HOLOMORPHIC AUTOMORPHIC FORMS AND COHOMOLOGY

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ABSTRACT. We investigate the correspondence between holomorphic automorphic forms on the upper half-plane with complex weight and parabolic cocycles. For integral weights at least 2 this correspondence is given by the Eichler integral. We use Knopp's generalization of this integral to real weights, and apply it to complex weights that are not an integer at least 2. We show that for these weights the generalized Eichler integral gives an injection into the first cohomology group with values in a module of holomorphic functions, and characterize the image. We impose no condition on the growth of the automorphic forms at the cusps. Our result concerns arbitrary cofinite discrete groups with cusps, and covers exponentially growing automorphic forms, like those studied by Borcherds, and like those in the theory of mock automorphic forms.

For real weights that are not an integer at least 2 we similarly characterize the space of cusp forms and the space of entire automorphic forms. We give a relation between the cohomology classes attached to holomorphic automorphic forms of real weight and the existence of harmonic lifts.

A tool in establishing these results is the relation to cohomology groups with values in modules of "analytic boundary germs", which are represented by harmonic functions on subsets of the upper half-plane. It turns out that for integral weights at least 2 the map from general holomorphic automorphic forms to cohomology with values in analytic boundary germs is injective. So cohomology with these coefficients can distinguish all holomorphic automorphic forms, unlike the classical Eichler theory.

CONTENTS

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Introduction

Classically, the interpretation of holomorphic modular forms of integral weight on the complex upper half-plane $\mathfrak{H}$ in terms of group cohomology has provided a tool that has had many important applications to the geometry of modular forms, the study of their periods, the arithmetic of special values of their L-functions, for instance in [112, 66, 86, 73].

A similar interpretation for Maass forms had to wait until the introduction of periods of Maass forms given by Lewis and Zagier [82, 84]. The analogue of Eichler cohomology and the Eichler-Shimura isomorphism for Maass forms of weight zero was established in [15].

We recall that Eichler [43] attached a cocycle ψ to meromorphic automorphic forms F of weight $k \in 2\mathbb{Z}_{\geq 1}$ by

$$\psi_{F,\gamma}(t) \;=\; \int_{\gamma^{-1}z_0}^{z_0} F(z)\,(z-t)^{k-2}\,dt\,. \tag{1}$$

This cocycle has values in the space of polynomial functions of degree at most $k-2$, with the action of weight $2-k$. The action of a Fuchsian group Γ is induced by the action $|_{2-k}$ on functions $f : \mathfrak{H} \to \mathbb{C}$, and is given by

$$(f|_{2-k}\gamma)(z) := (cz+d)^{k-2} f(\gamma z).$$

The class of the cocycle does not depend on the base point z_0 in $\mathfrak{H}$. To get independence of the integrals on the path of integration F is supposed to have zero residue at all its singularities. This is the case for cusp forms F. In that case we can put the base point z_0 at a cusp, and arrive at so-called parabolic cocycles.

For cusp forms for the modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ one takes z_0 at ∞. Then the cocycle is determined by its value on $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One calls $\psi_{F,S}$ a *period polynomial* of F, whose coefficients are values of the L-function of F at integral points in the critical strip.

Knopp [66] generalized this approach to automorphic forms with arbitrary real weight. Then a multiplier system is needed in the transformation behavior of holomorphic automorphic forms. The factor $(z-t)^{k-2}$ becomes ambiguous if one replaces the positive even weight k by a real weight r. Knopp solves this problem by replacing t by $\bar{t}$ for points $t \in \mathfrak{H}$, and restoring holomorphy by complex conjugation of the whole integral. The values of the resulting cocycle are holomorphic functions on the upper half-plane. Knopp [66] shows that for cusp forms F they have at most polynomial growth as t approaches the boundary. In this way he obtains an antilinear map between the space of cusp forms and the first cohomology group with values in a module of holomorphic functions with polynomial growth. He showed, [66], that for many real weights, this map is a bijection, and conjectured this for all $r \in \mathbb{R}$. Together with Mawi [71] he proved it for the remaining real weights.

For positive even weights this seems to contradict the classical results of Eichler [43] and Shimura [112], which imply that the parabolic cohomology with values in the polynomials of degree at most $k-2$ is isomorphic to the direct sum of the space of cusp forms of weight k and its complex conjugate. The apparent contradiction is explained by the fact that Knopp uses a larger module for the cohomology. Half of the cohomology classes for the classical situation do not survive the extension of the module.

In the modular case the period function of a modular cusp form of positive even weight satisfies functional equations (*Shimura-Eichler relations*). Zagier noticed that a functional equation with a similar structure occurs in Lewis's discussion in [82] of holomorphic functions attached to even Maass cusp forms. Together [84] they showed that there is a cohomological interpretation. In [15] this relation is extended to arbitrary cofinite discrete groups of motions in the upper half-plane and Maass forms of weight zero with spectral parameters in the vertical strip $0 < \mathrm{Re}\, s < 1$. It gives an isomorphism between spaces of Maass cusp forms of weight 0 and a number of parabolic cohomology groups, and for the spaces of all invariant eigenfunctions to larger cohomology groups.

In this paper we study relations between the space of automorphic forms without growth condition at the cusps and various parabolic cohomology groups. We use the approach of [15] in the context of holomorphic automorphic forms for cofinite discrete groups of motions in the upper half-plane that have cusps. Like in [15] we do not need to impose growth conditions at the cusps, and speak of *unrestricted* holomorphic automorphic forms. We take the module of holomorphic functions in which the cocycles take their values as small as possible. That means the classical space of polynomials of degree at most $k-2$ for weights $k \in \mathbb{Z}_{\geq 2}$. Also for other weights we use a smaller module than Knopp [66]. To avoid the complex conjugation we use modules of holomorphic functions on the lower half-plane $\mathfrak{H}^-$. The flexibility gained by considering functions on the lower half plane was recognized by the last two authors [28] in the context of Eichler cohomology associated to non-critical values of L-functions. It turns out that, for the main results, working with arbitrary weights in $\mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ takes no more effort than working with real weights; so that is the generality that we choose where possible. We shall show that the definition in (1), suitably interpreted, gives a bijection between the spaces of unrestricted holomorphic automorphic forms and several isomorphic parabolic cohomology groups.

There are several motivations and potential applications for this. Knopp's approach could "see" only cusp forms, we work with smaller modules of analytic vectors in a highest weight subspace of a principal series representation, and obtain a cohomological description of all automorphic forms. In particular, this covers the case of automorphic forms with exponential growth at the cusps. This case is important especially because of its prominent role in Borcherds's theory [7] and in the theory of mock modular forms.

In the same way that representation theory has provided an important unified setting for holomorphic and Maass forms, our construction reflects a common framework for the cohomology of holomorphic and Maass forms.

There are a lot of important relations between the theory of cohomology of modular forms and various problems in number theory. For instance, Zagier [120] gives a new elementary proof of the Eichler-Selberg trace formula using the explicit description of the action of Hecke operators on the space of cohomology groups. In the same paper Zagier connects cocycles with double zeta values, in which many interesting further results are developed recently ([59], [125]). Another application is the possibility of an interpretation of the higher Kronecker limit formula in terms of cohomology group [117].

Finally, we note that one of striking applications of Eichler cohomology concerns algebraicity results for critical values of L-functions of classical (integral weight) cusp forms, eg, Manin's periods theorem [86], or [85]. The results obtained were later extended, at least conjecturally, to other values and to values of derivatives in a manner eventually formalized in the conjectures of Deligne, Beilinson, Bloch-Kato and others. See [74].

In the case of values of derivatives, the main pathway to such results did not involve directly Eichler cohomology. However, for f of weight 2, in [51], (resp. [39]), $L_f'(1)$ (resp. $L_f^{(n)}(1)$) is expressed in terms of a "period" integral similar to an Eichler cocycle, when $L_f(1) = 0$. Despite the similarity, this "period" integral does not seem at first to have a direct cohomological interpretation. Nevertheless, in §9.4 we are able to show that, $L_f'(1)$ can be expressed as a derivative with respect to a the parameter of a family of parabolic cocycles $r \mapsto \psi^{\infty}_{f_r,\cdot}$ associated to a family $r \mapsto f_r$

of automorphic forms. With [39], similar expressions can be proved for higher derivatives. We hope that better insight into the cohomology whose foundations we establish here should yield information about the algebraic structure of derivatives of L-functions along the lines of the algebraicity results for critical values derived with the help of classical Eichler cohomology.

We now proceed with a discussion of the results of this paper. We avoid many technicalities, and state the main theorems giving only rough descriptions of the cohomology groups and coefficient modules involved. In the next sections we define precisely all objects occurring in the statements.

Let Γ be a cofinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with cusps. We take a *complex* weight $r \in \mathbb{C}$ and an associated multiplier system $v : \Gamma \to \mathbb{C}^*$. We denote by $A_r(\Gamma, v)$ the space of all holomorphic functions $F : \mathfrak{H} \to \mathbb{C}$ such that

$$F(\gamma z) \;=\; v(\gamma)\,(cz+d)^r\,F(z) \qquad \text{for all } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma,\; z \in \mathfrak{H}\,.$$

For a fixed $z_0 \in \mathfrak{H}$ and an $F \in A_r(\Gamma, v)$ consider the map $\psi_F^{z_0} : \gamma \mapsto \psi_{F,\gamma}^{z_0}$ on Γ, where $\psi_{F,\gamma}^{z_0}$ is the function of $t \in \mathfrak{H}^-$ given by t

$$\psi_{F,\gamma}^{z_0}(t) \;:=\; \int_{\gamma^{-1}z_0}^{z_0} (z-t)^{r-2}\,F(z)\,dz\,. \tag{2}$$

We take the branch of $(z-t)^{r-2}$ with $-\frac{\pi}{2} < \arg(z-t) < \frac{3\pi}{2}$.

Our first main theorem is:

Theorem A. *Let Γ be a cofinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with cusps. Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, and let v be an associated multiplier system.*

i) *The assignment $\psi_F^{z_0} : \gamma \mapsto \psi_{F,\gamma}^{z_0}$ is a cocycle, and $F \mapsto \psi_F^{z_0}$ induces an injective linear map*

$$\mathbf{r}_r^{\omega} : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}_{v,2-r}^{\omega})\,. \tag{3}$$

Here $\mathcal{D}_{v,2-r}^{\omega}$ denotes a space of holomorphic functions on the lower half-plane $\mathfrak{H}^-$ that are holomorphically continuable to a neighbourhood of $\mathfrak{H}^- \cup \mathbb{R}$, together with an action depending on v.

ii) *The image $\mathbf{r}_r^{\omega} A_r(\Gamma, v)$ is equal to the mixed parabolic cohomology group $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}_{v,2-r}^{\omega}, \mathcal{D}_{v,2-r}^{\omega^0,\mathrm{exc}})$, which consists of elements of $H^1(\Gamma; \mathcal{D}_{v,2-r}^{\omega})$ represented by cocycles whose values on parabolic elements of Γ satisfy certain additional conditions at the cusps.*

This result is comparable to Theorem C in Bruggeman, Lewis, Zagier [15] where a linear injection of Maass forms of weight 0 into a cohomology group is established.

The proof of Theorem A will require many steps, and will be summarized in Subsection 10.5.

We characterize the images under $\mathbf{r}_r^{\omega}$ of the spaces $S_r(\Gamma, v)$ of cusps forms and $M_r(\Gamma, v)$ of entire automorphic forms:

Theorem B. *Let Γ be a cofinite discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *with cusps. Let $r \in \mathbb{R}$ and let v be a* unitary *multiplier system on Γ for the weight r.*

i) *If $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$*
$$\mathbf{r}_r^\omega\, S_r(\Gamma, v) \;=\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})\,,$$
where $\mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$ is a subspace of $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$ defined by a smoothness condition.

ii) *If $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 1}$*
$$\mathbf{r}_r^\omega\, M_r(\Gamma, v) \;=\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp},\mathrm{exc}}_{v,2-r})\,,$$
with the Γ-module $\mathcal{D}^{\omega^0,\mathrm{smp},\mathrm{exc}}_{v,2-r} \supset \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$ also contained in $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$.

Here we give only a result for real weights. It seems that for non-real weights the cusp forms do not form a very special subspace of the space of all automorphic forms. There is, as far as we know, no nice bound for the Fourier coefficients and it seems hard to define L-functions in a sensible way.

