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The Clifford algebra and its Spin subgroup

BACHELOR THESIS

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TWIN: Wis- en Natuurkunde

$$
\begin{array}{ccc}
\text{(S)Pin}(p,q) & \xrightarrow{\;\rho(x)(v)=-\alpha(x)\cdot v\cdot x^{-1}\;} & (S)O(p,q) \\
\Big\uparrow\exp\;\Big\downarrow\text{Lie} & & \Big\downarrow\text{Lie} \\
C\ell^2(p,q)=[V,V] & \xrightarrow[\cong]{\;\phi([v,w])=\big[[v,w],\,\cdot\,\big]\;} & \mathfrak{so}(p,q)
\end{array}
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Abstract

This thesis gives an introduction to the Clifford algebra and its Spin subgroup. We first construct the Clifford algebra before using the spinorial representation to proof its (semi)simplicity. We then construct the Spin group and determine its irreducible representations using weight theory. The reader is expected to have prior knowledge about rings and representations of groups, but all required theory will be recalled in the preliminaries.

Contents

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1 Introduction

The Clifford algebra and its substructures are among the most well-known and well-studied algebraic structures in history. Named after the British mathematician and philosopher William Kingdon Clifford, the algebra generalises Grassman's exterior algebra while simultaneously generalising the complex space and Hamilton's quaternion algebra.[1] In modern day, the Clifford algebra still is an interesting structure as it connects multiple disciplines of mathematics: the Clifford algebra itself combines geometry and algebra, while its Spin subgroup is a Lie group doubly covering the special orthogonal group. The theory of Clifford algebras is also closely linked with both general relativity and particle physics.

The goal of this thesis is to give an introduction to the Clifford algebra, the Spin group and their special properties. We will start with some preliminaries, followed by some general theory of semisimple algebras. In chapter 3, we first construct the Clifford algebra and apply the general theory of semisimple algebras to find the spinorial representation. Later in the same chapter, we construct the Spin subgroup and its Lie algebra. Finally in chapter 4 we study the representation theory of the Spin group, using the spinorial representation to indirectly determine all irreducible representations.

2 Preliminaries

In this section, we will recall some preliminary notions as well as theorems that will be used (but not proven) in further sections.

2.1 Representations and modules [2]

Firstly, we recall the definition of a representation. Intuitively, a representation of a group G gives a map from G to matrices, such that the matrix multiplication on the image has similar properties to the group multiplication. More rigorously, we define a representation as follows:

Definition 2.1.1. Let V be a (finite dimensional) vector space over $\mathbb{R}$ or $\mathbb{C}$. A representation of G over V is a group-homomorphism $\rho : G \to \mathrm{GL}(V)$.

Here $\mathrm{GL}(V)$ denotes the 'general linear' group of invertible linear maps $V \to V$, with composition as the multiplication. By choosing any basis for V, we can make the linear maps into matrices in a natural way. Two different choices of bases will give different matrices, but we consider these matrix-representations to be equivalent.

Since the image of the representation are linear maps, we can consider how these maps act on the vector space V. A group element $g \in G$ becomes a linear map $\rho(g)$ which acts on a vector $v \in V$ to give a vector $\rho(g)(v) \in V$. Using this, we can define an action of G on V as $G \times V \to V; (g, v) \mapsto g \cdot v := \rho(g)(v)$. This action is linear in v and inherits all homomorphism-properties of ρ, such as $g_1 \cdot (g_2 \cdot v) = (g_1 g_2) \cdot v$ for $g_1, g_2 \in G, v \in V$. Using these properties, we define a module:

Definition 2.1.2. Let V be a (finite dimensional) vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then V is a G-module if there is an action $G \times V \to V$ such that for any $g, h \in G, u, v \in V, \lambda \in \mathbb{K}$ the following conditions are satisfied:

- $g \cdot v \in V$,

- $g \cdot (h \cdot v) = (gh) \cdot v$,

- $e \cdot v = v$ where e denotes the identity in G,

- $g \cdot (\lambda v + u) = \lambda(g \cdot v) + g \cdot u$.

We have seen above that each representation induces a G-module; similarly each G-module V induces a representation by $\rho(g)(v) = g \cdot v, v \in V, g \in G$. This means that representations and G-modules are in a one-to-one correspondence and we can freely switch between the two. We will use representations or G-modules depending on which is more useful at that point.

Note that if $\rho : G \to \mathrm{GL}(V)$ is a representation and $A \in \mathrm{GL}(V)$ an invertible linear map, then $\tau : G \to \mathrm{GL}(V)$ defined by $\tau(g) = A^{-1} \circ \rho(g) \circ A$ for $g \in G$ is also a homomorphism: for any $g, h \in G$ we have $\tau(g)\tau(h) = (A^{-1} \circ \rho(g) \circ A) \circ (A^{-1} \circ \rho(h) \circ A) = A^{-1} \circ \rho(gh) \circ A = \tau(gh)$. In this manner, any number of representations can be made. It's therefore common to look at isomorphism classes of representations (or modules) instead. The exact isomorphism-condition is the following:

Definition 2.1.3. The representations $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ are equivalent if there is an intertwining isomorphism between V and V', i.e. a vector space isomorphism $\phi : V \to V'$ such that $\phi(\rho(g)(v)) = \rho'(g)(\phi(v))$ for any $g \in G, v \in V$.

Finally, there are some properties a representation or G-module can have. Note that if ρ, ρ' are equivalent, they have the same properties.

Definition 2.1.4. A G-submodule is linear vector subspace of G-module which is closed under the action the group G. A G-module is irreducible if there are no proper submodules. A G-module is faithful if $g \cdot v = v$ implies that g is the unit element.

Note that these definitions carry over to representations. The last definition is equivalent to saying that the representation is injective.

2.2 Manifolds, Lie groups and Lie algebras [3]

Secondly, we recall the definition of a (smooth) manifold. Intuitively, a manifold is a (topological) space such that the space locally resembles $\mathbb{R}^n$, $n \in \mathbb{N}$. For instance, the n-sphere resembles $\mathbb{R}^n$ when zoomed in 'far enough'. This can be made precise as follows:

Definition 2.2.1. A n-manifold is a topological space M together with charts $(U, \chi : U \to \mathbb{R}^n)$ such that each U is open, each χ is a homeomorphism, the union of all U's covers M, and for any two charts $(U, \chi), (V, \eta)$ the map $\chi \circ \eta^{-1} : \eta(U \cap V) \to \chi(U \cap V)$ is smooth (as a map $\mathbb{R}^n \to \mathbb{R}^n$). The number n is called the dimension of M.

The advantage of manifolds is that we can define smooth functions between them. We do this by using the charts to write the function as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, where the meaning of 'smooth' is well-defined.

On a manifold, one defines the tangent (vector) space by means of derivatives. This is intuitively best illustrated by the idea that velocity is always a vector, while position is a point on the manifold. Therefore, for any element of the tangent space in a given point there is a curve passing through the point with that element as its velocity. However, the derivative of the curve has not been defined yet (as it depends on the chosen chart), so we have to define the tangent space in an abstract manner, similar to a dual space. We will not give this abstract definition since it is not important for this thesis. The tangent space of the manifold M in the point p is given the notation T_pM. Note that if M locally looks like $\mathbb{R}^n$, then the tangent space also resembles $\mathbb{R}^n$, hence the dimension of T_pM (as a vector space) is equal to the dimension of M (as a manifold).

Assuming we have defined the tangent space, we can define the tangent bundle as the disjoint union of the tangent spaces: $TM = \coprod_{p \in M} T_pM$. The elements of TM are written as (p, v) where $p \in M$, $v \in T_pM$. Note that the tangent bundles can again be seen as a manifold, hence we can speak of 'smooth functions to TM'. This is used to give a notion of a smoothly varying vector field, by defining vector fields as smooth functions $X : M \to TM$ such that $X(p) \in T_pM$.

Now consider a Lie group, that is, a manifold G with a group structure, such that the operations of multiplication $(G \times G \to G, (g, h) \mapsto gh)$ and inversion $(G \to G, g \mapsto g^{-1})$ are smooth. It follows that the

left-multiplication operations $l_g : G \to G, h \mapsto gh$ for $g \in G$ are smooth, so one can consider its derivative in the point h: $(dl_g)_h : T_h G \to T_{gh} G$, or in general $dl_g : TG \to TG$. The second form allows us to define the post-composition of a vector field with dl_g as follows. Let $X : G \to TG$ be a vector field and let $g \in G$. Since l_g is smooth, so is dl_g, so the map $dl_g \circ X : G \to TG$ is smooth. Therefore, $dl_g \circ X$ is a vector field.

We call a vector field X left-invariant if $dl_g \circ X = X$ for each $g \in G$. We can uniquely determine a left-invariant vector field by its value at the identity e of the group, as $X(g) = dl_g(X(e))$ for any $g \in G$. Using this relation, it turns out that the set of left-invariant vector fields is, as an vector space, isomorphic to $T_e G$. However, the set of left-invariant vector fields has additional structure; it naturally admits a non-associative antisymmetric bilinear multiplication $[\cdot, \cdot]$ which satisfies the identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any left-invariant vector fields X, Y, Z. This last identity is called the Jacobi identity, and a vector space with the given properties is called a Lie algebra. We see that any Lie group induces a Lie algebra isomorphic to the tangent space in its identity. This Lie algebra is denoted by $\mathrm{Lie}(G)$, although its also commonly typeset using lower-case gothic letters, so $\mathrm{Lie}(G)$ can also be denoted by $\mathfrak{g}$. Note that the dimension of $\mathrm{Lie}(G)$ is equal to $\dim(T_e G) = \dim(G)$.

There are a multitude of proposition which follow from this result; we won't go into detail for all of them, but will call upon them when necessary. Some main results are:

- For any Lie algebra $\mathfrak{g}$, there is a Lie group G such that $\mathfrak{g} = \mathrm{Lie}(G)$.

- Let G, H be Lie groups and let $F : G \to H$ be a (group) homomorphism, then there is an unique push-forward $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ such that F_* is an algebra homomorphism.

- (Ado's Theorem) Every finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.

Here, a Lie algebra representation of a Lie algebra $\mathfrak{g}$ is a vector space V together with a map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ where $\mathfrak{gl}(V)$ is $\mathrm{Lin}(V)$ endowed with the commutator brackets $[A, B] = A \circ B - B \circ A$ as (Lie) multiplication, and a faithful representation is again defined as an injective representation.

In particular, we note that $\mathrm{GL}(V)$ is a Lie group with Lie algebra $\mathfrak{gl}(V)$.

2.3   (Special) Orthogonal group

Thirdly, we recall the definition of the (special) orthogonal group. The orthogonal group can be seen as the group of all rotations and reflections on $\mathbb{R}^n$, while the special orthogonal group is the subgroup of rotations.

Definition 2.3.1. The <u>orthogonal group</u> $O(n)$ is the group of all real $n \times n$–matrices A with the property that $A^\mathsf{T} A = AA^\mathsf{T} = \mathbb{1}$. The <u>special orthogonal group</u> $SO(n)$ is the subgroup $\{A \in O(n) \mid \det(A) = 1\}$.

Since all $n \times n$–matrices can be associated with vectors in $\mathbb{R}^{n^2}$, there is a natural topology on $O(n)$ and $SO(n)$. It turns out both groups are compact Lie groups of dimension $n(n-1)/2$. The group $O(n)$ has two connected components, one being $SO(n)$ and the other $\{A \in O(n) \mid \det(A) = -1\}$.

Since $O(n)$ and $SO(n)$ are Lie groups, we can calculate their Lie algebras. The easiest way to do this calculation is by considering $O(n)$ and $SO(n)$ as Lie subgroups of $\mathrm{GL}(n) := \mathrm{GL}(\mathbb{R}^n)$ while noting that the Lie algebra of $\mathrm{GL}(n)$ is $\mathfrak{gl}(n) = \mathrm{Lin}(n)$. This implies that the Lie algebras $\mathfrak{o}(n) = \mathrm{Lie}(O(n))$ and $\mathfrak{so}(n) = \mathrm{Lie}(SO(n))$ are Lie subalgebras of $\mathrm{Lin}(n)$. The orthogonality condition ($AA^\mathsf{T} = 1$) becomes the condition $B + B^\mathsf{T} = 0$ for B in the algebra, while the condition $\det A = 1$ becomes $\mathrm{Tr}\, B = 0$. Therefore, $\mathfrak{o}(n)$ is the Lie algebra of skew-symmetric linear maps, while $\mathfrak{so}(n)$ is the Lie algebra of skew-symmetric linear maps with trace 0. Since all skew-symmetric maps have trace 0, it turns out that $\mathfrak{o}(n) = \mathfrak{so}(n)$.

3   Associative Algebras

In order to understand Clifford Algebras, we will first have to understand more about general associative algebras. With that in mind, we define simple and semi-simple algebras and investigate their relations with representations. We start, however, with some more general definitions, such as the definition of an algebra. The definitions are all based on Hermann [4]. Note that some of these definitions do not fully agree with other sources.

3.1   General notions

We define the 'algebra' structure, its substructures and homomorphisms.

Definition 3.1.1. An <u>algebra</u> A is an vector space over a field $\mathbb{K}$ together with a ($\mathbb{K}$-)bilinear product $\cdot : A \times A \to A, (a,b) \mapsto a \cdot b =: ab$. An algebra is said to be <u>associative</u> if the product is associative, i.e. for any $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

We will assume that A has finite dimension over $\mathbb{K}$ and that $\mathbb{K}$ is a field of characteristic 0. Remark that an associative algebra resembles a ring, but does not necessarily have an multiplicative identity. As we have defined a structure, we naturally want to define a corresponding substructure. Since we already have the notion of a vector-subspace and a subring, we get a natural definition for the subalgebra of an algebra.
Let A be an associative algebra.

Definition 3.1.2. For any $V, W \subseteq A$ let $VW := \{v \cdot w \mid v \in V, w \in W\}$. A <u>subalgebra</u> of A is a linear subspace $A' \subseteq A$ such that $A'A' \subseteq A'$.

Since the multiplication is associative, we can also consider terms like BCD for $B, C, D \subseteq A$.
Recall that in group theory, one cannot take quotients over any subgroup. Instead, one has to define normal subgroups. Similarly, there is a stronger notion which allows us to take quotients over algebras.

Definition 3.1.3. Let $A' \subseteq A$ be a subalgebra. We call A' a left-handed <u>ideal</u> if $AA' \subseteq A'$; we call A' a right-handed ideal if $A'A \subseteq A'$. If A' is both left-handed and right-handed, we call A' two-sided. An ideal A' is said to be non-trivial if $A' \neq 0, A$.

When we refer to an ideal A', we mean a two-sided ideal, unless otherwise specified. A 'left-handed ideal' is usually shortened to 'left ideal', similarly for right-handed. Finally, we have to define the homomorphisms between algebras.

Definition 3.1.4. Let A, A' be algebras. A <u>algebra homomorphism</u> is a $\mathbb{K}$-linear map $\phi : A \to A'$ such that $\phi(ab) = \phi(a)\phi(b)$ for $a, b \in A$. If ϕ is bijective it is called an <u>isomorphism</u>, and A, A' are said to be isomorphic. (Notation: $A \cong A'$.) An isomorphism from A to itself is called an <u>automorphism</u>, while an <u>anti-automorphism</u> is a $\mathbb{K}$-linear bijection ϕ from A to itself such that $\phi(ab) = \phi(b)\phi(a)$ for $a, b \in A$.

The image of an algebra homomorphism $\phi : A \to A'$ is by linearity of ϕ a linear subspace of A'. Moreover, $\phi(ab) = \phi(a)\phi(b)$ for $a, b \in A$, so $\phi(A)$ is a subalgebra of A'.

Example 3.1.5. Let V be a ($\mathbb{K}$-)vector space. The linear algebra $\mathrm{Lin}(V)$ is the (vector) space of $\mathbb{K}$-linear maps (endomorphisms) $V \to V$ equipped with $(+, \circ)$ such that $(A \circ B)(v) = A(B(v))$ and $(A + B)(v) = A(v) + B(v)$ for any $v \in V$. This algebra is associative (as $((A \circ B) \circ C)(v) = A(B(C(v))) = (A \circ (B \circ C))(v)$ for $v \in V$) and has an identity $\mathbb{1}_V$ defined by $\mathbb{1}_V(v) = v$ for $v \in V$. $\qquad \triangle$

Using the linear algebra, we can define representations of an associative algebra as follows:

Definition 3.1.6. Let A be an associative algebra over $\mathbb{K}$, and V be a (finite dimensional) vector space over $\mathbb{K}$. A (linear) <u>representation</u> of A in V is an algebra homomorphism $\rho : A \to \mathrm{Lin}(V)$. V together with a bilinear map (action) $\cdot : A \times V \to V, (a, v) \mapsto a \cdot v$ such that $(ab) \cdot v = a \cdot (b \cdot v)$ is called an <u>A-module</u>.

Similar to the group representation and (G-)modules, algebra representations and (A-)modules are equivalent.

Theorem 3.1.7. *Modules and (linear) representations are in an one-to-one correspondence.*

Proof. Let A be an associative algebra over a field $\mathbb{K}$. Let V be a vector space over $\mathbb{K}$ and $\rho : A \to \text{Lin}(V)$ be a representation. Define the bilinear map $\cdot : A \times V \to V$ by $(a, v) \mapsto \rho(a)(v)$. Now $b \cdot (a \cdot v) = \rho(b)(\rho(a)(v)) = (\rho(b) \circ \rho(a))(v) = \rho(ba)(v) = (ba) \cdot v$ for $a, b \in A, v \in V$, so V equipped with $\cdot$ is an A-module.
Conversely, assume that V is an A-module. Define $\rho : A \to \text{Lin}(V)$ by $\rho(a)(v) = a \cdot v$. Now $\rho(ab)(v) = (ab) \cdot v = a \cdot (b \cdot v) = \rho(a)(\rho(b)(v)) = (\rho(a) \circ \rho(b))(v)$ for $a, b \in A, v \in V$, so $\rho(ab) = \rho(a) \circ \rho(b)$. Since the action is bilinear, it follows that ρ is indeed an algebra homomorphism, so ρ is a representation.
We conclude that representations and modules are one-to-one. $\qquad\square$

Example 3.1.8. Let V be a n-dimensional vector space over $\mathbb{K}$, and let V^* be its dual space. Equip the tensor product $A = V \otimes V^*$ with the multiplication $A \times A \to A, (v_1 \otimes \phi_1)(v_2 \otimes \phi_2) = \phi_1(v_2)(v_1 \otimes \phi_2)$. Let $\{w_i \mid i = 1, \ldots, n\}$ be a basis of V and $\{\omega_i\}$ its dual basis, i.e. $\omega_i(w_j) = \delta_{ij}$. Remark that $\{w_i \otimes \omega_j\}$ forms a basis for A as a vector space. Define the action of A on V by $(w_i \otimes \omega_j) \cdot v = \omega_j(v) w_i$ for $v \in V$ and $i, j \in \{1, \ldots, n\}$; this is indeed an action, as it is linear and $(w_i \otimes \omega_j) \cdot ((w_k \otimes \omega_l) \cdot v) = (w_i \otimes \omega_j) \cdot (\omega_l(v) w_k) = \omega_l(v)\omega_j(w_k) w_i = \omega_j(w_k)(w_i \otimes \omega_l) \cdot (v) = (\omega_j(w_k)(w_i \otimes \omega_l)) \cdot v = ((w_i \otimes \omega_j)(w_k \otimes \omega_l)) \cdot v$.
Using the above correspondence, we get a representation $\rho : A \to \text{Lin}(V)$. This representation is injective: Let $a = (w_i \otimes \omega_j), b = (w_k \otimes \omega_l) \in A$ and assume $\rho(a) = \rho(b)$. Then $\rho(a)(w_j) = \rho(b)(w_j)$, so $w_i = \omega_l(w_j)w_k$. By definition of the dual basis, it follows that $j = l$ and $i = k$, so $a = b$. Since $\dim A = \dim V \cdot \dim V^* = n^2 = \dim \text{Lin}(V)$, we find that ρ is an bijective homomorphism, so ρ is an isomorphism. We conclude that $V \otimes V^* \cong \text{Lin}(V)$. $\qquad\triangle$

We have now defined algebra representations. We define irreducible algebra representations analogous to irreducible group representations. In the following definitions and examples, A denotes an arbitrary associative algebra.

Definition 3.1.9. Let V be a vector space, $\rho : A \to \text{Lin}(V)$ a representation. Let $U \subseteq V$ be a linear subspace of V such that $\rho(A)(U) \subseteq U$. Then U is called <u>invariant</u>. Define the representation of A on U created by restricting each linear map in $\rho(A)$ to U, this is the <u>subrepresentation</u> of A in U. The subrepresentation is also denoted by ρ.
Let $L \subseteq \text{Lin}(V)$. We say that L <u>acts irreducibly</u> on V if there is no non-trivial invariant subspace V' of V. For a representation $\rho : A \to \text{Lin}(V)$, we say that the representation is <u>irreducible</u> if $\rho(A) \subseteq \text{Lin}(V)$ acts irreducibly; we say that an A-module is irreducible if its corresponding representation is irreducible.
A representation is called <u>faithful</u> if ρ is injective, i.e. if the kernel of ρ is 0.
We call V <u>completely reducible</u> if there are $k \in \mathbb{N}$ and V_i for $i = 1, \ldots, k$ such that $V = V_1 \oplus \cdots \oplus V_k$, $\rho(A)(V_i) \subseteq V_i$ and $\rho(A)$ acts irreducibly on each V_i, i.e. the subrepresentations are irreducible.

Example 3.1.10. Consider the map $A \times A \to A, (a, b) \mapsto ab$. Since A is a vector space, we can consider this as an action of A onto itself. From the properties of the multiplication, it follows that A is an A-module under this action. This induces a representation which is called the (left) regular representation or standard representation and denoted by $\rho_{reg} : A \to \text{Lin}(A)$.
Consider an minimal left ideal B of A, i.e. a left ideal such that for any non-trivial left ideal $B' \subseteq B$ we have $B' = B$. We see that the action of A leaves B invariant, as for any $a \in A$ we have $a \cdot B \subseteq B$. Therefore, consider the subrepresentation of A in B.

Proposition 3.1.11. *Let A be an associative algebra, B an ideal in A. The regular representation $\rho_{reg} : A \to \text{Lin}(B)$ is irreducible.*

Proof. Assume there is a $C \subseteq B$ such that $\rho_{reg}(A)(C) \subseteq C$. Then $aC \subseteq C$ for any $a \in A$, so $AC \subseteq C$. This gives us that C is a left ideal of A. But $C \subseteq B$, so from minimality it follows that $C = 0$ or $C = B$. We find that there are no non-trivial invariant subspaces of B, so ρ_{reg} is irreducible in B. $\qquad\square$

As B was arbitrary, we see that the regular representation is irreducible on all minimal left ideals. Therefore, if A is the direct sum of its minimal left ideals, the regular representation will be completely reducible. We will later see other properties which relate to A being the direct sum of its minimal left ideals. $\triangle$

We also want to have a sense for the equivalence of representations.

Definition 3.1.12. Let V, V' be two vector spaces and let $\rho : A \to \mathrm{Lin}(V), \rho' : A \to \mathrm{Lin}(V')$ be representations. A linear map $\phi : V \to V'$ is said to <u>intertwine</u> ρ and ρ' if $\phi(\rho(a)(v)) = \rho'(a)(\phi(v))$ for $a \in A, v \in V$. If ϕ is an intertwining isomorphism, ρ and ρ' are said to be <u>equivalent</u>.
Two A-modules are said to be equivalent if the corresponding representations are equivalent.

Example 3.1.13. Let $\rho : A \to \mathrm{Lin}(V)$ be a representation of A in a vector space V, and let X be an invertible map in $\mathrm{Lin}(V)$. Define $\rho' : A \to \mathrm{Lin}(V)$ by $\rho'(a) = X\rho(a)X^{-1}$. Remark that ρ' is a representation, as $\rho'(a)\rho'(b) = X\rho(a)X^{-1}X\rho(b)X^{-1} = X\rho(ab)X^{-1} = \rho'(ab)$ for $a, b \in A$. Now $\rho'(a)X = X\rho(a)X^{-1}X = X\rho(a)$ for all $a \in A$, so X intertwines ρ and ρ'. Since X is an invertible linear map from V to itself, it is an isomorphism, so ρ and ρ' are equivalent. $\triangle$

We can now prove the following theorem, which is known as 'Schur's lemma'.

Theorem 3.1.14 (Schur's Lemma). *Let V, V' be vector spaces and ρ, ρ' irreducible representations of A in V respectively V'.*

- *Let $\phi : V \to V'$ intertwine ρ and ρ'. Then either $\phi = 0$ or ϕ^{-1} exists.*

- *Let $\mathbb{K} = \mathbb{C}$ and let $\phi : V \to V$ intertwine ρ with itself. Then there is a $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$.*

Proof. Let $N \subseteq V$ be the kernel of ϕ, let $v \in N$ and $a \in A$. Now $\phi(\rho(a)(v)) = \rho'(a)(\phi(v)) = \rho'(a)(0) = 0$, so $\rho(a)(v) \in N$ for any $a \in A$. We see that N is an invariant subspace of V. However, V is irreducible, so $N = 0$ or $N = V$. For $N = V$, it immediately follows that $\phi(V) = 0$ so $\phi = 0$. For $N = 0$, the kernel of the linear map ϕ is 0 so ϕ is injective. In this case, consider $\phi(V) \subseteq V'$. For all $a \in A, v \in V$ we have $\rho'(a)(\phi(v)) = \phi(\rho(a)(v)) \in \phi(V)$ so $\phi(V)$ is an invariant subspace of V'. Since ρ' is irreducible and $\phi(V)$ is non-zero (by injectivity) we have $\phi(V) = V'$, so ϕ is surjective. We find that ϕ is a linear bijection, so ϕ^{-1} exists.
Let $\mathbb{K} = \mathbb{C}$ and $V = V'$. Since ϕ is a complex linear map from V to itself, is has an eigenvalue λ. Let $U = \{v \in V \mid \phi(v) = \lambda v\}$ be the set of eigenvectors for λ. For $a \in A, v \in U$ we have $\phi(\rho(a)(v)) = \rho(a)(\phi(v)) = \rho(a)(\lambda v) = \lambda\rho(a)(v)$, so $\rho(A)(U) \subseteq U$. As ρ is irreducible, we find that $U = V$, so $\phi(v) = \lambda v$ for all $v \in V$. $\square$

3.2   Properties of associative algebras

In the following, A is an associative algebra over the field $\mathbb{K}$ with characteristic 0.
We consider some properties of associative algebras concerning the existence of ideals. We start by defining a property which guarantees that all ideals are 'nice', where we define a 'non-nice' ideal using nil-potency:

Definition 3.2.1. Both an element of A and a subset can be nilpotent.

- Let $a \in A$. The element a is said to be <u>nilpotent</u> if there is an $n \in \mathbb{N}$ such that $a^n = 0$.

- Let $A' \subseteq A$. The subset A' is called <u>nilpotent</u> if there is an $n \in \mathbb{N}$ such that for any $a_1, \dots, a_n \in A'$, $a_1 \cdot a_2 \cdot \dots \cdot a_n = 0$ holds. Note that this condition is equivalent to $(A')^n = 0$.

We can now define an associative algebra with 'nice' ideals, or no ideals at all.

Definition 3.2.2. The associative algebra A is called <u>semisimple</u> if any non-trivial ideal is not nilpotent. It is called <u>simple</u> if there are no non-trivial ideals.

Remark 3.2.3. The given definitions of simplicity and semi-simplicity come from Hermann [4] and do not agree with all other authors, who might additionally require A to be non-nilpotent or a similar condition. The given definition, however, is strong enough for our use since we work in finite dimensions. In fact, if a non-zero algebra A were to be nilpotent and $m \in \mathbb{N}$ be the smallest integer such that $A^m = 0$, then any subset of A^{m-1} would be a nilpotent ideal, hence a semisimple algebra must be non-nilpotent.

Example 3.2.4. Consider $\mathbb{C}$ as an algebra over $\mathbb{R}$. Since $\mathbb{C}$ is a field, each non-zero element has an inverse. This implies that each ideal in $\mathbb{C}$ is either 0, or contains 1 in which case the ideal is equal to $\mathbb{C}$. We see that $\mathbb{C}$ is simple as an algebra over $\mathbb{R}$.
Another example of a semisimple algebra is the quaternion algebra. Consider the quaternion group $\{\pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = ijk = -1\}$. The group algebra $\mathbb{H}$ of the quaternion group over $\mathbb{R}$ is defined as the vector space over $\mathbb{R}$ spanned by e_1, e_i, e_j, e_k with the multiplication induced by $e_1^2 = e_1, e_i^2 = e_j^2 = e_k^2 = -e_1 = e_i e_j e_k$. For any element $h = ae_1 + be_i + ce_j + de_k$ we can define a ('quaternion') conjugate $h^* = ae_1 - be_i - ce_j - de_k$. Calculating hh^* and h^*h gives $hh^* = (a^2 + b^2 + c^2 + d^2)e_1 = h^*h$. For any non-zero $h \in \mathbb{H}$, we have $a^2 + b^2 + c^2 + d^2 > 0$ since $a, b, c, d \in \mathbb{R}$. We therefore find an inverse of h by dividing h^* by $a^2 + b^2 + c^2 + d^2$. We find that any non-zero element has an inverse, so $\mathbb{H}$ is a division ring. Using the same arguments as for $\mathbb{C}$, we see that $\mathbb{H}$ is simple. $\triangle$

In the earlier example with the regular representation, we saw that minimal left ideals are in some sense irreducible. This works more generally: if V is a vector space, $\rho : A \to \mathrm{Lin}(V)$ a representation and B a minimal left ideal of A, then $\rho(A)\rho(B)(V) = \rho(AB)(V) = \rho(B)(V)$, so $\rho(B)(V)$ is an invariant subspace. Similarly, minimal left ideals have more nice properties. In the following theorems, we investigate a few of these properties.

Theorem 3.2.5. *Let A be an associative algebra and let B be a non-nilpotent, minimal left ideal of A. Then there is an $e \in A$ such that $e^2 = e$ and $B = Ae$.*

Proof. Consider B^2. Since $AB \subseteq B$, we know that $B^2 \subseteq B$ and that $AB^2 \subseteq B^2$, so B^2 is a left ideal contained in B. By minimality, either $B^2 = B$ or $B^2 = 0$. Since $B^2 = 0$ implies that B is nilpotent, we have $B^2 = B$. Consider $Bb \subseteq B^2 = B$ for a $b \in B$. This is once again a left ideal, so by minimality either $Bb = 0$ or $Bb = B$. Since $B^2 = B$, there must be a $b \in B$ with $Bb = B$. Therefore, the map $B \to B$ given by right multiplication by b is surjective. Since the dimension of B is finite, the map is also injective. The surjectivity of the map implies there is an $e \in B$ such that $eb = b$. Multiplying by e gives $e^2 b = eb$, so $(e^2 - e)b = 0$. By injectivity $e^2 - e = 0$, so $e^2 = e$. Finally, Ae is a left ideal with $Ae \subseteq AB \subseteq B$, so by minimality $Ae = B$. We conclude that e has the required properties. $\square$

Corollary 3.2.6. *Let A be a semisimple associative algebra and B as in the previous theorem such that $B = Ae$ and $e^2 = e$. Then eA is a minimal right ideal.*

Proof. It is clear that eA is a right ideal. We will prove that it is minimal. For this, we first need that eAe is a division ring, i.e. $eAe \setminus \{0\}$ is a group.
First remark that $eAe \subseteq Ae$ and $(eAe)(eAe) = e(Ae^2A)e \subseteq eAe$ is closed under multiplication, and that for any non-zero $x \in eAe$ we have $ex = xe = x$. Consider such a non-zero $x \in eAe$. Then $x \in Ae$ so the left ideal Ax is contained in Ae. As Ae is minimal, $Ax = Ae$. As $e \in A$, there is an $a \in A$ such that $ax = e^2 = e$. Define $y = eae \in eAe$, then $yx = eaex = eax = ee^2 = e$, so y is a left inverse of x. Consider this y. Similarly, there is an $z \in eAe$ such that $zy = e$. Now $xy = exy = zyxy = z(yx)y = zey = zy = e$, so y is also the right inverse of x.
We conclude eAe is closed under multiplication and each non-zero element has an inverse, so eAe is a division ring. Let $I \subseteq eA$ be a non-zero right ideal, let $t \in I$. Now $t \in eA$, so $te \in eAe$, so te has an inverse y or

$te = 0$. There must be an element $t \in I$ such that $te \neq 0$, since $te = 0$ for all $t \in I$ implies $I^2 \subseteq IeA = 0$ which means I is nilpotent. Since A is semisimple, I cannot be nilpotent.

Choose an element t such that te has the inverse y, hence $ty = tey = e$. But $y \in A$, so $e = ty \in IA \subseteq I$. We conclude that $e \in I$ so $I = eA$, Since I was arbitrary, we conclude that eA is a minimal right ideal. $\square$

The above theorem tells us that certain minimal left ideals are generated by an element $e \in A$. We give this e a name.

Definition 3.2.7. An $e \in A$ such that $e^2 = e$ is called an <u>idempotent</u>; if Ae is a minimal left ideal, e is called a <u>minimal idempotent</u>.

Minimal idempotents will be written as a or e.

To apply the above theorem, we need a minimal left ideal that is non-nilpotent. Proving that a given left ideal is non-nilpotent is (usually) not overly complicated, but a theorem which guarantees that all left ideals are non-nilpotent is still very useful.

Theorem 3.2.8. *Let A be a semisimple associative algebra and $B \neq 0$ a left ideal of A. Then B is non-nilpotent.*

Proof. Assume B is nilpotent, and define $N = BA$. By definition $AN = ABA \subseteq BA = N$ and $NA = BA^2 \subseteq BA = N$, so N is a two-sided ideal. Moreover, $N^n = (BA)^n = BABA\ldots BA \subseteq BB\ldots BA = B^n A$ by $AB \subseteq B$, so the nilpotency of B gives that N is nilpotent. Since A is semisimple, we get that $N = 0$. Hence $BA = 0 \subseteq B$, so B is a two-sided ideal. We find that $B = 0$. Using contraposition, we conclude that $B \neq 0$ is non-nilpotent. $\square$

We see that if A is semisimple, then all non-trivial left ideals are non-nilpotent, which can be useful. However, we still lack ways to prove that A is semisimple. Naturally, that is our next step.

