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STOCHASTIC INFLATIONARY SCALAR ELECTRODYNAMICS

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ABSTRACT

We stochastically formulate the theory of scalar quantum electrodynamics on a de Sitter background. This reproduces the leading infrared logarithms at each loop order. It also allows one to sum the series of leading infrared logarithms to obtain explicit, nonperturbative results about the late time behavior of the system. One consequence is confirmation of the conjecture by Davis, Dimopoulos, Prokopec and Törnkvist that super-horizon photons acquire mass during inflation. We compute $M_\gamma^2 \simeq 3.2991 \times H^2$. The scalar stays perturbatively light with $M_\varphi^2 \simeq 0.8961 \times 3e^2 H^2 / 8\pi^2$. Interestingly, the induced change in the cosmological constant is negative, $\delta\Lambda \simeq -0.6551 \times 3GH^4/\pi$.

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1 Introduction

Gravitons and massless, minimally coupled (MMC) scalars are unique in being massless without classical conformal invariance. The combination of these properties causes the accelerated expansion of spacetime during inflation to tear long wavelength virtual quanta out of the vacuum [1, 2, 3]. As more and more gravitons and MMC scalars emerge from the vacuum, the metric and MMC scalar field strengths experience a slow growth. The effect can be felt by any quantum field theory which involves either the undifferentiated metric or an undifferentiated MMC scalar.

A typical example is afforded by the MMC scalar with a quartic self-interaction,

$$\mathcal{L} = -\frac{1}{2}(1 + \delta Z)\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu}\sqrt{-g} - \frac{1}{2}\delta\xi\varphi^2 R\sqrt{-g} - \frac{1}{4!}(\lambda + \delta\lambda)\varphi^4\sqrt{-g}. \quad (1)$$

Consider this theory quantized on a nondynamical, locally de Sitter background,

$$ds^2 = -dt^2 + a^2 d\vec{x} \cdot d\vec{x} \quad \text{where} \quad a(t) = e^{Ht}. \quad (2)$$

If the finite parts of the renormalization constants (and the cosmological counterterm) are chosen to make the expectation value of the stress tensor vanish at $t = 0$, then an explicit two loop computation using dimensional regularization reveals the following results for the induced energy density $\rho(t)$ and pressure $p(t)$ [4, 5],

$$\rho(t) = \frac{\lambda H^4}{(2\pi)^4} \left\{ \frac{1}{8} \ln^2(a) \right\} + O(\lambda^2), \quad (3)$$

$$p(t) = \frac{\lambda H^4}{(2\pi)^4} \left\{ -\frac{1}{8} \ln^2(a) - \frac{1}{12} \ln(a) \right\} + O(\lambda^2). \quad (4)$$

The factors of $\ln(a) = Ht$ in expressions (3-4) are known as *infrared logarithms*. They derive from the slow growth of the scalar field amplitude that is apparent even in the free theory [6, 7, 8],

$$\langle \Omega_0 | \varphi^2(x) | \Omega_0 \rangle = \text{Divergent Constant} + \frac{H^2}{(2\pi)^2} \ln(a). \quad (5)$$

The two loop expectation value of the stress tensor acquires two such factors coming from the $-\frac{1}{4!}\lambda\varphi^4 g_{\mu\nu}$ term.

Any quantum field theory which involves undifferentiated MMC scalars or metrics will show similar infrared logarithms in some of its Green's functions. They arise at one and two loop orders in the scalar self-mass-squared of this same theory [9, 10]. In scalar quantum electrodynamics they have been seen in the one loop vacuum polarization [11, 12] and the two loop expectation values of certain scalar bilinears [13]. In Yukawa theory they show up in the one loop fermion self-energy [14, 15] and in the two loop coincident vertex function [16]. In pure quantum gravity they occur in the one loop graviton self-energy [17] and in the two loop expectation value of the metric [18]. When quantum gravity is coupled to a massless, Dirac fermion they occur in the one loop fermion self-energy [19, 20]. They even contaminate loop corrections to the power spectrum of cosmological perturbations [21, 22, 23, 24, 25, 26] and other fixed-momentum correlators [27].

Infrared logarithms introduce a fascinating secular element into the usual, static results of quantum field theory. For example, without infrared logarithms, the expectation value of the stress tensor of (1) would be a constant times $g_{\mu\nu}$. With the same renormalization conventions [4, 5] in computing (3-4), the constant would actually be zero!

The most intriguing property of infrared logarithms is their ability to compensate for powers of the loop counting parameter which suppress quantum loop effects. Indeed, the continued growth of $\ln(a) = Ht$ must eventually *overwhelm* the loop counting parameter, no matter how small it is. However, this does not necessarily mean that quantum loop effects become strong. The correct conclusion is rather that perturbation theory breaks down past a given point in time. One must employ a nonperturbative technique to follow what happens at later times.

Certain models lend themselves to resummation schemes such as the $1/N$ expansion [28, 29] but a more general technique is suggested by the form of the expansion for $\rho(t)$ in (1),

$$\rho(t) = H^4 \sum_{\ell=2}^{\infty} \lambda^{\ell-1} \left\{ c_{\ell,0} [\ln(a)]^{2\ell-2} + c_{\ell,1} [\ln(a)]^{2\ell-3} + \dots + c_{\ell,2\ell-2} \ln^2(a) \right\}. \quad (6)$$

Here the constants $c_{\ell,k}$ are pure numbers which are assumed to be of order one. The term in (6) involving $[\lambda \ln^2(a)]^{\ell-1}$ is the *leading logarithm* contribution at ℓ loop order; the other terms are *subdominant logarithms*. Perturbation theory breaks down when $\ln(a) \sim 1/\sqrt{\lambda}$, at which point the leading infrared logarithms at each loop order contribute numbers of order one times

H^4 . In contrast, the subleading logarithms are all suppressed by at least one factor of the small parameter $\sqrt{\lambda} \ll 1$. So it makes sense to retain only the leading infrared logarithms,

$$\rho(t) \longrightarrow H^4 \sum_{\ell=2}^{\infty} c_{\ell,0} \left[\lambda \ln^2(a) \right]^{\ell-1}. \quad (7)$$

This is known as the *leading logarithm approximation*.

Starobinskiĭ has developed a simple stochastic formalism [30] which reproduces the leading infrared logarithms at each order for any scalar potential model of the form,

$$\mathcal{L} = -\frac{1}{2}(1 + \delta Z) \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} - V(\varphi) \sqrt{-g}. \quad (8)$$

Probabilistic representations of inflationary cosmology have been much studied in order to understand initial conditions [31, 32] and global structure [33, 34]. However, we wish here to focus on Starobinskiĭ’s technique as a wonderfully simple way of recovering the most important secular effects of inflationary quantum field theory [35, 36, 37, 38]. It is of particular importance for us that Starobinskiĭ and Yokoyama have shown how to take the late time limit of the series of leading infrared logarithms whenever the potential $V(\varphi)$ is bounded below [39]. This is the true analogue of what the renormalization group accomplishes in flat space quantum field theory and statistical mechanics.

The solution of Starobinskiĭ and Yokoyama is an amazing achievement, but it only gives us nonperturbative control over the infrared logarithms which arise in scalar potential models (8). The most general models which show infrared logarithms possess two complicating features:

- Couplings to fields other than MMC scalars and gravitons; and
- Interactions which involve differentiated MMC scalars and gravitons.¹

An important step forward was a recent leading log solution for the model comprised by a MMC scalar which is Yukawa-coupled to a massless, Dirac

¹Of course there would be no infrared logarithms if *all* the MMC scalars and gravitons were differentiated. However, infrared logarithms must arise, in the expectation values of some operators, from interactions which involve at least one undifferentiated MMC scalar or graviton. Examples include the $h''\partial h\partial h$ interaction of pure quantum gravity [17, 18] and scalar interactions of the form $\varphi^2\partial\varphi\partial\varphi$ [21, 41].

fermion [16]. That model possesses the first complicating feature but not the second. In this paper we derive a similar leading log solution for MMC scalar quantum electrodynamics (SQED),

$$\begin{aligned} \mathcal{L} = & -(1 + \delta Z_2) (\partial_\mu - ieA_\mu)\phi^* (\partial_\nu + ieA_\nu)\phi g^{\mu\nu} \sqrt{-g} - \delta\xi \phi^* \phi R \sqrt{-g} \\ & - \frac{1}{4}(1 + \delta Z_3) F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} - \frac{\delta\lambda}{4} (\phi^* \phi)^2 \sqrt{-g} \ , \end{aligned} \quad (9)$$

Although this model has derivative interactions we will see that a gauge choice permits one to avoid them at leading log order. We still do not have a full understanding of how to treat derivative interactions.

Section 2 of this paper summarizes a recent all-order derivation [40, 41] of the Starobinskiĭ formalism for scalar potential models. Of course it is crucial to understand why the technique works in order to apply it to more general theories. That problem is discussed in section 3, reaching the conclusion that one deals with other fields by integrating them out and then stochastically simplifying the resulting effective action. Section 4 accomplishes this for the vector potential of SQED. One must also integrate out the vector potential from any operator whose expectation value is desired. The resulting, purely scalar operator is then stochastically simplified before computing the leading log contribution to its expectation value using Starobinskiĭ's technique. We do this in section 5 for the various constituents of the SQED stress tensor. Because the leading logarithm limit of SQED gives a model of the form (8), with a potential which is bounded below, one can exploit the solution of Starobinskiĭ and Yokoyama to make explicit, nonperturbative predictions. We do this in section 6 for the expectation values of $\varphi^*(x)\varphi(x)$, $F_{\mu\nu}(x)F_{\rho\sigma}(x)$ and $T_{\mu\nu}(x)$. We also confirm the remarkable conjecture of Davis, Dimopoulos, Prokopec and Tornkvist that super-horizon photons acquire mass during inflation [42, 43]. The eventual, nonperturbative photon mass-squared turns out to be about a hundred times larger than perturbative estimates.

2 Deriving Starobinskiĭ's Formalism

Infrared logarithms arise in explicit perturbative computations of Green's functions formed from quantum field operators. It is an amazing fact that one can reproduce the leading infrared logarithms in any model of the form (8) using a formalism in which the fields are classical random variables. One wonders, what became of the Uncertainty Principle? What became of the

ultraviolet divergences and the counterterms? And how did the stochastic jitter emerge?

In order to apply Starobinskiĭ's formalism to more general models one must understand why it works. That is the task of this section. We begin by expressing the dimensionally regulated, Heisenberg field equations in Yang-Feldman form [44]. We then explain how infrared logarithms arise and, of crucial importance, the conditions for any expectation value of undifferentiated fields to receive a leading logarithm contribution. Based upon this understanding, we identify a series of simplifications that can be made to the Yang-Feldman equation without in any way affecting the leading infrared logarithms. Taking the time derivative of this simplified Yang-Feldman equation results in Starobinskiĭ's Langevin equation, with the white noise emerging as the time derivative of the simplified free field. The section closes with a review of the nonperturbative solution of Starobinskiĭ and Yokoyama obtained [39] for the late time limit of any model with a potential which is bounded below.

2.1 The Free Field Expansion

The dimensionally regulated, Heisenberg operator equation for (8) is,

$$\ddot{\varphi} + (D-1)H\dot{\varphi} - \frac{\nabla^2}{a^2}\varphi + \frac{V'(\varphi)}{1+\delta Z} = 0. \quad (10)$$

Integrating it results in the Yang-Feldman equation [44],

$$\varphi(t, \vec{x}) = \varphi_0(t, \vec{x}) - \int_0^t dt' a'^{D-1} \int d^{D-1}x' G(x; x') \frac{V'(\varphi(x'))}{1+\delta Z}. \quad (11)$$

A number of quantities in (11) require definition. The *free field* $\varphi_0(x)$ and the *retarded Green's function* $G(x; x')$ are,

$$\varphi_0(t, \vec{x}) \equiv \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \theta(k-H) \left\{ u(t, k) e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k}) + u^*(t, k) e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k}) \right\}, \quad (12)$$

$$G(x; x') \equiv i\theta(\Delta t) \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\Delta\vec{x}} \left\{ u(t, k) u^*(t', k) - u^*(t, k) u(t', k) \right\}. \quad (13)$$

Here $\Delta t \equiv t - t'$, $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$ and the mode function $u(t, k)$ is,

$$u(t, k) = i\sqrt{\frac{\pi}{4Ha^{D-1}}} H_{\frac{D-1}{2}}^{(1)}\left(\frac{k}{Ha}\right) = \frac{\Gamma(\frac{D-1}{2})}{\sqrt{4\pi H}} \left(\frac{2H}{k}\right)^{\frac{D-1}{2}} \left\{ 1 + O\left(\frac{k^2}{H^2 a^2}\right) \right\}. \quad (14)$$

The nonzero commutation relations of the canonically normalized creation and annihilation operators are,

$$[\alpha(\vec{k}), \alpha^\dagger(\vec{k}')] = (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}'). \quad (15)$$

It follows that the retarded Green's function can be expressed as the commutator of two free fields,²

$$G(x; x') = i\theta(\Delta t) [\varphi_0(x), \varphi_0(x')]. \quad (16)$$

Iterating the Yang-Feldman equation generates the usual interaction picture expansion of the field, in this case expressed in terms of a free field $\varphi_0(x)$ which agrees with the full field and its first time derivative at $t = 0$. Without worrying about operator ordering, we can write out the first few terms of this expansion,

$$\begin{aligned} \varphi(x) = & \varphi_0(x) - \int d^D x' \sqrt{-g(x')} G(x; x') \frac{V'(\varphi_0(x'))}{1 + \delta Z} \\ & + \int d^D x' \sqrt{-g(x')} G(x; x') \frac{V''(\varphi_0(x'))}{1 + \delta Z} \\ & \times \int d^D x'' \sqrt{-g(x'')} G(x'; x'') \frac{V'(\varphi_0(x''))}{1 + \delta Z} + \dots \quad (17) \end{aligned}$$

A nice diagrammatic representation for this expansion has recently been given by Musso [47].

The integrals over x'^μ and x''^μ are known as *vertex integrations*. The expectation value of any operator which involves $\varphi(x)$ — for example, $\varphi^N(x)$ or $\varphi(x)\varphi(x')$ — can obviously be reduced to a sum of terms, each one of which consists of a number of vertex integrations of Green's functions times the expectation value of some number of free fields. The expectation value of free fields can be further reduced to a sum of products of expectation values of two free fields.

²The restriction to $k \equiv \|\vec{k}\| \geq H$ in the free field mode sum (12) is imposed to avoid an infrared singularity in the free propagator [45]. The physical reason for this singularity is that no causal process would allow an experimenter to prepare the initial state in coherent Bunch-Davies vacuum over an infinite spatial section. Sensible physics can be regained either by employing an initial state for which the super-horizon modes are less strongly correlated [46], or else by working on a compact spatial manifold such as T^{D-1} for which there are initially no super-horizon modes [1]. In both cases the modes with $k < H$ are effectively absent. Note also that one typically removes the cutoff on any mode sum, such as (13), which is not singular at $k = 0$.

2.2 Genesis of Infrared Logarithms

Infrared logarithms derive from two sources: the expectation values of pairs of free fields and the vertex integrations.

