Chapter 26
Uses of Technology in K–12 Mathematics Education: Concluding Remarks

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Abstract The aim of this closing chapter is to reflect on the content of this book and on its overall focus on the development of mathematical proficiencies through the design and use of digital technology and of teaching and learning with and through these tools. As such, rather than making an attempt to provide an overview of the field as a whole, or trying to define overarching theoretical approaches, we chose to follow a bottom-up approach in which the chapters in this monograph form the point of departure. To do so, we reflect on the book's content from four different perspectives. First, we describe a taxonomy of the use of digital tools in mathematics education, and set up an inventory of the different book chapters in terms of these types of educational use. Second, we address the learning of mathematics with and through technology. Third, the way in which the assessment of mathematics with and through digital technology is present in this monograph is reflected upon. Fourth, the topic of teachers teaching with technology is briefly addressed. We conclude with some final reflections, including suggestions for a future research agenda.

Keywords Teaching mathematics · Learning mathematics · Digital technology Digital tools · Assessment · Taxonomy

26.1 Taxonomy of Educational Use of Digital Tools

The title of this book mentions tools, topics and trends, so it seems appropriate in this closing chapter to look back at the different types of digital tools that are used throughout the chapters, and the role these tools play in mathematics education.

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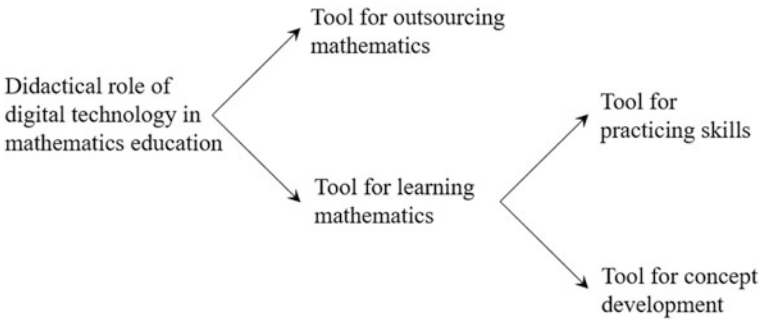

Fig. 26.1 Taxonomy of didactical roles of digital technology for mathematics

To frame this synthesis, a taxonomy might be useful. As a point of departure, we take the model shown in Fig. 26.1, which is adapted from earlier work (Arcavi, Drijvers, & Stacey, 2017; Drijvers, 2015). In this model, three different didactical roles for digital technology in mathematics education are distinguished: as a tool for outsourcing mathematics, as a tool for practicing skills, and as a tool for concept development.

In the first role, the didactical function of the digital tool comes down to outsourcing mathematics. The tool acts as a 'mathematical assistant' who takes over a (often more procedural or low-level) part of the work from the student, so that he can concentrate on the core issues at stake. Even if this may sound easy, the student constantly has to take decisions on what kind of job to outsource to the tool, how to do this, and how to critically interpret the results. As such, there is certainly a learning aspect involved in this type of use and in the process of instrumental genesis (Artigue, 2002).

The second didactical role focuses on practicing mathematical skills. For example, online applets may be used to practice skills such as equation solving, or algebraic expansion and factorization. Students can practice as long as they want, in private, and don't need a teacher. Randomization generates many tasks and automated feedback can be delivered. As is shown in the chapter by Drijvers (Chap. 9), however, online practice does not always lead to better achievement.

The third didactical role of digital tools concerns concept development, and probably is the most subtle one. Through graphical and dynamic representations, digital tools offer expressive power to students. This may lead them to engage in activities that support concept development. In many cases, this type of use includes outsourcing basic work to the technology, so this third role may encompass the first role. Also, we should point out that these different didactical roles do not just rely on the opportunities and constraints of the digital technology, but also on the type of task or activity, and its orchestration in the learning process. As such, we should not assign different roles too tightly to different digital tools.

How do the different chapters in this volume address these didactical roles? The first one, digital technology as a tool for outsourcing mathematics, is apparent in many contributions, even if it is sometimes addresses implicitly. Dick's chapter (Chap. 14) focuses on digital assessment. It is suggested that assessment may be enhanced through the use of digital technology such as dynamic geometry systems. Basic procedural skills, such as graphing lines and parabolas, or changing triangles and quadrilaterals by moving points, are taken care of by the digital environment.