Theorem 3.2.9. *Let A be an associative algebra. The following are equivalent:*

1. *A is semisimple.*

2. *A is the direct sum of minimal left ideals (as a vector space).*

3. *There exists a vector space V and a representation $\rho : A \rightarrow \mathrm{Lin}(V)$ which is faithful and completely reducible.*

Proof. We proof "$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$". The proof "$1 \rightarrow 2$" and "$3 \rightarrow 1$" are based on proofs in [4].

"$1 \rightarrow 2$": Let A be semisimple, and let B be a minimal left ideal. Then there is an $a \in A$ such that $B = Aa$. Define $A' = \{b - ba \mid b \in A\}$. It is clear A' is a left ideal of A. We show A is the direct sum of B with A'. Let $x \in B \cap A'$. Then $x = ca = d - da$ for some $c, d \in A$. Multiplying the right by a gives $ca^2 = da - da^2$, or $ca = da - da = 0$. We find that $x = 0$, so $B \cap A' = 0$. For any $a' \in A$ we have $a' = a'a + (a'a - a)$, and since $a'a \in B, (a'a - a) \in A'$ we find that $A = B + A'$. Combining this with $B \cap A' = 0$, we conclude that $A = B \oplus A'$.

We can repeat this process for A'. Since B has at least dimension 1 and A is finite dimensional, this process stops after finite iterations. We conclude that A is the direct sum of minimal left ideals.

"$2 \rightarrow 3$": Let A be the direct sum of k minimal left ideals. Since the ideal 0 does not contribute to a direct sum, we may assume each minimal left ideal is non-zero. Each minimal left ideal can be written as Aa_i for a minimal idempotent a_i, for $1 \leq i \leq k$, so $A = \bigoplus_{i=1}^{k} Aa_i$. Let $\rho_{reg} : A \rightarrow \mathrm{Lin}(A)$ be the regular representation, and define by $\rho_{reg}^i : A \rightarrow \mathrm{Lin}(Aa_i)$ the irreducible subrepresentations. We will show ρ_{reg}^i is faithful. Assume, to the contrary, that $\rho_{reg}^i(b) = \rho_{reg}^i(c)$ for unequal $b, c \in Aa_i$. Define $N = Aa_i(b - c)$. Now N is a left ideal with $N \subseteq Aa_i Aa_i \subseteq Aa_i$. Since $b - c \in N$, we have $N \neq 0$, so minimality gives $N = Aa_i$.

However, $a_i \in Aa_i$, so there is a $d \in Aa_i$ such that $d(b - c) = a_i$. Taking the ρ^i_{reg} on both sides gives $\rho^i_{reg}(d) \cdot 0 = \rho^i_{reg}(d(b-c)) = \rho^i_{reg}(a_i)$, so $\rho^i_{reg}(a_i) = 0$. But then $a_i = a_i a_i = \rho^i_{reg}(a_i)a_i = 0$, so $Aa_i = 0$. This contradicts the assumption that each minimal left ideal is non-zero. We conclude that ρ^i_{reg} is faithful, so the kernel of ρ^i_{reg} is 0. It follows that the kernel of ρ_{reg} is 0, so the regular representation is faithful. We see that the regular representation suffices.

"$3 \to 1$": Let V be the vector space and $\rho : A \to \text{Lin}(V)$ the faithful and completely reducible representation. There are $V_1, \ldots, V_n$, $n \in \mathbb{N}$ such that $V = V_1 \oplus \cdots \oplus V_n$ and ρ acts irreducibly on each V_i. Assume, to the contrary, that N is a nilpotent ideal in V. Let $k \in \mathbb{N}$ such that $N^k = 0$. By definition $AN \subseteq N$, so $\rho(N)V_i \subseteq V_i$ is an invariant subspace. As V_i is irreducible, this implies that either $\rho(N)V_i = V_i$ or $\rho(N)V_i = 0$. However $N^k = 0$ implies $\rho(N)^k = 0$, so $\rho(N)^k V_i = 0$. This excludes $\rho(N)V_i = V_i$, so $\rho(N)V_i = 0$. Since V is the direct sum of the V_i, we find that $\rho(N) = 0$. Using the injectivity of ρ (ρ is faithful), we conclude that $N = 0$. $\qquad\square$

Remark 3.2.10. The proofs of Theorem 3.2.5 and 3.2.9 "$1 \to 2$" also show that any semisimple A is the direct sum of minimal right ideals, by writing $B = aA$ with a similar argument and considering $A' = \{b - ab \mid b \in A\}$. We will use this for the next two corollaries.

Corollary 3.2.11. *Let A be a semisimple associative algebra. Then A is the direct sum of simple ideals.*

Proof. By the previous theorem, A is the direct sum of left ideals $A = \bigoplus_{i=1}^k A_i = \bigoplus_{i=1}^k Ae_i$ for some $k \in \mathbb{N}$, where the e_i are the minimal idempotents. For each Ae_i, consider Ae_iA. This is, by construction, a two-sided ideal in A. We will show this ideal is simple.
First of all, let $i \in \{1, \ldots, k\}$ and note that for any $j \in \{1, \ldots, k\}$, Ae_j either satisfies $Ae_j \subseteq Ae_iA$ or $Ae_iA \cap Ae_j = \varnothing$, as Ae_iA is a left ideal. Therefore, Ae_iA is the direct sum of some Ae_j for $j \in \{1, \ldots, k\}$. By the above theorem, Ae_iA is semisimple.
Let $B \subseteq Ae_iA$ be a non-zero ideal of Ae_iA, let $b = ae_ia'$ with $a, a' \in A$ be an element of B. Now $ae_i \in Ae_i$ which is a minimal left ideal of A, so $A(ae_i) = Ae_i$ which gives us

$$Ae_iA(ae_ia') = Ae_iAe_ia' = Ae_ia'$$

as $(Ae_i)^2 = Ae_i$ for minimal left ideals. We find that $Ae_ia' \subseteq B$.
By Remark 3.2.10, we find that A is the sum of minimal right ideals $A = \bigoplus_{i=1}^k e_iA$. Therefore, the above steps repeated for right ideals gives that $Ae_iA \subseteq B$, so $B = Ae_iA$ hence Ae_iA is simple.
For each Ae_i, we consider the Ae_iA. Since the intersection of two ideals is an ideal, we see that for $i, j \in \{1, \ldots, k\}$ either $Ae_iA = Ae_jA$ or $Ae_iA \cap Ae_jA = 0$ as the ideals are simple. We conclude that the direct sum of left ideals induces a direct sum of simple ideals if we substitute $Ae_i \mapsto Ae_iA$ and remove all doubles. $\quad\square$

Corollary 3.2.12. *Let A be a semisimple associative algebra. Then A has an unit.*

Proof. By Theorem 3.2.9, A is the direct sum of minimal left ideals. Moreover, by Theorem 3.2.5 we have that each minimal left ideal can be written as Ae for a (minimal) idempotent e. Therefore, write $A = \bigoplus_{i=1}^k Ae_i$ for a $k \in \mathbb{N}$. We have also seen that the intersection of two left ideals is again a left ideal, so for any left ideal B and any i we find that $Ae_i \cap B$ is a left ideal contained in Ae_i. Using the minimality of Ae_i, we find $Ae_i \cap B = 0$ or $Ae_i \cap B = Ae_i$ for any i. Since A is the direct sum of these minimal left ideals, it follows that any minimal left ideal of A is of the form Ae_i or 0. In Corollary 3.2.6 we saw that the corresponding e_iA are minimal right ideals.
By Remark 3.2.10, we see that A is the direct sum of minimal right ideals. Likewise, we see that each right ideal can be written eA for a minimal idempotent e, i.e. such that Ae is a minimal left ideal. From these considerations, it follows that $A = \bigoplus_{i=1}^k e_iA$.
Consider the map $A \to Ae_1, a \mapsto ae_1$. This is the right-unit on Ae_1 since $(e_1)^2 = e_1$, and is the null-map on each other Ae_i since $e_1 \in Ae_1$ and A is the direct sum of all Ae_i. Similarly, e_i is the right-unit on Ae_i

and null on each other minimal left ideal. Therefore, consider $e := \sum_{i=1}^{k} e_i$. This e acts as a right-unit on A. Since we can write $A = \bigoplus_{i=1}^{k} e_i A$ as well, we similarly get that e is a left-unit on A. We conclude that $e = \sum_{i=1}^{k} e_i$ is an unit, so A has an unit. $\qquad \square$

We have thus found that a semisimple algebra is the direct sum of minimal left ideals, so one can find the structure of A by determining its minimal left ideals. We have also seen that the regular representation is irreducible on minimal left ideals. We use this to show that all irreducible representations of A can be found in A as a minimal left ideal.

Theorem 3.2.13. *Let A be a semisimple associative algebra, V a vector space and $\rho : A \to \mathrm{Lin}(V)$ an irreducible representation. Then there exists a minimal left ideal $B \subseteq A$ such that ρ is equivalent to the restriction $\rho_{reg} : A \to \mathrm{Lin}(B)$ of the regular representation.*

Proof. Remark that if $V = 0$, then $B = 0$ has the required properties. We assume $V \neq 0$.

Using Theorem 3.2.9, write A as a direct sum $A_1 \oplus \cdots \oplus A_n$ of minimal left ideals for some $n \in \mathbb{N}$. Choose a non-zero $v \in V$ and define $B' = \{a \in A \mid \rho(a)v = 0\}$. For any $b, c \in B'$ we have that $\rho(ab)v = \rho(a)\rho(b)v = 0$ and $\rho(\lambda b + c)v = \lambda \rho(b)v + \rho(c)v = 0$ for all $a \in A, \lambda \in \mathbb{K}$, so B' is a left ideal of A. The intersection $B' \cap A_i \subseteq A_i$ is a left ideal, so by minimality $B' \cap A_i = 0$ or $B' \cap A_i = A_i$ for all $i = 1, \ldots, n$. Therefore, B' is the (direct) sum of A_i for some values of i. Take all the direct sum of all other values of i and call that B, such that $B \oplus B' = A$. We will show that ρ is equivalent to ρ_{reg} acting on B, and that B is minimal.

Define $\phi : B \to V$ by $b \mapsto \rho(b)(v)$. By construction, each $b \in B$ is not in B' except 0, so $\rho(b)(v) \neq 0$ unless $b = 0$ hence ϕ is injective. Since ρ is irreducible, V has no non-trivial invariant subspaces, so $\rho(A)(v) = V$ hence ϕ is surjective. We find that ϕ is bijective. Because

$$\phi(\rho_{reg}(a)(b)) = \phi(ab) = \rho(ab)(v) = \rho(a)(\rho(b)(v)) = \rho(a)(\phi(b))$$

for $a \in A, b \in B$, we see that ϕ intertwines ρ_{reg} and ρ. We conclude that $\phi : B \to V$ is an (intertwining) isomorphism, $B \cong V$.

Since ρ is irreducible, ρ_{reg} is irreducible in B. But we have seen before that ρ_{reg} is irreducible in each minimal left ideal, and that B is the direct sum of minimal left ideals. Since each minimal left ideal is an invariant subspace of B, we conclude that B is a minimal left ideal. $\qquad \square$

This theorem essentially tells us that all irreducible representations of a semisimple associative algebra can be found as subrepresentations of its regular representation. For simple algebras, we can prove a similar but stronger statement.

Theorem 3.2.14. *Let A be a simple associative algebra and let B, B' be non-zero minimal left ideals of A. Then there exists an isomorphism $\phi : B \to B'$ intertwining $\rho_{reg}(A)$, i.e. $\rho_{reg}(A)|_B \cong \rho_{reg}(A)|_{B'}$. (This means that B and B' are equivalent as A-modules.)*

Proof. Since B is a left ideal, BA is a two-sided ideal, so $BA = A$ as A is simple. Therefore, using that A has finite dimension, we can find a $k \in \mathbb{Z}$ and $a_i \in A$ for each $i \in \{1, \ldots, k\}$ such that the (not necessarily direct) sum of Ba_i is A. Choose the smallest k for which it is possible to write A as the sum of Ba_i, $1 \leq i \leq k$, and choose a corresponding set of a_i such that $A = Ba_1 + \cdots + Ba_k$. By our choice of k, the Ba_i are non-zero. Consider the map $\psi_i : B \to Ba_i, b \mapsto ba_i$. For any $a \in A$, $a \cdot \psi_i(b) = a(ba_i) = (ab)a_i = \psi_i(a \cdot b)$, so ψ_i intertwines the action of $\rho_{reg}(A)$ on B and Ba_i. Additionally, ψ_i is by construction surjective, and it is injective since B is minimal (and the kernel of ψ_i is an invariant subspace). We conclude that the ψ_i are isomorphisms, so each Ba_i is isomorphic to B (as A-modules). This implies the Ba_i are minimal, so $Ba_i \cap Ba_j = 0$ for any $i \neq j$ since k was minimal. We see that the sum of the Ba_i is direct, so $A = \bigoplus_{i=1}^{k} Ba_i$. Define $\phi_i : A \to Ba_i$ as the projection map of the direct sum. The map ϕ_i intertwines $\rho_{reg}(A)$, as for any $a \in A, b = b_1 a_1 + \cdots + a_k b_k \in A$, $a \cdot \phi_i(b) = a \cdot (b_i a_i) = ab_i a_i = \phi_i(ab_1 a_1 + \cdots + ab_k a_k) = \phi_i(a \cdot b)$. Consider $\phi_i|_{B'} : B' \to Ba_i$. As both B' and Ba_i are minimal, this map will either be the null-map or an isomorphism

(as the kernel is B' or 0 respectively). As B' is non-zero, there must be a map $\phi_i|_{B'}$ which is an intertwining isomorphism. Therefore, the composition $\phi := (\phi_i|_{B'})^{-1} \circ \psi_i : B \to B'$ is an intertwining isomorphism from B to B'. We conclude that B and B' are equivalent as A-modules. $\qquad\square$

We find that all minimal left ideals of a simple algebra are isomorphic, and that the restrictions of the regular representation to each minimal left ideal is equivalent. Combining this with Theorem 3.2.13, we naturally get the following result.

Corollary 3.2.15. *Let A be a simple associative algebra, V, V' vector spaces and let $\rho : A \to \mathrm{Lin}(V)$, $\rho' : A \to \mathrm{Lin}(V')$ be irreducible representations. Then ρ and ρ' are equivalent.*

Summarising this section, we have found a few ways to prove an associative algebra is (semi)simple and seen properties that are implied by (semi)simplicity, like the existence of an unit and that the vector space is the direct sum of left ideals. This essentially tells us semisimple algebras are 'nice enough' for our purposes.

3.3 Tensor product of associative algebras

As we will see in the next chapter, some (associative) algebras are difficult but can be simplified by taking the product with another algebra. As algebras are vector spaces, this product will be defined such that the vector space of the product is the product of the vector spaces. This naturally brings us to the tensor product.
In this section, $\mathbb{K}$ will again denote a field with characteristic 0.

Definition 3.3.1. Let A_1, A_2 be associative algebras over the field $\mathbb{K}$. The <u>tensor product</u> $A_1 \otimes_{\mathbb{K}} A_2$ of A_1 with A_2 is the vector space $A_1 \otimes A_2$ equipped with the multiplication defined by $(a \otimes a')(b \otimes b') = (ab) \otimes (a'b')$ which is extended linearly to non-pure tensors.

Proposition 3.3.2. *The vector space $A_1 \otimes A_2$ equipped with the given multiplication is an associative algebra.*

Proof. First of all, we have to prove that the extension of the multiplication is well-defined. Let $a, a', c \in A_1$ and $b, b', d \in A_2$. Remark that $(a + a') \otimes b = (a \otimes b) + (a' \otimes b)$ and similarly $a \otimes (b + b') = (a \otimes b) + (a \otimes b')$. Multiplying by $c \otimes d$ gives:

$$\big((a + a') \otimes b\big)(c \otimes d) = \big((a + a')c\big) \otimes bd = (ac + a'c) \otimes bd = (ac \otimes bd) + (a'c \otimes bd)$$
$$\big((a \otimes b) + (a' \otimes b)\big)(c \otimes d) = (a \otimes b)(c \otimes d) + (a' \otimes b)(c \otimes d) = (ac \otimes bd) + (a'c \otimes bd)$$

We see that the multiplication gives the same result in both cases. The same steps also give

$$\big(a \otimes (b + b')\big)(c \otimes d) = (ac \otimes bd) + (a'c \otimes bd) = \big((a \otimes b) + (a \otimes b')\big)(c \otimes d).$$

We conclude the extension is well-defined.
Secondly, we have to prove $\mathbb{K}$-bilinearity and associativity of the multiplication. Let $a, b, c \in A_1$, $a', b', c' \in A_2$ and $\lambda, \mu \in \mathbb{K}$. We have

$$\big(\lambda(a \otimes a')\big)\big(\mu(b \otimes b')\big) = \big((\lambda a) \otimes a\big)\big((\mu b) \otimes b'\big) = \big((\lambda a)(\mu b)\big) \otimes (a'b') = (\lambda\mu ab) \otimes (a'b') = \lambda\mu\big((ab) \otimes (a'b')\big)$$

and

$$\big((a \otimes a')(b \otimes b')\big)(c \otimes c') = ((ab) \otimes (a'b'))(c \otimes c') = (abc) \otimes (a'b'c')$$
$$= (a \otimes a')\big((bc) \otimes (b'c')\big) = (a \otimes a')\big((b \otimes b')(c \otimes c')\big)$$

so the multiplication is both $\mathbb{K}$-bilinear and associative. $\qquad\square$

Note that for any $a \in A_1, a' \in A_2$ we have that $a \otimes 0 = 0(a \otimes a') = 0 \otimes a' = 0 \otimes 0$. This is consistent with the fact that $0 \otimes 0$ is the zero of the vector space $A_1 \otimes A_2$.

Remark 3.3.3. Let $A, B \in A_1$, $A', B' \in A_2$ be subalgebras. The above proposition naturally gives that $A \otimes_{\mathbb{K}} A'$ and $B \otimes_{\mathbb{K}} B'$ are subalgebras of $A_1 \otimes_{\mathbb{K}} A_2$. Note that $(A \otimes_{\mathbb{K}} A')(B \otimes_{\mathbb{K}} B') = \{(ab) \otimes (a'b') \mid a \in A, b \in B, a' \in A', b' \in B'\} = AB \otimes_{\mathbb{K}} A'B'$.

In particular, we have considered (left) ideals. This leads us to the following result.

Proposition 3.3.4. *Let A_1, A_2 be associative algebras over $\mathbb{K}$ and let $B \subseteq A_1, C \subseteq A_2$ be subalgebras. If B and C are left ideals, then $B \otimes_{\mathbb{K}} C$ is a left ideal of $A_1 \otimes_{\mathbb{K}} A_2$. If B and C are two-sided ideals, then so is $B \otimes_{\mathbb{K}} C$. If B and C are two-sided ideals and B or C is nilpotent, then $B \otimes_{\mathbb{K}} C$ is nilpotent.*

Proof. Assume B and C are left ideals. By definition we have $A_1 B \subseteq B$ and $A_2 C \subseteq C$, so $A_1 B \otimes_{\mathbb{K}} A_2 C \subseteq B \otimes_{\mathbb{K}} C$. We conclude $(A_1 \otimes_{\mathbb{K}} A_2)(B \otimes_{\mathbb{K}} C) \subseteq B \otimes_{\mathbb{K}} C$, so by definition $B \otimes_{\mathbb{K}} C$ is a left ideal of $A_1 \otimes_{\mathbb{K}} A_2$. The same argument holds for right ideals, so $B \otimes_{\mathbb{K}} C$ is a two-sided ideal if B, C are two-sided ideals. Finally, without loss of generality, let B be nilpotent. From the definition of nilpotency, let $n \in \mathbb{N}$ such that $B^n = 0$. Then $(B \otimes_{\mathbb{K}} C)^n = B^n \otimes_{\mathbb{K}} C^n = 0 \otimes_{\mathbb{K}} C^n = 0 \otimes_{\mathbb{K}} 0$ as $0 \otimes c = 0 \otimes 0$ for any $c \in C$. We conclude that $B \otimes_{\mathbb{K}} C$ is nilpotent if B or C is nilpotent. $\qquad\square$

Corollary 3.3.5. *Let A_1, A_2 be associative algebras over $\mathbb{K}$ and let $A_1 \otimes_{\mathbb{K}} A_2$ be semisimple. Then A_1 and A_2 are semisimple.*

Proof. Assume, to the contrary, that A_1 is not semisimple. Then there exists a nilpotent ideal $I \subset A_1$. Now $I \otimes_{\mathbb{K}} A_2$ is a nilpotent ideal of $A_1 \otimes_{\mathbb{K}} A_2$, which is in contradiction with the fact that $A_1 \otimes_{\mathbb{K}} A_2$ is semisimple. We find that A_1 must be semisimple. Using the same argument, we find that A_2 must be semisimple. We conclude A_1 and A_2 are semisimple. $\qquad\square$

Note that this corollary does not imply that $A_1 \otimes_{\mathbb{K}} A_2$ is semisimple if A_1 and A_2 are semisimple, as there might be a nilpotent ideal generated by a non-pure element of $A_1 \otimes_{\mathbb{K}} A_2$.
We derive a similar corollary for simple algebras.

Corollary 3.3.6. *Let A_1, A_2 be associative algebras over $\mathbb{K}$ and let $A_1 \otimes_{\mathbb{K}} A_2$ be simple. Then A_1 and A_2 are simple.*

Proof. Assume, to the contrary, that A_1 is not simple. Then there exists a non-trivial ideal $I \subset A_1$. Now $I \otimes_{\mathbb{K}} A_2$ is an ideal of $A_1 \otimes_{\mathbb{K}} A_2$. By contradiction, we find that A_1 is simple. Without loss of generality, we conclude that A_1 and A_2 are simple. $\qquad\square$

Example 3.3.7. In Example 3.2.4 we have seen that $\mathbb{H}$ and $\mathbb{C}$ are simple as algebras over $\mathbb{R}$. Consider the product $\mathbb{H}_{\mathbb{C}} := \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$. The vector space $\mathbb{H} \otimes \mathbb{C}$ is generated by $e_1 \otimes 1, e_1 \otimes \mathrm{i}, e_i \otimes 1$, etc. We shorten these notations to $1, \mathrm{i}, i, i\mathrm{i}$, etc. Although $\mathbb{H}$ and $\mathbb{C}$ are both division rings, their product isn't. For instance, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ has the element $1 + j\mathrm{i}$ and its (quaternion) conjugate $1 - j\mathrm{i}$. It is clear that $(1 + j\mathrm{i})(1 - j\mathrm{i}) = 0$, so $1 + j\mathrm{i}$ is a zero divisor and therefore has no inverse.
Consider the left ideal generated by $1 + j\mathrm{i}$, so the ideal $I := \mathbb{H}_{\mathbb{C}}(1 + j\mathrm{i})$. Remark that $1(1 + j\mathrm{i}) = 1 + j\mathrm{i}$, $i(1 + j\mathrm{i}) = i + k\mathrm{i}$, $j(1 + j\mathrm{i}) = j - \mathrm{i}$ and $k(1 + j\mathrm{i}) = k - i\mathrm{i}$, so out of the 8 dimensions of $\mathbb{H} \otimes \mathbb{C}$, at least 4 are in this ideal. In particular, $-\mathrm{i}(i + k\mathrm{i}) = k - i\mathrm{i}$ and $-\mathrm{i}(1 + j\mathrm{i}) = j - \mathrm{i}$, so all elements of the ideal are of the form $a(i + k\mathrm{i}) + b(1 + j\mathrm{i})$ where a and b are complex numbers hence the ideal is 4 dimensional.
We will show the ideal I is minimal. Assume, to the contrary, there is a non-zero element $x \in I$ such that $X = \mathbb{H}_{\mathbb{C}} x$ is a smaller ideal (note that X is contained in I). Now x and $x\mathrm{i}$ are in X, so if there is any $y \in X$ such that $(\mathrm{i})^n x$ and y are linearly independent for $n = 1, 2, 3, 4$ then X at least contains $x, x\mathrm{i}, y$ and $y\mathrm{i}$. It follows the existence of such an y implies that X has 4 dimensions, which implies $X = I$. Therefore, there can be no such y, so for each $y \in X$, y must be in the linear span of x and $x\mathrm{i}$.
Therefore, there must be complex numbers c_i, c_j, c_k such that $ix = c_i x$, $jx = c_j x$, $kx = c_k x$. (One can think about this as x being an 'eigenvector' for i, j and k.) However, $iji = ki = j$, so $ijix = jx$ which gives us

$c_i^2 c_j = c_j$ since complex numbers commute. Because j is not a null-divisor and x is non-zero, the complex number c_j cannot be equal to 0, so $c_i^2 = 1$ or $c_i = \pm 1$. Similarly, we find $c_j = \pm 1, c_k = \pm 1$.

Write x in the form $x_0 + ix_1 + jx_2 + kx_3$ for $x_0, \ldots, x_3 \in \mathbb{C}$. Since $c_i = \pm 1$, we have

$$-x_1 + ix_0 - jx_3 + kx_2 = \pm(x_0 + ix_1 + jx_2 + kx_3)$$

so $-x_1 = \pm x_0$, $x_0 = \pm x_1$, $-x_3 = \pm x_2$ and $x_2 = \pm x_3$, where the plus-minuses are either all plus or all minus. But now $-x_1 = \pm x_0 = (\pm)^2 x_1 = x_1$, so $x_1 = 0$ and $x_0 = 0$. Similarly, $x_2 = x_3 = 0$. We conclude $x = 0$. But we assumed x to be non-zero, so we see that there are no smaller ideals inside I. We conclude that I is minimal.

By Theorem 3.2.5 we know that I can be written as $\mathbb{H}_{\mathbb{C}}e$ for a minimal idempotent e. Since we have $I = \mathbb{H}_{\mathbb{C}}(1 + j\mathrm{i})$, we will use $1 + j\mathrm{i}$. Remark that $(1 + j\mathrm{i})^2 = 2 + 2j\mathrm{i}$, so $\frac{1}{2}(1 + j\mathrm{i})$ is idempotent. It is clear that $\frac{1}{2}(1 + j\mathrm{i})$ generates I, so we have found the minimal idempotent.

In the same way as above, we find that $\mathbb{H}_{\mathbb{C}}(1 - j\mathrm{i})$ is a minimal left ideal with $\frac{1}{2}(1 - j\mathrm{i})$ as minimal idempotent. Note that $\frac{1}{2}(1 + j\mathrm{i}) + \frac{1}{2}(1 - j\mathrm{i}) = 1$, so we have found a decomposition of $\mathbb{H}_{\mathbb{C}}$ into minimal left ideals, such that $\mathbb{H}_{\mathbb{C}}$ is the direct sum of the left ideals. We conclude that $\mathbb{H}_{\mathbb{C}}$ is semisimple.

Remark that $\mathbb{H}_{\mathbb{C}}(1 + j\mathrm{i})\mathbb{H}_{\mathbb{C}} = \mathbb{H}_{\mathbb{C}}(1 - j\mathrm{i})\mathbb{H}_{\mathbb{C}}$ as $-k(1 + j\mathrm{i})k = 1 - j\mathrm{i}$ (using (anti-)commutation relations). Therefore, using the construction we found in the proof of Corollary 3.2.11, we find that $\mathbb{H}_{\mathbb{C}}$ is the direct sum of a single simple ideal, so $\mathbb{H}_{\mathbb{C}}$ is simple.

We conclude $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ is simple. This implies that $\mathbb{H}$ and $\mathbb{C}$ are simple. This is consistent with our earlier findings. $\triangle$

In the above example, we have seen that the tensor product with $\mathbb{C}$ makes the space $\mathbb{H}$ slightly more complicated, but also gives us more options. Since $\mathbb{C}$ commutes, the spaces $\mathbb{H}_{\mathbb{C}}$ is implicitly identified with "$\mathbb{H}$ where all real numbers have been replaced by complex numbers". This is a common technique, so we give it a name.

Definition 3.3.8. Let A be an associative algebra over $\mathbb{R}$. The space $A_{\mathbb{C}} := A \otimes_{\mathbb{R}} \mathbb{C}$ is called the complexification of A. In general, if A is an associative algebra over a field $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ is a larger field containing $\mathbb{K}$, we call the construction of $A \otimes_{\mathbb{K}} \widetilde{\mathbb{K}}$ the extension of the ground field from $\mathbb{K}$ to $\widetilde{\mathbb{K}}$.

Note that the complexification of a real algebra is still a real algebra, but since complex multiplication is commutative, there is a natural way to extend the scalar multiplication to the complex numbers. When using complexifications in later chapters, we will always use the complexification as a complex algebra.

In the next chapter, we will consider a particular kind of associative algebras. We will find that the complex case is much more intuitive compared to the real case, which shows a clear advantage of complexifications.

4 Clifford Algebras

In this section, we will introduce the concepts of Clifford Algebras and the (S)Pin group. Intuitively, a Clifford algebra over a vector space V with a (non-degenerate) quadratic form $Q : V \to \mathbb{K}$ is the smallest associative algebra generated by elements of V with the property that $v \cdot v = Q(v)$; however, 'smallest' is not directly well-defined. We therefore start with a construction of the Clifford algebra. In this construction we roughly follow the construction of Hermann [4].

4.1 Construction

Let V be a $\mathbb{K}$-vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. First of all, we define the tensor algebra over V as the algebra generated by V such that product can not be simplified. This means we start with V and take the (associative) tensor product of all elements of V for a set $V \otimes V$. We repeat this for sets $V \otimes V \otimes V$, etc. to get $V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$. We want our algebra to have an identity for multiplication, so we add an '1' to get $\{1\} \oplus V \oplus (V \otimes V) \oplus \dots$; however this is not yet closed under the scalar multiplication of $\mathbb{K}$ so we arrive at:

$$\mathbb{K}(\{1\} \oplus V \oplus (V \otimes V) \oplus \dots) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \dots$$

since V is closed under scalar multiplication. We have constructed the following algebra:

Definition 4.1.1. We define the <u>tensor algebra</u> TV as $\mathbb{K} \oplus \bigoplus_{k=1}^{\infty} \bigotimes_{j=1}^{k} V$.

Remark 4.1.2. In this context, it is common to denote the tensor product of k copies of V by V^k and to define $V^0 := \mathbb{K}$. This allows us to write the tensor algebra as $TV = \bigoplus_{k=0}^{\infty} V^k$.

We want to be able to simplify the product, intuitively we do that by 'defining' $v \otimes v = Q(v)$, for a quadratic form Q. We do this by considering the tensor algebra TV modulo terms $v \otimes v - Q(v)$. This can be made rigorous be defining the ideal I generated by $\{v \otimes v - Q(v)1 \mid v \in V\}$, where 1 is the identity element we added earlier, and taking the quotient TV/I.

Definition 4.1.3. Let $Q : V \to \mathbb{K}$ be a non-degenerate quadratic form, i.e. let $\{v_1, \dots, v_n\}$ be a basis for V and let A be a symmetric matrix with non-zero determinant such that $Q(v) = v^{\mathsf{T}} \cdot A \cdot v$ for $v \in V$. We define the <u>Clifford algebra</u> $C\ell(V, Q)$ as TV/I where I is the ideal generated by $\{v \otimes v - Q(v)1 \mid v \in V\}$. We identify V and $\mathbb{K}$ with the images $V^1 + I$ and $V^0 + I$.

There are a couple of remarks attached to this definition.

Remark 4.1.4. For every quadratic form Q, the function $P : V \to \mathbb{K}, P(v) = -Q(v)$ is also a quadratic form. Therefore, the expressions $v \otimes v - P(v)1$ and $v \otimes v + Q(v)1$ are equivalent. This causes some authors to instead use the ideal generated by $\{v \otimes v + Q(v)1 \mid v \in V\}$, effectively giving their quadratic form an additional minus. In this thesis, the convention will always be that the ideal has a minus.

Remark 4.1.5. Instead of a quadratic form, one might use a non-degenerate symmetric bilinear form $\beta : V \times V \to \mathbb{K}$. In this case, the ideal generated by $\{v \otimes u + u \otimes v - 2\beta(u, v) \mid u, v \in V\}$ is used. Note, however, that a non-degenerate bilinear form naturally induces a quadratic form by $Q(v) := \beta(v, v)$ while a quadratic form induces a non-degenerate bilinear form by the polarisation identity $\beta(u, v) := \frac{1}{2}(Q(u+v) - Q(u) - Q(v))$. If we write the quadratic form as $Q(v) = v^{\mathsf{T}} \cdot A \cdot v$ with A a symmetric matrix with non-zero determinant, then $\beta(u, v) = u^{\mathsf{T}} \cdot A \cdot v$ and vice-versa. The convention used in this thesis uses the quadratic form and the bilinear form interchangeably.

We have now constructed the Clifford algebra. From here on, we won't explicitly write $a \otimes b$ the whole time, but shorten it to ab, for a, b elements of the Clifford algebra.

4.2   Signature and grading

Of particular interest are the Clifford algebras over $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$ with standard quadratic forms; that is to say, quadratic forms without cross-terms. These forms all rely on a basis $\{e_1, e_2, \ldots, e_n\}$ of V to be written as $Q(v) = v_1^2 + v_2^2 + \cdots + v_p^2 - (v_{p+1}^2 + \cdots + v_{p+q}^2)$, for $p, q \in \mathbb{N}_0$ with $p + q = n$.

Remark 4.2.1. Let $Q(v) = v^{\mathsf{T}} \cdot A \cdot v$ be a quadratic form with A a symmetric matrix with non-zero determinant. Then we can diagonalise A using eigenvectors to get a diagonal matrix. For each element of that matrix, we can then rescale that element to either 1 or -1, by writing that element as $\pm 1 \cdot \lambda^2$ for some positive real λ and rescaling the respective basis element by λ. This way, we will always end up with a new basis, such that the quadratic form Q can be written as $Q(v) = v_1^2 + v_2^2 + \cdots + v_p^2 - (v_{p+1}^2 + \cdots + v_{p+q}^2)$ in that basis, where we have p elements scaled to 1 and q scaled to -1. This basis is, by construction, orthogonal. We conclude that for any quadratic form, we can choose an orthogonal basis such that the quadratic form takes a standard form.