• *Free Field Expectation Values:* Let us first consider the expectation value of two free fields in the presence of the state which obeys $\alpha(\vec{k})|\Omega\rangle = 0$,

$$\langle \Omega | \varphi_0(x) \varphi_0(x') | \Omega \rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \theta(k-H) e^{i\vec{k}\cdot\Delta\vec{x}} u(t,k) u^*(t',k), \quad (18)$$

$$= \int_H^\infty dk k^{D-2} \frac{J_{\frac{D-3}{2}}(k\Delta x) u(t,k) u^*(t',k)}{2^{D-2} \pi^{\frac{D-1}{2}} \left(\frac{k\Delta x}{2}\right)^{\frac{D-3}{2}}}. \quad (19)$$

At high k the Bessel functions in (19) oscillate for $x^\mu \neq x'^\mu$, which makes the integral converge. Even at coincidence there is no possibility of an a -dependent ultraviolet divergence,

$$\langle \Omega | \varphi_0^2(x) | \Omega \rangle = \frac{1}{2^D \pi^{\frac{D-3}{2}} \Gamma(\frac{D-1}{2}) H a^{D-1}} \int_H^\infty dk k^{D-2} \left\| H_{\frac{D-1}{2}}^{(1)}\left(\frac{k}{Ha}\right) \right\|^2, \quad (20)$$

$$= \frac{H^{D-2}}{2^D \pi^{\frac{D-3}{2}} \Gamma(\frac{D-1}{2})} \int_{\frac{1}{a}}^\infty dz z^{D-2} \left\| H_{\frac{D-1}{2}}^{(1)}(z) \right\|^2. \quad (21)$$

The only possible a dependence is the finite contribution from the infrared.

The small k expansion of the integrand in (19) is,

$$k^{D-2} \frac{J_{\frac{D-3}{2}}(k\Delta x) u(t,k) u^*(t',k)}{2^{D-2} \pi^{\frac{D-1}{2}} \left(\frac{k\Delta x}{2}\right)^{\frac{D-3}{2}}} = \frac{\Gamma(\frac{D-1}{2}) H^{D-2}}{2\pi^{\frac{D+1}{2}} k} \left\{ 1 + O\left(\frac{k^2}{H^2 a^2}, \frac{k^2}{H^2 a'^2}, k^2 \Delta x^2\right) \right\}. \quad (22)$$

Had the lower limit not been cut off at $k = H$ this would give a logarithmic divergence. With the infrared cutoff there is no divergence, but one does get a large logarithm. It derives exclusively from the first term of (22), integrated up to the point where the expansion breaks down and the integrand begins to oscillate. This point is $k \simeq \text{Min}(Ha, Ha', 1/\Delta x)$. By taking account of causality,

$$\Delta x \leq \left| \frac{1}{Ha} - \frac{1}{Ha'} \right| \implies \frac{1}{\Delta x} \geq \text{Min}(Ha, Ha'), \quad (23)$$

we see that the upper limit is actually $k = \text{Min}(Ha, Ha') \equiv \alpha$,³

$$\frac{\Gamma(\frac{D-1}{2})H^{D-2}}{2\pi^{\frac{D+1}{2}}} \int_H^{H\alpha} \frac{dk}{k} = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} 2\ln(\alpha) \longrightarrow \frac{H^2}{4\pi^2} \ln(\alpha) \quad (\text{in } D = 4). \quad (24)$$

To summarize, *infrared logarithms from $\varphi_0(t, \vec{x})$ derive exclusively from the range $H \lesssim k \lesssim Ha(t)$. Further, only the first term in the long wavelength expansion (14) of the mode functions contributes.*

• *Vertex Integrations:* We turn now to the second source of infrared logarithms, which is vertex integrations. It is apparent from expressions (13-14) that the most infrared-singular part of the mode function drops out of the retarded Green's function,

$$\begin{aligned} i\theta(\Delta t) [u(t, k)u^*(t', k) - u^*(t, k)u(t', k)] \\ = \frac{\theta(\Delta t)}{(D-1)H} \left[\frac{1}{a'^{D-1}} - \frac{1}{a^{D-1}} \right] \left\{ 1 + O\left(\frac{k^2}{H^2 a^2}, \frac{k^2}{H^2 a'^2}\right) \right\}. \end{aligned} \quad (25)$$

Hence the retarded Green's function cannot contribute infrared logarithms. However, consider the vertex integration of $G(x; x')$ against n powers of $\ln(a')$,

$$\begin{aligned} \int_0^t dt' a'^{D-1} \int d^{D-1}x' G(x; x') [\ln(a')]^n \\ = \frac{1}{(D-1)H} \int_0^t dt' \left[1 - \left(\frac{a'}{a}\right)^{D-1} \right] [\ln(a')]^n, \end{aligned} \quad (26)$$

$$= \frac{1}{(n+1)(D-1)H^2} \left\{ [\ln(a)]^{n+1} + O([\ln(a)]^n) \right\}. \quad (27)$$

The temporal vertex integration has increased the number of infrared logarithms from n to $n+1$.

A temporal vertex integration can only produce an additional infrared logarithm when it receives nearly equally weighted contributions from its full range. This requires that no factors of a' should remain after multiplying by

³We have also exploited the doubling formula [48] to write,

$$\Gamma\left(\frac{D-1}{2}\right) = \frac{\sqrt{\pi}}{2^{D-2}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}.$$

the a'^{D-1} from the measure and performing the spatial vertex integrations. For example, consider the two terms from the right hand side of (26),

$$\int_0^t dt' [\ln(a')]^n = \int_0^t dt' (Ht')^n = \frac{[\ln(a)]^{n+1}}{(n+1)H}, \quad (28)$$

$$\int_0^t dt' \left(\frac{a'}{a}\right)^{D-1} [\ln(a')]^n = e^{-\beta Ht} \left(\frac{\partial}{\partial \beta}\right)^n \left[\frac{e^{\beta Ht} - 1}{\beta H} \right]_{\beta=D-1}, \quad (29)$$

$$= \frac{[\ln(a)]^n}{(D-1)H} - \frac{n[\ln(a)]^{n-1}}{(D-1)^2 H} + \dots \quad (30)$$

To summarize, *infrared logarithms from vertex integrations derive entirely from the part of the long wavelength expansion of the Green's function which goes like $1/a'^{D-1}$.*

2.3 Conditions for a Leading Log Contribution

We have seen the ways in which infrared logarithms originate from the expectation values of pairs of free fields, and from vertex integrations. It is now time to consider the crucial issue of how many infrared logarithms must derive from each source in order to reach leading log order. Although the answer is completely general, it is easier to explain in the context of the quartic self-interaction,

$$V(\varphi) = \frac{1}{2} \delta \xi \varphi^2 R + \frac{1}{4!} (\lambda + \delta \lambda) \varphi^4. \quad (31)$$

Let us first establish that the various counterterms cannot make leading log contributions. This follows from the number of fields they carry and the number of factors of λ they involve in this model [4, 5, 9],

$$\delta Z = O(\lambda) \quad , \quad \delta \xi = O(\lambda) \quad , \quad \delta \lambda = O(\lambda^2). \quad (32)$$

The coupling constant renormalization, $\delta \lambda$, multiplies the same φ^4 term as λ , so contributions involving it produce the same structure of infrared logarithms as contributions from $\lambda \varphi^4$. Because contributions from $\delta \lambda \varphi^4$ have at least one extra factor of λ , with no more infrared logarithms, they can never be leading order. The same argument applies to the field strength renormalization, δZ , on account of the fact that it enters the field equations in the form, $V'(\varphi)/(1 + \delta Z)$. While both $\delta \xi$ and λ go like λ , the former

multiplies φ^2 while the latter multiplies φ^4 . So no contribution involving $\delta\xi$ can produce as many infrared logarithms, at the same order in λ , as contributions involving only the $\lambda\varphi^4$ term. We therefore get exactly the same leading log contributions from the simplified Yang-Feldman equation without the counterterms,

$$\varphi(x) = \varphi_0(x) - \frac{\lambda}{6} \int_0^t dt' a'^{D-1} \int d^{D-1}x' G(x; x') \varphi^3(x'). \quad (33)$$

Let us consider the generic form of the free field expansion derived from (33), keeping track only of the number of λ 's, the number of vertex integrations, and the number of fields at any point. We already have symbols for the coupling constant λ and the field φ_0 . Let us employ the symbol “ I ” to denote generic vertex integrations,

$$I \equiv \int_0^t dt' a'^{D-1} \int d^{D-1}x' G(x; x'). \quad (34)$$

In this notation we might render (33) as follows,

$$\varphi \sim \varphi_0 + \lambda I \varphi^3. \quad (35)$$

Note that we do not worry about signs or numerical factors such as $1/6$, nor do we worry about which spacetime points the various fields reside. In this generic language it is simple to iterate (35) to exhibit the generic form of the free field expansion,

$$\varphi \sim \varphi_0 + \lambda I \varphi_0^3 + \lambda^2 I^2 \varphi_0^5 + \lambda^3 I^3 \varphi_0^7 + \dots. \quad (36)$$

In other words, *each additional factor of λ involves one more vertex integration and two more free fields.*

The same generic form (36) applies to any operator whose VEV we might wish to compute. For example, the product of N φ 's — even at all different points — would be rendered,

$$\varphi^N \sim (\varphi_0)^N \{1 + \lambda I \varphi_0^2 + \lambda^2 I^2 \varphi_0^4 + \lambda^3 I^3 \varphi_0^6 + \dots\}. \quad (37)$$

Note that we do not worry, at this level, about which spacetime points the various free fields reside, or which of the parent full fields gives rise to corrections in the free field expansion. The only interesting things are the number

of factors of λ , the number of vertex integrations, and the number of free fields.

Now recall that leading log corrections to any result in this theory must produce two infrared logarithms for each extra factor of the loop-counting parameter λ [4, 5, 9, 10]. From the generic free field expansion (37) we see that each additional factor of λ is accompanied by one new vertex integration and two new free fields. Because a vertex integration can add at most a single infrared logarithm, as can any pair of free fields, it follows that *leading log contributions to the expectation value of any combination of undifferentiated full fields φ can only result when each vertex integration and each free field contributes to an infrared logarithm.* That turns out to be a general conclusion for any potential $V(\varphi)$.

2.4 Starobinskii's Langevin Equation

It is well to summarize what we have learned in the previous sub-sections:

1. For the expectation value of any combination of undifferentiated full fields to receive a leading log contribution, every $\langle \varphi_0(x)\varphi_0(x') \rangle$ and every vertex integration must contribute an infrared logarithm.
2. The infrared logarithm from $\varphi_0(t, \vec{x})$ derives exclusively from modes in the range $H < k < Ha(t)$, and from just the leading term in the small k expansion of $u(t, k)$. That is, the following replacement makes no change at leading log order,

$$\begin{aligned} \varphi_0(t, \vec{x}) \longrightarrow & \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \theta(k - H)\theta(Ha - k) \\ & \times \frac{\Gamma(\frac{D-1}{2})}{\sqrt{4\pi H}} \left(\frac{2H}{k}\right)^{\frac{D-1}{2}} \left\{ e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k}) \right\}. \end{aligned} \quad (38)$$

3. The infrared logarithm from a vertex integration derives exclusively from the term in the small k expansion of $u(t, k)u^*(t', k) - u^*(t, k)u(t', k)$ which goes like $1/a'^{D-1}$. That is, the following replacement makes no change at leading log order,

$$G(x; x') \longrightarrow \frac{\theta(\Delta t)\delta^{D-1}(\vec{x} - \vec{x}')}{(D - 1)Ha'^{D-1}}. \quad (39)$$

Because of point 1 we can make the simplifications (38-39) in the Yang-Feldman equation. Because the infrared truncation of (38) removes any possibility for ultraviolet divergences, we can set $D = 4$ to write the infrared truncated free field as,

$$\Phi_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \theta(k - H)\theta(Ha - k) \frac{H}{\sqrt{2k^3}} \left\{ e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k}) \right\}. \quad (40)$$

Note that we have used a new symbol, $\Phi_0(x)$, to distinguish it from the original free field, $\varphi_0(x)$. Whereas $\varphi_0(x)$ embodies the Uncertainty Principle, the infrared truncated free field does not,

$$\left[\Phi_0(x), \Phi_0(x') \right] = 0. \quad (41)$$

Note also that, whereas the VEV of $\varphi_0^2(x)$ diverges, the VEV of $\Phi_0^2(x)$ does not. We have proved that these very different quantities nevertheless produce precisely the same infrared logarithms.

We have already seen that counterterms can be dropped at leading log order. The resulting, simplified Yang-Feldman equation is accordingly,

$$\Phi(t, \vec{x}) = \Phi_0(t, \vec{x}) - \frac{1}{3H} \int_0^t dt' V'(\Phi(t', \vec{x})). \quad (42)$$

Note that we have called the field $\Phi(x)$ to distinguish it from $\varphi(x)$. This field $\Phi(x)$ commutes with $\Phi(x')$ for all x'^μ , just like the free field, $\Phi_0(x)$. Further, VEV's involving it are completely free of ultraviolet divergences. They nevertheless agree exactly with VEV's of $\varphi(x)$ at leading log order.

Taking the time derivative of (42) gives Starobinskiĭ's Langevin equation [39],

$$\dot{\Phi}(t, \vec{x}) = \dot{\Phi}_0(t, \vec{x}) - \frac{1}{3H} V'(\Phi(t, \vec{x})). \quad (43)$$

Starobinskiĭ's stochastic noise term is the time derivative of the infrared truncated free field,

$$\dot{\Phi}_0(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \delta(Ha - k) \frac{H^2}{\sqrt{2k}} \left\{ e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k}) \right\}. \quad (44)$$

A simple calculation reveals that it behaves like white noise,

$$\langle \Omega | \dot{\Phi}_0(t, \vec{x}) \dot{\Phi}_0(t', \vec{x}') | \Omega \rangle = \frac{H^3}{4\pi^2} \delta(t - t'). \quad (45)$$

2.5 Nonperturbative Solution

Langevin equations of the form (43) have been much studied [49]. Expectation values of functionals of the stochastic field can be computed in terms of a probability density $\varrho(t, \phi)$ as follows,

$$\langle \Omega | F[\Phi(t, \vec{x})] | \Omega \rangle = \int_{-\infty}^{+\infty} d\phi \varrho(t, \phi) F(\phi) . \quad (46)$$

The probability density satisfies a Fokker-Planck equation whose first term is given by the interaction in (43) and whose second term is fixed by the normalization of the white noise (45):

$$\dot{\varrho}(t, \phi) = \frac{1}{3H} \frac{\partial}{\partial \phi} [V'(\phi) \varrho(t, \phi)] + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \left[\frac{H^3}{4\pi^2} \varrho(t, \phi) \right] . \quad (47)$$

To recover the nonperturbative late time solution of Starobinskiĭ and Yokoyama [39] one makes the ansatz,

$$\lim_{t \rightarrow \infty} \varrho(t, \phi) = \varrho_{\infty}(\phi) , \quad (48)$$

because the $-V'(\varphi)$ force should eventually balance the tendency of inflationary particle production to push the scalar up its potential. This ansatz results in a first order equation,

$$\frac{d\varrho_{\infty}(\phi)}{\varrho_{\infty}(\phi)} = -\frac{8\pi^2}{3H^4} V'(\phi) d\phi . \quad (49)$$

The solution is straightforward,

$$\varrho_{\infty}(\phi) = N \exp \left[-\frac{8\pi^2}{3H^4} V(\phi) \right] . \quad (50)$$

3 Generalizing Starobinskiĭ's Formalism

In this section we discuss the problem of generalizing Starobinskiĭ's formalism beyond scalar potential models. We begin by explaining the two complicating features: derivative interactions and passive fields which do not produce infrared logarithms. We then derive a formula which gives the leading logarithm series for a general interaction. By applying this formula to simple

models it emerges that ultraviolet divergences can contaminate even the leading infrared logarithms. This is because passive fields — and differentiated fields of any type — make contributions of order one which multiply the leading logarithms contributed by other fields. Unlike the infrared logarithms, these order one contributions derive from all portions of the free field mode sum, and from the full free field mode functions. Hence it is not valid to stochastically simplify passive fields as we did in the previous section. The correct procedure is to integrate them out, and then stochastically simplify the resulting effective field equations. We show that this amounts to computing the effective potential.