In Theorems A and B automorphic forms of weight r are related to cohomology with values in a module with the "dual weight" $2 - r$.

The characterization in Theorems A and B of the images of spaces of automorphic forms is one of several possibilities given in Theorem E, which we state in Subsection 1.7, after some Γ-modules containing $\mathcal{D}^\omega_{v,2-r}$ have been defined. There we see that the map $\mathbf{r}_r^\omega$ in Theorem A is far from surjective. In Section 14 we discuss a space of *quantum automorphic forms*, for which there is, if $r \notin \mathbb{Z}_{\geq 1}$, a surjection to the space $H^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$.

In §2.3 we will compare Part i) of Theorem B to the main theorem of Knopp and Mawi [71], which gives an isomorphism $S_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^{-\infty}_{v,2-r})$ for some larger Γ-module $\mathcal{D}^{-\infty}_{v,2-r} \supset \mathcal{D}^\omega_{v,2-r}$. The combination of the theorem of Knopp and Mawi with Theorem A shows that there are many automorphic forms $F \in A_r(\Gamma, v)$ for which $\mathbf{r}_r^\omega F$ is sent to zero by the natural map $H^1(\Gamma; \mathcal{D}^\omega_{v,2-r}) \to H^1(\Gamma; \mathcal{D}^{-\infty}_{v,2-r})$. This means that the cocycle $\gamma \mapsto \psi^{z_0}_{F,\gamma}$ becomes a coboundary when viewed over the module $\mathcal{D}^{-\infty}_{v,2-r}$, i.e., that there is $\Phi \in \mathcal{D}^{-\infty}_{v,2-r}$ such that $\psi^{z_0}_{F,\gamma} = \Phi|_{v,2-r}(\gamma - 1)$ for all $\gamma \in \Gamma$.

The following result relates the vanishing of the cohomology class of $\gamma \mapsto \psi^{z_0}_{F,\gamma}$ over a still larger module $\mathcal{D}^{-\omega}_{v,2-r} \supset \mathcal{D}^{-\infty}_{v,2-r}$ to the existence of harmonic lifts, a concept that we will discuss in Subsections 1.8 and 5.2.

Theorem C. *Let Γ be a cofinite discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *with cusps. Let $r \in \mathbb{C}$ and let v be a multiplier system for the weight r. The following statements are equivalent for $F \in A_r(\Gamma, v)$:*

a) *The image of $\mathbf{r}_r^\omega F$ under the natural map $H^1(\Gamma; \mathcal{D}^\omega_{v,2-r}) \to H^1(\Gamma; \mathcal{D}^{-\omega}_{v,2-r})$ vanishes.*

b) *The automorphic form F has a harmonic lift in* $\mathrm{Harm}_{2-\bar r}(\Gamma, \bar v)$*; ie, F is in the image of the antilinear map* $\mathrm{Harm}_{2-\bar r}(\Gamma, \bar v) \to A_r(\Gamma, v)$ *given by* $H \mapsto 2iy^{2-r}\,\overline{\partial_{\bar z} H}$.

We prove this theorem in Subsection 5.2. Combining the theorem of Knopp and Mawi [71] with Theorem C we obtain the existence of harmonic lifts in many cases. See Theorem 5.3 and Corollary 5.2.

Boundary germs. An essential aspect of the approach in [15] is the use of "analytic boundary germs". These germs form Γ-modules isomorphic to the modules in [15] corresponding to $\mathcal{D}^{\omega}_{v,2-r}$ and $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$ in our case. In [15] the boundary germs are indispensable for the proof of the surjectivity of the map from Maass forms of weight zero to cohomology. The same holds for this paper.

In Sections 6–8 we study the spaces of boundary germs that are relevant for our present purpose. In particular we define spaces $\mathcal{E}^{\omega}_{v,r}$ and $\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}$ that are for weights in $\mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ isomorphic to $\mathcal{D}^{\omega}_{v,2-r}$ and $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$, respectively. In Theorem 10.18 we obtain, for all complex weights r, an injective map

$$\mathbf{q}^{\omega}_r : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{E}^{\omega}_{v,r}) \tag{4}$$

and study the image.

For weights $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ we use Theorem 10.18 in the proof of Theorem A. Theorem 10.18 is also valid for weights in $\mathbb{Z}_{\geq 2}$. For these weights it leads to the following result:

Theorem D. *Let $r \in \mathbb{Z}_{\geq 2}$, let Γ be a cofinite discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *with cusps, and let v be a multiplier system on Γ with weight r.*

i) *Put $c_r = \frac{i}{2\,(r-1)!}$, let ρ_r denote the natural morphism $\mathcal{E}^{\omega}_{v,r} \to \mathcal{D}^{\omega}_{v,2-r}$, and let $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$ denote the submodule of $\mathcal{D}^{\omega}_{v,2-r}$ consisting of polynomial functions of degree at most $r-2$. The following diagram commutes:*

$$\begin{tikzcd}
H^1(\mathcal{D}^{\omega}_{v,r}) \arrow[r] & H^1(\mathcal{E}^{\omega}_{v,r}) \arrow[r, "\rho_r"] & H^1(\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[r] & 0 \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[u, hook] \arrow[r] & H^1_{\mathrm{pb}}(\mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[u, hook] \arrow[r, "\rho_r"] & H^1_{\mathrm{pb}}(\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[u, hook] & \\
A_{2-r}(v) \arrow[u, "\mathbf{r}^{\omega}_{2-r}", "\cong"'] \arrow[r, "c_r\,\partial_\tau^{r-1}"] & A^0_r(v) \arrow[u, "\mathbf{q}^{\omega}_r", "\cong"'] \arrow[r, hook] & A_r(v) \arrow[uul, hook, bend right, "\mathbf{q}^{\omega}_r"'] \arrow[uu, bend right, "\mathbf{r}^{\omega}_r"'] &
\end{tikzcd} \tag{5}$$

(To save space the group Γ is suppressed in the notation.)

ii) *The top row and the middle row are exact.*

iii) *The maps $H^1(\Gamma; \mathcal{D}^{\omega}_{v,r}) \to H^1(\Gamma; \mathcal{E}^{\omega}_{v,r})$ in the top row and the map and $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r})$ in the middle row are injective, unless $r = 2$ and v is the trivial multiplier system. In that exceptional case both maps have a kernel isomorphic to the trivial Γ-module $\mathbb{C}$.*

Remarks. (a) For $r \in \mathbb{Z}_{\geq 1}$ the Γ-module $\mathcal{D}^{\omega}_{v,r}$ can be considered as a submodule of $\mathcal{E}^{\omega}_{v,r}$. The space $A^0_r(\Gamma, v)$ is the space of unrestricted holomorphic automorphic form for which the Fourier terms of order zero at all cusps vanish.

(b) We note that automorphic forms both of weight r and of the dual weight $2 - r$ occur in the diagram. The theorem shows that boundary germ cohomology in some sense interpolates between the cohomology classes attached to automorphic forms of weight $2 - r$ and of weight r, with $r \in \mathbb{Z}_{\geq 2}$.

(c) The second line in diagram (5) has no closing $\to 0$. In §11.5 we will discuss how this surjectivity can be derived by classical methods, provided that we assume that the multiplier system v is unitary.

Comparison with [15]. This paper has much in common with the notes [15]. Both give isomorphisms between spaces of functions with automorphic transformation behavior and mixed parabolic cohomology groups. The main difference is in the modules in which the cohomology groups have their values. The Γ-modules in [15] are spherical principal series representations. The linear map in [15] analogous to our map $\mathbf{r}^{\omega}_r$ sends Maass forms of weight zero to cohomology classes in $H^1(\Gamma; \mathcal{V}^{\omega}_s)$, where $\mathcal{V}^{\omega}_s$ is the space of analytic vectors in the principal series representation of $\mathrm{PSL}_2(\mathbb{R})$ with spectral parameter s. The assumption $0 < \mathrm{Re}\, s < 1$ ensures that the representation $\mathcal{V}^{\omega}_s$ is irreducible. Holomorphic automorphic forms of weight $r \in \mathbb{C}$ correspond to a spectral parameter $\frac{r}{2}$, for which the corresponding space of analytic vectors is reducible. Hence here we work with the highest weight subspace. It is irreducible precisely if $r \notin \mathbb{Z}_{\geq 2}$, which explains that in this paper weights in $\mathbb{Z}_{\geq 2}$ require a special treatment.

Another complication arises as soon as the weight is not an integer. This means that we deal with highest weight subspace of principal series representations of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. In the main text of these notes we have avoided use of the covering group. We discuss it in the Appendix.

Although the main approach of this paper relies heavily on methods from [15], and also on ideas in [84], it was far from trivial to handle the complications not present in [15].

Overview of the paper. In Sections 1–4 we discuss results that can be formulated with the modules $\mathcal{D}^{*}_{v,2-r}$. Here the proof of Theorem B is reduced to that of Theorem A. Sections 5–7 give results for harmonic functions and boundary germs. In section 5 one finds the proof of Theorem C. We use the boundary germs in Sections 8–11 to determine the image of automorphic forms in cohomology, and prove Theorems A and D. Sections 12 and 13 give the proof of Theorem E (which itself is stated on page 18). The map $\mathbf{r}^{\omega}_r$ in Theorem A is not surjective. In Section 14 we discuss how quantum automorphic forms are mapped, under some conditions, onto $H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$.

At the end of most sections we mention directly related literature. In Section 15 we give a broader discussion of literature related to the relation between automorphic forms and cohomology. In the Appendix we give a short discussion of the universal covering group and its principal series representations.

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Part I. **Cohomology with values in holomorphic functions**

1. Definitions and notations

We work with the *upper half-plane* $\mathfrak{H} = \{z \in \mathbb{C} \ : \ \operatorname{Im} z > 0\}$ and the *lower half-plane* $\mathfrak{H}^-$ defined by $\operatorname{Im} z < 0$. For $z \in \mathfrak{H} \cup \mathfrak{H}^-$ we will often use without further explanation $y = \operatorname{Im} z$, $x = \operatorname{Re} z$. Both half-planes are disjoint open sets in the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, with the real projective line $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ as their common boundary.

1.1. **Operators on functions on the upper and lower half-plane.**

Let $r \in \mathbb{C}$. For functions f on the upper or lower half-plane

$$f|_r g\,(z) \ := \ (cz+d)^{-r}\, f\Bigl(\frac{az+b}{cz+d}\Bigr) \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\,, \tag{1.1}$$

where we compute $(cz+d)^{-r}$ according to the *argument convention* to take

$$\arg(cz+d) \in (-\pi,\pi] \text{ if } z \in \mathfrak{H}\,, \qquad \arg(cz+d) \in [-\pi,\pi) \text{ if } z \in \mathfrak{H}^-\,. \tag{1.2}$$

These operators $|_r g$ do not define a representation of $\mathrm{SL}_2(\mathbb{R})$. (One may relate it to a representation of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. See the Appendix, §A.1.1.) There are however two useful identities. Set

$$G_0 \ := \ \Bigl\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \ : \ -\pi < \arg(ci+d) < \pi \Bigr\}. \tag{1.3}$$

Then, for all $g \in G_0$ and $p = \bigl(\begin{smallmatrix} y & x \\ 0 & y^{-1} \end{smallmatrix}\bigr)$ with $x \in \mathbb{R}$ and $y > 0$:

$$(f|_r g^{-1})|_r g \ = \ (f|_r g)|_r g^{-1} \ = \ f\,, \tag{1.4}$$

$$f|_r gpg^{-1} \ = \ ((f|_r g)|_r p)|_r g^{-1}\,. \tag{1.5}$$

To interchange functions on the upper and the lower half-plane we use the antilinear involution ι given by

$$(\iota f)(z) \;:=\; \overline{f(\bar z)}\,. \tag{1.6}$$

It maps holomorphic functions to holomorphic functions, and satisfies

$$\iota\,(f|_r g) \;=\; (\iota f)|_{\bar r} g \qquad (g\in \mathrm{SL}_2(\mathbb{R}))\,. \tag{1.7}$$

1.2. **Discrete group.** Everywhere in this paper we denote by Γ a cofinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with cusps, containing $\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)$. *Cofinite* means that the quotient $\Gamma\backslash\mathfrak{H}$ has finite volume with respect to the hyperbolic measure $\frac{dx\,dy}{y^2}$. The presence of cusps implies that the quotient is not compact. The standard example is the *modular group* $\Gamma(1)=\mathrm{SL}_2(\mathbb{Z})$.

