Representation Theory in Quantum Mechanics: A First Step Towards the Wigner Classification

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Abstract

In this thesis, we focus on the manifestation of representation theory in quantum mechanics, mainly from a mathematical perspective. The symmetries of a spacetime manifold M form a Lie group G, of which there must exist a projective representation on the state space of a quantum system located in this spacetime. The possible distinct irreducible projective representations are associated with different types of elementary particles. This approach was first outlined by Wigner [20] in his 1939 paper, in which he also explicitly performed the calculation of all irreducible projective representations for the symmetry group of Minkowski space, the spacetime manifold in Einstein's theory of special relativity. To get an idea of its implications, we quote [16]:

"*It is difficult to overestimate the importance of this paper, which will certainly stand as one of the great intellectual achievements of our century. It has not only provided a framework for the physical search for elementary particles, but has also had a profound influence on the development of modern mathematics, in particular the theory of group representations.*"

Chapter 1 serves as an introduction to the theory of Lie groups and their representations. In Chapter 2, we prove a theorem regarding unitary representations of non-compact simple Lie groups, of which the Lorentz group is an example of particular interest to us. Namely, it is the group of spacetime symmetries fixing the origin, and therefore a natural subgroup of the Poincaré group, which is the full symmetry group of special relativity. Hence, Wigner's calculations involve precisely this group. A complete and rigorous treatment for more general symmetry groups was achieved by Mackey [14]. We will probe some aspects of his theory of systems of imprimitivity in Chapter 3, but only in a heavily simplified context. Nevertheless, it will shed light on the steps involved in the broader case. Assuming the appropriate generalization of the result obtained in Chapter 3, we are able to provide an exposition of Wigner's work in Chapter 4, along with some of its historical implications.

Contents

Chapter 1

Preliminaries

Here we introduce some fundamental language which is important throughout the course of this thesis. This consists of elementary representation theory, in addition to some aspects from the theory of Lie groups, Lie algebras and their representations. We will try to avoid using too much of the general machinery of differential geometry. For more background, the reader may consult [13]. In most cases we do attempt to be fully rigorous in our presentation: although some steps may be omitted in the arguments, one can always consult the references mentioned in the proofs in order to fill in the gaps if needed.

1.1 Basic Notions in Lie Theory: Lie Groups

Here we introduce the concept of a Lie group and analyze some of its basic properties. As we will see later, they arise naturally in many areas of physics. Although we expect the reader to be familiar with general group theory (i.e. homomorphisms, kernels, quotients by normal subgroups, cosets, group actions etc.) we still wish to recall the definition of a group:

Definition 1.1: *A group G is a set G with a product map $m : G \times G \to G, (g, h) \mapsto gh = m(g, h)$ and an inversion map $i : G \to G, g \mapsto g^{-1} = i(g)$ satisfying the following axioms:*

(i) *Associativity*: $(fg)h = f(gh)$ for all $f, g, h \in G$.

(ii) *Unital element*: $\exists e \in G : ge = eg = g$ for all $g \in G$.

(iii) *Inversion*: $gg^{-1} = g^{-1}g = e$ for all $g \in G$.

A good way to think of a group is as a structure which describes the symmetries of a certain object. Every element of the group transforms the object in a certain way so that some of its properties remain the same. As an example from linear algebra, the *orthogonal group* with respect to the Euclidean inner product $(\cdot,\cdot)$ on $\mathbb{R}^n$ is the group of all $n \times n$ matrices A such that $(Ax, Ay) = (x, y) \quad \forall x, y \in \mathbb{R}^n$. This is a group under matrix multiplication, and it acts on the space $\mathbb{R}^n$ in such a way that distances are preserved. It turns out that this is also an example of a Lie group.

Definition 1.2: *A Lie group G is a smooth manifold which is endowed with a group structure such that multiplication $m : G \times G \to G, (g, h) \mapsto gh$ and inversion $i : G \to G, g \mapsto g^{-1}$ are smooth maps.*

We will not introduce the notion of a smooth manifold here, but the intuition is that it is a topological space M which locally looks like (is homeomorphic to) $\mathbb{R}^n$, together with some additional data called a *smooth structure* which enables us to talk about derivatives of maps of manifolds. If n is as above, we say that M is n-dimensional. Almost all of the Lie groups that we encounter throughout this thesis will be subgroups of some general linear group over a finite dimensional vector space, hence they can be understood as submanifolds of some $\mathbb{R}^m$. This makes it easier to understand at least the outcomes of some of the calculations without knowledge of general manifold theory.

Definition 1.3: *A homomorphism of Lie groups is a smooth group homomorphism $\phi : G \to H$ where G, H are Lie groups.*

Definition 1.4: *Let G be a Lie group. A one-parameter subgroup of G is a smooth map $\alpha : \mathbb{R} \to G$ which is also a group homomorphism from $\mathbb{R}$ to G: $\alpha(s+t) = \alpha(s)\alpha(t)$.*

That is, it is a Lie group homomorphism from $(\mathbb{R}, +)$ to G. As an example, the map $\alpha : \mathbb{R} \to S^1, t \mapsto e^{it}$ is a one-parameter subgroup of the circle. Whenever we are dealing with a smooth manifold M and a smooth map of manifolds $F : M \to N$, we denote the tangent space at $p \in M$ by T_pM and the differential or tangent map of F at p by $T_pF : T_pM \to T_{F(p)}N$ (except in the case of the exterior derivative acting on a function, in which case we just use $\mathrm{d}F_p$). In the case of Lie groups, we will be particularly interested in the tangent space at the identity T_eG.

It turns out that there is an extremely important correspondence between tangent vectors in T_eG and one-parameter subgroups. This yields the definition of the so-called exponential map:

Proposition 1.5: *For each $X \in T_eG$, there is a unique one-parameter subgroup α_X satisfying $\alpha'(0) = T_0\alpha(1) = X$. The assignment $X \mapsto \alpha_X$ is a bijection from the set of tangent vectors T_eG to the set of one-parameter subgroups.*

Proof: This is standard in the literature. We refer to Theorem 20.1 in [13]. $\square$

Definition 1.6: *With the notation of Proposition 1.5, we define the exponential map by* $\exp : T_eG \to G, X \mapsto \alpha_X(1)$.

It follows from the theory of ordinary differential equations that exp is actually a smooth map. We will now summarize the most important properties of exp in the next proposition:

Proposition 1.7: *Following the notation of Definition 1.6, the following properties hold:*

(i) *The tangent map* $T_0\exp : T_eG \to T_eG$ *of* exp *equals the identity* I_{T_eG} *on* T_eG.

(ii) *There exist open neighborhoods* $\Omega \subset T_eG$ *of* 0 *and* $U \subset G$ *of* e *such that* $\exp : \Omega \to U$ *is a diffeomorphism.*

(iii) *We have* $\exp(sX) = \alpha_X(s)$ *for all* $s \in \mathbb{R}$.

(iv) *The identity component* G_e *of* G *is generated by elements of the form* $\exp(X)$ *with* $X \in T_eG$.

Proof: We refer to Propositions 7.15 and 20.8 in [13]. □

The exponential map will be important in establishing the connection between the theory of Lie groups and the theory of Lie algebras, as we shall see in Section 1.3.

Any Lie group acts on itself by multiplication (sometimes also called left translation). We denote left multiplication by g by the map $l_g : G \to G, h \mapsto gh$. This map is a diffeomorphism with inverse $l_{g^{-1}}$. Indeed, this follows from restricting m to the embedded submanifold $\{g\} \times G$ of $G \times G$. In practice, we can use this general fact to compare structures of Lie groups at different points. As an example, we state the following theorem:

Theorem 1.8: *Let* G *and* H *be Lie groups. Then any Lie group homomorphism* $\phi : G \to H$ *has constant rank. In particular, if it is immersive (resp. submersive) at* e*, then it is everywhere immersive (resp. submersive).*

Proof: Suppose ϕ has rank k at e, i.e. its tangent map at e is a linear map of rank k. Let $g \in G$ be arbitrary. Then by assumption, $\rho \circ l_g = l_{\rho(g)} \circ \rho$, hence taking tangent maps at e we obtain: $T_g\rho \circ T_el_g = T_el_{\rho(g)} \circ T_e\rho$. These two maps must have equal rank. Composing with invertible maps does not change the rank (because it does not change the dimension of the kernel) so we are done. □

Definition 1.9: *Let* G *be a Lie group. A Lie subgroup* H *of* G *is a subset of* G *which is a subgroup, i.e.* $h_1h_2 \in H \quad \forall h_1, h_2 \in H$ *and has a topology and smooth structure making it into a Lie group of its own such that the inclusion is smooth.*

A natural question would be whether a general subgroup H of a Lie group G 'sits nicely' inside the ambient manifold G as an embedded submanifold. Unfortunately, this may not be the case. That is, a Lie subgroup need not be an embedded submanifold as it need not carry the subspace topology inherited from G. For examples of this we refer to the so-called dense curve on the torus (this is Example 4.20 in [13]) where the topology on the image is finer than the subspace topology, so that the manifold is only immersed. However, we do have the result below.

Theorem 1.10: *Let G be a Lie group and H be a subgroup of G. Then the following are equivalent:*

(i) *H is a closed subset of G.*

(ii) *H is an embedded submanifold and a Lie subgroup of G.*

Proof: We refer to Theorems 7.21 and 20.12 in [13]. □

Definition 1.11: *Let G be a Lie group. Its center $Z(G)$ consists of the elements that commute with all other elements, i.e. $Z(G) = \{g \in G \mid xg = gx \text{ for all } x \in G\}$.*

One readily verifies that the kernel is a subgroup and furthermore, it is closed (essentially because group multiplication is smooth and hence continuous). Hence the kernel is an embedded submanifold and a Lie subgroup by the above assertion.

Aside from translation, a group G also acts on itself by conjugation: denote for $x \in G$ the map $g \mapsto xgx^{-1}$ by C_x. It has inverse equal to $C_{x^{-1}}$ and is readily verified to be a bijective homomorphism and a diffeomorphism, i.e. it as an *isomorphism of Lie groups*. This gives rise to the following definition:

Definition 1.12: *Let G be a Lie group and let C_x be as above. We define $\mathrm{Ad}(x) : T_eG \to T_eG$ to be the tangent map of C_x. It is called the adjoint representation at x. Moreover, we can define the adjoint representation of G by $\mathrm{Ad} : G \to GL(T_eG), x \mapsto \mathrm{Ad}(x)$.*

Proposition 1.13: *Let G be a connected Lie group and let the adjoint representation be defined as above. This is a homomorphism of Lie groups, and its kernel equals precisely the center $Z(G)$.*

Proof: Note that we already mentioned the identity $C_xC_y = C_{xy}$. Differentiating at e on the left and right hand side and using the chain rule, we obtain $T_{C_y(e)}C_x \circ T_eC_y = T_eC_{xy}$, whence $\mathrm{Ad}(x) \circ \mathrm{Ad}(y) = T_eC_x \circ T_eC_y = T_{C_y(e)}C_x \circ T_eC_y = T_eC_{xy} = \mathrm{Ad}(xy)$. We postpone the proof of the second assertion to Section 1.3. □

Definition 1.14: *We define the linear map* $\mathrm{ad}\colon T_eG \to \mathrm{End}(T_eG), \mathrm{ad}(X) = (T_e\mathrm{Ad})(X)$.

In the next section, we will encounter another map defined on Lie algebras for which we use the same notation. This will be justified in Section 1.3.

1.2 Basic Notions in Lie Theory: Lie Algebras

We first introduce Lie algebras as objects in their own right, along with some of their structural properties. Then, we will see that (at least in the finite-dimensional case) we can relate them to Lie groups.

In our entire exposition, k denotes a field equal to either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.15: A Lie algebra $\mathfrak{g}$ is a finite-dimensional vector space $\mathfrak{g}$ over a field k with an additional binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following axioms:

(i) *Bilinearity*: $[X, aY + bZ] = a[X, Y] + b[X, Z]$ *for all* $X, Y, Z \in \mathfrak{g}$ *and* $a, b \in k$.

(ii) *Antisymmetry:* $[X, Y] = -[Y, X]$ *for all* $X, Y \in \mathfrak{g}$.

(iii) *Jacobi Identity:* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$.

We see that a Lie algebra is typically not commutative, but rather anticommutative: $[X, Y] = -[Y, X]$. If $\mathfrak{g}$ is commutative, then the bracket is identically zero because the ground field k has characteristic zero.

Observe that a Lie algebra has no unit element in the algebra sense, unless it is zero-dimensional. The Jacobi identity can be viewed as a kind of 'generalized associativity'.

In Lie algebras, there are notions of ideals and subalgebras similar to those in the theory of rings:

Definition 1.16: *Let* $\mathfrak{g}$ *be a Lie algebra over* k. *A Lie subalgebra is a vector subspace* $\mathfrak{h}$ *which is closed under the bracket:* $[X, Y] \in \mathfrak{h}$ *for any* $X, Y \in \mathfrak{h}$. *An ideal* $\mathfrak{a}$ *is a vector subspace which satisfies* $[X, Y] \in \mathfrak{a}$ *for all* $X \in \mathfrak{a}$ *and* $Y \in \mathfrak{g}$.

Note that because of (ii) in the definition of a Lie algebra, there is no distinction between left and right ideals. Any ideal clearly is a subalgebra, and we can view ideals and subalgebras as Lie algebras in their own right.

Definition 1.17: *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over k. A Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ which also satisfies $\psi[X, Y] = [\psi(X), \psi(Y)]$ for all $X, Y \in \mathfrak{g}$. In the case where $\mathfrak{g} = \mathfrak{h}$, we call an isomorphism an automorphism.*

As an example, the image of a homomorphism is a Lie subalgebra of the target space, and the kernel (in the linear sense) is an ideal and hence a subalgebra of the domain. A Lie algebra is called *simple* if the only ideals are $\mathfrak{g}$ and $\{0\}$, and it is nonabelian. It is called *semisimple* if it is isomorphic (i.e. there is a bijective homomorphism) to a direct sum of simple Lie algebras.

As an example of a homomorphism, note that any associative algebra can be turned into a Lie algebra by defining the bracket to be the commutator of elements, i.e. $[a, b] = ab - ba$. For example, k is a Lie algebra with bracket identically equal to zero. We now introduce the adjoint map:

Definition 1.18: *Let $\mathfrak{g}$ be a Lie algebra over k. The adjoint map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is defined by setting* $\mathrm{ad}(X)(Y) = [X, Y]$.

It is very easy to check that this is a homomorphism of Lie algebras. It is linear and well defined by bilinearity of the bracket, and $\mathrm{ad}[X, Y] = [\mathrm{ad}(X), \mathrm{ad}(Y)]$ by the Jacobi identity, where on the right hand side we have the commutator of matrices. This can be seen as a motivation for imposing the Jacobi identity.

We would like to be able to find convenient methods to decide whether two given Lie algebras are isomorphic or not. To this end, we will define some important notions which we will also encounter later on. For a start, we introduce the Killing form of a Lie algebra:

Definition 1.19: *Let $\mathfrak{g}$ be a Lie algebra over k. We define its Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$ by $\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$.*

Note that this definition is basis independent, because the trace operator is. It turns out that the Killing form has several important properties which give information about the structure of $\mathfrak{g}$. They are summarized in the proposition below. We give proofs of (i) and (ii) and refer to [9] for details on the other parts.

Proposition 1.20: *Let $\mathfrak{g}$ be a Lie algebra over k and let κ be its Killing form. Then:*

(i) *The Killing form is invariant under automorphisms of $\mathfrak{g}$: $\kappa(\psi(X), \psi(Y)) = \kappa(X, Y)$ for all $\psi \in \mathrm{Aut}(\mathfrak{g})$.*

(ii) *The Killing form satisfies $\kappa([X, Y], Z) = \kappa(X, [Y, Z])$.*

(iii) *Cartan's criterion: $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

(iv) *If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then the Killing form of $\mathfrak{a}$ is the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{a}$.*

Proof of (i): For (i), note that $\mathrm{ad}(\psi(X))(Y) = [\psi(X), Y] = \psi([X, \psi^{-1}(Y)])$, so $\mathrm{ad}(\psi(X)) = \psi \circ \mathrm{ad}(X) \circ \psi^{-1}$ as maps from $\mathfrak{g}$ to itself. Hence we have $\kappa(\psi(X), \psi(Y)) = \mathrm{tr}(\mathrm{ad}(\psi(X))\mathrm{ad}(\psi(Y))) = \mathrm{tr}(\psi \circ \mathrm{ad}(\psi(X))\mathrm{ad}(\psi(Y)) \circ \psi^{-1})$. By cyclicity of the trace, this equals $\mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$ and we are done. □

Proof of (ii): For (ii), observe that $\kappa([X, Y], Z) = \mathrm{tr}(\mathrm{ad}([X, Y])\mathrm{ad}(Z))$ and $\mathrm{ad}([X, Y]) = \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X)$. Then we get $\mathrm{tr}((\mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X))\mathrm{ad}(Z))$. By cyclicity of the trace this equals $\mathrm{tr}((\mathrm{ad}(X)\mathrm{ad}(Y)\mathrm{ad}(Z) - \mathrm{ad}(X)\mathrm{ad}(Z)\mathrm{ad}(Y)) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}([Y, Z])) = \kappa(X, [Y, Z])$. □

In most specific cases, the Killing form is not too hard to compute. We can then for example use (i) to conclude something about the Lie algebra of interest. It turns out that properties like simplicity and semisimplicity are important in analysing representations of Lie algebras. We return to this later.

Similar to the theory of groups and rings, a Lie algebra also possesses a notion of a centre:

Definition 1.21: *Let $\mathfrak{g}$ be a Lie algebra over k. Its center is denoted by $Z(\mathfrak{g})$ and consists of all elements which commute with all of $\mathfrak{g}$, i.e. $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0$ for all $X, Y \in \mathfrak{g}\}$.*

One readily verifies that the centre is always an ideal by virtue of the Jacobi identity, and in fact it equals precisely the kernel of the adjoint map. As a consequence, any homomorphism from a simple Lie algebra to another Lie algebra is either injective or trivial.

Finally, we introduce the derived algebra as a subalgebra of a Lie algebra. It is not hard to see that this is actually an ideal.

Definition 1.22: *If $\mathfrak{g}$ is a Lie algebra over k, the derived algebra of $\mathfrak{g}$ is defined as $[\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{[X, Y] \mid X, Y \in \mathfrak{g}\}$ where* Span *denotes linear span.*

Lemma 1.23: *For a semisimple Lie algebra $\mathfrak{g}$, we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof: This is seen by using (i) and (iv) of Proposition 1.20. Consider the space $[\mathfrak{g}, \mathfrak{g}]^\perp = \{X \in \mathfrak{g} \mid \kappa(X, Y) = 0 \text{ for all } Y \in [\mathfrak{g}, \mathfrak{g}]\}$. Then $X \in [\mathfrak{g}, \mathfrak{g}]^\perp$ if and only if $\kappa([X, Y], Z) = 0$ for all $Y, Z \in \mathfrak{g}$. Hence by nondegeneracy, $[X, Y] = 0$ for all $Y \in \mathfrak{g}$, so X is an element of the center of $\mathfrak{g}$. But the center of a semisimple Lie algebra is trivial, because if $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ is the decomposition of $\mathfrak{g}$ into simple ideals $\mathfrak{g}_i$, we know that

$Z(\mathfrak{g}) \cap \mathfrak{g}_i$ is an ideal in $\mathfrak{g}_i$. Indeed, $[h,x] = 0$ for $h \in \mathfrak{g}_i, x \in Z(\mathfrak{g})$. Hence $Z(\mathfrak{g}) \cap \mathfrak{g}_i = 0$ for all i. This proves the assertion. $\square$

1.3 The Lie Group - Lie Algebra Correspondence

In this section we examine some of the connections between Lie groups and Lie algebras. It turns out that the tangent space at the identity T_eG of a Lie group G has a natural 'bracket' structure turning it into a Lie algebra. In this setting, a homomorphism of Lie groups can be differentiated to yield a homomorphism of Lie algebras. Philosophically, this means that any result on the level of Lie groups will have a corresponding result on the level of Lie algebras. For a start, we have the following extremely important theorem involving the exponential map:

Theorem 1.24: *Let G, H be Lie groups and let $\phi : G \to H$ be a Lie group homomorphism. Denote by ϕ_* its tangent map at e. Then for any $X \in T_eG$, we have $\phi((\exp_G(X)) = \exp_H(\phi_* X)$. That is, the diagram below commutes:*

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp_G"] \arrow[r, "\phi_*"'] & \mathfrak{h} \arrow[u, "\exp_H"'] \end{tikzcd}$$

Proof: We will need to use the bijective correspondence between one-parameter subgroups and tangent vectors in T_eG given in Proposition 1.5, along with the results of Proposition 1.7. For a given $X \in T_eG$, denote $Y = \phi_* X \in T_eH$. The map $\alpha_Y : \mathbb{R} \to H, t \mapsto \exp_H(Y)$ is the unique one-parameter subgroup of H which has velocity equal to Y at $t = 0$. But the curve $\beta : t \mapsto \phi((\exp_G(tX))$ also satisfies these properties by a straightforward application of the chain rule, and noting that it is a homomorphism because ϕ is. Hence the two maps must be equal by uniqueness. The assertion follows by substituting $t = 1$. $\square$

Consequently, we can derive some relations between maps we introduced in Section 1.1. For example, taking $\phi = C_x$, we see that $x(\exp X)x^{-1} = \exp(\mathrm{Ad}(x))X$.

End of proof of Proposition 1.13: We see that if $\mathrm{Ad}(x) = I$, then x commutes with elements of the form $\exp(X)$, and hence with all of $G_e = G$. Conversely, if x commutes with $G_e = G$, then $\exp(t\mathrm{Ad}(x)Y) = x\exp(tY)x^{-1} = \exp(tY)$. Differentiating at $t = 0$ yields $\mathrm{Ad}(x) = I$. $\square$

Furthermore, we can apply Theorem 1.24 to $\phi = \mathrm{Ad}$ as well. We note that in matrix groups, conjugation by A is a linear map. From this it can be deduced that the exponential of $B \in M_n(\mathbb{R}) = T_I GL(n, \mathbb{R})$ is just given by the usual matrix expansion of e^B. As a result, we see that $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$.

Proposition 1.25: *Let $\phi : G \to H$ be a homomorphism of Lie groups. Denote* $\mathrm{ad}(X)(Y) = [X, Y]$. *Then the tangent map $\phi_* : T_eG \to T_eH$ satisfies $[\phi_* X, \phi_* Y] = \phi_*[X, Y]$*

Proof: Let $\mathrm{Ad}_G, \mathrm{Ad}_H$ denote the adjoint representations of the groups in question. Since ϕ is a homomorphism, we have for arbitrary $x \in G$ the equality $\phi \circ C_x = C_{\phi(x)} \circ \phi$ as maps from G to H. The chain rule for differentiation yields $T_e\phi \circ \mathrm{Ad}_G(x) = \mathrm{Ad}_H(\phi(x)) \circ T_e\phi$ as maps from T_eG to T_eH. Hence for a fixed $X \in T_eG$, we can consider these as maps from G to T_eH if we vary x. Differentiating with respect to x yields that $\phi_* \circ \mathrm{ad}_G(X) = \mathrm{ad}_H(\phi_* X) \circ \phi_*$. Applying both maps to an arbitrary $Y \in T_eG$ yields $[\phi_* X, \phi_* Y] = \phi_*[X, Y]$ which is precisely what we wanted. $\square$

Proposition 1.26: *Let* ad*: $T_eG \to \mathrm{End}(T_eG)$ be defined as in Definition 1.14. With the bracket as defined in Proposition 1.25, the vector space T_eG has the structure of a Lie algebra over $k = \mathbb{R}$ as given in Definition 1.15.*

Proof: Bilinearity of ad is obvious. Note that if $X \in \mathfrak{g}$ is arbitrary, we have the equalities $\exp(tX) = \exp(sX)\exp(tX)\exp(-sX) = \exp(\mathrm{Ad}(\exp(sX)tX))$. Take the derivative with respect to t at zero to see that this implies $X = \mathrm{Ad}(\exp(sX)X)$. Again differentiating at 0 yields $\mathrm{ad}(X)(X) = 0$. Let $V, W \in \mathfrak{g}$ be arbitrary. Then, $0 = [V + W, V + W] = [V, V] + [W, V] + [V, W] + [W, W] = [W, V] + [V, W]$ as desired. Since Ad is a Lie group homomorphism, $\mathrm{ad}[X, Y] = [\mathrm{ad}(X), \mathrm{ad}(Y)]$. But the bracket on $\mathrm{End}(T_eG)$ is just the commutator of matrices, so $\mathrm{ad}[X, Y] = \mathrm{ad}(X)\mathrm{ad}(Y) - \mathrm{ad}(Y)\mathrm{ad}(X)$. Applying this to arbitrary Z yields, upon rearrangement and using antisymmetry, the Jacobi identity. $\square$

From now on, we denote the tangent space at the identity of a Lie group using the corresponding German letter, in order to carry over the notation used in Section 1.2. As a simple corollary of Proposition 1.26 we note that isomorphic Lie groups have isomorphic Lie algebras. Indeed, if G and H are diffeomorphic via an isomorphism of Lie groups, then the tangent map is also bijective, so $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$ precisely via the tangent map of the diffeomorphism at e.

A much more interesting and subtle question would be whether two Lie groups with isomorphic Lie algebras are necesserily isomorphic. It turns out that this need not be the case: for example, the (unique) one-dimensional real Lie algebra (which is easily seen to be abelian) belongs to two distinct Lie groups, namely $\mathbb{R}$ and S^1. Note that the former covers the latter, which stresses the fact that they are locally the same, but on a global level they are topologically different. Both of these groups are abelian. A less straightforward example is the famous double covering $SU(2) \to SO(3)$, where the former is simply connected but the latter is not. Fortunately, we can say a little more in the case where we are dealing with a subgroup H of a given Lie group G.