Note that in $\mathbb{C}^n$ the basis can be substituted with $\{e_1, \ldots, e_p, \mathrm{i}e_{p+1}, \ldots, \mathrm{i}e_n\}$ to change the form into $Q(v) = v_1^2 + v_2^2 + \cdots + v_n^2$, so for $\mathbb{C}^n$ we can always assume $q = 0$. We conclude that complex Clifford algebras of the same dimension are isomorphic.

This brings us to the following definitions:

Definition 4.2.2. Let $Q_{p,q}$ denote the standard quadratic form with p positive signs and q negative signs. The tuple (p, q) is called the <u>signature</u> of the vector space. We define $C\ell(p, q) = C\ell(\mathbb{R}^n, Q_{p,q})$ and $C\ell(n, \mathbb{C}) = C\ell(\mathbb{C}^n, Q_{n,0})$. In the real case for $q = 0$, we write $C\ell(n)$ instead.

We say that the Clifford algebra has indefinite signature if $q \neq 0$ and that is has definite signature otherwise. Note that $Q_{p,q}(v) = -Q_{q,p}(v)$ for any $v \in V$, so for any $Q_{p,q}$, $Q_{q,p}$ takes the role of P in the earlier remark. The given basis of V is orthogonal in $C\ell(p, q)$ and $C\ell(n, \mathbb{C})$ in the sense that $\beta(e_i, e_j) = \pm\delta_{ij}$ where $i, j \in \{1, \ldots, n\}$, β is the associated bilinear form and δ_{ij} is the Kronecker delta. This gives us a natural extension of the basis of V to a basis of the Clifford algebra by $\{1, e_i, e_i e_j, \ldots, e_1 e_2 \ldots e_n\}$ for $i, j \in \{1, \ldots, n\}$.

Theorem 4.2.3. *The set $B = \{1, e_i, e_i e_j, \ldots, e_1 e_2 \ldots e_n\}$ for ordered indices $i, j, \cdots \in \{1, \ldots, n\}, i < j < \ldots$ forms a basis for $C\ell(V, Q)$ as a vector space.*

Proof. We will prove the given set is linearly independent and spans the whole space. Assume there are numbers $k, k_i, k_{ij}, \ldots, k_{12\ldots n} \in \mathbb{K}$ such that

$$k + \sum_{i=1}^{n} k_i e_i + \sum_{i<j} k_{ij} e_i e_j + \cdots + k_{12\ldots n} e_1 e_2 \ldots e_n = 0,$$

where we sum over ordered indices. Since 0 commutes with every e_i, the left-hand can be split up into terms that commute with e_i and terms that anti-commute with e_i, each part adding up to 0. Using this for each e_i, we find that we only have to prove the linear independence of terms with the same commutation relations. We therefore consider the commutative properties of each element of B.

Let $f = e_{i_1} e_{i_2} \ldots e_{i_l} \in B$ consist of l different e_i's. If l is odd, f commutes with each $e_{i_j}, j = 1, \ldots, l$ while it anti-commutes with all other e_k. In this case, we say that f has odd length. If l is even, f commutes with each e_m where $m \neq i_j$ for $j = 1, \ldots, l$, while it anti-commutes with each $e_{i_j}, j = 1, \ldots, l$. In this case, we say that f has even length.

We see that two element of B with an odd length have the same commutation relations iff they are equal, and the same holds for two elements of even length. This gives us that each element of B is either unique in the way it commutes with the e_i, or there is exactly one other element with a length of opposite parity. Which of these is true depends on the parity of n. We separate the cases of n even and n odd.

Let n be even, and let $f = e_{i_1} \ldots e_{i_l} \in B$ be of odd length. We will show there is no $g \neq f \in B$ with the same commutation relations. Assume, to the contrary, that $g \neq f$ has the same commutation relations. By earlier

remarks, g must have even length. Let e_i commute with f, then e_i is one of $e_{i_1}, \ldots, e_{i_l}$. Since e_i commutes with g, it follows that e_i does not occur in g. Let e_j anti-commute with f, then e_j does not occur in f while it does occur in g. We conclude that g is the product of all e_j that do not occur in f. However, there are $n - l$ of such e_j, which is an odd number, so g has odd length. This is in contradiction with the fact that g has even length. We conclude that each f is unique in B with respect commutation relations, so all $k_\bullet$ must be zero, hence the elements of B are linearly independent.

Let n be odd, and let $f \in B$. Let g be the element of B that is the product of all e_j that do not occur in f. Now f and g have the exact same commutation relations. To show that B is linearly independent, we therefore have to show that f, g are linearly independent. Consider $f = 1$, now $g = e_1 e_2 \ldots e_n$. These two elements are independent since for each non-zero k, $k_{12\ldots n} \in \mathbb{K}$, $k + k_{12\ldots n} e_1 e_2 \ldots e_n$ is not an element of the ideal I. Therefore, $k = k_{12\ldots n} = 0$. Now let $f \in B$ arbitrary, and g defined as before and let k_f, k_g be their corresponding scalars. If $k_f f + k_g g = 0$, then so is $k_f f \cdot f + k_g f \cdot g = \pm k_f \pm k_g e_1 e_2 \ldots e_n$. But that directly implies $k_f = k_g = 0$. We conclude that all $k_\bullet$ are zero, so all elements of B are linearly independent.

We see that in both cases, it follows that all elements of B are linearly independent. We only have to show that the elements of B span the whole space. For this, we remark that free tensor products of e_i's give a basis for TV, by construction. By taking the quotient over I, we add anti-commutation relations and replace repeated e_i's by ± 1. The image of each free tensor product of the e_i's is therefore either an element of B, or minus an element of B. We conclude that the elements of B indeed span the space, so B is a basis for $C\ell(V, Q)$ as a vector space. $\qquad\square$

In the above proof we separated two cases for even and odd dimension. We saw that n odd, there is a twofold degeneracy in the commutation relations. We will later see that this case distinction has more relevance than is shown here.

From the theorem, it follows that $\dim_\mathbb{R} C\ell(p, q) = \dim_\mathbb{C} C\ell(n, \mathbb{C}) = 2^n$, where we take the real respectively the complex dimension.

Using this basis, we can make the graded structure of the Clifford algebra (as a vector space) explicit. For $k \in \mathbb{N}_0$ we denote the vector space generated by elements $e_{i_1} e_{i_2} \ldots e_{i_k}$ with $i_j \in \{1, \ldots, n\}$ by $C\ell^k(V, Q)$; we say that elements of $C\ell^k(V, Q)$ have grade k. We also say that an elements has even (respectively odd) grade if k is even (respectively odd), and denote corresponding vector spaces by $C\ell^{even}(V, Q)$ respectively $C\ell^{odd}(V, Q)$. Now $C\ell(V, Q) = C\ell^0(V, Q) \oplus C\ell^1(V, Q) \oplus \cdots = C\ell^{even}(V, Q) \oplus C\ell^{odd}(V, Q)$. Remark that $C\ell^0(V, Q) = \mathbb{K}$ and $C\ell^1(V, Q) = V$ under the aforementioned identification, and note that "$even \cdot even = odd \cdot odd = even$" and "$odd \cdot even = even \cdot odd = odd$" hold since 0 is both odd and even. Finally, note that 0 is the only element of $C\ell(V, Q)$ that has both even and odd grade, because of the direct sum.

4.3 Properties of the Clifford algebra

We want to be able to use theorems such as Theorem 3.2.13 and Theorem 3.2.14. Therefore, we try to determine whether Clifford algebras are (semi)simple. We will first consider complex Clifford algebras, as they only depend on the dimension. In the proof of Theorem 4.2.3 we have seen that the odd dimensional Clifford spaces might have an ideal of the form $1 \pm e_1 e_2 \ldots e_n$ or similar. We therefore start with even dimension.

4.3.1 Complex Clifford algebras of even dimension

Consider the Clifford algebra $C\ell(2n, \mathbb{C})$. To determine (semi)simplicity, we could hunt for (nilpotent) ideals or try to proof that there are no such ideals. There is an easier way, however, which is based on Physics. Even-dimensional Clifford algebras appear in particle physics, where they are used to describe fermion states.[1] In that context, a faithful irreducible representation of even-dimensional Clifford algebras has been found.

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[1]In particular, physicists use the special case in which the vector space is a Hilbert space such that the inner product is consistent with the bilinear form. See also CAR algebras and (anti-symmetric) Fock spaces.

We will construct this representation and proof its faithfulness and irreducibility. Theorem 3.2.9 implies this representations is semisimple; we will use the stronger condition of irreducibility (instead of complete reducibility) to proof $C\ell(2n, \mathbb{C})$ is simple.

Let V be an n-dimensional complex vector space, and ΛV the exterior algebra of V. Recall that the exterior algebra is the algebra generated by (anti-commuting) wedges, where $v \wedge v = 0$ for all $v \in V$. The construction of the exterior algebra is similar to that of the Clifford algebra but with 0 as the bilinear form. We write $\Lambda V = \bigoplus_{i=0}^{\infty} \Lambda^i V$, where $\Lambda^0 V = \mathbb{K}$, $\Lambda^1 V = V$, $\Lambda^2 V = V \wedge V$ etc. Elements of V^i are said to have grade i. For any $v \in V$ define the 'creation operator' $a^+(v)$ as

$$a^+(v) : \Lambda V \to \Lambda V, x \mapsto v \wedge x.$$

Note that this is for any $x \in \Lambda V$ as the $\wedge$ operation is distributive over addition. It is clear that the creation operator $a^+(v)$ increases the grade of a wedged product $v_1 \wedge v_2 \wedge \cdots \wedge v_k, k \in \mathbb{N}$ by one, unless v isn't linearly independent with the vectors v_i as that causes the result to be 0.
If we 'create', we also need to 'annihilate'. Let V^* be the dual space of V and let $\theta \in V^*$. We define the 'annihilation operator' $a^-(\theta)$ as

$$a^-(\theta) : \Lambda V \to \Lambda V, w \mapsto \theta \lrcorner w.$$

Here the interior product $\lrcorner$ is defined by

$$\theta \lrcorner (v_1 \wedge \cdots \wedge v_j) = \sum_{i=1}^{j} (-1)^{i-1} \theta(v_i)(v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_j)$$

for $j > 0$ and $\theta \lrcorner \lambda = 0$ for $\lambda \in \mathbb{K}$ ($j = 0$), extended linearly. The annihilation operator decreases the grade by one, unless the sum $\theta(v_i)$ is 0 and the terms are added in the 'right way' (e.g. $\theta(v_i) = 0$ for all i).

Remark 4.3.1. In physics, the creation and annihilation operators are scaled with an additional factor $\sqrt{j}$ (or $\sqrt{j \pm 1}$), where j is the grade. This comes from the normalisation of the wave functions on which the operators act. However, this complication is unneeded in our construction.

Let $v, w \in V$ and $\theta, \phi \in V^*$. We calculate the anti-commutators of the corresponding operators.

$$\begin{aligned}
\{a^+(v), a^+(w)\}(\dots) &= (v \wedge w \wedge (\dots) + w \wedge v \wedge (\dots)) \\
&= (v \wedge w + w \wedge v) \wedge (\dots) = 0 \\
\{a^-(\theta), a^-(\phi)\}(\dots) &= (\theta \lrcorner \phi \lrcorner (\dots) + \phi \lrcorner \theta \lrcorner (\dots)) = 0 \\
\{a^-(\theta), a^+(v)\}(\dots) &= \theta \lrcorner (v \wedge (\dots)) + v \wedge (\theta \lrcorner (\dots)) = \theta(v)(\dots) - v \wedge (\theta \lrcorner (\dots)) + v \wedge (\theta \lrcorner (\dots)) \\
&= \theta(v)(\dots)
\end{aligned}$$

where we used $\theta \lrcorner \phi \lrcorner (\dots) = -\phi \lrcorner \theta \lrcorner (\dots)$, which follows from its definition.

We use these anti-commutations relations to define a structure on $V \oplus V^*$.
Let a be the map

$$a : V \oplus V^* \to \mathrm{Lin}(\Lambda V), v + \theta \mapsto a^+(v) + a^-(\theta).$$

Let $v + \theta \in V \oplus V^*$ and $w + \phi \in V \oplus V^*$. Now $a(v + \theta)a(w + \phi) + a(w + \phi)a(v + \theta) = \theta(w) + \phi(v)$ according to the anti-commutation relations. Since the anti-commutator is bilinear, this induces a bilinear map

$$\beta : (V \oplus V^*) \times (V \oplus V^*) \to \mathbb{C}, \beta(v + \theta, w + \phi) = \theta(w) + \phi(v).$$

Note that this bilinear map is non-degenerate, as for any $v + \theta$ we can choose the corresponding dual elements $v^* + \theta^*$ where $v^*(v) = 1$ and $\theta(\theta^*) = 1$, such that $\beta(v + \theta, \theta^* + v^*) = 2$. Also note that the induced quadratic

map is given by $Q(v + \theta) = 2\theta(v)$. This implies that $Q(v) = 0$ and $Q(\theta) = 0$ for any $v \in V, \theta \in V^*$, so the product of elements of V which are not linearly independent is 0.

Using this bilinear map, we define the Clifford algebra $C\ell(V \oplus V^*)$. Since the vector space V is n-dimensional, the vector space V^* is also n-dimensional, so $V \oplus V^*$ is a $2n$-dimensional vector space.

We now extend a to a representation $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ as follows:

$$\begin{aligned} \rho : \lambda &\mapsto \lambda \mathbb{1} & \text{for } \lambda \in \mathbb{C} \\ v + \theta &\mapsto a^+(v) + a^-(\theta) & \text{for } v \in V,\, \theta \in V^*. \\ \rho(xy) &= \rho(x)\rho(y) & \text{otherwise} \\ &\text{Extended linearly} \end{aligned}$$

where $\mathbb{1}$ denotes the identity map on ΛV. Here we used that any element of $C\ell(V \oplus V^*)$ can be written as a finite product of elements of $V \oplus V^*$ and elements of $\mathbb{C}$.

Proposition 4.3.2. $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ *is a well-defined representation.*

Proof. To prove that ρ is well-defined, first consider $\rho' : T(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ defined by $\rho'(\lambda) = \lambda \mathbb{1}$ for $\lambda \in \mathbb{C}$, $\rho'(v + \theta) = a^+(v) + a^-(\theta)$ for $v \in V, \theta \in V^*$ and $\rho(xy) = \rho(x)\rho(y)$ for elements $xy \in (V \oplus V^*)^k, k \geq 2$. This is well-defined, as each element of $T(V \oplus V^*)$ has a unique factorisation into elements of $V \oplus V^*$.

Now to go from $T(V \oplus V^*)$ to $C\ell(V \oplus V^*)$, we quotient over the ideal, such that $(v+\theta)(w+\phi)+(w+\phi)(v+\theta) = \theta(w) + \phi(v)$. Therefore, the only additional condition for ρ to be well-defined is that $\rho(v+\theta)\rho(w+\phi)+\rho(w+\phi)\rho(v+\theta) = \theta(w) + \phi(v)$. However, $\rho(v + \theta) = a(v + \theta)$ and $\rho(w + \phi) = a(w + \phi)$, so we have already proven the above condition in our construction of $C\ell(V \oplus V^*)$. We conclude that ρ is well-defined. $\qquad\square$

$$\begin{array}{c} T(V \oplus V^*) \\ \downarrow \qquad \searrow^{\rho'} \\ C\ell(V \oplus V^*) \xrightarrow{\ \rho\ } \mathrm{Lin}(\Lambda V) \end{array}$$

We now have the representation $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$. We will show that it's faithful and irreducible. We start with proving the faithfulness.

Proposition 4.3.3. *The representation $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ is faithful.*

Proof. First of all, remark that a creation operator is never equal to the null-map, as $a^+(v)(\lambda) = \lambda v$ for any $\lambda \in \mathbb{C}$. We see that a product of creation operators $a^+(v_i), 1 \leq i \leq k$ for $k \in \mathbb{N}$ can only be zero if $\lambda v_1 \wedge \cdots \wedge v_k = 0$ for $\lambda \in \mathbb{C}$, which is true if and only if the v_i are not linearly independent. However, that means we can write v_k as a linear combination of $v_1, \dots v_{k-1}$. Since the $v_i \cdot v_i = Q(v_i) = 0$ an the v_i can be brought next to each other using (anti)commutation, we conclude that $v_1 \cdot \ldots \cdot v_k = 0$. We see that a product of creation operators can only be equal to 0 if the product of the corresponding vectors is also 0. This implies that any element in the kernel of ρ must contain annihilation operators.

Secondly, the anti-commutation relations imply we can always sort our products of operators into creation and annihilation operators, although that will separate a product of operators into a sum of products. We see this as follows. For any $v \in V, \theta \in V^*$ and X, Y product of operators, we have

$$Xa^-(\theta)a^+(v)Y = -Xa^+(v)a^-(\theta)Y + \theta(v)XY.$$

Now on the right we have two terms, but both are a step closer to being sorted. Using this step repeatedly[2], we find that after finite steps any product is equal to a sum of sorted products of the form A^+A^-, with A^+ a product of creation operators and A^- a product of annihilation operators. Using the exact same reasoning,

---

[2] The specific sorting algorithm we use is 'Bubble sort'.

we can sort elements of $C\ell(V \oplus V^*)$ into a sum of products starting with elements of V and ending with elements of V^*.

Finally, let $x \in \ker \rho$. Assume, to the contrary, that x is non-zero. Using the above consideration, we write x as a sum of sorted products, call these products 'words'. Remark that this implies that $\rho(x)$ is also a sum of sorted products. Now $\rho(x) = 0$, so $\rho(x)(1) = 0$. This implies that each sorted product of $\rho(x)$ ends with at least one annihilation operator, so each word ends in at least 1 element of V^*. Moreover, $\rho(x)(v) = 0$ for all $v \in V$, so each sorted product ends in two annihilation operators, so each word ends with at least two elements of V^*. This continues in a inductive manner with elements $v_1 \wedge \cdots \wedge v_k$ giving us that each word ends in $k + 1$ annihilation operators, until we arrive at $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and find that each word of x should end with at least $n + 1$ elements of V^*. However, V^* is n-dimensional, so the $n + 1$ elements are not linearly independent, which in turn implies that $x = 0$. (Recall that $Q(\theta) = 0$ for any $\theta \in V^*$.) We find that $x = 0$, which is in contradiction with the assumption that x is non-zero. We conclude that ρ is injective, so it is faithful. $\square$

Remark 4.3.4. The dimension of $C\ell(V \oplus V^*)$ is 2^{2n} while the dimension of ΛV is 2^n so $\dim \mathrm{Lin}(\Lambda V) = (2^n)^2 = 2^{2n} = \dim C\ell(V \oplus V^*)$. Together with the injectivity proven in Proposition 4.3.3, we find that the representation ρ is surjective.

Next, we prove that ρ is irreducible.

Proposition 4.3.5. *The representation* $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ *is irreducible.*

Proof. We show that any non-zero invariant subspace U contains the unit, from which it follows immediately that $U = \Lambda V$ as ρ is surjective.

Let $e_1, \ldots, e_n$ be a basis for V. Now $1, e_i, e_i \wedge e_j, e_i \wedge e_j \wedge e_k, \ldots, e_1 \wedge \cdots \wedge e_n$ for $i < j < k < \ldots$ is a basis for ΛV. Let $a = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}, i_1 < i_2 < \cdots < i_j, j \in \mathbb{N}$ be an arbitrary element of the basis. Now there is a corresponding element b of the basis such that b is the product of all other e_i of the basis of V, so $a \wedge b = \pm e_1 \wedge e_2 \wedge \cdots \wedge e_n$.

Consider the linear map which sends $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ to 1 and each other element of the basis to 0. Since ρ is surjective, there is an element c of $C\ell(V \oplus V^*)$ such that $\rho(c)$ is this linear map.

Consider also the linear map which sends each element x of ΛV to $b \wedge x$. This sends any element of the basis of ΛV with length j or longer to 0, except a which it sends to $\pm e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Since this map is also linear, there is once again a $d \in C\ell(V \oplus V^*)$ such that the linear map is $\rho(d)$.

The composition $\rho(cd)$ now is a map which sends a to ± 1 and any other element to 0.

Let $u \in U$ be non-zero. Write u in terms of the basis,

$$u = \sum_{j=1}^{n} \sum_{i_1 < i_2 < \cdots < i_j} \lambda_{i_1, i_2, \ldots, i_j} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}.$$

Because u is non-zero, there must be j and $e_{i_1}, \ldots, e_{i_j}$ such that $\lambda_{i_1, i_2, \ldots, i_j} \neq 0$. Let $a = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ and $\lambda = \lambda_{i_1, i_2, \ldots, i_j}$ for those j and e_i. Choose c, d as in the above. Now $\rho(cd)$ sends u to $\pm \lambda$, so $\lambda \in U$. But that implies $1 \in U$, as λ spans $\mathbb{C}$ because it is non-zero. We conclude that $U = \Lambda V$, so ΛV has no non-trivial invariant subspaces. $\square$

We have found that $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ is faithful and irreducible. It certainly follows that ρ is faithful and completely reducible, so we can use Theorem 3.2.9 to show that $C\ell(V \oplus V^*)$ is simple.

Theorem 4.3.6. *The Clifford algebra* $C\ell(V \oplus V^*)$ *is simple.*

Proof. We first apply Theorem 3.2.9 '3 $\to$ 1' to directly conclude that $C\ell(V \oplus V^*)$ is semisimple. Now assume that I is an ideal of $C\ell(V \oplus V^*)$. Since ρ is a isomorphism, $\rho(I)$ is an ideal of $\mathrm{Lin}(\Lambda V)$. However, that implies that $\rho(I)$ is an invariant subspace, so $\rho(I) = 0$ or $\rho(I) = \mathrm{Lin}(\Lambda V)$. In the first case $I = 0$, while in the second

case $I = C\ell(V \oplus V^*)$ since ρ is faithful. We conclude that any ideal I is either 0 or the whole algebra, so $C\ell(V \oplus V^*)$ is simple. $\qquad\square$

Corollary 4.3.7. *The even-dimensional Clifford algebra $C\ell(2n, \mathbb{C})$ is simple.*

Proof. By Remark 4.2.1 we find that the complex Clifford algebra $C\ell(V \oplus V^*)$ is isomorphic to $C\ell(2n, \mathbb{C})$ as both have $2n$ dimensions. Here the isomorphism is given by a change of basis from $V \oplus V^*$ to $\mathbb{C}^{2n}$. This isomorphism sends (nilpotent) ideals in $C\ell(V \oplus V^*)$ to (nilpotent) ideals in $C\ell(2n, \mathbb{C})$ and the inverse change of basis does the same in the reverse direction. We conclude that $C\ell(2n, \mathbb{C})$ must be simple since $C\ell(V \oplus V^*)$ is simple. $\qquad\square$

We have found that $C\ell(2n, \mathbb{C})$ is simple. Additionally, we have an isomorphism to $C\ell(V \oplus V^*)$ and a faithful irreducible representation $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$, so the composition is a faithful irreducible representation $C\ell(2n, \mathbb{C}) \to \mathrm{Lin}(\Lambda V)$. This representation is often called the <u>spinorial representation</u>, although others use that name for the restriction of this representation to a certain substructure. (See Section 4.4 and Chapter 5.)
Now Corollary 3.2.15 implies that any other irreducible representation is equivalent to $C\ell(2n, \mathbb{C}) \to \mathrm{Lin}(\Lambda V)$ and therefore faithful. With this, we have determined the only irreducible representations of $C\ell(2n, \mathbb{C})$.

4.3.2 Complex Clifford algebras of odd dimension

Now consider odd dimensional complex Clifford algebras $C\ell(2n + 1, \mathbb{C})$ for $n \in \mathbb{N}$ or $n = 0$. The proof of Theorem 4.2.3 already implies that odd dimensional Clifford algebras might have non-trivial ideals, generated by terms like $k + k_{12\ldots n}e_1 e_2 \ldots e_n$. We use such an ideal to show that $C\ell(2n + 1, \mathbb{C})$ is not simple.

Proposition 4.3.8. *The complex Clifford algebra $C\ell(2n + 1, \mathbb{C})$ is not simple.*

Proof. Consider $e_0' = e_1 e_2 \ldots e_{2n+1}$. Note that $(e_0')^2 = \pm 1$, as

$$
\begin{aligned}
(e_0')^2 &= e_1 e_2 \ldots e_{2n+1} e_1 e_2 \ldots e_{2n+1} \\
&= (-1)^{2n}(e_1)^2 \cdot e_2 e_3 \ldots e_{2n+1} e_2 e_3 \ldots e_{2n+1} \\
&= (-1)^{2n}(-1)^{2n-1}(e_1)^2(e_2)^2 \cdot e_3 e_4 \ldots e_{2n+1} e_3 e_4 \ldots e_{2n+1} \\
&= (-1)^{2n+(2n-1)+\cdots+0}(e_1)^2(e_2)^2 \ldots (e_{2n+1})^2 \\
&= (-1)^{n(2n+1)}(1)^{2n+1} = (-1)^{n(2n+1)} \\
&= \pm 1
\end{aligned}
$$

so either $(e_0')^2 = 1$ (if n is even) or $(e_0'\mathrm{i})^2 = 1$ (if n is odd). Let e_0 be e_0' or $e_0'\mathrm{i}$ such that $e_0^2 = 1$. We will show that $1 + e_0$ generates a non-trivial ideal of $C\ell(2n + 1, \mathbb{C})$.
First of all, note that e_0 commutes with e_i for every i, as e_0 has $2n + 1$ terms, one of which is e_i and all the other $e_j, i \neq j$, anti-commute with e_i. Therefore, e_0 commutes with every element of $C\ell(2n + 1, \mathbb{C})$. We find that $1 + e_0$ commutes with every element of $C\ell(2n + 1, \mathbb{C})$.
Now consider the ideal $I = C\ell(2n + 1, \mathbb{C}) \cdot (1 + e_0) \cdot C\ell(2n + 1, \mathbb{C})$. Since $1 + e_0$ commutes, this is the same as $I = C\ell(2n + 1, \mathbb{C}) \cdot (1 + e_0)$. Now

$$
I(1 - e_0) = C\ell(2n + 1, \mathbb{C}) \cdot (1 + e_0)(1 - e_0) = C\ell(2n + 1, \mathbb{C}) \cdot 0 = 0.
$$

But $1 \cdot (1 - e_0) = (1 - e_0) \neq 0$, so 1 is not an element of I. Moreover, $1 + e_0$ is in I, so I is non-zero. We conclude that I is a non-trivial ideal of $C\ell(2n + 1, \mathbb{C})$. $\qquad\square$

In the above proof, we have constructed an e_0 and generated an ideal with $(1 + e_0)$. Call this ideal I_+. It is clear that I_+ is also generated by the idempotent $\frac{1}{2}(1 + e_0)$, and that a similar ideal I_- is generated by the

idempotent $\frac{1}{2}(1-e_0)$. Since $\frac{1}{2}(1+e_0)+\frac{1}{2}(1-e_0) = 1$ and $\frac{1}{2}(1+e_0)\cdot\frac{1}{2}(1-e_0) = \frac{1}{2}(1-e_0)\cdot\frac{1}{2}(1+e_0) = 0$, we find that $Cl(2n+1,\mathbb{C})$ is the direct sum of the ideals generated by $(1+e_0)$ and $(1-e_0)$; $Cl(2n+1,\mathbb{C}) = I_+ \oplus I_-$. Also note that $1\cdot(1+e_0) = 1+e_0 = e_0(1+e_0)$, so 1 and e_0 are 'the same' in I_+. Therefore, the ideal generated by $(1+e_0)$ is isomorphic to the quotient $Cl(2n+1,\mathbb{C})/(1-e_0) := Cl(2n+1,\mathbb{C})/I_-$. This also follows from the direct sum. The explicit isomorphism is given by $Cl(2n+1,\mathbb{C})/(1-e_0) \to I_+, x+I_- \mapsto x(1+e_0)$.

In the space $Cl(2n+1,\mathbb{C})/(1-e_0)$ we have the relation $1 = e_0$ (modulo I_-). This implies that $e_1,\ldots,e_{2n+1}$ are no longer independent, but bound by a single condition. However, for any $e_1\ldots e_{2n}$ we can always replace e_{2n+1} by $\mathrm{i}^k e_{2n}e_{2n-1}\ldots e_1$ for a certain $k \in \{0,1,2,3\}$, which guarantees that $e_0 = 1$. This implies that we can make a map from $Cl(2n+1,\mathbb{C})/(1-e_0)$ to $Cl(2n,\mathbb{C})$ by sending $e_i \mapsto d_i$ for $1 \le i \le 2n$ and $e_{2n+1} \mapsto \iota d_{2n}d_{2n-1}\ldots d_1$ with $\iota = \pm 1$ or $\iota = \pm\mathrm{i}$. Here we use $d_1,\ldots,d_{2n}$ as the basis of $\mathbb{C}^{2n}$ (instead of e_i) to avoid confusion.

Proposition 4.3.9. *The map $\phi : Cl(2n+1,\mathbb{C})/(1-e_0) \to Cl(2n,\mathbb{C})$ which sends e_i to d_i for $1 \le i \le 2n$ and e_{2n+1} to $\iota d_{2n}\ldots d_1$ where $\iota = 1$ if n is even and $\iota = -\mathrm{i}$ if n is odd, is a well-defined isomorphism.*

Proof. Let $\iota = 1$ if n is even and $\iota = -\mathrm{i}$ if n is odd, and remark that $\iota e_0 = e_1 e_2 \ldots e_{2n+1}$ by construction. First consider the map $\Phi : Cl(2n+1,\mathbb{C}) \to Cl(2n,\mathbb{C})$ defined by $e_i \mapsto d_i$ and $e_{2n+1} \mapsto \iota d_{2n}\ldots d_1$. This sends each of the (independent) generators of $Cl(2n+1,\mathbb{C})$ to elements of $Cl(2n,\mathbb{C})$, and extends linearly. Moreover, by definition of the map, $\Phi(ab) = \Phi(a)\Phi(b)$ for $a,b \in Cl(2n+1,\mathbb{C})$, so this map is a well-defined homomorphism.
Consider the image of ιe_0 under this map. We have

$$\Phi(\iota e_0) = \Phi(e_1)\Phi(e_2)\ldots\Phi(e_{2n})\Phi(e_{2n+1}) = d_1 d_2 \ldots d_{2n}(\iota d_{2n}\ldots d_2 d_1) = \iota 1^{2n} = \iota,$$

so $\Phi(e_0) = 1$. Also, Φ sends $e_1 \mapsto d_1$, so $1 = e_1^2 \to 1 = d_1^2$ i.e. $\Phi(1) = 1$. We see that $\Phi(1) = \Phi(e_0)$, so $1 - e_0$ is in the kernel of Φ. It follows that I_- is in the kernel of Φ. Therefore, Φ induces a well-defined map $\phi : Cl(2n+1,\mathbb{C})/(1-e_0) \to Cl(2n,\mathbb{C})$. This map is precisely defined by $e_i \mapsto d_i$ and $e_{2n+1} \mapsto \iota d_{2n}\ldots d_1$, as required.
Secondly, note that for any element y of $Cl(2n,\mathbb{C})$, there is a corresponding element x of $Cl(2n+1,\mathbb{C})$ which we get by substituting all d_j with e_j for $1 \le j \le n$. Clearly, $\Phi(x) = y$ as the e_j are once again send to d_j. We find that Φ is surjective. Because ϕ is induced by Φ and the domain of ϕ is the quotient of the domain of Φ by the kernel, we see that ϕ has the same image as Φ. It follows that ϕ is surjective. Note that $\dim Cl(2n+1,\mathbb{C}) = 2^{2n+1}$ and $\dim Cl(2n,\mathbb{C}) = 2^{2n}$, so the dimension of the kernel of Φ is $2^{2n+1} - 2^{2n} = 2^{2n}$. Now consider the same map Φ as above, but we send $e_{2n+1} \mapsto -\iota e_{2n}\ldots e_1$ instead. Now $(1+e_0)$ replaces the role of $(1-e_0)$, and the corresponding ideal is now I_+. In this new map, we once again find that the dimension of the kernel is 2^{2n}. Since $I_+ \oplus I_- = Cl(2n+1,\mathbb{C})$, the sum of the dimensions is 2^{2n+1}. The only way that this is possible is that $\dim I_+ = \dim I_- = 2^{2n}$. We find that the kernel of Φ is equal to I_-, so the kernel of ϕ is 0. We conclude that ϕ is injective.
Now ϕ is injective, surjective and a homomorphism, so ϕ is an isomorphism. $\qquad\square$

Corollary 4.3.10. *The Clifford algebra $Cl(2n+1,\mathbb{C})$ has two simple ideals, both isomorphic to $Cl(2n,\mathbb{C})$.*

Proof. We have found that I_+ is isomorphic to $Cl(2n+1,\mathbb{C})/(1-e_0)$, which is isomorphic to $Cl(2n,\mathbb{C})$, so I_+ is isomorphic to $Cl(2n,\mathbb{C})$. Similarly, I_- is isomorphic to $Cl(2n,\mathbb{C})$. We conclude that $Cl(2n+1,\mathbb{C})$ has two simple ideals, both isomorphic to $Cl(2n,\mathbb{C})$. $\qquad\square$

We have found our decomposition of $Cl(2n+1,\mathbb{C})$ into simple ideals. Since any irreducible representation corresponds to a minimal left ideal and $Cl(2n,\mathbb{C})$ had a single irreducible representation (up to equivalence), we find that $Cl(2n+1,\mathbb{C})$ has two irreducible representations up to equivalence: one for each simple ideal, given by the composition of the isomorphism with the representation ρ from the previous subsection. For a

n-dimensional space V, we can write the two irreducible representation as

$$\rho_+ : C\ell(2n+1, \mathbb{C}) \to \text{Lin}(\Lambda V), \ e_0 \mapsto \mathbb{1}$$

and

$$\rho_- : C\ell(2n+1, \mathbb{C}) \to \text{Lin}(\Lambda V), \ e_0 \mapsto -\mathbb{1}$$

where $\mathbb{1}$ denotes the identity map on ΛV and all other elements are fixed by the choice of e_0.