The second didactical role, digital technology as a tool for practicing skills, is addressed only to a limited extent. Moyer-Packenham et al. (Chap. 2) studied the use of apps in a touch screen environment with young children addressing a variety of skills, appropriate to the children's age, like count skipping. Drijvers (Chap. 9) describes an experiment with an applet on practicing equation solving, the results which were slightly disappointing. Some possible causes are discussed.

The third didactical role, digital technology as a tool for concept development, dominates the book, even if it in many cases builds upon the outsourcing role. In Chap. 3, Tucker studied the use of apps to foster the concept of a number line, and its infinite density in particular—a concept known to be challenging for students. Note that the students were only 11 years old. Larsen and colleagues (Chap. 4) focus on discourse and communication that foster higher-order skills such as problem solving. The contributions by Voltolini and the one by Maschietto (Chaps. 5 and 11, respectively) both focus on conceptual understanding in geometry through the use of "duo's of artefacts", consisting of material and digital tools. In the case by Maschietto, this concerns Pythagorean theorem, whereas Voltolini addresses the concept of triangle. Both see the two types of tools as complementary and the articulation of the two as productive for concept development. In Chap. 6, the active learning cluster was the strongest in the cluster analysis carried out by Larkin and Milford. Even if active learning can address different kinds of skills, our impression is that the focus, again, is on conceptual understanding in most cases. In her overview, Heid (Chap. 10) discusses in somewhat more detail a study by Parnafes and Disessa (2004), in which the focus was on reasoning with different computer-based representations of motion. Ball and Barzel's examples in Chap. 12 focus on the concept of variable, as it is manifest in the type of representations and interactions that are provided by computer algebra devices and spreadsheets. In the contribution by Trgalova and colleagues (Chap. 15), the tool draws geometrical elements for the student, and in this sense it may address the first function of outsourcing; yet the whole idea of the c-book is to allow for an expressive way to relate to geometrical concepts so that the students will be creative in experiencing links and general properties. As such, the chapter focuses on concept development. In the chapter by Ng and Sinclair (Chap. 16), the 3D pen is used to conceptualize ideas based on "drawing" a three-dimensional object. The opportunities to create tangible, movable and rotatable drawings is exploited to enhance conceptual learning. In a similar approach, but with different technological tools, Moreno-Armella and Corey Brady (Chap. 19) draw on the dynamic and connected representational power of technological tools to foster conceptual understanding. In Chap. 20 by Donevska-Todorova, the emphasis on outsourcing is more prevalent than in the other chapters mentioned in this paragraph. However, also in this case the main motivation is to use dynamic representations to support the learning of abstract concepts.

To summarize this brief inventory, the first conclusion is that many contributions in this book focus on the didactical opportunities digital technology offers for concept development. Apparently, this is the role that researchers find the most relevant or urgent to investigate. In many cases, the concept development activity is rooted in some functionality that can be outsourced to the digital technology, so from this perspective the outsourcing role is often implicitly present. A second conclusion is that the didactical role of tool for practicing skills is addressed only to a limited extent. Taking into account the popularity of apps, online tutors, exercisers and repositories that focus on skill mastery among students, this is slightly surprising. Is the mathematics education research community not so much interested in skill training? Are there no pressing research issues in this domain? Does this limited attention relate to the preference for qualitative research, addressed in the contribution by Heid? These are interesting questions to further explore.

26.2 Learning With and Through Technology

One of the enduring focuses of research concerning the use of digital technologies in mathematics education has been the design and use of digital tools with regard to their impact on student learning and the ways in which students might engage with mathematics through their use of these tools. Much of this research activity arises because the continual development of digital technology such as tablets, touch screens, virtual manipulatives and screen casting which opens up new possibilities for the use of digital tools and apps for mathematics learning. These developments highlight the need for further research on the interdependence of tools, tasks, pedagogical approaches and learning outcomes (Hoyles & Lagrange, 2010).