Multiplier system. A *multiplier system* on Γ for the weight $r\in\mathbb{C}$ is a map $v:\Gamma\to\mathbb{C}^*$ such that the function on $\Gamma\times\mathfrak{H}$ given by

$$j_{v,r}\Bigl(\begin{pmatrix}a&b\\c&d\end{pmatrix},z\Bigr) \;=\; v\begin{pmatrix}a&b\\c&d\end{pmatrix}\,(cz+d)^r \tag{1.8}$$

satisfies the following conditions:

$$\begin{aligned} j_{v,r}(\gamma\delta,z) \;&=\; j_{v,r}(\gamma,\delta z)\,j_{v,r}(\delta,z) &&\text{for } \gamma,\delta\in\Gamma\,,\\ j_{v,r}\Bigl(\begin{pmatrix}-a&-b\\-c&-d\end{pmatrix},z\Bigr) \;&=\; j_{v,r}\Bigl(\begin{pmatrix}a&b\\c&d\end{pmatrix},z\Bigr) &&\text{for } \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\,. \end{aligned} \tag{1.9}$$

We call a multiplier system *unitary* if $\bigl|v(\gamma)\bigr|=1$ for all $\gamma\in\Gamma$.

Action of the discrete group. Let v be a multiplier system on Γ for the weight r. For functions on $\mathfrak{H}$ and $p\equiv r \bmod 2$ we put for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$:

$$f|_{v,p}\gamma\,(z) \;:=\; v(\gamma)^{-1}\,(cz+d)^{-p}\,f(\gamma z) \;=\; j_{v,p}(\gamma,z)^{-1}\,f(\gamma z)\,, \tag{1.10}$$

and for functions on $\mathfrak{H}^-$ and $p\equiv -r \bmod 2$

$$f|_{v,p}\gamma\,(z) \;:=\; v(\gamma)^{-1}\,(cz+d)^{-p}\,f(\gamma z)\,. \tag{1.11}$$

The operator $|_{v,p}$ defines a holomorphy-preserving action of Γ on the spaces of functions on $\mathfrak{H}$ and on $\mathfrak{H}^-$, *ie.*, $(f|_{v,p}\gamma)|_{v,p}\delta=f|_{v,p}\gamma\delta$ for all $\gamma,\delta\in\Gamma$. Furthermore, $f|_{v,p}\left(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right)=f$, hence we have an action of $\bar\Gamma:=\Gamma/\{1,-1\}\subset\mathrm{PSL}_2(\mathbb{R})$. Finally,

$$\iota(f|_{v,r}\gamma) \;=\; (\iota f)|_{\bar v,\bar r}\gamma \qquad \text{for } \gamma\in\Gamma\,. \tag{1.12}$$

1.3. **Automorphic forms.** We consider automorphic forms without any growth condition.

Definition 1.1. A *unrestricted holomorphic automorphic form* on Γ with weight $r\in\mathbb{C}$ and multiplier system v on Γ for the weight r is a holomorphic function $F:\mathfrak{H}\to\mathbb{C}$ such that

$$F|_{v,r}\gamma \;=\; F \qquad \text{for all } \gamma\in\Gamma\,. \tag{1.13}$$

We use $A_r(\Gamma, v)$ to denote the space of all such unrestricted holomorphic automorphic forms. We often abbreviate *unrestricted holomorphic automorphic form* to *holomorphic automorphic form* or to *automorphic form*.

Cusps. A cusp of Γ is a point $\mathfrak{a} \in \mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ such that the *stabilizer* $\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma \ : \ \gamma\,\mathfrak{a} = \mathfrak{a}\}$ is infinite and has no other fixed points in $\mathbb{P}^1_{\mathbb{C}}$. This group is of the form $\Gamma_{\mathfrak{a}} = \{\pm\pi_{\mathfrak{a}}^n \ : \ n \in \mathbb{Z}\}$ for an element $\pi_{\mathfrak{a}} \in \Gamma$ that is conjugate to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$. The elements $\pi_{\mathfrak{a}}^n$ have trace 2, and are, for $n \neq 0$, called *parabolic*. The elements $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{a}}^{-1}$ are *primitive parabolic* since they are not of the form γ^n with $\gamma \in \Gamma$ and $n \geq 2$.

For each cusp $\mathfrak{a}$ there are (non-unique) $\sigma_{\mathfrak{a}} \in G_0$ such that $\pi_{\mathfrak{a}} = \sigma_{\mathfrak{a}} T \sigma_{\mathfrak{a}}^{-1}$. We arrange the choice such that for all $\gamma \in \Gamma$ we have $\sigma_{\gamma\mathfrak{a}} = \pm\gamma\sigma_{\mathfrak{a}} T^n$ for some $n \in \mathbb{Z}$.

The set of cusps of a given discrete group Γ is a finite union of Γ-orbits. Each of these orbits is an infinite subset of $\mathbb{P}^1_{\mathbb{R}}$.

Fourier expansion. Each $F \in A_r(\Gamma, v)$ has at each cusp $\mathfrak{a}$ of Γ a *Fourier expansion*

$$F|_r\sigma_{\mathfrak{a}}\,(z) \;=\; \sum_{n \equiv \alpha_{\mathfrak{a}} \bmod 1} a_n(\mathfrak{a}, F)\, e^{2\pi i n z}\,, \tag{1.14}$$

with $\alpha_{\mathfrak{a}}$ such that $v(\pi_{\mathfrak{a}}) = e^{2\pi i \alpha_{\mathfrak{a}}}$. The Fourier coefficients $a_n(\mathfrak{a}, F)$ depend (by a non-zero factor) on the choice of $\sigma_{\mathfrak{a}}$ in $\mathrm{SL}_2(\mathbb{R})$. In general, $\sigma_{\mathfrak{a}} \notin \Gamma$, so we have to use the operator $|_r\sigma_{\mathfrak{a}}$, and not the action $|_{v,r}$ of Γ.

If the multiplier system is not unitary, it may happen that $\left|v(\pi_{\mathfrak{a}})\right| \neq 1$ for some cusps $\mathfrak{a}$. Then $\alpha_{\mathfrak{a}} \in \mathbb{C} \smallsetminus \mathbb{R}$, and the Fourier term orders n in (1.14) are not real.

Definition 1.2. We define the following subspaces of $A_r(\Gamma, v)$:

i) The space of *cusp forms* is

$$S_r(\Gamma, v) \;:=\; \Bigl\{F \in A_r(\Gamma; v) \ : \ \forall_{\mathfrak{a}\text{ cusp}}\, \forall_{n \equiv \alpha_{\mathfrak{a}}(1)}\ \mathrm{Re}\, n \leq 0 \Rightarrow a_n(\mathfrak{a}, F) = 0\Bigr\}.$$

ii) The space of *entire automorphic forms* is

$$M_r(\Gamma, v) \;:=\; \Bigl\{F \in A_r(\Gamma, v) \ : \ \forall_{\mathfrak{a}\text{ cusp}}\, \forall_{n \equiv \alpha_{\mathfrak{a}}(1)}\ \mathrm{Re}\, n < 0 \Rightarrow a_n(\mathfrak{a}, F) = 0\Bigr\}.$$

If $v(\pi_{\mathfrak{a}}) \neq 1$ the name "entire" is not very appropriate, since then the Fourier expansion at $\mathfrak{a}$ in (1.14) needs non-integral powers of $q = e^{2\pi i z}$.

This implies that $F \in S_r(\Gamma, v)$ has *exponential decay at all cusps*:

$$\begin{gathered} \forall_{\mathfrak{a}\text{ cusp of }\Gamma}\, \forall X > 0\, \exists_{\varepsilon > 0}\, \forall_{x \in [-X, X]} \\ F(\sigma_{\mathfrak{a}}(x + iy)) \;=\; \mathrm{O}(e^{-\varepsilon y}) \text{ as } y \to \infty\,. \end{gathered} \tag{1.15}$$

If v is not unitary we need to restrict x to compact sets. Similarly, functions $F \in M_r(\Gamma, v)$ have at most *polynomial growth* at the cusps:

$$\begin{gathered} \forall_{\mathfrak{a}\text{ cusp of }\Gamma}\, \forall X > 0\, \exists_{a > 0}\, \forall_{x \in [-X, X]} \\ F(\sigma_{\mathfrak{a}}(x + iy)) \;=\; \mathrm{O}(y^a) \text{ as } y \to \infty\,. \end{gathered} \tag{1.16}$$

1.4. **Cohomology and mixed parabolic cohomology.** We recall the basic definitions of group cohomology. Let V be a right $\mathbb{C}[\Gamma]$-module. Then the first cohomology group $H^1(\Gamma;V)$ is

$$H^1(\Gamma;V) \;=\; Z^1(\Gamma;V) \bmod B^1(\Gamma;V)\,, \tag{1.17}$$

where $Z^1(\Gamma;V)$ is the space of 1-*cocycles* and $B^1(\Gamma;V)\subset Z^1(\Gamma;V)$ the space of 1-*coboundaries*. A 1-cocycle is a map $\psi:\Gamma\to V:\gamma\mapsto\psi_\gamma$ such that $\psi_{\gamma\delta}=\psi_\gamma|\delta+\psi_\delta$ for all $\gamma,\delta\in\Gamma$ and a 1-coboundary is a map $\psi:\Gamma\to V$ of the form $\psi_\gamma=a|\gamma-a$ for some $a\in V$ not depending on γ.

Definition 1.3. Let $V\subset W$ be right Γ-modules. The *mixed parabolic cohomology group* $H^1_{\mathrm{pb}}(\Gamma;V,W)\subset H^1(\Gamma;V)$ is the quotient $Z^1_{\mathrm{pb}}(\Gamma;V,W)/B^1(\Gamma;V)$, where

$$Z^1_{\mathrm{pb}}(\Gamma;V,W) \;=\; \{\psi\in Z^1(\Gamma;V)\;:\;\psi_\pi\in W|(\pi-1)\text{ for all parabolic }\pi\in\Gamma\}\,. \tag{1.18}$$

The space $H^1_{\mathrm{pb}}(\Gamma;V):=H^1_{\mathrm{pb}}(\Gamma;V,V)$ is the usual *parabolic cohomology group*.

We call cocycles in $Z^1_{\mathrm{pb}}(\Gamma;V,W)$ *mixed parabolic cocycles*, and *parabolic cocycles* if $V=W$.

In (1.18) it suffices to check the condition $\psi_\pi\in W|(\pi-1)$ for $\pi=\pi_{\mathfrak{a}}$ with $\mathfrak{a}$ in a (finite) set of representatives of the Γ-orbits of cusps.

1.5. **Modules.** The coefficient modules that we will use for cohomology are based on the following spaces:

Definition 1.4. Let $r\in\mathbb{C}$. For functions φ define the function $\mathsf{prj}_{2-r}\varphi$ by

$$(\mathsf{prj}_{2-r}\varphi)(t) \;:=\; (i-t)^{2-r}\,\varphi(t)\,, \tag{1.19}$$

where $(i-t)^{2-r}$ denotes the branch with $\arg(i-t)\in(-\frac{\pi}{2},\frac{3\pi}{2})$.

i) $\mathcal{D}^{-\omega}_{2-r}:=\Bigl\{\varphi:\mathfrak{H}^-\to\mathbb{C}\;:\;\varphi\text{ is holomorphic}\Bigr\}$.

ii) $\mathcal{D}^{-\infty}_{2-r}:=\Bigl\{\varphi\in\mathcal{D}^{-\omega}_{2-r}\;:\;\exists_{B>0}\;\varphi(t)=\mathrm{O}(|\mathrm{Im}\,t|^{-B})+\mathrm{O}(|t|^B)\text{ on }\mathfrak{H}^-\Bigr\}$, the space of functions with at most *polynomial growth*.

iii) $\mathcal{D}^{\infty}_{2-r}\;=\;\Bigl\{\varphi\in\mathcal{D}^{-\omega}_{2-r}\;:\;\mathsf{prj}_{2-r}\varphi\in C^\infty(\mathfrak{H}^-\cup\mathbb{P}^1_{\mathbb{R}})\Bigr\}$

iv) $\mathcal{D}^{\omega}_{2-r}\;=\;\mathsf{prj}^{-1}_{2-r}\varinjlim\mathcal{O}(U)$ where U runs over the open neighbourhoods of $\mathfrak{H}^-\cup\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, and $\mathcal{O}$ denotes the sheaf of holomorphic functions on $\mathbb{P}^1_{\mathbb{C}}$.

v) For $r\in\mathbb{Z}_{\geq2}$ we put $\mathcal{D}^{\mathrm{pol}}_{2-r}:=\Bigl\{\varphi\in\mathcal{D}^{\omega}_{2-r}\;:\;\varphi\text{ is given by a polynomial function on }\mathbb{C}\text{ of degree at most }r-2\Bigr\}$.