4.3.3   Real Clifford algebras

We now know that all complex Clifford algebras are semisimple and that the even-dimensional ones are simple. Like we said in Section 3.3, we want to use the complexification of real Clifford algebras to show that they are also semisimple. However, the complexification of a real Clifford algebra is certainly a complex algebra, but it's not trivial that the complexification is once again a Clifford algebra. We will need to show that first.

Proposition 4.3.11. *The complexification of a real Clifford algebra is isomorphic to a complex Clifford algebra.*

Proof. Let V be a real vector space and $Q : V \to \mathbb{R}$ be a non-degenerate quadratic form on V. Let $V_\mathbb{C} = V \otimes \mathbb{C}$ be the complexified vector space (which we see as a vector space over $\mathbb{C}$) and let $Q_\mathbb{C} : V_\mathbb{C} \to \mathbb{C}$ be the complex extension of the quadratic form, defined by $Q_\mathbb{C}(v \otimes z) = z^2 Q(v)$ extended linearly to the rest of $V_\mathbb{C}$. Remark that $Q_\mathbb{C}$ now corresponds to a complex bilinear form $\beta_\mathbb{C} : V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}, \beta_\mathbb{C}(v \otimes z_1, w \otimes z_2) = z_1 z_2 \beta(v, w)$, again extended linearly.

Now consider the complex Clifford algebra $C\ell(V_\mathbb{C}, Q_\mathbb{C})$. This is, by definition, equal to $TV_\mathbb{C}/I_\mathbb{C}$ where $I_\mathbb{C}$ is the ideal generated by

$$\{(v \otimes z) \otimes (v \otimes z) - Q_\mathbb{C}(v \otimes z)(1 \otimes 1) \mid v \in V, z \in \mathbb{C}\},$$

so the entire space is

$$\bigoplus_{k=0}^{\infty} (V \otimes \mathbb{C})^k / \{(v \otimes z) \otimes (v \otimes z) - Q_\mathbb{C}(v \otimes z)(1 \otimes 1) \mid v \in V, z \in \mathbb{C}\}.$$

Remark that $1 \otimes 1$ is the unit of $C\ell(V_\mathbb{C}, Q_\mathbb{C})$.

Also consider the complexified Clifford algebra $C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C}$, which is given by

$$\Big(\bigoplus_{k=0}^{\infty} V^k / \{v \otimes v - Q(v)1 \mid v \in V\} \Big) \otimes_\mathbb{R} \mathbb{C}.$$

We first define $\Phi : \bigoplus_{k=0}^{\infty} (V \otimes \mathbb{C})^k \to C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C}$ by

$$(v_1 \otimes z_1) \otimes (v_2 \otimes z_2) \otimes \ldots \otimes (v_k \otimes z_k) \mapsto (v_1 \cdot v_2 \cdot \ldots \cdot v_k) \otimes (z_1 z_2 \ldots z_k),$$

extended linearly. Since the quotient map $TV \to TV/I$ is a well-defined homomorphism, Φ is also a well-defined homomorphism. Now consider the image of $(v \otimes z) \otimes (v \otimes z)$ for $v \in V, z \in \mathbb{C}$:

$$\Phi((v \otimes z) \otimes (v \otimes z)) = (v \cdot v) \otimes z^2 = Q(v)(1 \otimes z^2) = z^2 Q(v)(1 \otimes 1) = Q_\mathbb{C}(v \otimes z)(1 \otimes 1).$$

It follows that $(v \otimes z) \otimes (v \otimes z) - Q_\mathbb{C}(v \otimes z)(1 \otimes 1)$ is in the kernel of Φ for each $v \in V, z \in \mathbb{C}$. Since the kernel of a homomorphism is an ideal, it follows that $I_\mathbb{C}$ is contained in the kernel, so we can quotient out $I_\mathbb{C}$ in the

domain. This induces a map

$$\phi : TV_{\mathbb{C}}/I_{\mathbb{C}} = C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}}) \to C\ell(V, Q) \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

This map is surjective, since for any $v_1 \cdot v_2 \cdot \ldots \cdot v_k \in C\ell(V, Q), k \in \mathbb{N}$ we have $\phi((v_1 \otimes 1) \cdot (v_2 \otimes 1) \cdot \ldots \cdot (v_k \otimes 1)) = (v_1 \cdot v_2 \cdot \ldots \cdot v_k) \otimes 1$, $\phi(1 \otimes 1) = 1 \otimes 1$ and ϕ is linear. Since $\dim_{\mathbb{C}} C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}}) = 2^{\dim_{\mathbb{C}} V_{\mathbb{C}}} = 2^{\dim V}$ and $\dim_{\mathbb{C}} C\ell(V, Q) \otimes_{\mathbb{R}} \mathbb{C} = \dim_{\mathbb{C}} C\ell(V, Q) \cdot \dim_{\mathbb{C}} \mathbb{C} = 2^{\dim V}$, we find that ϕ is also injective.
Now ϕ is injective, surjective and a homomorphism, so $\phi : C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}}) \to C\ell(V, Q) \otimes \mathbb{C}$ is an isomorphism. We conclude that a complexified Clifford algebra is isomorphic to a complex Clifford algebra. $\square$

We use this to determine whether $C\ell(p, q)$ is (semi)simple.
First let $p, q \in \mathbb{N}$, and consider $C\ell(p, q)$. We have just seen that $C\ell(p, q) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $C\ell(p + q, \mathbb{C})$, so if $p + q$ is even it follows that $C\ell(p, q)$ is simple. Since $C\ell(2n, \mathbb{C})$ and $\mathrm{Lin}(\Lambda V)$ are isomorphic as complex algebras, they are also isomorphic as real algebras, so the case that $p + q$ is even, $C\ell(p, q)$ is isomorphic to its image in $\mathrm{Lin}(\Lambda V)$. Therefore, the unique irreducible representation of $C\ell(p, q)$ can be found in the restriction of the representation of $C\ell(2n, \mathbb{C})$.

Secondly consider $C\ell(p, q)$ with $p + q$ odd, so $p + q = 2k + 1$. Since $C\ell(2k + 1, \mathbb{C})$ is semisimple, it directly follows that $C\ell(p, q)$ is semisimple. However, the ideals of $C\ell(2k + 1, \mathbb{C})$ don't have to appear in $C\ell(p, q)$, as they might be generated by mixed elements. The only two ideals of $C\ell(2k + 1, \mathbb{C})$ are generated by $1 \pm e_0$, where $e_0 = e_1 e_2 \ldots e_{2k+1}$ if k is even and $e_0 = i e_1 e_2 \ldots e_{2k+1}$ if k is odd. It is clear that i is not an element of $C\ell(p, q)$. However, the square of $e_0 := e_1 e_2 \ldots e_{2k+1}$ no longer just depends on the anti-commutativity of the e_j, but also on the fact that $e_i^2 = -1$ for $p + 1 \le i \le p + 1$. If $e_0^2 = 1$, $1 + e_0$ generates a non-trivial ideal, while if $e_0^2 = -1$, the ideal generated by $1 + e_0$ is trivial. This is the case because $(1 + e_0)(1 - e_0) = 1 - e_0^2$ has an inverse if and only if $e_0^2 = -1$. Therefore, we calculate the square:

$$\begin{aligned}
(e_0)^2 &= e_1 e_2 \ldots e_n e_1 e_2 \ldots e_n \\
&= (-1)^{n-1}(e_1)^2 \cdot e_2 e_3 \ldots e_n e_2 e_3 \ldots e_n \\
&= (-1)^{n-1}(-1)^{n-2}(e_1)^2(e_2)^2 \cdot e_3 e_4 \ldots e_n e_3 e_4 \ldots e_n \\
&= (-1)^{(n-1)+(n-2)+\cdots+0}(e_1)^2(e_2)^2 \ldots (e_n)^2 \\
&= (-1)^{n(n-1)/2}(1)^p(-1)^q \\
&= (-1)^{(p+q)(p+q-1)/2+q}.
\end{aligned}$$

We separate four cases: p odd, q even and $(p + q - 1)/2$ odd; p even, q odd and $(p + q - 1)/2$ odd; p odd, q even and $(p + q - 1)/2$ even; and p even, q odd and $(p + q - 1)/2$ even. These cases give $(e_0)^2$ equal to -1, 1, 1, and -1 respectively. It turns out these cases are most easily characterised by $p - q \mod 4$, where they correspond to 3, 1, 1 and 3. We see that $p - q \equiv 1 \mod 4$ implies $(e_0)^2 = 1$ while $p - q \equiv 3 \mod 4$ implies $(e_0)^2 = -1$ Therefore, if $p - q \equiv 1 \mod 4$ then $C\ell(p, q)$ is only semisimple, while if $p - q \equiv 3$ then $C\ell(p, q)$ is simple. The irreducible representations of $C\ell(p, q)$ for $p + q$ odd can be found as a part of the spinorial representations ρ_+, ρ_- of $C\ell(p + q, \mathbb{C})$.
Note that we have seen that in the case $p - q \equiv 3 \mod 4$ there is an e_0 which squares to -1 and commutes with all other elements. This e_0 can be seen as the i that generates $\mathbb{C}$. This allows us to define a scalar product $\mathbb{C} \times C\ell(p, q) \to C\ell(p, q), (a + bi, x) \mapsto (a + be_0)x$ for $a, b \in \mathbb{R}$. Under this scalar product, $C\ell(p, q)$ with $p - q \equiv 3 \mod 4$ becomes a complex algebra.
Our conclusions are in Table 1. This table, along with the earlier representations ρ, ρ_+, ρ_- for the complex cases, determines all irreducible representations of Clifford algebras of the form $C\ell(p, q)$ or $C\ell(n, \mathbb{C})$. Since any Clifford algebra is isomorphic to a Clifford algebra of this form, we have indirectly determined all irreducible representations of real or complex Clifford algebras.

$p - q \mod 4$	$C\ell(p, q)$	$C\ell(p + q, \mathbb{C})$
0	simple	simple
1	semisimple	semisimple
2	simple	simple
3	simple	semisimple

Table 1: Simplicity of Clifford algebras

The case distinction to determine the precise form of the real Clifford algebras is tedious (it relies on the case separation of $p - q \mod 8$) but not terribly complicated, so will not go into that. The complete derivation can be found in Bilge, Kocak, and Uguz [5]. To give an idea of the process, we will show the most important theorem (Lemma 2.4 in [5]).

Theorem 4.3.12. *There are isomorphisms*

$$C\ell(2, 0) \underset{\mathbb{R}}{\otimes} C\ell(p, q) \cong C\ell(q + 2, p)$$

$$C\ell(1, 1) \underset{\mathbb{R}}{\otimes} C\ell(p, q) \cong C\ell(p + 1, q + 1)$$

$$C\ell(0, 2) \underset{\mathbb{R}}{\otimes} C\ell(p, q) \cong C\ell(q, p + 2)$$

For integers $p, q \geq 0$.

Proof. Call the first p elements of the basis of $C\ell(p, q)$ a_i for $1 \leq i \leq p$ and call the next q elements b_j for $1 \leq j \leq q$, so $(a_i)^2 = 1$ and $(b_j)^2 = -1$. Similarly, let $(d_1)^2 = (d_2)^2 = 1$, $(e_1)^2 = (e_2)^2 = -1$ such that $C\ell(2, 0)$ is generated by d_1, d_2, $C\ell(1, 1)$ by d_1, e_1 and $C\ell(0, 2)$ by e_1, e_2. Now $(d_1 e_1)^2 = 1$ due to anti-commutativity, while $(d_1 d_2)^2 = (e_1 e_2)^2 = -1$.

First consider $C\ell(1, 1) \otimes_{\mathbb{R}} C\ell(p, q)$. This algebra is generated by the tensor product of both bases. We want to show an isomorphism to $C\ell(p + 1, q + 1)$, we do this via the basis. We therefore need to find $p + 1$ elements such that their square is equal to 1 and $q + 1$ elements with a square equal to -1, which all anti-commute. A first attempt could be $1 \otimes a_i, d_1 \otimes 1$ as the first $p + 1$ elements and $1 \otimes b_i, e_1 \otimes 1$ as the last $q + 1$ elements, but these elements do not anti-commute as required. We know, however, that $(d_1 e_1)^2 = 1$ and d_1, e_1 both anti-commute with $d_1 e_1$. We therefore choose $(d_1 e_1) \otimes a_i, d_1 \otimes 1$ for $1 \leq i \leq p$ as the first $p + 1$ elements and $(d_1 e_1) \otimes b_i, e_1 \otimes 1$ for $1 \leq i \leq q$ as the last $q + 1$ elements. It is clear that these elements anti-commute and have correct squares, therefore they are isomorphic to the basis of $C\ell(p + 1, q + 1)$. This induces our isomorphism.

Secondly consider $C\ell(2, 0) \otimes_{\mathbb{R}} C\ell(p, q)$. We want to do something similar here, but we now have $(d_1 d_2)^2 = -1$ instead. Therefore, the square of $(d_1 d_2) \otimes a_i$ is -1, which means it is now part of the 'last (...)' elements. We find that there are now $q + 2$ elements with square 1, which are $(d_1 d_2) \otimes b_j, 1 \leq j \leq q$ and $1 \otimes d_1, 1 \otimes d_2$, and that there are p elements with square -1, namely $(d_1 d_2) \otimes a_i, 1 \leq i \leq p$. This gives us an isomorphism to $C\ell(q + 2, p)$.

Finally, we do the same for $C\ell(0, 2) \otimes_{\mathbb{R}} C\ell(p, q)$ to get q first elements $(e_1 e_2) \otimes b_j, 1 \leq j \leq q$ and $p + 2$ last elements $(e_1 e_2) \otimes a_i, 1 \leq i \leq p$ and $1 \otimes e_1, 1 \otimes e_2$. We conclude there is an isomorphism to $C\ell(q, p + 2)$. $\square$

Using these isomorphisms, we only need to know a few small Clifford algebras ($p, q \leq 3$) to generate all real Clifford algebras inductively [5]. The irreducible representations of real Clifford algebras can be generated using a similar construction, where we take a representation of $C\ell(p, q)$ and tensor it with the representation of $C\ell(1, 1)$, $C\ell(2, 0)$, or $C\ell(0, 2)$ (the Pauli matrices).

We now know the (semi-)simplicity of both real and complex Clifford algebras, and know how to determine their exact structures. This means we can conclude we know 'everything' about the Clifford algebras and their representations.

Although a representation of a Clifford algebra induces a representation for all its substructures, there might still be substructures with more irreducible representations. We will investigate them next.

4.4   Substructures of the Clifford algebra

We will now consider some relevant substructures of the Clifford algebra. We will not consider any subalgebras, as we have already determined the ideals. Instead, we will find groups and Lie algebras inside the Clifford algebra.

As before, let V be a n-dimensional vector space ($n \geq 1$) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and Q a non-degenerate quadratic form on V. We will use the orthogonality of the basis of V, so we assume $C\ell(V, Q)$ is in standard form ($C\ell(p, q)$ or $C\ell(n, \mathbb{C})$). We will first consider the case $C\ell(n)$, then $C\ell(p, q)$ and finally $Cl(n, \mathbb{C})$.

4.4.1   (S)Pin groups in $C\ell(n)$

We first consider $C\ell(n)$, so $V = \mathbb{R}^n$. In this case, β corresponds to the inner product on $\mathbb{R}^n$ while Q corresponds to the square of the norm.

It is clear that all invertible elements of the Clifford algebra form a group $C\ell^\times(n)$ and that any subgroup G of $C\ell(n)$ must be a subgroup of $C\ell^\times(n)$. We follow the construction from Atiyah, Bott, and Shapiro [6], which uses homomorphisms to find subgroups of $C\ell^\times(n)$. For these homomorphisms, we first define an automorphism and anti-automorphism on $C\ell(n)$.

Recall that $C\ell(n)$, as a vector space, had a graded structure $C\ell(n) = C\ell^0(n) \oplus C\ell^1(n) \oplus \cdots = C\ell^{even}(n) \oplus C\ell^{odd}(n)$.

Definition 4.4.1. Let $\cdot^t$ denote the map

$$\cdot^t : C\ell^k(n) \to C\ell^k(n), \ (v_1 v_2 \ldots v_k)^t = v_k v_{k-1} \ldots v_1,$$

extended to $C\ell(n)$ by the direct sum. We call this map the <u>transpose</u> map. The transpose map $\cdot^t$ sends the 'empty product' $\lambda \in \mathbb{R}$ to itself.

Note that the transpose map sends elements $vw + wv$ to itself, and also sends $\beta(v, w)$ to itself, for any $v, w \in V$. Therefore, the elements of the ideal (which defines the Clifford algebra) are all invariant under the transpose map. This guarantees that the transpose map is well-defined on products $x \cdot y$ where x, y contain linearly dependent vectors.

It is clear that the transpose map is an anti-homomorphism since

$$\big((v_1 v_2 \ldots v_l)(w_1 w_2 \ldots w_k)\big)^t = w_k w_{k-1} \ldots w_1 v_l v_{l-1} \ldots v_1 = (w_1 w_2 \ldots w_k)^t (v_1 v_2 \ldots v_l)^t$$

for $l, k \in \mathbb{N}$ and v_i, w_j elements of V, while 'empty products' commute with products. Moreover, $(x^t)^t = x$ for any $x \in C\ell(n)$, so it is bijective. We see that $\cdot^t$ is an anti-automorphism.

Definition 4.4.2. The <u>canonical automorphism</u> is the map

$$\alpha : C\ell^{even}(n) \oplus C\ell^{odd}(n) \to C\ell(n), \ \alpha(x + y) = x - y$$

where x is a sum of elements of even grade and y a sum of elements of odd grade.

The canonical automorphism can also be described by $1 \mapsto 1, v \mapsto -v$ for $v \in V$. We again see that $\alpha(\alpha(x)) = x$ for $x \in C\ell(n)$. Additionally, if $x, x' \in C\ell^{even}(n)$ and $y, y' \in C\ell^{odd}(n)$ then $xx', yy' \in C\ell^{even}(n)$ and $xy', x'y \in C\ell^{odd}(n)$, such that

$$\alpha\big((x + y)(x' + y')\big) = \alpha((xx' + yy') + (xy' + x'y)) = (xx' + yy') - (xy' + x'y)$$
$$= (x - y)(x' - y') = \alpha(x + y)\alpha(x' + y'),$$

which shows us that α is a homomorphism. We find that α is an automorphism.

From the formulas of $\cdot^t$ and α, we directly see that $\alpha(x^t) = \alpha(x)^t$ for $x \in C\ell(n)$ which implies α and $\cdot^t$ commute. For clarity, we introduce the shorter notation $\bar{x} = \alpha(x)^t$ for $x \in C\ell(n)$. Because the composition of an anti-automorphism with an automorphism is an anti-automorphism, the operation $x \mapsto \bar{x}$ is an anti-automorphism.

Remark 4.4.3. For any $x \in C\ell(n)$, if x is invertible, then $\alpha(x)$, x^t and $\bar{x}$ are invertible, with inverses given by $\alpha(x^{-1})$, $(x^{-1})^t$ respectively $\overline{x^{-1}}$. Therefore, α, $\cdot^t$ and $\bar{}$ are well-defined (anti-)automorphisms on $C\ell^\times(n)$.

We now define our first subgroup, which uses the equality $V = C\ell^1(n)$ to multiply elements of V with elements of $C\ell(n)$.

Definition 4.4.4. Let Γ be the subgroup of $C\ell^\times(n)$ of elements x such that $\alpha(x) \cdot v \cdot x^{-1} \in V$ for all $v \in V$.

This is a well-defined group because α is an automorphism and $x \mapsto x^{-1}$ is an anti-automorphism on $C\ell^\times(n)$. Remark that Γ is non-empty, as $\lambda \in \Gamma$ for non-zero $\lambda \in \mathbb{R}$. It also contains other elements, as we will show.

Proposition 4.4.5. *For any invertible $v \in V$, we have $v \in \Gamma$.*

Proof. Let $v \in V$ have inverse v^{-1} and let $w \in V$ be arbitrary. Remark that $Q(v) \neq 0$ as $v = v \cdot v \cdot v^{-1} = Q(v)v^{-1}$. Note that $\alpha(v) \cdot w \cdot v^{-1} = -v \cdot w \cdot v^{-1}$ by definition of α. Now $w \cdot v + v \cdot w = 2\beta(v,w)$ where β is the bilinear form of the Clifford algebra, so $v \cdot w \cdot v^{-1} + w \cdot v \cdot v^{-1} = 2\beta(v,w)v^{-1}$. Using that $v \cdot v^{-1} = 1$ and $v^{-1} = \frac{1}{Q(v)}v$, we find

$$-v \cdot w \cdot v^{-1} = w - 2\beta(v,w)v^{-1} = w - 2\frac{\beta(v,w)}{Q(v)}v,$$

which is a linear combination of v and w and therefore an element of V. Since w was arbitrary, we conclude $\alpha(v) \cdot w \cdot v^{-1} \in V$ for all $w \in V$, and therefore $v \in \Gamma$. $\qquad\square$

Since the basis-vectors $e_i, 1 \leq i \leq n$ are invertible (with inverse $\pm e_i$) we conclude that $e_i \in \Gamma, 1 \leq i \leq n$.

By construction, it is clear that the map $\Gamma \times V \to V, (x,v) \mapsto \alpha(x) \cdot v \cdot x^{-1}$ is well-defined. We also see that $\alpha(y)(\alpha(x) \cdot v \cdot x^{-1})y^{-1} = \alpha(yx) \cdot v \cdot (yx)^{-1}$ for $x, y \in \Gamma, v \in V$, so the map $\Gamma \to \mathrm{GL}(V)$ is a homomorphism. This gives us a well-defined representation of Γ.

Corollary 4.4.6. *The map $\rho : \Gamma \to \mathrm{GL}(V), \rho(x) = \big(v \mapsto \alpha(x) \cdot v \cdot x^{-1}\big)$ is a well-defined group-representation of Γ over V.*

We call this representation the <u>twisted adjoint representation</u>. We want to ascertain whether it is faithful and determine its reducibility. Unfortunately, the representation is clearly not faithful, as $\rho(\lambda)(v) = v$ for any non-zero $\lambda \in \mathbb{R}, v \in V$. In order to find a faithful representation, we calculate the kernel.

Proposition 4.4.7. *The kernel of $\rho : \Gamma \to \mathrm{GL}(V)$ is equal to $\mathbb{R}^* := \mathbb{R} \setminus 0$.*

Proof. Let $x \in C\ell^{even}(n), y \in C\ell^{odd}(n)$ such that $x+y \in \ker \rho$. Now $\rho(x+y) = \mathbb{1}_V$, so $\alpha(x+y) \cdot v \cdot (x+y)^{-1} = v$. We rewrite this as $x \cdot v - y \cdot v = v \cdot x + v \cdot y$ where we used the definition of α. Now $x \in C\ell^{even}(n)$ and $v \in C\ell^1(n) \subset C\ell^{odd}(n)$, so $x \cdot v \in C\ell^{odd}(n)$. Similarly $x \cdot v \in C\ell^{odd}(n)$, while $v \cdot y$ and $y \cdot v$ are in $C\ell^{even}(n)$. Therefore, the earlier condition is equivalent to $x \cdot v = v \cdot x$ and $-y \cdot v = v \cdot y$.

If $x \cdot v = v \cdot x$ for any $v \in V$, then it certainly holds for the elements $v = e_i, 1 \leq i \leq n$ of the basis of V. This gives $x \cdot e_i = e_i \cdot x$ for $1 \leq i \leq n$, so x commutes with all e_i. But as we have seen in the proof of Theorem 4.2.3, the fact that 1 has even length and that x has the same commutation relations as 1 implies that each term of x is linear in 1 or has odd grade. Now each term of x with grade 2 or higher has both odd and even grade, so those terms are 0. We find that x is linear in 1, so $x \in \mathbb{R}$.

Similarly, $-y \cdot e_i = e_i \cdot y$ for $1 \leq i \leq n$, so y anti-commutes with each e_i. Fix an e_i, and write y out in terms of the basis. Now each term must anti-commute with e_i, so each term contains an odd number of $e_j, i \neq j$.

But each term has odd grade, so that implies e_i does not occur in y when y is written out in terms of the basis. Since e_i was arbitrary, it follows that $y \in \mathbb{R}$. But y anti-commutes with e_1, so the only option is $y = 0$. We conclude that $x + y = \lambda$ for $\lambda \in \mathbb{R}$ that are in Γ. We already knew that $\mathbb{R}^*$ is in the kernel, and it is clear that 0 has no inverse, so the kernel of ρ is equal to $\mathbb{R}^*$. $\square$

We want to find a subgroup of Γ such that this representation is faithful, which means we want to 'divide Γ by $\mathbb{R}^*$'. We determine the subgroup using a homomorphism with image $\mathbb{R}^*$.

Let $N : C\ell(n) \to C\ell(n)$ be the map $x \mapsto x \cdot x^t$. For $v \in V$ we have $N(v) = v \cdot v^t = v \cdot v = Q(v)$ while for $\lambda \in \mathbb{R}^*$ we have $N(\lambda) = \lambda \cdot \lambda^t = \lambda^2$. This suggests that $N(\Gamma) \subset \mathbb{R}^*$. To prove this, however, we first need that $x^t \in \Gamma$ for any $x \in \Gamma$, to make sure N is well-defined as a map from Γ to Γ.

Proposition 4.4.8. *Let $x \in \Gamma$. Then $\alpha(x), x^t, \bar{x} \in \Gamma$.*

Proof. The condition '$\alpha(x) \cdot v \cdot x^{-1} \in V$ for any $v \in V$' is the same as saying $\alpha(x)Vx^{-1} \subseteq V$. We apply α to both sides and use that it is a homomorphism to find $xV\alpha(x)^{-1} \subseteq V$ or $\alpha(x) \in \Gamma$.
Since $v^t = v$ for any $v \in V$, taking the transpose on both sides gives $(x^{-1})^t V \alpha(x)^t \subseteq V$, so $(x^t)^{-1}$ and $\alpha(x^t)^{-1}$ are in V. Using that Γ is a group, we have $\left((x^t)^{-1}\right)^{-1} = x^t \in \Gamma$. Now $\bar{x} = \alpha(x^t)$ is also in Γ. We conclude that $\alpha(x), x^t, \bar{x} \in \Gamma$ for $x \in \Gamma$. $\square$

We can now talk about the map $N : \Gamma \to \Gamma$, which we use in the following proposition.

Proposition 4.4.9. *For any $x \in \Gamma$ we have $N(x) \in \mathbb{R}^*$.*

Proof. Let $x \in \Gamma$ and let $v \in V$ be arbitrary. We will prove $N(x) \in \ker \rho$. By definition of Γ, we have $\alpha(x) \cdot v \cdot x^{-1} = v'$ for some $v' \in V$. Taking the transpose and canonical automorphism of both sides and using that $\bar{v} = -v, \bar{v}' = -v'$, we have $\overline{x^{-1}}(-v)\overline{\alpha(x)} = (-v')$, or

$$(\bar{x})^{-1} \cdot v \cdot x^t = v' = \alpha(x) \cdot v \cdot x^{-1}.$$

Since v was arbitrary, this holds for all $v \in V$.
Now let $w \in V$ be arbitrary and consider $\rho(N(x))(w) = \alpha(N(x)) \cdot w \cdot N(x)^{-1}$. Since $N(x) = x \cdot x^t$, we have $\alpha(N(x)) = \alpha(x)\bar{x}$ and $N(x)^{-1} = (x^t)^{-1}x^{-1}$, so

$$\alpha(N(x)) \cdot w \cdot N(x)^{-1} = \alpha(x)\left(\bar{x} \cdot w \cdot (x^t)^{-1}\right)x^{-1} = (\bar{x})^{-1}\bar{x} \cdot w \cdot (x^t)^{-1}x^t = w$$

where we used the earlier equation for $v = \bar{x} \cdot w \cdot (x^t)^{-1}$. We conclude that $N(x)$ is in the kernel of ρ. But $\ker \rho = \mathbb{R}^*$, so $N(x) \in \mathbb{R}^*$. As x was arbitrary, we have proven that $N(x) \in \mathbb{R}^*$ for any $x \in \Gamma$. $\square$

We once again change the codomain of N to get $N : \Gamma \to \mathbb{R}^*$, which is well-defined by the above. We now directly see that $N(xy) = xyy^tx^t = xN(y)x^t = N(x)N(y)$ for $x, y \in \Gamma$, and that $N(\alpha(x)) = \alpha(x)\bar{x} = \alpha(xx^t) = \alpha(N(x)) = N(x)$ for $x \in \Gamma$, so $N : \Gamma \to \mathbb{R}^*$ is a homomorphism. We use these properties of N to show that $Q(\rho(x)(v)) = Q(v)$ for $x \in \Gamma, v \in V$.

Proposition 4.4.10. *Let $x \in \Gamma$ and $v \in V$, then $Q(\rho(x)(v)) = Q(v)$.*

Proof. By definition, $\rho(x)(v) = \alpha(x) \cdot v \cdot x^{-1}$. We have also seen that $N(v) = Q(v)$ for $v \in V$. Now $N(\alpha(x) \cdot v \cdot x^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = Q(v)N(x)N(x^{-1}) = Q(v)$, so $Q(\rho(x)(v)) = Q(v)$. $\square$

Recall that $Q(v) = \|v\|^2$ for $v \in V$. Since the norm is non-negative, we see that $\|\rho(x)(v)\| = \|v\|$, so $\rho(x)$ is a linear isometry for any $x \in \Gamma$.
Since $N : \Gamma \to \mathbb{K}^*$ is a homomorphism, we can take the kernel, which is a subgroup of Γ.

Definition 4.4.11. Let $\mathrm{Pin}(n)$ be the kernel of $N : \Gamma \to \mathbb{R}^*$. We call $\mathrm{Pin}(n)$ the <u>Pin group</u>.

Note that $1, -1 \in \text{Pin}(n)$ and $v \in \text{Pin}(n)$ for any v with $Q(v) = \|v\|^2 = 1$, as $vv^t = Q(v) = 1$ so v is in the kernel of N.

Let v be such an element in $\text{Pin}(n)$. In the proof of Proposition 4.4.5 we have seen that v acts on V by sending $w \mapsto w - 2\frac{\beta(v,w)}{Q(v)}v$ for $w \in V$. Using that β is the inner product and $Q(v) = 1$, we find

$$w \mapsto w - 2\langle v, w \rangle v.$$

We recognise this formula as the reflection of w in the hyperplane orthogonal to v. These reflections generate the group $O(n)$, so $\text{Pin}(n)$ acts on $\mathbb{R}^n$ via reflections and rotations. We already saw that each element of $\text{Pin}(n)$ acts as an isometry, so the image of $\text{Pin}(n)$ is precisely contained in $O(n)$.

However, this map from $\text{Pin}(n)$ to $O(n)$ will not be an isomorphism. After all, the formula shows that v and $-v$ act in the same way. This is also intuitively clear: the unique hyperplane orthogonal to v will certainly be orthogonal to $-v$. However, if we can prove that the map is surjective and determine its kernel, we will still know a lot about the exact structure of $\text{Pin}(n)$. We will show that the kernel of the map $\text{Pin}(n) \mapsto O(n)$ is given by $\{1, -1\}$.

Theorem 4.4.12. *There is a (well-defined) short exact sequence*

$$1 \to Z_2 \to \text{Pin}(n) \xrightarrow{\rho} O(n) \to 1$$

where $Z_2 \cong \{1, -1\} \subseteq \text{Pin}(n)$. In other words, the kernel of $\rho : \text{Pin}(n) \to O(n)$ is isomorphic to Z_2, while ρ is surjective.

Proof. First of all, note that the map $\rho : \text{Pin}(n) \to O(n)$ is well-defined, as we have seen that $\rho(x)$ is an isometry for any $x \in \text{Pin}(n)$ and $O(n)$ is the group of all isometries on $\mathbb{R}^n$. Furthermore, for any hyperplane there is an unit vector orthogonal to it. Since any unit vector is in $\text{Pin}(n)$, it is clear that ρ is surjective.

Now we calculate the kernel of $\rho : \text{Pin}(n) \to O(n)$. We have seen before that the kernel of $\rho : \Gamma \to \text{GL}(\mathbb{R}^n)$ is $\mathbb{R}^*$, so the kernel of $\rho : \text{Pin}(n) \to O(n)$ should be a subset of $\mathbb{R}^*$. The only two elements λ of $\mathbb{R}^*$ that satisfy $N(\lambda) = 1$ are $\lambda = 1$ and $\lambda = -1$, so we find that the kernel of ρ is $\{1, -1\}$. Since $\{1, -1\}$ is isomorphic to Z_2 as a group, we arrive the short exact sequence, where $1 \to Z_2$ is the trivial map, $Z_2 \to \text{Pin}(n)$ sends Z_2 to $\{1, -1\}$ and $O(n) \to 1$ is once again trivial. $\square$

It can be show that $\text{Pin}(n)$ is the universal cover of $O(n)$ for $n \geq 3$, but I will not go into that, as it uses a lot of preliminaries from algebraic topology. The main idea is that Z_2 is finite, so by defining a topology on $\text{Pin}(n)$ such that each x and $-x$ have disjoint neighbourhoods, we find that ρ is a double covering which induces a Lie structure on $\text{Pin}(n)$. This implies that the Lie algebra of $\text{Pin}(n)$ is the Lie algebra $\mathfrak{o}(n)$ of $O(n)$.

We now consider the separation of $\text{Pin}(n)$ in even and odd elements. We have seen that the $v \in V$ with $Q(v) = 1$ are send to reflections, and we know that an even number of reflections is a rotation, while an odd number of reflections is once again an reflection. Therefore $\text{Pin}^{even}(n)$ maps to $SO(n)$, the group of rotations.

Definition 4.4.13. We define $\text{Spin}(n) = \text{Pin}(n) \cap C\ell^{even}(n)$. This group is called the <u>Spin group</u>.