3.1 Active, Passive and Differentiated Fields

We use the term *active* to denote a field whose mode functions are right to produce infrared logarithms. Fields whose mode functions cannot produce infrared logarithms are called *passive*. A typical passive field is the massless, conformally coupled scalar,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\psi\partial_\nu\psi g^{\mu\nu}\sqrt{-g} - \frac{1}{8}\left(\frac{D-2}{D-1}\right)\psi^2 R\sqrt{-g}. \quad (51)$$

The plane wave mode functions for this field can be worked out for any scale factor,

$$v(t, k) = \frac{a^{1-\frac{D}{2}}}{\sqrt{2k}} \exp\left[-ik \int_0^t \frac{dt'}{a(t')}\right]. \quad (52)$$

We shall henceforth make the specialization to de Sitter so that,

$$\text{de Sitter} \quad \implies \quad v(t, k) = \frac{a^{1-\frac{D}{2}}}{\sqrt{2k}} \exp\left[-\frac{ik}{H} + \frac{ik}{Ha}\right]. \quad (53)$$

These mode functions are not singular enough for the expectation value of two free fields to produce an infrared logarithm for $D > 2$,

$$\langle \Omega | \psi(x)\psi(x') | \Omega \rangle = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\cdot\Delta\vec{x}} v(t, k)v^*(t', k), \quad (54)$$

$$= \frac{(aa')^{1-\frac{D}{2}}}{(4\pi)^{\frac{D-1}{2}}} \int_0^\infty dk k^{D-3} \frac{J_{\frac{D-3}{2}}(k\Delta x) e^{\frac{ik}{Ha} - \frac{ik}{Ha'}}}{\left(\frac{k\Delta x}{2}\right)^{\frac{D-3}{2}}}. \quad (55)$$

Nor, again for $D > 2$, can one get an infrared logarithm from the vertex integration of the retarded conformal Green's function,

$$\int_0^t dt' a'^{D-1} \int d^{D-1} x' G_{\text{cf}}(x; x') = \frac{1}{H} \int_0^t dt' \left[\left(\frac{a'}{a} \right)^{\frac{D}{2}-1} - \left(\frac{a'}{a} \right)^{\frac{D}{2}} \right]. \quad (56)$$

The absence of infrared logarithms from either source for the conformally coupled scalar is obviously related because its mode functions obey the same Wronskian as do the minimally coupled scalar mode functions,

$$v(t, k)\dot{v}^*(t, k) - \dot{v}(t, k)v^*(t, k) = \frac{i}{a^{D-1}} = u(t, k)\dot{u}^*(t, k) - \dot{u}(t, k)u^*(t, k). \quad (57)$$

This relation requires that a total of $D - 1$ factors of a must be shared between any two linearly independent solutions. For example, the real and imaginary parts obey,

$$-2\text{Re}(v)\text{Im}(\dot{v}) + 2\text{Re}(\dot{v})\text{Im}(v) = \frac{1}{a^{D-1}} = -2\text{Re}(u)\text{Im}(\dot{u}) + 2\text{Re}(\dot{u})\text{Im}(u). \quad (58)$$

For the VEV of a pair of free fields to produce an infrared logarithm requires that the far infrared ($\vec{k} \approx 0$) mode function should approach a phase divided by $k^{\frac{D-1}{2}}$, with no dependence upon a . If we make the phase zero then this fixes the small k dependence of the real part and we can use the Wronskian to infer how the imaginary part depends upon k and a ,

$$\text{Re}(u) \longrightarrow \frac{\#}{k^{\frac{D-1}{2}}}, \quad \text{Im}(u) \longrightarrow \frac{1}{2\#(D-1)H} \frac{k^{\frac{D-1}{2}}}{a^{D-1}}. \quad (59)$$

In this same limit the measure factor times retarded Green's function is the Fourier transform of the combination,

$$a'^{D-1} i [u(t, k)u^*(t', k) - u^*(t, k)u(t', k)] \longrightarrow \frac{1}{(D-1)H} \left[1 - \left(\frac{a'}{a} \right)^{D-1} \right]. \quad (60)$$

In contrast, the leading small k behavior of the conformally coupled mode functions is less singular than $u(t, k)$ by a factor of $(k/Ha)^{\frac{D}{2}-1}$. This precludes getting an infrared logarithm from the VEV of a pair of free fields. It also shifts scale factors from the real part of v to the imaginary part,

$$\text{Re}(v) \longrightarrow \frac{\#}{k^{\frac{1}{2}} a^{\frac{D}{2}-1}}, \quad \text{Im}(v) \longrightarrow \frac{1}{2\#H} \frac{k^{\frac{1}{2}}}{a^{\frac{D}{2}}}. \quad (61)$$

Hence the measure factor times retarded Green's function goes to the Fourier transform of,

$$a'^{D-1}i[v(t, k)v^*(t', k) - v^*(t, k)v(t', k)] \longrightarrow \frac{1}{H} \left[\left(\frac{a'}{a} \right)^{\frac{D}{2}-1} - \left(\frac{a'}{a} \right)^{\frac{D}{2}} \right]. \quad (62)$$

As we saw in the previous section, positive powers of a'/a weight temporal vertex integrations overwhelmingly at their upper limits, which prevents them from producing infrared logarithms. Similar considerations apply to all passive fields.

Derivatives also suppress infrared logarithms. For if a free active field is differentiated with respect to space then its mode sum contains an extra factor of k and the logarithmic singularity evident in (22) is absent. A time derivative is even worse. In $D = 4$ it gives rise to two extra factors of k ,

$$\dot{u}(t, k) = \frac{\partial}{\partial t} \left\{ \frac{H}{\sqrt{2k^3}} \left[1 - \frac{ik}{Ha} \right] e^{\frac{ik}{Ha}} \right\} = \frac{H}{\sqrt{2k^3}} \left[-\frac{k^2}{Ha^2} \right] e^{\frac{ik}{Ha}}. \quad (63)$$

Of course it is possible that a differentiated field in the Lagrangian will give rise to a differentiated vertex integration, rather than a differentiated free field, in the free field expansion. This also prevents the appearance of an extra infrared logarithm because the undifferentiated vertex integration is a function only of time at leading logarithm order. Hence a spatial derivative of it gives zero, whereas a time derivative would lower the number of infrared logarithms by one.

Even though passive fields do not cause infrared logarithms, they can still transmit an infrared logarithm acquired through interaction with an undifferentiated active field. Fig. 1 depicts a two loop contribution to the VEV of $F_{\mu\nu}(x)F_{\rho\sigma}(x)$ in which an infrared logarithm from the coincident scalar loop at x' is propagated through the virtual photon loop from x' to x .

Passive fields can also induce interactions between active fields. For example, the photon loop in Fig. 2 induces an effective $(\varphi^*\varphi)^2$ interaction, The same comments apply to differentiated active fields. For example, Fig. 3 shows the differentiated scalar and the photon of the 3-point vertex (78) inducing an interaction between undifferentiated active fields. This is part of the full 1PI 2-point function which has recently been computed at one loop order [50].

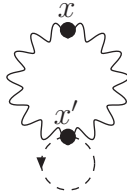


Figure 1: Two loop contribution to $\langle \Omega | F_{\mu\nu}(x) F_{\rho\sigma}(x) | \Omega \rangle$.

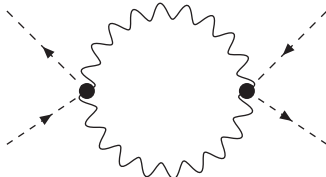


Figure 2: Effective $(\varphi^* \varphi)^2$ coupling in SQED.

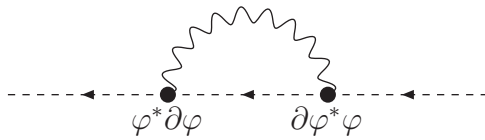


Figure 3: Effective $\varphi^* \varphi$ coupling in SQED.

3.2 Reaching Leading Logarithm Order

One can understand the relation between coupling constants and leading infrared logarithms directly from the Lagrangian. Consider a term in the potential involving N undifferentiated, active scalars, $V(\varphi) \sim c_N \varphi^N$. An elementary exercise in diagram topology reveals that it requires two such vertices to add $N-2$ loops to any diagram. Hence the loop counting parameter is $(C_N)^{\frac{2}{N-2}}$. Now consider the $2N$ extra fields associated with two N -point vertices. In the free field expansion some of these $2N$ fields would contribute to retarded Green's functions while others would remain as free fields. It requires two free fields to produce a retarded Green's function whose associated vertex integration can result in an infrared logarithm. An infrared logarithm can also come from the VEV of a pair of free fields. It follows that the $2N$ extra fields from two vertices can produce at most N infrared logarithms. Hence the leading logarithm contributions to any VEV represent an expansion in powers of the parameter,

$$C_N \varphi^N \quad \Longrightarrow \quad \left[C_N^2 \times \ln^N(a) \right]^{\frac{1}{N-2}}. \quad (64)$$

For $\lambda \varphi^3$ the series would be in powers of $\lambda^2 \ln^3(a)$; for $\lambda \varphi^4$ we have already seen that the series is in powers of $\lambda \ln^2(a)$; for $\lambda \varphi^5$ the series would be in powers of $\lambda^2 \ln^5(a)$, and so on.

Now consider a model which consists of an active field $\varphi(x)$ and a passive field $\psi(x)$ that interact through a potential of the form, $K \psi^\ell (\partial \varphi)^m \varphi^n$. Because the interaction contains $N = \ell + m + n$ fields, the addition of two vertices to any diagram increases the number of loops by $N - 2 = \ell + m + n - 2$. However, only the $2n$ active fields from these two vertices can contribute to infrared logarithms. Hence leading logarithm contributions to any VEV represent an expansion in the parameter,

$$K \psi^\ell (\partial \varphi)^m \varphi^n \quad \Longrightarrow \quad \left[K^2 \times \ln^n(a) \right]^{\frac{1}{\ell+m+n-2}}. \quad (65)$$

For example, the $\kappa h \partial h \partial h$ vertex of quantum gravity has $\ell = 0$, $m = 2$ and $n = 1$, which produces a series in powers of $\kappa^2 \ln(a)$. The same result follows for any of the $\kappa^n h^n \partial h \partial h$ interactions of quantum gravity.

A model whose leading logarithm solution has already been obtained is Yukawa theory [16],

$$\mathcal{L} = -\frac{1}{2}(1 + \delta Z) \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} - \frac{\delta \xi}{2} \varphi^2 R \sqrt{-g} - \frac{\delta \lambda}{4!} \varphi^4 \sqrt{-g}$$

$$+(1 + \delta Z_2) i \bar{\psi} e_b^\mu \gamma^b \left(\partial_\mu + \frac{i}{2} A_{\mu cd} J^{cd} \right) \psi \sqrt{-g} - (f + \delta f) \varphi \bar{\psi} \psi \sqrt{-g} . \quad (66)$$

The scalar φ is active whereas the massless fermion ψ is passive. The basic interaction vertex is $-f \varphi \bar{\psi} \psi \sqrt{-g}$ and comparison with (65) shows that leading logarithm contributions represent an expansion in powers of $f^2 \ln(a)$. Contrast this with a $-f \varphi^3 \sqrt{-g}$ vertex which would produce an expansion in powers of $f^2 \ln^3(a)$. So we see that *trading in an active field for a passive field, or for a differentiated active field, reduces the number of infrared logarithms per coupling constant. However, there will still be infrared logarithms as long as the vertex contains any undifferentiated active fields.*

3.3 The Role of the Ultraviolet

It is instructive to examine the behavior of counterterms in Yukawa theory. Recall that they completely drop out at leading logarithm order when only undifferentiated active fields are present. The various counterterms of Yukawa theory behave as follows,

$$\delta Z \sim f^2 \quad , \quad \delta Z_2 \sim f^2 \quad , \quad \delta f \sim f^3 \quad , \quad \delta \xi \sim f^2 \quad , \quad \delta \lambda \sim f^4 . \quad (67)$$

Field strength renormalization is irrelevant because the incorporation of either a δZ or a δZ_2 vertex would add a factor of f^2 without any undifferentiated active fields. The 3-point counterterm δf is also subleading because it carries three extra factors of f with only one undifferentiated active field. However, the one loop contribution to the conformal counterterm $\delta \xi$ adds a factor of f^2 with two undifferentiated active fields. These two fields could produce the extra $\ln(a)$ needed to remain at leading logarithm order. Similarly, the one loop contribution to the 4-point counterterm $\delta \lambda$ can also give leading logarithm corrections. Note that in neither case do we need to worry about higher loop counterterms because these would bring more factors of f^2 with no additional active fields. So we see that *leading logarithm order in Yukawa theory requires one loop conformal and 4-point counterterms, but no other counterterms, and no renormalization at all beyond one loop.*

Passive fields engender ultraviolet divergences, even at leading logarithm order, precisely because they do not produce infrared logarithms. What they give instead is factors of order one which multiply the infrared logarithms from active fields. It is important to understand that these factors of order one derive from the full range of the free field mode sum and from the full

structure of the mode functions. Therefore, it is not possible to stochastically simplify Heisenberg operator equations which contain passive fields. The same considerations apply to differentiated active fields.

3.4 Purging the Passive Fields

It might seem as if there is no tractable formalism that describes the leading logarithm limit of models which contain passive fields. This is false. By integrating out the passive fields one obtains effective field equations which involve only active fields. Although these equations are hideously nonlocal, we will explain how they are equivalent to simple, local equations at leading logarithm order.

Effective field equations are notoriously difficult on account of the field-dependent, inverse differential operators they involve. For example, consider integrating out a conformal scalar $\psi(x)$ whose Lagrangian is,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\psi\partial_\nu\psi g^{\mu\nu}\sqrt{-g} - \frac{1}{8}\left(\frac{D-2}{D-1}\right)\psi^2 R\sqrt{-g} - \frac{\lambda}{4}\varphi^2\psi^2\sqrt{-g}. \quad (68)$$

The resulting contribution to the $\varphi(x)$ equation of motion takes the form,

$$-\lambda\varphi(x)\left\langle x\left|\frac{i}{\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{4}\left(\frac{D-2}{D-1}\right)R\sqrt{-g} - \frac{\lambda}{2}\varphi^2\sqrt{-g}}\right|x\right\rangle, \quad (69)$$

where the quantity at the right is the coincidence limit of the ψ propagator in an arbitrary $\varphi(x)$ background.⁴ We shall probably never know this Green's function for arbitrary $\varphi(x)$. It can be expanded in terms of the conformal propagator $i\Delta_{\text{cf}}(x; x')$ for $\varphi = 0$,

$$\begin{aligned} \left\langle x\left|\frac{i}{\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{4}\left(\frac{D-2}{D-1}\right)R\sqrt{-g} - \frac{\lambda}{2}\varphi^2\sqrt{-g}}\right|x\right\rangle &= i\Delta_{\text{cf}}(x; x) \\ &- \frac{i\lambda}{2} \int d^D x' \sqrt{g(x')} \varphi^2(x') [i\Delta_{\text{cf}}(x; x')]^2 + \left(-\frac{i\lambda}{2}\right)^2 \int d^D x' \sqrt{-g(x')} \varphi^2(x') \\ &\quad \times \int d^D x'' \sqrt{-g(x'')} \varphi^2(x'') i\Delta_{\text{cf}}(x; x') i\Delta_{\text{cf}}(x'; x'') i\Delta_{\text{cf}}(x''; x) + \dots \quad (70) \end{aligned}$$

⁴It should properly be the coincidence limit of the $++$ propagator of the Schwinger-Keldysh formalism [51] so that it depends only upon fields $\varphi(x')$ in the past light-cone of x^μ . Giving this a proper explication would require a substantial digression which we shall forgo.