Discussion. (a) The largest of these space, $\mathcal{D}^{-\omega}_{2-r}$, consists of all holomorphic functions on the lower half-plane. The subspace $\mathcal{D}^{-\infty}_{2-r}$ is determined by behavior of $\varphi(t)$ as t approaches the boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}^-$. The real-analytic function $Q(t)=\frac{|\mathrm{Im}\,t|}{|t-i|^2}$ on $\mathbb{P}^1_{\mathbb{C}}\smallsetminus\{i\}$ satisfies $0<Q(t)\leq1$ on the lower half-plane and zero on its boundary. A more uniform definition of polynomial growth requires that functions f satisfy $f(t)\ll Q(t)^{-B}$ for some $B>0$. In Part ii) we use Knopp's formulation in [66], transformed to the lower half-plane. Both are equivalent. To see this, we use in one direction that (for $t \in \mathfrak{H}^-$)

$$\frac{|\mathrm{Im}\, t|^{-B} + |t|^B}{Q(t)^{-B}} \;\le\; \frac{1 + |t|^{2B}}{|t-i|^{2B}} \;\le\; 1 + 1\,.$$

In the other direction we carry out separate estimates for the following three cases (1) $|t| \le 1$, with $Q(t)^{-B} \le |\mathrm{Im}\, t|^{-2B}$; (2) $|t| \ge 1$, $|\mathrm{Im}\, t| \ge \frac{1}{2}$ with $Q(t)^{-B} \ll |t|^{2B} + 1$; (3) $|t| \ge 1$, $|\mathrm{Im}\, t| \le \frac{1}{2}$, with $Q(t)^{-B} \le \frac{|t|^{2B}}{|\mathrm{Im}\, t|^B} + |\mathrm{Im}\, t|^{-B} \;\le\; |t|^{4B} + |\mathrm{Im}\, t|^{-2B} + |\mathrm{Im}\, t|^{-B}$.

(b) With $t \in \mathfrak{H}^-$, the factor $(i-t)^{2-r}$ in (1.19) is $\mathrm{O}(1)$ if $\mathrm{Re}\, r \ge 2$ and $\mathrm{O}(|t|^{2-\mathrm{Re}\, r})$ if $\mathrm{Re}\, r \le 2$, and its inverse $(i-t)^{r-2}$ satisfies similar estimates. So the function φ on $\mathfrak{H}^-$ has at most polynomial growth if and only $\mathsf{prj}_{2-r}\varphi$ has polynomial growth. So we could formulate the definition of $\mathcal{D}^{-\infty}_{v,2-r}$ with $\mathsf{prj}_{2-r}\varphi$ instead of φ.

(c) The polynomial growth in Part ii) concerns the behavior of $\varphi(t)$ as t approaches the boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}^-$ at any point. The polynomial growth at the cusps in (1.16) concerns the approach of $F(z)$ as z approaches cusps in the boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}$.

(d) For some holomorphic φ on $\mathfrak{H}^-$ it may happen that $\mathsf{prj}_{2-r}\varphi$ extends from $\mathfrak{H}^-$ to yield a function that is smooth on $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$. Then $\mathsf{prj}_{2-r}\varphi$ satisfies near $\xi \in \mathbb{R}$ a Taylor approximation of any order N

$$\mathsf{prj}_{2-r}\varphi(t) \;=\; \sum_{n=0}^{N-1} a_n\,(t-\xi)^n + \mathrm{O}\big((t-\xi)^N\big)$$

as t approaches ξ through $\mathfrak{H}^- \cup \mathbb{R}$. Near ∞ we have a Taylor approximation in $-1/t$. This defines the space in Part iii) as a subspace of $\mathcal{D}^{-\omega}_{2-r}$.

These Taylor expansions imply that $\mathsf{prj}_{2-r}\varphi$ has at most polynomial growth at the boundary. So $\mathcal{D}^{\infty}_{2-r}$ is in fact a subspace of $\mathcal{D}^{-\infty}_{2-r}$.

(e) Instead of Taylor expansions of any order, we may require that $\mathsf{prj}_{2-r}\varphi$ is near each $\xi \in \mathbb{P}^1_{\mathbb{R}}$ given by a convergent power series expansion. Then it extends as a holomorphic function to a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. That defines the space $\mathcal{D}^{\omega}_{2-r}$ in Part iv).

The formulation with an inductive limit implies that we consider two extensions to be equal if they have the same restriction to $\mathfrak{H}^-$.

(f) If $r \in \mathbb{Z}_{\ge 2}$, and φ is a polynomial function of degree at most $r-2$ the function $\mathsf{prj}_{2-r}\varphi(t)$ extends holomorphically to $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i\}$.

(g) We have defined a decreasing sequence of spaces of holomorphic functions on the lower half-plane: $\mathcal{D}^{-\omega}_{2-r} \supset \mathcal{D}^{-\infty}_{2-r} \supset \mathcal{D}^{\infty}_{2-r} \supset \mathcal{D}^{\omega}_{2-r} \supset \mathcal{D}^{\mathrm{pol}}_{2-r}$ (the last one only if $r \in \mathbb{Z}_{\ge 2}$).

One can show that the spaces $\mathcal{D}^{*}_{2-r}$ arise as highest weight subspaces occurring in principal series representations of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. Then $\mathcal{D}^{\omega}_{2-r}$ corresponds to a space of *analytic vectors*, $\mathcal{D}^{\infty}_{2-r}$ to a space of C^∞-vectors, $\mathcal{D}^{-\infty}_{2-r}$ to a space of *distribution vectors*, and $\mathcal{D}^{-\omega}_{2-r}$ to a space of *hyperfunction vectors*. This motivates the choice of the superscripts ω, ∞, $-\infty$ and $-\omega$. See §A.2 in the Appendix.

(h) The vector spaces $\mathcal{D}^{\omega}_{2-r}$ and $\mathcal{D}^{\infty}_{2-r}$ depend on r, the spaces $\mathcal{D}^{-\infty}_{2-r}$ and $\mathcal{D}^{-\omega}_{2-r}$ do not.

Projective model. We have characterized the spaces $\mathcal{D}^{*}_{2-r}$ in iii) and iv) in Definition 1.4 by properties of $\mathsf{prj}_{2-r}\varphi$, not of φ itself, and could also equally well use $\mathsf{prj}_{2-r}\varphi$ in i) and ii).

We call $\mathsf{prj}_{2-r}\mathcal{D}^{*}_{2-r}$ the *projective model* of $\mathcal{D}^{*}_{2-r}$. Advantages of the projective model are the simpler definitions and the fact that none of the spaces $\mathsf{prj}_{2-r}\mathcal{D}^{*}_{2-r}$ depends on r. Moreover, the projective model focuses our attention to the behavior of the functions near the boundary $\mathbb{P}^1_{\mathbb{R}}$ of the lower half-plane.

A big advantage of the spaces $\mathcal{D}^{*}_{2-r}$ themselves is the simple form of the operators $|_{2-r}g$ with $g \in \mathrm{SL}_2(\mathbb{R})$. We will mostly work with these spaces, and use the projective model only where it makes concepts or proofs easier.

The formula in (1.1) for the operators $|_r g$ is the usual formula when one works with holomorphic automorphic forms. Of course these operators can be formulated in the projective model, as is done in Proposition 1.5 below. At first sight that description looks rather complicated. However, even this formula has its advantage, as will become clear in the proof of Proposition 1.6.

Proposition 1.5. *Let $r \in \mathbb{C}$. Under the linear map prj_{2-r} the operators $|_{2-r}g$ with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ correspond to operators $|^{\mathrm{prj}}_{2-r}g$ given on h in the projective model by*

$$h|^{\mathrm{prj}}_{2-r}g(t) \;=\; (a-ic)^{r-2}\Bigl(\frac{t-i}{t-g^{-1}i}\Bigr)^{2-r} h(gt)\,, \tag{1.20}$$

for $t \in \mathfrak{H}^-$ and the choice $\arg(a-ic) \in [-\pi, \pi)$.

Proof. We want to determine the operator $|_{2-r}g$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{prj}_{2-r}\mathcal{D}^{-\omega}_{2-r} \arrow[r, "|^{\mathrm{prj}}_{2-r}g"] & \mathsf{prj}_{2-r}\mathcal{D}^{-\omega}_{2-r} \\
\mathcal{D}^{-\omega}_{2-r} \arrow[u, "\mathsf{prj}_{2-r}"'] \arrow[r, "|_{2-r}g"] & \mathcal{D}^{-\omega}_{2-r} \arrow[u, "\mathsf{prj}_{2-r}"]
\end{tikzcd}$$

For $\varphi \in \mathcal{D}^{-\omega}_{2-r}$ put $h = \mathsf{prj}_{2-r}\varphi$. So $\varphi(t) = (i-t)^{r-2}\, h(t)$. Then $h|^{\mathrm{prj}}_{2-r}g\,(t)$ should be given by

$$(\mathsf{prj}_{2-r}(\varphi|_{2-r}g))(t) \;=\; (i-t)^{2-r}\,(ct+d)^{r-2}\,(i-gt)^{r-2}\,h(gt)$$

So we need to check that

$$(i-t)^{2-r}\,(ct+d)^{r-2}\,(i-gt)^{r-2} \;=\; (a-ic)^{r-2}\Bigl(\frac{t-i}{t-g^{-1}i}\Bigr)^{2-r}.$$

For g near to the identity in $\mathrm{SL}_2(\mathbb{R})$ and t near $-i$ this can be done by a direct computation. The equality extends by analyticity of both sides to $(t, g) \in \mathfrak{H}^- \times G_0$. (See (1.3) for G_0.)

All factors are real-analytic in (t, g) on $\mathfrak{H}^- \times \mathrm{SL}_2(\mathbb{R})$, except $(ct+d)^{r-2}$ and $(a-ic)^{r-2}$. So we have to check that the arguments of these two factors tend to the same limit as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} -p & q \\ 0 & -p^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \smallsetminus G_0$, with $p > 0$ and $q \in \mathbb{R}$. We have indeed $\arg(ct+d) \to -\pi$, and $\arg(a-ic) \to -\pi$. □

Proposition 1.6. *Each of the spaces $\mathcal{D}^{\mathrm{pol}}_{2-r}$, $\mathcal{D}^{\omega}_{2-r}$, $\mathcal{D}^{\infty}_{2-r}$, $\mathcal{D}^{-\infty}_{2-r}$ and $\mathcal{D}^{-\omega}_{2-r}$ is invariant under the operators $|_{2-r}g$ with $g \in \mathrm{SL}_2(\mathbb{R})$.*

Proof. We work with the projective model. The factor $\left(\frac{t-i}{t-g^{-1}i}\right)^{2-r}$ and its inverse are holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus p$, were p is a path in $\mathfrak{H}$ from i to $g^{-1}i$ in $\mathfrak{H}$. Multiplication by this factor preserves the projective models of each of the last four spaces. The invariance of $\mathcal{D}^{\mathrm{pol}}_{2-r}$ is easily checked without use of the projective model. □

Definition 1.7. Let Γ be a cofinite subgroup of $\mathrm{SL}_2(\mathbb{R})$ and let v be a multiplier system for the weight $r \in \mathbb{C}$. For each choice of $* \in \{-\omega, -\infty, \infty, \omega, \mathrm{pol}\}$, we define $\mathcal{D}^{*}_{v,2-r}$ as the space $\mathcal{D}^{*}_{2-r}$ with the action $|_{v,2-r}$ of Γ, defined in (1.11).