Proposition 1.27: *Let G be a Lie group and H a subset. Suppose H admits the structure of a Lie subgroup. Then it must be an immersed submanifold of G.*

Proof: The inclusion $\iota : H \hookrightarrow G$ is an injective Lie group homomorphism. It has constant rank by Theorem 1.8. Since it is injective, it must be an immersion everywhere. This is precisely the definition of an immersed submanifold. □

This allows us to identify the Lie algebra $\mathfrak{h}$ of H as a subspace (and hence subalgebra) of the Lie algebra $\mathfrak{g}$ of G. The natural converse questions in this context would be: given a Lie subalgebra of the Lie algebra of G, can we find a Lie subgroup H with precisely this Lie algebra? Is it unique? Can we find such an *embedded* H? We have the following two results:

Theorem 1.28: *Let G be a Lie group and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there exists exactly one connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$. It may not be an embedded submanifold of G.*

Proof: We refer to Theorem 7.11 of [3]. □

Theorem 1.29: *Let G be a Lie group and H a Lie subgroup. Then, the Lie algebra $\mathfrak{h}$ of H is characterized by $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H \quad \forall t \in \mathbb{R}\}$.*

Proof: This is Proposition 20.9 in [13]. □

As a final illustrative example, we compute the Lie algebra of the center subgroup, and show that it coincides with the center of the Lie algebra in the case where G is connected. We will use this later.

Lemma 1.30: *Let G be a connected Lie group and $Z(G)$ its center. Then the Lie algebra of $Z(G)$ is precisely $Z(\mathfrak{g})$.*

Proof: We use the characterization of Theorem 1.29. If $[X, Y] = 0$ for all Y, then $e^{\mathrm{ad}(tX)}Y = Y$, so $\mathrm{Ad}(\exp(tX))Y = Y$, whence $\exp(tX)\exp(Y)\exp(-tX) = \exp(Y)$, but the identity component of G is generated by elements of the form $\exp(Y)$, so $\exp(tX) \in Z(G)$. The other inclusion is similar. □

1.4 Some Representation Theory

Since we are interested in representations of finite groups, Lie groups and Lie algebras, the definition of a representation in these contexts is slightly different. It should always be obvious which case we consider when we mention a representation. Specific results for finite groups are postponed to Chapter 3. We only treat the basic notions needed to understand the arguments in Chapter 2. As usual, $k = \mathbb{R}$ or $\mathbb{C}$.

Roughly speaking, the general aim of representation theory is to 'realize' abstract objects like groups and algebras by describing their possible linear actions on (not necessarily finite dimensional) vector spaces up to some notion of equivalence. To make this precise, we introduce the following definition:

Definition 1.31: *Let V be a finite-dimensional vector space over k. We have the following three definitions:*

(i) *A representation of a finite group G is a pair (π, V) where $\pi : G \to GL(V)$ is a homomorphism of groups.*

(ii) *A representation of a Lie group G is a pair (π, V) where $\pi : G \to GL(V)$ is a Lie group homomorphism.*

(iii) *A representation of a Lie algebra $\mathfrak{g}$ is a pair (π, V) where $\pi : G \to \mathrm{End}(V)$ is a Lie algebra homomorphism.*

A representation is said to be k-dimensional if dim $V = k$. It is possible to extend these definitions to vector spaces of infinite dimension, but we will not do this here. We will sometimes just write gv for the action of an element g of G on a vector $v \in V$ in the cases (i), (ii). In case (iii), we write Xv for this action.

Definition 1.32: *An invariant subspace of a representation (π, V) of a group, Lie group or Lie algebra is a subspace W such that $gW \subset W$ for all $g \in G$ (resp. $XW \subset W$ for all $X \in \mathfrak{g}$). A representation is called irreducible if the only invariant subspaces are $\{0\}$ and V.*

Definition 1.33: *Let $(\pi_1, V_1), (\pi_2, V_2)$ be representations the same group, Lie group or Lie algebra. An intertwiner is a linear map $\psi : V_1 \to V_2$ which satisfies $\pi_2(g)(\psi(v_1)) = \psi(\pi_1(g)(v_1))$ for all $v_1 \in V_1, g \in G$ in the group cases, with the obvious analog (replace g by X) in the Lie algebra case. For example, for the group case the following diagram of vector space isomorphisms commutes for all $g \in G$:*

$$\begin{tikzcd} V_1 \arrow[r, "\psi"] & V_2 \\ V_1 \arrow[u, "\pi_1(g)"] \arrow[r, "\psi"'] & V_2 \arrow[u, "\pi_2(g)"'] \end{tikzcd}$$

Lemma 1.34: *Let* $\mathfrak{g}$ *be a simple Lie algebra. Then the adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is irreducible.*

Proof: Suppose ad admits an invariant subspace W. Then $[X, W] \subset W \quad \forall X \in \mathfrak{g}$, but this precisely means that W is an ideal. By virtue of simplicity, this can only be in one of the two extreme cases: $W = \{0\}$ or $W = \mathfrak{g}$. $\square$

Lemma 1.35: *Let* G *be a simple Lie group, i.e. one with a simple Lie algebra. The adjoint representation* Ad*:* $G \to GL(\mathfrak{g})$ *is an irreducible representation of* G*.*

Proof: Suppose the the representation Ad of G on its Lie algebra $\mathfrak{g}$ admits a nontrivial invariant subspace $V \subset \mathfrak{g}$. Then for any $t \in \mathbb{R}, X \in \mathfrak{g}$, $\mathrm{Ad}(\exp(tX)V = e^{\mathrm{ad}(tX)}V \subset V$. Differentiating at 0 we infer that $\mathrm{ad}(X)V \subset V$. But then V is a nontrivial ideal in $\mathfrak{g}$, contradiction. Hence Ad is irreducible. $\square$

Two representations are called isomorphic if there is an intertwiner between them which is also an isomorphism of vector spaces. The only elementary result we need to introduce is Schur's lemma on irreducible representations. To this end, we need some preliminaries:

Lemma 1.36: *Let* $(\pi_1, V_1), (\pi_2, V_2)$ *be representations of the same Lie group or Lie algebra. If* ψ *is an intertwiner between these representations, then* $\ker(\psi)$ *is an invariant subspace of* V_1 *and* $\mathrm{Im}(\psi)$ *is an invariant subspace of* V_2*. As a consequence, if* V_1 *is irreducible then* $\psi = 0$ *or* ψ *in injective and if* V_2 *is irreducible, then* $\psi = 0$ *or* ψ *is surjective.*

Proof: Suppose $v \in \ker(\psi)$. Then if $g \in G$, we have $\psi(gv) = g(\psi v) = g(0) = 0$ hence $gv \in \ker\psi$. This proves the first part. Next, if $v \in \mathrm{Im}(\psi)$, then $v = \psi(w)$, and $gv = g\psi(w) = \psi(gw)$ so $gv \in \mathrm{Im}(\psi)$. In the case of a Lie algebra, the proof is identical. The consequences are immediate, because ($\ker(\psi) = V$ resp. $\{0\}$) $\iff$ ($\psi = 0$ resp. ψ is injective), and furthermore ($\mathrm{Im}(\psi) = \{0\}$ resp. V) $\iff$ ($\psi = 0$ resp. ψ is surjective). $\square$

Lemma 1.37 (Schur's lemma): *Let* (π, V) *be an irreducible representation over* k *of a group, Lie group or Lie algebra. If* ψ *is an intertwiner from* V *to* V *and it admits an eigenvalue* $\lambda \in k$*, then* $\psi = \lambda \cdot \mathrm{Id}$.

Proof: Since Id is certainly an intertwiner, we easily see that $\psi - \lambda \cdot \mathrm{Id}$ is an intertwiner as well. But then by Lemma 1.36, it must be either zero or an isomorphism. It cannot be an isomorphism because its kernel is nontrivial (it contains the eigenspace for λ). Hence it is zero, from which the assertion follows. $\square$

Observe that for $k = \mathbb{C}$, the existence of an eigenvalue is guaranteed as a complex polynomial always has a zero somewhere.

Definition 1.38: *Let $(\pi_1, V_1), (\pi_2, V_2)$ be representations of a group, Lie group or Lie algebra. We can form their direct sum, denoted $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ where $(\pi_1 \oplus \pi_2)(g)(v_1 \oplus v_2) = \pi_1(g)(v_1) \oplus \pi_2(g)(v_2)$.*

Definition 1.39: *Let $(\pi_1, V_1), (\pi_2, V_2)$ be representations of a group or Lie group. Then we can also form the tensor product of two representations, denoted $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ where $(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2$. If $(\pi_1, V_1), (\pi_2, V_2)$ are representations of a Lie algebra, their tensor product $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ is defined by $(\pi_1 \otimes \pi_2)(X)(v_1 \otimes v_2) = \pi_1(X)v_1 \otimes v_2 + v_1 \otimes \pi_2(X)v_2$.*

Throughout the rest of this thesis, we will be interested in representations on finite-dimensional Hilbert spaces. Recall the definition of a Hilbert space:

Definition 1.40: *A Hilbert space is a vector space V over k with a complete inner product, i.e. a positive definite symmetric bilinear (or sesquilinear) form $(\cdot, \cdot)$ which induces a complete metric.*

Given a linear map U on a finite-dimensional Hilbert space V, we have the notion of an adjoint map or Hermitian conjugate, which is the unique linear map $U^\dagger : V \to V$ satisfying $(v, Uw) = (U^\dagger v, w)$ for all $v, w \in V$. The extra structure of an inner product on the vector space V naturally leads us to consider representations which preserve this inner product:

Definition 1.41: *Let V be a finite-dimensional Hilbert space. A unitary representation of a finite group or Lie group is a representation (π, V) where $\pi : G \to \mathcal{U}(V)$. Here $\mathcal{U}(V)$ denotes the subgroup of $GL(V)$ consisting of all unitary maps from V to V, i.e. $\mathcal{U}(V) = \{U \in GL(V) \mid U^\dagger U = I\}$.*

In the case of a Lie group, smoothness still makes sense because $\mathcal{U}(V)$ is readily seen to be smoothly embedded in $GL(V)$.

Given a representation (π, V) of a group or Lie group on a finite-dimensional vector space over k, one might ask if there exists a complete inner product on V which makes (π, V) into a unitary representation. If this is the case, the representation is called *unitarizable.*

Definition 1.41: *Let (π, V) be a representation of a group, Lie group or Lie algebra. We say that (π, V) is completely reducible if V is, as a representation, isomorphic to a direct sum of irreducible representations.*

We will see in Chapter 3 that unitarizability is closely related to the concept of complete reducibility. This concludes our first discussion on representations.

Chapter 2

Unitary Representations of Non-Compact Groups

This chapter is devoted to proving that all unitary representations of a non-compact connected simple Lie group are infinite-dimensional. This is a first step towards appreciating the complexity of the physically relevant representations of the non-compact Lorentz group. In particular, this theorem can be applied to the connected component $SO(1,3)_e$ of the Lorentz group. We start with some algebraic properties of simple Lie algebras. In the second section, we address the closedness of the adjoint map using some of our general knowledge of Lie groups from Chapter 1. In the third and final section, we employ some covering space theory and a result of H. Weyl to provide the remainder of the proof.

2.1 Formal Statement

We prove the following:

Theorem 2.1: *Let G be a non-compact connected Lie group whose Lie algebra $\mathfrak{g}$ is simple and suppose ρ is a unitary representation of G. Then ρ is not finite dimensional.*

This shows that the unitary representations of the non-compact Lorentz group are not so easy to understand. In order to prove this, we will need a few preliminary results. First, we establish some facts about simple Lie algebras. Although some of the results have straighforward generalizations, we omit them here for the sake of clarity.

Lemma 2.2:. *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. Then its Killing form κ cannot be positive definite.*

Proof:. Assume the contrary. Then since κ is positive definite, symmetric and real bilinear, it can be viewed as an inner product on $\mathfrak{g}$. Let $O(\mathfrak{g}, \kappa)$ be the group of orthogonal transformations with respect to κ. Note that by automorphism invariance of the Killing form, any automorphism of Lie algebras belongs to this group. By picking a basis for $\mathfrak{g}$ which is orthonormal with respect to κ, we obtain an identification with the familiar orthogonal group $O(n, \mathbb{R})$ where $n = \dim(\mathfrak{g})$. In particular, we see that for all $t \in \mathbb{R}$ and all $X, Y, Z \in \mathfrak{g}$,

$$\kappa(\mathrm{Ad}(\exp tX)Y, \mathrm{Ad}(\exp tX)Z) = \kappa(Y, Z) \tag{2.1}$$

Evaluating the derivative of the above expression on both sides at t=0 we obtain

$$\kappa(\mathrm{ad}(X)Y, Z) + \kappa(Y, \mathrm{ad}(X)Z) = 0 \tag{2.2}$$

This simply means that the matrix representation of $\mathrm{ad}(X)$ is antisymmetric w.r.t. an orthonormal basis for $\mathfrak{g}$ (recall that the Lie algebra of $O(n, \mathbb{R})$ consists of precisely the antisymmetric matrices, i.e. those satisfying $B^T = -B$). Suppose X is nonzero. Observe that we now know that $\mathrm{ad}(X)$ commutes with its transpose and hence by familiar linear algebra it is diagonalizable. Denoting its eigenvalues by λ_i, we can compute κ on its diagonal:

$$\kappa(X, X) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(X)) = -\mathrm{tr}(\mathrm{ad}(X)^T\mathrm{ad}(X)) = -\sum \lambda_i^2 < 0. \tag{2.3}$$

Here we used the fact that not all eigenvalues can be zero, because if this were the case, $\mathrm{ad}(X) = 0$ contradicting the fact that ad is injective for semisimple Lie algebras. We obtain a contradiction, hence κ cannot be positive definite. $\square$

Phrased differently, for any Lie algebra over $\mathbb{R}$ with definite Killing form, this Killing form is necessarily negative definite.

Proposition 2.3: *Let G be a Lie group. Suppose that its Lie algebra $\mathfrak{g}$ is simple and furthermore G admits a nontrivial finite dimensional unitary representation ρ on some Hilbert space V. Then the Killing form on $\mathfrak{g}$ is negative definite.*

Proof: Note that by picking an orthonormal basis for V, we can identify $\mathcal{U}(V)$ with the unitary group $U(n)$ where $n = \dim(V)$. We can differentiate the representation ρ at the identity to obtain a group homomorphism $\rho_* : \mathfrak{g} \to \mathfrak{u}(n)$. The kernel of this homomorphism is an ideal of $\mathfrak{g}$, so it is 0 or $\mathfrak{g}$. By the nontriviality assumption, it must be 0 so ρ_* is injective.

Define a form $(\cdot, \cdot)$ on $M_n(\mathbb{C})$ by $(X, Y) := \mathrm{tr}(X^\dagger Y)$. This is clearly sesquilinear (we will always adopt the convention that it is conjugate linear in the first argument). Let us restrict this form to $\mathfrak{u}(n)$. One can verify that $(\cdot, \cdot)|_{\mathfrak{u}(n)}$ is real bilinear (i.e. it maps into $\mathbb{R}$). Using that $X \in \mathfrak{u}(n) \iff X^\dagger = -X$, we infer that all elements are diagonalizable and hence this form is positive definite, so it is an inner product. We define

$$\beta(X,Y) := (\rho_*(X), \rho_*(Y)) \tag{2.4}$$

This is an inner product on $\mathfrak{g}$ (it is positive definite because ρ_* is injective). We claim that it is $\mathrm{Ad}(G)$-invariant. To see this, note that ρ is certainly a homomorphism of groups and so for arbitrary $x \in G$ we have

$$\rho \circ C_x = C_{\rho(x)} \circ \rho \tag{2.5}$$

as maps from G to itself, where C_x denotes conjugation by x. Taking the tangent map at the identity element e of G, we obtain

$$\rho_* \circ \mathrm{Ad}(x) = \mathrm{Ad}(\rho(x)) \circ \rho_*. \tag{2.6}$$

Also, one may verify that the inner product $(\cdot,\cdot)$ is invariant under conjugation by elements of $U(n)$. This follows from the cyclicity of the trace operator: $\mathrm{tr}(UYU^{-1}) = \mathrm{tr}(Y)$. Also, for a matrix Lie group H we see that conjugation by A is given by $C_A(B) = ABA^{-1}$, which is a linear map. Hence for $B \in T_eH$, $\mathrm{Ad}(A)(B) = T_eC_A(B) = ABA^{-1}$. We apply this to infer that

$$\begin{aligned}
\beta(\mathrm{Ad}(x)X, \mathrm{Ad}(x)Y) &= (\rho_*(\mathrm{Ad}(x)X), \rho_*(\mathrm{Ad}(x)Y)) \\
&= (\rho(x)\rho_*(X)\rho(x)^{-1}, \rho(x)\rho_*(Y)\rho(x)^{-1}) \\
&= (\rho_*(X), \rho_*(Y)) \\
&= \beta(X,Y)
\end{aligned} \tag{2.7}$$

establishing the assertion. If we denote the Killing form on $\mathfrak{g}$ by κ again, we see that simplicity of $\mathfrak{g}$ implies its nondegeneracy by Cartan's criterion. This means that it defines an isomorphism $\tilde{\kappa}$ from $\mathfrak{g}$ to its dual $\mathfrak{g}^*$. Explicitly, we define $\tilde{\kappa}(X)(Y) := \kappa(X,Y)$. Since we already proved that β is nondegenerate as well, we can define $\tilde{\beta}$ in a similar way. We obtain an isomorphism $\tilde{\kappa}^{-1} \circ \tilde{\beta} : \mathfrak{g} \to \mathfrak{g}$. We now need the following fact:

Lemma 2.4: *The isomorphism $\tilde{\kappa}^{-1} \circ \tilde{\beta} : \mathfrak{g} \to \mathfrak{g}$ is an* Ad*-intertwiner.*

Proof: Unwinding the definitions, we see that for arbitrary $x \in G$ and $X, Y \in \mathfrak{g}$ the following holds:

$$\begin{aligned}
&\tilde{\kappa}^{-1} \circ \tilde{\beta} \circ \mathrm{Ad}(x)(X) = Y \\
\iff &\tilde{\beta} \circ \mathrm{Ad}(x)(X)(Z) = \kappa(Y,Z) \quad \forall Z \in \mathfrak{g} \\
\iff &\beta(\mathrm{Ad}(x)X, Z) = \kappa(Y,Z) \quad \forall Z \in \mathfrak{g} \\
\iff &\beta(\mathrm{Ad}(x)X, \mathrm{Ad}(x)W) = \kappa(Y, \mathrm{Ad}(x)W) \quad \forall W \in \mathfrak{g} \\
\iff &\beta(X,W) = \kappa(\mathrm{Ad}(x)^{-1}Y, W) \quad \forall W \in \mathfrak{g} \\
\iff &\tilde{\kappa}^{-1} \circ \tilde{\beta}(X) = \mathrm{Ad}(x)^{-1}Y \\
\iff &\mathrm{Ad}(x) \circ \tilde{\kappa}^{-1} \circ \tilde{\beta}(X) = Y
\end{aligned} \tag{2.8}$$

We conclude that the two maps on the top and bottom lines are equal. This establishes the claim. □

End of proof of Proposition 2.3: By Lemma 1.36, Ad is irreducible. But since $\tilde{\kappa}^{-1} \circ \tilde{\beta} : \mathfrak{g} \to \mathfrak{g}$ admits a real eigenvalue (it is symmetric), we can apply Schur's lemma. It follows that $\tilde{\kappa}^{-1} \circ \tilde{\beta} : \mathfrak{g} \to \mathfrak{g}$ is a nonzero scalar multiple of the identity. But then β must be a nonzero scalar multiple of κ, the former of which is definite. Hence κ is definite. By Lemma 2.2, it is negative definite. This proves Proposition 2.3. □

We state the following theorem and use it to prove Theorem 2.1:

Theorem 2.5: *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ with negative definite Killing form. Then if G is a Lie group whose Lie algebra equals $\mathfrak{g}$, the group G is necessarily compact.*

Proof of Theorem 2.1: This is a matter of combining our previous results: Suppose ρ is finite dimensional. Then by Proposition 2.4, its Killing form is negative definite. By Theorem 2.5, this implies that G is compact which is obviously a contradiction. This proves Theorem 2.1. □

2.2 Closedness of the Adjoint Image

It remains to prove Theorem 2.5. The rest of this chapter will be devoted to this (in addition to the appendix on more advanced algebraic notions). There are several ways to do this. For example, one can use methods from Riemannian geometry in order to prove that G can be given the structure of a complete Riemannian manifold with Ricci curvature bounded below by a strictly positive constant. By the Bonnet-Myers theorem, G is then necessarily compact. For details, see [6]. We will not pursue this method here. Instead, we take a different route and first prove the following:

Proposition 2.6: *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ is simple and has negative definite Killing form. Then the* Ad*-image of G is a compact subgroup of $GL(\mathfrak{g})$, the space consisting of invertible linear maps from $\mathfrak{g}$ to itself.*

The proof of this proposition will require a preliminary algebraic result, for which we need the notion of derivations of a Lie algebra.

Definition 2.7: *Let $\mathfrak{g}$ be a Lie algebra. The space of derivations of $\mathfrak{g}$ denoted by* $\mathrm{Der}(\mathfrak{g})$ *is the linear subspace of* $\mathrm{End}(\mathfrak{g})$ *consisting of precisely those endomorphisms D satisfying the Leibniz rule $D[X,Y] = [DX,Y] + [X,DY]$ for $X, Y \in \mathfrak{g}$ arbitrary.*

The space $\mathrm{Der}(\mathfrak{g})$ defines a Lie subalgebra of $\mathrm{End}(\mathfrak{g})$, as is easily verified.

Lemma 2.8: *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then* $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$*.*

Proof: The ad-image of any vector in $\mathfrak{g}$ is a derivation, by straightforward application of the Jacobi identity. The nontrivial part is the other inclusion. For this, note that the Killing form of $\mathfrak{g}$ is nondegenerate by Cartan's criterion. The ad-image I of $\mathfrak{g}$ is an ideal in $\mathrm{Der}(\mathfrak{g})$. Indeed,

$$\begin{aligned} [D, \mathrm{ad}X](Y) &= D(\mathrm{ad}X(Y)) - \mathrm{ad}X(DY) \\ &= [DX, Y] + [X, DY] - [X, DY] \\ &= \mathrm{ad}(DX)(Y) \end{aligned} \tag{2.9}$$

for arbitrary $D \in \mathrm{Der}(\mathfrak{g})$ and $X, Y \in \mathfrak{g}$. Hence the Killing form κ on $\mathrm{Der}(\mathfrak{g})$ restricted to I really is the Killing form on I. But I is isomorphic to $\mathfrak{g}$ because ad is injective, so the Killing form is nondegenerate on $I \subset \mathrm{Der}(\mathfrak{g})$. Hence the orthocomplement $I^\perp$ (cf. Lemma 1.23) of I in $\mathrm{Der}(\mathfrak{g})$ w.r.t. κ satisfies $I^\perp \cap I = \{0\}$ (we only need nondegeneracy on I for this). But $I^\perp$ is an ideal by invariance of κ, so $[I, I^\perp] \subseteq I \cap I^\perp = \{0\}$. But then if D' is not an inner derivation, we may assume without loss of generality that $D' \in I^\perp$, so that $\mathrm{ad}(D'X) = [D', \mathrm{ad}X] = 0$ for all $X \in \mathfrak{g}$, so by injectivity of ad, D' is the zero derivation. □

Lemma 2.9: *Let $\mathfrak{g}$ be a Lie algebra and let* $\mathrm{Aut}(\mathfrak{g})$ *be its automorphism group. Then* $\mathrm{Aut}(\mathfrak{g})$ *a closed subset of $GL(\mathfrak{g})$ and has Lie algebra equal to* $\mathrm{Der}(\mathfrak{g})$*.*

Proof: Let $D \in \mathrm{End}(\mathfrak{g})$. Then $D \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) \iff \exp(tD) \in \mathrm{Aut}(\mathfrak{g})$ for $t \in \mathbb{R}$. Simply expanding the definitions, $\exp(tD)[X, Y] = [\exp(tD)X, \exp(tD)Y]$ for $t \in \mathbb{R}$. We differentiate at $t = 0$ to obtain the defining property of a derivation. For the converse inclusion, note that an induction argument yields the binomial identity

$$D^n[X, Y] = \sum_{j=0}^{n} [D^j X, D^{n-j} Y] \binom{n}{j} \tag{2.10}$$

By summing over all n and changing the summation indices, it is not hard to see that

$$\begin{aligned} \exp(tD)[X, Y] &= \sum_{n=0}^{\infty} \frac{D^n[X, Y]}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} [D^j X, D^{n-j} Y] \binom{n}{j} \frac{1}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} [\frac{D^j X}{j!}, \frac{D^{n-j} Y}{(n-j)!}] \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} [\frac{D^k X}{k!}, \frac{D^m Y}{m!}] = [\sum_{k=0}^{\infty} \frac{D^k X}{k!}, \sum_{m=0}^{\infty} \frac{D^m Y}{m!}] \\ &= [\exp(tD)X, \exp(tD)Y] \end{aligned} \tag{2.11}$$

To see that the automorphism group is closed, let $X, Y \in \mathfrak{g}$. Then the set $K_{X,Y} = \{\phi \in GL(\mathfrak{g}) \mid \phi[X,Y] = [\phi(X), \phi(Y)]\}$ is closed in $GL(\mathfrak{g})$, because the left hand side is linear in ϕ and the right hand side the restriction of a bilinear function to its diagonal, so that $\phi \mapsto \phi[X,Y] - [\phi(X), \phi(Y)]$ is a continuous function of ϕ. This means that $\mathrm{Aut}(\mathfrak{g}) = \cap_{X,Y} K_{X,Y}$ is closed in $GL(\mathfrak{g})$ as well. This proves Lemma 2.9. □

We see that the automorphism group sits nicely as an embedded submanifold inside $GL(\mathfrak{g})$. This enables us to finally prove Proposition 2.6.