We do know $i\Delta_{\text{cf}}(x; x')$ — in conformal coordinates it is just $(aa')^{1-\frac{D}{2}}$ times the massless, flat space propagator — but there is no way of doing the integrals and summing the series for arbitrary $\varphi(x')$.

That we can obtain a tractable formalism at leading logarithm order derives from two facts:

1. To reach leading logarithm order, every active field must contribute to an infrared logarithm; and
2. Integrating over passive field Green's functions cannot produce infrared logarithms.

The first fact means we can ignore the spatial dependence of all the φ 's in expansion (70). Of course their temporal dependence matters because this is the ultimate source of infrared logarithms. However, integrating infrared logarithms against a passive field Green's function does not change the leading infrared logarithm. For example, compare the result of having infrared logarithms inside such an integral with the result of placing them outside,

$$\begin{aligned} & \int_0^t dt' a'^{D-1} \int d^{D-1} x' G_{\text{cf}}(x; x') \ln^N(a') \\ &= \frac{1}{H} \int_0^t dt' \left[\left(\frac{a'}{a}\right)^{\frac{D}{2}-1} - \left(\frac{a'}{a}\right)^{\frac{D}{2}} \right] \ln^N(a') = \frac{4 \ln^N(a)}{D(D-2)H^2} \left\{ 1 + O\left(\frac{1}{\ln(a)}\right) \right\}, \end{aligned} \quad (71)$$

$$\begin{aligned} & \ln^N(a) \int_0^t dt' a'^{D-1} \int d^{D-1} x' G_{\text{cf}}(x; x') \\ &= \frac{\ln^N(a)}{H} \int_0^t dt' \left[\left(\frac{a'}{a}\right)^{\frac{D}{2}-1} - \left(\frac{a'}{a}\right)^{\frac{D}{2}} \right] = \frac{4 \ln^N(a)}{D(D-2)H^2} \left\{ 1 + O\left(a^{1-\frac{D}{2}}\right) \right\}. \end{aligned} \quad (72)$$

To leading logarithm order there is no difference!

From the preceding discussion we see that the leading logarithms are not changed by moving all the φ 's of expansion (70) from inside the various integrals to outside,

$$\begin{aligned} & \langle x | \frac{i}{\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{4}(\frac{D-2}{D-1})R\sqrt{-g} - \frac{\lambda}{2}\varphi^2\sqrt{-g}} | x \rangle \longrightarrow i\Delta_{\text{cf}}(x; x) \\ & - \frac{i\lambda}{2}\varphi^2(x) \int d^D x' \sqrt{g(x')} [i\Delta_{\text{cf}}(x; x')]^2 + \left(-\frac{i\lambda}{2}\varphi^2(x)\right)^2 \int d^D x' \sqrt{-g(x')} \\ & \quad \times \int d^D x'' \sqrt{-g(x'')} i\Delta_{\text{cf}}(x; x') i\Delta_{\text{cf}}(x'; x'') i\Delta_{\text{cf}}(x''; x) + \dots \end{aligned} \quad (73)$$

This last expression can be recognized as the coincidence limit of the propagator of a conformally coupled scalar with mass $m^2 = \frac{\lambda}{2}\varphi^2(x)$. A compact result for this can be given in terms of the parameter $\nu \equiv \frac{1}{2}\sqrt{1 - 2\lambda\varphi^2(x)/H^2}$ [52, 53],

$$\langle x | \frac{i}{\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{1}{4}(\frac{D-2}{D-1})R\sqrt{-g} - \frac{\lambda}{2}\varphi^2\sqrt{-g}} | x \rangle$$

$$\rightarrow \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D-1}{2} + \nu)\Gamma(\frac{D-1}{2} - \nu)}{\Gamma(\frac{D}{2})} {}_2F_1\left(\frac{D-1}{2} + \nu, \frac{D-1}{2} - \nu; \frac{D}{2}; 1\right), \quad (74)$$

$$= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(1 - \frac{D}{2})\Gamma(\frac{D-1}{2} + \nu)\Gamma(\frac{D-1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu)\Gamma(\frac{1}{2} - \nu)}. \quad (75)$$

It remains to substitute (75) in (69) and recognize the result as minus the derivative of the unrenormalized, one loop effective potential,

$$-V'_{\text{eff}}(\varphi) = -\lambda\varphi(x) \times \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(1 - \frac{D}{2})\Gamma(\frac{D-1}{2} + \nu)\Gamma(\frac{D-1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu)\Gamma(\frac{1}{2} - \nu)}. \quad (76)$$

The factor of $\Gamma(1 - \frac{D}{2})$ represents a one loop, ultraviolet divergence which can be absorbed by renormalizing the quartic conformal and quartic self-couplings. The model that results is of the scalar potential type (8) already solved by Starobinskii.

A little reflection reveals that the technique we have just sketched must always work. Of course we can integrate out any subset of fields. The nonlocality engendered by doing this to passive fields will necessarily be restricted to passive field Green's functions. By definition of the field being passive, such Green's functions cannot produce infrared logarithms. Hence it is always valid to extract undifferentiated active fields from inside the nonlocal effective field equations. The result must always give minus the derivative of the unrenormalized effective potential.

A final point should be noted concerning expectation values of operators, in the original theory, which involve passive fields. In computing the leading logarithm result for such an expectation value one must functionally integrate out the passive fields. This will produce a potentially divergent expression involving only active fields. The stochastic expectation value of this expression can then be computed using the effective potential.

4 Effective Potential of SQED

The point of this paper is to obtain a leading logarithm solution for SQED and that is largely accomplished in this section. We begin by applying the procedure of section 3.2 to show that SQED gives a series in powers of $e^2 \ln(a)$ at leading logarithm order. We also work out which counterterms make leading log contributions. We then integrate out the photon field and renormalize the scalar effective potential. The section closes with large field and small field expansions of the effective potential.

We repeat the SQED Lagrangian (9) from the Introduction,

$$\begin{aligned} \mathcal{L} = & -(1 + \delta Z_2) (\partial_\mu - ieA_\mu)\varphi^* (\partial_\nu + ieA_\nu)\varphi g^{\mu\nu} \sqrt{-g} - \delta\xi \varphi^* \varphi R \sqrt{-g} \\ & - \frac{1}{4}(1 + \delta Z_3) F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} - \frac{\delta\lambda}{4} (\varphi^* \varphi)^2 \sqrt{-g} , \end{aligned} \quad (77)$$

The complex scalar $\varphi(x)$ is active whereas the photon $A_\mu(x)$ is passive. The primitive 3-point interaction involves a passive field, a differentiated active field and an undifferentiated active field, so we have the case of (64) with $\ell = m = n = 1$,

$$ieA_\mu [\varphi^* \partial_\nu \varphi - \partial_\nu \varphi^* \varphi] g^{\mu\nu} \sqrt{-g} \implies e^2 \ln(a) . \quad (78)$$

The primitive 4-point interaction represents $\ell = n = 2$ with $m = 0$, so the result is the same,

$$-e^2 A_\mu A_\nu \varphi^* \varphi g^{\mu\nu} \sqrt{-g} \implies e^2 \ln(a) . \quad (79)$$

The various counterterms have the following dependences upon e^2 ,

$$\delta Z_2 \sim e^2 \quad , \quad \delta Z_3 \sim e^2 \quad , \quad \delta\xi \sim e^2 \quad , \quad \delta\lambda \sim e^4 . \quad (80)$$

Hence field strength renormalization cannot contribute at leading logarithm order because the δZ_2 and δZ_3 interactions add a factor of e^2 with no undifferentiated active fields. On the other hand, the conformal and quartic counterterms correspond to $\ell = m = 0$, with $n = 2$ and $n = 4$, respectively. We therefore conclude that *leading logarithm SQED requires one loop conformal and 4-point counterterms, but no other counterterms and no renormalization at all beyond one loop.*

It is now time to integrate out the vector potential. Ever since the classic work of Coleman and Weinberg [54], it has been realized that this is greatly facilitated in Lorentz gauge,

$$\partial_\mu (\sqrt{-g} g^{\mu\nu} A_\nu) = 0 . \quad (81)$$

The simplifications turn out to be even greater in de Sitter background so we will follow the usual practice, although other gauges can of course be employed [55]. Dropping the field strength renormalizations, partially integrating, using the gauge condition and expanding the Lagrangian in powers of the vector potential gives,

$$\mathcal{L} \longrightarrow \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (82)$$

$$\mathcal{L}_0 = -\partial_\mu \varphi^* \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} - \delta\xi \varphi^* \varphi R \sqrt{-g} - \frac{\delta\lambda}{4} (\varphi^* \varphi)^2 \sqrt{-g}, \quad (83)$$

$$\mathcal{L}_1 = -[\partial_\mu \varphi^* \varphi - \varphi^* \partial_\mu \varphi] i e A_\nu g^{\mu\nu} \sqrt{-g} \equiv J^\nu A_\nu, \quad (84)$$

$$\mathcal{L}_2 = \frac{1}{2} A_\mu [\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}] A_\nu \sqrt{-g}. \quad (85)$$

Here $\square^{\mu\nu}$ is the vector d'Alembertian defined by $\square^{\mu\nu} A_\nu = A^{\mu;\rho}{}_\rho$. The effective action is determined by the equation,

$$e^{i\Gamma[\varphi^*, \varphi]} \equiv \int [dA_\mu] \delta[\partial_\mu(\sqrt{-g} g^{\mu\nu} A_\nu)] e^{iS[\varphi^*, \varphi, A]}. \quad (86)$$

Because \mathcal{L} is quadratic in the vector potential we can obtain the following explicit expression for $\Gamma[\varphi^*, \varphi]$,

$$\Gamma[\varphi^*, \varphi] = S_0[\varphi^*, \varphi] + \frac{i}{2} \ln \left\{ \det \left[\sqrt{-g} (\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}) \right] \right\} \\ \frac{i}{2} \int d^D x J^\mu(x) \int d^D y \langle x | \frac{i}{\sqrt{-g} [\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}]} | y \rangle J^\nu(y). \quad (87)$$

Expression (87) consists of three terms: the purely scalar parts of the bare action, a determinant factor, and the ‘‘current-current’’ term from completing the square in the exponential of the functional integral. One of the great things about Lorentz gauge is that this third term drops out at leading logarithm order. Recall from the previous section that leading logarithm contributions are unchanged by moving undifferentiated scalars around to different sides of inverse differential operators. But then a partial integration gives a gradient of Lorentz gauge propagator, which vanishes. For example, consider the leading logarithm contribution from the second term in $J^\mu(x)/ie$,

$$\int d^D x \varphi^*(x) \partial_\rho \varphi(x) \sqrt{-g(x)} g^{\rho\mu}(x) \langle x | \frac{i}{\sqrt{-g} [\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}]} | y \rangle \Big|_{\text{lead log}}$$

$$= \varphi^*(y) \int d^D x \partial_\rho \varphi(x) \sqrt{-g(x)} g^{\rho\mu}(x) \langle x | \frac{i}{\sqrt{-g}[\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}]} | y \rangle, \quad (88)$$

$$= -\varphi^* \int d^D x \varphi \partial_\rho \left[\sqrt{-g} g^{\rho\mu} \langle x | \frac{i}{\sqrt{-g}[\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}]} | y \rangle \right] = 0. \quad (89)$$

The same cancellation also occurs for the flat space effective potential [54].

The determinant term in (87) makes a leading logarithm contribution to the effective field equations, but only in the form of the coincidence limit of a massive photon propagator, with $m^2 \equiv 2e^2 \varphi^* \varphi$ treated as if it were constant. The massive photon propagator and its coincidence limit have recently been worked out in de Sitter background [56] and the result is,

$$\frac{\delta\Gamma[\varphi^*, \varphi]}{\delta\varphi^*(x)} \longrightarrow \frac{\delta S_0[\varphi^*, \varphi]}{\delta\varphi^*(x)} - e^2 \varphi(x) \sqrt{-g} g^{\mu\nu} \langle x | \frac{i}{\sqrt{-g}[\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu}]} | x \rangle, \quad (90)$$

$$\begin{aligned} &\longrightarrow \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) - \delta\xi \varphi R \sqrt{-g} - \frac{\delta\lambda}{2} \varphi^* \varphi^2 \sqrt{-g} - e^2 \varphi \sqrt{-g} g^{\mu\nu} \\ &\quad \times g_{\mu\nu} \left(\frac{D-1}{2} \right) \frac{H^2}{m^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2}) \Gamma(\frac{D+1}{2} + \nu) \Gamma(\frac{D+1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)} \right\}. \end{aligned} \quad (91)$$

The parameter ν has the following definition,

$$\nu \equiv \sqrt{\left(\frac{D-3}{2}\right)^2 - \frac{m^2}{H^2}} \longrightarrow \sqrt{\left(\frac{D-3}{2}\right)^2 - \frac{2e^2 \varphi^* \varphi}{H^2}}. \quad (92)$$

We can recognize the derivative of the unrenormalized effective potential from expression (102),

$$\begin{aligned} V'_{\text{eff}}(\varphi^* \varphi) &= \delta\xi D(D-1)H^2 + \frac{\delta\lambda}{2} \varphi^* \varphi + \frac{e^2}{2} D(D-1) \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \\ &\quad \times \frac{H^2}{m^2} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \Gamma\left(-\frac{D}{2}\right) \frac{\Gamma(\frac{D+1}{2} + \nu) \Gamma(\frac{D+1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)} \right\}. \end{aligned} \quad (93)$$

Renormalization is accomplished by first setting $D = 4 - \epsilon$ and expanding the final term,

$$\begin{aligned} \frac{H^2}{m^2} \left\{ \frac{\Gamma(3-\epsilon)}{\Gamma(3-\frac{\epsilon}{2})} - \Gamma\left(-2+\frac{\epsilon}{2}\right) \frac{\Gamma(\frac{5}{2} + \nu - \frac{\epsilon}{2}) \Gamma(\frac{5}{2} - \nu - \frac{\epsilon}{2})}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)} \right\} &= -\left(1 + \frac{e^2 \varphi^* \varphi}{H^2}\right) \frac{2}{\epsilon} \\ &+ \frac{1}{2} + \left(1 + \frac{e^2 \varphi^* \varphi}{H^2}\right) \left[\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) - \frac{3}{2} + \gamma \right] + O(\epsilon). \end{aligned} \quad (94)$$

The digamma function $\psi(z)$ has the following definition and expansions for small z and large z [48],

$$\psi(z) \equiv \frac{d}{dz} \ln(\Gamma(z)) , \quad (95)$$

$$= -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) (z-1)^{n-1} , \quad (96)$$

$$= \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} + O\left(\frac{1}{z^6}\right) . \quad (97)$$

We make the following choices for the two relevant counterterms,

$$\delta\xi = \frac{e^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{1}{4-D} + \frac{\gamma}{2} + O(D-4) \right\} , \quad (98)$$

$$\delta\lambda = \frac{(D-1)D e^4 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{2}{4-D} + \gamma - \frac{3}{2} + O(D-4) \right\} . \quad (99)$$

Substituting (94) and (98-99) in (93) and taking the limit $\epsilon \rightarrow 0$ gives,

$$V'_{\text{eff}}(\varphi^* \varphi) = \frac{e^2}{H^2} \times \frac{3H^4}{8\pi^2} \left\{ -1 + 2\gamma + (-3 + 2\gamma) \frac{e^2 \varphi^* \varphi}{H^2} \right. \\ \left. + \left(1 + \frac{e^2 \varphi^* \varphi}{H^2} \right) \left[\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) \right] \right\} . \quad (100)$$

Of course the divergent parts of $\delta\xi$ and $\delta\lambda$ are fixed. Our choices for the finite parts are motivated to make the $(\varphi^* \varphi)^0$ and $(\varphi^* \varphi)^1$ terms in $V'_{\text{eff}}(\varphi \varphi^*)$ vanish, which keeps the scalar light as long as possible. Interestingly, the same choice for $\delta\xi$ cancels the leading infrared logarithm in the two loop expectation value of $\varphi^*(x)\varphi(x)$ [13], and also keeps the one loop scalar mode functions from receiving any significant late time correction [57].

