressed in a local form, we would like to point out that this yields a local expression for the stress-energy tensor too. Although quantum fluctuations of the conformal field may still act nonlocally on the background spacetime, at the classical level the trace anomaly affects the spacetime only locally. Again, when working in quaside Sitter spacetime where ϵ is both small and time independent, we can truncate this expression for the quantum density finding

$$\rho_Q = -6b'[2\dot{H}H^2 + H^4] + 18b''\dot{H}H^2. \tag{23}$$

Again, when $b'' = 2b'/3$, this analysis becomes exact.

In the next sections, we solve the Einstein field equations in FLRW spacetimes for a universe in the presence of (a) a nonzero cosmological constant, (b) the trace anomaly as a contribution to the quantum stress-energy tensor, and (c) matter with constant equation of state $\rho_M = wp_M$, where $w > -1$. We thus consider nontachyonic matter only, which does not exclude [46,47], where the effect of a finite period with $w < -1$ is investigated.

The trace anomaly enters the Einstein field equation naturally in the trace equation (12). We can thus study the effect of the trace anomaly on the evolution of a universe with a cosmological constant and matter. Hence, the trace equation of the Einstein field equations is given by

$$R - 4\Lambda = -8\pi G\{T_Q + T_M\}, \tag{24}$$

where

$$T_M = -\rho_M + 3p_M = \rho_M(3\omega - 1). \tag{25}$$

Now, we can employ the (00) Einstein field equation from (11) to express ρ_M in terms of ρ_Q yielding

$$9(1+\omega)H^2(t) + 6\dot{H}(t) - 3(1+\omega)\Lambda = -8\pi G[T_Q + (1-3\omega)\rho_Q]. \tag{26}$$

The above equation governs the dynamics for the Hubble parameter that we will solve in various interesting cases. For the anomalous trace we can either take the exact expression (6) containing higher derivatives of the Hubble parameter or its truncated version (8). Likewise, for the quantum density we can either insert the full expression (22) or the truncated one (23). The reason for truncating the expression for the quantum trace and density as outlined above is the realization that higher derivative contributions in an equation of motion generally have the tendency to destabilize a system.

In the literature (see e.g. [42]), one only has considered an empty universe with a cosmological term and trace anomaly. In reality, one solved for a radiation dominated universe, for only radiation ($\omega = 1/3$) does not contribute to the trace of the Einstein field equations. We incorporate matter with constant equation of state parameter ω.

Independently on whether one truncates the expressions for the anomalous trace or quantum density, one can solve for the asymptotes yielding the late time behavior. Setting all time derivatives of the Hubble parameter equal to zero, one can easily solve for the two late time constants:

$$(H_0^{C,A})^2 = \frac{-1 \pm \sqrt{1 + 64\pi G b'\Lambda/3}}{32\pi G b'}. \tag{27}$$

Here, H_0^C turns out to be the classical de Sitter attractor, whereas H_0^A is a new, quantum anomaly driven attractor. This result for the late time constants is identical to the case where matter is absent [41]. We can write the above expression in a somewhat more convenient form by defining the dimensionless parameter λ:

$$\lambda = \frac{G\Lambda}{3}, \tag{28}$$

that sets the scale for the cosmological constant Λ. In the current epoch, λ is extremely small, which allows us to expand (27) finding[3]:

$$H_0^C = \sqrt{\frac{\Lambda}{3}}[1 - 8\pi b'\lambda] \tag{29a}$$

$$H_0^A = \sqrt{\frac{-1}{16\pi G b'} - \frac{\Lambda}{3}}. \tag{29b}$$

In the absence of a cosmological constant, the trace anom-

[3]Note the nomenclature in the literature is somewhat misleading. Rather than calling (29a) the classical de Sitter attractor, it would be more natural to denote it with the quantum corrected classical attractor. Hence, the quantum attractor (29b) should preferably be denoted by anomaly driven attractor or Planck scale attractor. We will nevertheless adopt the nomenclature existing in the literature.

aly can thus provide us with an inflationary scenario which has already been appreciated by [35,36,38,39]. Finally, note these asymptotes are independent of b''.

III. THE EFFECTIVE ACTION GENERATING THE TRACE ANOMALY

The authors of [32] are correct in saying that the conformal anomaly cannot be generated by a finite effective action built out of local quadratic geometric curvature invariants only. We show that the trace anomaly can be generated by an infinite effective action in flat FLRW spacetimes, that consists only of local quadratic geometric curvature invariants. Although infinite at the level of the effective action, we generate a finite on shell contribution. We first write the trace anomaly in conformal time $dt = a(\eta)d\eta$ such that the full conformal anomaly (6) reads

$$T_{\rm Q} = 24b'\left(\frac{a''}{a^3}\left(\frac{a'}{a^2}\right)^2 - \left(\frac{a'}{a^2}\right)^4\right) - 6(b'' - 2b'/3) \times \left(\frac{a''''}{a^5} - 4\frac{a'''}{a^4}\frac{a'}{a^2} - 3\left(\frac{a''}{a^3}\right)^2 + 6\frac{a''}{a^3}\left(\frac{a'}{a^2}\right)^2\right). \qquad (30)$$

Here, dashes denote conformal time derivatives. In general, the trace of a stress-energy tensor can be written as

$$T = -\frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{4}{a^3(\eta)V}\frac{\delta S}{\delta a}. \qquad (31)$$

Here, V is the (spatial) volume. The correct effective action $\Gamma_{\rm an}$ that generates the trace anomaly in spatially flat FLRW spacetimes is given by

$$\Gamma_{\rm an} = \int d^D x\sqrt{-g}\{\beta_{\rm D}E - 12(b'' - 2b'/3)R^2\}, \qquad (32)$$

where

$$\beta_{\rm D} = b'\frac{1}{(D-4)} + \beta_0. \qquad (33)$$

Here D is the dimension of the spacetime and β_0 is an undetermined finite (and physically irrelevant) constant. Only after variation we let $D \to 4$. The numerical factors in (32) are chosen in accordance with the trace of the Einstein field equations, i.e.: the trace of the variation:

$$g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}S = g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}[S_{\rm EH} + \Gamma_{\rm an}] = 0, \qquad (34)$$

indeed yields

$$R - 4\Lambda = -8\pi G T. \qquad (35)$$

The effective action (32) merits some clarifying remarks. First, the variation of R^2 straightforwardly yields the $\Box R$ term in the conformal anomaly. This term is not unique for one could have equally well taken $R^{\mu\nu}R_{\mu\nu}$ into account.[4]

[4]We can easily split $R^{\mu\nu}R_{\mu\nu}$ in an R^2 contribution and a Gauss-Bonnet term, that yields a vanishing surface term in four dimensions.