Since the product of two even elements is even, and since the inverse of an even element is also even, we see that $\text{Spin}(n)$ is indeed a subgroup of $\text{Pin}(n)$.

Because the earlier sequence was exact, we find that $\text{Spin}(n) \to SO(n)$ is surjective and has kernel $\{1, -1\}$.

Corollary 4.4.14. *The sequence*

$$1 \to Z_2 \to \text{Spin}(n) \xrightarrow{\rho} SO(n) \to 1$$

is exact.

We see that $\mathrm{Spin}(n)$ is the double cover of $SO(n)$, which implies $\mathrm{Lie}(\mathrm{Spin}(n)) = \mathfrak{so}(n) = \mathfrak{o}(n) = \mathrm{Lie}(\mathrm{Pin}(n))$. Therefore, $\mathrm{Spin}(n)$ has the same Lie algebra as $\mathrm{Pin}(n)$.

We have found two short exact sequences describing the subgroups (S)$\mathrm{Pin}(n)$ of $C\ell(n)$. We now want to generalise these sequences to subgroups of $C\ell(p,q)$. The groups (S)$\mathrm{Pin}(p,q)$ can be defined in a way similar to (S)$\mathrm{Pin}(n)$. However, we haven't defined a group such as "$O(p,q)$" yet, so we have to do that first.

4.4.2   Interlude: indefinite orthogonal groups

We define a generalisation of $O(n)$ in order to describe the image of $\mathrm{Pin}(p,q)$ in $\mathrm{GL}(\mathbb{R}^n)$.[3]

Definition 4.4.15. Let $\mathbb{R}^{p,q}$ with $p + q = n$ be $\mathbb{R}^n$ equipped with the non-degenerate quadratic form $\tilde{Q}(v) = v_1^2 + v_2^2 + \ldots + v_p^2 - v_{p+1}^2 - \ldots - v_{p+q}^2$. We call (p,q) the <u>signature</u> and say that this space is the <u>real space with signature (p,q)</u>.

As a vector space, we identify $\mathbb{R}^{p,q}$ with $\mathbb{R}^p \times \mathbb{R}^q$ such that the non-degenerate quadratic form can be written as $\tilde{Q}(v) = \tilde{Q}(u,w) = \|u\|^2 - \|w\|^2$ for $v \in \mathbb{R}^{p,q}, u \in \mathbb{R}^p, w \in \mathbb{R}^q$. We now define the group of transformations of $\mathbb{R}^{p,q}$.

Definition 4.4.16. Let $O(p,q)$ be the group of matrices $A \in \mathrm{GL}(\mathbb{R}^n)$ that leave $\tilde{Q}$ invariant, so $\tilde{Q}(Av) = \tilde{Q}(v)$. We call this group the <u>indefinite orthogonal group of signature (p,q)</u>.

The group $O(p,q)$ consists of rotations and reflections on the first p dimensions, rotations and reflections on the last q dimensions, and (Lorentz) boosts between the first p and the last q dimensions. A well-known example of an indefinite orthogonal group is the Lorentz group, which is $O(1,3)$ or $O(3,1)$ depending on the convention used.

The corresponding definition for the symmetric bilinear form $\tilde{\beta}$ on $\mathbb{R}^{p,q}$ is $\tilde{\beta}(v,w) = v_1 w_1 + \ldots + v_p w_p - v_{p+1} w_{p+1} - \ldots - v_{p+q} w_{p+q}$. Note that this symmetric bilinear form induces the non-degenerate quadratic form $\tilde{Q}$ and conversely this symmetric bilinear form can be defined via the polarisation identity (see Remark 4.1.5). Therefore, we can alternatively define $O(p,q)$ as the group of matrices $A \in \mathrm{GL}(\mathbb{R}^n)$ such that $\tilde{\beta}(Av, Aw) = \tilde{\beta}(v,w)$ for any $v,w \in \mathbb{R}^{p,q}$.

Let $I_{p,q} = \mathrm{diag}(1,1,\ldots,1,-1,\ldots,-1)$ denote the diagonal matrix with the first p elements equal to 1 and the next q equal to -1. Now the $\tilde{\beta}(v,w)$ can be written as $v^\mathsf{T} I_{p,q} w$ for $v,w \in \mathbb{R}^{p,q}$. The condition $\tilde{\beta}(Av, Aw) = \tilde{\beta}(v,w)$ then becomes $(Av)^\mathsf{T} I_{p,q}(Aw) = v^\mathsf{T} I_{p,q} w$ for all v, w, or

$$v^\mathsf{T} A^\mathsf{T} I_{p,q} A w = v^\mathsf{T} I_{p,q} w.$$

By choosing $v = e_i, w = e_j$ for $1 \leq i, j \leq n$ we conclude this is only possible if $A^\mathsf{T} I_{p,q} A = I_{p,q}$, which gives us a condition similar to $A^\mathsf{T} A = \mathbb{1}$ for $A \in O(n)$.

The condition $A^\mathsf{T} I_{p,q} A = I_{p,q}$ conversely implies $\tilde{Q}(Av) = \tilde{Q}(v)$, so we find that we can alternatively define $O(p,q)$ by $\{A \in \mathrm{GL}(\mathbb{R}^n) \mid A^\mathsf{T} I_{p,q} A = I_{p,q}\}$.

Remark 4.4.17. The equation $A^\mathsf{T} I_{p,q} A = I_{p,q}$ implies that the determinants on both sides are equal, so $\det(A^\mathsf{T}) \det(A) = 1$. Since $\det(A^\mathsf{T}) = \det(A)$, we find $\det(A) = \pm 1$. This agrees with $\det A' = \pm 1$ for any $A' \in O(n)$.

The final definition makes it clear that $O(p,q)$ is a Lie group, since it is the kernel of the map $A \mapsto A^\mathsf{T} I_{p,q} A - I_{p,q}$ from $\mathrm{GL}(\mathbb{R}^n)$ to itself. From the condition, it is easy to calculate the Lie algebra of $O(p,q)$ as the condition for the Lie algebra is given by the derivative of the condition for the group. We therefore get the following.

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[3]These definitions are considered well-known as they stem from physics, so I give them without source.

Corollary 4.4.18. *The Lie algebra of $O(p, q)$ is given by $\mathfrak{o}(p, q) := \{B \in M(n) \mid B^\mathsf{T} I_{p,q} = -I_{p,q} B\}$, where $M(n)$ denotes all real n by n matrices.*

Note that $I_{p,q}$ is its own inverse, so we can write the condition as $B^\mathsf{T} = -I_{p,q} B I_{p,q}$, or $B_{ji} = -\eta(i) B_{ij} \eta(j)$, where $\eta(i) = 1$ for $1 \le i \le p$ and $\eta(i) = -1$ for $p + 1 \le i \le p + q$. For $i, j \le p$ this gives $B_{ji} = -B_{ij}$ and similarly for $i, j \ge p + 1$, but for $i \le p < j$ or $j \le p < i$ this gives $B_{ji} = B_{ij}$. Therefore, the matrices in $\mathfrak{o}(p, q)$ are anti-symmetric on the first $p \times p$ block and on the last $q \times q$ block, but symmetric on the rest of the matrix (the $p \times q$ and the $q \times p$ blocks off the diagonal are each others transpose). This gives $n(n-1)/2$ free choices, so the dimension of $\mathfrak{o}(p, q)$ is $\frac{1}{2} n(n - 1)$. The diagonal is covered by the $p \times p$ and $q \times q$ block and therefore anti-symmetric, so the trace of elements of $\mathfrak{o}(p, q)$ is always 0.
Much like $O(n)$, we define a special subgroup.

Definition 4.4.19. The indefinite special orthogonal group $SO(p, q)$ is the group $\{A \in O(p, q) \mid \det(A) = 1\}$.

It is clear that $SO(p, q)$ is once again a Lie group. Since the trace of all matrices in $\mathfrak{o}(p, q)$ is 0, it follows that $\mathfrak{so}(p, q) = \mathfrak{o}(p, q)$.
Finally, we give the following lemma. It is a specific case of the Cartan-Dieudonné Theorem. We will not give or prove the theorem, as the proof is too complicated and the theorem is not related to the rest of this thesis. A proof can be found in Fukshansky [7].

Lemma 4.4.20. *The group $O(p, q)$ is completely generated by reflections. In this context, 'reflections' are maps of the form $w \mapsto w - 2\frac{\tilde{\beta}(v, w)}{\tilde{\beta}(v, v)} v$ with $v \in V$ non-isotropic and β the symmetric bilinear form.*

Proof. This follows directly from the Cartan-Dieudonné theorem. □

4.4.3   (S)Pin groups in $C\ell(p, q)$

We now consider $C\ell(p, q)$, assume $q \ge 1$. We will construct (S)Pin(p, q) in a way similar to the construction of (S)Pin(n), following the construction from Gallier [8]. We will only see a difference when taking the kernel of N, everything before that is the same.
Let $C\ell^\times(p, q)$ be the group of invertible elements. We define the transpose map $\cdot^t$ and canonical automorphism α in the same way as before, and define the same notation $\bar{x} = \alpha(x^t)$ for $x \in C\ell(p, q)$. It can easily be checked that $\cdot^t$ and α are again an anti-automorphism respectively automorphism. Next, we define Γ is the same way as before along with the twisted adjoint representation. In particular, note that the proof of Proposition 4.4.5 stays the same, so we once again have

$$\rho(v)(w) = w - 2\frac{\beta(v, w)}{Q(v)} v$$

for $v, w \in V$ with v invertible ($Q(v) \ne 0$). Once again, the kernel of ρ is $\mathbb{R}^*$.
We once again define $N : \Gamma \to \Gamma, N(x) = x \cdot x^t$, and see that $N(x) \in \mathbb{R}^*$ for $x \in \Gamma$. Note that $Q(\rho(x)(v)) = Q(v)$ for $x \in \Gamma, v \in V$ is still true, as the proof does not use $p = n, q = 0$.
However, Q is no longer the square of the norm, so $N(e_{p+1}) = Q(e_{p+1}) = -1$. Therefore, e_{p+1} is not in the kernel of N. In order to make sure that our Pin-group contains all e_i, $1 \le i \le p + q$, we have to take the pre-image of $\{1, -1\}$ instead. Fortunately, $\{1, -1\}$ is a normal subgroup of $\mathbb{R}^*$, so the pre-image is still a subgroup of Γ.

Definition 4.4.21. Let $\text{Pin}(p, q) = N^{-1}(\{1, -1\})$ be the pre-image. We call $\text{Pin}(p, q)$ the (indefinite) Pin group.

We once again define the Spin group as the even subgroup of the Pin groups. Note that $C\ell^{even}(p, q)$ is closed under multiplication, and that the inverse of an even element is once again an even element.

Definition 4.4.22. We define $\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap C\ell^{even}(p,q)$. We call this group the (indefinite) Spin group.

We now consider the image of the maps $\rho : \mathrm{Pin}(p,q) \to \mathrm{GL}(V)$ and its restriction to $\mathrm{Spin}(p,q)$. We have seen that $Q(\rho(x)(v)) = Q(v)$ for $x \in \Gamma$, so the same is true for $x \in \mathrm{Pin}(p,q)$. Therefore, $\rho(x)$ is an isometry with respect to Q for $x \in \mathrm{Pin}(p,q)$. Now remark that $Q = \tilde{Q}$ on $V = \mathbb{R}^{p+q}$ and $\beta = \tilde{\beta}$, so $\rho(x), x \in \mathrm{Pin}(p,q)$ leaves $\tilde{Q}$ invariant. By definition of $O(p,q)$, it follows that $\rho(x) \in O(p,q)$ for $x \in \mathrm{Pin}(p,q)$.

We also want to show that $\rho(x) \in SO(p,q)$ for $x \in \mathrm{Spin}(p,q)$. To do that, we show that $\det(\rho(v)) = -1$ for $v \in V$ with $Q(v) = \pm 1$. From $\det(\rho(v)) = -1$ for $v \in V$ and the fact that $v \in V = C\ell^1(p,q)$ are the only elements of grade 1, it follows that the determinant of the image of even-graded terms is 1, so indeed $\rho(\mathrm{Spin}(p,q)) \subseteq SO(p,q)$.

Proposition 4.4.23. *For any $v \in V$ with $Q(v) = \pm 1$, we have $\det(\rho(v)) = -1$.*

Proof. Let $v = \sum_{i=1}^{n} v_i e_i$ with $Q(v) = \sum_{i=1}^{p} v_i^2 - \sum_{j=1}^{q} v_{p+j}^2 = \pm 1$. Call $\alpha = Q(v)$. We calculate the determinant by writing it out. In the proof of Proposition 4.4.5 we have found that $\rho(v)$ is given by the formula

$$w \mapsto w - 2\frac{\beta(v,w)}{Q(v)}v.$$

By choosing $w = e_i$ for $1 \le i \le n$, we can find the matrix of $\rho(v)$. First consider an e_i with $i \le p$. Then the map becomes $e_i \mapsto e_i - 2\frac{v_i}{\alpha}v$, which describes the first p columns of our matrix. Next consider an e_j with $j \ge p+1$. Then we have $e_j \mapsto e_j - 2\frac{-v_j}{\alpha}v$, so we also know the last q columns of our matrix. Writing it out, we get:

$$\begin{pmatrix} 1 - 2\alpha v_1^2 & -2\alpha v_2 v_1 & \dots & -2\alpha v_p v_1 & 2\alpha v_{p+1} v_1 & \dots & 2\alpha v_{p+q} v_1 \\ -2\alpha v_1 v_2 & 1 - 2\alpha v_2^2 & \dots & -2\alpha v_p v_2 & 2\alpha v_{p+1} v_2 & \dots & 2\alpha v_{p+q} v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -2\alpha v_1 v_p & -2\alpha v_2 v_p & \dots & 1 - 2\alpha v_p^2 & 2\alpha v_{p+1} v_p & \dots & 2\alpha v_{p+q} v_p \\ -2\alpha v_1 v_{p+1} & -2\alpha v_2 v_{p+1} & \dots & -2\alpha v_p v_{p+1} & 1 + 2\alpha v_{p+1}^2 & \dots & 2\alpha v_{p+q} v_{p+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -2\alpha v_1 v_{p+q} & -2\alpha v_2 v_{p+q} & \dots & -2\alpha v_p v_{p+q} & 2\alpha v_{p+1} v_{p+q} & \dots & 1 + 2\alpha v_{p+q}^2 \end{pmatrix}$$

We see that the matrix has a lot of repeated terms. We will assume $v_1 \ne 0$ for the Gauss-elimination; the method is similar for each other v_i. For the rows $2 \le i \le p$, we subtract the first row times v_i/v_1; for the last rows $p+1 \le i \le p+q$, we add the first row times v_i/v_1. This gives us the matrix

$$\begin{pmatrix} 1 - 2\alpha v_1^2 & -\frac{v_2}{v_1} & \dots & -\frac{v_p}{v_1} & \frac{v_{p+1}}{v_1} & \dots & \frac{v_{p+q}}{v_1} \\ -2\alpha v_1 v_2 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -2\alpha v_1 v_p & 0 & \dots & 1 & 0 & \dots & 0 \\ -2\alpha v_1 v_{p+1} & 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -2\alpha v_1 v_{p+q} & 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}$$

which clearly has determinant

$$(1 - 2\alpha v_1^2) - \left(\frac{-v_2}{v_1} \cdot -2\alpha v_1 v_2\right) - \dots - \left(\frac{-v_p}{v_1} \cdot -2\alpha v_1 v_p\right) + \frac{v_{p+1}}{v_1} \cdot -2\alpha v_1 v_{p+1} + \dots + \frac{v_{p+q}}{v_1} \cdot -2\alpha v_1 v_{p+q}$$

$$= 1 - 2\alpha(v_1^2 + v_2^2 + \dots + v_p^2 - v_{p+1}^2 - \dots - v_{p+1}^2) = 1 - 2\alpha^2 = -1.$$

Since v was a general element of V with $Q(v) = \pm 1$, we conclude that all elements with odd grade have determinant -1. $\square$

As mentioned, it follows that $\rho(x) \in SO(p,q)$ for $x \in \mathrm{Spin}(p,q)$. We are finally ready to prove the short exact sequences.

Theorem 4.4.24. *Let* $\mathrm{Pin}(p,q)$ *and* $\mathrm{Spin}(p,q)$ *as before. There are (well-defined) short exact sequences*

$$1 \to Z_2 \to \mathrm{Pin}(p,q) \xrightarrow{\rho} O(p,q) \to 1$$

and

$$1 \to Z_2 \to \mathrm{Spin}(p,q) \xrightarrow{\rho} SO(p,q) \to 1$$

where $Z_2 \cong \{1, -1\} \subseteq \mathrm{Pin}(p,q)$. *In other words, the kernel of* $\rho : (\mathrm{S})\mathrm{Pin}(p,q) \to (S)O(p,q)$ *is isomorphic to* Z_2, *while* ρ *is surjective.*

Proof. First, consider the kernel of $\rho : \mathrm{Pin}(p,q) \to O(p,q)$. We already know that the kernel should be a subset of $\mathbb{R}^*$ such that $N(\lambda) = \pm 1$ for each $\lambda \in \mathbb{R}$, so the kernel is $\{1, -1\}$. Simlarly, the kernel of $\rho : \mathrm{Spin}(p,q) \to SO(p,q)$ is $\{1, -1\}$.
Next, we need to show surjectivity. Using Lemma 4.4.20 we see that $O(p,q)$ is generated by reflections. Since for any reflection there is a $v \in \mathbb{R}^n$ such that this v generates the reflections, we once again conclude that $\rho : \mathrm{Pin}(p,q) \to O(p,q)$. The surjectivity of $\rho : \mathrm{Spin}(p,q) \to SO(p,q)$ then follows from the the fact that the determinant of $\rho(x)$ is 1 if and only if $x \in \mathrm{Spin}(p,q)$.
We conclude that the short exact sequences are well-defined. $\square$

We see that the maps $(\mathrm{S})\mathrm{Pin}(p,q) \to (S)O(p,q)$ are again double covers, and that $\mathrm{Lie}(\mathrm{Spin}(p,q)) = \mathrm{Lie}(\mathrm{Pin}(p,q)) = \mathfrak{so}(p,q)$. We conclude that we can determine the structure of $(\mathrm{S})\mathrm{Pin}(p,q)$ by studying $(S)O(p,q)$.
Finally, we consider the complex case.

4.4.4   (S)Pin groups in $C\ell(n, \mathbb{C})$

Let $V = \mathbb{C}^n$ and let the Clifford algebra be of standard form, $C\ell(n, \mathbb{C})$. Our construction is based on the previous two, but the definitions change slightly because we are now working with complex numbers.
We first define the (anti)-automorphisms.

Definition 4.4.25. Let $\cdot^t : C\ell^k(n, \mathbb{C}) \to C\ell^k(n, \mathbb{C})$ denote the transpose map

$$(v_1 v_2 \dots v_k)^t \mapsto (v_k v_{k-1} \dots v_1)^c,$$

extended to $C\ell(n, \mathbb{C})$ by the direct sum. Here $\cdot^c$ denotes complex conjugation.

Note that the complex conjugation does not change the fact that $\cdot^t$ is an anti-automorphism.
We define the canonical automorphism the same as before, and define $\Gamma^{\mathbb{C}}$ as the subgroup of $C\ell^{\times}(n, \mathbb{C})$ such that $\alpha(x) \cdot v \cdot x^{-1} \in V$ for all $v \in V, x \in \Gamma^{\mathbb{C}}$. However, we instead use the subgroup Γ of $\Gamma^{\mathbb{C}}$ such that $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for $v \in \mathbb{R}^n \subset V$, as this Γ is much more similar to the earlier groups.
We again find the twisted adjoint representation ρ, but we see that it's kernel now is $\mathbb{C}^* = \mathbb{C} \setminus 0$. We also define $N : C\ell(n, \mathbb{C}) \to C\ell(n, \mathbb{C})$.

Definition 4.4.26. Let $N : \Gamma \to \Gamma$ be the map $N(x) = x \cdot x^t$.

The map N looks the same as before, and is certainly a well-defined homomorphism. However, it includes the complex conjugation from earlier, so $N(\lambda) = |\lambda|^2$ for any $\lambda \in \mathbb{C}^*$.
We continue following the construction of the real case, and see that the complex version of Proposition 4.4.9 implies that $N(x) \in \ker \rho$, so $N(x) \in \mathbb{C}^*$ for any $x \in \Gamma$. However, the definition of N is $N(x) = x \cdot x^t$ so $N(x)^t = (x \cdot x^t)^t = x \cdot x^t$. Therefore, if $N(x) \in \mathbb{C}^*$, then $N(x) = N(x)^t = N(x)^c$ where $\cdot^c$ denotes complex conjugation, so $N(x) \in \mathbb{R}^*$. We see that we can again define $N : \Gamma \to \mathbb{R}^*$. Since this has the same form as in the $C\ell(n)$ case, we use the same definition for the Pin group.

Definition 4.4.27. Let $\text{Pin}(n, \mathbb{C}) = \text{Pin}^{\mathbb{C}}(n)$ be the kernel of $N : \Gamma \to \mathbb{R}^*$. We call $\text{Pin}^{\mathbb{C}}(n)$ the (complex) Pin group. We additionally define $\text{Spin}(n, \mathbb{C}) = \text{Spin}^{\mathbb{C}}(n)$ as $\text{Pin}^{\mathbb{C}}(n) \cap C\ell^{even}(n, \mathbb{C})$, and call that group the (complex) Spin group.

We now consider the image of $\text{Pin}^{\mathbb{C}}(n)$ in $\text{GL}(\mathbb{C}^n)$. We want to to make an argument like earlier, where we used that $\rho(x), x \in \Gamma$ is a linear isometry. However, $N(v) \neq Q(v)$ for a general $v \in V$, so we can no longer guarantee that $v \in \text{Pin}^{\mathbb{C}}(n)$ for every $v \in V$ with $Q(v) = 1$. Fortunately, $N(v) = Q(v)$ if v is a $\mathbb{R}$-linear combination of the e_i, $1 \leq i \leq n$. Therefore, we can say $\text{Pin}(n) \subset \text{Pin}^{\mathbb{C}}(n)$ as sets, and the image of $\text{Pin}^{\mathbb{C}}(n)$ in $\text{GL}(\mathbb{C}^n)$ contains $O(n)$.

We also know $N(\lambda) = 1$ for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ i.e. for $\lambda \in U(1)$, so $U(1) \subset \text{Pin}^{\mathbb{C}}(n)$. (Recall that $U(1) \cong S^1$ is the group of all $\lambda \in \mathbb{C}$ with norm 1 under multiplication.) However, the elements of $U(1)$ are all send to 1 in the twisted adjoint representation, meaning they well replace the Z_2 in the short exact sequence, rather than appearing as part of the image.

Since we have seen that $\text{Pin}(n) \subset \text{Pin}^{\mathbb{C}}(n)$ and $U(1) \subset \text{Pin}^{\mathbb{C}}(n)$, it follows that product of elements of $U(1)$ and elements of $\text{Pin}(n)$ are in $\text{Pin}^{\mathbb{C}}(n)$. In fact, such products generate the whole group.

Proposition 4.4.28. *For any $x \in \text{Pin}^{\mathbb{C}}(n)$, we can write $x = \lambda y$ with $\lambda \in \mathbb{C}$, $y \in \text{Pin}(n)$.*

Proof. Let $x \in \text{Pin}^{\mathbb{C}}(n)$. Now $x \cdot x^t = 1$ and $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for any $v \in \mathbb{R}^n$. Note that $x \in \text{Pin}^{\mathbb{C}}(n)$ implies $x^t \in \text{Pin}^{\mathbb{C}}(n)$ and that $\text{Pin}^{\mathbb{C}}(n)$ is closed under complex conjugation. First consider x^c, where $\cdot^c$ denotes complex conjugation (which is an automorphism). Since $(\alpha(x) \cdot v \cdot x^{-1})^c = \alpha(x^c) \cdot v \cdot (x^c)^{-1}$ for $v \in \mathbb{R}^n$ and $w^c = w$ for all $w \in \mathbb{R}^n$, we have

$$\alpha(x^c) \cdot v \cdot (x^c)^{-1} = \alpha(x) \cdot v \cdot x^{-1}$$

for all $v \in \mathbb{R}^n$, $x \in \text{Pin}^{\mathbb{C}}(n)$. Next consider $x \cdot (x^c)^t$, where $\cdot^c$ denotes complex conjugation. We have

$$\alpha(x \cdot (x^c)^t) \cdot v \cdot (x \cdot (x^c)^t)^{-1} = \alpha(x) \cdot \left(\bar{x}^c \cdot v \cdot ((x^c)^t)^{-1}\right) \cdot x^{-1}.$$

But we have just seen that x^c and x have the same action, so $(x^c)^t$ and x^t have the same action, so this is equal to

$$\alpha(x) \cdot \bar{x} \cdot v \cdot (x^t)^{-1} \cdot x^{-1} = \alpha(N(x)) \cdot v \cdot (N(x))^{-1} = v.$$

We conclude that $x \cdot (x^c)^t$ is in the kernel of ρ for any $x \in \text{Pin}^{\mathbb{C}}(n)$, so $x \cdot (x^c)^t$ is a complex non-zero number. Let $x \in \text{Pin}^{\mathbb{C}}(n)$ and let $\lambda \in \mathbb{C}$ such that $\lambda^2 = x \cdot (x^c)^t$. Note that λ is non-zero, and consider $y = \lambda^{-1}x$. Now $y \cdot y^t = 1$ is clear, and $\alpha(y) \cdot v \cdot y^{-1} = \alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$, so $y \in \text{Pin}^{\mathbb{C}}(n)$. But we also find that $y \cdot (y^c)^t = 1$. So $y \cdot y^t = y \cdot (y^c)^t$. Using that y is invertible, we find $y^t = (y^c)^t$ or $y = y^c$. We find that y is a real linear combination of the basis-vectors, so $y \in \text{Pin}(n)$. We conclude that $x = \lambda y$, with $\lambda \in \mathbb{C}$ and $y \in \text{Pin}(n)$. $\square$

Remark 4.4.29. The λ and y are not unique for a given x, as $(-\lambda)(-y) = x$ also works. This degeneracy is caused by the square root in the definition of λ.

Remark 4.4.30. The proof of the above proposition would not have worked for $\Gamma^{\mathbb{C}}$ instead of Γ, as the equality $v^c = v$ was used. This shows why we have chosen to define $\text{Pin}^{\mathbb{C}}(n)$ as a subgroup of Γ.

We have found that $\rho : \text{Pin}^{\mathbb{C}}(n) \to \text{GL}(\mathbb{C}^n)$ has kernel $U(1)$ and image $O(n)$. This brings us to the complex equivalent of 4.4.12:

Theorem 4.4.31. *There are exact sequences*

$$1 \to U(1) \to \text{Pin}^{\mathbb{C}}(n) \xrightarrow{\rho} O(n) \to 1$$

and

$$1 \to U(1) \to \text{Spin}^{\mathbb{C}}(n) \xrightarrow{\rho} SO(n) \to 1$$

where $U(1) \cong \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \subseteq \text{Pin}^{\mathbb{C}}(n)$.

Proof. The short exact sequence of $\text{Pin}^{\mathbb{C}}(n)$ follows directly from the fact that $\rho : \text{Pin}^{\mathbb{C}}(n) \to \text{GL}(\mathbb{C}^n)$ has kernel $U(1)$ and image $O(n)$. The complex Spin group $\text{Spin}^{\mathbb{C}}(n)$ contains all even elements. Remark that all elements of $U(1)$ are even, while the even elements of $\text{Pin}(n)$ form $\text{Spin}(n)$. Therefore, the Spin group also has kernel $U(1)$, while the image of $\text{Spin}^{\mathbb{C}}(n)$ is equal to the image of $\text{Spin}(n)$ which is $SO(n)$. We conclude that both sequence are exact. $\qquad\square$

We now have described $(\text{S})\text{Pin}^{\mathbb{C}}(n)$ by a short exact sequence. However, unlike the sequences of $(\text{S})\text{Pin}(n)$ and $(\text{S})\text{Pin}(p,q)$ which both started with the finite group Z_2, the sequence of $(\text{S})\text{Pin}^{\mathbb{C}}(n)$ starts with an infinite group. Therefore, we cannot conclude that $(\text{S})\text{Pin}^{\mathbb{C}}(n)$ is the double cover of $O(n)$ or a similar statement. Because of this, we try to extend ρ in some way such that the kernel of the new map is equal to Z_2. In order to do this, we will have to make sure that complex numbers are no longer all send to 1. Since a representation is a homomorphism, this is only possible if λx is no longer send to the same value as x, for $\lambda \in U(1), x \in \text{Pin}(n)$. Therefore, we have to distinguish between λx and x. This brings us to the following definition.

Definition 4.4.32. Let $\tilde{Q} : \text{Pin}^{\mathbb{C}}(n) \to U(1)$ be the map $Q(x) = x \cdot (x^c)^t$.

This map is a well-defined, as we have seen in the proof of Proposition 4.4.28. It is also clear that this map is a homomorphism. Also note that $\tilde{Q}(v) = v \cdot v = Q(v)$ for any $v \in V = \mathbb{C}^n$, so $\tilde{Q}$ is an extension of Q to the whole complex Pin group, which explains the notation $\tilde{Q}$. Also recall that $Q(\lambda) = \lambda^2$ for $\lambda \in U(1)$, which implies that Q is surjective.
We use this homomorphism to extend our representation ρ to a (more general) homomorphism.

Definition 4.4.33. Let $\tilde{\rho} : \text{Pin}^{\mathbb{C}}(n) \to U(1) \times O(n)$ be the map

$$\tilde{\rho}(x) = (Q(x), \rho(x)).$$

Finally, we use the extension of ρ for another short exact sequence.

Theorem 4.4.34. *There are exact sequences*

$$1 \to Z_2 \to \text{Pin}^{\mathbb{C}}(n) \xrightarrow{\tilde{\rho}} U(1) \times O(n) \to 1$$

and

$$1 \to Z_2 \to \text{Spin}^{\mathbb{C}}(n) \xrightarrow{\tilde{\rho}} U(1) \times SO(n) \to 1$$

where $Z_2 \cong \{1, -1\}$ *and* $U(1) \cong \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *are subgroups of* $\text{Pin}^{\mathbb{C}}(n)$.

Proof. We have seen that ρ and Q are surjective, so the surjectivity or $\tilde{\rho}$ is clear. We calculate the kernel of $\tilde{\rho}$. Since the kernel of ρ was $U(1)$, the kernel of $\tilde{\rho}$ is the subset of elements $\lambda \in U(1)$ such that $Q(\lambda) = \lambda^2 = 1$. The only options are $\lambda = \pm 1$, so the kernel of $\tilde{\rho}$ is $\{1, -1\} \cong Z_2$.
Similarly, the surjectivity of $\tilde{\rho} : \text{Spin}^{\mathbb{C}}(n) \to U(1) \times SO(n)$ follows from the surjectivity of ρ and Q, and the injectivity is the same subset of $U(1)$. We conclude that the second sequence is also exact. $\qquad\square$

Since we now have short exact sequences starting with a finite group, we can conclude that $(\text{S})\text{Pin}^{\mathbb{C}}(n)$ is a double cover of $U(1) \times (\text{S})O(n)$. We can also determine which double cover it is, by comparing the short exact sequences.

Corollary 4.4.35. *There is a Lie group isomorphism*

$$U(1) \times_{Z_2} (\text{S})\text{Pin}(n) \cong (\text{S})\text{Pin}^{\mathbb{C}}(n)$$

where $U(1) \times_{Z_2} (\text{S})\text{Pin}(n)$ *is* $U(1) \times (\text{S})\text{Pin}(n)$ *where* Z_2 *causes* $(-1, -1)$ *to be identified with* $(1, 1)$.

Proof. We could prove this by comparing the exact sequences, and using that Q restricted to $U(1)$ gives a double covering of $U(1)$. However, from Proposition 4.4.28 we already know that each element of $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n)$ can be written as λx with $\lambda \in U(1)$, $x \in (\mathrm{S})\mathrm{Pin}(n)$. We have also seen that the product $(-\lambda)(-x)$ works as well, but these are the only two options. This gives us a short exact sequence

$$1 \to Z_2 \to (\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n) \to U(1) \times (\mathrm{S})\mathrm{Pin}(n) \to 1,$$

where $Z_2 = \{(1,1),(-1,-1)\}$. We conclude that $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n) \cong U(1) \times_{Z_2} (\mathrm{S})\mathrm{Pin}(n)$. $\square$

We conclude that $\mathrm{Spin}^{\mathbb{C}}(n) \cong U(1) \times_{Z_2} \mathrm{Spin}(n)$ and $\mathrm{Pin}^{\mathbb{C}}(n) \cong U(1) \times_{Z_2} \mathrm{Pin}(n)$. This completely determines $\mathrm{Spin}^{\mathbb{C}}(n)$ and $\mathrm{Pin}^{\mathbb{C}}(n)$ in terms of $(\mathrm{S})\mathrm{Pin}(n)$.

Finally, we consider the Lie algebras of $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n)$. Since double covers have the same Lie algebras as the Lie group they cover, we see that

$$\mathrm{Lie}(\mathrm{Pin}^{\mathbb{C}}(n)) = \mathrm{Lie}(\mathrm{Spin}^{\mathbb{C}}(n)) = \mathrm{Lie}\big(U(1) \times \mathrm{Spin}(n)\big) = (\mathbb{R}\mathrm{i}) \times \mathfrak{so}(n),$$

since the Lie algebra of $U(1) = S^1$ is the line $\mathbb{R}\mathrm{i}$.

To recapitulate: we have defined the groups $(\mathrm{S})\mathrm{Pin}(n)$, $(\mathrm{S})\mathrm{Pin}(p,q)$ and $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n)$ and shown that they are double covers of $(S)O(n)$, $(S)O(p,q)$ respectively $U(1) \times (S)O(n)$. We have also seen that $\mathrm{Lie}(\mathrm{Spin}(p,q)) = \mathrm{Lie}(\mathrm{Pin}(p,q)) = \mathfrak{so}(p,q)$ and similarly for $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n)$. Therefore, we can determine the structure of $(\mathrm{S})\mathrm{Pin}(p,q)$ and $(\mathrm{S})\mathrm{Pin}^{\mathbb{C}}(n)$ by studying $(S)O(p,q)$, with particular focus on $(S)O(n)$.