Proof of Proposition 2.6: We know that the image of any Lie group homomorphism can be given the structure of a Lie subgroup (see Theorem 21.27 of [13]). We apply this to the case of $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$. The center of G is a closed Lie subgroup of G, as it is the kernel of Ad. Its Lie algebra is given precisely by the center of $\mathfrak{g}$, which is trivial by assumption. It follows that the center of the group is zero-dimensional (and hence discrete). We infer that the dimension of the image equals that of G itself, so by injectivity of ad we obtain dim ad $(\mathfrak{g})$ = dim Lie $(\mathrm{Ad}(G))$. It is clear that $\mathrm{e}^{(\mathrm{ad}_{tX})} = \mathrm{Ad}(\exp tX)$ so that $\mathrm{ad}(\mathfrak{g})$ is contained in the Lie algebra of $\mathrm{Ad}(G)$. For dimensional reasons this is an equality. Summarizing, $\mathrm{Ad}(G)$ and the identity component $\mathrm{Aut}(\mathfrak{g})_e$ have equal Lie algebras and are connected. By uniqueness, they are equal: $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})_e$. But the latter is clearly closed, being a path component of a closed subgroup. In particular, $\mathrm{Ad}(G)$ is an embedded submanifold (it carries the subspace topology, which was the trickiest part of this proof). We will make explicit use of this fact in the topological argument given below.

We employ some properties of the Killing form κ: it is Ad-invariant, so the image $\mathrm{Ad}(G)$ is contained in $O(\mathfrak{g}, -\kappa)$, the orthogonal group defined by the inner product $-\kappa$. The latter is clearly compact, and closed subsets of compacts are compact. □

2.3 Using Covering Space Theory

In order to finish the proof of Theorem 2.5, we will need to make use of some of basic facts about the theory of covering spaces. Additionally, a deeper result on the universal covering group of a compact Lie group is needed. This is a result due to H. Weyl, which we now formulate:

Proposition 2.10 (Weyl): *Let G be a connected compact semisimple Lie group. Then, the universal covering manifold of G, which is a Lie group in its own right, is compact as well.*

The proof of this is postponed to Appendix A in order to avoid long digressions not following the path we intend to take here. This appendix also contains the results on covering spaces we need in order to prove Theorem 2.5. We need two more preliminary results. They are stated below and allow us to immediately provide a proof of Theorem 2.5.

Lemma 2.11: *If H is a discrete subgroup of a Lie group G, there exists a neighborhood U of $e \in G$ such that $hU \cap U = \emptyset$ for all $h \neq e$.*

Proof: We argue by contradiction. Since G is first countable, we can pick a countable basis of neighborhoods $\{U_n \mid n \in \mathbb{N}\}$ of the identity element. Then for all n there is an element $h_n \in H \backslash e$ for which there is an element $y_n \in h_n U_n \cap U_n \neq \emptyset$. But then since the U_n are a basis, the sequence y_n must converge to e. But $h_n^{-1} y_n \in U_n$, so this sequence also converges to e. It follows that h_n^{-1} and hence also h_n must converge to e. But this is only possible if $h_n = e$ for large enough n, because by discreteness of H there is a neighborhood V of e whose intersection with H is just e. We reach a contradiction. $\square$

Lemma 2.12: *For G a simple Lie group, the map* Ad*: $G \to \mathrm{Ad}(G)$ is a smooth covering map.*

Proof: It is clear from the proof of Proposition 1.6 that Ad is still smooth when we restrict the codomain to the image (after all, it is an embedded submanifold). It is a homomorphism of Lie groups. Tautologically, it is surjective. We saw before that its kernel is given precisely by the center of G, which is zero-dimensional. We know that the tangent map at $e \in G$ of Ad is given by ad, which is injective. It follows that Ad is immersive at $e \in G$. By Theorem 1.8, it is immersive everywhere and it must be a submersion as well (alternatively, one can use that a surjective map of constant rank is a submersion). It then follows from the inverse function theorem that Ad must be a local diffeomorphism everywhere.

We still need to find a trivializing neighborhood for each point in $\mathrm{Ad}(G)$. By another homogeneity argument, it suffices to show this for $\mathrm{Id} \in \mathrm{GL}(\mathfrak{g})$. As a trivializing neighborhood, we can pick U as in the result of Lemma 2.11 (just put $H = Z(G)$). Then since $yU \cap U = \emptyset \quad \forall y \neq e, y \in H$, we also have $yU \cap zU = z(z^{-1}yU \cap U) = \emptyset \quad \forall z \neq y \in H$. This means that $\mathrm{Ad}^{-1}(U) = \sqcup_{y \in Z(G)} yU$, which is a disjoint union of open sets. Ad restricted to each of these open sets is a diffeomorphism, because $yU \cap Z(G) = \{y\}$, so that Ad is injective on yU. $\square$

Proof of Theorem 2.5: By Lemma 2.12, the map Ad: $G \to \mathrm{Ad}(G)$ is a smooth covering map. Also, we know that the universal cover $\widetilde{\mathrm{Ad}(G)}$ of $\mathrm{Ad}(G)$ is compact by Proposition 2.10 (note that $\mathrm{Ad}(G)$ is simple because G is, and their Lie algebras are isomorphic precisely because one covers the other). By the characteristic property of the universal cover, the universal cover also covers G. This implies in particular that there is a surjective continuous map $\tilde{G} \to G$ (see Lemma A.21 in Section 3 of the appendix). By compactness of $\tilde{G}$, the Lie group G is itself compact. $\square$

Chapter 3

Mackey Theory

In this chapter we probe some of the more specific results of the representation theory of finite groups. In this chapter, all groups are finite. In general, given a group G, one would like to construct and classify all irreducible representations of G. It turns out that in the case of finite groups (or more generally, compact Lie groups) all representations can be constructed as *direct sums* of irreducibles! Hence if we manage to find all irreducibles, we essentially understand all representations. However, this is generally quite difficult. In certain cases, it is possible to construct all irreducibles of a group G from irreducibles of certain subgroups H of G by a process called *induction*: given a representation of a subgroup H, we can construct from it a representation of G in a canonical way.

First, we formulate some general theory on the decomposition of arbitrary representations. Then, we apply this to groups which are semidirect products of an arbitrary group H and an abelian group N. It turns out that here, all irreducibles of $G = H \ltimes N$ are induced from certain representations of its subgroups.

Once this theory is suitably formulated, this gives rise to the notion of a *system of imprimitivity* which is a tool for analyzing representations that are obtained by induction of representations of subgroups, not just in the finite case. In Chapter 4, we will see how this has a profound significance in theoretical physics. The reader may also consult Chapter 6 of [17] for a broad treatment of this concept.

3.1 Decompositions of Representations

We begin by establishing an important relation between unitarizability and complete reducibility, as mentioned at the end of Section 1.4.

Lemma 3.1: *Let G be a group and (π, V) a representation of G. If π is unitarizable, then π is completely reducible.*

Proof: Denote the invariant form by β. The key idea is that if W is an invariant subspace of V, then so is its orthocomplement with respect to the inner product. Indeed, if w' is such that $\beta(w, w') = 0$ for all $w \in W$, then $\beta(w, gw') = \beta(g^{-1}w, w') = 0$. Now, suppose the result is proven for $\dim(V) < n$. Then if $\dim(V) = n$, we either have that V is irreducible, in which case we are done, or V admits an invariant subspace W, in which case we write $V = W \oplus W^{\perp}$ and apply the hypothesis to W and $W^{\perp}$. Also, the case $n = 1$ is clear. This concludes the argument. $\square$

Lemma 3.2: *Let G be a finite group and (π, V) a representation of G. Then π is unitarizable.*

Proof: Endow V with any (positive definite) inner product (for example, we can pick a basis and pull back the Euclidean inner product from $\mathbb{R}^n$ or $\mathbb{C}^n$). Denote it by $(\cdot, \cdot)$. Define a new *averaged* inner product by $\langle v, w\rangle = \sum_{g \in G}(gv, gw)$. It is easily checked that $\langle hv, hw\rangle = \sum_{g\in G}(ghv, ghw) = \sum_{g\in G}(gv, gw) = \langle v, w\rangle$ for any $h \in G$, which is what we wanted. Furthermore this averaged inner product is obviously bilinear (or sesquilinear in the complex case) and $\langle v, v\rangle = 0$ implies $(gv, gv) = 0$ for all $g \in G$, so that $v = 0$. Hence we indeed have an inner product. $\square$

As a simple corollary, finite dimensional representations of finite groups are completely reducible. This means we can write any representation (π, V) as a direct sum of irreducible components.

From now on, we assume for simplicity that $k = \mathbb{C}$. Denote by $\mathbb{C}(G)$ the space of complex-valued functions on G.

It turns out that up to isomorphism, a finite group only admits finitely many irreducible representations (see Lemma 3.8). Denote them by $W_1, \ldots, W_k$. Then we can write

$$V \cong \bigoplus_{i=1}^{k} W_i^{\oplus n_i} \quad ; \quad W_i^{\oplus n_i} = \underbrace{W_i \oplus W_i \cdots \oplus W_i}_{n_i \text{ times}} \tag{3.1}$$

It turns out that this decomposition is unique (up to ordering of the irreducible representations). We do not go into details here. One may consult Chapter 2 of [15].

A convenient way to study representations is by means of their *characters*. We will later see that representations of finite groups are completely determined by their characters, hence the name. It turns out that we can investigate the irreducibility of a representation by looking at its character.

Definition 3.3: *Let G be a group and let (π, V) be a representation of G. The character of this representation, denoted χ_V or simply χ if the representation is understood,* *is the element of* $\mathbb{C}(G)$ *defined by*

$$\chi(g) = \chi_V(g) = \mathrm{Tr}_V \pi(g) \qquad (g \in G) \tag{3.2}$$

Lemma 3.4: *Let* χ_1, χ_2 *be two characters of (representations of) a group* G*. Then for any* $g, h \in G$ *the following elementary properties hold:*

(i) $\chi(g) = \chi(hgh^{-1})$

(ii) $\chi(e) = \dim(V)$, *the dimension of the representation* V*.*

(iii) $\chi(g^{-1}) = \chi(g)^*$ *(here* z^* *is the complex conjugate of* $z \in \mathbb{C}$*)*

Proof: Property (i) follows from cyclicity of the trace. For (ii), note that $\mathrm{Tr}(\mathrm{Id}) = \dim(V)$. Lastly, (iii) follows from unitarizability of the representation. A unitary matrix U satisfies $U^{-1} = (U^T)^*$ and transposition leaves the trace invariant, so we are done. □

Definition 3.5: *Let* $\mathbb{C}(G)$ *denote the space of complex-valued functions on* G*. Let* $\phi, \psi \in \mathbb{C}(G)$*. Define a form* $\langle \cdot \mid \cdot \rangle$ *on* $\mathbb{C}(G)$ *by setting*

$$\langle \phi \mid \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)^* \psi(g) \tag{3.3}$$

By straightforward verification, this is a Hermitian inner product on $\mathbb{C}(G)$. Note that by finiteness of G, this vector space is finite dimensional. We now come to an important statement on irreducibility:

Proposition 3.6: *Let* (π, V) *and* (ρ, W) *be any two representation of a group* G*. Then we have the following*:

(i) *If* V, W *are non-isomorphic irreducibles, then* $\langle \chi_V \mid \chi_W \rangle = 0$.

(ii) *If* V *is irreducible, then* $\langle \chi_V \mid \chi_V \rangle = 1$.

(iii) *The number of times an irreducible representation* W *occurs in the decomposition of* V *equals* $\langle \chi_V \mid \chi_W \rangle$.

(iv) *If* $\langle \chi_V \mid \chi_V \rangle = 1$*, then* V *is irreducible.*

Proof: First, let $\Psi : V \to W$ be any linear map. We define a new linear map by

$$\Psi_0 = \frac{1}{|G|} \sum_{g \in G} \pi(g) \circ \Psi \circ \rho(g)^{-1} \tag{3.4}$$

This map intertwines the representations π and ρ. Indeed, for arbitrary $h \in G$ we have

$$\pi(h) \circ \Psi_0 \circ \rho(h)^{-1} = \frac{1}{|G|} \sum_{g \in G} \pi(h)\pi(g) \circ \Psi \circ \rho(g)^{-1} \rho(h)^{-1} \tag{3.5}$$

$$= \frac{1}{|G|} \sum_{g \in G} \pi(hg) \circ \Psi \circ \rho(hg)^{-1} = \frac{1}{|G|} \sum_{g \in G} \pi(g) \circ \Psi \circ \rho(g)^{-1} = \Psi_0 \tag{3.6}$$

Note that if $\pi = \rho$, then $\mathrm{Tr}(\Psi_0) = \mathrm{Tr}(\Psi)$. We can use this in order to compute the inner product between two characters. A computation shows that in orthonormal bases $\{e_i\}, \{f_j\}$ with respect to the inner product (20) for V, W respectively we have

$$\begin{aligned}
\langle \chi_V \mid \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)^* \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^{-1}) \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} \langle e_i | \pi(g) | e_i \rangle \langle f_j | \rho(g^{-1}) | f_j \rangle \\
&= \sum_{i,j} \langle e_i | \left(\frac{1}{|G|} \sum_{g \in G} \pi(g) | e_i \rangle \langle f_j | \rho(g^{-1}) \right) | f_j \rangle
\end{aligned} \tag{3.7}$$

The map $\frac{1}{|G|} \sum_{g \in G} \pi(g) | e_i \rangle \langle f_j | \rho(g^{-1})$ is an intertwiner for each value of i, j, so if V and W are non-isomorphic irreducibles it follows that this is the zero map. This proves (i). For (ii), we take $\pi = \rho$ in (3.7) and $e_i = f_i$ and see that again by Schur's lemma, $\frac{1}{|G|} \sum_{g \in G} \pi(g) | e_i \rangle \langle f_j | \pi(g^{-1})$ is a multiple of Id_V. Using the above, we get

$$\delta_{ij} = \mathrm{Tr}(|e_i\rangle\langle e_j|) = \mathrm{Tr}\left(\frac{1}{|G|} \sum_{g \in G} \pi(g) | e_i \rangle \langle e_j | \pi(g^{-1}) \right) = \mathrm{Tr}(\lambda_{ij} \mathrm{Id}_V) = \lambda_{ij} \dim(V) \tag{3.8}$$

whence $\lambda_{ij} = \frac{\delta_{ij}}{\dim(V)}$. Inserting this back into (3.7) and carrying out the sum over i, j yields (ii). For (iii), we decompose the representation V according to (3.1) and note that $W = W_j$ for some j. Using Lemma 3.4, we have $\chi_V = \sum_{i=1}^k n_i \chi_{W_i}$ and so

$$\langle \chi_V \mid \chi_W \rangle = \langle \chi_V \mid \chi_{W_j} \rangle = \langle \sum_{i=1}^k n_i \chi_{W_i} \mid \chi_{W_j} \rangle = \sum_{i=1}^k n_i \delta_{ij} = n_j \tag{3.9}$$

as desired. Finally, (iv) follows from the fact that $\langle \chi_V \mid \chi_W \rangle = \sum_{i=1}^k n_i^2$ which is 1 if and only if all terms are zero except for one index j for which $n_j = 1$. □

As a simple consequence, two representations are isomorphic if and only if they have the same character. Lastly, one can derive the following convenient criterion for finding *all* irreducibles of a given group:

Lemma 3.7 *Let $\{W_i\}_{1\leq i\leq k}$ be a complete set of irreducible, pairwise non-isomorphic representations of a group G (i.e. any nonzero irreducible representation of G is isomorphic to W_i for some i). Then we have*

$$\sum_{i=1}^{k} \dim(W_i)^2 = |G| \tag{3.10}$$

Consequently, if $\{W_i\}_{1\leq i\leq k}$ is any set of pairwise non-isomorphic irreducibles of G, then it is complete if and only if the above equality holds.

Proof: This can be found in [15], Corollary 2(a) and Remark 1 in Chapter 2. □

Lemma 3.8: *Let G be a group. Then the number of distinct irreducible representations equals the number of conjugacy classes in G.*

Proof: This is Theorem 7 of Chapter 2 in [15]. □

It should be noted that Lemmata 3.7 and 3.8 admit no generalization to the context of Lie groups, although the lemma below does. We will use it in Section 3.3.

Lemma 3.9: *Let A be an abelian group. Then, all irreducible representations of A are one-dimensional. Furthermore, the irreducible characters of A form a group $A^\vee = \mathrm{Hom}(A, \mathbb{C}^*)$, the group of all group homomorphisms from A to $\mathbb{C}^*$. We have $|A^\vee| = |A|$.*

Proof: Let (π, V) be any irreducible representation of A. Then it is easily seen that $\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$, whence $\pi(h)$ is an intertwiner. But then it must be a scalar by Schur's lemma. Since h was arbitrary, this can only be if the representation is one-dimensional. Next, if χ is its character then $\chi(g) = \mathrm{Tr}_V(\rho(g)) = \rho(g)$ because the representation is one-dimensional. This shows that the characters are homomorphisms.

We would have to prove that the product of two characters again corresponds to an irreducible representation. For this, see [15]. It relies on the classification of finite abelian groups (which is a special case of Proposition A.15 in the appendix: there is only a torsion part). Since A is abelian, the number of conjugacy classes is simply $|A|$. By Lemma 3.8, $|A^\vee| = |A|$. □

Lemma 3.10 (Orbit-Stabilizer): *Let G be a group acting on a finite set X, that is, we have a group homomorphism $G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the group of all bijections from X to X. Denote the orbit of an element $x \in X$ by O_x and the stabilizer of x by G_x. Then $|O_x||G_x| = G$.*

Proof: See Theorem 17.2 of [1].

3.2 Restriction and Induction

Given a representation (π, V) of a group G and a subgroup $H \subset G$, it is easy to come up with a representation H: we can *restrict* the representation of G to H. From now on, we denote this representation of H by $\mathrm{Res}_H^G(\pi)$. Interestingly, given a representation (ρ, W) of H, there is also a canonical way to construct one of G. This process is called *induction*. The resulting representation of G is denoted by $\mathrm{Ind}_H^G(\rho)$. We can compute its character in terms of that of ρ. It turns out that the operations Res and Ind are closely related via the principle of *Frobenius reciprocity*, which we address next. Then we develop a convenient criterion for irreducibility of induced representations which is due to Mackey.

Definition 3.11: *Let G be a group and H a subgroup. Suppose (π, W) is a representation of H. The induced representation, denoted $\mathrm{Ind}_H^G(\pi)$, is defined on the vector space of functions*

$$\mathrm{Ind}_H^G(W) = \{\phi : G \to W \mid \phi(hg) = \pi(h)\phi(g) \quad \forall g \in G, h \in H\} \tag{3.11}$$

Here, the action of G is given by

$$(g \cdot \phi)(g') = \phi(g'g) \tag{3.12}$$

Indeed, if $\phi \in \mathrm{Ind}_H^G(W)$, then $(g \cdot \phi)(hg') = \phi(hg'g) = \pi(h)\phi(g'g) = \pi(h)(g \cdot \phi)(g')$ so this is a well-defined action. Note that (30) is a left action of G, i.e. $g_1 \cdot (g_2 \cdot \phi) = (g_1 g_2) \cdot \phi$.

Recall that a subgroup H of a group G induces a partition of G into cosets. The number of cosets r (called the index of H in G) is seen to be equal to $r = \frac{|G|}{|H|}$. Denote the (right) cosets of H in G by $\{Hg\}_{g \in G}$.

Lemma 3.12: *In the setting of Definition 3.11, we have that for any system of representatives $\{g_1, \ldots g_r\}$ of the right cosets of G in H, the induced representation decomposes as a vector space according to*

$$\mathrm{Ind}_H^G(W) = \bigoplus_r W_r \quad ; \quad W_r = \{\phi \in \mathrm{Ind}_H^G(W) \mid \mathrm{supp}(\phi) \subseteq Hg_r\} \tag{3.13}$$

In particular, $\dim \mathrm{Ind}_H^G(W) = \frac{|G|}{|H|} \dim(W)$.

Proof: It is clear that the sum (3.13) is direct because different cosets are disjoint, and any function can be written as a linear combination of functions supported on specific cosets. By inspecting (3.11), we can see that once ϕ is known on any member of a coset, it is known on the entire coset. Conversely, any vector $w \in W$ defines an element of W_r by setting $\phi(g_r) = w$ and $\phi(g_{r'}) = 0$ for $r \neq r'$. This shows that $\dim(W_r) = \dim(W)$ for all r. □

Definition 3.13: *Let G be a group. Let (π, V) and (ρ, W) be representations of G. Then we define*

$$\mathrm{Hom}_G(V,W) = \{\phi \in \mathrm{Lin}(V,W) \mid \phi(\pi(g)v) = \rho(g)(\phi(v)) \text{ for } g \in G\} \tag{3.14}$$

i.e. it is just the vector space of intertwiners (sometimes called G-equivariant maps) from V to W.

Lemma 3.14: *Let V and V' be representations of a group G. Then we have*

$$\dim(\mathrm{Hom}_G(V,V')) = \langle \chi_V \mid \chi_{V'} \rangle \tag{3.15}$$

Proof: We can decompose V and V' as direct sums of irreducibles according to (3.1). Suppose that $V = \oplus_i n_i W_i$ and $V' = \oplus_i m_i W_i$. Then by Proposition 3.6, the inner product $\langle \chi_V \mid \chi_{V'} \rangle$ equals $\sum_i n_i m_i$. Note that by Schur's lemma, any intertwiner $\phi : V \to V'$ can only map a copy of W_i into a copy of the same W_i. Indeed, the set $\phi(W_i) \cap W_j$ is an invariant subspace of each W_j in the image. By irreducibility of W_j, this can only be $\{0\}$ or W_j. If it is equal to W_j, then W_j must be isomorphic to W_i. This means that we can write $\phi = \oplus_i \phi_i$ where $\phi_i : n_i W_i \to m_i W_i$. Again by Schur's lemma, ϕ_i acts by a scalar on each W_i. Hence each ϕ_i is a linear combination of the maps $\phi_{i,kl} : n_i W_i \to m_i W_i$ which map the k-th copy of W_i in the domain to the l-th copy of W_i in the image by the identity. Hence $\dim(\mathrm{Hom}_G(V,V')) = \sum_i n_i m_i$. This proves the assertion. $\square$

As a remark, another way of saying this is that if $\{W_i\}_i$ is a complete set of irreducible representations of a group G and V is any other representation, then the map

$$\Omega : \bigoplus_i \mathrm{Hom}_G(W_i, V) \otimes W_i \to V \quad ; \quad \phi \otimes w \mapsto \phi(w) \tag{3.16}$$

is an isomorphism.

From now on, if we are given representations V, W of a group G we will write

$$\langle V \mid W \rangle = \langle \chi_V \mid \chi_W \rangle \tag{3.17}$$

We clearly have $\langle V \mid W \rangle = \langle W \mid V \rangle$ because both are (real) integers.

Proposition 3.15: *Let G be a group, H a subgroup. Suppose (π, V) is a representation of G and (ρ, W) a representation of H. Then there is a natural correspondence*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \tag{3.18}$$

The isomorphism $\Psi : \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \to \mathrm{Hom}_H(\mathrm{Res}_H^G(V), W)$ is given by $\Psi(\phi)(v) = \phi(v)(e)$ for $\phi \in \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W))$ and $v \in V$. This is known as Frobenius reciprocity.

Proof: There are a number of things to check. For simplicity, we will denote the action without writing π, ρ each time. First of all, note that $\Psi(\phi)$ is an H-intertwiner, because

$$\begin{aligned}\Psi(\phi)(h\cdot v) &= \phi(h\cdot v)(e) = (h\cdot\phi(v))(e) = \phi(v)(eh)\\ &= \phi(v)(he) = h\cdot(\phi(v)(e)) = h\cdot(\Psi(\phi)(v)).\end{aligned}\tag{3.19}$$

Note that in the fifth equality, we used the definition of the induced representation. Let $\beta\in\mathrm{Hom}_H(\mathrm{Res}^G_H(V),W)$. W define an inverse Ψ' for Ψ by setting $(\Psi'(\beta)v)(x)=\beta(x\cdot v)$ for $v\in V$ and $x\in G$ arbitrary. First we check that $(\Psi'(\beta)v)$ is indeed in $\mathrm{Ind}^G_H(W)$. This follows from

$$(\Psi'(\beta)v)(hx) = \beta((hx)\cdot v) = \beta(h\cdot(x\cdot v)) = h\cdot(\beta(x\cdot v)) = h\cdot(\Psi'(\beta)v)(x).\tag{3.20}$$

Also, $\Psi'(\beta)$ is a G-intertwiner. Indeed,

$$\begin{aligned}\Psi'(\beta(g\cdot v))(x) &= \beta(x\cdot(g\cdot v)) = \beta((xg)\cdot v)\\ &= (\Psi'(\beta)v)(xg) = (g\cdot\Psi'(\beta)v)(x).\end{aligned}\tag{3.21}$$

In the last equality we again made use of the induced action of G. Finally, we need to check that Ψ and Ψ' are mutual inverses. First, we compute

$$\Psi(\Psi'(\beta))(v) = (\Psi'(\beta)v)(e) = \beta(e\cdot v) = \beta(v).\tag{3.22}$$

Finally, we also see that

$$(\Psi'(\Psi(\beta))v)(x) = \Psi(\beta)(x\cdot v) = \beta(x\cdot v)(e) = (x\cdot\beta(v))(e) = \beta(v)(x).\tag{3.23}$$

This concludes the proof. $\square$

Using the above and Lemma 3.14, we have (following the notation as in Proposition 3.15) as a simple corollary that

$$\langle V \mid \mathrm{Ind}^G_H(W)\rangle_G = \langle \mathrm{Res}^G_H(V) \mid W\rangle_H\tag{3.24}$$

where the subscripts G, H indicate that the characters are of representations in those groups, respectively. Hence on the left, we sum over all $g\in G$ and on the right over all $h\in H$.

There is a way of decomposing the restriction of an induced representation (say, (ρ, W) from a subgroup H) to any subgroup K of G. We later only apply this to the case where $K=H$. The statement is a bit involved, so we first introduce some notation.