Second, note that the local effective action depends solely on the scale factor. If one were to rewrite this effective action covariantly in terms of the full metric, the effective action would become nonlocal [8,33], also see [48,49]. It is well known that the effective action of the conformal factor can be written in a local form [45,50]. However, we have been able to rewrite this expression in terms of the Gauss-Bonnet invariant multiplying an infinite constant. Finally, note that the coefficient (33) diverges in four dimensions. However, the Gauss-Bonnet invariant in D dimensions reads

$$E = \left(\frac{a'}{a^2}\right)^4[(D-8)(D-3)(D-2)(D-1)] + \frac{a''}{a^3}\left(\frac{a'}{a^2}\right)^2[4(D-3)(D-2)(D-1)], \qquad (36)$$

which can be determined from the D dimensional generalizations of Eq. (5). Note

$$-\frac{2}{\sqrt{-g}}g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^D x\sqrt{-g}E = D(D-4)E. \qquad (37)$$

Clearly, only in four dimensions the Gauss-Bonnet term corresponds to a total derivative that can be neglected. Hence, we choose the divergent coefficient $\beta_{\rm D}$ such that the factor $(D-4)$ cancels yielding a nonvanishing and finite contribution at the level of the equation of motion when $D \to 4$. The contribution from β_0 is identically zero when we let $D \to 4$. Although divergent in the action, we have shown that at the level of the equation of motion the term involving $\beta_{\rm D}$ yields a finite contribution. All physical measurements are performed on shell, hence we cannot exclude a counterterm of the form $\beta_{\rm D}E$ in the effective action.

This procedure can be debated. In the literature one usually prefers a finite effective action. We abandon this assumption and require a finite equation of motion only. We cannot think of a physical measurement that distinguishes the two approaches and therefore we argue that one should consider and examine all possible effects the counterterms might have on the trace anomaly.

In homogeneous cosmology, R^2 and E are the only local quadratic geometric curvature invariants.[5] Divergences up to one loop in perturbative quantum gravity can be canceled only by counterterms of the form R^2 and E. The physical coefficients multiplying R^2 and E in the renormalized one-loop effective action receive contributions which one can write as

[5]Because $F = 0$, we can express $R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}$ in terms of $R_{\mu\nu}R^{\mu\nu}$ and R^2. Hence, R^2 and E are linearly independent. Note furthermore that if one investigates inhomogeneities in the Universe, one measures (statistical) correlation functions. They are translation invariant as a consequence of the symmetry of the vacuum state and therefore respect the symmetry of the background spacetime.

$$\alpha_{\rm phys} R^2 = [\alpha_{\rm anom}(m) + \alpha_{\rm ct}] R^2 \quad (38a)$$

$$\beta_{\rm phys} E = [\beta_{\rm anom}(m) + \beta_{\rm ct}] E, \quad (38b)$$

where $\alpha_{\rm anom}(m)$ is the mass dependent finite contribution from the trace anomaly, where $m = \{m_i\}$ denotes the mass of the particles i. It is in principle uniquely defined by the requirement that $\alpha_{\rm anom} \to 0$ as $m \to \infty$. Similarly, $\beta_{\rm anom}(m)$ is the (infinite) anomalous contribution determined by the requirement that $\beta_{\rm anom} \to 0$ as $m \to \infty$. These requirements in principle fix $\beta_{\rm anom}(m)$ and $\alpha_{\rm anom}(m)$ uniquely. While there is agreement in the literature on the value of $\alpha_{\rm anom}(m)$ (also used in this paper), disagreement exists on the value of $\beta_{\rm anom}(m)$. Therefore we leave it unspecified. The contributions $\alpha_{\rm ct}$ and $\beta_{\rm ct}$ correspond to the parts of the counterterms that remain when eventual one loop divergences are cancelled.[6] It is however only the sum of these terms, yielding $\alpha_{\rm phys}$ and $\beta_{\rm phys}$, that is physical, i.e. measurable [24]. Since these parameters have not been measured, we cannot simply assume that the coefficients in the trace anomaly are just given by (4). We should allow these coefficients to vary in order to examine the full effect of the trace anomaly on the evolution of our FLRW universe. This is precisely what we pursue in the following sections.

IV. THE TRACE ANOMALY IN QUASI DE SITTER SPACETIME

In this section we work in quasi-de Sitter spacetime, where we treat ϵ as a small and time independent constant which allows us to neglect higher order derivative contributions. The reason for discarding higher order derivative contributions is that they tend to destabilize a dynamical system, formally known as the theorem of Ostrogradsky [44].

We have to distinguish two cases separately. In the spirit of [32], the numerical value of the parameter b'' occurring in the trace anomaly is not fixed because it is a regularization scheme dependent parameter. Generally, however, we cannot exclude the presence of this term and we therefore allow it to take different values. First, we allow for an unrestricted value of b'' and secondly we set $b'' = 2b'/3$. This case is particularly interesting as this value of b'' sets the total coefficient multiplying the $\Box R$ contribution in the trace anomaly to zero.