It remains to work out the effective potential and expand it for large and small field strengths. From (100) and the definition (92) of ν we see that the result depends upon the combination,

$$z \equiv \frac{e^2 \varphi^* \varphi}{H^2} . \quad (101)$$

Integrating (100) gives,

$$V_{\text{eff}} = \frac{3H^4}{8\pi^2} \left\{ (-1 + 2\gamma)z + \left(-\frac{3}{2} + \gamma\right)z^2 \right. \\ \left. + \int_0^z dx (1+x) \left[\psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-8x}\right) + \psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-8x}\right) \right] \right\} . \quad (102)$$

An explicit power series expansion can be obtained for V_{eff} in terms of the parameter,

$$\Delta z \equiv \frac{1}{2} - \frac{1}{2}\sqrt{1-8z} = 2z + 4z^2 + 16z^3 + O(z^4). \quad (103)$$

Substituting (96) in (102) and performing the integral gives,

$$V_{\text{eff}} = \frac{3H^4}{8\pi^2} \left\{ \frac{1}{2} \ln(1 - \Delta z) + \frac{1}{2} \Delta z + \frac{1}{4} \Delta z^2 + \frac{7}{12} \Delta z^3 - \frac{3}{8} \Delta z^4 \right. \\ \left. + \sum_{m=1}^{\infty} \zeta(2m+1) \left[-\frac{\Delta z^{2m+1}}{2m+1} + \frac{\frac{3}{2} \Delta z^{2m+2}}{2m+2} + \frac{\frac{3}{2} \Delta z^{2m+3}}{2m+3} - \frac{\Delta z^{2m+4}}{2m+4} \right] \right\}, \quad (104)$$

$$= \left[\frac{5}{12} - \frac{1}{3} \zeta(3) \right] \Delta z^3 - \left[\frac{1}{2} - \frac{3}{8} \zeta(3) \right] \Delta z^4 + O(\Delta z^5). \quad (105)$$

As already stated, our choices for the finite parts of $\delta\xi$ and $\delta\lambda$ cancel the order z and z^2 terms in the small field expansion.

The large field expansion derives from substituting (97) in (102),

$$V_{\text{eff}} = \frac{3H^4}{8\pi^2} \left\{ \frac{1}{2} z^2 \ln(z+1) + \left[-\frac{7}{4} + \frac{1}{2} \ln(2) + \gamma \right] z^2 + z \ln(z+1) \right. \\ \left. + \left[-\frac{13}{6} + \ln(2) + 2\gamma \right] z + \frac{19}{60} \ln(z+1) + O(1) \right\}. \quad (106)$$

Because z grows for small H , as well as for large $\varphi^*\varphi$, the $(\varphi^*\varphi)^2 \ln(\varphi^*\varphi)$ term in (106) should also agree with the flat space result of Coleman and Weinberg [54, 58]. From equation (4.5) of their paper we see that it does. The full asymptotic expansion of (106) up to order one also gives a generally accurate approximation of the potential, even for $z < 1$, as can be seen from Fig. 4 and Fig. 5.

5 SQED Stress Tensor

The stress tensor of SQED is a composite operator which involves the passive field A_μ . One must therefore integrate A_μ out and simplify the resulting functional of φ^* and φ before its expectation value can be computed stochastically. That is the task of this section. As a bonus we obtain independent results for the two gauge invariant operators which principally comprise $T_{\mu\nu}$:

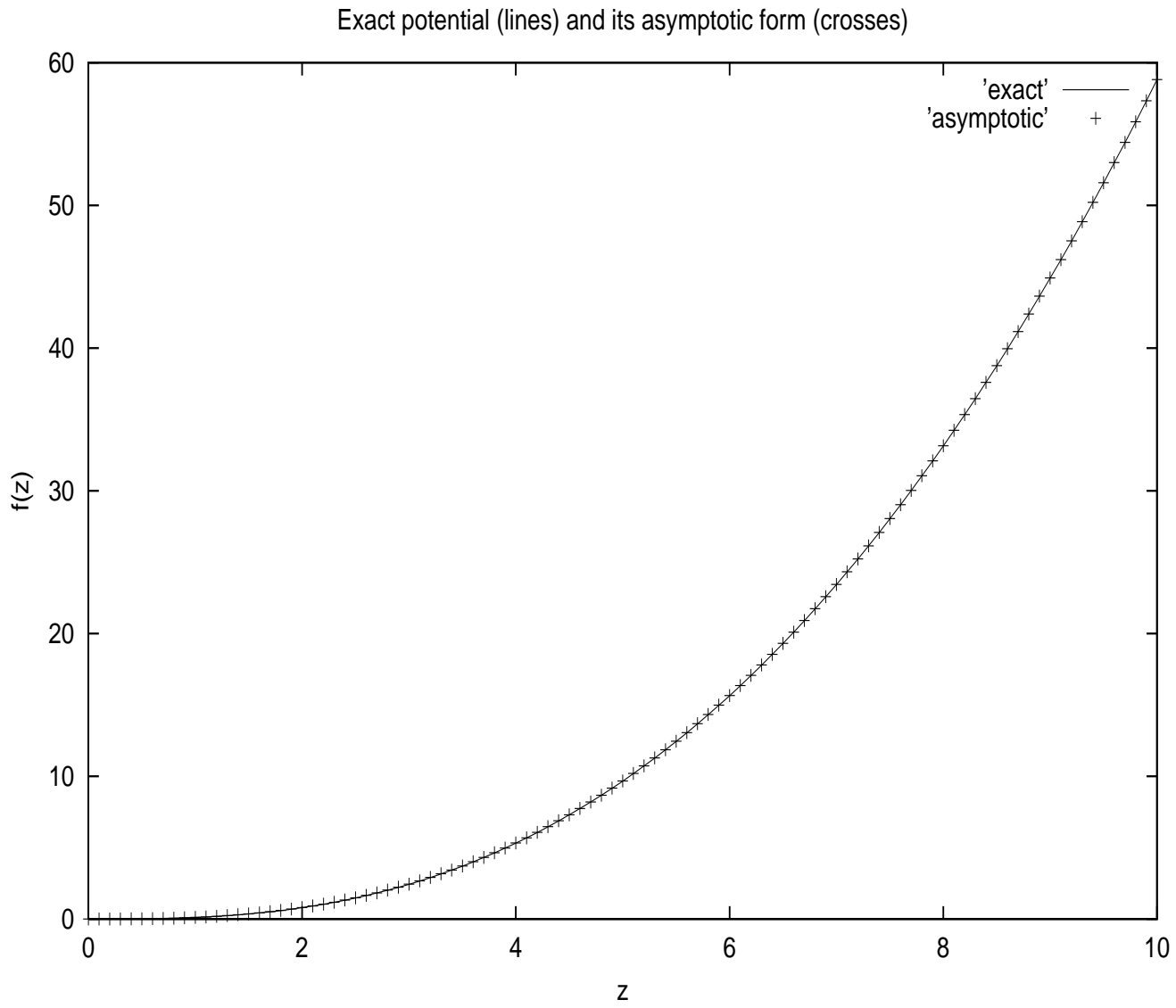


Figure 4: $V_{\text{eff}} = \frac{3H^4}{8\pi^2} f\left(\frac{e^2 \phi^* \varphi}{H^2}\right)$ (solid line) and its asymptotic form (crosses).

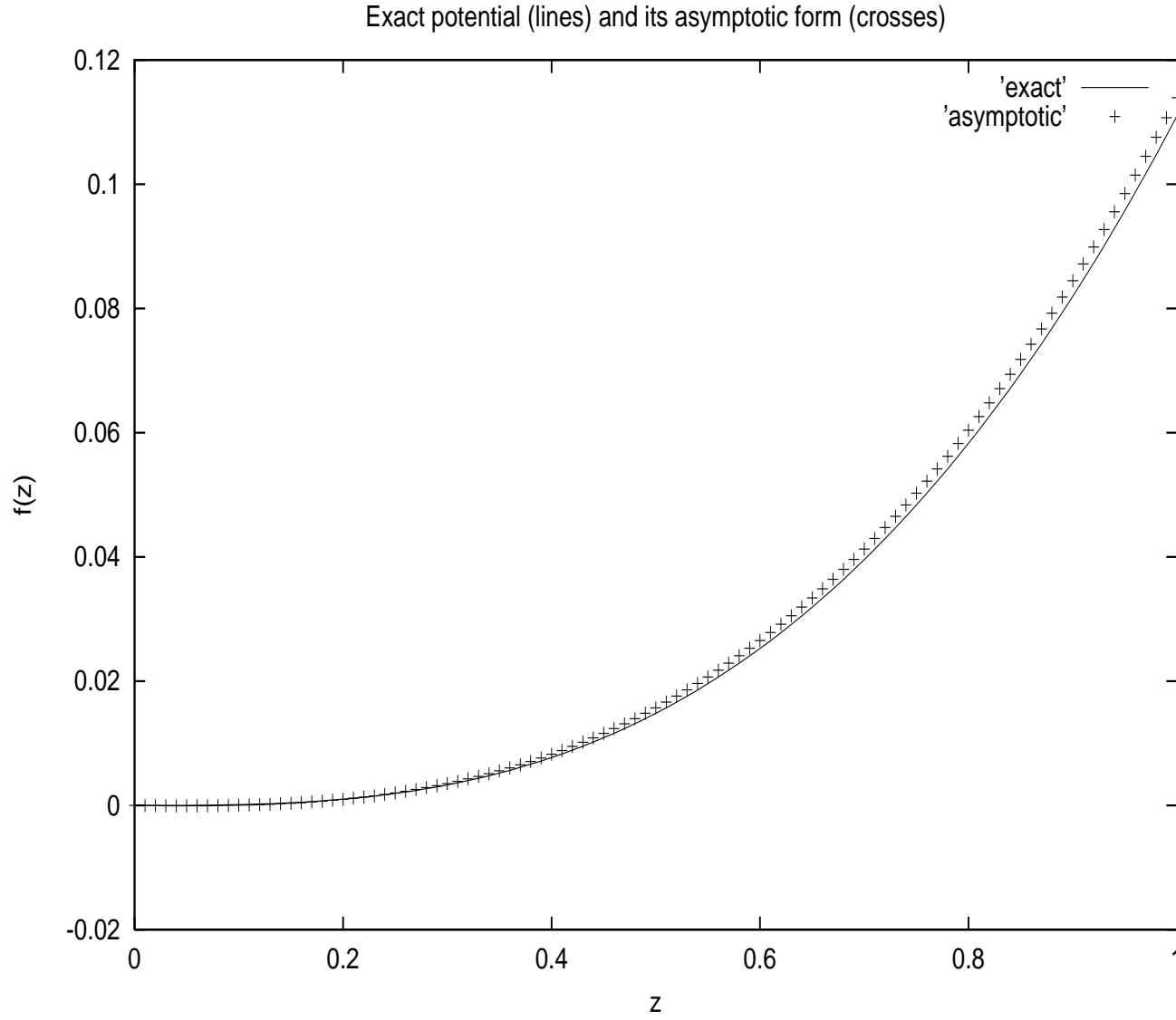


Figure 5: Expanded small field behavior of $V_{\text{eff}} = \frac{3H^4}{8\pi^2} f\left(\frac{e^2\phi^* \varphi}{H^2}\right)$ (solid line) and its asymptotic form (crosses).

the field strength bilinear and the scalar kinetic bilinear. The section closes with a discussion of the curious fact that the simplified stress tensor is not quite $-g_{\mu\nu}V_{\text{eff}}(\varphi^*\varphi)$.