In the above setting, we obtain a partition of G into *double cosets* denoted HsK, where $s\in S\subset G$ and S is a set of representatives for these double cosets, i.e. we pick exactly one s in each double coset. Let us assume without leFor $s\in S$, let

$H^s = s^{-1}Hs \cap K$ and $\rho^s(h) = \rho(shs^{-1})$ for $h \in H^s$. Then ρ^s is a representation of H^s in W. To make a clear distinction, we write (ρ^s, W^s) for this representation. Noting that $H^s \subset K$, we can then induce (ρ^s, W^s) to obtain a representation of K, whose vector space is $\mathrm{Ind}_{H^s}^K(W^s)$. We now assert the following:

Proposition 3.16: *With the notation from the preceding paragraph, we have an isomorphism of K-representations*

$$\mathrm{Res}_K^G(\mathrm{Ind}_H^G(W)) \cong \bigoplus_{s\in S} \mathrm{Ind}_{H^s}^K(W^s) \tag{3.25}$$

Proof: As a shorthand, put $V = \mathrm{Ind}_H^G(W)$. Let $s \in S$ be arbitrary. We define $V(s) = \mathrm{Span}\{W_x \mid x \in HsK\}$, where W_x is defined as in Lemma 3.12. First, we claim that $V(s) \cap V(s') = \{0\}$ if $s \neq s'$. Indeed, suppose that $s \neq s'$. Then if $x \in HsK$ and $x' \in Hs'K$, we must prove that Hx and Hx' are disjoint. Write $x = hsk$. If Hx and Hx' are not disjoint, then we must have $x'x^{-1} = h' \in H$. But this yields

$$x' = (x'x^{-1})x = h'(hsk) \in HsK \tag{3.26}$$

which is a contradiction since $HsK \cap Hs'K = \emptyset$. Hence (again by Lemma 3.12) we are allowed to write V as a direct sum of vector spaces

$$V = \bigoplus_{s\in S} V(s) \tag{3.27}$$

Next, one can observe that each $V(s)$ is actually K-invariant. Indeed, let $\phi \in W_x$ and $x = hsk \in HsK$. Then for arbitrary $k' \in K$, we see that $(k' \cdot \phi)(g) = \phi(gk')$ which can only be nonzero if $gk' \in Hx$, or equivalently $g \in Hx(k')^{-1} = Hsk(k')^{-1} \subset HsK$, which proves what we wanted. In other words, $V(s)$ consists precisely of those functions which are supported on a specific double coset HsK.

It remains to prove that each $V(s)$ is isomorphic to $\mathrm{Ind}_{H^s}^K(W^s)$ as a K-representation. To this end, note that

$$\begin{aligned} \mathrm{Ind}_{H^s}^K(W^s) &= \{\phi : K \to W \mid \phi(yk) = \rho^s(y)\phi(k)\} \\ &= \{\phi : K \to W \mid \phi(yk) = \rho(sys^{-1})\phi(k)\} \end{aligned} \tag{3.28}$$

for $y \in H^s$, so that we can define an intertwiner $\Psi : \mathrm{Ind}_{H^s}^K(W^s) \to V(s)$ by $\Psi(\phi)(hsk) = \rho(h)\phi(k)$. We need to check that this is well-defined. Let $x = hsk = h'sk'$. Then $y = k'k^{-1} = s^{-1}(h')^{-1}hs \in H^s$. But then

$$\rho(h')\phi(k') = \rho(h')\phi(s^{-1}(h')^{-1}hsk) = \rho(h')\rho((h')^{-1}h)\phi(k) = \rho(h)\phi(k) \tag{3.29}$$

Also, it is clear that $\Psi(\phi)(h'x) = \rho(h')\Psi(\phi)(x)$. One may also verify that Ψ intertwines the K-actions, but this is rather obvious. To see that Ψ is bijective, we define an inverse $\Psi' : V(s) \to \mathrm{Ind}_{H^s}^K(W^s)$ by $\Psi'(\phi)(k) = \phi(sk)$. Indeed, this is supported on HsK and

$$\Psi'(\Psi(\phi))(hsk) = \rho(h)\Psi'(\phi)(k) = \rho(h)\phi(sk) = \phi(hsk) \tag{3.30}$$

$$\Psi(\Psi'(\phi))(k) = \Psi(\phi)(sk) = \rho(e)\phi(k) = \phi(k) \tag{3.31}$$

This concludes the proof. □

We now come to our criterion for irreducibility of an induced representation. Remarkably, the proof is reduced to just a few lines using the tools we have developed so far.

Theorem 3.17 (Mackey): *Let G be a group and H a subgroup. Suppose (ρ, W) is a representation of H. Then the representation $\mathrm{Ind}_H^G(\rho)$ is irreducible if and only if ρ is irreducible and the representations ρ^s and $\mathrm{Res}_{H^s}^H(\rho)$ from Proposition 3.16 are disjoint for $s \in G - H$, or in other words $s \neq e$.*

Proof: Following Proposition 3.6, we wish to compute the inner product of the character of the induced representation with itself. To do this, we put $K = H$ in Proposition 3.16. But then the inner product is

$$\begin{aligned}\langle \mathrm{Ind}_H^G(W) \mid \mathrm{Ind}_H^G(W)\rangle_G = \langle \mathrm{Res}_H^G(\mathrm{Ind}_H^G(W)) \mid W\rangle_H &= \langle \bigoplus_{s\in S} \mathrm{Ind}_{H^s}^H(W^s) \mid W\rangle_H \\ = \sum_{s\in S}\langle \mathrm{Ind}_{H^s}^H(W^s) \mid W\rangle_H &= \sum_{s\in S}\langle W \mid \mathrm{Ind}_{H^s}^H(W^s)\rangle_H \\ = \sum_{s\in S}\langle \mathrm{Res}_{H^s}^H(W) \mid W^s\rangle_{H^s} &\end{aligned} \tag{3.32}$$

But according to Proposition 3.6, this is equal to 1 if and only if there is only one term equal to 1 in the last expression. But for $s = e$ we have $W^s = W$, with the non-vanishing term $\langle \chi_W \mid \chi_W\rangle$. In conclusion, the whole expression sums to 1 if and only if W is irreducible and all the other terms are zero, which is precisely the statement we were after. □

3.3 Representations of Semidirect Products

Here we apply the theory we developed in Sections 3.1 and 3.2 to a special case.

Definition 3.18: *Let H and N be groups and suppose there is a left H-action on N by automorphisms, i.e. a group homomorphism $\phi : H \to \mathrm{Aut}(N)$. Then, we can form a new group called semidirect product of H by N, denoted $H \ltimes N$ which has $H \times N$ as underlying set, and multiplication defined by $(h, n) \cdot (h', n') = (hh', n\phi(h)(n'))$.*

Note that the semidirect product depends on the map ϕ, but it is typically omitted in the notation. Sometimes one uses $H \ltimes_\phi N$ to emphasize this, but we will not.

Hence from now on we just write $h(n) = \phi(h)(n)$. The identity element is (e_H, e_N). The inverse of an element is given by $(h,n)^{-1} = (h^{-1}, h^{-1}(n^{-1}))$ as is readily verified. The groups H and N are naturally embedded in the semidirect product by $h \mapsto (h, e)$ and $n \mapsto (e, n)$.

Lemma 3.19: *The semidirect product $H \ltimes N$ of two groups H, N acts naturally on N again via $(h,n)n' = n(h(n'))$.*

Proof: We just need to check that this action defines a group homomorphism into $\mathrm{Aut}(N)$. Indeed, $(e,e)n' = e(e(n')) = n'$ and $(h_1,n_1)((h_2,n_2)n') = (h_1,n_1)(n_2(h_2(n')) = n_1(h_1(n_2(h_2(n')))) = n_1(h_1(n_2))(h_1h_2(n')) = (h_1h_2, n_1(h_1(n_2)))n'$. This proves the claim. $\square$

Remark: Suppose we have again a (left) action of a group H on another group N by automorphisms. In literature, one often sees different definitions of the 'semidirect product'. For example, in [13] the underlying set is taken $N \times H$ with multiplication $(n,h)(n',h') = (hh', nh(n'))$ (so the arguments are flipped). The semidirect product is then denoted $N \rtimes H$. A third convention is to keep $H \times N$ as underlying set and define $(h,n)(h',n') = (hh', h'^{-1}(n)n')$. In this case one also writes $H \ltimes N$ for the semidirect product, which then acts on N by $(h,n)n' = h(nn')$. Semidirect products often arise as symmetry groups of vector spaces with a metric. In Chapter 4, we will encounter the Poincaré group as an example. The 'rotational' part $H < H \ltimes N$ fixes the origin and the 'translational' part $N \lhd H \ltimes N$ does not. A more concrete example would be *special Euclidean group* $SE(2) = SO(2) \ltimes \mathbb{R}^2$, which is the set of all orientation-preserving isometries of $\mathbb{R}^2$. Here $SO(2)$ acts on $\mathbb{R}^2$ by applying the matrix to the column vector (this is indeed a left action by automorphisms). The action given in Lemma 3.19 can then be interpreted as saying that $(A, w) \in SE(2)$ acts as a symmetry transformation on $\mathbb{R}^2$ by first rotating by A and then translating by w. Alternatively, one can first translate and then rotate, in which case the semidirect product would be defined according to the third option listed above.

Now suppose we have an abelian group N. In this case, the irreducible representations of N are all one-dimensional. We are in the setting of Lemma 3.9.

If H is another group acting on N, we see that there is a natural left action of H on $N^\vee$, namely $(h \cdot \chi)(n) = \chi(h^{-1}(n))$. The stabilizer of a particular element χ is a subgroup of H which we denote by H_χ. For each orbit O_i of the action we pick a representative $\chi_i \in O_i$. Denote $H_i = H_{\chi_i}$. We will now construct all irreducible representation of $H \ltimes N$ as follows:

(i) Pick for each i an irreducible representation π_i of H_i, acting on a vector space W.

(ii) Extend π_i to a representation of $H_i \ltimes N$ by setting $\pi_i(h,n) = \pi_i(h)$. Note that we still use the same symbol π_i.

(iii) Extend the character χ_i to a representation of $H_i \ltimes N$ by setting $\chi_i(h,n) = \chi_i(n)$. Crucially, this a well-defined representation because h stabilizes χ_i.

(iv) Consider now the tensor product representation $\pi_i \otimes \chi_i$ of $H_i \ltimes N$. Upon identification of $W \otimes \mathbb{C}$ with W, the action on $w \in W$ is given by $(\pi_i \otimes \chi_i)(h,n)w = \chi_i(n)\pi_i(h)w$.

(v) Finally, we *induce* to obtain a representation $V_{i,\pi} = \mathrm{Ind}_{H_i \ltimes N}^{H \ltimes N}(\pi_i \otimes \chi_i)$.

Theorem 3.20: *Let N be an abelian group and let H be a group acting on N by automorphisms. Let $\{O_i\}_{1 \leq i \leq k}$ denote the orbits of the action of H on $N^\vee$ as described above. Pick a set of representatives $\chi_i \in O_i$ for each i from 1 to k. Let R_i denote the set of (isomorphism classes of) irreducible representations of H_i, the stabilizer of χ_i. Then if we set $V_{i,\pi} = \mathrm{Ind}_{H_i \ltimes N}^{H \ltimes N}(\pi_i \otimes \chi_i)$ as above, the following holds:*

(i) *$V_{i,\pi}$ is irreducible for all i and all $\pi \in R_i$.*

(ii) *Any two different elements of $S = \{V_{i,\pi}\}_{1 \leq i \leq k; \pi \in R_i}$ are nonisomorphic. In other words, if $V_{i,\pi} \cong V_{i',\pi'}$ then $i = i'$ and $\pi_i \cong \pi_i'$.*

(iii) *Any irreducible representation of $H \ltimes N$ is isomorphic to some $V_{i,\pi} \in S$.*

In other words, we have classified all irreducible representations of the semidirect product $H \ltimes N$ without double occurrences.

Proof of (i): We use Mackey's criterion, which is Theorem 3.17. We have *two* representations of H^s, which is now a subgroup of H. One is given by restriction of $\rho = \pi_i \otimes \chi_i$, the other by conjugation by s and then applying $\pi_i \otimes \chi_i$. We need to prove that these are disjoint if $s \neq e$. If they are not disjoint, then they are also not disjoint as representations of $N \subset H^s$. Indeed, any H^s-representation decomposes as an N-representation, so if $\langle W^s \mid W \rangle_{H^s} \neq 0$ then also $\langle W^s \mid W \rangle_N \neq 0$. It therefore suffices to prove that W^s and W are disjoint as N-representations. Let $s = (h,n)$. Then it is easy to check that

$$\rho^s(n')w = (\pi_i \otimes \chi_i)(e, n(h(n'))n^{-1})w = \chi_i(h(n'))w \tag{3.33}$$

We conclude that as an N-representation, ρ^s is just given by multiplication with the character $h^{-1} \cdot \chi_i$ (it decomposes as a direct sum of $\dim(W)$ irreducibles $h^{-1} \cdot \chi_i$, and its character is therefore $\chi_{\rho^s} = \dim(W)(h \cdot \chi_i)$). But the restricted representation ρ is given by multiplication by χ_i so its character is $\chi_\rho = \dim(W)(\chi_i)$. Since $h^{-1} \cdot \chi_i \neq \chi_i$, these representations are disjoint. □

Proof of (ii): We already established the irreducibility of the representations. To prove the assertion, we prove that given an element $V_{i,\pi} \in S$, we can retrieve a unique orbit O_i and we can also recover the representation π_i up to isomorphism. To this end we decompose $V_{i,\pi}$ as

$$V_{i,\pi} = \bigoplus_{1\le i\le k} \bigoplus_{\chi\in O_i} V_\chi \quad ; \quad V_\chi = \{\phi \in V_{i,\pi} \mid (e,n)\phi = \chi(n)\phi\} \tag{3.34}$$

The claim is that the inner summand $\bigoplus_{\chi\in O_i} V_\chi$ is $H \ltimes N$-stable. To verify this, note that any V_χ is N-invariant as N is abelian. Additionally, an element $h \in H$ sends V_χ to $V_{h\cdot\chi}$ since $(e,n)(h,e) = (h,n) = (h,e)(e,h^{-1}(n))$, which for $\phi \in V_\chi$ yields

$$\begin{aligned} n\cdot(h\cdot\phi) &= (e,n)(h,e)\phi = (h,n)\phi = (h,e)(e,h^{-1}(n))\phi \\ &= (h,e)(\chi(h^{-1}(n))\phi) = (h\cdot\chi)(n)(h\cdot\phi) \end{aligned} \tag{3.35}$$

By irreducibility of $V_{i,\pi}$, the outer summand can contain only one nontrivial term: there is precisely one index i (that is, precisely one orbit) for which $\bigoplus_{\chi\in O_i} V_\chi \neq \{0\}$. We then recognize that $O_i = \{\chi \in N^\vee \mid V_\chi \neq \{0\}\}$. This shows that we can recover i from $V_{i,\pi}$.

Next, we consider the set $V_i = V_{\chi_i} = \{\phi \in V_{i,\pi} \mid (e,n)\phi = \chi_i(n)\phi\}$ in $V_{i,\pi}$. We claim that it is stable under H_i. Indeed, this follows immediately from the above argument. Hence we obtain a representation of H_i on V_i. Moreover, this is isomorphic to π_i by the intertwiner $\Psi : V_i \to W, \phi \mapsto \phi(e,e)$. Indeed,

$$\begin{aligned} \Psi(h\cdot\phi) &= (h\cdot\phi)(e,e) = \phi(h,e) = \chi_i(e)\pi_i(h)\phi(e,e) \\ &= \pi_i(h)\phi(e) = \pi_i(h)\Psi(\phi) \end{aligned} \tag{3.36}$$

Evidently, Ψ is linear. We need to prove that it is injective and surjective. To see this, note that $\phi(h,n) = \chi_i(n)\pi(h)\phi(e,e)$ for $(h,n) \in H_i \ltimes N$. Also, from $(n\cdot\phi)(h',n') = \chi_i(h'(n))\phi(h',n') = \chi_i(n)\phi(h',n')$ we deduce that ϕ must be zero outside $H_i \ltimes N$, i.e. it must be *supported* on $H_i \ltimes N$. But then ϕ is completely determined by $\Psi(\phi)$. Hence Ψ is injective. On the other hand, given $w \in W$, we can set $\phi(h,n) = \chi_i(n)\pi_i(h)w$ to obtain a vector $\phi \in V_i$. Hence Ψ is also surjective and we are done. $\square$

Proof of (iii): We need to prove that the sum of the squares of the dimensions of the $V_{i,\pi}$ add up to $|H \ltimes N| = |H| \cdot |N|$. Using previous results, we compute

$$\begin{aligned}
\sum_{1\leq i\leq k;\pi\in R_i} \dim(V_{i,\pi})^2 &= \sum_{1\leq i\leq k;\pi\in R_i} \left(\frac{|H \ltimes N|}{|H_i \ltimes N|}\right)^2 \dim(W)^2 \\
&= \sum_{1\leq i\leq k;\pi\in R_i} \left(\frac{|H|}{|H_i|}\right)^2 \dim(W)^2 \\
&= \sum_{1\leq i\leq k} \left(\frac{|H|}{|H_i|}\right)^2 |H_i| \\
&= |H| \sum_{1\leq i\leq k} \frac{|H|}{|H_i|} \\
&= |H| \sum_{1\leq i\leq k} |O_i| = |H||N^\vee| = |H||N|
\end{aligned} \tag{3.37}$$

where we also used Lemmata 3.9 and 3.10. □

3.4 Generalization to Systems of Imprimitivity

We are now ready to introduce the concept of a *system of imprimitivity*. We will prove that given a group G and a subgroup H, a (unitary) representation of H will give rise to a system of imprimitivity for G based on the space of cosets $H\backslash G$. Conversely, any system of imprimitivity based on this coset space is canonically inherited from some unitary representation of H, which is unique up to isomorphism.

Definition 3.21: *Let X be a finite set and V a finite-dimensional Hilbert space. A projection-valued measure P on X is a map $P : \mathcal{P}(X) \to \mathrm{Pr}(V)$, where $\mathrm{Pr}(V)$ denotes the set of projections of V, i.e. $\mathrm{Pr}(V) = \{A \in \mathrm{End}(V) \mid A^\dagger = A = A^2\}$, which satisfies:*

(i) $P(\emptyset) = 0$

(ii) $P(X) = \mathrm{Id}_V$

(iii) $P(S_1 \cap S_2) = P(S_1) \circ P(S_2)$ *for all* $S_1, S_2 \subset X$.

(iv) $P(S_1 \cup S_2) = P(S_1) + P(S_2)$ *for all* $S_1, S_2 \subset X$ *such that* $S_1 \cap S_2 = \emptyset$.

Definition 3.22: *Let G be a group and X a set equipped with a left G-action. A system of imprimitivity for G based on X consists of a triple (ρ, V, P) where ρ is a unitary representation of H on V, which is a finite-dimensional Hilbert space. The* *operator $P : \mathcal{P}(X) \to \mathrm{Pr}(V)$ is a projection-valued measure on X. Furthermore, we have the following compatibility condition for $g \in G$ and $S \in \mathcal{P}(X)$:*

$$P(g \cdot S) = \rho(g) \circ P(S) \circ \rho(g^{-1}) \tag{3.38}$$

Definition 3.23: *Let (ρ, V, P) be a system of imprimitivity for G based on X. An invariant subspace of (ρ, V, P) is a subspace $W \subset V$ which is an invariant subspace of (ρ, V) and also of P, in the sense that $P(S)V \subset V$ for all $S \in \mathcal{P}(X)$. The system of imprimitivity is called irreducible if the only invariant subspaces are $\{0\}$ and V.*

Observe that if W is an invariant subspace of (ρ, V, P), then (ρ, W, P) is a system of imprimitivity in its own right if we consider ρ a representation on W. Clearly, if ρ is irreducible, then so is (ρ, V, P).

Definition 3.24: *Let (ρ, V, P) and (ρ', V', P') be systems of imprimitivity for G based on the same set X. They are called equivalent if there exists an isometric isomorphism $\alpha : V \to V'$ which intertwines ρ and ρ', and also P and P' in the sense that for all $S \in \mathcal{P}(X)$ and $v \in V$ we have $P'(S)\alpha(v) = \alpha(P(S)v)$.*

Proposition 3.25: *Let G be a group and H a subgroup. Suppose (π, W) is a unitary representation of H. Then (ρ^π, V^π, P^π) is a system of imprimitivity for G based on the coset space $H\backslash G$, where the coset space is understood to have the left action $g \cdot Hg' = Hg'g^{-1}$, and*

(i) $\rho^\pi = \mathrm{Ind}_H^G(\pi)$

(ii) $V^\pi = \mathrm{Ind}_H^G(W)$, *where* $\langle \phi \mid \psi \rangle_{V^\pi} = \frac{1}{|H|} \sum_{g \in G} \langle \phi(g) \mid \psi(g) \rangle_W$.

(iii) $P^\pi(S)\phi = \mathbf{1}_S \cdot \phi$, *where $\mathbf{1}_S$ is the indicator function of $S \subset G$.*

Proof: It is clear that the formula in (ii) defines a Hermitian inner product on V^π. We need to check that P^π is a projection-valued measure and that the compatibility condition is satisfied. Since $\mathbf{1}_S$ is a constant on each coset, we see that $\mathbf{1}_S\phi \in V^\pi$ for $\phi \in V^\pi$. Let $S, S' \in \mathcal{P}(H\backslash G)$. Then indeed $\mathbf{1}_{S \cap S'} = \mathbf{1}_S \circ \mathbf{1}_{S'}$ and if S, S' are disjoint then $\mathbf{1}_{S \cup S'} = \mathbf{1}_S + \mathbf{1}_{S'}$. The other two properties are immediate as well. Note furthermore that $(P^\pi)^2 = P^\pi$, and

$$\begin{aligned} \langle \phi \mid \mathbf{1}_S\psi \rangle_{V^\pi} &= \frac{1}{|H|} \sum_{g \in G} \langle \phi(g) \mid \mathbf{1}_S\psi(g) \rangle_W = \frac{1}{|H|} \sum_{s \in S} \langle \phi(s) \mid \psi(s) \rangle_W \\ &= \frac{1}{|H|} \sum_{g \in G} \langle \mathbf{1}_S\phi(g) \mid \psi(g) \rangle_W = \langle \mathbf{1}_S\phi \mid \psi \rangle_{V^\pi} \end{aligned} \tag{3.39}$$

so that $P^\pi(S)$ is Hermitian and hence a projection. Finally, for the compatibility condition note that $\mathbf{1}_S(g^{-1} \cdot \phi)(k)$ equals 0 for k not in S, and $\phi(kg^{-1})$ for $k \in S$. Hence,

$(g \cdot \mathbf{1}_S(g^{-1} \cdot \phi))(k)$ equals 0 for kg not in S, and $\phi(k)$ for $kg \in S$. But $kg \in S$ is equivalent to $k \in Sg^{-1} = g \cdot S$. Therefore, $(g \cdot \mathbf{1}_S(g^{-1} \cdot \phi))(k) = (\mathbf{1}_{g \cdot S}\phi)(k)$. This concludes the proof. $\square$

Next, we note that a projection-valued measure P based on any G-set X also induces a linear map $P : \mathbb{C}(X) \to \mathrm{End}(V)$ on functions $f : X \to \mathbb{C}$, by setting

$$P(f) = \sum_{s \in X} f(s)P(\{s\}) \tag{3.40}$$

If the set X possesses a natural left action by a group G, then the set of functions from X to $\mathbb{C}$ also possesses a left G-action defined as $(g \cdot f)(s) = f(g^{-1} \cdot s)$. It is then straightforward to check that the compatibility condition is equivalent to

$$P(g \cdot f) = \rho(g)P(f)\rho(g^{-1}) \tag{3.41}$$

for all $f \in \mathbb{C}(X)$. Indeed, P is linear so it is necessary and sufficient to check this for the basis of indicator functions on single elements $\mathbf{1}_{\{s\}}$, where $s \in S$. But it is clear that $g \cdot \mathbf{1}_{\{s\}} = \mathbf{1}_{\{g \cdot s\}}$.

We claim that P is actually an algebra homomorphism, i.e. $P(fg) = P(f) \circ P(g)$ and $P(\mathbf{1}_X) = \mathrm{Id}_V$, where $\mathbf{1}_X$ is the multiplicative unit of $\mathbb{C}(X)$, i.e. the function which is identically equal to 1. Indeed, $P(\mathbf{1}_X) = \sum_{s \in X} P(\{s\}) = P(X) = \mathrm{Id}_V$ by (iv) in Definition 3.21. Observe also that $P(\{s\}) \circ P(\{s'\}) = 0$ unless $s = s'$, in which case it is $P(\{s\})$ again. Then it follows that

$$P(f) \circ P(g) = \sum_{s \in X} \sum_{s' \in X} f(s)P(\{s\})g(s')P(\{s'\}) = \sum_{s \in X} f(s)g(s)P(\{s\}) = P(fg) \tag{3.42}$$

In the case where $P = P^\pi$ is the induced projection-valued measure from Proposition 3.25, one may check that $P^\pi(f)\phi = f\phi$ for $\phi \in \mathrm{Ind}_H^G(W)$ and $f \in \mathbb{C}(H\backslash G)$.
In the following, we retain the notation of Proposition 3.25. Suppose we are given a system of imprimitivity (ρ, V, P) for G based on $H\backslash G$. We will construct a representation π of H such that $(\rho^\pi, V^\pi, P^\pi) \cong (\rho, V, P)$ in the sense of Definition 3.24.
If W is an arbitrary vector space, denote its dual by W^*. Define a function $B : V \times V \to \mathbb{C}^*(H\backslash G)$ by $B(v, w)(f) = \langle P(f)v \mid w\rangle_V$. Straightforward verifications show that B is sesquilinear (for this, we need the inner product to be complex linear in the first variable). We claim that, for any two vectors $v, w \in V$, there exists a unique function $\mu_{v,w} : H\backslash G \to \mathbb{C}$ such that

$$B(v, w)(f) = \sum_{Hg} f(Hg)\mu_{v,w}(Hg) \tag{3.43}$$

where the sum is over all cosets. Extending this function to all of $\mathcal{P}(H\backslash G)$ by requiring additivity under unions of sets, it can be thought of as a *measure* on the coset space.