Remark. The finite-dimensional module $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$ is the coefficient module used by Eichler [43]. Knopp [66] used an infinite-dimensional module isomorphic (under ι in (1.6)) to $\mathcal{D}^{-\infty}_{v,2-r}$ for the cocycles attached to cusp forms of real weight. In our approach $\mathcal{D}^{\omega}_{v,2-r}$ will be the basic Γ-module.

1.6. **Semi-analytic vectors.** For a precise description of the image of the map $\mathbf{r}^{\omega}_{r}$ from automorphic forms to cohomology with values in $\mathcal{D}^{\omega}_{v,2-r}$, we need more complicated modules, in spaces where we relax the conditions in Part iv) of Definition 1.4 in a finite number of points of $\mathbb{P}^1_{\mathbb{R}}$.

Definition 1.8. Semi-analytic vectors.

i) Let $\xi_1, \ldots, \xi_n \in \mathbb{P}^1_{\mathbb{R}}$.

$$\mathcal{D}^{\omega}_{2-r}[\xi_1, \ldots, \xi_n] \ := \ \mathsf{prj}^{-1}_{2-r} \varinjlim O(U)\,, \tag{1.21}$$

where U runs over the open sets in $\mathbb{P}^1_{\mathbb{C}}$ that contain $\mathfrak{H}^-$ and $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}$.

ii) $\mathcal{D}^{\omega^*}_{2-r} := \varinjlim \mathcal{D}^{\omega}_{2-r}[\xi_1, \ldots, \xi_n]$, where $\{\xi_1, \ldots, \xi_n\}$ runs over the finite subsets of $\mathbb{P}^1_{\mathbb{R}}$.

iii) $\mathcal{D}^{\omega^0}_{2-r} := \varinjlim \mathcal{D}^{\omega}_{2-r}[\mathfrak{a}_1, \ldots, \mathfrak{a}_n]$, where $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_n\}$ runs over the finite sets of cusps of Γ.

iv) For $\varphi \in \mathcal{D}^{\omega^*}_{2-r}$ we define the set of *boundary singularities* $\mathsf{BdSing}\,\varphi$ as the minimal set $\{\xi_1, \ldots, \xi_n\}$ such that $\varphi \in \mathcal{D}^{\omega}_{2-r}[\xi_1, \ldots, \xi_n]$.

Conditions on the singularities. The elements of the spaces in Definition 1.8 can be viewed as real-analytic functions on $\mathbb{R} \smallsetminus E$ for some finite set E, without conditions on the nature of the singularities at the exceptional points in E. We will define subspaces by putting restrictions on the singularities that we allow.

If $\varphi \in \mathcal{D}^{\omega}_{2-r}$ then $h = \mathsf{prj}_{2-r}\varphi$ is holomorphic at each point $\xi \in \mathbb{P}^1_{\mathbb{R}}$, hence its power series at ξ represents h on a neighbourhood of ξ in $\mathbb{P}^1_{\mathbb{C}}$:

$$h(t) \;=\; \sum_{n\geq 0} a_n\,(t-\xi)^n \quad (\xi\in\mathbb{R})\,, \qquad h(t) \;=\; \sum_{n\geq 0} a_n\,t^{-n} \quad (\xi=\infty)\,. \tag{1.22}$$

If φ is in the larger space $\mathcal{D}^{\infty}_{2-r}$, then there need not be a power series that converges to the function $h = \mathsf{prj}_{2-r}\varphi$, but only an *asymptotic series*

$$h(t) \;\sim\; \sum_{n\geq 0} a_n\,(t-\xi)^n \quad (\xi\in\mathbb{R})\,, \qquad h(t) \;\sim\; \sum_{n\geq 0} a_n\,t^{-n} \quad (\xi=\infty)\,, \tag{1.23}$$

valid as t approaches ξ through $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$. By this formula we mean that for any order $N \geq 1$ we have

$$h(t) \;=\; \sum_{n=0}^{N-1} a_n\,(t-\xi)^n + \mathrm{O}((t-\xi)^N)$$

as $t \to \xi$ through $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$, and analogously for $\xi = \infty$.

Smooth semi-analytic vectors. The first condition on the singularities that we define is rather strict:

Definition 1.9. $\mathcal{D}^{\omega,\infty}_{2-r}[\xi_1,\ldots,\xi_n] := \mathcal{D}^{\omega}_{2-r}[\xi_1,\ldots,\xi_n] \cap \mathcal{D}^{\infty}_{2-r}$. We call it a space of *smooth semi-analytic vectors*.

Semi-analytic vectors with simple singularities. We may also allow the asymptotic expansions in (1.23) to run over $n \geq -1$. This gives the following space of semi-analytic vectors with *simple singularities*:

Definition 1.10. We define spaces of *semi-analytic vectors with simple singularities* by

$$\begin{aligned}\mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\xi_1,\ldots\xi_n] \;:=\; \Bigl\{\varphi \in \mathcal{D}^{\omega}_{2-r}[\xi_1,\ldots,\xi_n] \;:\;& \\ t\mapsto (t-\xi_j)\,(\mathsf{prj}_{2-r}\varphi)(t) \text{ is in } C^\infty(\mathfrak{H}^-\cup\mathbb{R}) \text{ if } \xi_j\in\mathbb{R}\,,& \\ t\mapsto t^{-1}\,(\mathsf{prj}_{2-r}\varphi)(t) \text{ is in } C^\infty(\mathfrak{H}^-\cup\mathbb{P}^1_{\mathbb{R}}\smallsetminus\{0\}) \text{ if } \xi_j=\infty&\Bigr\}.\end{aligned} \tag{1.24}$$

Example. Elements of $\mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\cdots]$ turn up naturally. Often we will be interested in equations like the following one:

$$h(t+1) - h(t) \;=\; \varphi(t)\,,$$

where φ is given. In the case $\varphi \in \mathcal{D}^{\mathrm{pol}}_{2-r}$ with $r \in \mathbb{Z}_{\geq 2}$, we cannot find a solution h in $\mathcal{D}^{\omega}_{2-r}$ if φ is a (nonzero) polynomial with degree equal to $r-2$. If there is a solution h of the equation given by a polynomial, then $\deg h = r-1$, and h cannot be in $\mathcal{D}^{\mathrm{pol}}_{2-r}$. Further note that such a solution h is even not in $\mathcal{D}^{\omega}_{2-r}$, since $(\mathsf{prj}_{2-r}h)(t) = (i-t)^{2-r}\,h(t)$ is not holomorphic at ∞. However, $t \mapsto t^{-1}\,(\mathsf{prj}_{2-r}h)(t)$ is holomorphic at ∞, hence $h \in \mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\infty]$.

Semi-analytic vectors supported on an excised neighbourhood. Much more freedom leaves the last condition that we define. It does not work with asymptotic expansions, but with the nature of the domain to which the function can be holomorphically extended.

Definition 1.11. A set $\Omega \subset \mathbb{P}^1_{\mathbb{C}}$ is an *excised neighbourhood* of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$, if it contains a set of the form

$$U \smallsetminus \bigcup_{\xi \in E} W_\xi \,,$$

where U is a standard neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, where E is a finite subset of $\mathbb{P}^1_{\mathbb{R}}$, called the *excised set*, and where W_ξ has the form

$$W_\xi \;=\; \{h_\xi z \in \mathfrak{H} \;:\; |\mathrm{Re}\, z| \leq a \text{ and } \mathrm{Im}\, z > \varepsilon\}\,,$$

with $h_\xi \in \mathrm{SL}_2(\mathbb{R})$ such that $h_\xi \infty = \xi$, and $a, \varepsilon > 0$.

Instead of "excised neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ with excised set E" we shall often write *E-excised neighbourhood.*

A typical excised neighbourhood Ω of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ with excised set $E = \{\infty, \xi_1, \xi_2\}$ looks as indicated in Figure 1.

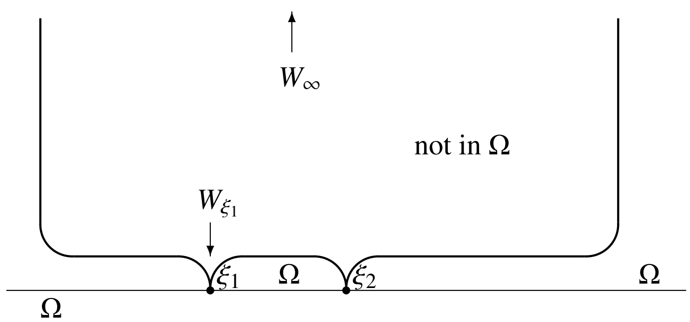

Figure 1. An $\{\infty, \xi_1, \xi_2\}$-excised neighbourhood.

Definition 1.12. For $\xi_1, \ldots, \xi_n \in \mathbb{P}^1_{\mathbb{R}}$ we define spaces of *excised semi-analytic vector*

$$\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\xi_1, \ldots, \xi_n] \;:=\; \mathsf{prj}^{-1}_{2-r} \varinjlim \mathcal{O}(\Omega)\,, \tag{1.25}$$

where Ω runs over the $\{\xi_1, \ldots, \xi_n\}$-excised neighbourhoods.

Definition 1.13. For cond $\in \{\infty, \mathrm{smp}, \mathrm{exc}\}$ we define

$$\begin{aligned} \mathcal{D}^{\omega^*,\mathrm{cond}}_{2-r} &= \varinjlim \mathcal{D}^{\omega,\mathrm{cond}}_{2-r}[\xi_1, \ldots, \xi_n]\,, \\ \mathcal{D}^{\omega^0,\mathrm{cond}}_{2-r} &= \varinjlim \mathcal{D}^{\omega,\mathrm{cond}}_{2-r}[\mathfrak{a}_1, \ldots, \mathfrak{a}_n]\,, \end{aligned} \tag{1.26}$$

where $\{\xi_1,\dots\xi_n\}$ runs over the finite subsets of $\mathbb{P}^1_{\mathbb{R}}$, and $\{\mathfrak{a}_1,\dots\mathfrak{a}_n\}$ over the finite sets of cusps of Γ.

Notation. The conditions ∞, and 'smp' can be combined with 'exc'. For instance, by $\mathcal{D}^{\omega^*,\infty,\mathrm{exc}}_{2-r}$ we mean $\mathcal{D}^{\omega^*,\infty}_{2-r}\cap\mathcal{D}^{\omega^*,\mathrm{exc}}_{2-r}$.

Proposition 1.14. i) *$\mathcal{D}^{\omega}_{2-r}[\xi_1,\dots,\xi_n]|_{2-r}g=\mathcal{D}^{\omega}_{2-r}[g^{-1}\xi_1,\dots,g^{-1}\xi_n]$ for each $g\in\mathrm{SL}_2(\mathbb{R})$. Hence*

- a) *The space $\mathcal{D}^{\omega^*}_{2-r}$ is invariant under the operators $|_{2-r}g$ with $g\in\mathrm{SL}_2(\mathbb{R})$.*
- b) *The space $\mathcal{D}^{\omega^0}_{2-r}$ is invariant under the operators $|_{2-r}\gamma$ for $\gamma\in\Gamma$.*

ii) *The same holds for the corresponding spaces with condition ∞,* smp *or* exc *at the singularities.*

iii) *$\mathsf{BdSing}\,(\varphi|_{2-r}g)=g^{-1}\,\mathsf{BdSing}\,\varphi$ for $\varphi\in\mathcal{D}^{\omega^*}_{2-r}$ and $g\in\mathrm{SL}_2(\mathbb{R})$.*

Proof. Most is clear. For Part ii) we check that the conditions are stable under the operators $|_{2-r}g$. □

Notation. We denote for each of these spaces $\mathcal{D}^{\omega^*}_{2-r}$, $\mathcal{D}^{\omega^0}_{2-r}$, $\mathcal{D}^{\omega^*,\mathrm{cond}}_{2-r}$, $\mathcal{D}^{\omega^0,\mathrm{cond}}_{2-r}$, by $\mathcal{D}^{*}_{v,2-r}$ that space provided with the action $|_{v,2-r}$ of Γ.

1.7. **Isomorphic cohomology groups.** Theorems A and B give one characterization of the images of $A_r(\Gamma,v)$, $S_r(\Gamma,v)$ and $M_r(\Gamma,v)$ under the map $\mathbf{r}^{\omega}_r$ in Theorem A to the analytic cohomology group $H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r})$. At this point we have available all Γ-modules to give several more characterizations of these images, thus extending Theorems A and B.