A. Case I: unrestricted value of b″

We thus insert the truncated expression for the trace anomaly (8) and the quantum density (23) into the Einstein field equation (26). This yields

[6]For a calculation involving anomaly calculations around Minkowski spacetime, we refer to [51,52]. For the calculation of the $\alpha_{\rm anom}(m)$ function for a scalar field in de Sitter spacetime, we refer to [6].

$$\begin{aligned} 9(1+\omega)H^2(t) + 6\dot{H}(t) - 3(1+\omega)\Lambda \\ = -8\pi G[\{12b'(5+3\omega) - 54b''(1+\omega)\}\dot{H}H^2 \\ + 18b'(1+\omega)H^4]. \end{aligned} \quad (39)$$

This differential equation can be solved exactly. Separation of variables yields

$$t - t' = \frac{1}{(1+\omega)} \int_{H(t')}^{H(t)} dH \frac{2 + \alpha H^2}{\Lambda - 3H^2 - 48\pi G b' H^4}, \quad (40)$$

where α is conveniently defined as

$$\alpha = 8\pi G\{4b'(5+3\omega) - 18b''(1+\omega)\}. \quad (41)$$

One can perform the integral in terms of logarithms where one has to take the signs of the occurring parameters carefully into account. The integral above gives

$$\begin{aligned} t - t' = &\frac{1}{-48\pi G b'(1+\omega)\{(H_0^{\rm A})^2 - (H_0^{\rm C})^2\}} \\ &\times\Big[-\frac{1+\frac{1}{2}\alpha(H_0^{\rm A})^2}{H_0^{\rm A}}\Big\{\log\Big(\frac{H(t)+H_0^{\rm A}}{H(t)-H_0^{\rm A}}\Big) \\ &- \log\Big(\frac{H(t')+H_0^{\rm A}}{H(t')-H_0^{\rm A}}\Big)\Big\} + \frac{1+\frac{1}{2}\alpha(H_0^{\rm C})^2}{H_0^{\rm C}} \\ &\times\Big\{\log\Big(\frac{H(t)+H_0^{\rm C}}{H(t)-H_0^{\rm C}}\Big) - \log\Big(\frac{H(t')+H_0^{\rm C}}{H(t')-H_0^{\rm C}}\Big)\Big\}\Big]. \end{aligned} \quad (42)$$

Note the asymptotes of this analytic solution coincide with the asymptotes obtained earlier in (29) as expected.

In Fig. 1, we numerically calculate the dynamics of the Hubble parameter for various initial conditions. The two asymptotes divide this graph into three distinct regions that are not connected for finite time evolution. The region bounded by the two asymptotes contains initial conditions

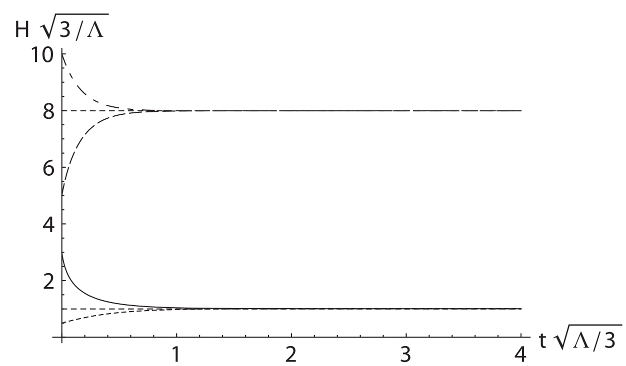

FIG. 1. Dynamics of the Hubble parameter in quasi-de Sitter spacetime in the presence of a nonzero cosmological constant, the trace anomaly, and matter ($\omega = 0$). Depending on the initial conditions, the Hubble parameter evolves to either the classical de Sitter or the quantum anomaly driven attractor. We have used $\lambda = 1/50$, $b' = -0.015$ (standard model value) and $b'' = 0$.

for $H(t)$ such that $H(t)$ grows for late times towards H_0^A and initial conditions such that $H(t)$ asymptotes to the de Sitter attractor H_0^C. In Figs. 2 and 3 we examine whether our approximation that ϵ is both small and a constant is valid for calculating the dynamics. In Fig. 2 one can clearly see that $\epsilon \ll 1$ for late times. If $\dot{\epsilon} = 0$, we should have $\dot{\epsilon}/(H\epsilon) \ll 1$, an assumption that is violated as depicted in Fig. 3. However, already for a classical cosmological constant dominated universe with matter, a similar violation occurs.

Furthermore, note the existence of a branching point, an initial condition for $H(t)$ such that for $H(0) > H_{\rm BP}$ the Hubble parameter asymptotes to the quantum attractor H_0^A and for $H(0) < H_{\rm BP}$ the Hubble parameter decreases to the classical de Sitter attractor H_0^C. When rewriting equation of motion (39) in terms of ϵ and noting that ϵ should diverge exactly at the branching point, one finds

$$H_{\rm BP} = \frac{1}{\sqrt{8\pi G\{9(1+\omega)b'' - 2(5+3\omega)b'\}}}. \tag{43}$$

We probe the dependence on scale by changing the numerical value attached to λ in Eq. (28). If we decrease λ, then also Λ decreases which results in a smaller H_0^C. Also, it turns out that both of the asymptotes are already reached much faster. This improves the validity of the assumption $\epsilon \ll 1$, whereas the assumption $\dot{\epsilon}/(H\epsilon) \ll 1$ is still seriously violated at all times.

Let us study the analytical solution (42) more closely. In particular, it is interesting to derive the high and low energy limits of this solution. Of course, one naively expects in the high or low energy limit to flow towards the quantum or classical attractor, respectively. However, the analysis turns out to be somewhat more subtle. We will show that under a certain condition, the quantum anomaly driven attractor becomes unstable. Although solution (42) looks complicated at first glance, it simplifies when defining

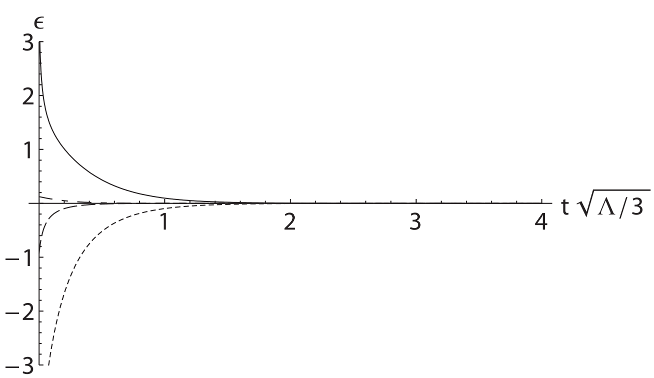

FIG. 2. Validity of the assumption $\epsilon \ll 1$. For the various initial conditions in Fig. 1, one can clearly see that this approximation is well justified.