Since both sides are linear in f, it suffices to show this for indicator functions again. But for such functions $\mathbf{1}_{Hg}$, we can simply infer that $\mu_{v,w}(Hg) = B(v,w)(\mathbf{1}_{Hg})$, as only one term in the summand is nonzero. From this, it also obvious that $\mu_{v,w}$ is sesquilinear as a function of v and w. Now define $\beta(v,w) = \mu_{v,w}(He)$. This is a sesquilinear form on V satisfying the following important properties:

Lemma 3.26: *In the notation of the above paragraph, the sesquilinear form $\beta : V \times V \to C$ is positive semidefinite. Its kernel $\ker(\beta) = \{v \in V \mid \beta(v,w) = 0 \quad \forall w \in V\}$ is an H-invariant linear subspace of V. Finally, β descends to a Hermitian inner product β' on the quotient vector space $V/\ker(\beta)$.*

Proof: Let $v \in V$ be arbitrary. We have that $\beta(v,v) = B(v,v)(\mathbf{1}_{He})$ by the above paragraph, so that $\beta(v,v) = \langle P(\mathbf{1}_{He})v \mid v\rangle_V$ by definition of B. Observing that $\mathbf{1}_{He}$ is real and equal to its square, we infer that $P(\mathbf{1}_{He}) = P(\mathbf{1}_{He})^2 = P(\mathbf{1}_{He})^\dagger$ (this is the adjoint as in Section 1.4). Consequently,

$$\beta(v,v) = \langle P(\mathbf{1}_{He})v \mid v\rangle_V = \langle P(\mathbf{1}_{He})^2 v \mid v\rangle_V = \langle P(\mathbf{1}_{He})v \mid P(\mathbf{1}_{He})v\rangle_V \geq 0 \tag{3.44}$$

which establishes the first assertion. Next, for arbitrary $h \in H$ and $v, w \in V$ we have

$$\begin{aligned}\beta(hv,hw) &= \langle P(\mathbf{1}_{He})hv \mid hw\rangle_V = \langle P(\{He\})hv \mid hw\rangle_V \\ &= \langle hP(\{He\})v \mid hw\rangle_V = \langle P(\{He\})v \mid w\rangle_V \\ &= \langle P(\mathbf{1}_{He})v \mid w\rangle_V = \beta(v,w)\end{aligned} \tag{3.45}$$

where we used $P(\{He\}) = P(\{h \cdot He\}) = hP(\{He\})h^{-1}$ so that $P(\{He\})hv = hP(\{He\})v$. The second assertion from the lemma follows, as H acts by invertible linear transformations on V. Lastly, we need to prove that the induced form β' is well-defined and nondegenerate on the quotient. Indeed, if $v_1 = v_2 + v$ with $v \in \ker(\beta)$, then for any $w \in V$ we get $\beta(v_1,w) = \beta(v_2,w) + \beta(v,w) = \beta(v_2,w)$.

It remains to show that β' is in fact nondegenerate. Suppose $\beta'([v],[v]) = 0$. Then $\beta(v,v) = 0$. We must prove that $\beta(v,w) = 0$ for arbitrary $w \in V$. For this, recall from (61) that this implies $P(\mathbf{1}_{He})v = 0$, so that $\beta(v,w) = \langle P(\mathbf{1}_{He})v \mid w\rangle_V = 0$ for arbitrary w. This finishes the proof of the lemma. □

From now on we denote $W = V/\ker(\beta)$. Crucially, the fact that $\ker(\beta)$ is an H-invariant subspace means that the representation $\rho|_H$ factors through the quotient. This means we can define a representation π of H on W by $\pi(h)w = [\rho(h)v]$ where v is such that $[v] = w$. The proof of the previous lemma also shows that π is in fact unitary with respect to the inner product β' on W. We are now ready to prove the promised result:

Theorem 3.27 (Mackey): *Let (ρ, V, P) be a system of imprimitivity for G based on $H\backslash G$. Then, with the notation above, the induced system of imprimitivity (ρ^π, V^π, P^π) of the representation (π, W) of H is unitarily equivalent to (ρ, V, P). Moreover, any* *other representation of H whose induced system of imprimitivity is unitarily equivalent to (ρ, V, P) is canonically isomorphic to (π, W). The system (ρ, V, P) is irreducible if and only if (π, W) is.*

Proof: Define a map $T : V \to V^\pi$ by $T(v)(g) = [\rho(g)v] \in W$. We need to check a number of things. First, observe that T really maps into V^π, because for $h \in H$ and $g \in G$ we have

$$T(v)(gh) = [\rho(hg)(v)] = [\rho(h)\rho(g)v] = \rho(h)[\rho(g)v] = \pi(h)T(v)(g) \tag{3.46}$$

Next, T intertwines the G-actions of both representations, because

$$(g \cdot T(v))(g') = T(v)(g'g) = [\rho(g'g)v] = [\rho(g')\rho(g)v] = [\rho(g')(g \cdot v)] = T(g \cdot v)(g') \tag{3.47}$$

We can also show, using the definition of the induced inner product from (ii), Proposition 3.25, that T is in fact an isometry. Indeed, for $v, w \in V$ we get

$$\begin{aligned}
\langle Tv \mid Tw \rangle_{V^\pi} &= \frac{1}{|H|} \sum_{g \in G} \beta'([\rho(g)v], [\rho(g)w]) = \frac{1}{|H|} \sum_{g \in G} \beta(g \cdot v, g \cdot w) \\
&= \frac{1}{|H|} \sum_{g \in G} \langle P(\{He\}(g \cdot v) \mid g \cdot w \rangle_V = \frac{1}{|H|} \sum_{g \in G} \langle gP(\{Hg\})v \mid gw \rangle_V \\
&= \frac{1}{|H|} \sum_{g \in G} \langle P(\{Hg\})v \mid w \rangle_V = \frac{1}{|H|} \langle \sum_{g \in G} P(\{Hg\})v \mid w \rangle_V \\
&= \frac{1}{|H|} \langle |H| \mathrm{Id}_V v \mid w \rangle_V = \langle v \mid w \rangle_V
\end{aligned} \tag{3.48}$$

This is also why we put the prefactor $\frac{1}{|H|}$ in the induced inner product. Summing $P(\{Hg\})$ over all *cosets* yields the identity on V, so summing over all $g \in G$ yields a factor of $|H|$, which cancels the prefactor.

It is then automatic that T is injective, by nondegeneracy of the inner products involved. Since all vector spaces are finite-dimensional in our discussion, surjectivity is an immediate consequence. Hence T is a linear isomorphism.

We need to verify that T also intertwines the projection-valued measures P and P^π. We will do this by showing that T intertwines the operators $P(f)$ and $P^\pi(f)$ for all $f \in \mathbb{C}(H \backslash G)$. As we remarked before, $P^\pi(f)\phi = f\phi$. We know that T is surjective. Let $v' \in V^\pi$ be arbitrary, and write $T(w) = v'$. Similar to the previous computation, we see that

$$\begin{aligned}\langle f(Tv) \mid Tw\rangle_{V^\pi} &= \frac{1}{|H|}\sum_{g\in G}\beta'(f(Hg)[\rho(g)v],[\rho(g)w]) = \frac{1}{|H|}\sum_{g\in G}\beta(f(Hg)g\cdot v, g\cdot w)\\ &= \frac{1}{|H|}\sum_{g\in G}\langle f(Hg)P(\{He\})(g\cdot v) \mid g\cdot w\rangle_V\\ &= \frac{1}{|H|}\sum_{g\in G}\langle gP(\{Hg\})f(Hg)v \mid gw\rangle_V\\ &= \frac{1}{|H|}\sum_{g\in G}\langle f(Hg)P(\{Hg\})v \mid w\rangle_V = \langle P(f)v \mid w\rangle_V = \langle TP(f)v \mid v'\rangle_{V^\pi}\end{aligned} \tag{3.49}$$

The prefactor is cancelled again for the same reason. Since this holds for arbitrary v', the two elements $P^\pi(f)(Tv) = f(Tv)$ and $TP(f)v$ must be equal, by nondegeneracy of the induced inner product. This concludes the first part of the proof.

For uniqueness of π up to isomorphism, note that if (η, U) is another representation of H such that $(\rho^\eta, V^\eta, P^\eta) \cong (\rho, V, P) \cong (\rho^\pi, V^\pi, P^\pi)$, we have a unitary isomorphism $R : V \to V^\eta$. Let $v, w \in V$. Then, using the definition of the inner product on V^η, we get

$$\begin{aligned}\langle P(He)v \mid w\rangle_V &= \frac{1}{|H|}\sum_{g\in G}\langle (RP(He)v)(g) \mid Rw(g)\rangle_\eta\\ &= \frac{1}{|H|}\sum_{g\in G}\langle P^\eta(He)(Rv)(g) \mid Rw(g)\rangle_\eta\\ &= \frac{1}{|H|}\sum_{g\in G}\langle \mathbf{1}_{He}(Rv)(g) \mid Rw(g)\rangle_\eta\\ &= \frac{1}{|H|}\sum_{h\in H}\langle Rv(h) \mid Rw(h)\rangle_\eta = \langle (Rv)(e) \mid (Rw)(e)\rangle_\eta\end{aligned} \tag{3.50}$$

But $\beta(v, w) = \langle P(He)v \mid w\rangle_V$, so $\mathrm{ev}_e \circ R$ factors through the kernel $\ker(\beta)$ to a map $\epsilon : W = V/\ker(\beta) \to U$. It is evident that ev is an H-intertwiner, so ϵ is also an intertwiner for H. It is an isometry, whence it is bijective.

Lastly, we must show that (ρ, V, P) is irreducible if and only if (π, W) is. Suppose that (π, W) is reducible. Write $W = W_1 \oplus W_2$. Let $\pi_i = \pi|_{W_i}$ for $i = 1, 2$. Then $\pi \cong \pi_1 \oplus \pi_2$. The direct sum of the induced systems of imprimitivity of the π_i then equals (ρ, V, P), whence (ρ, V, P) is reducible. Conversely, if (ρ, V, P) is reducible it splits into two nontrivial subsystems which we abbreviate by P_i where $i = 1, 2$. By the above, each of the P_i is induced from a representation π_i of H. Put $\pi' = \pi_1 \oplus \pi_2$. By the above, the direct sum of the P_i is equal to P again. But then by uniqueness, $\pi_1 \oplus \pi_2 = \pi' \cong \pi$, so that π is reducible. Hence we are done. $\square$

3.5 Systems of Imprimitivity for Semidirect Products

Armed with the machinery from Section 3.4, we can now readdress the representation theory of semidirect products by abelian groups as seen in Section 3.3. In particular, it will be possible to establish the same classification without relying on specific results from Section 3.2. The derivation here is very similar to the general case of a semidirect product of a Lie group H by an abelian Lie group N, albeit that the analysis below does not require any topology, measure theory or functional analysis. For the general case, one may consult [14]. Throughout this section, we write $G = H \ltimes N$ for a semidirect product of a finite group H by a finite abelian group N.

Definition 3.28: *Let (π, V) be a unitary representation of an abelian group N. Given $f \in \mathbb{C}(N)$, we define $\pi(f)$ by*

$$\pi(f) = \sum_{n \in N} f(n)\pi(n) \tag{3.51}$$

Definition 3.29: *Let N be an abelian group and let $f \in \mathbb{C}(N)$. Then we define its (discrete) Fourier transform as the function $f^\vee : N^\vee \to \mathbb{C}$, where*

$$f^\vee(\chi) = \sum_{n \in N} f(n)\chi(n) \tag{3.52}$$

Our first result is a simple finite-dimensional *spectral theorem* for abelian groups: its result relies on the orthogonality relations between characters we established in Section 3.1.

Proposition 3.30: *Let N be an abelian group and (π, V) a unitary representation of N. Then there exists a unique projection valued measure P_π based on $N^\vee$ such that $\pi(f) = P_\pi(f^\vee)$ for all $f \in \mathbb{C}(N)$. Furthermore, two unitary representations (π_i, V_i) (i=1,2) are unitarily equivalent if and only if their associated projection valued measures are.*

Proof: Let us label the irreducible characters of N by an index i. It is obvious from the definitions that P_π must satisfy the relation

$$\sum_{n \in N} f(n)\pi(n) = \sum_{n \in N} \sum_{i=1}^{|N|} f(n)\chi_i(n)P_\pi(\{\chi_i\}) \tag{3.53}$$

As usual, it is necessary and sufficient to prove this for indicator functions, so that we must have

$$\pi(n) = \sum_{i=1}^{|N|} \chi_i(n)P_\pi(\{\chi_i\}) \tag{3.54}$$

But this means that

$$\begin{aligned}\sum_{n\in N}\pi(n)\chi_j^*(n) &= \sum_{i=1}^{|N|}\sum_{n\in N}\chi_j^*(n)\chi_i(n)P_\pi(\{\chi_i\})\\ &= |N|\sum_{i=1}^{|N|}\delta_{ji}P_\pi(\{\chi_j\}) = |N|\cdot P_\pi(\{\chi_j\})\end{aligned}\tag{3.55}$$

so that if P_π exists, it is unique. Note that any projection valued measure is completely determined by its values on singleton sets, and if we can prove that $P_\pi(\chi_j)\in \mathrm{Pr}(V)$ for all j and $P_\pi(N^\vee)=\mathrm{Id}_V$, then we are done. We compute

$$P_\pi(\{\chi_j\})^\dagger = \frac{1}{|N|}\sum_{n\in N}\pi(n)^\dagger\chi_j(n) = \frac{1}{|N|}\sum_{n\in N}\pi(n^{-1})\chi_j^*(n^{-1}) = P(\{\chi_j\})\tag{3.56}$$

$$\begin{aligned}P_\pi(\{\chi_j\})^2 &= \frac{1}{|N|^2}\sum_{n\in N}\sum_{n'\in N}\pi(n)\pi(n')\chi_j^*(n)\chi_j^*(n')\\ &= \frac{1}{|N|^2}\sum_{n\in N}\sum_{n'\in N}\pi(nn')\chi_j^*(nn')\\ &= \frac{1}{|N|}\sum_{n\in N}\pi(n)\chi_j^*(n) = P_\pi(\{\chi_j\})\end{aligned}\tag{3.57}$$

Lastly, we need to invoke another fact that follows from studying the regular representation of a group: for any group N, the sum over all irreducible characters evaluated at $n\in N$ equals $|N|$ for $n=e$ and zero otherwise. See [15], Corollary 2b) of Chapter 2. This means that

$$P_\pi(N^\vee) = \sum_{j=1}^{|N|}P_\pi(\{\chi_j\}) = \frac{1}{|N|}\sum_{j=1}^{|N|}\sum_{n\in N}\pi(n)\chi_j^*(n) = |N|\cdot\frac{1}{|N|}\pi(e) = \mathrm{Id}_V\tag{3.58}$$

This proves that P_π is indeed a projection valued measure and concludes the first part of the proposition. Next, we deduce from the above that an isomorphism $T:V\to V'$ intertwines $\pi(n)$ and $\pi'(n)$ for all n if and only if it intertwines $P_\pi(\chi_i)$ and $P_{\pi'}(\chi_i)$ for all i, so that the last assertion is immediate. This ends the proof. □

Using this, we are now able to establish a one-to-one correspondence between unitary representations of a semidirect product $G=H\ltimes N$ and systems of imprimitivity for H based on $N^\vee$. This is the content of the following theorem:

Theorem 3.31: *Let $G=H\ltimes N$. Suppose (π,V) is a unitary representation of G. Then, if we denote by P_π the projection valued measure on $N^\vee$ corresponding to $\pi|_N$ as in Proposition 3.30, the triple $(\pi|_H,V,P_\pi)$ is a system of imprimitivity for H based on $N^\vee$. This system is irreducible if and only if (π,V) is. The assignment*

$(\pi, V) \mapsto (\pi|_H, V, P_\pi)$ induces (up to equivalence) a bijection between irreducible unitary representations of $H \ltimes N$ and irreducible systems of imprimitivity for H on $N^\vee$.

Proof: To prove that $(\pi|_H, V, P_\pi)$ is a system of imprimitivity, we only need to check the compatibility condition. The Fourier transform $f \mapsto f^\vee$ is bijective, as is easily verified using orthogonality relations of characters. First we compute

$$\begin{aligned}(h \cdot f)^\vee(\chi) &= \sum_{n \in N} (h \cdot f)(n)\chi(n) = \sum_{n \in N} f(h^{-1}n)\chi(n) \\ &= \sum_{n \in N} f(n)\chi(h(n)) = \sum_{n \in N} f(n)(h^{-1} \cdot \chi)(n) \\ &= f^\vee(h^{-1} \cdot \chi) = (h \cdot f^\vee)(\chi)\end{aligned} \tag{3.59}$$

But then for $\phi = f^\vee \in \mathbb{C}(N^\vee)$ we have

$$\begin{aligned}\pi(h)P_\pi(f^\vee)\pi(h^{-1}) &= \pi(h)\pi(f)\pi(h^{-1}) = \sum_{n \in N} f(n)\pi(h)\pi(n)\pi(h^{-1}) \\ &= \sum_{n \in N} f(n)\pi(hnh^{-1}) = \sum_{n \in N} f(n)\pi(h \cdot n) \\ &= \sum_{n \in N} f(h^{-1} \cdot n)\pi(n) = \pi(h \cdot f) \\ &= P_\pi((h \cdot f)^\vee) = P_\pi(h \cdot (f^\vee))\end{aligned} \tag{3.60}$$

To verify the remark on irreducibility, note that if $W \subset V$ is an invariant subspace for $(\pi|_H, V, P_\pi)$, then certainly $h \cdot W \subset W$ for all $h \in H$ and also $P_\pi(f^\vee)W \subset W$ for all $f \in \mathbb{C}(N)$. Taking f an indicator function $\mathbf{1}_n$ and noting that in that case $\pi(n) = \pi(f) = P_\pi(f^\vee)$, we see that $n \cdot W \subset W$ for all $n \in N$. Because in the semidirect product we have $(e, n) \cdot (h, e) = (h, n)$, the space W is invariant under all of $H \ltimes N$. The converse is nearly identical.

Using Proposition 3.30, the final remark also follows: given a system (ρ, V, P) on $N^\vee$ with ρ a representation of H, simply *define* $\pi(n)$ according to (3.54) and extend it to the entire group by putting $\pi(h, n) = \pi(n)\rho(h)$. A straightforward computation indeed yields that this is a homomorphism $H \ltimes N \to GL(V)$. The assignments $(\rho, V, P) \mapsto (\pi, V)$ and $(\pi, V) \mapsto (\pi|_H, V, P_\pi)$ are inverses of each other, and irreducibility is preserved under these assignments. □

We continue by proving a result which, both in its statement and in its proof, is very similar to what we have seen in Section 3.3.

Lemma 3.32: *If (π, V) is an irreducible representation for G, then there exists a unique orbit O of the H-action in $N^\vee$ such that $P_\pi = 0$ on $N^\vee \backslash O$.*

Proof: We show that every orbit gives rise to an invariant subspace. More precisely, consider for a specific orbit $O \subset N^\vee$ the vector space $P_\pi(O)(V)$. It is H-invariant, because we know that $h \cdot O = O$ and $(h \cdot P_\pi(O))(V) = P_\pi(h \cdot O)(h \cdot V) = P_\pi(O)(V)$. By looking at (3.54) again, it is easy to see that $P_\pi(O)$ is N-invariant as well. Hence this space is invariant under all of G. Since orbits form a partition of $N^\vee$ and hence are mutually disjoint, we obtain a decomposition

$$V = \bigoplus_O P_\pi(O)(V) \tag{3.61}$$

The claim from the lemma now follows immediately by the irreducibility assumption together with the above discussion. □

We would now like to transfer the system of imprimitivity of H as mentioned above to a system of imprimitivity based on a coset space of a subgroup of $H \ltimes N$, so that we are in the setting of Theorem 3.27.

Lemma 3.33: *Keeping the notation and hypotheses from Lemma 3.32, let $\chi \in O$ where O is the unique orbit on which P_π is supported. Define H_χ to be the stabilizer of χ under the action of H. Let $G_\chi = H_\chi \ltimes N$. Define a map $i : G \to N^\vee$ by $(h, n) \mapsto h^{-1} \cdot \chi$. Then i descends to a bijection $\iota : G_\chi \backslash G \to O$ and induces a natural system of imprimitivity for G based on $G_\chi \backslash G$.*

Proof: Note that $h_1 \cdot \chi = h_2 \cdot \chi$ if and only if $h_2 h_1^{-1} \in H_\chi$, so that ι exists and is injective. For surjectivity, note that i has image O. Next, we have a natural G-action on $G_\chi \backslash G$ and also on $N^\vee$, because the projection $G \to H$ is a homomorphism. It is readily verified that ι intertwines these actions: indeed, for $g = (h, n)$ and $g_1 = (h_1, n)$ we have

$$\iota(g \cdot (G_\chi g_1))(n') = \iota(G_\chi g_1 g^{-1})(n') = \chi((h_1 h^{-1})(n')) \tag{3.62}$$

$$(g \cdot \iota(G_\chi g_1))(n') = \iota(G_\chi g_1)(g^{-1} n') = \chi((h_1 h^{-1})(n')) \tag{3.63}$$

Let now $S \in \mathcal{P}(G_\chi \backslash G)$. Define the operator $\iota^* P_\pi$ by $\iota^* P_\pi(S) = P_\pi(\iota(S))$. Then it is obvious from the above that $\iota^* P_\pi$ is a projection-valued measure on $G_\chi \backslash G$ and it satisfies the compatibility condition because of the intertwining property of ι. This completes the proof. □

We can also observe that there is a canonical bijection $\alpha : H_\chi \backslash H \to O$ constructed in a similar fashion, which intertwines H-actions. It then follows that given an irreducible representation (π, V) of $G = H \ltimes N$, there is a system of imprimitivity $(\pi|_H, V, \alpha^* P_\pi)$ of H based on $H_\chi \backslash H$ which is irreducible. By Theorem 3.27, there exists an essentially unique irreducible representation (ξ, W) of H_χ such that $\pi|_H \cong \mathrm{Ind}_{H_\chi}^H(\xi)$.

Define as in Section 3.3 the representation $(\xi \otimes \chi)(h, n)w = \chi(n)\xi(h)w$ of $G_\chi = H_\chi \ltimes N$ in W.

Proposition 3.34: *With the above notation, the G-representation $\operatorname{Ind}_{G_\chi}^G(\xi \otimes \chi)$ is unitarily equivalent to π.*

Proof: The representation π is realized on V. As seen from the imprimitivity theorem, we have an isometry $T : V \to V^\xi$ which intertwines the H-actions of π and π^ξ. Hence we can realize π on V^ξ by conjugation with T: define $\rho(h,n)\phi = (T \circ \pi(h,n) \circ T^{-1})\phi$, for $\phi \in V^\xi$. Trivially, ρ is unitary. Since T intertwines the H-actions, $\rho|_H \cong \pi|_H \cong \pi^\xi$. Note that

$$\begin{aligned}\pi(n) &= \sum_i \gamma_i(n) P_\pi(\{\gamma_i\}) \\ &= \sum_{H_\chi h} (h^{-1} \cdot \chi)(n) P_\pi(\{h^{-1} \cdot \chi\}) \\ &= \sum_{H_\chi h} \chi(h(n))(\alpha^* P_\pi)(\{H_\chi h\})\end{aligned} \tag{3.64}$$

This implies that the N-action of ρ on the space V^ξ is given by

$$\begin{aligned}\rho(n)\phi &= \sum_{H_\chi h} \chi(h(n))(T \circ \alpha^* P_\pi(\{H_\chi h\}) \circ T^{-1})(\phi) \\ &= \sum_{H_\chi h} \chi(h(n)) P^\xi(\{H_\chi h\})\phi \\ &= \sum_{H_\chi h} \mathrm{ev}_{\chi,n}(H_\chi h) P^\xi(\{H_\chi h\})\phi \\ &= P^\xi(\mathrm{ev}_{\chi,n})\phi = (\mathrm{ev}_{\chi,n})\phi\end{aligned} \tag{3.65}$$

where $\mathrm{ev}_{\chi,n} : H_\chi \backslash H \to \mathbb{C}$ maps $H_\chi h$ to $\chi(h(n))$. Note how we used that T intertwines the projections and $P^\xi(f)\phi = f\phi$. We conclude that π is unitarily equivalent to the given action ρ of G on V^ξ, precisely via conjugation by T.