Recall from the previous section that field strength renormalization counterterms do not contribute at leading logarithm order whereas the conformal and quartic counterterms do. The relevant part of the SQED stress tensor is therefore,

$$T_{\mu\nu} = \left[\delta_\mu^\alpha \delta_\nu^\rho - \frac{1}{4} g_{\mu\nu} g^{\alpha\rho} \right] g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} + \left[\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\nu^\rho - g_{\mu\nu} g^{\rho\sigma} \right] (D_\rho \varphi)^* D_\sigma \varphi \\ + 2\delta\xi \left[\varphi^* \varphi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + g_{\mu\nu} (\varphi^* \varphi)^{;\rho}{}_\rho - (\varphi^* \varphi)_{;\mu\nu} \right] - \frac{\delta\lambda}{4} (\varphi^* \varphi)^2 g_{\mu\nu}, \quad (107)$$

where the covariant derivative is $D_\mu \varphi \equiv (\partial_\mu \varphi + ie A_\mu \varphi)$. In de Sitter background the Ricci tensor is $R_{\mu\nu} = (D-1)H^2 g_{\mu\nu}$, and differentiated scalars with the same power of e^2 as undifferentiated scalars are guaranteed to be subleading logarithm. Hence we can simplify the counterterms to,

$$2\delta\xi \left[\varphi^* \varphi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + g_{\mu\nu} (\varphi^* \varphi)^{;\rho}{}_\rho - (\varphi^* \varphi)_{;\mu\nu} \right] - \frac{\delta\lambda}{4} (\varphi^* \varphi)^2 g_{\mu\nu} \\ \longrightarrow - \left[(D-1)(D-2)\delta\xi H^2 \varphi^* \varphi + \frac{\delta\lambda}{4} (\varphi^* \varphi)^2 \right] g_{\mu\nu}. \quad (108)$$

It remains to integrate the vector potential out of the field strength and scalar kinetic bilinears,

$$F_{\alpha\beta} F_{\rho\sigma} \quad \text{and} \quad (D_\rho \varphi)^* D_\sigma \varphi. \quad (109)$$

Consider the general case of integrating the vector potential out of some operator $\mathcal{O}[\varphi^*, \varphi, A]$ to obtain a new operator $\tilde{\mathcal{O}}[\varphi^*, \varphi]$ depending only upon the scalar,

$$\int [dA_\mu] \delta \left[\partial_\mu (\sqrt{-g} g^{\mu\nu} A_\nu) \right] e^{iS[\varphi^*, \varphi, A]} \times \mathcal{O}[\varphi^*, \varphi, A] = e^{i\Gamma[\varphi^*, \varphi]} \times \tilde{\mathcal{O}}[\varphi^*, \varphi]. \quad (110)$$

The functional integration is trivial because the Lagrangian of SQED is quadratic in the vector potential. After some partial integrations and applications of the Lorentz gauge condition (81), it consists of a part \mathcal{L}_0 which depends only upon the scalar, a linear part of the form $\mathcal{L}_1 = J^\nu A_\nu$, and a quadratic part $\mathcal{L}_2 = \frac{1}{2} A_\mu \mathcal{D}^{\mu\nu} A_\nu$. We can read off the current J^ν from (84) and the differential operator $\mathcal{D}^{\mu\nu}$ from (85),

$$J^\nu \equiv -ie \left[\partial_\mu \varphi^* \varphi - \varphi^* \partial_\mu \varphi \right] g^{\mu\nu} \sqrt{-g}, \quad (111)$$

$$\mathcal{D}^{\mu\nu} \equiv \sqrt{-g} \left[\square^{\mu\nu} - R^{\mu\nu} - 2e^2 \varphi^* \varphi g^{\mu\nu} \right]. \quad (112)$$

One evaluates the functional integral (110) by completing the square,

$$\frac{1}{2}A_\mu \mathcal{D}^{\mu\nu} A_\nu + A_\nu J^\nu = \frac{1}{2} \left[A_\mu + \frac{1}{\mathcal{D}^{\mu\rho}} J^\rho \right] \mathcal{D}^{\mu\nu} \left[A_\nu + \frac{1}{\mathcal{D}^{\nu\sigma}} J^\sigma \right] - \frac{1}{2} J^\mu \frac{1}{\mathcal{D}^{\mu\nu}} J^\nu. \quad (113)$$

The operator $\tilde{\mathcal{O}}[\varphi^*, \varphi]$ will therefore be the sum of terms from $\mathcal{O}[\varphi^*, \varphi, A]$ in which all combinations of the following replacements are made,

$$A_\mu(x) \longrightarrow - \int d^D x' \langle x | \frac{1}{\mathcal{D}^{\mu\nu}} | x' \rangle J^\nu(x'), \quad (114)$$

$$A_\mu(x) A_\nu(x') \longrightarrow \langle x | \frac{i}{\mathcal{D}^{\mu\nu}} | x' \rangle. \quad (115)$$

As we saw from expressions (88-89) of the previous section, the Lorentz gauge condition means that there is never a leading logarithm contribution from replacement (114). Hence we need only consider replacement (115).

At this stage we must digress to discuss the photon propagator. We shall never know the inverse of the differential operator (112) for arbitrary $\varphi(x)$. However, leading logarithm results only involve this operator evaluated for the special case where $2e^2\varphi^*(x)\varphi(x)$ is a constant we shall call m^2 . That Green's function we do know [56]. It can be expressed in terms of the de Sitter invariant function of conformal coordinates $x^\mu = (\eta, \vec{x})$,

$$y(x; x') \equiv aa' H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\epsilon)^2 \right]. \quad (116)$$

The massive, Lorentz gauge photon propagator takes the form [56],

$$\langle x | \frac{i}{\mathcal{D}^{\mu\nu}} | x' \rangle = B(y) \frac{\partial^2 y}{\partial x^\mu \partial x'^\nu} + C(y) \frac{\partial y}{\partial x^\mu} \frac{\partial y}{\partial x'^\nu}. \quad (117)$$

The functions $B(y)$ and $C(y)$ can be expressed in terms of a single function $\gamma(y)$,

$$B(y) \equiv \frac{1}{4(D-1)H^2} \left[-(4y - y^2)\gamma'(y) - (D-1)(2-y)\gamma(y) \right], \quad (118)$$

$$C(y) \equiv \frac{1}{4(D-1)H^2} \left[(2-y)\gamma'(y) - (D-1)\gamma(y) \right]. \quad (119)$$

The function $\gamma(y)$ is,

$$\begin{aligned} \gamma(y) = & - \left(\frac{D-1}{2} \right) \frac{H^2}{m^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} {}_2F_1 \left(D-1, 2; \frac{D}{2}+1; 1-\frac{y}{4} \right) \right. \\ & \left. + \frac{\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{D}{2}+1)} {}_2F_1 \left(\frac{D+1}{2}+\nu, \frac{D+1}{2}-\nu; \frac{D}{2}+1; 1-\frac{y}{4} \right) \right\}. \quad (120) \end{aligned}$$

Here the parameter ν is,

$$\nu \equiv \sqrt{\left(\frac{D-3}{2}\right)^2 - \frac{m^2}{H^2}}. \quad (121)$$

Because our work is limited to coincidence limits we require only the integer powers in the Laurent expansion of $\gamma(y)$,

$$\begin{aligned} \gamma(y) = & -\left(\frac{D-1}{2}\right) \frac{H^2}{m^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \left\{ -(n+1) \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2}+1)} \left(\frac{y}{4}\right)^n \right. \\ & \left. + \frac{\Gamma(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{\Gamma(n+\frac{D}{2}+1)n!} \frac{\Gamma(n+\frac{D+1}{2}+\nu)\Gamma(n+\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \left(\frac{y}{4}\right)^n + O\left(y^{n-\frac{D}{2}+1}\right) \right\}. \end{aligned} \quad (122)$$

When $x'^{\mu} = x^{\mu}$, the function $y(x; x')$ vanishes. Because dimensional regularization ignores all D -dependent powers of zero, the coincidence limits of $\gamma(y)$ and its derivatives derive from factors of $y^0 = 1$. The two we require are,

$$\gamma(0) = \left(\frac{D-1}{2}\right) \frac{H^2}{m^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \right\}, \quad (123)$$

$$\gamma'(0) = \frac{(D-1)^2}{2(D+2)} \frac{H^2}{m^2} \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+3}{2}+\nu)\Gamma(\frac{D+3}{2}-\nu)}{2(D-1)\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \right\}. \quad (124)$$

And the two coincidence limits we need of the photon propagator are,

$$\lim_{x' \rightarrow x} \langle x | \frac{i}{\mathcal{D}^{\mu\nu}} | x' \rangle = \gamma(0) g_{\mu\nu}, \quad (125)$$

$$\begin{aligned} \lim_{x' \rightarrow x} D_{\rho} D'_{\sigma} \langle x | \frac{i}{\mathcal{D}^{\mu\nu}} | x' \rangle = & H^2 \left[-2 \left(\frac{D+1}{D-1} \right) \gamma'(0) + \gamma(0) \right] g_{\mu\nu} g_{\rho\sigma} \\ & + H^2 \left[\frac{2}{D-1} \gamma'(0) \right] g_{\mu\rho} g_{\nu\sigma} + H^2 \left[\frac{2}{D-1} \gamma'(0) - \gamma(0) \right] g_{\mu\sigma} g_{\nu\rho}. \end{aligned} \quad (126)$$

Here D_{ρ} is the covariant derivative operator defined by $D_{\rho} A_{\mu} \equiv A_{\mu;\rho}$.

We can now integrate the vector potential out of the field strength and scalar kinetic bilinears. First, recall that the ordinary derivatives in the field strength tensor can be replaced by covariant derivatives,

$$F_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} = D_{\mu} A_{\nu} - D_{\nu} A_{\mu}. \quad (127)$$

Combining this with relations (126) and (123-124), and with our earlier insight about the J^{ν} terms dropping, we see that the leading logarithm form

of the field strength bilinear is,

$$\begin{aligned} & \int [dA_\mu] \delta \left[\partial_\mu (\sqrt{-g} g^{\mu\nu} A_\nu) \right] e^{iS[\varphi^*, \varphi, A]} \times F_{\alpha\beta}(x) F_{\rho\sigma}(x) \\ & \longrightarrow e^{i\Gamma[\varphi^*, \varphi]} \times \lim_{x' \rightarrow x} \left\{ D_\alpha D'_\rho \langle x | \frac{i}{\mathcal{D}\beta\sigma} | x' \rangle - D_\alpha D'_\sigma \langle x | \frac{i}{\mathcal{D}\beta\rho} | x' \rangle \right. \\ & \qquad \qquad \qquad \left. - D_\beta D'_\rho \langle x | \frac{i}{\mathcal{D}\alpha\sigma} | x' \rangle + D_\beta D'_\sigma \langle x | \frac{i}{\mathcal{D}\alpha\rho} | x' \rangle \right\}, \end{aligned} \quad (128)$$

$$= e^{i\Gamma[\varphi^*, \varphi]} \times (g_{\alpha\rho} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\rho}) H^2 \left[-4 \left(\frac{D+2}{D-1} \right) \gamma'(0) + 4\gamma(0) \right], \quad (129)$$

$$= e^{i\Gamma[\varphi^*, \varphi]} \times (g_{\alpha\rho} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\rho}) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-\frac{D}{2}) \Gamma(\frac{D+1}{2} + \nu) \Gamma(\frac{D+1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)}. \quad (130)$$

The analogous result for the scalar kinetic bilinear is,

$$\begin{aligned} & e^{-i\Gamma[\varphi^*, \varphi]} \times \int [dA_\mu] \delta \left[\partial_\mu (\sqrt{-g} g^{\mu\nu} A_\nu) \right] e^{iS[\varphi^*, \varphi, A]} \times (D_\rho \varphi(x))^* D_\sigma \varphi(x) \\ & \longrightarrow e^2 \varphi^*(x) \varphi(x) \times \langle x | \frac{i}{\mathcal{D}\rho\sigma} | x \rangle, \end{aligned} \quad (131)$$

$$= e^2 \varphi^* \varphi \times g_{\rho\sigma} \gamma(0), \quad (132)$$

$$= g_{\rho\sigma} \left(\frac{D-1}{4} \right) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2}) \Gamma(\frac{D+1}{2} + \nu) \Gamma(\frac{D+1}{2} - \nu)}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)} \right\}. \quad (133)$$

Before computing the stress tensor we should comment on the explicit perturbative computations which have been done to check the field strength bilinear (130) and the scalar kinetic bilinear (133). Of course there is no way to check the nonperturbative information these expressions contain! However, we can compare against explicit one and two loop computations by expanding the parameter ν in powers of $m^2 = 2e^2 \varphi^* \varphi$,

$$\nu = \sqrt{\left(\frac{D-3}{2} \right)^2 - \frac{m^2}{H^2}} \equiv \left(\frac{D-3}{2} \right) - \Delta\nu = \left(\frac{D-3}{2} \right) - \frac{1}{D-3} \frac{m^2}{H^2} + O\left(\frac{m^4}{H^4} \right). \quad (134)$$

From (130) we see that the leading logarithm result for the field strength bilinear should be,

$$\langle F_{\alpha\beta}(x) F_{\rho\sigma}(x) \rangle_{\text{lead log}}$$

$$= (g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\rho}) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \Gamma\left(-\frac{D}{2}\right) \left\langle \frac{\Gamma(D-1-\Delta\nu)\Gamma(2+\Delta\nu)}{\Gamma(\frac{D}{2}-1-\Delta\nu)\Gamma(2-\frac{D}{2}+\Delta\nu)} \right\rangle, \quad (135)$$

$$= (g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\rho}) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} \left\{ 1 + \left[-\psi(D-1) + \psi(2) + \psi\left(\frac{D}{2}-1\right) - \psi\left(2-\frac{D}{2}\right) \right] \langle \Delta\nu \rangle + \langle O(\Delta\nu^2) \rangle \right\}. \quad (136)$$

Now substitute the leading stochastic result using relation (24),

$$\langle \Delta\nu \rangle = \frac{2e^2}{(D-3)H^2} \langle \varphi^* \varphi \rangle + O(e^4 \langle (\varphi^* \varphi)^2 \rangle), \quad (137)$$

$$\longrightarrow \frac{4e^2 H^{D-4}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{(D-3)\Gamma(\frac{D}{2})} \ln(a) + O(e^4 \ln^2(a)). \quad (138)$$

Because the two loop result is divergent we report only the divergent part,

$$\begin{aligned} & \langle F_{\alpha\beta}(x) F_{\rho\sigma}(x) \rangle_{\text{lead log}} \\ &= (g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\sigma}g_{\beta\rho}) \frac{H^4}{16\pi^2} \left\{ 1 - \frac{1}{D-4} \times \frac{e^2}{\pi^2} \ln(a) + O(e^4 \ln^2(a)) \right\}. \end{aligned} \quad (139)$$

The one loop (order one) result is trivial and is not, in any case, a check of the technique because it contains no infrared logarithm. The two loop (order e^2) result agrees with the diagram given in Fig. 1 but we have not yet proved that the other two loop diagrams fail to contribute divergent leading logarithms.

The scalar kinetic bilinear has been more thoroughly checked. Our stochastic prediction for it is,

$$\begin{aligned} & \langle (D_\rho \varphi(x))^* D_\sigma \varphi(x) \rangle_{\text{lead log}} = g_{\rho\sigma} \left(\frac{D-1}{4} \right) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \\ & \quad \times \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \Gamma\left(-\frac{D}{2}\right) \left\langle \frac{\Gamma(D-1-\Delta\nu)\Gamma(2+\Delta\nu)}{\Gamma(\frac{D}{2}-1-\Delta\nu)\Gamma(2-\frac{D}{2}+\Delta\nu)} \right\rangle \right\}, \quad (140) \\ &= g_{\rho\sigma} \left(\frac{D-1}{4} \right) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} \left\{ \left[\psi(D-1) - \psi(2) - \psi\left(\frac{D}{2}-1\right) + \psi\left(2-\frac{D}{2}\right) \right] \langle \Delta\nu \rangle + \langle O(\Delta\nu^2) \rangle \right\}, \quad (141) \end{aligned}$$

$$\begin{aligned} \longrightarrow g_{\rho\sigma} \frac{e^2 H^{2D-4}}{(4\pi)^D} \frac{\Gamma(D-1)\Gamma(D)}{(D-3)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)} & \left[\psi\left(2-\frac{D}{2}\right) \right. \\ & \left. - \psi\left(\frac{D}{2}-1\right) + \psi(D-1) - \psi(2) \right] \ln(a) + O\left(e^4 \ln^2(a)\right). \end{aligned} \quad (142)$$

This agrees exactly, and for arbitrary dimension D , with the infrared logarithm in equation (146) of our recent, explicit two loop computation of the scalar kinetic bilinear [13].