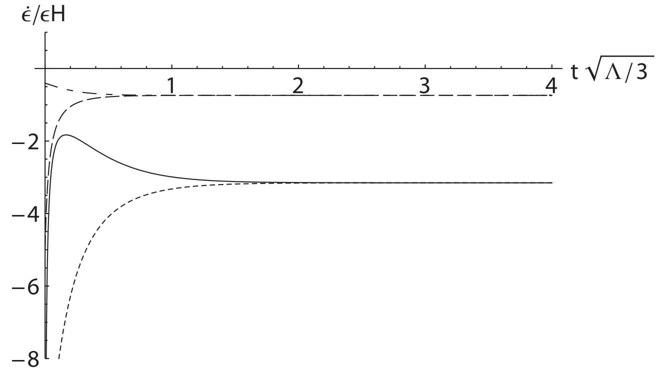

FIG. 3. Validity of the assumption $\dot{\epsilon} = 0$. For the various initial conditions in Fig. 1, we have calculated $\dot{\epsilon}/(H\epsilon)$. Clearly, this condition is violated at all times.

$$\Omega = 3(1+\omega)[1 + 32\pi b'\lambda] \tag{44a}$$

$$A = \frac{1 + \frac{1}{2}\alpha(H_0^A)^2}{H_0^A} \tag{44b}$$

$$B = \frac{1 + \frac{1}{2}\alpha(H_0^C)^2}{H_0^C}. \tag{44c}$$

We define the initial conditions at t' as

$$c_1 = \log\left(\frac{H(t') + H_0^A}{|H(t') - H_0^A|}\right) \tag{44d}$$

$$c_2 = \log\left(\frac{H(t') + H_0^C}{|H(t') - H_0^C|}\right). \tag{44e}$$

Note that $\Omega > 1$, generically. With these definitions, Eq. (42) reduces to

$$\Omega(t - t') = -A\left[\log\left(\frac{H(t) + H_0^A}{|H(t) - H_0^A|}\right) - c_1\right] + B\left[\log\left(\frac{H(t) + H_0^C}{|H(t) - H_0^C|}\right) - c_2\right], \tag{45}$$

In the high energy limit, we set

$$\delta(t) = \frac{H(t) - H_0^A}{H_0^A}, \tag{46}$$

such that $\delta(t) \ll 1$. Equation (45) thus modifies to

$$\Omega(t - t') = -A\left[\log\left(\frac{2 + \delta(t)}{|\delta(t)|}\right) - c_1\right] + B\left[\log\left(\frac{1 + \delta(t) + H_0^C/H_0^A}{|1 + \delta(t) - H_0^C/H_0^A|}\right) - c_2\right], \tag{47}$$

We can expand the second logarithm making use of $\lambda \ll 1$. The leading order contribution [in $\delta(t)$] is given by the denominator in the logarithm, because this term diverges as $\delta(t)$ approaches zero. We can thus exponentiate the equation and solve for the Hubble parameter:

$$|H_>(t) - H_0^{\rm A}| = 2H_0^{\rm A} \exp\left[\frac{\Omega}{A}(t - t' + \Delta t_>)\right], \qquad (48)$$

where the time shift $\Delta t_>$ is given by

$$\Omega \Delta t_> = B\left(c_2 - 2\frac{H_0^{\rm C}}{H_0^{\rm A}}\right) - Ac_1. \qquad (49)$$

As $\Omega > 0$, this solution converges whenever $A < 0$. This provides a stability condition on b'' in terms of b' and λ. The quantum anomaly driven attractor is stable, whenever the following inequality is satisfied:

$$b'' > \frac{2}{9}b' \frac{4 + 3\omega + 16\pi\lambda b'(5 + 3\omega)}{(1 + \omega)(1 + 16\pi\lambda b')}. \qquad (50)$$

In Fig. 4, we have numerically calculated the evolution of the Hubble parameter in a radiation dominated universe when this inequality is not satisfied. We used $b'' = 7b'/6 < 5b'/6$. For initial conditions above the attractor, the Hubble parameter increases to even higher energies, whereas for initial conditions below the quantum attractor, the Hubble parameter evolves towards the classical attractor. Hence even in quasi de Sitter spacetimes, physically questionable solutions occur. However, note that when $b'' = 2b'/3$, the specific case under consideration in subsection IV B, the above inequality is satisfied.

When the above inequality is satisfied, the Hubble parameter in the high energy limit decays exponentially towards the quantum anomaly driven attractor, where some "frequency dependence" through Ω/A and a time shift $\Delta t_>$ can be recognized. The time shift can without observational consequences be absorbed in the initial time t'.

The low energy limit reveals less surprising behavior. Here, we set

$$\tilde{\delta}(t) = \frac{H(t) - H_0^{\rm C}}{H_0^{\rm C}}, \qquad (51)$$

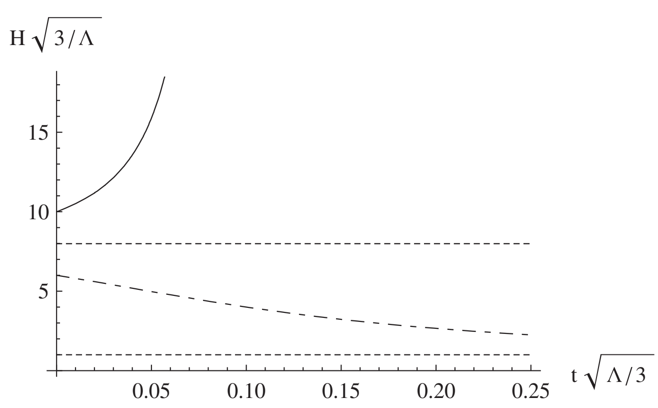

FIG. 4. Instability of the quantum anomaly driven attractor in quasi-de Sitter spacetimes. We have used $\lambda = 1/50$, $\omega = 1/3$, $b'' = 7b'/6 < 5b'/6$.

and $\tilde{\delta}(t) \ll 1$. Again we use $\lambda \ll 1$ in order to capture the leading order dynamics. This yields

$$|H_<(t) - H_0^{\rm C}| = 2H_0^{\rm C} \exp\left[-\frac{\Omega}{B}(t - t' + \Delta t_<)\right]. \qquad (52)$$

Because $B > 0$, this solution converges. In the low energy limit the time shift $\Delta t_<$ is slightly different as compared to (49):

$$\Omega \Delta t_< = Bc_2 - A\left(c_1 - 2\frac{H_0^{\rm C}}{H_0^{\rm A}}\right). \qquad (53)$$

Finally, we examine the phase space flow in quasi de Sitter spacetime. In Fig. 5, we show a parametric plot of H versus ϵ. The phase space basically consists of two lines. The phase space flow is towards either the classical or the quantum anomaly driven attractor as indicated by the arrows. Note that we have chosen both attractors to be stable. Furthermore, we also include the branching point (43) and the classical evolution, that is, the evolution of a universe with $b'' = b' = 0$. As expected, the flow is towards the classical attractor in this case. Although the analysis performed above is for generic values of b'', we set it to zero in Fig. 5 and b' takes its standard model value.