Recent reviews of research have reported on the transformation of learning with and through digital technology (Bartolini & Borba, 2010; Hoyles & Lagrange, 2010; Larkin & Calder, 2016; Trouche & Drijvers, 2010). A number of studies included in this monograph focus on the way in which learners interact with digital tools for their learning of mathematics or reported on what students learned when using digital technology. Previous research on learning with and through technology has been dominated by instrumental theories informing studies as the researchers focused on the interplay between the learner and the tool for the learning of mathematics and instrumental genesis theory that is more explicit and focused on the way in which students' thinking is shaped by the tool and shapes their interaction with the tool (Drijvers et al., 2010). However socio-cultural theories of learning have also been evident, for example Borba and Villarreal (2005) *humans-with-media* theory. Beatty and Geiger (2010) reported that as digital technology continued to develop, new possibilities for student investigation and for communication and collaboration between students and between the teacher and students opened up. Studies of tools and learning environments taking advantage of these affordance tend to be informed by Vygotsky's socio-cultural theories of learning. These theories of learning shift the focus from the individual learner and their interaction with the tool to the learner and their communication with and through the tool. Other factors aside from the affordance of the tool such as the classroom environment and the teacher's pedagogical practice become critical to the investigation of student learning in this context.

In this section of the chapter, we consider "learning with" and "learning through" technologies and the learning theories informing these studies as they relate to the purpose of the study or the features of the digital technologies used. We also consider the learning objectives of the mathematical tasks and tools in the research with respect to mathematics proficiencies to consider whether there has been a shift in the learning objectives of research of the impact of digital technologies on students' learning.

26.2.1 Distinguishing the With and the Through

Previous research has noted the different ways in which teachers intend the digital tool(s) to be used and students to interact with and use digital tools for their learning of mathematics. Goos, Galbraith, Renshaw, and Geiger (2003) provided a model with which to frame the purpose and way in which students used or interacted with digital technology. They identified four relationships: *master* where the student become dependent on the digital tool to do the mathematics where their knowledge and skills are limited and thus become more limited; *servant* where the student uses it as an efficient and speedy tool to replace calculating and reasoning by pen and paper; *partner* when the student uses the tool to experience and explore different representations or perspectives or to mediate communication; and *extension of self* when students use the digital technology autonomously to investigate and problem solve. The relationships of servant and partner might be described as concerning learning with digital tools, and depending on the tool, partner along with extension of self could be described as learning through digital tools.

As researchers, we are often concerned at the prevalence of digital tools being the master for some students and the prevalence of pedagogical practices which result in digital tools being a servant for many students. Bowman (Chap. 24) conducted a large study of 14–17 year old secondary students' use of graphics calculators to graph quadratics and their learning in order to address many concerns that teachers have about the use of digital tools as the *master* or a *servant* in upper secondary mathematics. The pre-and post-testing found that students who were introduced to graphic calculators at the beginning of the topic demonstrated higher levels of engagement with mathematical knowledge and retained their acquired knowledge more than students who commenced the topic with traditional transmission teaching and pen and paper exercises. These findings support the notion that students were more inclined to use the graphic calculator as a partner when first being introduced to a topic.

Greefrath and Sillar (Chap. 21) report on 10th grade students' use of *GeoGebra* for mathematical modelling. They report that students used the tool to draw, construct, measure and experiment, and observed that the task and the tool enabled the students to shift their perception and use of the digital tool from *servant* to *partner*. Whilst establishing the use of digital software for a number of steps in the mathematical modelling process they did not relate their work to a particular learning theory and in this chapter did not report on what the students learned. A number of other chapters, Moyer-Packenham, Lister, Bullock, and Shumway (Chap. 2), Tucker (Chap. 3), Walter (Chap. 7) and Calder and Murphy (Chap. 8) also report on studies where the purpose of the study was either to evaluate a tool for its potential to enable to learner to interact with or through the tool as a *partner*, or to use the tool as a mediator for discussion. The teacher in Maschietto (Chap. 11) study used an interactive whiteboard (IWB) to orchestrate whole class discussion of proof of Pythagoras' Theorem using images of proofs using paper-folding. Whilst there may have been opportunity for this study to investigate the use of an IWB to shift students from using digital tools as a servant to that of a partner, the teacher remained in control of the technology and therefore the study did not explore this possibility.

Three research studies investigated students' interactions with touch screens (Moyer-Packenham et al., Chap. 2; Tucker, Chap. 3, and Walter, Chap. 7). Each of these studies investigated the way in which students interacted with these tools. They explored the potential of these tools to represent mathematical concepts in different ways as occurs when learners use tools as a partner. They followed the tradition of instrumentation and instrumental genesis as the theoretical frame informing their studies (Drijvers et al., 2010) though only one of these studies specified a particular learning theory. Tucker (Chap. 3) uses activity theory and case study methods to identify factors that influence students' mathematical thinking when interacting with a dynamic number line tool. According to activity theory learning occurs when students' interaction with an object that is, the mathematical concept represented by the tool or artefact, changes to indicate new understandings of the concept. The study found that the decreased technological distance that is, the touch screen and dynamic number line, enabled the creation of the images that supported students' mathematical thinking and learning.