Theorem E. *Let Γ be a cofinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ with cusps, and let v be a multiplier system for the weight $r\in\mathbb{C}$.*

i) *Suppose that $r\notin\mathbb{Z}_{\geq 2}$.*

- a) *The image $\mathbf{r}^{\omega}_r A_r(\Gamma,v)=H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$ is equal to*
$$H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}),$$
and canonically isomorphic to
$$H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}).$$
- b) *The codimension of $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$ in $H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r})$ is infinite.*
- c) *The natural map $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})\to H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ is injective, and its image has infinite codimension in $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$.*

ii) *Suppose that $r\in\mathbb{R}\smallsetminus\mathbb{Z}_{\geq 2}$.*
The image $\mathbf{r}^{\omega}_r S_r(\Gamma,v)=H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$ is equal to
$$H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^0,\infty}_{v,2-r}),\quad H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^*,\infty}_{v,2-r}),$$
and canonically isomorphic to
$$H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^0,\infty}_{v,2-r}),\quad H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega^*,\infty}_{v,2-r}).$$

iii) *Suppose that* $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 1}$.

a) *The image* $\mathbf{r}_r^\omega M_r n(\Gamma, v) = H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp,exc}}_{v,2-r})$ *is equal to*

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r}), \quad H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})$$

and canonically isomorphic to $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r})$.

b) *The space* $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp,exc}}_{v,2-r})$ *is canonically isomorphic to the space* $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})$ *if* $v(\gamma) \neq e^{-r\ell(\gamma)/2}$ *for all primitive hyperbolic elements* $\gamma \in \Gamma$*, where* $\ell(\gamma)$ *is the hyperbolic length of the closed geodesic associated to* γ

Remarks. (a) In the statement of the theorem we speak of equality of mixed parabolic cohomology groups, all contained in $H^1(\Gamma, \mathcal{D}^\omega_{v,2-r})$, and of canonical isomorphisms, given by natural maps in cohomology corresponding to inclusions of Γ-modules.

(b) Some of the isomorphisms underlying this theorem are valid for a wider class of weights. See the results in Sections 12 and 13.

(c) Proposition 13.5 gives some additional information concerning $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})$ if $v(\gamma) = e^{-r\ell(\gamma)/2}$ for some primitive hyperbolic $\gamma \in \Gamma$.

(d) We will obtain Theorem E in many steps. We recapitulate the proof in Subsection 13.3

1.8. **Harmonic lifts of holomorphic automorphic forms.** The spaces of holomorphic automorphic forms are contained in larger spaces of harmonic automorphic forms.

Definition 1.15. Let $r \in \mathbb{C}$.

i) If $U \subset \mathfrak{H}$ is open and the function F on U is twice continuously differentiable, then we call F an r-harmonic function on U if $\Delta_r F = 0$ for the differential operator

$$\Delta_r \;=\; -4y^2 \frac{\partial^2}{\partial z\, \partial \bar{z}} + 2iry \frac{\partial}{\partial \bar{z}}\,. \tag{1.27}$$

ii) An *r-harmonic automorphic form* with the multiplier system v is a function $F : \mathfrak{H} \to \mathbb{C}$ that satisfies

a) $F|_{v,r}\gamma = F$ for all $\gamma \in \Gamma$.

b) F is r-harmonic.

We denote the linear space of such forms by $\mathrm{Harm}_r(\Gamma, v)$.

Definition 1.16. Let $r \in \mathbb{C}$. We call the following map ξ_r the *shadow operator*:

$$(\xi_r F)(z) = 2i\, y^{\bar{r}}\, \overline{\frac{\partial}{\partial \bar{z}} F(z)}\,. \tag{1.28}$$

A useful property of the shadow operator, which allows us to detect r-harmonicity, is the following equivalence:

$$F \in C^2(U) \text{ is } r\text{-harmonic} \;\Leftrightarrow\; \xi_r F \text{ is holomorphic}\,. \tag{1.29}$$

This is based on the relation $\frac{\partial}{\partial \bar z}(\xi_r F) = -\frac{iy^{\bar r-2}}{2}\overline{\Delta_r F}$.

The shadow operator induces an antilinear map

$$\xi_r : \mathrm{Harm}_r(\Gamma, v) \to A_{2-\bar r}(\Gamma, \bar v)$$

because ξ_r sends elements in the kernel of Δ_r to holomorphic functions, and

$$\xi_r(F|_r g) \;=\; (\xi_r F)|_{2-\bar r} g \quad \text{for each } g \in \mathrm{SL}_2(\mathbb{R})\,. \tag{1.30}$$

We have an exact sequence of $\mathbb{R}$-linear maps

$$0 \to A_r(\Gamma, v) \to \mathrm{Harm}_r(\Gamma, v) \xrightarrow{\xi_r} A_{2-\bar r}(\Gamma, \bar v)\,. \tag{1.31}$$

Definition 1.17. Let $F \in A_{2-\bar r}(\Gamma, \bar v)$. We call H a *harmonic lift* of F if

$$H \in \mathrm{Harm}_r(\Gamma, v) \quad \text{and} \quad \xi_r H \;=\; F\,.$$

In §A.1.4 in the Appendix we discuss r-harmonic automorphic forms on the universal covering group.

Remark 1.18. The action $|_{v,r}$ of Γ in the functions on $\mathfrak{H}$ gives rise to various spaces of invariants, for instance:

$$\begin{aligned}
C^\infty_{v,r}(\Gamma\backslash\mathfrak{H}) \;&=\; \Big\{ f \in C^\infty(\mathfrak{H}) \;:\; f|_{v.r}\gamma = f \text{ for all } \gamma \in \Gamma \Big\}\,;\\
\ker\Big(\Delta_r - \lambda : C^\infty_{v,r}(\Gamma\backslash\mathfrak{H}) &\longrightarrow C^\infty_{v,r}(\Gamma\backslash\mathfrak{H})\Big) \quad \text{with } \lambda \in \mathbb{C}\,,\\
&\qquad \text{real-analytic automorphic forms}\,;\\
\mathrm{Harm}_r(\Gamma, v) = \ker\Big(\Delta_r : C^\infty_{v,r}(\Gamma\backslash\mathfrak{H}) &\longrightarrow C^\infty_{v,r}(\Gamma\backslash\mathfrak{H})\Big)\,,\\
&\qquad \text{harmonic automorphic forms}\,;\\
A_r(\Gamma, v) = \mathrm{Harm}_r(\Gamma, v) &\cap \mathcal{O}(\mathfrak{H})\,,\\
&\qquad \text{holomorphic automorphic forms}\,.
\end{aligned}$$

For each of these spaces growth conditions at the cusp give rise to subspaces.

Real-analytic. A function on an open set $U \subset \mathbb{R}$ is real-analytic if on an open neighbourhood of each $x_0 \in U$ it is given by a convergent power series of the form $\sum_{n\geq 0} c_n\,(x - x_0)^n$. This gives a holomorphic extension of the function to a neighbourhood of U in $\mathbb{C}$.

A function on an open set $U \subset \mathbb{C}$) is real-analytic if an open neighbourhood of each point $z_0 = x_0 + iy_0 \in U$ it is given by an absolutely convergent power series $\sum_{n,m\geq 0} c_{n,m}(x - x_0)^n\,(y - y_0)^m$, or equivalently by a convergent power series $\sum_{n,m\geq 0} d_{n,m}(z - z_0)^n\,\big(\overline{z - z_0}\big)^m$. The latter representations give a holomorphic extension of the function to some neighbourhood in $\mathbb{C}^2$ of the image of the domain of the function under the map $z \mapsto (z, \bar z)$.

On $\mathbb{P}^1_{\mathbb{C}}$ one proceeds similarly, using power series in $1/z$ and $1/\bar z$ on a neighbourhood of ∞.

2. Modules and cocycles

In Section 1 we fixed the notations and defined most of the modules occurring in the main theorems in the Introduction. Now we turn to the map from automorphic forms to cohomology induced by (2). We also discuss the relation with the theorem of Knopp and Mawi [71].

2.1. **The map from automorphic forms to cohomology.**

Definition 2.1. Let F be any holomorphic function on $\mathfrak{H}$.

$$\omega_r(F;t,z) \;:=\; (z-t)^{r-2}\,F(z)\,dz \tag{2.1}$$

for $z\in\mathfrak{H}$ and $t\in\mathfrak{H}^-$; we take $-\frac{\pi}{2}<\arg(z-t)<\frac{3\pi}{2}$.

This defines $\omega_r(F;t,z)$ as a holomorphic 1-form in the variable z. The presence of the second variable enables us to view it as a differential form with values in the functions on $\mathfrak{H}^-$.

Lemma 2.2. i) *The differential form $\omega_r(F;\cdot,z)$ has values in $\mathcal{D}^{\omega}_{2-r}$.*

ii) *If $r\in\mathbb{Z}_{\geq 2}$ it has values in the subspace $\mathcal{D}^{\mathrm{pol}}_{2-r}$.*

Proof. In the projective model the differential form looks as follows:

$$\omega_r^{\mathrm{prj}}(F;t,z) \;:=\; \bigl(\mathsf{prj}_{2-r}\omega_r(F;\cdot,z)\bigr)(t) \;=\; \Bigl(\frac{z-t}{i-t}\Bigr)^{r-2} F(z)\,dz\,, \tag{2.2}$$

where for $t\in\mathfrak{H}^-$ and $z\in\mathfrak{H}$ we have $\arg\frac{z-t}{i-t}\in(-\pi,\pi)$. The factor $\bigl(\frac{z-t}{i-t}\bigr)^{r-2}$ is holomorphic for $t\in\mathbb{P}^1_{\mathbb{C}}\smallsetminus p$, where p is a path in $\mathfrak{H}$ from i to z, which implies Part i). Part ii) is clear from (2.1). □

Lemma 2.3. *Let F be holomorphic on $\mathfrak{H}$.*

i) $\omega_r(F;\cdot,gz)|_{2-r}g=\omega_r(F|_rg;\cdot,z)$ *for each* $g\in\mathrm{SL}_2(\mathbb{R})$.

ii) $\omega_r(F;\cdot,\gamma z)|_{v,2-r}\gamma=\omega_r(F|_{v,r}\gamma;\cdot,z)$ *for each* $\gamma\in\Gamma$.

iii) $\int_{\gamma z_1}^{\gamma z_2}\omega_r(F;\cdot,z)|_{v,2-r}\gamma \;=\; \int_{z_1}^{z_2}\omega_r(F|_{v,r}\gamma;\cdot,z)$ *for* $\gamma\in\Gamma$ *and* $z_1,z_2\in\mathfrak{H}$. *The integral is independent of the choice of the path.*

Proof. i) The relation amounts for $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$ to

$$(ct+d)^{r-2}\Bigl(\frac{z-t}{(ct+d)\,(cz+d)}\Bigr)^{r-2}F(gz)\,\frac{dz}{(cz+d)^2} \;=\; (z-t)^{r-2}\,(cz+d)^{-r}\,F(z)\,dz\,.$$

With the argument conventions in (1.2) for $\arg(cz+d)$ with $z\in\mathfrak{H}$ and $z\in\mathfrak{H}^-$ this equality turns out to hold for $t=-i$ and $z=i$. It extends holomorphically for $t\in\mathfrak{H}^-$ and $z\in\mathfrak{H}$.

ii) With $g=\gamma\in\Gamma$ we multiply the relation in Part i) by $v(\gamma)^{-1}$.

iii) We note that

$$\int_{\gamma z_1}^{\gamma z_2}\omega_r(F;\cdot,z)|_{v,2-r}\gamma\;(t) \;=\; \int_{z_1}^{z_2}\omega_r(F;\cdot;\gamma z)|_{v,2-r}\gamma\;(t) \;=\; \int_{z_1}^{z_2}\omega_r(F|_{v,r}\gamma;t,z)\,.$$

The differential form is holomorphic, hence closed, and the integral does not depend on the path of integration, only on the end-points.