B. Case II: $b'' = 2b'/3$

As indicated earlier, we must consider the case when $b'' = 2b'/3$ separately because in this particular case the total coefficient in front of the $\Box R$ contribution to the trace anomaly vanishes. All higher derivative contributions precisely cancel and also the $\dot{H}^2$ contribution happens to

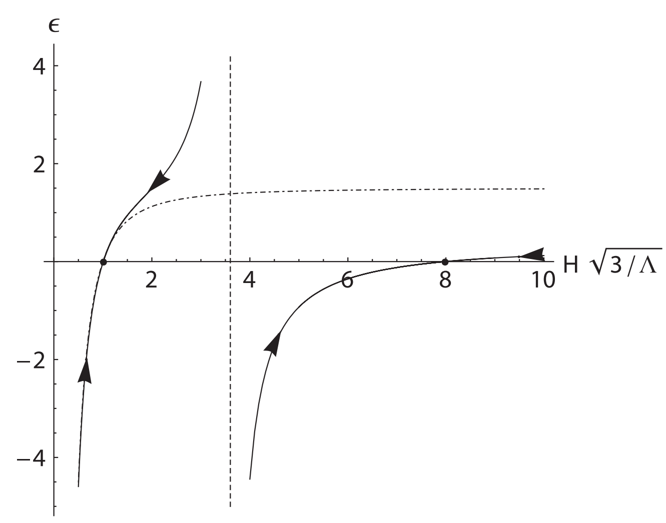

FIG. 5. Phase space flow in quasi-de Sitter spacetime. The phase space consists of two lines. The flow, indicated by the arrows, is towards the classical or quantum attractor represented by two dots. In this regime, both attractors are stable. The vertical dashed line indicates the branching point and the dashed-dotted line the classical evolution. We have used $\omega = 0$, $\lambda = 1/50$, $b' = -0.015$ (standard model value) and $b'' = 0$.

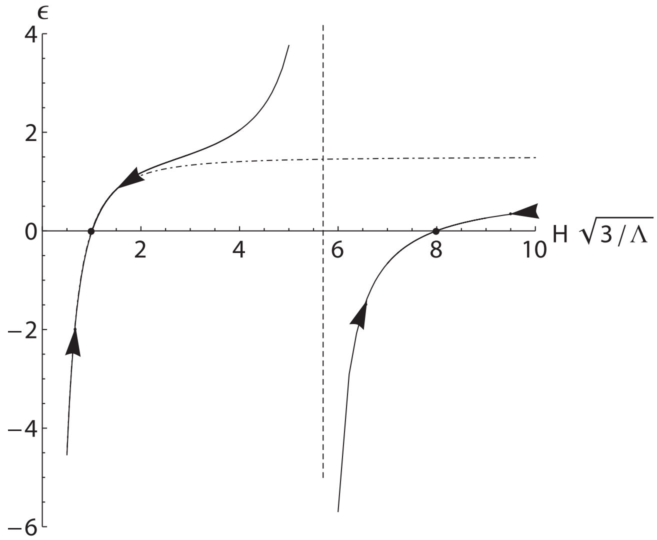

FIG. 6. Phase space flow in quasi de Sitter spacetime for $b'' = 2b'/3$ such that the $\Box R$ does not contribute in the trace anomaly. Qualitatively, the dynamics does not change when compared to Fig. 5. Again, both attractors are stable. Apart from b'', the value of the parameters are identical to Fig. 5.

cancel, such that we find ourselves immediately situated in quasi de Sitter spacetime. Albeit a simple case, we do take the full trace anomaly into account.

The analytic solution obtained in (42) still applies and moreover, it becomes exact. The branching point is still given by Eq. (43) for which we just have to insert $b'' = 2b'/3$. Clearly, in Fig. 6 one can see that qualitatively the dynamics has not changed compared to Fig. 5. The branching point has shifted somewhat to the right, and the way in which the Hubble parameter approaches its two late time asymptotes differs. However, the important features of Fig. 5, i.e.: two stable attractors, the occurrence of a branching point and the shape and dimension of the phase space, do not change.

V. THE TRACE ANOMALY IN FLRW SPACETIMES

We turn our attention to solving the full trace equation (26), where we truncate the expression neither for the anomalous trace (6) nor for the quantum density (22). Obviously, we cannot solve this equation analytically, for it contains all higher order derivative contributions, which forces us to rely on numerical methods.

First, we note that the asymptotes (29) do not change by including the higher derivative contributions. Keeping Ostrogradsky's theorem in mind, we expect to incur all kinds of issues related to the stability of our system and asymptotes, in particular. It is therefore essential to perform a stability analysis for small perturbations $\delta H(t)$ around both of the asymptotes. We insert

$$H(t) = H_0^{\rm C,A} + \delta H(t), \qquad (54)$$

in Eq. (26), where $H_0^{\rm C,A}$ can either denote the classical or the quantum attractor. For the small perturbations around these attractors we make the ansatz $\delta H(t) = c\exp[\xi t]$. We linearize the trace equation finding the characteristic equation from which we determine the eigenvalues ξ of our system:

$$\begin{aligned} 2(\mu - 3\nu/2)\xi^3 &+ 6(\mu - 3\nu/2)(2+\omega)H_0^{\rm C,A}\xi^2 \\ &+ (\{6\mu(5+3\omega) - 27\nu(1+\omega)\}(H_0^{\rm C,A})^2 - 3)\xi \\ &+ 9(4\mu(H_0^{\rm C,A})^2 - 1)(1+\omega)H_0^{\rm C,A} = 0, \end{aligned} \qquad (55)$$

where $\mu = -8\pi G b'$ and $\nu = -8\pi G b''$. Remarkably, the solutions of this third order equation are simple:

$$\xi^{(1)} = -3H_0^{\rm C,A}(1+\omega) \qquad (56a)$$

$$\xi^{(2)} = -\frac{3H_0^{\rm C,A}}{2} + \sqrt{\Delta} \qquad (56b)$$

$$\xi^{(3)} = -\frac{3H_0^{\rm C,A}}{2} - \sqrt{\Delta}, \qquad (56c)$$

where

$$\Delta = -\frac{3}{4(2\mu - 3\nu)}\{-4 + (H_0^{\rm C,A})^2(10\mu + 9\nu)\}. \qquad (57)$$