The other two studies involving touch screens and number sense might have framed their study using instrumental genesis or activity theory. Moyer-Packenham et al. (Chap. 2) argue that students need experiences with multiple representations of concepts to enhance learning. Their study used pre- and post-testing of pre-school to Grade 2 students' use of a number of different touch screen apps each using different representations of number concerning place value, number facts and derived addition strategies. They used pre- and post-testing to measure changes in students' performance and efficiency and found that there were not consistent improvements for both performance and efficiency for each grade level. They conclude that improvement in students' performance and efficiency occurred when there was both mathematical and structural alignment between the tool and the mathematics, and when the mathematical representations of the app accounted for and addressed common student misconceptions. Walter (Chap. 7) study might also have been framed by instrumental genesis or activity theory as he compared the learning that occurred when primary students with mathematic learning needs concerning efficient strategies for adding interacted with physical models and touch screen digital tools. He found that the affordance of the touch screen app provided potential for experiencing alternate representations of number but not all students took advantage of the tools' affordances to interpret the representation of number for learning to occur. So in spite of alternate representations of number that the touch screen tool provided, the tool was not influencing some students' mathematical interpretation of the digital representation and students were not using their mathematical understanding of number to create representations of number that showed evidence of number sense.

Calder and Murphy (Chap. 8) study provides a different perspective on learning with technology as a *partner*. In their study where Grade 4 students used an app to solve worded problems, and present and explain their solution, the digital technology acted as a mediator of classroom discourse. Not surprisingly, they used a social constructivist theory of learning, socio-materialism to frame their study. They cite Meyer (2015) and explain socio-materialism as recognising the complex and dynamic inter-relationships between people, communities such as classrooms, and tools where the dialogue and learning experience is changing and personalised. In their study, students created their own content, using their own language to create a movie to communicate their solution and explain their solution process. Screen casting along with audio recording provided a new mathematical learning experience for these students. Larsen et al. (Chap. 4) also investigated the use of screen casting technology in K–2 grade classrooms over a number of years and report that screen casting technology encourage students to communicate, self-assess and revise their mathematical ideas. Their study illustrated the way in which students in the early years of school interact with both digital tools as partners and with each other for their learning.

Only one study in the manuscript really explored the use of digital technology as an *extension of self*. Ng and Sinclair (Chap. 16) reported on Grade 12 calculus students' use of 3D pens to explore properties of quadratic functions. Their study can be argued to be concerned with extending self as they argued that using 3D pens enabled students to use gesture to explore and develop their understanding when using the 3D pens to 'draw' in 3D space. They framed their study using inclusive-materialism that is, the entwinement of humans, tools and concepts. They observed new gestural forms of thinking to make sense of the curve and tangent to the curve in both a physical and abstract sense.

26.2.2 Developing Mathematical Proficiencies with and Through Digital Technology

While there has been much research that focused on how students learn mathematics with and through digital technology and some studies have focused on mathematics achievement (see section of assessment this chapter), attention with respect to what students learn has mostly been directed at particular mathematics topics and concepts. With the increased emphasis given to the need for mathematics curriculum to include and develop students' problem solving and reasoning, however, it is worth considering which mathematical proficiencies can be supported with and through digital technology and whether or not the research is focused on this question. Mathematical proficiencies are included in the mathematics curriculum of many countries, most being influenced by the *Adding it Up* report (Kilpatrick et al., 2001). This report identified five proficiencies: conceptual understanding; procedural fluency that is, carrying out procedures efficiently, flexibly and accurately; strategic competence that is, formulating and solving problems; adaptive reasoning that is, logical thought, generalising, justifying and proving; and productive disposition that is, seeing mathematics as relevant and useful and makes sense (Kilpatrick, Swafford, & Findell, 2001).

We could argue that teachers in both primary and secondary school have been focused on using digital apps and tools to develop students' fluency (procedural knowledge) for practice and to target teaching and student practise of exercises using digital resources which provides automatic feedback for the student and records for the teacher. This negative perception of the use of digital tools—graphics calculators, was one of the main reasons for Bowman's study (Chap. 24). She was able to show that whilst graphics calculators may serve to assist with procedural fluency, using them at the introduction of the topic along with inquiry type activities promoted deeper engagement with the concepts.