The representation $\operatorname{Ind}_{G_\chi}^G(\xi \otimes \chi)$ is realized on $V^{\xi\otimes\chi}$. We will now also realize it on the space V^ξ. The natural inclusion $i : H \hookrightarrow G$ induces a vector space isomorphism $R : V^{\xi\otimes\chi} \to V^\xi$ given by $(R\psi)(h') = \psi(i(h')) = \psi(h,e)$. Again, it intertwines the H-actions on these spaces. It is unitary, as we have

$$\begin{aligned}\langle \psi \mid \psi' \rangle_{V^{\xi\otimes\chi}} &= \frac{1}{|H_\chi||N|} \sum_{(h,n)\in G} \langle \psi(h,n) \mid \psi'(h,n) \rangle_W \\ &= \frac{1}{|H_\chi||N|} \sum_{h\in H} \sum_{n \in N} \langle \chi(n)\psi(h,e) \mid \chi(n)\psi'(h,e) \rangle_W \\ &= \frac{1}{|H_\chi|} \sum_{h\in H} \langle \psi(h,e) \mid \psi'(h,e) \rangle_W = \frac{1}{|H_\chi|} \sum_{h \in H} \langle R\psi(h) \mid R\psi'(h) \rangle_W \\ &= \langle R\psi \mid R\psi' \rangle_{V^\xi}\end{aligned} \tag{3.66}$$

Like in the above paragraph, we can define a representation η of G in V^ξ by $\eta(h,n)\phi = R \circ \pi(h,n) \circ R^{-1}\phi$. Once more, we need to find the thus obtained action of N on V^ξ. Recall that $\psi \in V^{\xi\otimes\chi}$ transforms as $\psi(hh', nh(n')) = \chi(n)\xi(h)\psi(h', n')$ where $h \in H_\chi$. Let $\psi = R^{-1}\phi$, so that $\psi(h', e) = \phi(h')$ for $h' \in H$. We now compute

$$(\pi^{\xi\otimes\chi}(n)\cdot\psi)(h', e) = \psi(h', h'(n)) = \chi(h'(n))\xi(e)\psi(h', e) = \chi(h'(n))\psi(h', e) \tag{3.67}$$

which yields

$$(\eta(n)\phi)(h') = \chi(h'(n))\phi(h') = (\mathrm{ev}_{\chi,n}\phi)(h') \tag{3.68}$$

and this agrees with the ρ-action on V^ξ inherited from π. This concludes the proof. $\square$

Proposition 3.35: *Keeping the previous notation, let $\chi \in N^\vee$ and let ξ be an irreducible representation of H_χ. Then the induced representation $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ is irreducible.*

Proof: This relies on computations we already performed. In the proof of Proposition 3.34, we saw that $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ is unitarily equivalent to the representation η of G on V^ξ, where the H-action is the usual one and action of $n \in N$ is multiplication by $\mathrm{ev}_{\chi,n}$. If there is a nontrivial invariant subspace for η, then this is an invariant subspace of $N \subset G$ and also of $H \subset G$.

Any invariant subspace for $N \subset G$ yields an invariant subspace for P^ξ. To prove this claim, note that $\eta(n)\phi = P^\xi(\mathrm{ev}_{\chi,n})\phi$. Furthermore, the functions $\{\mathrm{ev}_{\chi,n}\}_{n\in N}$ span the space $\mathbb{C}(H_\chi\backslash H)$. Indeed, we note that H acts by automorphisms, so that $h(n)h(n^{-1}) = h(e) = e$ or $h(n)^{-1} = h(n^{-1})$, whence $(\mathrm{ev}_{\chi,n})^* = \mathrm{ev}_{\chi,n^{-1}}$. Furthermore, $\mathrm{ev}_{\chi,n}\mathrm{ev}_{\chi,n'} = \mathrm{ev}_{\chi,nn'}$ and $\mathrm{ev}_{\chi,e} = \mathbf{1}_{H_\chi\backslash H}$. It follows that the span of the set of functions $\{\mathrm{ev}_{\chi,n}\}_{n\in N}$ is a C^*-subalgebra of the C^*-algebra $\mathbb{C}(H_\chi\backslash H)$. It is point-separating, because if $H_\chi h \neq H_\chi h'$ then $h'h^{-1}$ does not stabilize χ, implying the existence of $n \in N$ such that $\chi(h'(n)) \neq \chi(h(n))$.

Viewing $H_\chi\backslash H$ as a compact topological space (endowed with the discrete topology, such that all functions on it are continuous), we apply the Stone-Weierstrass Theorem and observe that density of $\mathrm{Span}(\{\mathrm{ev}_{\chi,n}\}_{n\in N})$ in $\mathbb{C}(H_\chi\backslash H)$ with respect to the supremum metric is equivalent to $\mathrm{Span}(\{\mathrm{ev}_{\chi,n}\}_{n\in N}) = \mathbb{C}(H_\chi\backslash H)$. Hence an invariant subspace for $N \subset G$ corresponds to an invariant subspace for the projection-valued measure P^ξ.

Any invariant subspace for $H \subset G$ is an invariant subspace for $\mathrm{Ind}_{H_\chi}^H(\xi) = \pi^\xi$. This implies that the induced system of imprimitivity (π^ξ, V^ξ, P^ξ) is reducible. This contradicts Theorem 3.27, so we are done. $\square$

The last two results essentially give us a complete list of all irreducible unitary representations of G. We finally obtain the celebrated result of Mackey, now using the language from Sections 3.4 and 3.5. In Section 4.3, we will briefly discuss some aspects of the generalization to Lie groups which was used to formally derive Wigner's result.

Theorem 3.36 (Mackey): *Let $G = H \ltimes N$. Suppose $\chi \in N^\vee$ and take ξ an irreducible unitary representation of H_χ, the stabilizer subgroup of χ. Then the representation $\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi)$ is irreducible, and any irreducible representation of G is of this form. Furthermore, $\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi) \cong \mathrm{Ind}_{G_{\chi'}}^G(\xi' \otimes \chi')$ if and only if these data are conjugate by an element of G, that is, there exists $g \in G$ such that $g \cdot \chi = \chi'$ and $\xi \cong \xi' \circ C_g$.*

Proof: The first two parts have already been treated. For the last part, assume that $\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi) \cong \mathrm{Ind}_{G_{\chi'}}^G(\xi' \otimes \chi')$. Then the associated systems of imprimitivity are irreducible and unitarily equivalent. They are supported on unique orbits by Lemma 3.32, showing that there is a $g \in G$ such that $g \cdot \chi = \chi'$. The stabilizer subgroups $G_{\chi'}$ and G_χ are seen to be related by $G_{\chi'} = gG_\chi g^{-1}$. Indeed, $g_1 \in G_{\chi'}$ if and only if $\chi'(g_1^{-1}(n)) = \chi'(n)$ for all n, which is equivalent to $(g \cdot \chi)(g_1^{-1}(n)) = (g \cdot \chi)(n)$, or $g^{-1}g_1^{-1}g \in G_\chi$. Hence $C_g \circ G_\chi = G'_\chi$, so that $\xi' \circ C_g$ is indeed a representation of G_χ.

Next, denote $g \cdot \xi = \xi \circ C_{g^{-1}}$. We must prove that $g \cdot \xi \cong \xi'$. We can show that $\pi^{\xi\otimes\chi} \cong \pi^{(g\cdot\xi)\otimes\chi'}$. Indeed, write $g = (h, n)$ and recall that

$$\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi) = \{\phi : G \to W | \phi((h', n')g) = \chi(n')\xi(h')\phi(g)\} \quad (h', n') \in G_\chi \tag{3.69}$$

and analogously

$$\mathrm{Ind}_{G_{\chi'}}^G(\xi' \otimes \chi') = \{\psi : G \to W | \phi((h', n')g) = \chi(h^{-1}(n'))\xi(h^{-1}h'h)\phi(g)\} \tag{3.70}$$

where $(h', n') \in G_{\chi'}$. Define $R : \pi^{\xi\otimes\chi} \to \pi^{g\cdot\xi\otimes\chi'}$ by $(R\phi)(g') = \phi(g^{-1}g'g)$. Using the fact that $h^{-1}h'h \in H_\chi$ for $h' \in H_{\chi'}$, one can check that indeed $R\phi \in \mathrm{Ind}_{G_{\chi'}}^G(\xi' \otimes \chi')$. It can be verified directly from the definitions that R is an isometry and hence bijective. This yields a unitary equivalence $\pi^{\xi\otimes\chi} \cong \pi^{(g\cdot\xi)\otimes\chi'}$. But then also $\pi^{(g\cdot\xi)\otimes\chi'} \cong \pi^{\xi'\otimes\chi'}$. Equivalence of these representations yields equivalence of their systems of imprimitivity on $N^\vee$ (this is precisely the statement of Theorem 3.31). Pullback via the map $\alpha : H_{\chi'}\backslash H \to N^\vee$ from Lemma 3.33 yields an equivalence of systems of imprimitivity on $H_{\chi'}\backslash H$. By Theorem 3.27, it follows that $g \cdot \xi \cong \xi'$ which finishes the proof. $\square$

Chapter 4

The Wigner Classification

In this chapter, we apply a generalization of the result of Theorem 3.36 in the physical context considered by Wigner in [20]. We begin by looking at some structural properties of the Lorentz and Poincaré groups in the first section and discuss Wigner's classification in Section 4.2. For a more detailed overview of Mackey theory applied to the connected Poincaré group, see [2].

4.1 A Note on the Lorentz Group

Definition 4.1: *Let p and q be nonnegative integers. The generalized orthogonal group of type (p, q), denoted $O(p, q)$, is defined as the group of linear transformations on $\mathbb{R}^{p+q}$ which preserve the bilinear form β given by*

$$\beta(x, y) = \sum_{i=1}^{p} x_p y_p - \sum_{i=p+1}^{p+q} x_i y_i \quad x, y \in \mathbb{R}^n; \quad n = p + q \tag{4.1}$$

In other words, $O(p, q) = \{A \in GL(n, \mathbb{R}) \mid \beta(Ax, Ay) = \beta(x, y)\}$.

This bilinear form is symmetric, but it is also easily seen to be degenerate. It assumes both positive and negative values. We can observe that

$$O(p, q) = \{A \in GL(n, \mathbb{R}) \mid A^T J A = J\}, \quad J = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q\times q} \end{pmatrix} \tag{4.2}$$

Here the I's denote the identity matrices of size p and q, respectively. With this information, it becomes easy to prove that $O(p, q)$ is a closed subgroup of $GL(n, \mathbb{R})$. One can mimick the final part of the proof of Lemma 2.9. Hence it is an embedded submanifold. It is not compact for p, q both positive, because it is unbounded. What is not so easy to see is that for p, q both positive the group $O(p, q)$ has exactly four connected components. They correspond to spatial inversion and time reversal. A computation shows that the Lie algebra is given by

$$\mathfrak{o}(p,q) = \{X \in M(n,\mathbb{R}) \mid X^T J + JX = 0\} \tag{4.3}$$

with J as above. From now on we will restrict our attention to the case where $p = 1$ and $q = 3$. The group $O(1,3)$ is called the Lorentz group. The corresponding pseudo-metric β from Definition 3.1 is called the Minkowski metric. We denote its identity component by $SO(1,3)_e$. Similar to usual inner products, the form β induces a 'distance' function between two points x, y in the *spacetime manifold* $\mathbb{R}^4 = \mathbb{R}^{1,3}$, which we call the spacetime interval $\mathrm{d}s^2$. It is defined by

$$\mathrm{d}s^2(x,y) = \beta(x-y, x-y) = (x_0 - y_0)^2 - \sum_{i=1}^{3}(x_i - y_i)^2 \tag{4.4}$$

The spacetime-interval between two points is preserved by $O(1,3)$-transformations. However, since we only consider the difference of x and y when calculating the spacetime interval, $\mathrm{d}s^2$ is also invariant under translations in $\mathbb{R}^{1,3}$. This leads us to formulate the following definition:

Definition 4.2: *The Poincaré group $\mathcal{P}$ is the semidirect product of the translation group and the Lorentz group, i.e.*

$$\mathcal{P} := O(1,3) \ltimes \mathbb{R}^{1,3} \tag{4.5}$$

Here, the action of $O(1,3)$ on $\mathbb{R}^{1,3}$ is just by letting the matrix act on the vector. We see that this is similar to the setting of Chapter 3 in the sense that $\mathbb{R}^{1,3}$ is abelian. Observe that the Lie algebra of the Poincaré group is $\mathfrak{o}(1,3) \ltimes \mathbb{R}^4$. It should be noted that this is *not* a direct sum of Lie algebras. The curious reader may consult [2], Chapter 14 for more details. The important assertion is that $\mathcal{P}$ really is the full symmetry group of Minkowski spacetime. To make this precise, we need the following definition:

Definition 4.3: *By a pseudo-Riemannian manifold (M,β) we mean a smooth manifold M endowed with the structure of a pseudo-Riemannian metric β, i.e. a smooth symmetric covariant 2-tensor field. If $\mathrm{Diff}(M)$ denotes the diffeomorphism group of M, the automorphism group $\mathrm{Aut}(M)$ of M is then defined as*

$$\mathrm{Aut}(M) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\beta = \beta\} \tag{4.6}$$

Proposition 4.4: *Let M denote the space $\mathbb{R}^{1,3}$ with the structure of a pseudo-Riemannian manifold induced by the bilinear form β. There is a natural isomorphism Ψ between its automorphism group $\mathrm{Aut}(M)$ and the Poincaré group.*

Proof: The map Ψ is defined through the natural action of $\mathcal{P}$ on M as defined in Lemma 3.19. More concretely, if $(A, v) \in \mathcal{P}$ then we define $\Psi : \mathcal{P} \to \mathrm{Aut}(M)$ by $\Psi(A, v)w = v + Aw$ for $w \in M$. By the same lemma, this is indeed a group homomorphism. It is clear that $\Psi(A, v)$ is an affine map and hence a diffeomorphism. It maps into $\mathrm{Aut}(M)$. Indeed, under the canonical identification $T_{v+Aw}M \cong T_wM$ we have $\beta_{v+Aw}(AX_1, AX_2) = \beta_w(X_1, X_2)$ for $X_1, X_2 \in T_wM$ since $A \in O(1, 3)$.

It is rather obvious that Ψ is injective. It remains to prove surjectivity. Observe that M has a vector space structure, so we may view it as a commutative Lie group. Let $\phi \in \mathrm{Aut}(M)$. Let w be such that $\phi(w) = 0$. Denote by $T_w \in \mathbb{R}^4$ translation by w. Let $\phi' = \phi \circ T_w$. Then $\phi'(0) = 0$. Since $\phi' \in \mathrm{Aut}(M)$, we have $\phi'_* \in O(1, 3)$. Via the map Ψ we can view ϕ'_* as an element of $\mathrm{Aut}(M)$. It is then easy to verify that $(\phi'_*)^{-1} \circ \phi'$ fixes the origin. It has tangent map equal to Id_{T_0M} by construction. It follows from Proposition 5.9 and Lemma 5.10 in [12] that $(\phi'_*)^{-1} \circ \phi'$ equals the identity on an open and closed subset of M containing the origin. But M is connected, so we see that $(\phi'_*)^{-1} \circ \phi' = \mathrm{Id}_M$, whence $\phi = \phi'_* \circ T_{-w}$. This finishes the proof. $\square$

We will now analyze the Lie algebra of the Lorentz group, and in particular prove that it is simple. To this end, we will first need a preliminary result:

Proposition 4.5: *Let $\mathfrak{g}$ be a complex Lie algebra. We can also consider it to be a Lie algebra over $\mathbb{R}$ with the same bracket. Then $\mathfrak{g}$ is simple as an $\mathbb{R}$-Lie algebra if and only if it is simple as a $\mathbb{C}$-Lie algebra.*

Proof: Assume first that $\mathfrak{g}$ is simple as an $\mathbb{R}$-Lie algebra. Suppose $\mathfrak{g}$ has a nontrivial ideal $\mathfrak{a}$. Then in particular it is a vector space over $\mathbb{R} \subset \mathbb{C}$ as well, so this is also an $\mathbb{R}$-ideal in $\mathfrak{g}$. It follows that $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{g}$. Also since $[\mathfrak{g}, \mathfrak{g}] \neq 0$, we conclude that $\mathfrak{g}$ is nonabelian. Hence $\mathfrak{g}$ is simple as a $\mathbb{C}$-Lie algebra.

Assume now that $\mathfrak{g}$ is simple as a $\mathbb{C}$-Lie algebra. Define a map $J : \mathfrak{g} \to \mathfrak{g}, X \mapsto iX$. Then J is $\mathbb{R}$-bilinear and satisfies $[JX, Y] = J[X, Y]$ for arbitrary $X, Y \in \mathfrak{g}$. Let $\mathfrak{a}$ be a nontrivial $\mathbb{R}$-ideal of $\mathfrak{g}$. Then we can assume that it is minimal, i.e. it contains no strictly smaller ideals other than $\{0\}$. Indeed if this is not the case, we pick a strictly smaller ideal inside it and note that this process must terminate because the dimension reduces at each step. We can also assume that $\mathfrak{a}$ is non-abelian, i.e. $[\mathfrak{g}, \mathfrak{a}] \neq 0$. Indeed, if $[\mathfrak{g}, \mathfrak{a}] = 0$, then also $[\mathfrak{g}, J\mathfrak{a}] = 0$ so that $[\mathfrak{g}, \mathfrak{a} + J\mathfrak{a}] = 0$. But the space $\mathfrak{a} + J\mathfrak{a}$ is clearly a vector space over $\mathbb{C}$, which is then contained in the center of $\mathfrak{g}$. This is a contradiction. Now, $[\mathfrak{g}, \mathfrak{a}]$ is an ideal of $\mathfrak{g}$ which is contained in $\mathfrak{a}$. Hence $[\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$. But then $J\mathfrak{a} = J[\mathfrak{g}, \mathfrak{a}] = [J\mathfrak{g}, \mathfrak{a}] = [\mathfrak{g}, \mathfrak{a}] = \mathfrak{a}$. Hence $\mathfrak{a}$ is also a $\mathbb{C}$-ideal so $\mathfrak{a} = \mathfrak{g}$. $\square$

Proposition 4.6: *The identity component $SO(1, 3)_e$ of the Lorentz group is not simply connected. Its two-fold universal cover is the real Lie group $SL(2, \mathbb{C})$. The covering map is a Lie group homomorphism. As a consequence, the Lie algebra $\mathfrak{o}(1, 3)$ is isomorphic to the real Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.*

Proof: We will only do part of this calculation here. Let $A \in SL(2, \mathbb{C})$. Let $x \in M$. Let $H(\mathbb{C}^2)$ denote the real linear space of 2×2 Hermitian matrices. We have a (real) linear isomorphism $\phi : M \to H(\mathbb{C}^2)$ defined by

$$x = (x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \tag{4.7}$$

Observe that $\beta(x, x) = \det(\phi(x))$. Hence it makes sense to define $\Psi : SL(2, \mathbb{C}) \to SO(1, 3)_e$ by $\Psi(A)x = \phi^{-1}(A\phi(x)A^\dagger)$. It is readily verified that $A\phi(x)A^\dagger$ is again Hermitian and $\Psi(AB) = \Psi(A)\Psi(B)$. Also, $\beta(\Psi(A)x, \Psi(A)x) = \det(A\phi(x)A^\dagger) = \det(\phi(x)) = \beta(x, x)$. The map Ψ is smooth. As it turns out, $SL(2, \mathbb{C})$ is simply connected. This follows by considering its *polar decomposition*: there is a diffeomorphism $F : SU(2) \times H_0 \to SL(2, \mathbb{C})$ where H_0 is the space of 2×2 traceless Hermitian matrices. It is given by $(U, H) \mapsto U\exp(H)$. For more details, see [2]. Since $SU(2) \cong S^3$ is simply connected, $SL(2, \mathbb{C})$ deformation retracts onto a simply connected space and the assertion follows. The map Ψ has kernel $\{+I, -I\}$, which is discrete. Hence Ψ is a covering map. This realizes $SL(2, \mathbb{C})$ as the two-fold universal covering of the connected Lorentz group. The final part of the assertion follows easily. $\square$

Note then that $\Psi \otimes \mathrm{Id}$ is a two-to-one covering map from $SL(2, \mathbb{C}) \ltimes \mathbb{R}^{1,3}$ to $SO(1, 3)_e \ltimes \mathbb{R}^{1,3}$, realizing the former as the universal covering of the connected Poincaré group, with the semidirect product structure given by the action of $SL(2, \mathbb{C})$ on $M = \mathbb{R}^{1,3}$ via the map Ψ.

In particular, we can apply Proposition 4.5 to conclude that the Lorentz group is in fact simple. Indeed, it is well known that the complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is simple. Hence Theorem 2.1 applies and we conclude that it has no finite-dimensional unitary representations, except the trivial one. By restriction, the same applies to the Poincaré group and its universal covering. In the next section, we will see why the unitary representations of the universal covering are of great interest in theoretical physics.

4.2 Projective Representations of the Poincaré Group

In the seminal paper [20], twentieth-century theoretical physicist E.P. Wigner established a connection between elementary particles and certain irreducible representations of the universal cover of the Poincaré group. We will examine this idea merely in a qualitative manner.

In quantum mechanics, a physical system is described by a nonzero wave function $|\Psi\rangle$ living in some (generally infinite-dimensional) complex Hilbert space $\mathcal{H}$. Let us assume that Ψ is a pure state. Roughly speaking, an observable corresponds to a self-adjoint operator A acting on the Hilbert space $\mathcal{H}$. The eigenvalues of this operator correspond to the possible outcomes of an experiment measuring A. Writing Ψ as a superposition of eigenvectors of A then yields the probability of the system collapsing into a specific eigenstate. Explicitly, if $|\Phi\rangle$ is an eigenfunction of A, then the probability of the $|\Psi\rangle$ collapsing into the state $|\Phi\rangle$ upon measurement (called the *transition probability*) is given by

$$p(|\Psi\rangle, |\Phi\rangle) = p_{|\Psi\rangle\to|\Phi\rangle} = \frac{|\langle\Phi|\Psi\rangle|^2}{\langle\Psi|\Psi\rangle\langle\Phi|\Phi\rangle} \tag{4.8}$$

It can then be inferred that two wave functions $|\Psi\rangle$ and $|\Psi'\rangle$ yield the same probabilities if they differ by a nonzero complex scalar, i.e. $|\Psi'\rangle = \lambda|\Psi\rangle$ for some $\lambda \in \mathbb{C}^*$. That is to say, the systems described by $|\Psi\rangle$ and $|\Psi'\rangle$ cannot be distinguished upon any series of measurements, so they represent the same *physical state*. It is then natural to impose an equivalence relation $\sim$ on $\mathcal{H}^* = \mathcal{H} - \{0\}$ to obtain the true state space of the system, called *projectivized Hilbert space*. It carries the quotient topology inherited from $\mathcal{H}$. Explicitly, it is given by

$$\mathbb{P}(\mathcal{H}) = \mathcal{H}^*/\sim \quad ; \quad |\Psi\rangle \sim |\Psi'\rangle \iff \Big(|\Psi'\rangle = \lambda|\Psi\rangle; \quad \lambda \in \mathbb{C}^*\Big) \tag{4.9}$$

If we denote the equivalence class of $|\Psi\rangle$ by $[\Psi]$, we indeed see that (94) descends to a well-defined continuous map $\tilde{p}$ on $\mathbb{P}(\mathcal{H})$, i.e. $\tilde{p}([\Psi],[\Phi]) = p(|\Psi\rangle, |\Phi\rangle)$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$ satisfying $\tilde{p}(T[\Psi], T[\Phi]) = \tilde{p}([\Psi],[\Phi])$ is called a *projective automorphism*. The set of such maps forms a group under composition which is denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, the *projective automorphism group* of the state space. Its action leaves the transition probability invariant.

Suppose now that this particle is located in a flat Minkowski spacetime (that is, the manifold $M = \mathbb{R}^{1,3}$). As we saw earlier, the *symmetry* group of this manifold is precisely the Poincaré group $\mathcal{P}$. If two observers O and O' related via a transformation $\Lambda \in \mathcal{P}$ perform a quantum-mechanical experiment on a system, they will generally observe different states, say $[\Psi]$ and $[\Psi']$. It is expected that because P is the symmetry group of spacetime, the laws of physics should remain invariant under such transformations. Consequently, it seems reasonable to expect transition probabilities to be conserved upon passing from one reference frame to another. This implies that these states are related via a projective automorphism. Suppose then that $[\Psi] = T_\Lambda[\Psi']$ for some $T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. If $O = O'$, we should have $T_\Lambda = T_{\mathrm{Id}} = \mathrm{Id} : \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$. Lastly, if we have a third observer O'' related to O' via $\Gamma \in \mathcal{P}$, we should impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\circ\Gamma}$. Indeed, it should not matter whether the observer O'' communicates directly with O or via O'.

The above discussion demonstrates that a change of frame of reference induces an action of the Poincaré group by projective automorphisms on the quantum-mechanical state space. That is, we have a homomorphism $\Xi : \mathcal{P} \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. In literature, this map is required to satisfy a kind of continuity condition, but we will not dwell on this here. The pair $(\Xi, \mathcal{H})$ is called a *projective representation* of the Poincaré group. It is called irreducible if the only closed positive-dimensional subspace V of $\mathcal{H}$ descending to a $\mathcal{P}$-invariant subspace under Ξ is $\mathcal{H}$. That is to say, $\Xi(\mathcal{P})[V^*] \subset [V^*]$ if and only if $V = \mathcal{H}$. The closedness condition is necessary here: if we want to regard V as a Hilbert space in its own right, it should be complete and hence closed, but this is not automatically true in the infinite-dimensional case.

The key idea is that irreducible *projective* representations of the Poincaré group correspond to elementary particles within the quantum system under consideration. In fact, this was a first attempt at *defining* the notion of an elementary particle. As quantum field theory emerged throughout the decades following Wigner's paper, this definition was slightly modified. However, it still relied on the same intuition, which is as follows: an elementary particle state of a quantum system is represented by an element of $\mathbb{P}(\mathcal{H})$, which may differ between observers as we discussed before. The collection of all its possible states according to different observers constitutes a $\mathcal{P}$-invariant subspace of $\mathbb{P}(\mathcal{H})$, hence we obtain a subrepresentation of Ξ. Guided by the heuristic that the entire system should be a composition of elementary particles, the thus obtained subrepresentation should correspond to a subsystem, which is 'elementary' if it is irreducible (because otherwise it contains yet smaller subsystems).

This converts the physical question of classifying all relativistic elementary free particles in flat Minkowski spacetime to the mathematical task of finding all irreducible projective representations of the Poincaré group. If we restrict ourselves to its identity component $\mathrm{SO}(1,3)_e \ltimes \mathbb{R}^{1,3}$, it turns out that they can are in bijective correspondence with unitary representations of its universal covering group, which we calculated to be $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. As mentioned before, $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{R}^{1,3}$ via the covering homomorphism onto $\mathrm{SO}(1,3)_e$. Moreover, irreducibility is preserved by this correspondence. Such a correspondence does not hold for general connected Lie groups, but here we are lucky enough. For more details, see [2]. Once these claims are justified, the work of Mackey comes into play: the irreducible unitary representations of the semidirect product $\mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ can be constructed if we manage to compute the orbit structure of the $\mathrm{SL}(2,\mathbb{C})$-action, along with the irreducible unitary representations of the stabilizers. Now we turn to the computation of the aforementioned orbit structure.