Clearly, when $\mathrm{Re}\,\xi^{(i)} < 0$ for $i = 1, 2, 3$, the corresponding attractor is stable. Since we only consider nontachyonic matter, eigenvalue (56a) is negative. However, a finite period in which $w < -1$ is not excluded (see e.g.: [46,47]). If one were to consider other equations of state than the simple linear one $\rho_{\rm M} = \omega p_{\rm M}$, this statement might no longer hold [53]. Hence only for $\xi^{(2)}$ when $\Delta > 0$, we could encounter a potential instability. Surprisingly, the stability analysis does not depend on the equation of state ω because ω enters only through Eq. (56a). The condition for instability thus reads

$$\Delta > \left(\frac{3H_0^{\rm C,A}}{2}\right)^2. \qquad (58)$$

We can rewrite this equation to find

$$\begin{aligned} 4 + 8\pi G(H_0^{\rm C,A})^2(10b' + 9b'') \\ \gtrless 72\pi G(b'' - 2b'/3)(H_0^{\rm C,A})^2. \end{aligned} \qquad (59)$$

In the expression above, we should read the inequality $>$ or $<$ whenever $b'' - 2b'/3 > 0$ or $b'' - 2b'/3 < 0$, respectively. We can now insert either the classical or quantum asymptotes previously derived in Eq. (29) and verify which of the two above inequalities is satisfied. Upon inserting the expression for the classical attractor, Eq. (59) yields

$$1 + 32\pi\lambda b' \gtrless 0, \qquad (60)$$

where $\lambda = G\Lambda/3 \ll 1$ as before. Only the first inequality $>$ will be satisfied. Hence we conclude that the classical attractor is unstable if $b'' - 2b'/3 > 0$. The converse will be true if $b'' - 2b'/3 < 0$. Likewise, Eq. (59) for the quantum attractor after some algebra reads

$$-1-\frac{32\pi\lambda}{3}b' \gtrless 0, \tag{61}$$

Concluding, when using $\lambda \ll 1$, we unambiguously find

$$\text{If } b''-2b'/3>0, \text{ then} \begin{cases}\text{Classical attractor unstable}\\ \text{Quantum attractor stable}\end{cases} \tag{62a}$$

$$\text{If } b''-2b'/3<0, \text{ then} \begin{cases}\text{Classical attractor stable}\\ \text{Quantum attractor unstable}\end{cases} \tag{62b}$$

Let us first of all recall that it is precisely the combination $b''-2b'/3$ that multiplies the $\Box R$ contribution in the trace anomaly. This calculation thus proves the statements about stability made in e.g. [41] using the Routh-Hurwitz method. Our proof is more general because we include a constant but otherwise arbitrary equation of state parameter $\omega > -1$. Moreover, while the Routh-Hurwitz method can only guarantee stability of a solution (when certain determinants are all strictly positive), it does not tell anything about instability [41,54]. Furthermore, appreciate that the singular point in this analysis, $b''-2b'/3=0$, or equivalently $2\mu/3-\nu=0$, immediately directs us to the quasi-de Sitter spacetime analysis performed in Sec. IV B, where all higher derivative contributions precisely cancel, rendering both attractors stable.

Let us compare Figs. 7 and 9. In the former figure, we used $b''=0$ such that $b''-2b'/3>0$, yielding an unstable classical attractor. However, if $H(0) \leq H_0^{\mathrm{C}}$ the quantum anomaly driven asymptote is not an attractor and the Hubble parameter runs away to negative infinity. In the latter figure, we set $b''=b'$ such that $b''-2b'/3<0$ which gives us a stable classical attractor. Likewise, for initial conditions $H(0) \geq H_0^{\mathrm{A}}$ the de Sitter solution is not an attractor and the Hubble parameter rapidly blows up to positive infinity.

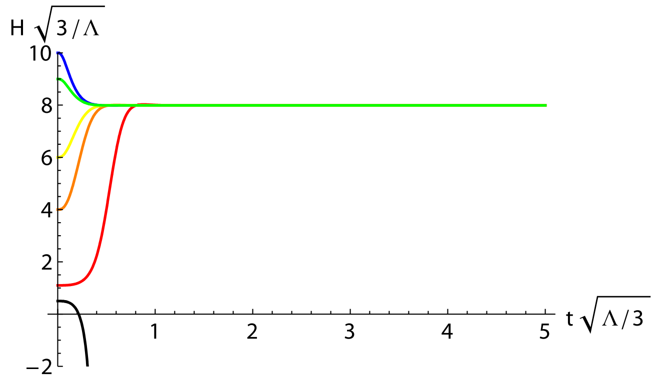

FIG. 7 (color online). Dynamics of the Hubble parameter taking the full trace anomaly into account. We took $b''=0$ such that $b''-2b'/3>0$ yielding an unstable classical attractor. We have used $\omega=0$, $\lambda=1/50$, $b'=-0.015$ (standard model value).