Much research has focused on developing apps and digital tools to develop student understanding, especially in primary school, hence the focus on affordances and constraints and trialing and evaluating of tools and apps with both small and large sample sizes. This is the case with studies reported in this monograph. Moyer-Packenham et al. (Chap. 2), Tucker (Chap. 3), Walter (Chap. 7), Calder and Murphy (Chap. 8), and Ng and Sinclair (Chap. 16) have all focused on students developing understanding with and through digital tools. For younger students this included developing number sense for numbers to 20 and using this sense for adding numbers to 20 (Moyer-Packenham et al., Chap. 11; Walter, Chap. 7) and developing number sense for whole numbers and decimals (Tucker, Chap. 3).

The assumption has been that secondary teachers would use digital tools, software and apps for inquiry to develop new understandings for example by exploring properties of shapes, functions and forming and testing conjectures about these properties. However, the focus of the secondary studies reported here concerned developing understanding. The study by Ng and Sinclair (Chap. 16) focused on students' understanding of tangents to quadratic functions rather than forming and testing conjectures or generalising and Maschietto (Chap. 11), who studied a lesson that used an IWB to explore proofs of Pythagoras' theorem, focused on developing conceptual understanding rather than adaptive reasoning.

Three studies where there was a focus on learning with or through technology were concerned with or reported on developing one or more of the other three proficiencies: strategic competence, adaptive reasoning and productive disposition.

Greefrath and Sillar (Chap. 21) set out to show that *GeoGebra* could be used by secondary students to solve a modelling problem and showed that digital tools and tasks could be used to develop students' strategic competence or problem solving. They did not however report on the learning outcomes in this chapter. In the study by Calder and Murphy (Chap. 8) students solved word problems and reported and explained their solution and solution process using an embedded screen casting technology. They reported that the tool and tasks provided a degree of student autonomy and motivated students to articulate and communicate their mathematical thinking. Likewise, Larsen et al. (Chap. 4) study illuminates the potential for screen casting technology to promote young students' mathematical reasoning. The teachers' learning goal for these mathematics lessons was to construct viable arguments and to critique others' arguments. The way in which these students were able to verbalise mathematical processes and make connections between concepts resulted in teachers improving their attitudes about their students' capabilities.

26.3 Conclusion

The studies reported at the 13th ICME conference and those published in this monograph concerning learners and learning tend to focus on how learning occurs with or through digital technology rather than providing evidence of learning with or through technology. These studies are also predominantly concerned with developing students' understanding of mathematics concepts rather than engaging students in problem solving and adaptive reasoning or developing productive dispositions. Two studies published here, Moyer-Packenham et al. (Chap. 2) and Bowman (Chap. 24), used a large sample, to evaluate the effectiveness of digital tools for learning, however further evidence may be needed to convince teachers that these tools can develop students' proficiencies regarding understanding, problem solving and reasoning in order for them to shift from the dominant practice of using these tools to develop fluency or as servants to do mathematics. One study focused on the learning of students with specific mathematics learning needs, however for the most part these studies do not consider the student cohort, their needs or funds of knowledge. Research is needed concerning particular cohorts of students, especially groups who are disadvantaged or where lower student outcomes have persisted overtime such as girls, low socio-economic and Indigenous students. Students who are differently abled are beginning to receive research attention, especially as digital technology enables other means of interacting with the tool and representations. Instrumental genesis theory and activity theory were the most used frameworks for these studies though others that drew upon socio-cultural theories of learning were evident in the research where students were expected to communicate solutions and problem solving processes or mathematical reasoning. Inevitably, theoretical frameworks will evolve as we deepen our understanding of the role and place of digital technology for learning mathematics.

26.4 Assessment with and Through Technology

Learning, specifically in formal institutions like schools, cannot do without assessment. The more than 20 years old *Assessment Standards for School Mathematics* (NCTM, 1995) are still valid for large-scale and classroom assessments in mathematics education: ensuring that assessments contain high quality mathematics; enhance student learning; reflect and support equitable practices; open and transparent; inferences made from assessments are appropriate to the assessment purpose; and finally the assessment—together with the curriculum and instruction, form a coherent whole. Often "assessments define what counts as valuable learning and assign credit accordingly" (Baird, Hopfenbeck, Newton, Stobart, & Steen-Utheim, 2014, p. 21). It is therefore clear that the integration of technology into mathematics teaching and learning should be coupled by integration of technology into both formative and summative mathematics assessments.