□

Proposition 2.4. *Let $F \in A_r(\Gamma, v)$.*

i) *The map $\psi_F^{z_0} : \gamma \mapsto \psi_{F,\gamma}^{z_0}$ defined in* (2) *in the introduction is an element of* $Z^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$.

ii) *The linear map $\mathbf{r}_r^\omega : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$ associating to F the cohomology class of $\psi_F^{z_0}$ is well defined.*

iii) *If $r \in \mathbb{Z}_{\geq 2}$ then $\mathbf{r}_r^\omega A_r(\Gamma, v) \subset H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r})$.*

Proof. i) Since we integrate over a compact set in $\mathfrak{H}$ the values $\psi_{F,\gamma}^{z_0}$ are in $\mathcal{D}^\omega_{2-r}$. For the cocycle relation we compute for $\gamma, \delta \in \Gamma$:

$$\psi_{F,\gamma\delta}^{z_0} - \psi_{F,\delta}^{z_0} = \left(\int_{\delta^{-1}\gamma^{-1}z_0}^{z_0} - \int_{\delta^{-1}z_0}^{z_0} \right) \omega_r(F; \cdot, z) = \int_{\delta^{-1}\gamma^{-1}z_0}^{\delta^{-1}z_0} \omega_r(F; \cdot z)$$

$$\overset{\text{Part iii) in Lemma 2.3}}{=} \int_{\gamma^{-1}z_0}^{z_0} \omega(F|_{v,r}\delta^{-1}; \cdot, z)|_{v,2-r}\delta = \psi_{F,\gamma}^{z_0}|_{v,2-r}\delta\,.$$

ii) To see that the cohomology class of $\psi_F^{z_0}$ does not depend on the choice of the base point z_0 we check that with two base points z_0 and z_1 the difference is a coboundary:

$$\psi_{F,\gamma}^{z_0} - \psi_{F,\gamma}^{z_1} = \left(\int_{\gamma^{-1}z_0}^{z_0} - \int_{\gamma^{-1}z_1}^{z_1} \right) \omega_r(F; \cdot, z)$$

$$= \left(\int_{\gamma^{-1}z_0}^{\gamma^{-1}z_1} - \int_{z_0}^{z_1} \right) \omega_r(F; \cdot, z) \overset{\text{Part iii) in Lemma 2.3}}{=} b|_{v,2-r}\gamma - b\,,$$

with $b = \int_{z_0}^{z_1} \omega_r(F; \cdot; z)$ in $\mathcal{D}^\omega_{v,2-r}$. Hence $\mathbf{r}_r^\omega$ is well defined.

iii) See Part ii) of Lemma 2.2. □

2.2. **Cusp forms.** A cusp form $F \in S_r(\Gamma, v)$ decays exponentially at each cusp $\mathfrak{a}$ of Γ, and we can define for the cusp $\mathfrak{a}$

$$\psi_F^{\mathfrak{a}} : \gamma \mapsto \psi_{F,\gamma}^{\mathfrak{a}}(t) := \int_{\gamma^{-1}\mathfrak{a}}^{\mathfrak{a}} \omega_r(F; t, z)\,. \tag{2.3}$$

We use $\sigma_{\mathfrak{a}}$ such that $\mathfrak{a} = \sigma_{\mathfrak{a}}\infty$ and $\pi_{\mathfrak{a}} = \sigma_{\mathfrak{a}} T \sigma_{\mathfrak{a}}^{-1}$ as in §1.3. If $|v(\pi_{\mathfrak{a}})| \neq 1$, then $F(\sigma_{\mathfrak{a}}(x + iy))$ may be unbounded as a function of $x \in \mathbb{R}$. Then it is important to approach the cusps $\mathfrak{a}$ and $\gamma^{-1}\mathfrak{a}$ along a geodesic half-line.

Remarks. (a) If $|v(\pi_{\mathfrak{a}})| \neq 1$ some care is needed in the choice of the path of integration in its approach of $\mathfrak{a}$. Now $F(\sigma_{\mathfrak{a}} z)$ may have exponential growth in $x = \operatorname{Re} z$, although for a given x it has exponential decay as $\operatorname{Im}(z) = y \uparrow \infty$. The integral converges uniformly if we restrict x to a suitable compact set, for instance by requiring that the path approaches $\mathfrak{a}$ along a geodesic half-line.

(b) Proposition 2.4 extends easily to the situation with $\mathfrak{a}$ as the base point, and we see that $\psi_F^{\mathfrak{a}}$ is a cocycle, and that a change in the choice of the cusp $\mathfrak{a}$ adds a coboundary. The following lemma prepares the identification of $\mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$ as a Γ-module in which $\psi^{\mathfrak{a}}_F$ takes its values.

Lemma 2.5. *Let* $\mathfrak{a} = g\infty$ *with* $g \in \mathrm{SL}_2(\mathbb{R})$. *Suppose that* F *is a holomorphic function on* $\mathfrak{H}$ *and that there is* $a > 0$ *such that* $F(gz) = \mathrm{O}(e^{-ay})$ *as* $\mathrm{Im}\,(z) = y \to \infty$ *for each value of* $x = \mathrm{Re}\,z$. *For* $z_0 \in \mathfrak{H}$ *and* $t \in \mathfrak{H}^-$ *we define* h *by*

$$h(t) \;=\; \int_{z_0}^{\mathfrak{a}} \omega_r(F; t, z)\,, \tag{2.4}$$

Then h *extends holomorphically across* $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\mathfrak{a}\}$ *and defines an element of the space* $\mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\mathfrak{a}]$.

Proof. By Part i) of Lemma 2.3 it suffices to consider the case $\mathfrak{a} = 0$ and $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Inspection of (2.2) shows that

$$(\mathsf{prj}_{2-r} h)(t) \;=\; \int_{z_0}^{0} \Bigl(\frac{z-t}{i-t}\Bigr)^{r-2} F(z)\, dz$$

extends holomorphically to $\mathbb{C} \smallsetminus p$, where p is path from z_0 to 0. Since we can take this path as a geodesic half-line, we have a holomorphic extension to a $\{0\}$-excised neighbourhood. Hence $h \in \mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[0]$.

To show that $h \in \mathcal{D}^{\infty}_{2-r}$ we need to show that $h(t)$ has Taylor expansions of any order at 0 valid on a region $\{t \in \mathbb{C} \;:\; \mathrm{Im}\, t \leq 0,\; |t| < \varepsilon\}$ for some $\varepsilon > 0$.

We can assume that the path of integration approaches 0 vertically, and hence

$$h(t) \;=\; -i \int_0^{\varepsilon} (iy - t)^{r-2}\, F(iy)\, dy + \text{ a contribution in } \mathcal{D}^{\omega}_{v,2-r}\,.$$

The contribution in $\mathcal{D}^{\omega}_{v,2-r}$ is automatically in $\mathcal{D}^{\infty}_{v,2-r}$, so we consider only the integral. For $y \in (0, \varepsilon]$, $|t| \leq \varepsilon$ and $\mathrm{Im}\, t \leq 0$ we have

$$(iy - t)^{r-2} \;=\; e^{\pi i r/2}\, y^{r-2}\, (1 + it/y)^{r-2}\,,$$

with $\mathrm{Re}\, it/y \geq 0$. Taylor expansion of the factor $(1 + it/y)^{r-2}$ is not completely standard, since it/y is unbounded for the values of t and y under consideration. We use the version of Taylor's formula in Lang [79, §6, Chap. XIII]. It shows that the error term in the Taylor expansion of order $N-1$ of $(1+q)^a$ is

$$\mathrm{O}_N\Bigl(\int_0^1 (1-x)^{N-1}\, (1+xq)^{a-N}\, q^N\, dx\Bigr) \;=\; \mathrm{O}_N(q^N)\,,$$

if $N > \mathrm{Re}\, a$. (The subscript N indicates that the implicit constant may depend on N.) For sufficiently large N this leads to

$$(1 + it/y)^{r-2} \;=\; \sum_{n=0}^{N-1} \binom{r-2}{n} i^n\, t^n\, y^{-n} + \mathrm{O}_N(t^N y^{-N})$$

and hence

$$\int_0^\varepsilon (iy-t)^{r-2}\,F(iy)\,dy \;=\; e^{\pi ir/2}\sum_{n=0}^{N-1}\binom{r-2}{n} i^n t^n \int_0^\varepsilon y^{r-2-n}\,F(iy)\,dy \\ + \mathrm{O}_N\Bigl(\int_0^\varepsilon y^{r-2-N}\,t^N\,F(iy)\,dy\Bigr).$$

The exponential decay of F implies that all integrals converge, and we obtain a Taylor expansion of the integral of any order that is valid for $t \in \mathfrak{H}^- \cup \mathbb{R}$ near 0. $\square$

Parabolic cohomology and mixed parabolic cohomology. For $z_0 \in \mathfrak{H}$ the cocycle $\psi_F^{z_0}$ for a cusp form F takes values in $\mathcal{D}^\omega_{v,2-r}$. The next result shows that $\psi_F^{\mathfrak{a}}$ is a parabolic cocycle in a larger module, and relates both cocycles.

Proposition 2.6. *Let $r \in \mathbb{C}$.*

i) *For each cusp $\mathfrak{a}$ of Γ and each $F \in S_r(\Gamma, v)$ the cocycle $\psi_F^{\mathfrak{a}}$ defined in* (2.3) *is a parabolic cocycle in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$.*

ii) *Associating to $F \in S_r(\Gamma, v)$ the cohomology class $[\psi_F^{\mathfrak{a}}]$ defines a linear map*

$$\mathbf{r}_r^\infty : S_r(\Gamma, v) \longrightarrow H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})\,. \tag{2.5}$$

iii) $\mathbf{r}_r^\omega\, S_r(\Gamma, v) \subset H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$.

iv) *The following diagram is commutative:*

$$\begin{tikzcd} S_r(\Gamma, v) \arrow[r, "\mathbf{r}_r^\omega"] \arrow[dr, "\mathbf{r}_r^\infty"'] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}) \arrow[r, hook] \arrow[d] & H^1(\Gamma; \mathcal{D}^\omega_{v,2-r}) \\ & H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}) & \end{tikzcd} \tag{2.6}$$

The vertical arrow denotes the natural map associated to the inclusion $\mathcal{D}^\omega_{v,2-r} \subset \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$.

Remark. For $r \in \mathbb{Z}_{\geq 2}$, the linear maps $\mathbf{r}_r^\omega$ and $\mathbf{r}_r^\infty$ take values in the much smaller Γ-module $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$.

Proof. We split the integral in (2.3) as $-\int_{z_1}^{\gamma^{-1}\mathfrak{a}} + \int_{z_1}^{\mathfrak{a}}$ for any $z_1 \in \mathfrak{H}$, and find with Lemma 2.5 that $\psi_F^{\mathfrak{a}} \in \mathcal{D}^{\omega^*,\infty,\mathrm{exc}}_{v,2-r} \cap \mathcal{D}^\omega_{v,2-r}[\mathfrak{a}, \gamma^{-1}\mathfrak{a}] \subset \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$. So $\psi_F^{\mathfrak{a}} \in Z^1(\Gamma; \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$.

Like in the proof of Proposition 2.4, replacing the cusp $\mathfrak{a}$ by another cusp means adding a coboundary in $B^1(\Gamma; \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$. We have $\psi^{\mathfrak{a}}_{F,\pi_{\mathfrak{a}}} = 0$, and hence for a cusp η there is $p \in \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}$ such that

$$\psi^{\mathfrak{a}}_{F,\pi_\eta} \;=\; \psi^{\eta}_{F,\pi_\eta} + p|_{v,2-r}(\pi_\eta - 1) \;\in\; 0 + \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}|_{v,2-r}(\pi_\eta - 1)\,.$$

So $\psi_F^{\mathfrak{a}}$ is a parabolic cocycle, and $F \mapsto [\psi_F^{\mathfrak{a}}]$ defines a linear map $\mathbf{r}_r^\infty$ as in Part ii).