In Fig. 9, we can observe another interesting phenomenon. In this case, the classical attractor is under-damped, resulting in decaying oscillations around the de Sitter attractor. In Fig. 7 these oscillations are not always present. The eigenvalues (56) develop an imaginary contribution resulting in oscillatory behavior whenever

$$\Delta < 0. \tag{63}$$

We thus find

$$4+8\pi G(H_0^{\mathrm{C,A}})^2(10b'+9b'') \lesseqgtr 0. \tag{64}$$

The inequality $<$ or $>$ holds whenever $b''-2b'/3>0$ or $b''-2b'/3<0$ applies, respectively. Again, we verify which of the two inequalities is actually satisfied. To study oscillations around a stable classical attractor, we should take the $>$ inequality (there are no oscillations around an unstable attractor). We thus find

$$1+2\pi\lambda(10b'+9b'')>0. \tag{65}$$

Clearly, this inequality is always satisfied because $\lambda \ll 1$. We thus conclude that whenever the classical de Sitter attractor is stable, oscillations occur. Furthermore, we can insert the quantum anomaly driven attractor in Eq. (64). Now, we should use the $<$ inequality in order to study oscillatory behavior around the (stable) quantum attractor. This yields

$$-1-\frac{9}{2}\frac{b''}{b'}-8\pi\lambda(10b'+9b'')<0. \tag{66}$$

Oscillatory behavior around the quantum attractor thus occurs when

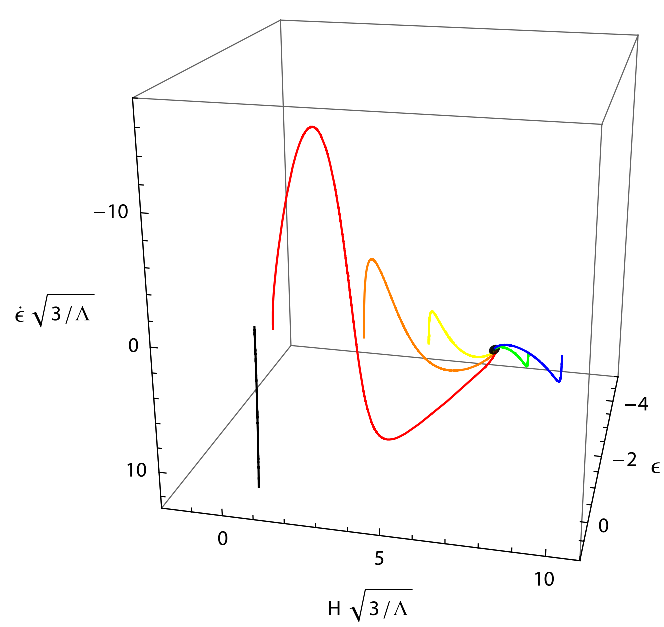

FIG. 8 (color online). Parametric phase space plot for Fig. 7 in which the classical attractor is unstable. We have indicated the quantum anomaly driven attractor as a small black sphere.

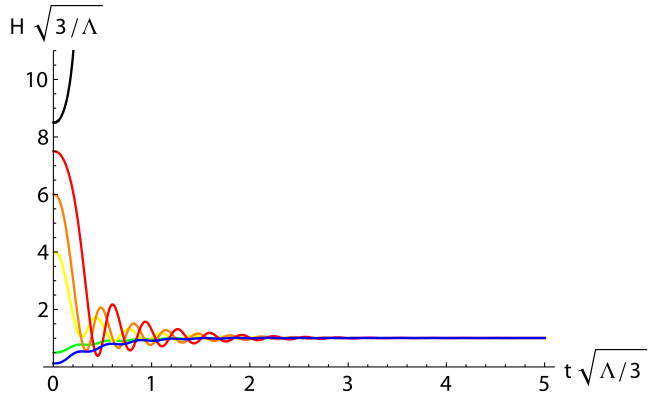

FIG. 9 (color online). Dynamics of the Hubble parameter taking the full trace anomaly into account. We took $b'' = b'$ such that $b'' - 2b'/3 < 0$ yielding a stable classical attractor. We have used $\omega = 0$, $\lambda = 1/50$, $b' = -0.015$ (standard model value). Clearly, the classical attractor is under-damped, resulting in various oscillations around H_0^C.

$$b'' < -\frac{2}{9}b'\left(\frac{1 + 8\pi\lambda b'}{1 + 8\pi\lambda}\right). \tag{67}$$

In Fig. 11 we depicted the parameter space (the b' versus b'' plane) resulting in oscillatory behavior around the quantum anomaly driven attractor. First of all, we should have $b'' - 2b'/3 > 0$ yielding a stable quantum attractor. Of course, one cannot have oscillations around an unstable attractor. Second, note that $b' < 0$ because of Eq. (4b). Finally, when the newly derived inequality (67) is satisfied, oscillations occur. These considerations divide the phase space into three regions: a region where oscillatory behavior occurs, another region which results in critically or over-damped behavior and a part of phase space that is forbidden, as shown in Fig. 11.

Let us now return to discussing the classical attractor that shows its oscillatory behavior manifestly. We calculate the frequency of oscillations around this attractor. Taking the square root of (57) and extracting an i we find the (quantum corrected) frequency for oscillations around the classical attractor:

$$\omega_C = \sqrt{\frac{1 + 2\pi\lambda(10b' + 9b'')}{8\pi G(2b'/3 - b'')}}. \tag{68}$$

Note that this frequency is independent on the equation of state parameter ω. The frequency of oscillations around the quantum attractor can be found analogously.

Let us analyze the phase space in the case of a stable quantum and classical attractor subsequently. In Fig. 8, we visualize the phase space flow for the former case parametrically in the $H(t)$, $\epsilon(t)$ and $\dot{\epsilon}(t)$ directions.[7] The small

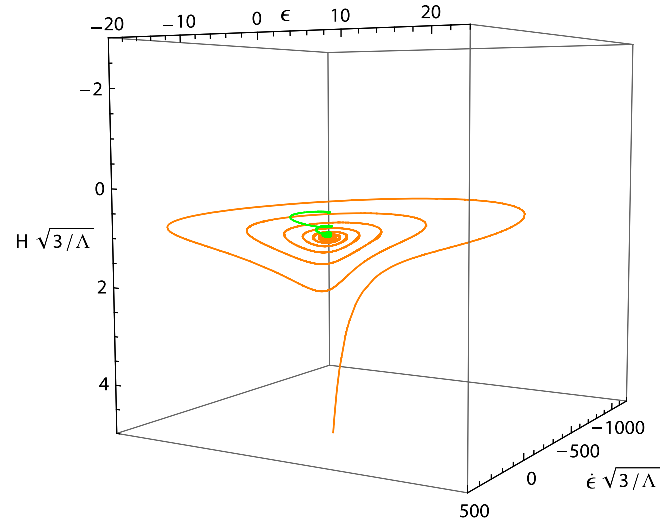

FIG. 10 (color online). Parametric phase space plot for Fig. 9 in which the classical attractor is stable. Because of under-damping, one spirals towards the classical attractor. For clarity, we have only included the flow for two initial conditions $H(0) = \sqrt{3/\Lambda}/2$ and $H(0) = 6\sqrt{3/\Lambda}$. The twister like structure is clearly visible. Qualitatively, the phase space flow resulting from other initial conditions is identical.

black sphere denotes the quantum anomaly driven attractor. For the latter case, we include in Fig. 10 the phase space flow for just two initial conditions for $H(t)$ for clarity. The under-damped oscillatory behavior results in the twister like structure visible in the $H(t)$, $\epsilon(t)$, and $\dot{\epsilon}(t)$ directions.