An important distinction between two types of technology-rich assessment, based on the work by Stacey and Wiliam (2013), was made by Drijvers et al. (2016): assessment with technology and assessment through technology. Assessment *with* digital technology concerns paper-and-pen written tests, during which students have access to digital technology such as (graphing or CAS) calculators or computers. In assessment *through* digital technology, the test is delivered and administered through digital means. Think, for example, of online tests, in which all student responses are entered in the digital test player environment. Both types of assessment may relieve students from computation and drawing, hence affects the type of skills assessed, the goals of the assessment, the tasks, and the validity and the reliability of the assessment. Students' mathematical literacy abilities can be assessed more easily, as well as their conceptual understanding, strategies, and modelling and problem-solving skills.

In their state-of-the-art survey about assessment, Suurtamm et al. (2016) raised several questions for further studies, which concern assessment and technology:

How does the use of technology influence the design of assessment items? What are the affordances of technology? What are the constraints? (p. 12).

What are some of the additional challenges in assessment when hand-held technologies are available (e.g., graphing calculators) or mobile technologies are easily accessible (e.g., smart phones with internet connections)? (p. 19).

Three papers in this volume address these questions and others, and provide initial answers which for the most part reflect three different directions—from theoretical and practical dimensions, at the middle and high school levels: Dick (Chap. 14) focused on geometry; Grugeon-Allys, Chenevotot-Quentin, Pilet, and Prévit (Chap. 13) focused on algebra; and Beck (Chap. 18) focused on written solutions in CAS allowed tests.

In the context of middle school geometry, Dick (Chap. 14) describes the development of an assessment with technology tool that will allow teachers to create their own assessments tasks. To go beyond the trivial, multiple-choice questions, which are checked against a pre-determined given key, he used DGE and

CAS. Students are presented with a geometric figure or graph, and are asked to modify the object so that it fulfils a given set of constrains. The system can check logical conditions or symbolic expressions, hence is able to accept a range of responses as valid. In this way, the system allows design of higher order tasks (Stein et al., 2000). The developers also created an interface by which the teacher can determined the specific tasks that his students will receive, and also retrieve his students' performances. The system was examined with some teachers and their classes, and is seen as a first promising step towards a fuller implementation. Dick's study highlighted both—the possibilities which open up while assessing with technology, as well as some constraints.

A very different example of the opportunities and limitations offered by assessment with technology is provided by Grugeon-Allys et al. (Chap. 13). The system developed by the researchers may be classified as assessing through technology. The aim was to develop a diagnostic tool—that will classify students' performance level using three specific sub-competencies in middle school algebra. The tasks in the diagnostic tool were design by the developers (and not the teachers) based on a comprehensive epistemological and cognitive analysis. This analysis is the basis for the classification of students' answers to three general levels of algebra competencies. It is important to note that the diagnostic tool is able to assess not only technics but also verbal answers by the students. Furthermore, the system also provides the teacher with a lesson or sequence of lessons that includes tasks on their chosen topic directed at learners that where classified to the three levels. First implementations of the system with teachers have already started.

The third study by Beck (Chap. 18) presents yet a different angle stemming from assessment with technology. When technology is used by the students as part of available tools in a test situation, the students may choose to use the technology at hand while answering the test items. Students have to decide what to do with the available technology, and to report on their uses as part of their paper and pencil report on the solution process. Beck used linguistic means to analyze students written reports on their solution process. His study highlighted the importance of communicating in class about how a written solution, which was partially based on open technology like CAS may look like to enable another person to make sense of the process. This is an example of a new challenge that arose as a result of incorporating technology with assessment.

26.5 Teachers Teaching with Technology

The mathematics-education research-community has a specific interest in teachers and technology. This is noted, for example, by the topic working group 15, focused on teaching mathematics with resources and technology during CERME10 (Clark-Wilson, Aldon, Kohanová, & Robutti, in press). The issues under study varied from professional development (PD) aimed at pre-service teachers to that of practicing teachers, possible gains from implementing technologies in teaching and learning mathematics and affective issues concerning implementation of that type. These issues and others are under ongoing study, as reflected also by four authors in the current volume.