First, we reduce the problem of finding the orbits of characters to finding orbits of the $SL(2,\mathbb{C})$-action in M instead of its character space:

Proposition 4.7: *Consider the commutative Lie group $\mathbb{R}^{1,3}$. The map $T : \mathbb{R}^{1,3} \to (\mathbb{R}^{1,3})^\vee$ defined by $T(v)x = e^{i\beta(v,x)}$ identifies $\mathbb{R}^{1,3}$ with its unitary dual (i.e. the space of unitary characters, or homomorphisms into S^1.) and intertwines the $SL(2,\mathbb{C})$-action on $M = \mathbb{R}^{1,3}$ with the action of $SL(2,\mathbb{C})$ on the character space. In particular, $v \in \mathbb{R}^{1,3}$ is $SL(2,\mathbb{C})$-stable if and only if $\chi = T(v) \in (\mathbb{R}^{1,3})^\vee$ is stable under $SL(2,\mathbb{C})$.*

Proof: It is easily seen that $T(v)(x+y) = T(v)x \cdot T(v)y$ and $T(v) \in S^1$. Obviously $T(v)$ is continuous. Now if $A \in SL(2,\mathbb{C})$ then $(A \cdot T(v))x = T(v)(A^{-1} \cdot x) = \mathrm{e}^{i\beta(v,\Psi(A)^{-1}x)} = \mathrm{e}^{i\beta(\Psi(A)v,x)} = T(\Psi(A)v)x$. If we can establish bijectivity of T, then indeed $A \cdot T(v) = T(v)$ for all A if and only if $\Psi(A)v = v$ for all A. This proves the final part. It thus remains to prove bijectivity of T.

Suppose $T(v) = T(v')$. Then, regarding both as smooth functions from $\mathbb{R}^4$ to S^1, we take the derivative at $x = 0$ to find $\beta(v,x) = \beta(v',x)$ for all $x \in T_0\mathbb{R}^4 \cong \mathbb{R}^4$. By nondegeneracy of the Lorentzian inner product, $v = v'$. For surjectivity, let χ be any unitary character. Note that for any real vector space V, the exponential map $\exp : T_0V \to V$ is just the identity under the canonical identification $T_0V \cong V$. We also have $T_0S^1 \cong i\mathbb{R}$ (upon regarding $S^1 \subset \mathbb{C}$). We then have the diagram

$$\begin{array}{ccc} \mathbb{R}^{1,3} & \xrightarrow{\chi} & S^1 \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow {\scriptstyle e^{\cdot}} \\ \mathbb{R}^{1,3} & \xrightarrow[\chi_*]{} & i\mathbb{R} \end{array}$$

But then $\chi = e^{\chi_*}$. Identifying $\mathrm{Lin}(\mathbb{R}^{1,3}, i\mathbb{R}) \cong i(\mathbb{R}^{1,3})^*$ we see that there is a (unique) linear functional $\alpha : \mathbb{R}^{1,3} \to \mathbb{R}$ such that $\chi = e^{i\alpha}$. Clearly, there is a $v \in \mathbb{R}^{1,3}$ such that $\beta(v,\cdot) = \alpha$, as β has full rank. This proves bijectivity and concludes the proof. $\square$

For the next calculation, we define

$$Y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.10}$$

It is straightforward to check (using (90)) that $Y \in \mathfrak{o}(1,3)$. By straightforward calculations, one may check that for $t \in \mathbb{R}$ arbitrary we have

$$\exp(tY) = \begin{pmatrix} \cosh(t) & 0 & 0 & \sinh(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(t) & 0 & 0 & \cosh(t) \end{pmatrix} \tag{4.11}$$

Lastly, note that $\cosh : \mathbb{R} \to \mathbb{R}$ attains its unique minimum at $t = 0$, with $\cosh(0) = 1$ and $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective. In preparation for the computation of the orbits, we need one more preliminary proposition:

Proposition 4.8: *The irreducible unitary representations of the circle group $G = SO(2)$ are all one-dimensional and are labeled by integers $n \in \mathbb{Z}$. Explicitly, they are given by*

$$\mathbb{C} \ni z \mapsto z^n \in \mathbb{C}; \quad n \in \mathbb{Z} \tag{4.12}$$

Proof: The fact that the irreducible representations are one-dimensional follows from Schur's lemma. They are identified with their characters, which can be assumed to be unitary because S^1 is compact, so that any representation is unitarizable. This bring us to the study of (smooth) homomorphisms from S^1 to itself. Like in the proof of Proposition 4.7, for any character χ of S^1 we have

$$\begin{tikzcd} S^1 \arrow[r, "\chi"] & S^1 \\ i\mathbb{R} \arrow[u, "e^{\cdot}"] \arrow[r, "\chi_*"'] & i\mathbb{R} \arrow[u, "e^{\cdot}"'] \end{tikzcd}$$

or equivalently, $\chi(e^{ix}) = e^{\chi_*(ix)}$ for $x \in \mathbb{R}$. Suppose that the real linear map χ_* is multiplication by $\lambda \in \mathbb{R}$. Then we must have $1 = \chi(1) = \chi(e^{2\pi i}) = e^{2\pi i\lambda}$, whence $\lambda \in \mathbb{Z}$. It is clear that this condition is sufficient. This concludes the proof. $\square$

From now on, we put $\beta(x, x) = m^2$. By invariance of β, each orbit O_x of the action must be a level set of β, i.e. $\beta(O_x, O_x) = m^2$ for a fixed $m \in \mathbb{C}$. Note that by connectedness of $SO(1, 3)_e$ (or $SL(2, \mathbb{C}$ for that matter) all orbits must be connected. This provides us with the following list:

(i) For $m^2 > 0$, we get a family of two-sheeted hyperboloids $m^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. The two sheets correspond to the two possible signs of x_0, i.e. $x_0 > 0$ and $x_0 < 0$. These two sheets are the path components of the hyperboloids. The sets $X^\pm = \{x \in M|\beta(x, x) = m^2; \quad x_0 \in \mathbb{R}^\pm\}$ are path connected, as may be verified. Let $v = (m, 0, 0, 0) \in X^+$. Then $\exp(tY)v = m(\cosh(t), 0, 0, \sinh(t))$. If $v' \in X^+$ we clearly have $v'_0 \geq v_0$. Hence there is a $t \in \mathbb{R}$ such that $v'_0 = v_0\cosh(t)$. Since $SO(3) \subset SO(1, 3)_e$ (embedded in the lower right corner) acts freely on a sphere of any radius, there is a rotation R such that $R \circ \exp(tY)v = v'$. Hence X^+ is an orbit. Similarly, X^- is an orbit. This gives us two families of orbits.

(ii) For $m^2 = 0$, the level set of β is the *light cone* $x_0^2 = x_1^2+x_2^2+x_3^2$. This space is path connected. The origin is a fixed point under all of $SL(2, \mathbb{C})$. The complement of the origin has two connected components, called the *future light cone* and the *past light cone*, for x_0 positive resp. negative. Denote them again by X^+ and X^-. We will now see that these are again full orbits. Let $v = (1, 0, 0, 1) \in X^+$. Then $\exp(tY)v = e^t v$, so that if $v' \in X^+$ we find an appropriate $t \in \mathbb{R}$ so as to obtain $e^t v_0 = v'_0$. Then we apply a suitable rotation as before to match the spatial components. Similarly, $v = (-1, 0, 0, 1)$ is seen to generate X^-. Hence we obtain two more families and one trivial orbit.

(iii) For $m^2 < 0$, the level set $\beta(x, x) = m^2$ is now path connected. Indeed, it consists of only one orbit. Let $v = (0, 0, 0, m)$ and let v' be an arbitrary element of the level set. Now, $\exp(tY)v = m(\sinh(t), 0, 0, \cosh(t))$. But sinh is surjective, so that for suitable t we have $m \sinh(t) = v'_0$. Again, we apply a suitable spatial rotation to see that $v' \in O_v$. This completes the computation of the orbits.

We now turn to the computation of the stabilizers. For this, we closely follow [16]. We once again identify vectors in M with 2×2 Hermitian matrices. Via conjugation, stabilizer subgroups of elements in the same orbit are isomorphic. We have the following exhaustive list:

(i) For $m^2 > 0$, we had two families corresponding to the signs of m. the stabilizer condition reads

$$A\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}A^\dagger = AA^\dagger\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \tag{4.13}$$

for both choices of sign. This clearly is equivalent to $A \in U(2)\cap SL(2,\mathbb{C}) = SU(2)$. This should not come as a surprise, as both sheets of the hyperboloid are invariant under rotations about any spatial axis and $SU(2)$ is the two-fold universal covering of $SO(3)$. Indeed, it can be checked that $\Psi(SU(2)) \cong SO(3) \subset SO(1,3)_e$.

(ii) For $m^2 = 0$, we still need to consider the future and past light cones. Labeling the entries of A by $a_{ij} = (A)_{ij}$, the condition (after dividing by m) is

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = A\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}A^\dagger = \begin{pmatrix} 2|a_{11}|^2 & 2a_{11}a_{21}^* \\ 2a_{21}a_{11}^* & 2|a_{21}|^2 \end{pmatrix} \tag{4.14}$$

so that $|a_{11}| = 1$ and $a_{21} = 0$. Note that the sign of m does not matter. Writing $a_{11} = e^{i\theta}$, we see that the stabilizer is the group G defined by

$$G = \{\begin{pmatrix} e^{i\theta} & b \\ 0 & e^{-i\theta} \end{pmatrix} \mid \quad \theta \in \mathbb{R} \quad b = x + iy \in \mathbb{C}\} \tag{4.15}$$

We claim that G is isomorphic to the semidirect product $SO(2) \ltimes \mathbb{R}^2$ with the action of $R_\theta \cdot (x,y) = R_{2\theta}(x,y)$. To see this, note that the elements for which to $\theta = 0$ in G form a normal subgroup N isomorphic to $\mathbb{C} \cong \mathbb{R}^2$. Indeed,

$$\begin{pmatrix} e^{i\theta} & b \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\theta} & -b \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} 1 & e^{2i\theta}c \\ 0 & 1 \end{pmatrix} \tag{4.16}$$

Furthermore, we can identify $SO(2)$ with the subgroup $H \subset G$ of elements with $b = 0$. Note that $HN = G$ and $H \cap N = \{I\}$. This shows that the map $SO(2) \ltimes \mathbb{R}^2 \ni (h,n) \mapsto hn \in G$ is an isomorphism, as can easily be verified. Hence the stabilizer of the past and future light cones is isomorphic to the double cover of the isometry group of $\mathbb{R}^2$.

(iii) For $m^2 < 0$, it turns out to be a bit more convenient to take $(0,0,m,0)$ as a representative. After dividing both sides by mi, the stabilizer condition reads

$$A\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}A^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.17}$$

Denoting the entries of A by $a_{ij} = (A)_{ij}$ again, we also have

$$A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & -(a_{11}a_{22} - a_{12}a_{21}) \\ (a_{11}a_{22} - a_{12}a_{21}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.18}$$

as $A \in SL(2, \mathbb{C})$ has unit determinant. So we must have $A^\dagger = A^T$, which is equivalent to all entries of A being real. Hence the stabilizer is $SL(2, \mathbb{R})$.

The final step in Mackey's procedure is to determine the irreducible unitary representations of the stabilizers. We have the following:

(i) For $m^2 > 0$, let us take without loss of generality $m > 0$. The number m is the *mass* of the particle. The irreducible unitary representations of $SU(2)$ are a textbook example: the group is compact, so all irreducible unitary representations are finite-dimensional. It is also simply connected, so in fact any representations of its Lie algebra lifts to a Lie group representation in a unique way. The irreducible unitary representations are labeled by a half-integer $s \in \frac{1}{2}\mathbb{N} \cup \{0\}$ called the *spin* of the particle. The representations are realized on $\mathbb{C}^{2s+1}$ and hence are $2s + 1$-dimensional, so there is one in each dimension. Common examples are leptons (like the electron) and quarks, which have half-integer spin, and the W and Z-bosons, which have integer spin.

(ii) For $m^2 = 0$, we have two cases. The first is the stabilizer $SL(2, \mathbb{C})$ of the origin. By what we proved in Chapter 2, all of its unitary representations are infinite-dimensional. Hence they are much harder to analyze, and we most certainly cannot do this here. In the second case, for the future light cone for example, we need the irreducible unitary representations of $SO(2) \ltimes \mathbb{R}^2$. We can use the Mackey machine for this case as well, of course. The orbits of the $SO(2)$-action are circles of positive radius, together with the origin. For positive radius, the stabilizer is trivial. For the origin, the stabilizer is all of $SO(2)$. The irreducible unitary representations of the abelian group $SO(2)$ are one-dimensional. As we saw before, they are labeled by an integer $n \in \mathbb{Z}$. It is customary to write $n = 2s$ with $s \in \frac{1}{2}\mathbb{Z}$. The absolute value of s is again called the spin of the particle (which is a bit unfortunate, because the corresponding stabilizer group is clearly *not* $SU(2)$). The sign corresponds to the *polarization*. Sometimes the integer s is called the helicity of the particle. Examples of massless particles are photons and gluons, as well as the hypothesized graviton.

(iii) For $m^2 < 0$, we saw the stabilizer to be equal to $SL(2, \mathbb{R})$. It is well known that this is also a simple Lie group. It is readily seen to be noncompact. It is connected, so we satisfy the hypotheses of Theorem 2.1 once more: all unitary representations are infinite-dimensional and this takes us beyond the scope of this thesis. These representations are sometimes called *tachyonic* and are typically not considered in physics literature. The main reason for this is that an imaginary mass corresponds to superluminal motion, which violates the causality principle.

We now also see how Wigner's work sparked a great deal of interest in the mathematics of representation theory as well: the study of the unitary representations of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ was unknown at the time the work was published and was later carried out by Bargmann [4]. Advances in a more general setting were made by Harish-Chandra, see [11].

4.3 The Mass-Squared Parameter

Wigner's classification is often loosely treated in physics textbooks on quantum field theory as well, with more emphasis on the physical interpretations and consequences of the result. It is the starting point of more advanced theories which for example try to explain why only particles with certain masses exist, or how one can incorporate other conserved quantities (like electric charge) into this framework. We refer to [19].

In the above reference, one focuses on calculating the stabilizer groups (which are called *little groups*) by using the Casimir elements of the enveloping algebra of the Poincaré group. Recall that, for a finite-dimensional Lie algebra $\mathfrak{g}$, its *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is uniquely defined by the property that for any associative algebra A and any Lie algebra homomorphism $\pi : \mathfrak{g} \to A$, there exists a unique homomorphism of associative algebras $\tilde{\pi} : \mathcal{U}(\mathfrak{g}) \to A$ such that $\pi = \tilde{\pi} \circ \iota$. Here, $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the canonical inclusion. Hence the following diagram commutes:

$$\begin{tikzcd} & \mathcal{U}(\mathfrak{g}) \arrow[d, dotted, "\tilde{\pi}"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[r, "\pi"'] & A \end{tikzcd}$$

Furthermore, a basis for the enveloping algebra can be given explicitly in terms of a basis for $\mathfrak{g}$ (by the well-known Poincaré-Birkhoff-Witt Theorem), from which we immediately infer that $\mathcal{U}(\mathfrak{g})$ has countable dimension. Recall that the Casimir elements commute with all generators of $\mathcal{U}(\mathfrak{g})$, or equivalently they are central in the enveloping algebra.

In the next part of our discussion, we see how this approach can show us that it is natural to associate to the parameter m the physical interpretation of *mass*. We need the following infinite-dimensional version of Schur's Lemma, which is due to Dixmier. The proof below is from [18], p.114.

Proposition 4.9: *Let A be an associative $\mathbb{C}$-algebra of countable dimension acting irreducibly on a complex vector space $\mathcal{H}$ of nonzero dimension. Then if L is an endomorphism of $\mathcal{H}$ commuting with the A-action, L must be a scalar.*

Proof: By the same arguments as in the finite-dimensional case, $\ker(L)$ and $\mathrm{Im}(L)$ are invariant subspaces, so that if $L \neq 0$, it must be an isomorphism. Suppose L is an algebraic element, that is, there exists a polynomial $P \in \mathbb{C}[X]$ such that $P(L) = 0 \in \mathrm{End}(\mathcal{H})$. Assume P to be of minimal degree. By the fundamental theorem of algebra, P admits a zero: there exists $\lambda \in \mathbb{C}$ such that $P(\lambda) = 0$. Write $P(X) = (X - \lambda)Q(X)$. Since $P(L) = 0$ we infer that $L - \lambda\mathrm{Id}$ cannot be invertible by minimality of the degree of P, so it must be zero. Thus, if L is algebraic we are done.

The situation is more subtle if L is transcendental. Clearly in this case, L is an isomorphism. This implies that L^{-1} exists and is also a transcendental intertwiner. Consequently, the elements $\{(L - \lambda\mathrm{Id})^{-1} | \lambda \in \mathbb{C}\}$ are all intertwiners. It can be shown that they are linearly independent over $\mathbb{C}$, so that the vector space of intertwiners $\mathrm{End}_A(\mathcal{H}) = \mathrm{Hom}_A(\mathcal{H}, \mathcal{H})$ is of uncountable dimension. But for nonzero $v \in \mathcal{H}$, evaluation at v is a homomorphism from A to $\mathcal{H}$. By irreducibility, it is surjective. Hence $\mathcal{H}$ is of countable dimension. Finally, since $\mathrm{End}_A(\mathcal{H})$ contains only isomorphisms, evaluation at v is also an injective homomorphism from $\mathrm{End}_A(\mathcal{H})$ into $\mathcal{H}$. Hence the former is of countable dimension, which is a contradiction. Hence our assumption that L is transcendental was false. This concludes the proof. □

As a simple corollary, if $\mathfrak{g}$ is a finite-dimensional complex Lie algebra, then any central element of $\mathcal{U}(\mathfrak{g})$ must act by a scalar in an irreducible representation.

In relativistic quantum mechanics, the four-momentum P_μ is the generator of translations in spacetime. That is, its four components ($\mu = 0, \ldots 3$) form the standard basis of the abelian part $\mathbb{R}^4$ of the Lie algebra $\mathfrak{o}(1,3) \ltimes \mathbb{R}^4$. Suppose we are dealing with a quantum elementary particle, i.e. an irreducible unitary representation π of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ on a Hilbert space $\mathcal{H}$. In the case of finite-dimensional representation theory, we can conclude that since the covering space is connected, the associated representation π_* of the Lie algebra $\mathfrak{o}(1,3) \ltimes \mathbb{R}^4$ must also be irreducible.

In the infinite-dimensional case, this parallel is more subtle. The reader may have noticed that in defining irreducibility of a representation of the group, we demanded non-existence of nontrivial invariant *closed* subspaces. However, in the proof of Proposition 4.9 on irreducible representations of the Lie algebra (and its envelope), the irreducibility was defined in a purely algebraic way. Indeed, the subspaces $\ker(L)$ and $\mathrm{Im}(L)$ are certainly invariant, but the latter may not be closed. Hence we could not conclude that L is zero or an isomorphism if we had taken the same definition as in the group case. For more details on this apparent discrepancy, the reader can consult [17].

In what follows, we will take for granted that the representation of the enveloping algebra on $\mathcal{H}$ is also irreducible, so that the central elements act by scalars according to Proposition 4.9. Under this action, the four-momentum is a vector whose components are Hermitian operators on $\mathcal{H}$. More precisely,

$$\pi(\exp(iP_\mu))|\Psi\rangle = e^{\pi_*(iP_\mu)}|\Psi\rangle = \pi(T_\mu)|\Psi\rangle \tag{4.19}$$

where $T_\mu \in \mathbb{R}^{1,3} \subset SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ is unit translation in the μ-direction. In the physicist's convention, the factor of i is inserted to ensure hermiticity of P_μ. We can now apply this to the Casimir element $P_\mu P^\mu = P_0^2 - P_1^2 - P_2^2 - P_3^2 \in \mathcal{U}(\mathfrak{g})$. By a straightforward calculation which we omit here, this is indeed central, so that our discussion applies. The interested reader may consult [19] (or any other textbook on quantum field theory for that matter) for an explicit description of the Poincaré algebra in terms of commutators.

The generalization of Mackey's work to Lie groups (which we treated in Chapter 3 for finite groups) tells us that any irreducible unitary representation of the double cover $G = SL(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$ of the Poincaré group is of induced type, so that we can take $\mathcal{H}$ to be the space of (continuous) functions on G with certain transformation properties under the representation of the little (stabilizer) group. That is,

$$\mathcal{H} = \mathrm{Ind}_{G_\chi}^{G}(\xi \otimes \chi) = \{\phi : G \to V_\xi | \phi((h,n)g) = \chi(n)\xi(h)\phi(g); \quad \phi \text{ continuous}\} \tag{4.20}$$

An irreducible representation π of G can be differentiated to yield an irreducible representation of the Lie algebra, i.e. a Lie algebra homomorphism $\pi_* : \mathfrak{g} \to \mathrm{End}(\mathcal{H})$. More concretely, $((\pi_* P_\mu)\phi)(g) = \frac{d}{dt}|_{t=0}\phi(g \cdot \exp(tP_\mu))$, where we used that evaluation at $g \in G$ is linear and hence commutes with the time derivative. This only makes sense for differentiable functions, but it turns out that we can restrict our attention to this subspace in the general framework (see Section 6 of [2]).

Following the remarks on the universal enveloping algebra, the map π_* can be extended to a homomorphism of associative algebras $\tilde{\pi}_* : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{H})$. Let d_0 denote the time derivative at $t = 0$. Then for $X \in \mathfrak{g}$,

$$\begin{aligned}(\pi_*(X)\phi)(e) &= d_0\phi(e\,\exp(tX)) = d_0\phi(\exp(tX)e) \\ &= d_0\chi(\exp(tX))\xi(e)\phi(e) = d_0\chi(\exp(tX))\phi(e) = \chi_*(X)\phi(e)\end{aligned} \tag{4.21}$$

Using the relation

$$\tilde{\pi}_*(P_\mu P^\mu) = \pi_*(P_0)^2 - \pi_*(P_1)^2 - \pi_*(P_2)^2 - \pi_*(P_3)^2 \tag{4.22}$$

we infer that the action of the Casimir element is given by

$$(\tilde{\pi}_*(P_\mu P^\mu)\phi)(e) = (\chi_*(P_0)^2 - \chi_*(P_1)^2 - \chi_*(P_2)^2 - \chi_*(P_3)^2)\phi(e) \tag{4.23}$$

But $\tilde{\pi}_*(P_\mu P^\mu)\phi$ is a multiple of ϕ, so by evaluating at the identity we deduce that the scalar λ by which the Casimir operator acts is equal to the complex number $\lambda = \chi_*(P_0)^2 - \chi_*(P_1)^2 - \chi_*(P_2)^2 - \chi_*(P_3)^2$.

Observe that for a given character χ_*, there is a unique $v \in \mathbb{R}^{1,3}$ such that $\chi_*(w) = i\beta(v,w)$ (this is Proposition 4.7). But then $\chi_*(P_\mu) = i\beta(v, P_\mu)$. Note that $P_\mu \in \mathbb{R}^4$ is really just the standard basis vector e_μ multiplied by $-i$ (because of the factor i in (4.19) in the exponential). This readily implies that

$$\lambda = (v_0)^2 - (-v_1)^2 - (-v_2)^2 - (-v_3)^2 = \beta(v,v) = m^2 \tag{4.24}$$

This is precisely what we wanted to show. In the language of the physicist, the 'relativistic length squared' $\tilde{\pi}_*(P_\mu P^\mu)$, which is usually just denoted $P_\mu P^\mu$, is equal to the rest mass of the particle.

Chapter 5

A: Universal Covers of Compact Simple Lie Groups

The purpose of this appendix is to prove Proposition 2.10. This will require more advanced methods from differential geometry and Lie algebra cohomology. We avoid the more general 'categoric' framework that goes with the functorial approach to this subject. We introduce the Chevalley-Eilenberg complex and its degree operator, the so-called Koszul differential. Then we compute the cohomology of degree 1 in the semisimple case.

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$. Roughly speaking, we divide the proof into the following steps:

(i) We introduce the Lie algebra cohomology groups $H^k(\mathfrak{g}, \mathbb{R})$ and prove that $H^1(\mathfrak{g}, \mathbb{R}) = 0$ for $\mathfrak{g}$ semisimple.

(ii) We recall the definition of de Rham cohomology, and prove that in the compact case it is identical to the Lie algebra cohomology via an averaging procedure: $H^k_{\text{dR}}(M) = H^k(\mathfrak{g}, \mathbb{R})$.

(iii) In particular, by the de Rham theorem (Proposition 18.14 in [13]) we obtain $\text{Hom}(H_1(G, \mathbb{Z}), \mathbb{R}) = 0$.

(iv) Using structure theory of finitely generated abelian groups, this implies that $H_1(M, \mathbb{Z})$ is finite. We give an algebraic proof for this.

(v) Thus, by familiar algebraic topology, the singular homology over $\mathbb{Z}$ is the abelianization of the fundamental group $\pi_1(G, e)$. By another well-known argument, the fundamental group of a topological group (and in particular a Lie group) is abelian, hence $\pi_1(G, e)$ is finite.

(vi) Finally, the universal cover of G is finite-sheeted, because the cardinality of the number of sheets equals that of the fundamental group. In particular it is compact (we give a topological proof for this).

5.1 Lie Algebra Cohomology

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Define for $k \in \mathbb{N} \cup \{0\}$ the vector space $M^k = \wedge^k \mathfrak{g}^*$, the space of all alternating multilinear maps on $\mathfrak{g}$. Let $d = \dim(\mathfrak{g})$.

We can define an operator δ on $M = \bigoplus_{k=0}^{d} M^k$ by defining its action on $\omega \in M^n$. Then the action of the $(n+1)$-form $\delta\omega$ on vectors $X_1, \ldots, X_{n+1} \in \mathfrak{g}$ is given by:

$$\delta\omega(X_1, \ldots, X_{n+1}) = \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{n+1}) \quad (5.1)$$

By convention, the hat indicates an omitted argument.