Finally, we would like to point out that including all higher derivative contributions, which thus corresponds to solving the trace equation exactly, modifies the dynamics

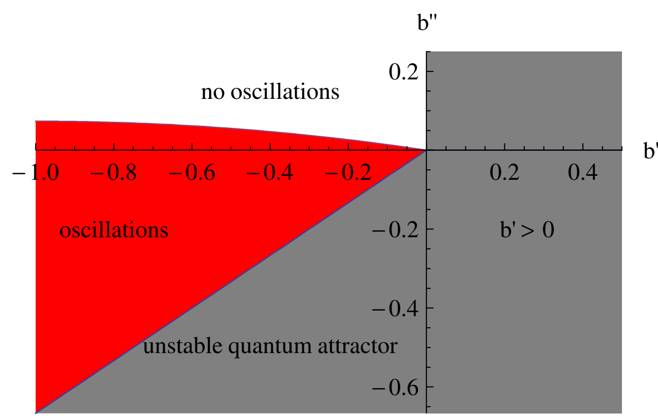

FIG. 11 (color online). Parameter space for oscillations around the quantum anomaly driven attractor. The gray region (light shaded) is excluded either because $b' > 0$ or because the quantum attractor has become unstable. The red region (dark shaded) in parameter space shows oscillatory behavior, whereas the white region is either critically damped or over-damped. We used $\lambda = 1/50$.

[7]Note we are not able to include the fourth dimension of the phase space, $\ddot{\epsilon}(t)$. However, also $\ddot{\epsilon}(t)$ rapidly approaches zero as time elapses.

of the Hubble parameter significantly. Attractors that were stable in the absence of higher derivatives under certain conditions destabilize. The reason for this, clearly, is attributable to the presence of the $\Box R$ term in the trace anomaly, generating these higher derivative contributions. We do not know whether or not incorporating the higher derivatives is a sensible thing to do. Usually higher order derivatives tend to destabilize a system signifying that some particular solutions are not physical. Therefore, in the spirit of Ostrogradsky's theorem, one can question whether the analysis where higher derivative contributions are discarded is correct, or the analysis taking the full trace anomaly into account.

VI. CONCLUSION

The trace of the Einstein field equations in cosmologically relevant spacetimes together with stress-energy conservation completely captures the dynamics of the Hubble parameter. We have derived the trace anomaly from an effective action in spatially flat FLRW spacetimes. It consists of the local quadratic geometric curvature invariants R^2 and the Gauss-Bonnet term E. Because of counterterms that are supposed to cancel divergences of the as yet unknown underlying fundamental theory, we expect the coefficients in the trace anomaly to change. The physical value of each of these coefficients receives contributions both from the anomalous trace and from these counterterms. Because we do not know the physical value these parameters will take, we must allow them to vary in order to examine all possibilities.

We have studied the dynamics of the Hubble parameter both in quasi-de Sitter and in FLRW spacetimes including matter, a cosmological term and the trace anomaly. In quasi-de Sitter spacetime, where we restrict the Hubble parameter to vary slowly in time, we find that for various initial conditions $H(t)$ asymptotes either to the classical de Sitter attractor, or to a quantum anomaly driven attractor. We find a region in parameter space where the quantum attractor destabilizes. Otherwise, both attractors are stable.

In FLRW spacetimes we include all higher derivative contributions in the trace anomaly. We perform a stability analysis for small perturbations around the two asymptotes. For $b'' - 2b'/3 > 0$, the quantum attractor is stable and the classical de Sitter attractor is unstable. On the contrary, for $b'' - 2b'/3 < 0$, the quantum attractor is unstable and the de Sitter attractor becomes stable. The singular point in this analysis, $b'' - 2b'/3 = 0$, immediately directs us to quasi-de Sitter spacetime in which the dynamics is much simpler. In this case, both attractors are stable. The classical de Sitter attractor always shows under-damped oscillatory behavior and we calculate the frequency of these oscillations. We analyze the phase space of the quantum attractor and conclude there is some region in parameter space for which oscillations occur.

There is no dynamical effect that influences the effective value of the cosmological constant, i.e.: the classical de Sitter attractor. Based on our semiclassical analysis we thus conclude that the trace anomaly does *not* solve the cosmological constant problem.

We have studied both the truncated and the exact expression of the trace anomaly in flat FLRW spacetimes. We do not know which of the two approaches is correct. Keeping Ostrogradsky's theorem in mind, higher derivative contributions usually have the tendency to destabilize a dynamical system. Discarding these higher derivatives and studying the trace anomaly in quasi de Sitter spacetime would thus seem plausible.

Finally, one could wonder whether the quantum anomaly driven attractor is physical. The quantum attractor is of the order of the Planck mass $M_{\rm pl}$, so only when matter in the early Universe is sufficiently dense, $H \simeq \mathcal{O}(M_{\rm pl})$. We then expect to evolve towards the quantum attractor. However, at these early times we also expect perturbative general relativity to break down. Hence, this attractor might even not be there or it may be seriously affected by quantum fluctuations. Quantum fluctuations present at that epoch might even induce tunneling towards the regime where $H(t)$ asymptotes to the classical attractor.

ACKNOWLEDGMENTS

We thank Tomas Janssen and Jan Smit for useful suggestions. The authors acknowledge financial support from FOM Grant No. 07PR2522 and by Utrecht University.

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