4.4.5   Lie subalgebras of the Clifford algebra

We now turn to Lie subalgebras of the Clifford algebra. Let $C\ell(V,Q)$ still be a (real or complex) Clifford algebra in standard form. Define the map $[\cdot,\cdot] : C\ell(V,Q) \times C\ell(V,Q) \to C\ell(V,Q)$ as the commutator $[v,w] = vw - wv$ for $v,w \in C\ell(V,Q)$. This is a Lie bracket, and using it we define the following Lie algebras.

Definition 4.4.36. We define $\mathbf{L}$ as $[V,V] = \{[v,w] \mid v,w \in V\}$, and $\mathbf{G}$ as $V \oplus [V,V]$. We call $\mathbf{L}$ the <u>bivector Lie subalgebra</u> and $\mathbf{G}$ the <u>orthogonal Lie subalgebra</u> of $C\ell(V,Q)$.

Of course, we have to prove the vector spaces $\mathbf{L}$ and $\mathbf{G}$ form Lie algebras together with the bracket $[\cdot,\cdot]$.

Proposition 4.4.37. *The vector spaces $\mathbf{L}$ and $\mathbf{G}$ form Lie algebras together with the commutator bracket.*

Proof. We first show that $\mathbf{L}$ is closed under the commutator bracket. Let β be the (induced) bilinear form of the Clifford algebra, and remark that for any $x,y \in V$ we have $xy + yx = 2\beta(x,y)$. Now $[x,y] = xy - yx = 2xy - (xy + yx) = 2xy - 2\beta(x,y)$. We will use this relation repeatedly. Also remark that for any $x,y,z \in V$ we have $[x,yz] = xyz - yzx = xyz - yxz + yxz - yzx = [x,y]z + y[x,z]$.

Let $[a, b], [c, d] \in \mathbf{L}, a, b, c, d \in V$. We have:

$$
\begin{aligned}
\big[[a, b], [c, d]\big] &= 4\big[ab - \beta(a, b), cd - \beta(c, d)\big] \\
&= 4[ab, cd] \\
&= 4\big(abcd - cdab\big) = 4\big(abcd - cabd + cabd - cdab\big) \\
&= 4\big([ab, c]d + c[ab, d]\big) \\
&= 4\big(a[b, c]d + [a, c]bd + ca[b, d] + c[a, d]b\big) \\
&= 8\big(abcd + acbd + cabd + cadb - ad\beta(b, c) - bd\beta(a, c) - ca\beta(b, d) - cb\beta(a, d)\big) \\
&= 8\big(a(bc + cb)d + ca(bd + db) - ad\beta(b, c) - bd\beta(a, c) - ca\beta(b, d) - cb\beta(a, d)\big) \\
&= 8\big(2ad\beta(b, c) + 2ca\beta(b, d) - ad\beta(b, c) - bd\beta(a, c) - ca\beta(b, d) - cb\beta(a, d)\big) \\
&= 8\big(ad\beta(b, c) + ca\beta(b, d) - bd\beta(a, c) - cb\beta(a, d)\big) \\
&= 4\bigg(\big([a, d] + 2\beta(a, d)\big)\beta(b, c) + \big([c, a] + 2\beta(c, a)\big)\beta(b, d) \\
&\qquad - \big([b, d] + 2\beta(b, d)\big)\beta(a, c) - \big([c, b] + 2\beta(c, b)\big)\beta(a, d) \bigg) \\
&= 4\big(\beta(b, c)[a, d] + \beta(b, d)[c, a] - \beta(a, c)[b, d] - \beta(a, d)[c, b]\big),
\end{aligned}
$$

which is a linear combination of terms in $\mathbf{L}$. We conclude that $\mathbf{L}$ is closed under the commutator bracket. Now we will show that $\mathbf{G}$ is closed under commutation. For $x, y \in V$ it should be clear that $[x, y] \in [V, V] \subset \mathbf{G}$, so we only have to consider the commutator of an element of $[V, V]$ with an element of V. Since commutation is anti-symmetric, we consider $[[x, y], z]$ with $x, y, z \in V$. We get the following:

$$
\begin{aligned}
\big[[x, y], z\big] &= [xy, z] - [yx, z] \\
&= x[y, z] + [x, z]y - y[x, z] - [y, z]x \\
&= x(2yz - 2\beta(y, z)) + (2xz - 2\beta(x, z))y - y(2xz - 2\beta(x, z)) - (2yz - 2\beta(y, z))x \\
&= 2xyz + 2xzy - 2yxz - 2yzx - 2\beta(y, z)x - 2\beta(x, z)y + 2\beta(x, z)y + 2\beta(y, z)x \\
&= 4\beta(y, z)x - 4\beta(x, z)y
\end{aligned}
$$

This is an element of V, so we see that $[A, x] \in V$ for $A \in [V, V], x \in V$. We conclude that $\mathbf{G}$ is also closed under commutation. $\square$

We have proven that $\mathbf{L}$ and $\mathbf{G}$ are Lie subalgebras of the Clifford algebra.

Remark 4.4.38. Let $e_1, \ldots, e_n$ generate a basis in the sense of Theorem 4.2.3. Now the basis of $\mathbf{L}$ is given by $[e_i, e_j] = e_i e_j - e_j e_i = 2e_i e_j$ for $1 \leq i < j \leq n$ and similarly for $i > j$. We see that $\mathbf{L}$ is as a vector space identical to $C\ell^2(V, Q)$, so all (non-zero) elements of $\mathbf{L}$ have grade 2. This explains the name 'bivector Lie subalgebra'. Similarly, we have $\mathbf{G} = C\ell^1(V, Q) \oplus C\ell^2(V, Q)$ but unlike $\mathbf{L}$, the elements of $\mathbf{G}$ can be odd, even, or linear combinations, so a general element has no well-defined grade.

Using the relatively easy structure of $\mathbf{L}$, we can calculate its dimension.

Proposition 4.4.39. *The dimension of $\mathbf{L}$ is $\frac{1}{2}n(n - 1)$, while the dimension of $\mathbf{G}$ is $\frac{1}{2}n(n + 1)$.*

Proof. First consider the vector space $V \wedge V$. We know that the dimension of $V \wedge V$ is $\frac{1}{2}n(n - 1)$. Consider the map $V \wedge V \to \mathbf{L}, v \wedge w \mapsto [v, w]$. We have $\mathbf{L} = [V, V] = \{[v, w] \mid v, w \in V\}$ and $v \wedge w = 0$ implies that $v = \lambda w$ (or $w = 0$) for some $\lambda \in \mathbb{K}$ which implies $[v, w] = 0$. Therefore, it is clear that the map is surjective. Conversely, if $[v, w] = 0$ then v and w commute, which is only possible if v and w are linear, so $v \wedge w = 0$. We conclude the map is also injective. We conclude that the map is bijective, so the dimension of $\mathbf{L}$ is $\frac{1}{2}n(n - 1)$. Since $\mathbf{G} = V \oplus \mathbf{L}$, we find that the dimension of $\mathbf{G}$ is $\frac{1}{2}n(n + 1)$. $\square$

In order to determine more of the structure of $\mathbf{L}$ and $\mathbf{G}$, we try to find homomorphisms from these Lie algebras to matrix Lie algebras. In the proof of Proposition 4.4.37, we have seen that for any $A \in [V, V], x \in V$ we have $[A, x] \in V$. This allows us to define an action of $\mathbf{L}$ on V by $\mathbf{L} \times V \to V, A \cdot x = [A, x]$.

Corollary 4.4.40. *The action* $\mathbf{L} \times V \to V$, $A \cdot x = [A, x]$ *is well-defined.*

Since this is a linear action, we can write the matrix corresponding to the map $[A, \cdot]$ for $A \in \mathbf{L}$. We use this to define a map of which we show that it is a homomorphism.

Definition 4.4.41. Let $\phi : \mathbf{L} \to M(n)$ denote the linear map defined by $\phi(A)v = [A, v]$ for all $v \in V$ where $M(n)$ are the real n by n matrices and $n = \dim V$.

Proposition 4.4.42. *The map ϕ is a injective Lie algebra homomorphism, i.e. $\phi([A, B]) = [\phi(A), \phi(B)]$ for $A, B \in \mathbf{L}$.*

Proof. We first show the injectivity.
Suppose $\phi([a, b])v = \phi([c, d])v$ for $a, b, c, d \in V$ and for all $v \in V$. Using the earlier formula, we find

$$4\beta(b, v)a - 4\beta(a, v)b = 4\beta(d, v)c - 4\beta(c, v)d$$

for all $v \in V$. Choose $v = e_i$, now $\beta(a, e_i) = e_i$ if $i \leq p$ or $-e_i$ if $i \geq p + 1$, and similarly for b, c, d, so

$$b_i a - a_i b = d_i c - c_i d.$$

Multiplying by e_i from the left, and summing over the i gives $ba - ab = dc - cd$ or $[a, b] = [c, d]$. We conclude that ϕ is injective.
In order to show that ϕ is a homomorphism, we have to show $\big[\phi([a, b]), \phi([c, d])\big] = \phi\big([[a, b], [c, d]]\big)$ for $a, b, c, d \in V$, i.e. to show
$$\big[\phi([a, b]), \phi([c, d])\big]v = \phi\big([[a, b], [c, d]]\big)v$$
for all $v \in V$. We use the identities $[[x, y], v] = 4\beta(y, v)x - 4\beta(x, v)y$ and $[[a, b], [c, d]] = 4\beta(b, c)[a, d] - 4\beta(a, c)[b, d] + 4\beta(a, d)[b, c] - 4\beta(b, d)[a, c]$. Extending the definition of ϕ, the right-hand-side becomes:

$$\begin{aligned}
\Big[[[a, b], [c, d]], v\Big] =&\, 4\beta(b, c)[[a, d], v] - 4\beta(a, c)[[b, d], v] + 4\beta(a, d)[[b, c], v] - 4\beta(b, d)[[a, c], v] \\
=&\, 16\beta(b, c)\beta(d, v)a - 16\beta(b, c)\beta(a, v)d - 16\beta(a, c)\beta(d, v)b + 16\beta(a, c)\beta(b, v)d \\
&+ 16\beta(a, d)\beta(c, v)b - 16\beta(a, d)\beta(b, v)c - 16\beta(b, d)\beta(c, v)a + 16\beta(b, d)\beta(a, v)c
\end{aligned}$$

while for $\big[\phi([a, b]), \phi([c, d])\big]v = \Big[[a, b], [[c, d], v]\Big] - \Big[[c, d], [[a, b], v]\Big]$ we get

$$\begin{aligned}
\Big[[a, b], [[c, d], v]\Big] =&\, 4\beta(d, v)\big[[a, b], c\big] - 4\beta(c, v)\big[[a, b], d\big] \\
=&\, 16\beta(b, c)\beta(d, v)a - 16\beta(a, c)\beta(d, v)b - 16\beta(b, d)\beta(c, v)a + 16\beta(a, d)\beta(c, v)b \\
\Big[[c, d], [[a, b], v]\Big] =&\, 16\beta(d, a)\beta(b, v)c - 16\beta(c, a)\beta(b, v)d - 16\beta(d, b)\beta(a, v)c + 16\beta(c, b)\beta(a, v)d.
\end{aligned}$$

By comparing the expressions, we see that $\big[\phi([a, b]), \phi([c, d])\big]v = \phi\big([[a, b], [c, d]]\big)v$ for any $a, b, c, d \in V$, so ϕ is a homomorphism.
We conclude that $\phi : \mathbf{L} \to M(n)$ is an injective homomorphism. $\square$

Now assume $C\ell(V, q)$ is the real algebra $C\ell(p, q)$. We want to find the image of the homomorphism. In order to do that, we consider the action of elements of $[V, V]$ on the basis vectors, and determine whether the matrices in the image are (skew-)symmetric, what their trace is, etc. Let $x, y \in V$ and consider $[[x, y], e_i]$. For $i \leq p$, we have $[[x, y], e_i] = 4y_i x - 4x_i y$, while we have minus that for $j \geq p + 1$, as $\beta(v, e_i) = v_i Q(e_i) = \pm v_i$.

The j-th component of this vector is $4y_i x_j - 4x_i y_j$, which is anti-symmetric in i, j. Therefore, the matrix $\phi([x, y])$ is anti-symmetric on the first $p \times p$ block. Similarly, if both $i \geq p+1$ and $j \geq p+1$ then the i, j-th is $4x_i y_j - 4y_i x_j$ while that j, i-th component is minus that, so the matrix is anti-symmetric on the last $q \times q$ block.

However, if $i \leq p < p+1 \leq j$, then $[[x, y], e_i]_j = (4y_i x - 4x_i y)_j = 4y_i x_j - 4x_i y_j = (4yx_j - 4xy_j)_i = (4(-y_j)x - 4(-x_j)y)_i = [[x, y], e_j]_i$, so the matrix is symmetric on the side $p \times q$ and $q \times p$ blocks.

This symmetry is exactly the symmetry of $\mathfrak{so}(p, q)$, as we have seen in Corollary 4.4.18. We conclude that the image of ϕ is contained is $\mathfrak{so}(p, q)$. This brings us to the following theorem.

Theorem 4.4.43. *The Lie algebra homomorphism $\phi : \mathbf{L} \to \mathfrak{so}(p, q)$ is a Lie algebra isomorphism.*

Proof. We know that ϕ is an injective homomorphism. Moreover, we have found in Proposition 4.4.39 that the dimension of $\mathbf{L}$ is $\frac{1}{2}n(n-1)$, while we found in Corollary 4.4.18 that the dimension of $\mathfrak{o}(p, q) = \mathfrak{so}(p, q)$ is also $\frac{1}{2}n(n-1)$. Therefore, it directly follows that ϕ is surjective. We conclude that $\phi : \mathbf{L} \to \mathfrak{so}(p, q)$ is a Lie algebra isomorphism. $\square$

Corollary 4.4.44. *The Lie algebras $\mathbf{L}$ and $\mathrm{Lie}(\mathrm{Spin}(p, q)) = \mathrm{Lie}(\mathrm{Pin}(p, q))$ are isomorphic.*

We have now found an indirect relation between $\mathbf{L}$ and $\mathrm{Spin}(p, q)$ using $(S)O(p, q)$. It turns out the relation can be made more direct. Let $e_i, e_j \in V$ be independent elements of the basis such that $e_i^2 = e_j^2 = \pm 1$. Now $(e_i e_j)^2 = -e_i^2 e_j^2 = -1$, $(e_i e_j)(e_j e_i) = 1$ and $(e_i e_j)^t = e_j e_i$. Therefore

$$
\begin{aligned}
N(\cos(\theta) + \sin(\theta)e_i e_j) &= (\cos(\theta) + \sin(\theta)e_i e_j)(\cos(\theta) + \sin(\theta)e_j e_i) \\
&= (\cos^2(\theta) + \sin^2(\theta)) + \sin(\theta)\cos(\theta)(e_i e_j + e_j e_i) = 1
\end{aligned}
$$

for any $\theta \in \mathbb{R}$, since $e_i e_j + e_j e_i = \beta(e_i, e_j) = 0$. It is easy to check that $\cos(\theta) + \sin(\theta)e_i e_j$ is in $\mathrm{Pin}(p, q)$, and therefore in $\mathrm{Spin}(p, q)$.

We can consider $\cos(\theta) + \sin(\theta)e_i e_j$ as a path (as function of θ) starting in the identity. By definition of the Lie algebra, the derivative of $\cos(\theta) + \sin(\theta)e_i e_j$ with respect to θ in $\theta = 0$ is an element of the Lie algebra, so $e_i e_j \in \mathrm{Lie}(\mathrm{Spin}(p, q))$. But we have seen that $e_i e_j \in \mathbf{L}$.

Similarly, for $e_i^2 = -e_j^2$ we find the path $\cosh(\theta) + \sinh(\theta)e_i e_j$ with derivative $e_i e_j$ so again $e_i e_j \in \mathrm{Lie}(\mathrm{Spin}(p, q))$. We see that for any e_i, e_j in the basis of V, the product is contained in $\mathrm{Lie}(\mathrm{Spin}(p, q))$. But these elements exactly generate $\mathbf{L}$. We see that $\mathrm{Lie}(\mathrm{Spin}(p, q))$ and $\mathbf{L}$ are not only isomorphic, but equal as vector spaces.

Remark 4.4.45. For any Lie group G the exponential map $\exp : \mathrm{Lie}(G) \to G$ is defined using the integral curve of elements of $\mathrm{Lie}(G)$. From the above findings, it follows that $\exp(\theta e_i e_j) = \cos(\theta) + \sin(\theta)e_i e_j$ for $\theta \in \mathbb{R}, e_i^2 = e_j^2$ and $\exp(\theta e_i e_j) = \cosh(\theta) + \sinh(\theta)e_i e_j$ for $e_i^2 = -e_j^2$. Considering the power series of $\cos, \sin, \sinh$ and $\cosh$, we find

$$
\exp(\theta e_i e_j) = \sum_{k=0}^{\infty} \frac{(e_i e_j)^k}{k!}\theta^k = \sum_{k=0}^{\infty} \frac{1}{k!}(\theta e_i e_j)^k = \begin{cases} \cos(\theta) + \sin(\theta)e_i e_j & \text{if } (e_i e_j)^2 = -1 \\ \cosh(\theta) + \sinh(\theta)e_i e_j & \text{if } (e_i e_j)^2 = 1 \end{cases}
$$

This is the same formula as the exponential of a number or a matrix.

Conversely, for a given e_i, e_j such that $e_i^2 = e_j^2$ we can choose a $\theta \in \mathbb{R}$ such that $\exp(\theta e_i e_j) = 1$ or similar for $\exp(\theta e_i e_j) = e_i e_j$. Therefore, the exponentials of elements $\theta e_i e_j, e_i^2 = e_j^2$ generate a part of $\mathrm{Spin}(p, q)$. However, for $e_i^2 = -e_j^2$ we cannot reach $e_i e_j$ as for large θ we instead get $\cosh(\theta) + \sinh(\theta)e_i e_j \approx e^{\theta}(1 + e_i e_j)$ but $1 + e_i e_j$ is not an element of $\mathrm{Spin}(p, q)$. Hence, the exponentials of $\theta e_i e_j \in \mathbf{L}$ generate a Lie subgroup of $\mathrm{Spin}(p, q)$ (called the identity component), which is equal to the entire group for $q = 0$.

Now consider $C\ell(n, \mathbb{C})$. Now $\mathbf{L} = [V, V]$ is complex. In this case, $\mathbf{L}$ is not equal to $\mathrm{Lie}(\mathrm{Spin}(n, \mathbb{C})) = (\mathbb{R}i) \times \mathfrak{so}(n)$. In fact, we cannot write any element of $\mathbf{L}$ as a product of a complex number with an element in

the real space $C\ell^2(n)$; we have to consider $\mathbf{L}$ as a complex Lie algebra. This gives us a fundamental difference between $(\mathbb{R}i) \times \mathfrak{so}(n)$ and $\mathbf{L}$, and implies there is no useful homomorphism between the two Lie algebras. Note that the equality $[[x, y], e_i] = 4y_i x - 4x_i y$ again holds, so the image of $\phi : \mathbf{L} \to M(n)$ consists of complex anti-symmetric matrices. Therefore, the image is contained in $\mathfrak{so}(n, \mathbb{C})$, the complex Lie algebra of the Lie group $SO(n, \mathbb{C})$ of complex matrices A such that $AA^{\mathsf{T}} = \mathbb{1}$, where the matrix is transposed but not (complex) conjugated. Since the dimension of $\mathfrak{so}(n, \mathbb{C})$ is equal to that of $\mathbf{L}$, we again get an isomorphism.

Remark 4.4.46. That $\mathfrak{so}(n, \mathbb{C})$ is isomorphic to a Lie subalgebra of the Clifford algebra $C\ell(n, \mathbb{C})$ implies that a cover of the group $SO(n, \mathbb{C})$ should be inside the Clifford algebra. Indeed, there are sources which instead define $\mathrm{Spin}^{\mathbb{C}}(n)$ as the double cover of $SO(n, \mathbb{C})$. This construction can be done by using the definition of the transpose map without complex conjugation, and following a construction similar to our construction for $\mathrm{Spin}^{\mathbb{C}}(n)$.

We have identified the relevant substructures of the Clifford algebras $C\ell(p, q)$ and $C\ell(n, \mathbb{C})$. In the next chapter, our aim is to determine the representations of these substructures.

5   Representations of the Spin group

We have seen a lot of relations relating the (real) Spin group with the complex Spin group or with the bivector algebra. Therefore, we try to determine all representations of the Spin group, as that will additionally gives us information about the representations of the other substructures of the Clifford algebra.
We do this by determining the representations of $\mathfrak{so}(n)$, i.e. Lie algebra homomorphisms to the general linear Lie algebra $\mathfrak{gl}(V) = \mathrm{Lie}(GL(V))$ (with V a vector space), and by using that the exponential of any Lie algebra homomorphism $\Phi : \mathfrak{so}(n) \to \mathfrak{gl}(V)$ gives a Lie group homomorphism $\phi : \mathrm{Spin}(n) \to GL(V)$ [see 9, pg. 116]. Conversely, any representation $\mathrm{Spin}(n) \to GL(V)$ induces a push-forward $\mathfrak{so}(n) \to \mathfrak{gl}(V)$.
Since we can extend any representation of $\mathfrak{so}(n)$ to a representation of $\mathfrak{so}(n, \mathbb{C})$ through complexification, we will determine all representations of $\mathfrak{so}(n, \mathbb{C})$. This has the added bonus that the representations of the identity component of $\mathrm{Spin}(p, q)$ can also be found, since $\mathfrak{so}(p, q)$ is isomorphic to a subalgebra of $\mathfrak{so}(n, \mathbb{C})$. We will follow the construction of Fulton and Harris [9], which writes any irreducible representation as a direct sum of eigenvector spaces of certain elements of the Lie algebra, and describes all other elements of the Lie algebra by how they permute these eigenvector spaces.

Remark 5.0.1. Note that this construction only works because $\mathfrak{so}(n, \mathbb{C})$ is a semisimple Lie algebra, i.e. for any $I \subset \mathfrak{so}(n, \mathbb{C})$ such that $[\mathfrak{so}(n, \mathbb{C}), I] \subseteq I$ we have $I = 0$. This semisimplicity also complies the complete reducibility of representations of $\mathfrak{so}(n, \mathbb{C})$, i.e. if $W \subset V$ is an invariant subspace of a module V then there is another invariant subspace W' such that $V = W \oplus W'$, and that if $\mathfrak{h} \subset \mathfrak{so}(n, \mathbb{C})$ is a set of simultaneously diagonalisable matrices, then the image of $\mathfrak{h}$ in $\mathfrak{gl}(V)$ is also simultaneously diagonalisable. The proofs for these statements can be found in Appendix C of Fulton and Harris [9].

Before construction the irreducible representations of $\mathfrak{so}(n, \mathbb{C})$, we first need to consider the representations of $\mathfrak{sl}(2, \mathbb{C})$, the Lie algebra of complex 2×2 matrices with trace 0.[4] This will not only clarify the process, but will actually be necessary when we use Lie subalgebras $\mathfrak{s} \subset \mathfrak{so}(n, \mathbb{C})$ which satisfy $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{C})$ to set up a lattice of weights.

5.1   Representations of $\mathfrak{sl}(2, \mathbb{C})$.

As mentioned, $\mathfrak{sl}(2, \mathbb{C})$ is the matrix of 2×2 matrices with trace 0. Therefore, the algebra is spanned by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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[4]This is the Lie algebra of the complex special linear group $SL(2, \mathbb{C})$ of 2×2 matrices with determinant 1.

Computing the commutation relations, we find $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. The first two equations essentially tell us that X, Y are 'eigenvectors' with respect to the action $[H, \cdot]$ of H on $\mathfrak{sl}(2, \mathbb{C})$. This action where an element of the Lie algebra acts by commutation is know as the <u>adjoint action</u> and notated by $\mathrm{ad}(H) = [H, \cdot]$. Since $[H, H] = 0$, we see that H, X, Y are eigenvectors of $\mathrm{ad}(H)$ with eigenvalues 0, 2 and -2 respectively. Since $\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C} \cdot X \oplus \mathbb{C} \cdot H \oplus \mathbb{C} \cdot Y$, we see that $\mathfrak{sl}(2, \mathbb{C})$ is split up into eigenvectors of $\mathrm{ad}(H)$. While this decomposition is not directly important in the 2 dimensional case, we will see its relevance when we consider $\mathfrak{so}(n, \mathbb{C})$.

We will now characterise the representations of $\mathfrak{sl}(2, \mathbb{C})$ by looking at the action of H on the vector space. Let $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V)$ be a representation, where V is a (finite dimensional) complex vector space. We will write $A \cdot v$ for the matrix multiplication $\phi(A)v$.

Since H is diagonal, the final statement of Remark 5.0.1 implies that we can choose a basis of V such that $\phi(H)$ is diagonal. This decomposes V into eigenspaces V_α such that $H \cdot v = \alpha v$ for $v \in V_\alpha$, where $\alpha \in \mathbb{C}$. Next consider the action of X, Y on such an eigenvector $v \in V_\alpha$. We cannot directly say anything about $X \cdot v$ or $Y \cdot v$, but we can consider $H \cdot (X \cdot v)$. Using the commutation relations, we find

$$H \cdot (X \cdot v) = (HX) \cdot v = [H, X] \cdot v + (XH) \cdot v = 2X \cdot v + X \cdot (\alpha v) = (\alpha + 2)(X \cdot v).$$

This shows that $X \cdot v$ is again an eigenvector of H, this time with eigenvalue $\alpha + 2$. Similarly $H \cdot (Y \cdot v) = (\alpha - 2)Y \cdot v$. This shows that X and Y send the spaces V_α to $V_{\alpha+2}$ respectively $V_{\alpha-2}$. Moreover, if $\alpha \neq 0$ we can exclude $X \cdot v = Y \cdot v = 0$, since $H = [X, Y]$, so $X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = \alpha v$.

Now additionally assume our representation is irreducible. Since H only sends each V_α to itself, the lack of invariant subspaces must come from the X and Y. Since each V_α can only be send to $V_{\alpha+2}$ and $V_{\alpha-2}$ by X and Y, it follows that $\bigoplus_{k \in \mathbb{Z}} V_{\alpha+2k}$ must be an invariant subspace, where we abuse notation by saying $V_\beta = \{0\}$ for eigenvalues β which do not occur. Therefore, the irreducibility of the representation gives that all eigenvalues must be equal modulo 2. Since V is finite dimensional, we see that there must be a 'smallest' eigenvalue α such that the other eigenvalues are $\alpha + 2, \alpha + 4, \ldots, \alpha + 2\ell$ for $\ell \in \mathbb{Z}$. We define $n := \alpha + 2\ell$. We now exactly know how X acts on V_n: $V_{n+2} = \{0\}$, so $X \cdot v = 0$ for $v \in V_n$. Since $H \cdot v = nv$, this implies $X \cdot (Y \cdot v) = nv$ so $Y \cdot v \neq 0$. This brings us to the following proposition.

Proposition 5.1.1. *Let $v \in V_n$ and let $Y^k \cdot v$ be short for $Y \cdot (\ldots (Y \cdot v))$ with k copies of Y for $k \in \mathbb{N}$. Then V is spanned by $v, Y \cdot v, Y^2 \cdot v, \cdots$.*

Proof. We will show the span W of these vectors is an invariant subspace; the irreducibility of the representation then gives that the whole vector space is spanned by the given vectors. First of all, note that $H \cdot (Y^k \cdot v) = (\alpha - 2k)Y^k \cdot v$, so $H \cdot W \subseteq W$. Secondly, $Y \cdot (Y^k \cdot v) = Y^{k+1} \cdot v$ so $Y \cdot W \subseteq W$. Now we only have to show that $X \cdot W \subseteq W$. We show this inductively over k for $Y^k \cdot v$.

For the basis, we see that $X \cdot v = 0 \in W$. Now if $X \cdot (Y^k \cdot v) = k(n - k + 1)(Y^{k-1} \cdot v)$, then

$$\begin{aligned}
X \cdot (Y^{k+1} \cdot v) &= (XY) \cdot (Y^k \cdot v) = [X, Y] \cdot (Y^k \cdot v) + Y \cdot (X \cdot (Y^k \cdot v)) \\
&= H \cdot (Y^k \cdot v) + Y \cdot (X \cdot (Y^k \cdot v)) = (n - 2k)(Y^k \cdot v) + Y \cdot (k(n - k + 1)Y^{k-1} \cdot v) \\
&= (k(n - k + 1) + n - 2k)(Y^k \cdot v) = (k + 1)(n - (k + 1) + 1)(Y^k \cdot v).
\end{aligned}$$

Using induction, we find $X \cdot (Y^k \cdot v) = k(n - k + 1)Y^{k-1} \cdot v$ for any $k \in \mathbb{N}$.

We find that X leaves W invariant, so W is an invariant subspace. We conclude that V is spanned by $Y^k \cdot v$, $k \in \mathbb{N}_0 := \mathbb{N} \cup 0$. $\square$

The proposition and its proof directly give us a few corollaries.

Corollary 5.1.2. *The eigenspaces V_β are one dimensional.*

Corollary 5.1.3. *The representation is uniquely determined by the eigenvalues $\alpha, \alpha + 2, \ldots, n$.*

Proof. We have found how X, Y and H act on the eigenspaces, and therefore how they act on V. Our description of these actions only depends on the eigenvalues. Hence, any two representations with the same eigenvalues must be equivalent. $\qquad\square$

Of course, when we talk about the uniqueness of representations, we mean uniqueness up to equivalence. The explicit formula $X \cdot (Y^k \cdot v) = k(n - k + 1)(Y^{k-1} \cdot v)$ combined with the finite dimension of V also gives us the following result.

Proposition 5.1.4. *The eigenvalues are integers and equal to $-n, -n+2, \ldots, n-2, n$.*

Proof. We first show that n is an integer. We know that the $Y^k \cdot v, k \in \mathbb{N}_0$ span the vector space. Moreover, the sequence $\{Y^k \cdot v\}_k$ must terminate after finite values, since each non-zero $Y^k \cdot v$ is linearly independent of the other non-zero eigenvectors. Therefore, there is a smallest $m \in \mathbb{N}_0$ such that $Y^m \cdot v = 0$. Now $X \cdot (Y^m \cdot v) = m(n - m + 1)(Y^{m-1} \cdot v) = 0$, so $m(n - m + 1) = 0$. Since $v \neq 0$ we have $m > 0$, so $n - m + 1 = 0$. We conclude that $n = m - 1 \in \mathbb{Z}$.
Conversely, we find that $m = n+1$ is the smallest integer such that $Y^m \cdot v = 0$, so there are $n+1$ eigenvalues: $n, n - 2, \ldots, -n$. $\qquad\square$

Together with Corollary 5.1.3 we find that the irreducible representation is uniquely determined by n.
We have now found that for any irreducible representation there is a highest eigenvalue $n \in \mathbb{N}_0$ such that the representation is uniquely determined by the value of n. This can be interpreted as an uniqueness theorem. Therefore, our next question is about the existence of the corresponding representations. Fortunately, the direct formula we have found for X, Y, H allow us to write them as matrices acting on the span of $e_1 := v, e_2 := Y \cdot v, \ldots, e_{n+1} := Y^n \cdot v$, which gives us an irreducible representation for each $n \in \mathbb{N}_0$.

Proposition 5.1.5. *Let $n \in \mathbb{N}_0$ and let $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^{n+1})$ be defined by*

$$\phi(H) = \begin{pmatrix} n & 0 & 0 & \ldots & 0 & 0 \\ 0 & n-2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & n-4 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -n+2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & -n \end{pmatrix}, \ \phi(X) = \begin{pmatrix} 0 & n & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2(n-1) & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & n \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

and

$$\phi(Y) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

extended linearly, where the coefficient of X are given by $k(n - k + 1)$ for $k \in \mathbb{N}_{\leq n}$. Then ϕ gives the (unique) irreducible representation with highest eigenvalue n.

Proof. Direct calculation shows that the $\phi(H), \phi(X), \phi(Y)$ satisfy the commutation relations $[\phi(H), \phi(X)] = 2\phi(X)$, $[\phi(H), \phi(Y)] = -2\phi(Y)$ and $[\phi(X), \phi(Y)] = \phi(H)$, so we find $[\phi(H), \phi(X)] = \phi([H, X])$ and similar for $[H, Y]$ and $[X, Y]$. This shows that ϕ is indeed a Lie algebra homomorphism.
Next, we have to show this representation is irreducible. Let $v = (v_0, v_1, \ldots, v_n) \in \mathbb{C}^{n+1}$ be non-zero. Then there is an $v_i \neq 0$. Now the first component of $\phi(X)^i v$ is a scalar multiple of v_i, where the scalar is non-zero (as it is a product of $k(n - k + 1)$ for $k = 1, \ldots, i$). Therefore, the invariant subspace containing v must also contain an element of the form $(1, w_1, w_2, \ldots, w_n), w_i \in \mathbb{C}$.

Now let $w = (1, w_1, w_2, \ldots, w_n)$, $w_i \in \mathbb{C}$. By definition of ϕ, we have $\phi(Y)w = (0, 1, w_1, \ldots, w_{n-1})$, $\phi(Y)^2 w = (0, 0, 1, \ldots, w_{n-2})$ up until $\phi(Y)^n w = (0, 0, \ldots, 1)$. It is clear that these vectors are all non-zero and linearly independent, so we conclude that the invariant subspace W containing w must at least be $n + 1$ dimensional. Since $W \subseteq \mathbb{C}^{n+1}$, we find that $W = \mathbb{C}^{n+1}$.

We conclude that the representation is irreducible. $\qquad\square$

This way of finding representations using the relations only works in this specific 2-dimensional case, as the eigenvalues are all on a line. More generally, we will determine representations by considering the standard representation and its symmetric and exterior powers. By the standard representation we mean the representation where H, X, Y directly act on $V = \mathbb{C}^2$, which is possible as they are 2×2 matrices, while they act on the symmetric powers $\operatorname{Sym}^k(V)$ and exterior powers $\Lambda^k(V)$ by extension.