We can now assemble the various constituents of the stress tensor. From (108) and (98) we that the conformal counterterm contributes,

$$-(D-1)(D-2)\delta\xi H^2 \varphi^* \varphi g_{\mu\nu} = -g_{\mu\nu} \times (D-1)(D-2) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{z}{\epsilon} + \frac{\gamma z}{2} + O(\epsilon) \right\}. \quad (143)$$

Recall that $z \equiv e^2 \varphi^* \varphi / H^2$ and $\epsilon \equiv 4 - D$. We shall keep the same form as (143) for the each of the four terms. From (108) and (99) the $(\varphi^* \varphi)^2$ counterterm gives,

$$-\frac{\delta\lambda}{4} (\varphi^* \varphi)^2 g_{\mu\nu} = -g_{\mu\nu} \times (D-1)(D-2) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{z^2}{\epsilon} + \frac{\gamma-1}{2} z^2 + O(\epsilon) \right\}. \quad (144)$$

The field strength contribution results from combining (107) with (130), and it turns out to be finite,

$$\begin{aligned} \left[\delta_\mu^\alpha \delta_\nu^\rho - \frac{1}{4} g_{\mu\nu} g^{\alpha\rho} \right] g^{\beta\sigma} \times \left(g_{\alpha\rho} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\rho} \right) \\ \times \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}, \end{aligned} \quad (145)$$

$$= \frac{1}{4} (D-4)(D-1) g_{\mu\nu} \frac{H^D}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}, \quad (146)$$

$$= -g_{\mu\nu} \times (D-1)(D-2) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{0}{\epsilon} - \frac{1}{2} (z + z^2) + O(\epsilon) \right\}. \quad (147)$$

The contribution of the scalar kinetic term comes from substituting (133) in (107) and then making use of the same expansion (94) as for the effective potential,

$$\left[\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\nu^\rho - g_{\mu\nu} g^{\rho\sigma} \right] \times g_{\rho\sigma}$$

$$\times \left(\frac{D-1}{4}\right) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \right\}, \quad (148)$$

$$= -\frac{(D-2)(D-1)}{4} \frac{g_{\mu\nu}H^D}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} - \frac{\Gamma(-\frac{D}{2})\Gamma(\frac{D+1}{2}+\nu)\Gamma(\frac{D+1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \right\}, \quad (149)$$

$$= -g_{\mu\nu} \times (D-1)(D-2) \frac{H^D}{(4\pi)^{\frac{D}{2}}} \left\{ -\left(\frac{z+z^2}{\epsilon}\right) + \left(-\frac{1}{2} + \frac{\gamma}{2}\right)z + \left(-\frac{3}{4} + \frac{\gamma}{2}\right)z^2 + \frac{1}{2}(z+z^2) \left[\psi\left(\frac{3}{2}+\nu\right) + \psi\left(\frac{3}{2}-\nu\right) \right] + O(\epsilon) \right\}. \quad (150)$$

Of course the divergences in (143) and (144) cancel those in (150), at which point we can take $D = 4$. The final result has the form $-g_{\mu\nu}V_s(z)$ where,

$$V_s = \frac{3H^4}{8\pi^2} \left\{ (-1 + \gamma)z + \left(-\frac{7}{4} + \gamma\right)z^2 + \frac{1}{2}z(1+z) \left[\psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-8z}\right) + \psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-8z}\right) \right] \right\}. \quad (151)$$

It is apparent from Fig. 6 that the stress tensor potential $V_s(\varphi^*\varphi)$ does not quite agree with the effective potential $V_{\text{eff}}(\varphi^*\varphi)$. This same sort of disagreement was also noted in the recent leading logarithm solution of Yukawa theory [16]. In both models the difference arises because the two potentials describe different physical processes: V_{eff} controls the scalar's evolution whereas V_s controls the gravitational back-reaction. The two are distinct because almost all the factors of H^2 in V_{eff} are $R/12$ for a general metric, and this changes the stress tensor, even in de Sitter background and at leading logarithm order.

To see the point, consider a contribution to the matter Lagrangian of the form,

$$\Delta\mathcal{L} = -F(R)\sqrt{-g}. \quad (152)$$

The corresponding contribution to the stress tensor is,

$$\Delta T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta\Delta S}{\delta g^{\mu\nu}}, \quad (153)$$

$$= 2R_{\mu\nu}F'(R) - g_{\mu\nu}F(R) + 2g_{\mu\nu}F(R)^{;\rho}{}_{\rho} - 2F(R)_{;\mu\nu}. \quad (154)$$

In $D = 4$ de Sitter background $R_{\mu\nu} = g_{\mu\nu}R/4$, and we can ignore the derivative terms at leading logarithm order,

$$\Delta T_{\mu\nu} \longrightarrow -g_{\mu\nu} \left[F(R) - \frac{1}{2}RF'(R) \right]. \quad (155)$$

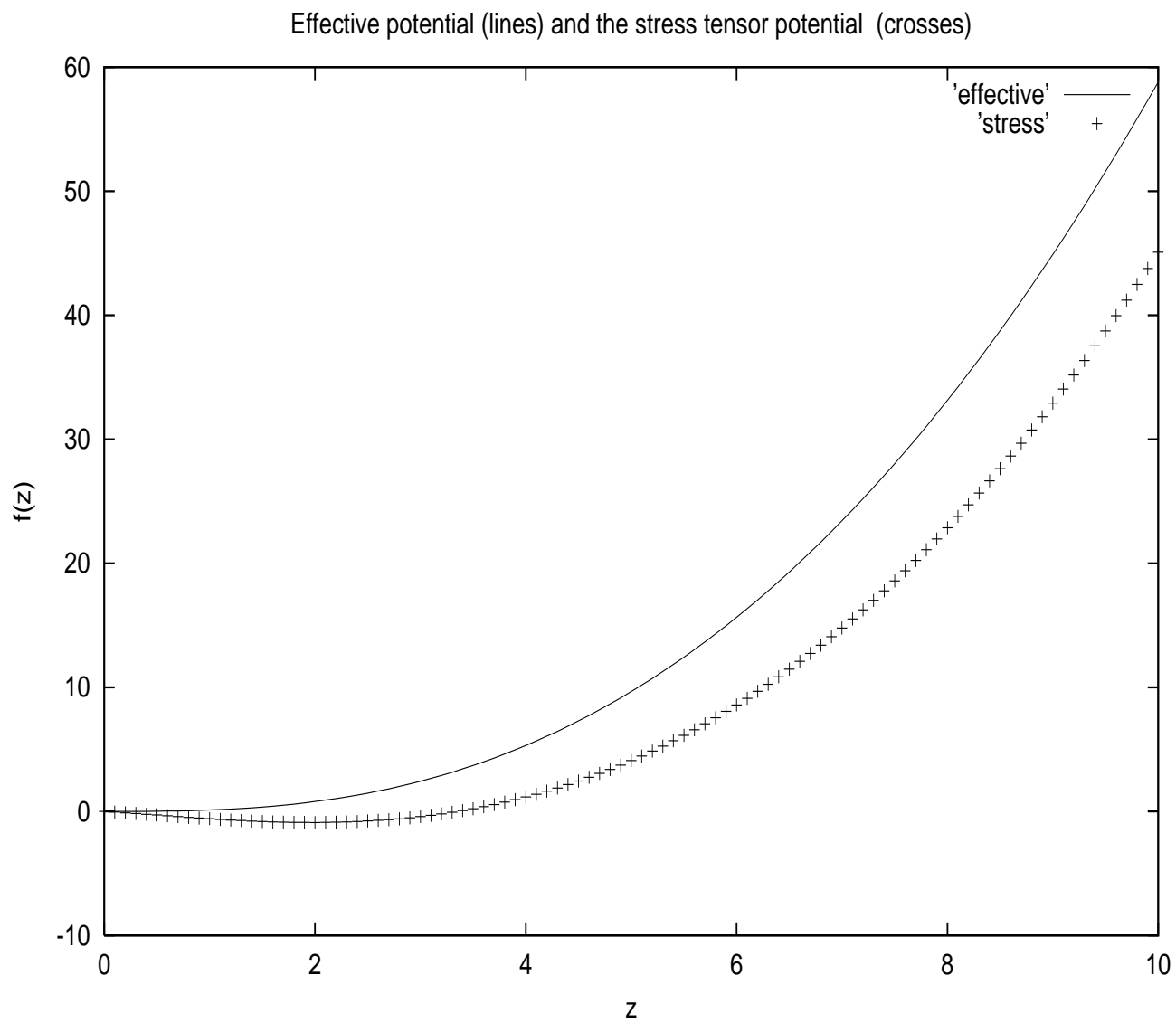


Figure 6: Effective potential $V_{\text{eff}} \equiv \frac{3H^2}{8\pi^2} f(z)$ (in solid lines) versus stress potential $V_s \equiv \frac{3H^2}{8\pi^2} f_s(z)$ (crosses).

If all the factors of H^2 in V_{eff} were $R/12$ for a general metric, then for de Sitter background we would have,

$$F(R) = \frac{3}{8\pi^2} \left(\frac{R}{12}\right)^2 f\left(\frac{12e^2\varphi^*\varphi}{R}\right) \implies \left[F(R) - \frac{1}{2}RF'(R)\right]_{\text{dS}} = \frac{3H^4}{8\pi^2} \times \frac{1}{2}zf'(z). \quad (156)$$

The actual relation between V_{eff} and V_s is tantalizingly close to (156). If we extract a factor of $3H^4/8\pi^2$ from each potential,

$$V_{\text{eff}}(\varphi^*\varphi) \equiv \frac{3H^4}{8\pi^2} f(z) \quad \text{and} \quad V_s(\varphi^*\varphi) \equiv \frac{3H^4}{8\pi^2} f_s(z), \quad (157)$$

then comparison of expressions (102) and (151) reveals the following relation between the two dimensionless functions of z ,

$$f_s(z) = \frac{1}{2}zf'(z) - \frac{1}{2}z - \frac{1}{4}z^2. \quad (158)$$

The reason for the extra contribution of $-z/2 - z^2/4$ is that a small portion of the H^2 dependence in V_{eff} is really the constant $\Lambda/3$, rather than $R/12$. These are the finite factors of $\ln(H^2)$ which derive from our counterterms (98-99) through the expansion,

$$\frac{H^{D-4}}{4-D} = \frac{1}{4-D} - \frac{1}{2}\ln(H^2) + O(4-D). \quad (159)$$

If we regard the arbitrary generalization of these factors of $\ln(H^2)$ as $\ln(\Lambda/3)$, rather than $\ln(R/12)$, it corresponds to adding the following term to $F(R)$,

$$\Delta F(R) = \frac{3}{8\pi^2} \left(\frac{R}{12}\right)^2 \left[\frac{12e^2\varphi^*\varphi}{R} + \frac{1}{2} \left(\frac{12e^2\varphi^*\varphi}{R}\right)^2 \right] \ln\left(\frac{R}{4\Lambda}\right). \quad (160)$$

Because the logarithm vanishes for $R = 12H^2$, the only change in $F - RF'/2$ in de Sitter background is precisely the required deficit term,

$$\left[\Delta F(R) - \frac{1}{2}R\Delta F'(R)\right]_{\text{dS}} = \frac{3H^4}{8\pi^2} \left\{ -\frac{1}{2}z - \frac{1}{4}z^2 \right\}. \quad (161)$$

6 Nonperturbative Predictions

It would be silly to stop without exploiting the formalism we have developed to answer nonperturbative questions about SQED. One would like to know:

1. How large does the scalar field strength become?
2. What is the asymptotic late time value of the photon mass?
3. Does the scalar remain light?
4. Does the vacuum energy increase or decrease, and by how much?
5. What becomes of the electric and magnetic field strengths?

Answering these questions is the task of this section. We begin by making the trivial generalization of Starobinskiĭ's formalism from a real scalar to a complex one. We then exploit the results of the previous two sections to compute explicit answers to each of the five questions.

The stochastic formalism of subsections 2.4 and 2.5 has a straightforward generalization to a complex scalar. One simply decomposes the complex field into two real scalars in the usual way,

$$\varphi(x) \equiv \frac{1}{\sqrt{2}}(\varphi_1(x) + i\varphi_2(x)) . \quad (162)$$

Now suppose expectation values of the quantum fields $\varphi_i(x)$ agree, at leading logarithm order, with those of the stochastic random variables $\Phi_i(x)$ which obey the Langevin equations,

$$\dot{\Phi}_i = \dot{\Phi}_{i0} - \frac{1}{3H} \frac{\partial V_{\text{eff}}}{\partial \Phi_i} . \quad (163)$$

The fields $\Phi_{i0}(t, \vec{x})$ are independent sources of Gaussian white noise,

$$\langle \Phi_{i0}(t, \vec{x}) \Phi_{j0}(t', \vec{x}') \rangle = \frac{H^3}{4\pi^2} \delta(t - t') \delta_{ij} . \quad (164)$$

Then the expectation value of any function of the Φ_i is given in terms of a probability density $\varrho(t, \phi_1, \phi_2)$,

$$\langle F[\Phi_1(t, \vec{x}), \Phi_2(t, \vec{x})] \rangle = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 \varrho(t, \phi_1, \phi_2) F(\phi_1, \phi_2) . \quad (165)$$

This probability density obeys the Fokker-Planck equation,

$$\dot{\varrho} = \frac{1}{3H} \sum_{i=1}^2 \frac{\partial}{\partial \phi_i} \left\{ \frac{\partial V_{\text{eff}}}{\partial \phi_i} \varrho \right\} + \frac{1}{2} \sum_{i=1}^2 \frac{\partial^2}{\partial \phi_i^2} \left\{ \frac{H^3}{4\pi^2} \varrho \right\} . \quad (166)$$

We now exploit the fact that the effective potential depends upon the φ_i only through the combination,

$$\varphi^* \varphi = \frac{1}{2}(\varphi_1^2 + \varphi_2^2) . \quad (167)$$

It follows that the probability density has the same form and that we can write the Fokker-Planck equation as,

$$\dot{\varrho} = \frac{1}{3H} \sum_{i=1}^2 \frac{\partial}{\partial \phi_i} \{ \phi_i V'_{\text{eff}} \varrho \} + \frac{H^3}{8\pi^2} \sum_{i=1}^2 \frac{\partial}{\partial \phi_i} \{ \phi_i \varrho' \} . \quad (168)$$

Here a prime denotes differentiation with respect to the variable $\phi^* \phi \equiv (\phi_1^2 + \phi_2^2)/2$.

It is apparent from the large field expansion (106), and from Fig. 4, that V_{eff} is bounded below. We can therefore make the ansatz of Starobinskiĭ and Yokoyama [39] that the probability density approaches a time independent form at late times,

$$\lim_{t \rightarrow \infty} \varrho(t, \phi_1, \phi_2) = \varrho_{\infty}(\phi^* \phi) . \quad (169)$$

Substituting this in our Fokker-Planck equation (168) and making a few simple inferences implies,

$$\varrho_{\infty}(\phi^* \phi) V'_{\text{eff}}(\phi^* \phi) = -\frac{3H^4}{8\pi^2} \varrho'_{\infty}(\phi^* \phi) \implies \varrho_{\infty}(\phi^* \phi) = N e^{-\frac{8\pi^2}{3H^4} V_{\text{eff}}(\phi^* \phi)} . \quad (170)$$

The asymptotic probability density can be more simply expressed in terms of the function $f(z)$ introduced in equation (157),

$$\varrho_{\infty}(\phi^* \phi) = N e^{-f(z)} , \quad (171)$$

where $z \equiv e^2 \phi^* \phi / H^2$ and,

$$f(z) = (-1 + 2\gamma)z + \left(-\frac{3}{2} + \gamma\right)z^2 + \int_0^z dx (1+x) \left[\psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-8x}\right) + \psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-8x}\right) \right] . \quad (172)$$

Hence the late time limit of any function of the operator $\varphi^*(x)\varphi(x)$ can be reduced to an ordinary integral,

$$\lim_{t \rightarrow \infty} \langle F(\varphi^*(x)\varphi(x)) \rangle = \int_{-\infty}^{\infty} d\phi_1 \int_{-\infty}^{\infty} d\phi_2 F(\phi^* \phi) \varrho_{\infty}(\phi^* \phi) , \quad (173)$$

$$= 2\pi N \int_0^{\infty} dz z F\left(\frac{H^2 z}{e^2}\right) e^{-f(z)} . \quad (174)$$

Operator	Expectation Value
$\varphi^*\varphi$	$1.6495 \times H^2/e^2$
$(\varphi^*\varphi)^2$	$3.3213 \times H^4/e^4$
$(\varphi^*\varphi)^3$	$7.6308 \times H^6/e^6$
$M_\gamma^2 \equiv 2e^2\varphi^*\varphi$	$3.2991 \times H^2$
$M_\varphi^2 \equiv V'_{\text{eff}}(\varphi^*\varphi)$	$.8961 \times 3e^2H^2/8\pi^2$
$V_{\text{eff}}(\varphi^*\varphi)$	$.7223 \times 3H^4/8\pi^2$
$V_s(\varphi^*\varphi)$	$-.6551 \times 3H^4/8\pi^2$
$(F_{\mu\nu}F_{\rho\sigma})_{\text{fin}}$	$-9.5246 \times H^4/8\pi^2 (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$

Table 1: Late time limits of expectation values of some important operators.