Definition A.1: *The pair (M, δ) is commonly called the Chevalley-Eilenberg complex. The map δ is referred to as the Koszul differential.*

Lemma A.2: *The Koszul differential is an antiderivation of degree +1 whose square is zero.*

Proof: By definition the first part of the assertion is clear. The proof that the Koszul differential squares to zero is the result of a somewhat tedious (but not very hard) proof by induction. We omit it here. □

In general, the Chevalley-Eilenberg complex can be constructed in a more systematic way instead of just introducing the definition without further context. If one does this, the proof that $\delta^2 = 0$ is more elegant and natural. We choose this approach because we stress again: we are not interested in general modules.

By the above, we can see that the map δ 'splits' into maps $\delta_k : M^k \to M^{k+1}$. In view of Lemma A.2, it turns out that they satisfy $\delta_{k+1} \circ \delta_k = 0$. Equivalently, $\mathrm{Im}(\delta_k) \subseteq \mathrm{Ker}(\delta_{k+1})$. We can now come to our first important definition:

Definition A.3: *Let $\mathfrak{g}$ be a Lie algebra. Define the Chevalley-Eilenberg complex and the Koszul differential as above. Then the Lie algebra cohomology groups of degree k denoted by $H^k(\mathfrak{g}, \mathbb{R})$ are defined as the quotient vector spaces*

$$H^k(\mathfrak{g}, \mathbb{R}) = \frac{\mathrm{Ker}(\delta_k)}{\mathrm{Im}(\delta_{k-1})} \quad (5.2)$$

As outlined in the introductory paragraph of this appendix, the first cohomology group is of particular interest to us. Luckily, it is not very hard to compute it:

Lemma A.4: *In the setting of the above definition, the cohomology group of degree 1 equals zero if $\mathfrak{g}$ is semisimple.*

Proof: We can make convenient use of the definition of δ_n: substituting $n = 0$ we see that the image of δ_0 is trivial, hence we want to prove that the kernel of δ_1 is trivial. We compute for $n = 1$: $\delta_1(\alpha)(X,Y) = -\alpha[X,Y]$. But $\alpha[\mathfrak{g},\mathfrak{g}] = \alpha(\mathfrak{g})$ as the Lie algebra is semisimple, by Lemma 1.23. So inevitably, α must be the zero map. This implies the assertion. $\square$

Recall that if M is a manifold, the de Rham cohomology is defined as the quotient vector space of closed forms modulo exact forms. Here the chain map is the exterior derivative d acting on the bundles $M \to \wedge^k(T^*M) = \sqcup_{p\in M} \wedge^k (T_p^*M)$, where $k = 0, 1, \ldots \dim(M)$. From now on we denote these by $\Omega^k(M)$. The reader familiar with these concepts will immediately notice a striking similarity between δ and the well-known invariant formula for the exterior derivative of an n-form in terms of its action on arbitrary smooth vector fields $X_1, \ldots X_{n+1}$:

$$\begin{aligned} \mathrm{d}\omega(X_1,\ldots,X_{n+1}) = \sum_{1\le i<j\le n+1} &(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{n+1}) \\ &+ \sum_{i=1}^{n+1}(-1)^{i+1}X_i(\omega(X_1,\ldots\hat{X}_i,\ldots X_{n+1})) \end{aligned} \tag{5.3}$$

The difference is that there is an additional term consisting of the vector fields acting on a function determined by ω. In the setting where we have M equal to a Lie group G, this term would vanish if we had a *left-invariant* form acting on *left-invariant* vector fields because then the function determined by ω is left-invariant and hence constant. We will make this heuristic precise.

Definition A.5:. *Let G a Lie group, ω a covariant tensor field on G and X a vector field on G. We call ω and X left-invariant if they are equal to their pullbacks under left multiplication l_g, where their pullbacks are defined as $(l_g^*X)_h = (dl_g)^{-1}(X_{gh})$ and $(l_g^*\omega)_h(v_1,\ldots,v_n) = \omega_{gh}(d(l_g)_h(v_1),\ldots,d(l_g)_h(v_n)$ respectively, where $v_1,\ldots,v_n \in T_gG$ are arbitrary.*

It now becomes straightforward to verify that if ω and $X_1,\ldots,X_{n+1}$ are all left-invariant, then the function $\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{n+1})$ is also left-invariant and hence constant, so that the second term in (5.3) vanishes (the action of a vector field on a function is just the differential of the function applied to that vector field in a pointwise manner).

Denote the space of all left-invariant k-forms by $\Omega^k(G)^G$. The nice thing about these objects is that they are completely determined by their action on vectors in $T_eG = \mathfrak{g}$. Left-invariance is preserved under sums and multiplication by scalars. If ω is left-invariant, then so is $d\omega$. This is an immediate consequence of the well-known fact that the exterior derivative commutes with pullbacks. This means that it makes sense to view d as a chain map on $\Omega(G)^G$. Hence we can define a cohomology on this complex:

Definiton A.6: *Let G be a Lie group and let $\Omega^*(G)^G$ be as above. The left-invariant de Rham cohomology group of G of degree k is defined as*

$$H^k_{G,\mathrm{dR}}(G,\mathbb{R}) = \frac{\mathrm{Ker}(d : \Omega^k(G)^G \to \Omega^{k+1}(G)^G)}{\mathrm{Im}(d : \Omega^{k-1}(G)^G \to \Omega^k(G)^G)} \tag{5.4}$$

We will denote the canonical quotient map by $[\cdot]$. In the light of the above discussion, it is now easy to formulate and prove the first important connection between de Rham cohomology and Lie algebra cohomology:

Proposition A.7: *The map $\epsilon : \Omega^k(G)^G \to M^k, \omega \mapsto \omega(e)$ is linear and intertwines the chain maps, i.e. $\epsilon(d\omega) = \delta(\epsilon\omega)$ or the diagram below makes sense and commutes. Furthermore, it is bijective. The induced map $\tilde{\epsilon} : H^k_{G,dR}(G,\mathbb{R}) \to H^k(\mathfrak{g},\mathbb{R}), [\omega] \mapsto [\epsilon\omega]$ defines an isomorphism between the cohomology groups of degree k, for all k.*

$$\begin{tikzcd}
\Omega^k(G)^G \arrow[r, "\epsilon"] \arrow[d, "{[\cdot]}"'] & M^k \arrow[d, "{[\cdot]}"] \\
H^k_{G,dR}(G,\mathbb{R}) \arrow[r, dotted, "\tilde{\epsilon}"'] & H^k(\mathfrak{g},\mathbb{R})
\end{tikzcd}$$

Proof: Linearity is obvious. As for intertwining of the chain maps, one way to see this is that given vectors $v_1, \ldots, v_n \in \mathfrak{g}$, there are unique left-invariant vector fields $X_1, \ldots, X_n$ whose values at e are precisely the v_i. Then one can substitute this into the definition of dω to see that the second term vanishes and apply ϵ on both sides. Finally, we can use left invariance to see that for any $u_1, \ldots, u_k \in T_gG$ we have

$$\omega_g(u_1, \ldots, u_k) = l^*_{g^{-1}}(\omega)(u_1, \ldots, u_k) = \epsilon\omega(d(l_{g^{-1}})_g(u_1), \ldots, d(l_{g^{-1}})_g(u_k)) \tag{5.5}$$

So if $\epsilon\omega = \epsilon\omega'$, then $\omega = \omega'$. We can also read this from right to left to see that an element of M^k defines a form ω which by construction will be left-invariant. Hence ϵ is surjective. Finally, if $[\omega] = [\omega']$ then $\omega - \omega' = d\alpha$ for some α, so that $\epsilon\omega - \epsilon\omega' = \epsilon(d\alpha) = \delta(\epsilon\alpha)$, whence $\tilde{\epsilon}$ is well defined. Now, ϵ has a two-sided inverse ψ, and the induced map $\tilde{\psi}$ is a two-sided inverse for $\tilde{\epsilon}$. Indeed, one can easily check that $\tilde{\psi}\tilde{\epsilon} = (\widetilde{\psi\epsilon}) = \tilde{\mathrm{Id}}$, and likewise $\tilde{\epsilon}\tilde{\psi} = \tilde{\mathrm{Id}}$. This proves Proposition A.7. $\square$

We now have to make use of the compactness assumption on G to ensure that the left-invariant cohomology actually coincides with the usual de Rham cohomology. To this end, we will need the famous de Rham theorem, which relates homology groups of degree k of a smooth manifold M (which we do not define here) to the de Rham cohomology groups of the same degree.

Proposition A.8 (de Rham): *Let M be a smooth manifold. Then there is a well-defined map from the de Rham cohomology group of degree k to* $\mathrm{Hom}(H_k(M,\mathbb{Z}),\mathbb{R})$, *the space of group homomorphisms from $H_k(M,\mathbb{Z})$ to $\mathbb{R}$ (the latter is a vector space under pointwise addition and multiplication). It is defined by*

$$\Psi : H^k_{\mathrm{dR}}(M) \ni [\omega] \mapsto \left(H_k(M,\mathbb{Z}) \ni [\sigma] \mapsto \int_\sigma \omega \in \mathbb{R} \right) \tag{5.6}$$

Furthermore, this map is always an isomorphism.

Proof: We refer to Proposition 18.14 of [13]. □

A familiar fact from smooth manifold theory is that if $F : M \to N$ is a smooth map between smooth manifolds M and N, then the natural pull-back $F^* : \Omega^k(N) \to \Omega^k(M)$ induces a well-defined map on cohomology classes for all k. This is because it commutes with the chain map, which is the exterior derivative d (cf. Proposition A.7).

Lemma A.9: *Let M and N be smooth manifolds and suppose that $F : M \to N$ and $G : M \to N$ are smoothly homotopic maps (i.e there exists a smooth homotopy of maps between them). Then, F and G induce the same map on de Rham cohomology: $F^* = G^*$.*

Proof: One needs to establish the existence of a cochain homotopy given a homotopy of maps. See Proposition 17.10 in [13] □

Definition A.10: *Let G be a Lie group. A right-invariant measure on G is a measure $d\mu$ (i.e. a linear functional) which for any real-valued continuous function f on G with compact support satisfies the following three conditions*:

(i) *Positivity: when $f(g) \geq 0$ for all $g \in G$, this implies $\int_G f \, \mathrm{d}\mu \geq 0$.*

(ii) *Nondegeneracy: if $f(g) \geq 0$ for all $g \in G$ and also $\int_G f \, \mathrm{d}\mu = 0$, then f is identically zero.*

(iii) *Invariance: for all $h \in G$ we have $\int_G f \circ r_h \, d\mu = \int_G f \, \mathrm{d}\mu$.*

Lemma A.11: *Let G be a compact Lie group. Then, there exists a unique normalized right-invariant measure $d\mu$ on G, i.e. a left-invariant measure such that $\int_G 1 \, d\mu = 1$. This is called the Haar measure on G.*

Proof: We refer to Chapter 9 of [8]. □

We can now prove the main result of this section, combining all the previous results.

Theorem A.12: *Let G be a connected, compact Lie group. Then the de Rham cohomology and the left-invariant cohomology from Definition A.6 coincide for all degrees: $H^k_{G,dR}(G,\mathbb{R}) = H^k_{dR}(G,\mathbb{R})$.*

Proof: Since G is connected, there exists for any $g \in G$ a path $\gamma : [0,1] \to G$ connecting e and g. It turns out that we can actually assume that this path is smooth. See for example Theorem 6.26 in [13], and note that any continuous path is easily seen to be smooth when restricted to the closed subset A of G containing the endpoints e and g. The path γ induces a smooth homotopy between l_g and $l_e = \mathrm{Id}_G$. By Lemma A.9, we conclude that for any $g \in G$ and any form ω, the forms $l_g^*\omega$ and ω are cohomologous.

We will only be able to sketch the remainder of the proof, which closely follows [7]. The interested reader may consult this reference for a more detailed exposition. By Lemma A.11 we can find a right Haar measure $\mathrm{d}\mu$ on G. To any differential k-form ω, we can associate its average $\mathcal{A}\omega$, which is another k-form defined by

$$(\mathcal{A}\omega)_h(v_1,\ldots,v_k) := \int_G (l_g^*\omega)_h(v_1,\ldots,v_k)\mathrm{d}\mu \tag{5.7}$$

where the integrand is the function which assigns to $g \in G$ the number $(l_g^*\omega)_h(v_1,\ldots,v_k)$. It is not so easy to check that this actually defines a smooth differential form. By compactness of G, the exterior derivative commutes with the integral. It also commutes with the pullback l_g^*, so that $d \circ \mathcal{A} = \mathcal{A} \circ d$. By previously established assertions, $\mathcal{A}$ descends to a map $\tilde{\mathcal{A}}$ on cohomology. Moreover, adopting the shorthand notation $\mathcal{A}\omega = \int_G l_g^*\omega \ \mathrm{d}\mu$, we can see that $\mathcal{A}\omega$ is left-invariant:

$$l_h^*(\mathcal{A}\omega) = l_h^* \int_G l_g^*\omega \ \mathrm{d}\mu = \int_G l_h^* l_g^*\omega \ \mathrm{d}\mu = \int_G l_{gh}^*\omega \ \mathrm{d}\mu = \int_G l_g^*\omega \ \mathrm{d}\mu = \mathcal{A}\omega \tag{5.8}$$

Hence we can view $\mathcal{A}$ as a map into the space of left-invariant forms $\Omega(G)^G$, so that $\tilde{\mathcal{A}}$ is a map from de Rham cohomology to left-invariant cohomology. The natural inclusion $\iota : \Omega(G)^G \to \Omega(G)$ descends to a map $\tilde{\iota}$ on cohomology as well. We claim that they are mutual inverses.

Indeed, it is readily verified that $\mathcal{A} \circ \iota = \mathrm{Id}$, so that $\tilde{\mathcal{A}} \circ \tilde{\iota} = \mathrm{Id}$. For the converse, we need to prove that if ω_1 and ω_2 are cohomologous in $H^k_{\mathrm{dR}}(G)$, then so are $\mathcal{A}\omega_1$ and $\mathcal{A}\omega_2$. We prove that in fact $[\mathcal{A}\omega] = [\omega]$ in $H^k_{\mathrm{dR}}(G)$ for arbitrary forms ω, which also implies the claim. We already know that $[\omega] = [l_g^*\omega]$ by the first part of the proof, so that by Proposition A.8 we obtain for arbitrary cycles σ:

$$\int_\sigma \omega = \int_\sigma l_g^*\omega \tag{5.9}$$

We can integrate the form $\mathcal{A}\omega$ over any cycle as well. This yields

$$\int_\sigma \mathcal{A}\omega = \int_\sigma \left(\int_G l_g^*\omega \ \mathrm{d}\mu \right) = \int_G \left(\int_\sigma l_g^*\omega \right) \mathrm{d}\mu = \int_G \left(\int_\sigma \omega \right) \mathrm{d}\mu = \left(\int_\sigma \omega \right) \left(\int_G 1 \ \mathrm{d}\mu \right) \tag{5.10}$$

But the measure $\mathrm{d}\mu$ is normalized, so

$$\int_\sigma \mathcal{A}\omega = \int_\sigma \omega \tag{5.11}$$

for arbitrary cycles σ. Again by Proposition A.8, this can only be the case if $[\mathcal{A}\omega] = [\omega]$ in $H^k_{\mathrm{dR}}(G)$. This proves Theorem A.12. $\square$

Before we go on to the transition from homology to the fundamental group, we summarize our achievements from this section.

Proposition A.13: *Let G be a connected, compact, semisimple Lie group. Then we have* $\mathrm{Hom}(H_1(G,\mathbb{Z}),\mathbb{R}) = 0$*.*

Proof: By Theorem A.12, the de Rham cohomology of degree 1 equals the left-invariant cohomology of degree 1. But by Proposition A.7, this also equals the Lie algebra cohomology of degree 1. But G is semisimple, so by Lemma A.4 this is the trivial group. By Proposition A.8, this means that $\mathrm{Hom}(H_1(G,\mathbb{Z}),\mathbb{R}) = 0$. This completes the proof. $\square$

5.2 Intermezzo on Abelian Groups

In the light of the result of Proposition A.13, we want to address the following question: given a finitely generated abelian group A which satisfies $\mathrm{Hom}(A,\mathbb{R}) = 0$, what can we say about A? To this end, we will first introduce the general statement and then apply it to our specific case.

Definition A.14: *Let G be a group. We say that G is finitely generated if there are finitely many elements $x_1, \dots x_n$ in G such that every element of G can be written as a (non-unique) product of integer powers of the x_i, i.e. $y = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$, where $n_i \in \mathbb{Z}$. We also say that the x_i generate the group G.*

Proposition A.15: *Let G be a finitely generated abelian group. Then G is isomorphic to a finite product of copies of $\mathbb{Z}$ and cyclic groups of prime order Z_{p_j}, i.e. there exist integers n, m and primes $p_1, \dots p_m$ such that*

$$G \cong \mathbb{Z}^n \bigoplus \left(\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_m} \right) \tag{5.12}$$

Proof: We refer to the literature. See for example [1], Theorem 21.1. $\square$.

The $\mathbb{Z}^n$-part is called the *free* part and the part between brackets is called the *torsion* part. Note that the torsion part has finite order equal to the product of the p_i. This can be applied to the singular homology groups of any degree of a compact manifold, as the following proposition ensures:

Theorem A.16: *Let M be a compact manifold. Then all singular homology groups $H_k(M,\mathbb{Z})$ are finitely generated abelian.*

Proof: A proof can be found in [5], Corollary E.5. □

Proposition A.17: *Let M be a compact manifold which satisfies* $\mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{R}) = 0$*. Then $H_1(M,\mathbb{Z})$ is finite.*

Proof: By Proposition A.16, the first singular homology group is finitely generated abelian. Hence we can apply Proposition A.15 to obtain the decomposition (33). But it is easy to see that $\mathrm{Hom}(\mathbb{Z},\mathbb{R})$ is nonzero because we can just inject $\mathbb{Z}$ into $\mathbb{R}$. Hence if $H_1(M,\mathbb{Z})$ has an element of infinite order, we can find a nontrivial homomorphism into $\mathbb{R}$. We conclude that the condition $\mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{R}) = 0$ holds if an only if the free part of $H_1(M,\mathbb{Z})$ is zero. □

5.3 Proof of Weyl's Theorem

We are now ready to prove Proposition 2.10. For the sake of completeness, we also provide a proof of the fact that the universal cover covers every other cover in a surjective way, so that all of the assertions used in Section 2.3 are justified. We adopt the convention that a covering map is always surjective.

Lemma A.18: *Let Y be a compact, connected manifold. Suppose that $\pi : X \to Y$ is a finite-sheeted covering. Then X is compact.*

Proof: Let $\{V_\alpha\}_{\alpha\in A}$ be a cover of X. We construct a finite subcover. For each $x \in X$, we can find some $\alpha(x)$ such that $x \in V_{\alpha(x)}$. Let now $y \in Y$. Then we find an evenly covered neighborhood W_y of y, so that we can write $\pi^{-1}(W_y) = \sqcup_{x\in\pi^{-1}(W_y)} S_x$. Hence because π is a homeomorphism on each of the components S_x, the set $\pi(V_{\alpha(x)} \cap S_x)$ is an open neighborhood of y, for each $x \in \pi^{-1}(y)$. But there are only finitely many such x, so the intersection $U_y = \cap_{x\in\pi^{-1}(y)}(\pi(V_{\alpha(x)} \cap S_x))$ is still an open neighborhood of y. Using the compactness of Y, we can cover Y by finitely many such neighborhoods U_{y_i}.

Finally,

$$\begin{aligned} X = \pi^{-1}(Y) &= \bigcup_{1\leq i\leq n} \pi^{-1}(U_y) \\ &\subset \bigcup_{1\leq i\leq n} \bigcup_{x\in\pi^{-1}(y_i)} \left(V_{\alpha(x)} \cap S_x\right) \\ &\subset \bigcup_{1\leq i\leq n} \bigcup_{x\in\pi^{-1}(y_i)} \left(V_{\alpha(x)}\right) \end{aligned} \tag{5.13}$$

which is clearly a union of finitely many V_α. This establishes the claim. □

Lemma A.19: *The fundamental group of a topological group G (and hence in particular a Lie group) is always abelian.*

Proof: There are two natural operations on $\pi_1(G, e)$: we can concatenate loops to produce a new loop, or we can use the group structure on G to define a product path through pointwise multiplication in the group: $(\gamma_1 \times \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$. Note that we pick the basepoint to be e so that the product path is again a loop based at e. For notational convenience, we do not distinguish between actual paths γ and their formal representatives $[\gamma]$. Indeed, $\times$ is well defined as an operation on $\pi_1(G, e)$. The homotopy between the products is just the product of the homotopies (note that multiplication is continuous).

We denote concatenation by a map $\star : \pi_1(G, e) \times \pi_1(G, e) \to \pi_1(G, e)$. The key observation is that

$$(\alpha \times \beta) \star (\gamma \times \delta) = (\alpha \star \gamma) \times (\beta \star \delta) \tag{5.14}$$

It is easy to prove this by plugging in the definitions and seeing where an element t of the unit interval is mapped to. The unit for both multiplications is given by the (class of the) constant path $\epsilon : t \mapsto e \quad \forall t$. We can now compute for two arbitrary loops α, β:

$$\begin{aligned} \alpha \times \beta &= (\alpha \star \epsilon) \times (\epsilon \star \beta) \\ &= (\alpha \times \epsilon) \star (\epsilon \times \beta) \\ &= \alpha \star \beta \\ &= (\epsilon \times \alpha) \star (\beta \times \epsilon) \\ &= (\epsilon \star \beta) \times (\alpha \star \epsilon) \\ &= \beta \times \alpha \end{aligned} \tag{5.15}$$

In particular, the operations coincide on $\pi_1(G, e)$ and both are commutative. □

Proposition A.20 *Let X be a topological space. Then the first homology group over $\mathbb{Z}$ is the abelianization of the fundamental group, i.e. $H_1(X, \mathbb{Z}) = \pi_1^{\text{ab}}$.*

Proof: This somewhat technical proof can be found in any text on algebraic topology, see for example [10], Theorem 2A.1.

Lemma A.21: *Suppose X is a connected manifold. Then X admits a universal cover $\tilde{X}$, i.e. a covering space which is simply connected. It is universal in the following sense: if Y is any other covering space for X, then there exists a covering $\tilde{X} \to Y$, which in particular is onto.*

Proof: Because X is a manifold, every point admits a basis of contractible neighborhoods and it is also path-connected. Hence a universal covering exists. A proof of this can be found in [10], pp. 63-65. We employ the so-called *lifting criterion* (see [10], Proposition 1.33): suppose we have any covering space $p : (Y, y_0) \to (X, x_0)$ and a map $\phi : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$. Then a lift $\tilde{\phi}$ of ϕ, i.e. a map such that $p \circ \tilde{\phi} = \phi$ (see the diagram below), exists if and only if $\phi_*(\pi_1(\tilde{X}, \tilde{x_0})) \subseteq p_*(\pi_1(Y, y_0))$. Here the lower star notation indicates the induced homomorphism on the fundamental groups given by composition.

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x_0}) \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi"'] & (X, x_0) \end{tikzcd}$$

As our notation suggests, we apply this proposition with $\tilde{X}$ the universal cover of X. Denote the covering map by ϕ. We know that $\tilde{X}$ is simply connected, so certainly $\phi_*(\pi_1(\tilde{X}, \tilde{x_0})) \subseteq p_*(\pi_1(Y, y_0))$. Hence by the lifting criterion we obtain a map $\tilde{\phi} : \tilde{X} \to Y$. It remains to prove that this map is a covering map.

For this, we employ the uniqueness lifts of paths: if we have a covering space $p : Y \to X$ and a path $\gamma : I \to X$ starting at x_0, then if $y_0 \in Y$ satisfies $p(y_0) = x_0$, there is a unique lift $\tilde{\gamma}$ of the entire path γ starting at y_0. Let $y \in Y$ be arbitrary. Then there is a path γ from y_0 to y. We can push it forward through p to a path in X, and then lift it to $\tilde{X}$. Denote the resulting path by $p \mathbin{\tilde{\circ}} \gamma$. By definition, this map satisfies $\phi \circ (p \mathbin{\tilde{\circ}} \gamma) = p \circ \gamma$, so $p \circ (\tilde{\phi} \circ (p \mathbin{\tilde{\circ}} \gamma)) = p \circ \gamma$. Hence both γ and $\tilde{\phi} \circ (p \mathbin{\tilde{\circ}} \gamma)$ are lifts of $p \circ \gamma$. They are then equal, so their endpoints agree. Hence $\tilde{\phi}$ is surjective.

Finally, $\tilde{\phi}$ satisfies the covering property. To see this, let $y \in Y$ be arbitrary again. Let $x = p(y)$. There is a neighborhood U of x which is evenly covered by both ϕ and p. There is a unique open $V \subset Y$ containing y which is homeomorphic to U via p. We clain that this is a trivializing neighborhood of y. To this end, write $\phi^{-1}(U) = \sqcup U_\alpha$. Then $\tilde{\phi}$ maps each of the U_α homeomorphically onto V. This proves the lemma. $\square$

If $\phi : \tilde{X} \to X$ is the path connected universal covering of a path connected space, the number of sheets equals the cardinality of the fundamental group of X. This can be inferred for example from [10], Proposition 1.32.

Proof of Proposition 2.10: Under the assumptions we made, we can apply Proposition A.13 to find $\mathrm{Hom}(H_1(G,\mathbb{Z}),\mathbb{R}) = 0$. But then by Proposition A.17, the group $H_1(G,\mathbb{Z})$ is finite. By Proposition A.20 and Lemma A.19, the fundamental group equals $H_1(G,\mathbb{Z})$ and hence is also finite. By the above remark combined with Lemma A.18, we see that the universal covering is finite-sheeted and hence compact. This proves the assertion. $\square$

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