Remark 5.1.6. Recall that

$$\operatorname{Sym}^k(V) = \{v_1 \cdot v_2 \cdot \ldots \cdot v_k \mid v_1, \ldots, v_k \in V, \ v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot \ldots \cdot v_{\sigma(k)} = v_1 \cdot v_2 \cdot \ldots \cdot v_k \text{ for all } \sigma \in S_k\}$$

where S_k is the symmetric group of k elements, and

$$\Lambda^k(V) = \{v_1 \wedge v_2 \wedge \ldots \wedge v_k \mid v_1, \ldots, v_k \in V, \ v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \ldots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma) v_1 \wedge v_2 \wedge \ldots \wedge v_k \text{ for all } \sigma \in S_k\}$$

where $\operatorname{sgn}(\sigma) = \pm 1$ is the sign of the permutation σ. The symmetric power $\operatorname{Sym}^k(V)$ corresponds to the homogeneous polynomials or order k; therefore, the representations of the form $\operatorname{Sym}^k(V)$ are known as polynomial representations in physics.

Example 5.1.7. Consider $\operatorname{Sym}^2(V)$ and $\Lambda^3(V)$. H acts on $v \cdot w \in \operatorname{Sym}^2(V)$ as $H(v \cdot w) = (Hv) \cdot w + v \cdot (Hw)$, and acts on $u \wedge v \wedge w \in \Lambda^3(V)$ as $H(u \wedge v \wedge w) = (Hu) \wedge v \wedge w + u \wedge (Hv) \wedge w + u \wedge v \wedge (Hw)$. $\qquad\triangle$

The most important advantage of the symmetric and exterior powers is that we can easily calculate their eigenvectors. Since H is the diagonal matrix with $1, -1$ on the diagonal, we can 'plot' the eigenvalues of the standard representation as $\{1, -1\} \subset \mathbb{R}$:

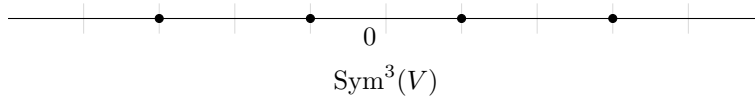Standard representation (V)
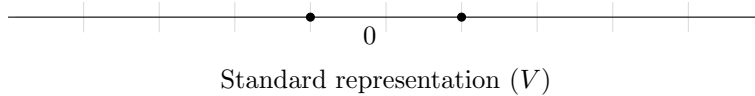
This is called a weight diagram, where the eigenvalues are weights. The non-zero weights of the adjoint representation are called the roots of the representation. For $\mathfrak{sl}(2, \mathbb{C})$ the roots are $\{2, -2\}$ corresponding to X respectively Y.

The weight diagram of $\operatorname{Sym}^2(V)$ has all products of roots, but without 'double counting', so

$\operatorname{Sym}^2(V)$
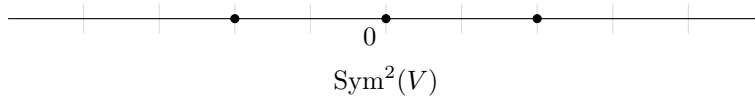
We see that this is the weight diagram for the irreducible representation with $n = 2$. Similarly, we have

$\operatorname{Sym}^3(V)$

for the irreducible representation with $n = 3$. Although we won't prove it here, it follows that the standard representation extended to its n-th symmetric power gives the irreducible representation with highest weight n.

We have seen all the main tools for analysing the representations. We will now generalise the same construction to the special orthogonal Lie algebras. In order to understand how we generalise the construction, we first consider the small cases $\mathfrak{so}(n, \mathbb{C})$, for $n = 4, 5$.

Remark 5.1.8. We ignore the cases $n = 1, 2, 3$, because they can be determined directly through the so-called 'accidental' isomorphisms. For instance, $\mathfrak{so}(1, \mathbb{C})$ is the group of 1×1 anti-symmetric matrices, i.e. $\mathfrak{so}(1, \mathbb{C}) \cong 0$. Similarly, we have $\mathfrak{so}(2, \mathbb{C}) \cong \mathbb{C}^*$ and $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. While we could take them as our examples, they have more structure then we can expect from $\mathfrak{so}(n, \mathbb{C})$ for general $n \in \mathbb{N}$, which makes the examples slightly misleading.

However, in the very first step of our construction, we used the diagonal matrix H. Since the matrices in $\mathfrak{so}(n, \mathbb{C})$ are anti-symmetric, we clearly cannot have any diagonal matrices in $\mathfrak{so}(n, \mathbb{C})$. We therefore first have to change the basis of our Lie algebra.

5.2 Interlude: alternative basis for $\mathfrak{so}(n, \mathbb{C})$

In order to change the basis of $\mathfrak{so}(n, \mathbb{C})$, we first return to its definition. While there are multiple equivalent definitions of $SO(n, \mathbb{C})$, one of the definitions is that $SO(n, \mathbb{C})$ is the group of $n \times n$ matrices which preserve the non-degenerate symmetric bilinear form $\langle x, y \rangle = x^\mathsf{T} y$ for $x, y \in \mathbb{C}^n$. Implicitly, we have $x^\mathsf{T} y = x^\mathsf{T} I_n y$ with I_n the identity, so the symmetric bilinear form is induced by the identity. However, we have seen that non-degenerate symmetric bilinear forms on $\mathbb{C}^n$ are all equivalent, so we could also define $SO(n, \mathbb{C})$ as the group of $n \times n$ matrices which preserve the (non-degenerate symmetric) bilinear form $\langle x, y \rangle = x^\mathsf{T} M y$ with M a symmetric matrix with non-zero determinant. This implies a basis in which $SO(n, \mathbb{C})$ consist of matrices A such that $A^\mathsf{T} M A = M$. In this basis, the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ is the Lie algebra of traceless matrices B such that $B^\mathsf{T} M + M B = 0$.

Our goal is to think of a M such that the corresponding basis of $\mathfrak{so}(n, \mathbb{C})$ is such that the matrices are in general not anti-symmetric. This can be done by no longer making M diagonal. Our choice for M is as follows, splitting the cases for even and odd:

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \text{ for } \mathfrak{so}(2n, \mathbb{C}) \text{ and } M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } \mathfrak{so}(2n + 1, \mathbb{C}).$$

Remark 5.2.1. Note that the bilinear form induced by M is exactly the bilinear form on $V \oplus V^*$ in Subsection 4.3.1.

Let us first consider the even case, $\mathfrak{so}(2n, \mathbb{C})$. We want to put a condition on the elements of $\mathfrak{so}(2n, \mathbb{C})$ similar to 'anti-symmetric'. Therefore, consider a $2n \times 2n$ matrix consisting of 4 $n \times n$ matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and assume this matrix is an element of $\mathfrak{so}(2n, \mathbb{C})$. Now

$$0 = \begin{pmatrix} A^\mathsf{T} & C^\mathsf{T} \\ B^\mathsf{T} & D^\mathsf{T} \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C^\mathsf{T} + C & A^\mathsf{T} + D \\ D^\mathsf{T} + A & B^\mathsf{T} + B \end{pmatrix},$$

so we find that B, C are anti-symmetric and A, D are minus each others transpose.

The odd case is similar. In this case, an arbitrary matrix in $\mathfrak{so}(2n + 1, \mathbb{C})$ can be written as

$$\begin{pmatrix} A & B & E \\ C & D & F \\ G & H & 0 \end{pmatrix}$$

where E, F are $n \times 1$ matrices and G, H are $1 \times n$ matrices. Note that the last component must be 0 because M has a 1 as final element of the diagonal. The condition becomes:

$$0 = \begin{pmatrix} A^\mathsf{T} & C^\mathsf{T} & G^\mathsf{T} \\ B^\mathsf{T} & D^\mathsf{T} & H^\mathsf{T} \\ E^\mathsf{T} & F^\mathsf{T} & 0 \end{pmatrix} \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & 0 \end{pmatrix} = \begin{pmatrix} C^\mathsf{T} + C & A^\mathsf{T} + D & G^\mathsf{T} + F \\ D^\mathsf{T} + A & B^\mathsf{T} + B & H^\mathsf{T} + E \\ F^\mathsf{T} + G & E^\mathsf{T} + H & 0 \end{pmatrix},$$

so C and B are anti-symmetric, $A^\mathsf{T} = -D$, $F^\mathsf{T} = -G$ and $E^\mathsf{T} = -H$.

With this description of the elements of $\mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{so}(2n+1, \mathbb{C})$, we are ready for $\mathfrak{so}(4, \mathbb{C})$ and $\mathfrak{so}(5, \mathbb{C})$.

5.3   Representations of $\mathfrak{so}(4, \mathbb{C})$

In this section, we consider $\mathfrak{so}(4, \mathbb{C})$, so $n = 2$. Our first goal is to generalise the H, X and Y from $\mathfrak{sl}(2, \mathbb{C})$. Therefore, we are looking for a subalgebra of commuting diagonal matrices, called an Cartan subalgebra and notated by $\mathfrak{h}$. From the previous description, it is clear that each element of $\mathfrak{so}(4, \mathbb{C})$ is of the form

$$\begin{pmatrix} a & b & 0 & e \\ c & d & -e & 0 \\ 0 & f & -a & -c \\ -f & 0 & -b & -d \end{pmatrix}$$

with $a, b, c, d, e, f \in \mathbb{C}$. Therefore, our Cartan subalgebra $\mathfrak{h}$ is generated by $H_1 = E_{1,1} - E_{3,3}$ (the a's) and $H_2 = E_{2,2} - E_{4,4}$ (the d's), where $E_{i,j}$ denotes the matrix with a 1 on the (i, j)-th place and zeroes otherwise. These two matrices obviously do not span the whole Lie algebra. We therefore define a few more matrices, corresponding to the b, c, e and f respectively in the general matrix; let $X_{1,2} = E_{1,2} - E_{4,3}$, $X_{2,1} = E_{2,1} - E_{3,4}$, $Y_{1,2} = E_{1,4} - E_{3,2}$ and $Z_{1,2} = E_{4,1} - E_{2,3}$. It is clear that H_1, H_2, $X_{1,2}$, $X_{2,1}$, $Y_{1,2}$ and $Z_{1,2}$ span $\mathfrak{so}(4, \mathbb{C})$ linearly. Calculating the commutators of the other elements with H_i, we get $[H_1, X_{1,2}] = X_{1,2}$, $[H_1, X_{2,1}] = -X_{2,1}, [H_1, Y_{1,2}] = Y_{1,2}$ and $[H_1, Z_{1,2}] = -Z_{1,2}$ for H_1, and $[H_2, X_{1,2}] = -X_{1,2}, [H_2, X_{2,1}] = X_{2,1}, [H_2, Y_{1,2}] = Y_{1,2}$ and $[H_2, Z_{1,2}] = -Z_{1,2}$. We see that $X_{1,2}, X_{2,1}, Y_{1,2}$ and $Z_{1,2}$ are eigenvectors of H_1 and H_2, but they do not have the same eigenvalue for H_1 and H_2. Therefore, in order to talk about 'the eigenvalue' we would have to talk about a vector of two eigenvalues, or a similar object. However, this becomes tedious when we generalise to higher dimensions. Instead, we define L_1 and L_2 to be the dual elements of H_1, H_2, so $L_1, L_2 \in \mathfrak{h}^*$ such that $L_i(H_j) = \delta_{i,j}$, and describe the eigenvalues as elements of $\mathfrak{h}^*$. Now the eigenvalue of $X_{1,2}$ is $L_1 - L_2$ while the eigenvalue of $X_{2,1}$ is $L_2 - L_1$. It is clear that the eigenvalues of $Y_{1,2}$ and $Z_{1,2}$ are $L_1 + L_2$ respectively $-L_1 - L_2$. We see that the eigenvalues are $\{\pm L_1 \pm L_2\}$. A plot of our non-zero eigenvalues is in Figure 1a.

We call the non-zero eigenvalues (now elements of $\mathfrak{h}^*$) the roots of our Lie algebra; it is clear that this definitions of the roots is a generalisation of the roots of $\mathfrak{sl}(2, \mathbb{C})$. The set of roots is normally denoted by R. We can also directly find the eigenvalues of the standard representation in terms of L_1, L_2; the vectors $(1, 0, 0, 0), \ldots, (0, 0, 0, 1)$ respectively give $L_1, L_2, -L_1$ and $-L_2$. We call these values the weights of the standard representation. A weight diagram of the standard representation, i.e. the plot of the weights of the standard representation, can be found in Figure 1b. The eigenvectors of (the elements of) the image of $\mathfrak{h}$ under the representation are called the weight vectors.

In the figure, we implicitly assume we can draw $L_1 \perp L_2$. The relevant bilinear form on $\mathfrak{h}^*$ is known as the Killing form [see 9, pg. 206]. Abusing the notation to write B for both the form on $\mathfrak{h}$ and the induced form on $\mathfrak{h}^*$, we can define this bilinear form on $\mathfrak{h}$ as $B(H, H') = \sum_{\alpha \in R} \alpha(H)\alpha(H')$. Since $R = \{\pm L_1 \pm L_2\}$, we find that $B(H_i, H_j) = \delta_{i,j}$. The induced bilinear form on $\mathfrak{h}^*$ therefore also has $B(L_i, L_j) = \delta_{i,j}$, so we indeed find $L_1 \perp L_2$.

(a) The roots of $\mathfrak{so}(4, \mathbb{C})$.   (b) The weights of the standard representation.
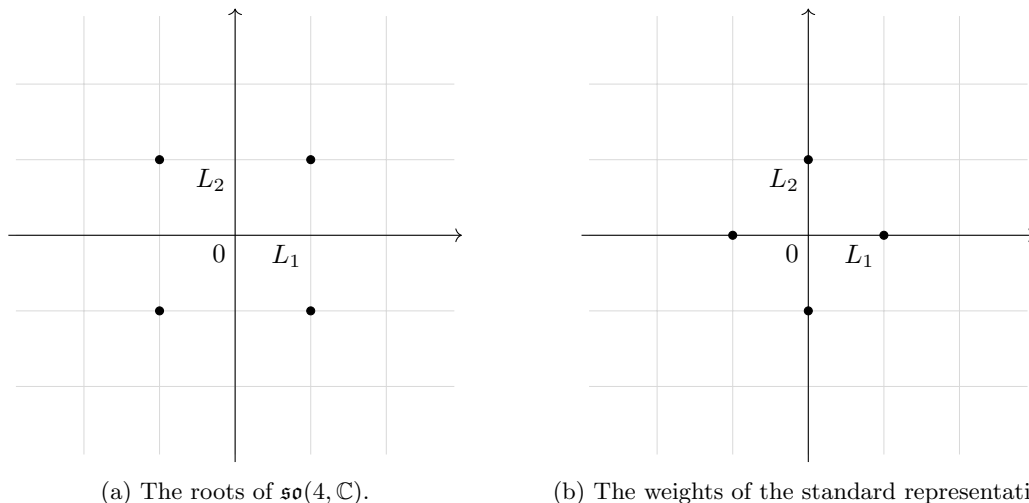
Figure 1

We have now seen two representations of $\mathfrak{so}(4, \mathbb{C})$. Our next step is to take any (irreducible) representation of $\mathfrak{so}(4, \mathbb{C})$ and show that it is uniquely determined by its weights. Moreover, we also want to show the existence of irreducible representations corresponding to certain given weights with given multiplicities.

Remark 5.3.1. In the case of $\mathfrak{sl}(2, \mathbb{C})$ and in the previous two representations, we have only seen weights with multiplicity 1, i.e. for every eigenvalue, the corresponding eigenspace was 1-dimensional. This will not be true in general.

We are currently missing a lot of information necessary to start calculating weights. The Figures 1ab are both symmetrical in a very nice way, so the symmetric and exterior powers will also be symmetrical. With that in mind, we want to prove that the weights of any representation are symmetrical. We also want to prove that the weights are discrete. We do this by finding copies of $\mathfrak{sl}(2, \mathbb{C})$ inside $\mathfrak{so}(4, \mathbb{C})$; after all, we have already found that the representations of $\mathfrak{sl}(2, \mathbb{C})$ are discrete and symmetric.
We first need the following definition.

Definition 5.3.2. Let $\alpha \in R$. Then $\mathfrak{g}_\alpha$ denotes the eigenspace corresponding to the root α. We call $\mathfrak{g}_\alpha$ a root space and the vectors in $\mathfrak{g}_\alpha$ root vectors.

We have already determined the roots and root vectors, so we know that $\mathfrak{g}_{L_1-L_2} = \mathbb{C} \cdot X_{1,2}$, $\mathfrak{g}_{L_2-L_1} = \mathbb{C} \cdot X_{2,1}$ and similar for $Y_{1,2}$ and $Z_{1,2}$. Now our Lie algebra decomposes into a direct sum as

$$\mathfrak{so}(4, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

This is called the Cartan decomposition of $\mathfrak{so}(4, \mathbb{C})$. Using this decomposition, we get the following proposition:

Proposition 5.3.3. *Let $\alpha \in R$ and define $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Then $\mathfrak{s}_\alpha$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. Note that $\mathfrak{s}_\alpha = \mathfrak{s}_{-\alpha}$, so we have two cases: $\alpha = L_1 + L_2$ or $\alpha = L_1 - L_2$. In the first case, we have $\mathfrak{s}_\alpha = \mathbb{C} \cdot Y_{1,2} \oplus \mathbb{C} \cdot Z_{1,2} \oplus \mathbb{C} \cdot [Y_{1,2}, Z_{1,2}]$. Direct calculation gives $[Y_{1,2}, Z_{1,2}] = -H_1 - H_2$. In order to prove the isomorphism, we now have to find $H_{L_1+L_2}, X_{L_1+L_2}$ and $Y_{L_1+L_2}$ satisfying the commutation relations $[H_{L_1+L_2}, X_{L_1+L_2}] = 2X_{L_1+L_2}$, $[H_{L_1+L_2}, Y_{L_1+L_2}] = -2Y_{L_1+L_2}$ and $[X_{L_1+L_2}, Y_{L_1+L_2}] = H_{L_1+L_2}$. It is clear that $X_{L_1+L_2}$ and $Y_{L_1+L_2}$ are not unique; we can e.g. multiply one by 3 and the other by 1/3 and the same commutation relations still hold. However, $H_{L_1+L_2}$ is fixed by the eigenvalues $2, -2$.

Since $H_{L_1+L_2}$ must be a multiple of $-H_1 - H_2$, we calculate $[-H_1 - H_2, Y_{1,2}] = (L_1 + L_2)(-H_1 - H_2)Y_{1,2} = -2Y_{1,2}$. This is minus the desired eigenvalue, so we find $H_{L_1+L_2} = H_1 + H_2$. We choose $X_{L_1+L_2} = Y_{1,2}$ and $Y_{L_1+L_2} = Z_{1,2}$ and note that the commutation relations are now satisfied. These $H_{L_1+L_2}, X_{L_1+L_2}, Y_{L_1+L_2}$ span $\mathfrak{s}_{L_1+L_2}$ in the exact same way as H, X, Y span $\mathfrak{sl}(2,\mathbb{C})$, hence we conclude the two Lie algebra are isomorphic.

The case $\alpha = L_1 - L_2$ is similar. Now $[X_{1,2}, X_{2,1}] = H_1 - H_2$ and $[H_1 - H_2, X_{1,2}] = X_{1,2}$, so we get $H_{L_1-L_2} = H_1 - H_2$ and choose $X_{L_1-L_2} = X_{1,2}$, $Y_{L_1-L_2} = X_{2,1}$ to find that $\mathfrak{s}_{L_1-L_2}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. $\qquad\square$

The copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}(2,\mathbb{C})$ inside $\mathfrak{so}(4,\mathbb{C})$ are called the distinguished copies. These copies tell us a lot about the weights of an arbitrary representation. After all, the restriction of the representation to $\mathfrak{s}_\alpha$ gives a representation of $\mathfrak{sl}(2,\mathbb{C})$. But we know that the weights of any representation of $\mathfrak{sl}(2,\mathbb{C})$ are integers.

Definition 5.3.4. Let Λ_W be the lattice of all $\beta \in \mathfrak{h}^*$ such that $\beta(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in R$. We call this lattice the weight lattice.

Corollary 5.3.5. *The weights of any representation of $\mathfrak{so}(4,\mathbb{C})$ must lie on the weight lattice.*

Since we know $H_{L_1+L_2} = H_1 + H_2$ and $H_{L_1-L_2} = H_1 - H_2$, we find

$$\Lambda_W = \{aL_1 + bL_2 \mid a + b \in \mathbb{Z} \text{ and } a - b \in \mathbb{Z}\} = \left\{aL_1 + bL_2 \,\middle|\, a, b \in \frac{1}{2}\mathbb{Z}\right\}.$$

We see that all half-integer combinations of L_1 and L_2 can appear as weights. Note that the roots and the weights of the standard representation are all integer in L_1 and L_2. This is related to the fact that $SO(n,\mathbb{C})$ has a double cover (similar to $\mathrm{Spin}(n)$); we will see the relation later in this section.

We finally assume that $\phi : \mathfrak{so}(4,\mathbb{C}) \to \mathfrak{gl}(V)$ is a representation with V a complex vector space. As noted in Remark 5.0.1, the semisimplicity of $\mathfrak{so}(4,\mathbb{C})$ implies that we can choose a basis of V such that $\phi(H_1), \phi(H_2)$ are diagonal. The matrix multiplications of $\phi(H_1), \phi(H_2)$ now splits V into a direct sum of eigenspaces $\bigoplus V_\beta$ where β are the weights of V. Note that the eigenspaces V_β do not have to be 1-dimensional.

We consider the action of the root vectors $X_{1,2}, X_{2,1}, Y_{1,2}$ and $Z_{1,2}$ on an eigenspace V_β. Let $i \in \{1,2\}$, let $v \in V_\beta$ be a weight vector and let A be the root vector corresponding to root α. Now

$$H_i \cdot (A \cdot v) = [H_i, A] \cdot v + A \cdot (H_i \cdot v) = \alpha(H_i)(A \cdot v) + \beta(H)(A \cdot v) = (\alpha + \beta)(H_i)(A \cdot v).$$

We either see that the weight vector $A \cdot v$ has weight $\alpha + \beta$, or $A \cdot v = 0$.

Corollary 5.3.6. *For any root $\alpha \in R$ and weight β of a given module V, the action of $\mathfrak{g}_\alpha$ on V sends V_β to a subspace of $V_{\alpha+\beta}$.*

This shows that if we restrict the representation of $\mathfrak{so}(4,\mathbb{C})$ to $\mathfrak{s}_\alpha$ for $\alpha = L_1 \pm L_2$, then the vector space V is guaranteed to split up into modules of the form $\bigoplus_{k \in \mathbb{Z}} V_{\beta+k\alpha}$ (here we again abuse notation by saying $V_\gamma = 0$ for any γ which is not a weight of V). By the symmetry of representations of $\mathfrak{sl}(2,\mathbb{C})$, any such module should be symmetrical in the line perpendicular to α. Since the $n = 0$ must also correspond to a weight which is perpendicular to α, we find that the modules are symmetrical in the line through $0 \in \mathfrak{h}^*$ perpendicular to α. For $\alpha = L_1 - L_2$ that means symmetry in $\mathbb{C} \cdot (L_1 + L_2)$ while for $\alpha = L_1 + L_2$ means symmetry in $\mathbb{C} \cdot (L_1 - L_2)$. See also Figure 2a.

We have found that the weight diagram of any representation should be symmetric in the line through $L_1 + L_2$ and $L_1 - L_2$. The group of these symmetries is known as the Weyl group.

Since our weight diagram is now symmetric, we only have to know a quarter of the weights of a given representation to determine all other weights. We will choose a direction and only consider weights in that direction. We also made this choice in the analysis of representations of $\mathfrak{sl}(2,\mathbb{C})$, but in a more implicit manner: we chose n to be the highest weight. We could have chosen $-n$, it would not have made a difference.

We first have to choose a similar idea to the 'highest weight'. In order to do that, we choose that L_1 is 'larger' than L_2. To make this precise, let $c, d \in \mathbb{R}$ such that $c > d > 0$. We say that the (possible) weight $aL_1 + bL_2 \in \Lambda_W$ is <u>positive</u> if $(aL_1 + bL_2)(cH_1 + dH_2) > 0$, i.e. if $ac + bd > 0$, and <u>negative</u> otherwise. Since $c > |d|$, we see that L_1, L_2, $L_1 + L_2$ and $L_1 - L_2$ are positive, but $L_2 - L_1$ is negative. This splits our set of roots R into two sets R^+ and R^- of positive respectively negative roots. For any root α, we note that $\alpha \in R^{\pm}$ implies $-\alpha \in R^{\mp}$. We shorten 'root vector with positive/negative weight' to 'positive/negative root vector'.

Using similar definitions, we can also compare two given weights. For two positive weights α_1, α_2, we say that α_1 is higher than α_2 if $\alpha_1(cH_1 + dH_2) > \alpha_2(cH_1 + dH_2)$.

We can now choose a quarter of the space, such that we can apply the symmetries on that quarter to determine the rest of the space. We will call this quarter the <u>Weyl chamber W</u>. It is natural to choose the chamber as a set of positive weights. Moreover, we choose our chamber such that the boundaries are symmetry axes. With these conditions in mind, there is only one natural choice: $W = \{aL_1 + bL_2 \mid a > |b| > 0\}$. Note that this definition of W does not include the boundaries, we will refer to its closure as the <u>closed Weyl chamber</u>. Our Weyl chamber is drawn in Figure 2c.

(a) The symmetry axes.   (b) The positive and negative weights.   (c) The Weyl chamber.
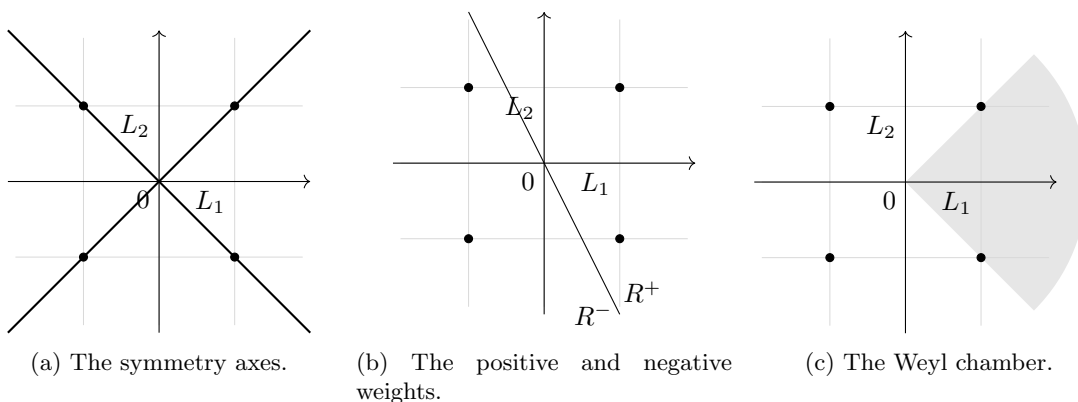
Figure 2: The symmetry axes, weights and Weyl chamber of $\mathfrak{so}(4, \mathbb{C})$. Each diagram has the roots for comparison.

We return to our (irreducible) module V. Since V is finite-dimensional, there must be a highest weight γ. Let $v \in V_\gamma$.

We already know that the action of $g_\alpha, \alpha \in R$ on V sends V_β to $V_{\alpha+\beta}$ for any weight β. For $\alpha \in R^+$, the weight $\alpha + \beta$ is higher than β. Therefore, the actions of $g_\alpha, \alpha \in R^+$ increase the weights, sending a weight vector to 0 or to another weights vector of higher weight. This corresponds to the action of X in $\mathfrak{sl}(2, \mathbb{C})$. Similarly, the actions of $g_\alpha, \alpha \in R^-$ decrease the weights, corresponding to $Y \in \mathfrak{sl}(2, \mathbb{C})$.

Since γ is the highest weight, we find that $A \cdot v = 0$ for any $A \in R^+$. This leads us to the following definition and proposition.

Definition 5.3.7. Let β be a weight of V and let $v \in V_\beta$ be non-zero. We call v a <u>highest weight vector</u> if $A \cdot v = 0$ for any positive root vector $A \in \mathfrak{so}(4, \mathbb{C})$.

Proposition 5.3.8. *Let $v \in V$ be a highest weight vector. Let W be the subspace of V generated by v and by the image of v under repeated applications of negative root vectors ($X_{2,1}$ and $Z_{1,2}$). Then W is an invariant subspace (i.e. W gives a subrepresentation) of V.*

Proof. First of all, note that W is a module, so we do not have to worry about the well-defined-ness of the action. Moreover, the action must be linear, so we only have to consider the action of elements of $\mathfrak{so}(4, \mathbb{C})$ on the elements spanning W.

By definition, it is clear that W is closed under applications of negative root vectors, and since every image of v is a weight vector or 0, W is also closed under the action $\mathfrak{h}$.

We note that $[Y_{1,2}, X_{2,1}] = 0 \in \mathfrak{h}$ and $[X_{1,2}, Z_{1,2}] = 0 \in \mathfrak{h}$, so the commutator of any negative root vector with any positive root vector is in $\mathfrak{h}$. Consider $v' = B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} \cdot v$, where $k \in \mathbb{N}_0$, $\alpha_i \in R^-$ and B_α is a root vector corresponding to the root α. We let a positive root vector A act on v'. Now

$$A \cdot (B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} \cdot v) = [A, B_{\alpha_1}] B_{\alpha_2} \cdots B_{\alpha_k} \cdot v + B_{\alpha_1}[A, B_{\alpha_2}] B_{\alpha_3} \cdots B_{\alpha_k} \cdot v + \ldots + B_{\alpha_1} \cdots [A, B_{\alpha_k}] \cdot v.$$

Now every $[A, B_{a_i}]$ is in $\mathfrak{h}$, so this reduces to

$$\begin{aligned}
A \cdot v' =& (\alpha - \alpha_1)([A, B_{\alpha_1}]) \cdot B_{\alpha_2} \cdots B_{\alpha_k} \cdot v \\
&+ (\alpha - \alpha_1 - \alpha_2)([A, B_{\alpha_2}]) \cdot B_{\alpha_1} B_{\alpha_3} \cdots B_{\alpha_k} \cdot v \\
&+ \ldots \\
&+ (\gamma)([A, B_{\alpha_k}]) \cdot B_{\alpha_1} \cdots B_{\alpha_{k-1}} \cdot v,
\end{aligned}$$

where $\alpha = \alpha_1 + \cdots + \alpha_k + \gamma$ is the weight of v'. The right hand side is a linear combination of elements in W, so we conclude $A \cdot v' \in W$. Since W is generated by elements with the form of v' and A was an arbitrary positive root vector, we conclude that W is invariant under the action of $\mathfrak{so}(4, \mathbb{C})$ on V. $\qquad\square$

Note that we could have avoided calculating the commutators by using induction over k. When we generalise to higher dimensions, we will see that the commutators are no longer necessarily in $\mathfrak{h}$.

We have the following corollaries. Note that these statements are not true for the trivial representation $\mathfrak{so}(4, \mathbb{C}) \to \{0\} = \mathfrak{gl}(\mathbb{C})$ since that representation has no non-zero vectors, hence no highest weight vectors.

Corollary 5.3.9. *Every non-trivial irreducible representation has exactly one highest weight vector up to scalar multiplication.*

Proof. Let $\mathfrak{so}(4, \mathbb{C})$ act on the irreducible module V and let w_1, w_2 be two highest weight vectors of V with weights β_1 respectively β_2. Proposition 5.3.8 tells us that w_1 and w_2 induce invariant subspaces W_1, W_2. Since V is irreducible, we find $W_1 = V = W_2$ hence $w_1 \in W_2, w_2 \in W_1$. This can only be true if $\beta_1 = \beta_2$ is the highest weight of V.

Since W_1 is generated by w_1 and by weight vectors with lower weights, it follows that w_2 is in the linear span of w_1 and vice-versa. We conclude that w_1 and w_2 are equal up to scalar multiplication. $\qquad\square$

Corollary 5.3.10. *Let V be a non-trivial irreducible module and let γ be the highest weight in V. Then V_γ is 1-dimensional.*

Proof. This follows directly, since every element of V_γ is a highest weight vector. $\qquad\square$

Note that any highest weight vector must lie in the closed Weyl chamber, since the weight in the chamber will always be higher than its mirror images.

Proposition 5.3.8 also tells us that irreducible representations with a given highest weight are unique, and tells us about the existence of an irreducible representation with given highest weight.

Proposition 5.3.11. *Let V_1, V_2 be irreducible modules and let their highest weights be equal. Then V_1, V_2 are equivalent.*

Proof. Let w_1, w_2 be the (up to scalars unique) highest weight vector of V_1 respectively V_2, with weight γ. Consider $W = V_1 \oplus V_2$. This is again a module, with the action defined by $A \cdot (u + v) = (A \cdot u) + (A \cdot v)$ for $u \in V_1, v \in V_2$ and $A \in \mathfrak{so}(4, \mathbb{C})$. Now $H \cdot (w_1 + w_2) = \gamma(H)w_1 + \gamma(H)w_2 = \gamma(H)(w_1 + w_2)$ and $A \cdot (w_1 + w_2) = A \cdot w_1 + A \cdot w_2 = 0 + 0 = 0$ for any positive weight vector A, so $w_1 + w_2$ is a highest weight vector with weight γ. Now $w_1 + w_2$ generates an irreducible submodule $U \subseteq V_1 \oplus V_2$.

Consider the projection maps $\pi_1 : U \to V_1$, $\pi_2 : U \to V_2$. These projection maps certainly intertwine the action of $\mathfrak{so}(4, \mathbb{C})$, by the construction of the direct sum. Moreover, $\pi_1(w_1 + w_2) = w_1$ and $\pi_2(w_1 + w_2) = w_2$, so the image of π_1 respectively π_2 is non-trivial. Since the image of a module under an intertwining operator gives an invariant subspace of the codomain and V_1, V_2 are irreducible, it follows that $\pi_1(U) = V_1$ and $\pi_2(U) = V_2$. Since U is also irreducible, we find that π_1 and π_2 are isomorphisms. We conclude that V_1 is isomorphic to V_2 as a module of $\mathfrak{so}(4, \mathbb{C})$. $\square$

We have found an uniqueness theorem, so our next step is to prove existence of each representation. We will need the following proposition.