It is of course impossible to obtain analytic expressions for integrals of the form (174) with the function $f(z)$ in (172). However, it is nothing these days to evaluate such integrals numerically. We have done this using the “NIntegrate” function of Mathematica [59]. The normalization factor is about,

$$2\pi N \equiv \left[\int_0^\infty dz z e^{-f(z)} \right]^{-1} \approx \frac{1}{2.16603}. \quad (175)$$

The scalar reaches a nonperturbatively large field strength,

$$\lim_{t \rightarrow \infty} \langle \varphi^*(x)\varphi(x) \rangle = \frac{H^2}{e^2} \times 2\pi N \int_0^\infty dz z^2 e^{-f(z)} \approx \frac{H^2}{e^2} \times \frac{3.57293}{2.16603}. \quad (176)$$

The photon mass-squared is $M_\gamma^2 \equiv 2e^2\varphi^*\varphi$, so (176) means that it reaches the asymptotic value,

$$\lim_{t \rightarrow \infty} M_\gamma^2 \approx 3.2991 \times H^2. \quad (177)$$

This is explicit, nonperturbative confirmation of the conjecture by Davis, Dimopoulos, Prokopec and Törnkvist [42, 43] that inflation induces a nonzero mass photon mass. Indeed, the nonperturbative result is about a hundred times larger than one loop computations [60, 11, 12, 61, 62, 63].

Because inflationary particle production would be quenched if the scalar were to develop a mass comparable to the Hubble parameter, it is important to check that the scalar remains light. The scalar mass-squared is the

derivative of the effective potential,

$$M_\varphi^2 = V'_{\text{eff}}(\varphi^* \varphi) . \quad (178)$$

Our result for its asymptotic value is,

$$\lim_{t \rightarrow \infty} \langle M_\varphi^2 \rangle = \frac{3e^2 H^2}{8\pi^2} \times 2\pi N \int_0^\infty dz z f'(z) e^{-f(z)} \approx \frac{3e^2 H^2}{8\pi^2} \times \frac{1.94103}{2.16603} . \quad (179)$$

Therefore the scalar is always light compared to the Hubble scale, and the approximation of treating it stochastically with the massless mode functions is justified.

Table 1 summarizes our results for late time limits of various operators. One operator of particular interest is the stress tensor potential, $V_s(\varphi^* \varphi)$, given in equation (151). In view of relations (157-158) we can express its late time limit as,

$$\lim_{t \rightarrow \infty} \langle V_s(\varphi^* \varphi) \rangle = \frac{3H^4}{8\pi^2} \times 2\pi N \int_0^\infty dz z \left(\frac{z}{2} f'(z) - \frac{z}{2} - \frac{z^2}{4} \right) e^{-f(z)} , \quad (180)$$

$$\approx \frac{3H^4}{8\pi^2} \times -\frac{1.41898}{2.16603} . \quad (181)$$

That the surprising sign is correct can be seen from Fig. 7 which gives an expanded view of the stress tensor potential. It should be noted that the sign is due to the two negative terms in the z integrand. With just the first term the result would be,

$$\frac{3H^4}{8\pi^2} \times 2\pi N \int_0^\infty dz \frac{z^2}{2} f'(z) e^{-f(z)} = +\frac{3H^4}{8\pi^2} . \quad (182)$$

The physical interpretation may be that inflationary particle production polarizes the vacuum, which lowers the energy of a charged particle in the medium provided the charge density is not too large.

Note from Table 1 that the expectation value of the effective potential is positive. As explained at the end of the previous section, this is no contradiction with our result for V_s because the two potentials answer slightly different physical questions. V_{eff} controls the evolution of the scalar field strength whereas V_s controls the gravitational response. In particular, our result for V_s implies that inflationary particle production induces a small

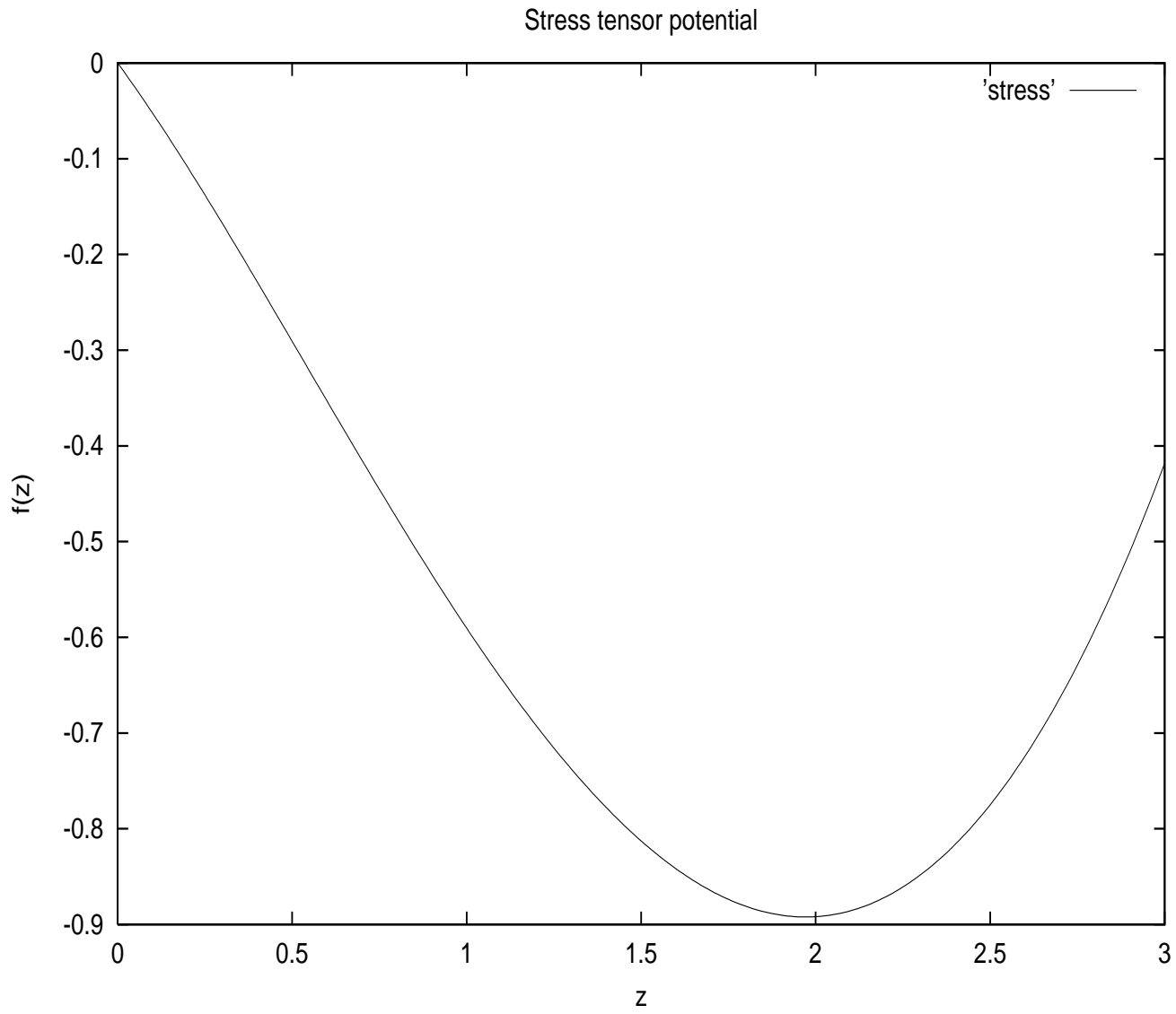


Figure 7: Stress tensor potential $V_s \equiv \frac{3H^4}{8\pi^2} f_s(z)$ showing the minimum.

fractional reduction of the asymptotic expansion rate,

$$3H_\infty^2 \approx 3H^2 - 8\pi G \times \frac{3H^4}{8\pi^2} \times .6551, \quad (183)$$

$$= 3H^2 \left\{ 1 - .6551 \times \frac{GH^2}{\pi} \right\}. \quad (184)$$

This is insignificant even for the highest scale inflation ($GH^2 \lesssim 10^{-12}$) consistent with the normalized CMB quadrupole and with the current upper bound on the scalar-to-tensor ratio [64]. The sign of the effect is nevertheless intriguing.

Because the field strength bilinear involves coincident passive fields it requires renormalization even at leading logarithm order. This is evident from the factor of $\Gamma(2 - \frac{D}{2})$ in (130). As might be expected, the divergence can be absorbed with terms proportional to $\varphi^*\varphi$ and $(\varphi^*\varphi)^2$,

$$\left(F_{\mu\nu} F_{\rho\sigma} \right)_{\text{div}} \equiv \frac{H^D}{(4\pi)^{\frac{D}{2}}} \Gamma\left(-\frac{D}{2}\right) 4z(1+z) (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \quad (185)$$

With this definition of the divergent part, the remaining finite part has the following $D = 4$ limit,

$$\begin{aligned} \left(F_{\mu\nu} F_{\rho\sigma} \right)_{\text{fin}} &\equiv \frac{H^4}{8\pi^2} \left\{ -z - z(1+z) \right. \\ &\times \left. \left[\psi\left(\frac{3}{2} - \frac{1}{2}\sqrt{1-8z}\right) + \psi\left(\frac{3}{2} + \frac{1}{2}\sqrt{1-8z}\right) \right] \right\} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \end{aligned} \quad (186)$$

The late time limit of this quantity is strongly negative,

$$\lim_{t \rightarrow \infty} \left\langle \left(F_{\mu\nu} F_{\rho\sigma} \right) \right\rangle \approx \frac{3H^4}{8\pi^2} \times -\frac{20.6305}{2.16603} \times (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \quad (187)$$

Of course the quadratic and quartic parts of (186) can be adjusted with the renormalization condition but the negative sign of the large field limiting form is unambiguous,

$$\left(F_{\mu\nu} F_{\rho\sigma} \right)_{\text{fin}} \longrightarrow \frac{H^2}{8\pi^2} \left[-z(z+1) \ln(2z+2) - \frac{1}{3}z + O(1) \right] (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \quad (188)$$

This indicates that the inflationary production of charged scalars increases the electric field strength (for example, $\mu = \rho = 0$ and $\nu = \sigma = i$), relative to

its vacuum value, while the magnetic field strength (for example, $\mu = \rho = i$ and $\nu = \sigma = j$) is decreased. This makes good physical sense. Although the average charge is zero, there is about one infrared scalar in each Hubble volume. The local electric field is necessarily dominated by this charge, so the square of the electric field strength should increase. On the other hand, the nonzero scalar field strength engenders a positive photon mass which drives down the magnetic field by the Meissner effect. From expression (147) we see that the net electromagnetic contribution to the stress tensor is that of a negative cosmological constant at leading logarithm order. This indicates that the damping of vacuum fluctuations in the magnetic field is greater than the enhancement of the electric field.

7 Epilogue

Infrared logarithms are the manifestation of enhanced quantum effects mediated by massless, minimally coupled scalars and gravitons. We call these *active fields*. The continued growth of infrared logarithms must eventually overwhelm even the smallest loop-counting parameter. At this point perturbation theory breaks down and one must employ some sort of nonperturbative technique to follow the subsequent evolution. A reasonable approach is to sum the series of leading infrared logarithms. Starobinskiĭ has developed a simple stochastic formalism which accomplishes this for any model of purely active fields with nonderivative interactions [30, 39].

More general models possess two sorts of complications: derivative interactions and couplings with *passive fields*, that is, fields which cannot cause infrared logarithms. We still do not have a general technique for handling derivative interactions. One deals with passive fields by integrating them out and then stochastically simplifying the resulting effective action of active fields. This amounts to computing the effective potential. The theory then reduces to the form that Starobinskiĭ has already solved. This reduction was previously accomplished for Yukawa theory [16], and we have done it here for SQED.

Note that one must integrate out passive fields from the VEV of any operator. This can result in ultraviolet divergences even at leading log order, as we found for both the field strength and the scalar kinetic bilinears. The reason for this is that passive fields contribute factors of order one which multiply the infrared logarithms contributed by active fields. Whereas in-

frared logarithms derive entirely from the long wavelength part of the free field mode sum, the factors of order one come as much from the ultraviolet as from the infrared.

The nontrivial role of the ultraviolet also shows up in the fact that certain renormalization counterterms can make leading order contributions. In both Yukawa theory [16] and SQED the one loop conformal and quartic counterterms contribute. However, no other counterterms contribute at leading logarithm order, nor do any higher loop counterterms matter.

We have obtained nonperturbative results for the VEV's of $\varphi^*\varphi$, $F_{\mu\nu}F_{\rho\sigma}$ and $T_{\mu\nu}$. Table 1 summarizes these. Our result is that the scalar approaches a nonperturbatively large field strength. This confirms the conjecture of Davis, Dimopoulos, Prokopec and Törnkvist [42, 43]. The scalar remains perturbatively light, which means the computation is self-consistent.

Our result for the stress tensor is curious in two ways. First, although it takes the form $T_{\mu\nu} \rightarrow -g_{\mu\nu}V_s$, the stress tensor potential V_s is not quite equal to the effective potential. This does not mean one potential is “right” and the other “wrong.” Rather, they are both the correct answers to slightly different physical questions. The effective potential controls how the scalar evolves at leading logarithm order, whereas the stress tensor potential describes how this evolution serves as a source for gravity.

As shown in section 5, the difference between V_s and V_{eff} arises because almost all the factors of H^2 in V_{eff} are actually $R/12$ for a general metric. One consequence is that V_{eff} represents a peculiar modified gravity theory which may have important implications for cosmology. Lagrangians of the form $F(R)$ seem to be theoretically viable, and they can be tuned to give any desired evolution for the scale factor [65]. However, there is no justification for such models from fundamental theory. In contrast, the modified gravity model we get is uniquely fixed and thoroughly justified — although it may not, of course, do anything interesting.

The second peculiar thing about our result for the stress tensor is that it reduces the vacuum energy. The physical interpretation for this may be that the inflationary production of charged scalars polarizes the vacuum, which lowers the energy of charged particles in this medium provided the charge density is not too large. Supporting this conjecture is the fact that the electromagnetic contribution to the stress tensor is that of a negative cosmological constant.

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