An example of possible gains from implementing technology with young learners is provided by Calder and Murphy (Chap. 8). Teachers who worked with young students in class with touch-screen technology commented on its similarity to interacting with physical representation of a mathematical object. Also, students were able to create their own mathematical objects and speak about them using their own terminology in a safe, non-judgmental environment. Such student behavior provided the teachers with access to students' ways of thinking.

A different perspective to understand teachers' frequent use of technology is linking it to teachers' beliefs. Effective aspects have a major influence on human behavior. Specifically, teachers' beliefs about technology are an important factor in the teachers' choices to implement mathematics teaching and learning in a technological environment. Thurm (Chap. 25) studied 160 high school mathematics teachers' responses to questions about the frequency of technology use in class and about teachers' beliefs about the value of such integration. By means of latent profile analysis, he was able to identify four distinct groups of teachers. Two groups acted in a consistent way with regard to technology use and beliefs: "positive beliefs—frequent users" and "negative beliefs—infrequent users". The two other groups acted in inconsistent ways: "positive beliefs—infrequent users" and "negative beliefs—frequent users". More research is needed to understand these teachers' actions. In any case, PD providers need to be aware of these four sub-groups, as they span a range of needs to be taken into account.

Yet another perspective to be considered with respect to teachers' work is suggested by Ball and Barzel (Chap. 12). The researchers pointed at three communicational roles of technology: communication *through* technology (e.g. social networks), communication *with* technology (e.g. syntax entry), and communication *of* technology (e.g. when technology displays are used as a stimulus for communication). From the teacher perspective, they call for professional development aimed first to help teachers to be confident in how to make personal use of the technology. Second, teachers need to learn how to use communication technologies to enable students learning in a technological environment. These two levels of knowledge are captured by the theoretical construct of *double instrumental genesis* (Haspekian, 2011). While the personal instrumental genesis is related to the development of a teacher's personal instrument for mathematical activity from a given artefact, the professional instrumental genesis yields a professional instrument for the teacher's didactical activity.

This same line of thought is further explored by Trgalová and Tabach (Chap. 23). Based on an extensive literature review of PD programs offered and implemented with high-school teachers, it turned out that most PD providers are not satisfied with teachers' knowledge by the end of their program. This dissatisfaction is stressed by the fact that in most cases the researches themselves are highly involved in providing the PD. Interestingly, a search for standards for teachers' knowledge while working in ICT environments at the international level yield only a few and very general recommendations. At the national level, there are variations between countries. The few available documents were analyzed by Trgalová and Tabach to identify if and how they address the double instrumental genesis, and to which of the seven knowledge-area suggested by Mishra and Koehler (2006) they relate.

26.6 Conclusion

To conclude this chapter, we now briefly revisit each of the sections and extrapolate some possible future research topics. In Sect. 26.2, it is remarkable that so many chapters in this monograph seem to focus on the use of digital technology to foster conceptual understanding. Of course, we don't want to argue against the importance of conceptual understanding in mathematics education, but we do recommend further research into the use of digital environments for other didactical purposes, such as practicing skills. It is our impression that online practicing environments are popular among students, so it might be worthwhile to know more about the conditions that make such practice most efficient and fruitful.

Section 26.3 also identified a need for studies that focus on using technology to develop problem solving and reasoning proficiencies. It ends with a plea for more attention to be given to the use of digital technology for students with special needs, which we repeat here. Are there specific characteristics or criteria for digital technology, as to make different types of students all benefit from the opportunities these tools offer? Much remains unknown on this topic.

A core distinction in Sect. 26.4 concerns assessment with and assessment through technology. Even if this distinction seems fruitful in many cases, the question still is how this with-through dimension impacts on different types of assessment, including formative assessment, peer assessment and self-assessment. How to set up these types of assessment in a fruitful way in digital environments?

In Sect. 26.5 on mathematics teachers' professional development with respect to the use of digital technology in their teaching, much is unknown about what exactly the knowledge and skills needed encompass. Also, the different theoretical approaches are not convergent. The TPACK model by Mishra and Koehler (2006), for example, is useful but does not include teachers' beliefs. As teachers' practices are crucial in education, further elaboration of successful models for teachers' professional development is needed.

Overall, the monograph on the one hand provides a wide range of relevant contributions to the knowledge in the field of using digital technology in mathematics education. On the other hand, much work needs to be done to fully exploit its potential in everyday teaching.

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