Proposition 5.3.12. *Let V_1, V_2 be modules with $w_1 \in V_1, w_2 \in V_2$ highest weight vectors with weight γ_1 respectively γ_2. Then $w_1 \otimes w_2$ is a highest weight vector of $V_1 \otimes V_2$ with weight $\gamma_1 + \gamma_2$.*

Proof. Recall that $V_1 \otimes V_2$ is a module, where the action is defined by $A \cdot (u \otimes w) = (A \cdot u) \otimes w + u \otimes (A \cdot w)$ for $A \in \mathfrak{so}(4, \mathbb{C})$, $u \in V_1, w \in V_2$. Direct calculation gives $A \cdot (w_1 \otimes w_2) = (A \cdot w_1) \otimes w_2 + w_1 \otimes (A \cdot w_2) = 0 \otimes w_2 + w_1 \otimes 0 = 0 \otimes 0$ for positive root vectors A, and $H \cdot (w_1 \otimes w_2) = (\gamma_1(H)w_1) \otimes w_2 + w_1 \otimes (\gamma_2(H)w_2) = (\gamma_1 + \gamma_2)H(w_1 \otimes w_2)$ for $H \in \mathfrak{h}$. We conclude that $w_1 \otimes w_2$ is a highest weight vector of weight $\gamma_1 + \gamma_2$. $\square$

Proposition 5.3.12 tells us that we can find any irreducible module as a submodule of a tensor product of smaller modules, assuming we have smaller modules whose weights add up in the right way. However, there is still a clear flaw here: we have found that any representation that is based on the standard representation or adjoint representation will have a highest weight of the form $aL_1 + bL_2$ with a, b integers. We have also seen that our weight lattice Λ_W instead allows a, b to be half-integers.

Since the intersection $\Lambda_W \cap \overline{W}$ of the weight lattice and the closed Weyl chamber gives us all possible highest weights, we see that we cannot yet generate an irreducible representation for each possible highest weight. We want representations with weight $\frac{1}{2}(L_1 + L_2)$ and $\frac{1}{2}(L_1 - L_2)$ lying along the boundaries of our Weyl chamber; it is clear that if we find representations with these weights, we have proven existence.

To find representations with these weights, we recall that the Lie algebra $\mathbf{L} = C\ell^2(n, \mathbb{C})$ is isomorphic to $\mathfrak{so}(n, \mathbb{C})$. Choosing $n = 4$ gives us an isomorphism $\mathbf{L} \cong \mathfrak{so}(4, \mathbb{C})$. But $\mathbf{L}$ comes with a natural representation: the spinorial representation $\rho : C\ell(V \oplus V^*) \to \mathrm{Lin}(\Lambda V)$ from Proposition 4.3.5 induces a Lie algebra representation $C\ell^2(V \oplus V^*) \to \mathfrak{gl}(\Lambda V)$ by taking the commutators on both sides. (This Lie algebra representation is also known under the name 'spinorial representation'; the same holds for the induced Lie group representation.) In order to see what this representation is, we follow the trace of the isomorphisms.

First of all, we had $\mathbf{L} \cong \mathfrak{so}(4, \mathbb{C})$ via $[x, y] \mapsto [[x, y], \cdot]$ (see Theorem 4.4.43 and the two successive remarks). We proved the isomorphism using the anti-symmetry of $[[x, y], \cdot]$, so we were in the 'normal' basis for both $\mathfrak{so}(4, \mathbb{C})$ and for $\mathbf{L}$. Now Remark 5.2.1 gives that $C\ell^2(V \oplus V^*)$ has the same basis as the 'new' basis of $\mathfrak{so}(4, \mathbb{C})$, so the isomorphism $C\ell^2(V \oplus V^*) \to \mathfrak{so}(4, \mathbb{C})$ will again be given by

$$C\ell^2(V \oplus V^*) \ni [u + \theta, v + \phi] \mapsto [[u + \theta, v + \phi], \cdot] \in \mathfrak{so}(V \oplus V^*) = \mathfrak{so}(4, \mathbb{C}).$$

Any representation of $C\ell^2(V \oplus V^*)$ will therefore directly give a representation of $\mathfrak{so}(4, \mathbb{C})$, so we no longer have to worry about the isomorphisms. We only have to know what subset of $C\ell^2(V \oplus V^*)$ corresponds to $\mathfrak{h} \subset \mathfrak{so}(4, \mathbb{C})$. Note that $C\ell^2(V \oplus V^*)$ has basis $\{e_1 e_2 - e_2 e_1, e_i e_j^* - e_j^* e_i, e_1^* e_2^* - e_2^* e_1^*\}$ for $i, j = \{1, 2\}$. Since $[e_i e_i^*, e_j] = 2\delta_{i,j} e_i$ and $[e_i e_i^*, e_k^*] = -2\delta_{i,k} e_i^*$ we find that H_1 maps to $\frac{1}{2}(e_1 e_1^* - e_1^* e_1)$ and H_2 to $\frac{1}{2}(e_2 e_2^* - e_2^* e_2)$. We similarly see that $X_{i,j}$ is send to $\frac{1}{2}(e_i e_j^* - e_j e_i^*)$, $Y_{1,2}$ to $\frac{1}{2}(e_1 e_2 - e_2 e_1)$ and $Z_{1,2}$ to $\frac{1}{2}(e_1^* e_2^* - e_2^* e_1^*)$.

Now recall the action of $C\ell(V \oplus V^*)$ on ΛV. The action was given by $(v + \theta) \cdot w = v \wedge w + \theta \lrcorner w$. The action of $e_i e_i^*$ on a basis vector $e_J = e_{j_1} \wedge \cdots \wedge e_{j_l}$, $l \in \mathbb{N}$ then is

$$e_i e_i^* \cdot e_J = e_i \wedge (e_i^* \lrcorner e_J) = e_i \wedge \left(\sum_{k=1}^{l} e_i^*(e_{j_k})(-1)^{i-1}(e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_l}) \right).$$

Now either i occurs as one of the j_k, or it doesn't. But if it occurs as j_k, then $e_i\, e_i^*(e_{j_k}) = e_i = e_{j_k}$ so $e_i e_i^* \cdot e_J = e_J$. We get:

$$e_i e_i^* \cdot e_J = \begin{cases} e_J & \text{There is a } k\colon j_k = i \\ 0 & \text{otherwise.} \end{cases}$$

We see that each e_J is an eigenvector of $e_i e_i^*$ with eigenvalue 1 if i occurs or 0 if it doesn't.

Now the eigenvalue of e_J with respect to $\frac{1}{2}(e_i e_i^* - e_i^* e_i)$ is $\frac{1}{2}$ if i occurs and $-\frac{1}{2}$ if i does not occur. Therefore, the weight of e_J is $\frac{1}{2}\sum_{i=j_k} L_i - \frac{1}{2}\sum_{i\neq j_k} L_i$.

Now consider $e_1 \wedge e_2$. It is clear that every $e_i e_j$ or $e_i e_j^*$ with $i \neq j$ acts on $e_1 \wedge e_2$ by sending it to zero (by repeated e_i in a single wedge product). Therefore, $e_1 \wedge e_2$ is a highest weight vector. This gives us a highest weight vector with weight $\frac{1}{2}(L_1 + L_2)$.

We also know that $\mathbf{L} = C\ell^2(4,\mathbb{C}) \subset C\ell^{even}(4,\mathbb{C})$, so the action of $\mathbf{L}$ on ΛV is even. Therefore, we find that the vectors of odd length should form an invariant subspace of ΛV. Consider e_1. We see that $X_{1,2}$ and $Y_{1,2}$ both send e_1 to 0, so e_1 is also a highest weight vector. Since e_1 has weight $\frac{1}{2}(L_1 - L_2)$, we have found our second highest weight vector.

We call these two representations the <u>half-spinorial representations</u>.
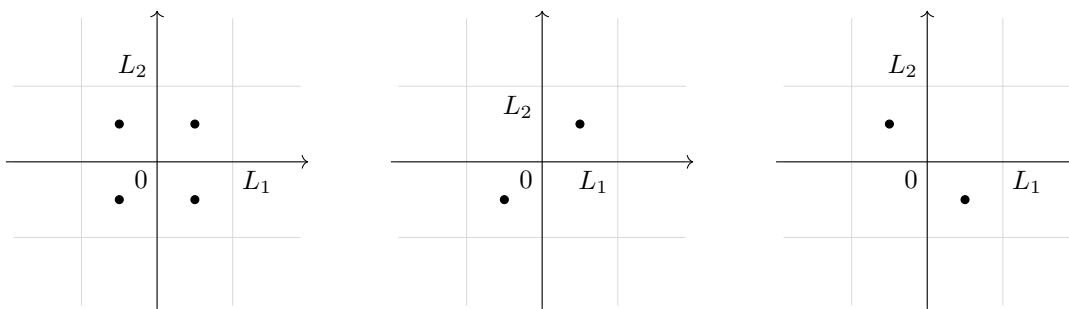
(a) The spinorial representation.   (b) The half-spinorial representation generated by $e_1 \wedge e_2$.   (c) The half-spinorial representation generated by e_1.

Figure 3: The spinorial representation and its decomposition into the half-spinorial representations.

Finally, we have proven uniqueness and have found two highest weight vectors with weights $\frac{1}{2}(L_1 + L_2)$ and $\frac{1}{2}(L_1 - L_2)$. Now the tensor products of these weight vectors span the intersection of the closed Weyl chamber and the weight lattice, which tells us that we have determined each representation.

We will now turn ourself to $\mathfrak{so}(5,\mathbb{C})$, before generalising to $\mathfrak{so}(2n,\mathbb{C})$ respectively $\mathfrak{so}(2n+1,\mathbb{C})$.

5.4   Representations of $\mathfrak{so}(5,\mathbb{C})$

The theory for $\mathfrak{so}(5,\mathbb{C})$ resembles $\mathfrak{so}(4,\mathbb{C})$ in many ways, but there are some small extra difficulties. We again start with writing a general element of $\mathfrak{so}(5,\mathbb{C})$:

$$\begin{pmatrix} a & b & 0 & e & g \\ c & d & -e & 0 & h \\ 0 & f & -a & -c & i \\ -f & 0 & -b & -d & j \\ -i & -j & -g & -h & 0 \end{pmatrix}$$

with $a,b,c,d,e,f,g,h,i,j \in \mathbb{C}$. We define $H_1, H_2, X_{1,2}, X_{2,1}, Y_{1,2}$ and $Z_{1,2}$ in the exact same way as before. However, we still have a g, h, i and j. Therefore, we define $U_1 = E_{1,5} - E_{5,3}$ ('$g = 1$'), $U_2 = E_{2,5} - E_{5,4}$ ('$h = 1$'), $V_1 = E_{3,5} - E_{5,1}$ ('$i = 1$') and $V_2 = E_{4,5} - E_{5,2}$ ('$j = 1$').

Simple calculations give that $[H_1, U_1] = [H_2, U_2] = 1$, $[H_2, U_1] = [H_1, U_2] = 0$ so U_1, U_2 are root vector with eigenvalues L_1 respectively L_2. Similarly, V_1, V_2 are root vectors for $-L_1$ and $-L_2$. Note that 0 is now a weight of the standard representation, since e_5 is send to 0 by both H_1 and H_2.

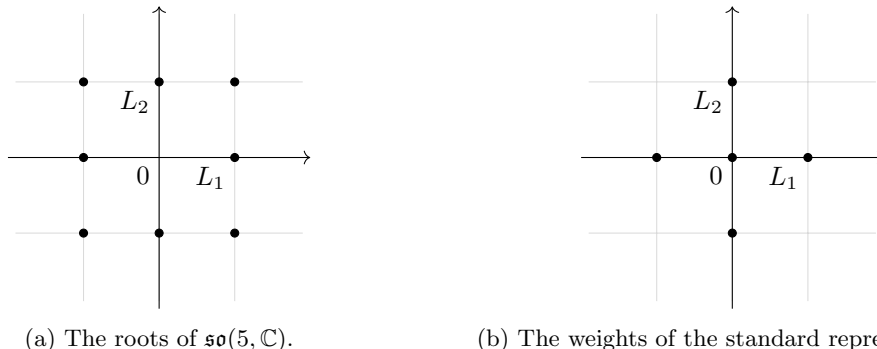The root diagram and weight diagram for the standard representation are thus as follows:

(a) The roots of $\mathfrak{so}(5, \mathbb{C})$.   (b) The weights of the standard representation.

Figure 4

Our next step is to determine the distinguished copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}(2, \mathbb{C})$. We again have $\mathfrak{s}_{L_1+L_2} = \mathfrak{g}_{L_1+L_2} \oplus \mathfrak{g}_{-L_1-L_2} \oplus [\mathfrak{g}_{L_1+L_2}, \mathfrak{g}_{-L_1-L_2}]$ generated by $Y_{1,2}$ and $Z_{1,2}$ with $H_{L_1+L_2} = H_1 + H_2$ and $\mathfrak{s}_{L_1-L_2} = \mathfrak{g}_{L_1-L_2} \oplus \mathfrak{g}_{-L_1+L_2} \oplus [\mathfrak{g}_{L_1-L_2}, \mathfrak{g}_{-L_1+L_2}]$ generated by $X_{1,2}$ and $X_{2,1}$ with $H_{L_1-L_2} = H_1 - H_2$. However, we now additionally have $\mathfrak{s}_{L_1}$ generated by U_1, V_1 and $\mathfrak{s}_{L_2}$ generated by U_2, V_2. Since $[U_i, V_i] = -H_i$ and $[-H_i, U_i] = -U_i$, we find that $H_{L_1} = 2H_1$ and $H_{L_2} = 2H_2$.

We see that the Weyl group is generated by reflection in $\mathbb{C} \cdot (L_1 + L_2)$, $\mathbb{C} \cdot (L_1 - L_2)$, $\mathbb{C} \cdot L_1$ and $\mathbb{C} \cdot L_2$, and that the weight lattice Λ_W is again $\{aL_1 + bL_2 \mid a, b \in \frac{1}{2}\mathbb{Z}\}$ since $H_{L_1} = 2H_1$ and $H_{L_2} = 2H_2$. We choose the same weights to be positive as in the case of $\mathfrak{so}(4, \mathbb{C})$ and define the Weyl chamber as $W = \{aL_1 + bL_2 \mid a > b > 0\}$. Note that we again have that for any weight, the mirror image in the closed Weyl chamber will have higher or equal weight. Therefore, all highest weights again lie in the closed Weyl chamber.

The positive roots, Weyl chamber and symmetry axes are given in Figure 5. We clearly see that all weights in the closed Weyl chamber are integer combinations of L_1 and $\frac{1}{2}(L_1 + L_2)$.

(a) The symmetry axes.   (b) The positive and negative   (c) The (closed) Weyl chamber.
                              weights.   The intersection with the weight
                                                                lattice is given by the small dots.
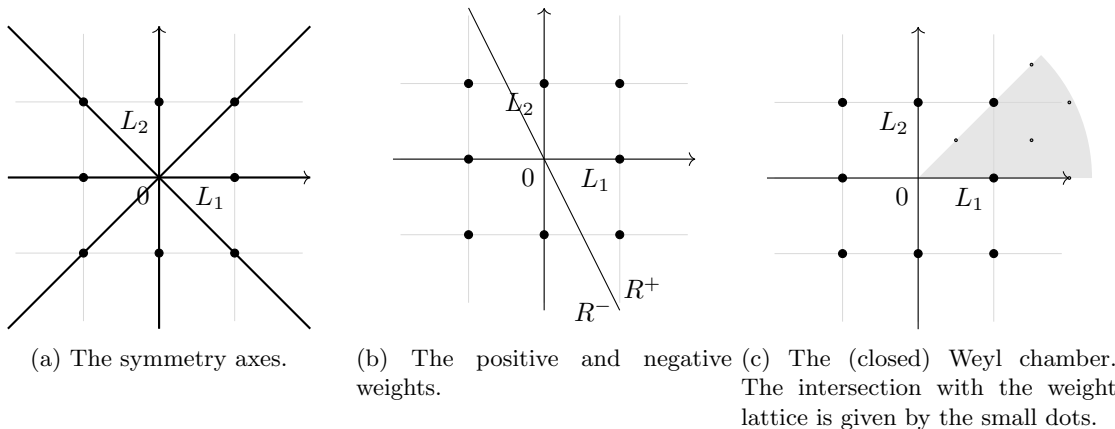
Figure 5: The symmetry axes, weights and Weyl chamber of $\mathfrak{so}(5, \mathbb{C})$.

We want to copy everything from Proposition 5.3.8 to Proposition 5.3.12. By going through the proofs, we find that all the proofs for $\mathfrak{so}(4, \mathbb{C})$ also work for $\mathfrak{so}(n, \mathbb{C}), n > 4$, except the proof of Proposition 5.3.8. We will have to proof that proposition through induction, as mentioned.

Proposition 5.4.1. *Let $v \in V$ be a highest weight vector. Let W be the subspace of V generated by v and by the image of v under repeated applications of negative root vectors. Then W is an invariant subspace of V.*

Proof. Again, V is a module so the action is well-defined, and v and its image under repeated applications of negative root vectors span W linearly, so we only have to consider the action on those elements.

W is clearly closed under the action of negative root vectors and under elements of $\mathfrak{h}$. We will prove that W is closed under the action positive root vectors, by letting a positive root vector A_α act on $v' = B_{\beta_1} B_{\beta_2} \ldots B_{\beta_l} \cdot v$, where $l \in \mathbb{N}_0$, $\beta_1, \ldots, \beta_l$ negative roots and B_β is the root vector corresponding to the negative root β. We prove that the result is again in W by induction over l.

First $l = 0$, so $v' = v$. Now $A_\alpha \cdot v = 0 \in W$, since v is a highest weight vector. This gives us the induction basis.

Now assume $A_\alpha \cdot B_{\beta'_1} B_{\beta'_2} \ldots B_{\beta'_k} \cdot v \in W$ for all $k < l$ and (arbitrary) negative weights $\beta'_1, \ldots, \beta'_k$. Consider $A_\alpha \cdot v'$. Using repeated commutation relations, we get

$$
\begin{aligned}
A_\alpha \cdot v' =& [A_\alpha, B_{\beta_1}] B_{\beta_2} \cdots B_{\beta_l} \cdot v \\
&+ B_{\beta_1} [A_\alpha, B_{\beta_2}] B_{\beta_3} \cdots B_{\beta_l} \cdot v \\
&+ \ldots \\
&+ B_{\beta_1} \cdots B_{\beta_{l-1}} [A_\alpha, B_{\beta_l}] \cdot v.
\end{aligned}
$$

Since the positive and negative root vectors and the elements of $\mathfrak{h}$ form a basis for $\mathfrak{so}(5, \mathbb{C})$, we can write each commutator as a linear combination of those elements. We already know that the negative root vectors and elements of $\mathfrak{h}$ leave W invariant, so we only have to consider the positive root vector components of each commutator. However, the positive root vectors act on $B_{\beta_2} \cdots B_{\beta_l} \cdot v$ or $B_{\beta_3} \cdots B_{\beta_l} \cdot v$ etc., so by the induction hypothesis we find that $A_\alpha \cdot v' \in W$. Since the β_i were arbitrary, we find that positive root vectors leave W invariant.

We conclude that W is invariant under the action of $\mathfrak{so}(5, \mathbb{C})$. $\square$

With this proposition proven, we again find the previous results of uniqueness and existence of irreducible representations of a given weight. We only need to find two representation which generate all other representations through tensor products. We are therefore looking for a representation with highest weight L_1 and one with weight $\frac{1}{2}(L_1 + L_2)$. We have already seen that the standard representation has highest weight L_1 (see Figure 1b). Therefore, we are looking for a representation with highest weight $\frac{1}{2}(L_1 + L_2)$. Considering the case of $\mathfrak{so}(4, \mathbb{C})$, we guess that this representation can be found using **L**.

We know that $C\ell(5, \mathbb{C})$ contains two copies of $C\ell(4, \mathbb{C})$ corresponding to $1 + e_0$ and $1 - e_0$ (see Proposition 4.3.8 for the definition of e_0). Therefore, each representation of $C\ell(4, \mathbb{C})$ is a representation of $C\ell(5, \mathbb{C})$ in two ways, giving the two spinorial representations. We can interpret the spinorial representation corresponding to $1 - e_0$ as sending $e_1, e_2, e_3, e_4 \in C\ell(5, \mathbb{C})$ to their counterparts in $C\ell(4, \mathbb{C})$ and sending $e_5 \in C\ell(5, \mathbb{C})$ to $e_1 e_2 e_3 e_4$. In that interpretation, we have $e_i e_j - e_j e_i$ mapping to their counterparts, and $e_1 e_5 - e_5 e_1 \mapsto e_2 e_3 e_4 - (-1)^3 e_2 e_3 e_4 = 2 e_2 e_3 e_4$ and similar for the other e_i. The image in $C\ell(V \oplus V^*)$ is then generated by

$$
\{ e_1 e_2 - e_2 e_1, \ e_i e_j^* - e_j e_i^*, \ e_1^* e_2^* - e_2^* e_1^*, \ e_2 e_1^* e_2^*, \ e_1 e_1^* e_2^*, \ e_1 e_2 e_1^*, \ e_1 e_2 e_2^* \}.
$$

We see that the corresponding representation on ΛV is an extension of the representation of $\mathfrak{so}(4, \mathbb{C})$ on ΛV to a larger Lie algebra.

The highest weight vector is again $e_1 \wedge e_2$, but e_1 is no longer a highest weight vector as $(e_1 e_2 e_1^*) \cdot e_1 = e_1 \wedge e_2$ which has higher weight. We find that ΛV is an irreducible representation with highest weight $\frac{1}{2}(L_1 + L_2)$. We call this the spinorial representation of $\mathfrak{so}(4, \mathbb{C})$. The weight diagram of the spinorial representation is equal to that of $\mathfrak{so}(4, \mathbb{C})$, see Figure 3.

We have now found irreducible representations with weight L_1 and $\frac{1}{2}(L_1 + L_2)$, spanning the intersection $\Lambda_W \cap \overline{W}$ of the weight lattice and the closed Weyl chamber. Hence, all irreducible representations of $\mathfrak{so}(5, \mathbb{C})$ have been classified.

We now generalise the last two sections to $\mathfrak{so}(2n, \mathbb{C})$ respectively $\mathfrak{so}(2n + 1, \mathbb{C})$.

5.5   Representations of $\mathfrak{so}(2n, \mathbb{C})$

We first consider the even dimensional case. We follow the same analysis as in the case of $\mathfrak{so}(4, \mathbb{C})$, but in higher dimensions. The given proofs of relevant results (uniqueness, existence) all generalise, so we only have to define our root vectors and Weyl chamber etc.

In the case of $\mathfrak{so}(4, \mathbb{C})$, we split the space into $H_1, H_2, X_{1,2}, X_{2,1}, Y_{1,2}$ and $Z_{1,2}$ along each letter. We can do the exact same for higher dimensions. Recall that elements of $\mathfrak{so}(2n, \mathbb{C})$ have the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with B, C anti-symmetric and $A^{\mathsf{T}} = -D$.

Let $H_i = E_{i,i} - E_{n+i,n+i}$ for $1 \leq i \leq n$. This gives us n diagonal matrices which all commute. These H_i span the Cartan subalgebra $\mathfrak{h}$. We will again call their dual elements L_i.

Also define $X_{i,j} = E_{i,j} - E_{n+j,n+i}$, $Y_{i,j} = E_{i,n+j} - E_{j,n+i}$ and $Z_{i,j} = E_{n+i,j} - E_{n+j,i}$, all with $i \neq j, 1 \leq i, j \leq n$. The $X_{i,j}$ again correspond to the off-diagonal elements of A (or D) while the $Y_{i,j}$ correspond to B and $Z_{i,j}$ to C.

Direct calculating gives that $X_{i,j}$ is a root vector with root $L_i - L_j$, while $Y_{i,j}$ is a root vector for $L_i + L_j$ and $Z_{i,j}$ for $-L_i - L_j$. Therefore, the roots are $\{\pm L_i \pm L_j\}_{1 \leq i < j \leq n}$.

Note that the Killing form once again shows that $L_i \perp L_j$ for $i \neq j$, so we can picture the weight diagram as hypercubes.

We have to choose a direction to be able to speak of positive and negative weights. We again choose that L_1 is 'larger' than L_2, which is 'larger' than L_3, etc. until L_n. To make this precise, we choose $c_1 > c_2 > \cdots > c_n > 0$, $c_i \in \mathbb{R}$ and say $H \in \mathfrak{h}$ is positive if $(c_1 L_1 + \cdots + c_n L_n)(H) > 0$. Now the L_i are positive, $L_i + L_j$ is positive and $L_i - L_j$ is positive if $i < j$. This gives us $R^+ = \{L_i + L_j, L_i - L_j\}_{1 \leq i < j \leq n}$ as positive roots.

The distinguished copies $\mathfrak{s}_\alpha$ are given by $X_{i,j}, X_{j,i}$ pairs and by $Y_{i,j}, Z_{i,j}$ pairs. General calculations give that $H_{L_i+L_j} = H_i + H_j$ for the $Y_{i,j}, Z_{i,j}$ pairs and $H_{L_i-L_j} = H_i - H_j$ for the $X_{i,j}, X_{j,i}$ pairs. Therefore, the weight lattice is again given by

$$\Lambda_W = \{\beta \in \mathfrak{h}^* \mid \beta(H_a) \in \mathbb{Z}, \alpha = \pm L_i \pm L_j\} = \{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n \mid a_1, \ldots, a_n \in \frac{1}{2}\mathbb{Z}\},$$

and the symmetry axes are the diagonals $\mathbb{C} \cdot (L_i \pm L_j)$. Therefore, we choose our Weyl chamber as $W = \{a_1 L_1 + \cdots + a_n L_n \mid a_1 > \cdots > a_{n-1} > |a_n| > 0\}$. The boundaries of the Weyl chamber are in the directions $L_1, L_1 + L_2, L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_{n-1} - L_n$ and $L_1 + \cdots + L_n$. Note that $L_1 + \cdots + L_{n-1}$ is not a boundary. The corresponding elements of Λ_W are thus $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}, \frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$ and $\frac{1}{2}(L_1 + \cdots + L_n)$.

Therefore, we have to find a total of n irreducible representations of the given weights. First consider the standard representation $V = \mathbb{C}^{2n}$. By direct calculation, we see that $e_1, \cdots, e_n$ have eigenvalues $L_1, \cdots, L_n$ respectively, while $e_{n+1}, \cdots, e_{2n}$ have eigenvalues $-L_1, \cdots, -L_n$. Hence, the highest weight vector is e_1, with weight L_1. We find an irreducible representation with weight L_1. Since the standard representation is irreducible, we find that the standard representation is the first of the n needed representations.

Next, consider the exterior product $\Lambda^2 V$ of the standard representation, consisting of elements $e_i \wedge e_j, i \neq j$. As mentioned before, $\mathfrak{so}(2n, \mathbb{C})$ acts on $\Lambda^2 V$ by $A \cdot (e_i \wedge e_j) = (A \cdot e_i) \wedge e_j + e_i \wedge (A \cdot e_j)$. $\Lambda^2 V$ has 'each

weight of V times each other weight' as weights, so $\{\pm L_i \pm L_j\}_{1 \leq i < j \leq n}$, each with multiplicity 1, and 0 with multiplicity n (as $0 = L_i + (-L_i)$). The highest weight is $L_1 + L_2$, corresponding to $e_1 \wedge e_2$. This gives us an irreducible representation with weight $L_1 + L_2$. It turns out that $\Lambda^2 V$ is irreducible, but we will not prove it here as it requires knowledge of the irreducible representations of $\mathfrak{sl}(n, \mathbb{C}), n \geq 3$.

Since $\Lambda^2 V$ has highest weight $L_1 + L_2$ and $V = \Lambda^1 V$ has highest weight L_1, we guess that $\Lambda^k V$ will have highest weight $L_1 + \cdots + L_k$ for $1 \leq k \leq n$. This is true: $\Lambda^k V$ has highest weight vector $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ with weight $L_1 + \cdots + L_k$. Moreover, it turns out that $\Lambda^k V$ is irreducible for $k < n$. This directly gives us the first $n - 2$ irreducible representations: $V, \Lambda^1 V, \ldots, \Lambda^{n-2} V$. Note that $\Lambda^{n-1} V$ has highest weight $L_1 + \cdots + L_{n-1}$, but that is not one of the boundaries of the Weyl chamber and therefore not one of the irreducible representations we were looking for.

Now for the final two irreducible representations we turn to the spinorial representation. The remarks for $\mathfrak{so}(4, \mathbb{C}) \cong C\ell^2(V \oplus V^*)$ are precisely the same for higher dimensions, so we find that H_i is send to $\frac{1}{2}(e_i e_i^* - e_i^* e_i)$, which acts on $\Lambda V, V = \mathbb{C}^n$ by sending each e_J to $\frac{1}{2} e_J$ if i occurs in J, or $-\frac{1}{2} e_J$ if it does not occur. Hence, the weights of ΛV are

$$\frac{1}{2} \sum_{j \in J} L_j - \frac{1}{2} \sum_{j \notin J} L_j.$$

Similarly to the $2n = 4$ case, consider the 'longest' vector in ΛV, $e_1 \wedge \cdots \wedge e_n$. This has weight $\frac{1}{2} \sum_{i=1}^n L_i$, which is the highest weight of ΛV. Therefore, $e_1 \wedge \cdots \wedge e_n$ generates a representation of $\mathfrak{so}(2n, \mathbb{C})$ with weight $\frac{1}{2}(L_1 + \cdots + L_n)$.

Again, $C\ell^2(V \oplus V^*)$ in contained in $C\ell^{even}(V \oplus V^*)$, and therefore has even elements. This gives us that the even and odd subspace of ΛV are left invariant. The even subspace has highest weight vector of even length, but the odd subspace has a highest weight vector of odd length. Therefore, $e_1 \wedge \cdots \wedge e_n$ is not the only highest weight vector. The next highest weight is $\frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$ corresponding to $e_1 \wedge \cdots \wedge e_{n-1}$. We find that this is the other highest weight vector.

We have therefore found two highest weight vectors with weights $\frac{1}{2}(L_1 + \cdots + L_n)$ and $\frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n)$, the <u>half-spinorial representations</u>. Therefore, we have determined all the needed representations and thus have found all representations of $\mathfrak{so}(2n, \mathbb{C})$.

Finally, we consider $\mathfrak{so}(2n + 1, \mathbb{C})$.

5.6 Representations of $\mathfrak{so}(2n + 1, \mathbb{C})$

The generalisation to $\mathfrak{so}(2n + 1, \mathbb{C})$ is almost identical to the combination of the generalisation from 4 to $2n$ and from 4 to 5.

We start with defining $H_i, X_{i,j}, Y_{i,j}, Z_{i,j}$ as in the previous section. Moreover, we additionally define $U_i = E_{i,2n+1} - E_{2n+1,n+i}$ with eigenvalues L_i and $V_i = E_{n+i,2n+1} - E_{2n+1,i}$ with eigenvalues $-L_i$, corresponding to the last column and row. This gives us additional distinguished copies of $\mathfrak{sl}(2, \mathbb{C})$, as the pair U_i, V_i gives $H_{L_i} = 2H_i$.

We again find $\Lambda_W = \{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n \mid a_1, \cdots, a_n \in \frac{1}{2}\mathbb{Z}\}$, but our Weyl group now also includes reflections in the axes. Therefore, our Weyl chamber becomes $W = \{a_1 L_1 + \cdots + a_n L_n \mid a_1 > \cdots > a_n > 0\}$. The intersections with the boundaries are given by $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-1}$ and $\frac{1}{2}(L_1 + \cdots + L_n)$.

The first $n - 1$ representations are again $\Lambda^k V$ for $1 \leq k \leq n - 1$, following the same line of thought as for the $2n$ case. As for the final representation, we again consider the spinorial representation. By sending $e_{2n+1} \mapsto \iota^{-1} e_1 e_2 \cdots e_{2n}$, where $\iota = 1$ if n is odd and $\iota = \mathrm{i}$ if n is even, we again find that the spinorial representation of $\mathfrak{so}(2n, \mathbb{C})$ gives a representation of $\mathfrak{so}(2n + 1, \mathbb{C})$ with highest weight $\frac{1}{2}(L_1 + \cdots + L_n)$.

We conclude that we once again find the needed irreducible representations. Therefore, we have characterised and determined all irreducible representations of $\mathfrak{so}(2n + 1, \mathbb{C})$.

5.7   Representations of $\mathfrak{so}(p,q)$ and $\mathrm{Spin}(n)$

As a last note, we return to $\mathfrak{so}(p,q)$ and $\mathrm{Spin}(n)$. It is clear that $\mathfrak{so}(p,q) \subset \mathfrak{so}(p+q,\mathbb{C})$ if $\mathfrak{so}(p+q,\mathbb{C})$ is in the right basis, because $\mathfrak{so}(p+q,\mathbb{C})$ does not depend on the bilinear form as long as it is symmetric and non-degenerate. Therefore, every representation of $\mathfrak{so}(p+q,\mathbb{C})$ induces a representation of $\mathfrak{so}(p,q)$. In fact, we can easily check that $\mathfrak{so}(p+q,\mathbb{C})$ is just the complexification of $\mathfrak{so}(p,q)$ (in the right basis), so any representation of $\mathfrak{so}(p,q)$ conversely induces a representation $\mathfrak{so}(p+q,\mathbb{C})$.

Now taking the exponential of the representation gives a representation of the identity component. While we have not talked about the identity component of $\mathrm{Spin}(p,q)$, we know that the exponential of $\mathfrak{so}(n)$ generates all of $\mathrm{Spin}(n)$. Therefore, the representations of $\mathrm{Spin}(n)$ are all exponentials of representations of $\mathfrak{so}(n)$. Since $\mathfrak{so}(n,\mathbb{C})$ is the complexification of $\mathfrak{so}(n)$, we conclude that all representations of $\mathrm{Spin}(n)$ have been determined, albeit indirectly.

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