For $F \in S_r(\Gamma, v)$ and $z_0 \in \mathfrak{H}$ we have for each cusp $\mathfrak{a}$ of Γ

$$\psi_{F,\pi_{\mathfrak{a}}}^{z_0} \;=\; \psi_{F,\pi_{\mathfrak{a}}}^{\mathfrak{a}} + h|_{v,2-r}(\pi_{\mathfrak{a}} - 1)\,,$$

with $h = \int_{z_0}^{\mathfrak{a}} \omega_r(F;\cdot,z)$. With Lemma 2.5 we have $\psi_{F,\pi_{\mathfrak{a}}}^{z_0} \in \mathcal{D}_{v,2-r}^{\omega^0,\infty,\mathrm{exc}}|_{v,2-r}(\pi_{\mathfrak{a}} - 1)$. This gives Parts iii) and iv). □

2.3. **The theorem of Knopp and Mawi.** Suppose that ∞ is a cusp of Γ, and that Γ_∞ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (This can be arranged by conjugation in $\mathrm{SL}_2(\mathbb{R})$.) The involution ι in (1.6) gives a parabolic cocycle $\iota\psi_F^\infty$ of the form

$$(\iota\psi_{F,\gamma}^\infty)(w) \;=\; \overline{\int_{\gamma^{-1}\infty}^{\infty} (z-\bar{w})^{r-2}\, F(z)\, dz} \;=\; \int_{\gamma^{-1}\infty}^{\infty} (\bar{z} - w)^{\bar{r}-2}\, \overline{F(z)}\, d\bar{z}\,. \tag{2.7}$$

This describes Knopp's cocycle [66, (3.8)]. In that paper the weight r is real and the multiplier system v unitary, so $\bar{v} = v^{-1}$. (Actually, in [66] the multiplier system for F is called $\bar{v}$, and the weight is called $r + 2$.)

The values of $\iota\psi_F^\infty$ are in the space $\iota\mathcal{D}_{2-r}^{\omega^0,\infty}$ which is contained in the space

$$\mathcal{P} \;:=\; \iota\mathcal{D}_{2-r}^{-\infty} \;=\; \Big\{\varphi \in O(\mathfrak{H}) \;:\; \exists_{A\in\mathbb{R}}\; \varphi(z) = \mathrm{O}(y^{-A}) + \mathrm{O}(|z|^A)\Big\} \tag{2.8}$$

(polynomial growth), which is invariant under the action $|_{\bar{v},\bar{r}}$ of Γ. (The notation $\mathcal{P}$ is taken from [66].)

Knopp [66] conjectured that the map $F \mapsto [\iota\psi_F^\infty]$ gives a bijection $S_r(\Gamma, v) \to H^1_{\mathrm{pb}}(\Gamma, \mathcal{P})$, and proved this for $r \in \mathbb{R} \smallsetminus (0, 2)$. He also gives a proof, by B.A. Taylor, that $H^1_{\mathrm{pb}}(\Gamma;\mathcal{P}) = H^1(\Gamma;\mathcal{P})$. In [71] Knopp and Mawi prove the isomorphism for all weights $r \in \mathbb{R}$ and unitary multiplier systems v. Transforming their result to the lower half-plane we obtain the following theorem:

Theorem 2.7. (Knopp, Mawi) *Let v be a unitary multiplier system on Γ for the weight $r \in \mathbb{R}$. Then*

$$S_r(\Gamma, v) \;\cong\; H^1(\Gamma;\mathcal{D}_{v,2-r}^{-\infty}) \;\cong\; H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}_{v,2-r}^{-\infty})\,. \tag{2.9}$$

In combination with the, not yet proven, Theorems A and B we obtain the following commuting diagram, valid for weights $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$ and unitary multiplier systems:

$$\begin{tikzcd} A_r(\Gamma, v) \arrow[r, hook, "\mathbf{r}_r^\omega"] & H^1(\Gamma;\mathcal{D}_{v,2-r}^\omega) \arrow[d] \\ S_r(\Gamma, v) \arrow[u, hook] \arrow[r, "\cong"] & H^1(\Gamma;\mathcal{D}_{v,2-r}^{-\infty}) \end{tikzcd} \tag{2.10}$$

This implies that there is a complementary subspace X giving a direct sum decomposition $A_r(\Gamma, v) = S_r(\Gamma, v) \oplus X$, such that for $F \in X$ the cocycle $\psi_F^{z_0}$ becomes a coboundary in $Z^1(\Gamma;\mathcal{D}^{-\infty}_{v,2-r})$. Then there is $H\in\mathcal{D}^{-\infty}_{v,2-r}$ such that $H|_{v,2-r}\gamma-H=\psi^{z_0}_{F,\gamma}$ for all $\gamma\in\Gamma$, in other words

$$\int_{\gamma^{-1}z_0}^{z_0}(z-t)^{r-2}\,F(z)\,dz \;=\; (H|_{v,2-r}\gamma)(t)-H(t)\,.$$

Remark 2.8. The operator ι can also be applied to the linear map $\mathbf{r}^\omega_r$. Thus we have two $\mathbb{R}$-linear maps from automorphic forms to cohomology:

$$\begin{aligned}\mathbf{r}^\omega_r &: A_r(\Gamma,v)\to H^1(\Gamma;\mathcal{D}^\omega_{v,2-r})\,,\\ \iota\mathbf{r}^\omega_r &: A_r(\Gamma,v)\to H^1(\Gamma;\iota\mathcal{D}^\omega_{v,2-r})\,.\end{aligned}\tag{2.11}$$

The second map is antilinear.

These two maps become interesting in the case $r\in\mathbb{Z}_{\geq 2}$ with a real-valued multiplier system v. Then $\mathcal{D}^\omega_{v,2-r}$ and $\iota\mathcal{D}^\omega_{v,2-r}$ have a nonzero intersection, namely $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$.

2.4. **Modular group and powers of the Dedekind eta-function.** The modular group $\Gamma(1)=\mathrm{SL}_2(\mathbb{Z})$ is generated by $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. In the quotient $\overline{\Gamma(1)}=\mathrm{SL}_2(\mathbb{Z})$ the relations are $\bar S^2=1$ and $(\bar S\bar T)^3=1$. There is a one-parameter family of multiplier systems parametrized by $r\in\mathbb{C}$ mod $12\mathbb{Z}$, determined by

$$v_r(T)\;=\;e^{\pi ir/6}\,,\qquad v_r(S)\;=\;e^{-\pi ir/2}\,.\tag{2.12}$$

It can be used for weights $p\equiv r$ mod 2. The complex power η^{2r} of the Dedekind eta-function can be chosen in the following way:

$$\eta^{2r}(z)\;:=\;e^{2r\,\log\eta(z)}\,,\quad \log\eta(z)\;=\;\frac{\pi i}{12}-\sum_{n\geq1}\sigma_{-1}(n)\,e^{2\pi inz}\,.\tag{2.13}$$

It defines $\eta^{2r}\in A_r(\Gamma(1),v_r)$. The Fourier expansion at the cusp ∞ has the form

$$\eta^{2r}(z)\;=\;\sum_{k\geq0}p_k(r)\,e^{2\pi i(12k+r)z/12}\,,\tag{2.14}$$

where the $p_k(r)$ are polynomials in r of degree k with rational coefficients. These polynomials have integral values at each $r\in\frac12\mathbb{Z}$. For $\mathrm{Re}\,r>0$ we have $\eta^{2r}\in S_r(\Gamma(1),v_r)$, and the parabolic cocycle $\psi^\infty_{\eta^{2r}}$ given by

$$\psi^\infty_{\eta^{2r},\gamma}(t)\;=\;\int_{\gamma^{-1}\infty}^{\infty}(z-t)^{r-2}\,\eta^{2r}(z)\,dz\,.\tag{2.15}$$

Convergence is ensured by the exponential decay of $\eta^{2r}(z)$ as $y\uparrow\infty$, and by the corresponding decay at other cusps by the invariance of η^{2r} under $|_{v_r,r}\gamma$.

Since $T\infty=\infty$ we have $\psi^\infty_{\eta^{2r},T}=0$. The cocycle $\psi^\infty_{\eta^{2r}}$ is determined by its value on the other generator

$$\psi^\infty_{\eta^{2r},S}(t)\;=\;\int_0^\infty(z-t)^{r-2}\,\eta^{2r}(z)\,dz\,.\tag{2.16}$$

We have $\psi^{\infty}_{\eta^{2r},S} \in \mathcal{D}^{\omega,\mathrm{exc}}_{v_r,2-r}[0,\infty] \subset \mathcal{D}^{\omega^0,\mathrm{exc}}_{v_r,2-r}$. This function is called the *period function* of η^{2r}. The relations between $\bar{S}$ and $\bar{T}$ imply

$$\psi^{\infty}_{\eta^{2r},S}|_{v_r,2-r}S \;=\; -\psi^{\infty}_{\eta^{2r},S}\,, \qquad \psi^{\infty}_{\eta^{2r},S}|_{v_r,2-r}(1+ST+STST) \;=\; 0\,, \tag{2.17}$$

which is equivalent to

$$\psi^{\infty}_{\eta^{2r},S}|_{v_r,2-r}S \;=\; -\psi^{\infty}_{\eta^{2r},S}\,, \quad \psi^{\infty}_{\eta^{2r},S} \;=\; \psi^{\infty}_{\eta^{2r},S}|_{v_r,2-r}(T+TST)\,. \tag{2.18}$$

Let us put

$$I(r,s) \;:=\; \int_0^\infty y^s\,\eta^{2r}(iy)\,\frac{dy}{y}. \tag{2.19}$$

The decay properties of η^{2r} imply that this function is holomorphic in (r,s) for $\mathrm{Re}\,r>0$ and $s\in\mathbb{C}$.

The reasoning in the proof of Lemma 2.5 gives that for a given $\varepsilon>0$ and $t\in\mathfrak{H}^-$ with $|t|<\varepsilon$ we have

$$i\int_0^\varepsilon (iy-t)^{r-2}\eta^{2r}(iy)\,dy \;=\; e^{\pi i r/2}\sum_{n=0}^{N-1}\binom{r-2}{n} i^n t^n \int_0^\varepsilon y^{r-2-n}\eta^{2r}(iy)\,dy + \mathrm{O}_n(t^N)$$

for all sufficiently large N. The integral over (ε,∞) can be computed by direct insertion of the Taylor series for $(iy-t)^{r-2}$. Since $t^N = \mathrm{O}(\varepsilon^N)$, this leads to the following asymptotic equality for the period function:

$$\psi^{\infty}_{\eta^{2r},S}(t) \;\sim\; e^{\pi i(r-1)/2}\sum_{n\geq 0} i^n\binom{r-2}{n} I(r,r-1-n)\,t^n\,, \tag{2.20}$$

for $\mathrm{Re}\,r>0$, $s\in\mathbb{C}$ and $t\to 0$ through $\mathfrak{H}^-$.

For a real weight $r>0$ one has the estimate $p_k(r) = \mathrm{O}(k^{r/2})$ from the fact that η^{2r} is a cusp form. For $\mathrm{Re}\,s > 1+r/12$ the integral $I(r,s)$ can be expressed in terms of the L-series

$$\begin{aligned} L(\eta^{2r},s) \;&=\; \sum_{k\geq 0}\frac{p_k(r)}{(r/12+k)^s}\,,\\ I(r,s) \;&=\; (2\pi)^{-s}\,\Gamma(s)\,L(\eta^{2r},s)\,. \end{aligned} \tag{2.21}$$

Usually one defines the analytic continuation of L-functions by the expressing it in the period integral (2.20).

If $\mathrm{Re}\,r\leq 0$, $\psi^{z_0}_{\eta^{2r}}$ is defined only with a base point $z_0\in\mathfrak{H}$. For instance, the case $r=0$ gives the constant function $1=\eta^0\in A_0(\Gamma(1),1)$, with the trivial multiplier system $v_0=1$, for which

$$\psi^{z_0}_{\eta^0,\gamma}(t) \;=\; \frac{1}{\gamma^{-1}z_0-t} - \frac{1}{z_0-t}\,. \tag{2.22}$$

It can be checked by a direct computation that $\psi^{z_0}_{1,\gamma} - \psi^{z_1}_{1,\gamma} \;=\; b|_{1,2}(\gamma-1)$, with $b(t) = \frac{1}{z_0-t} - \frac{1}{z_1-t}$.

We use this to find a substitute for the cocycle $\psi^{\infty}_{\eta^0}$. The rational function $b_\infty(t) = \frac{1}{z_0-t}$ is an element of $\mathcal{D}^{\omega,\mathrm{exc}}_{1,2}[\infty]$. Subtracting the coboundary $\gamma \mapsto b_\infty|_2(\gamma-1)$ from $\psi^{z_0}_{\eta^0}$ gives the parabolic cocycle $\tilde\psi \in Z^1(\Gamma(1); \mathcal{D}^{\omega^0,\mathrm{exc}}_{1,2})$ given on $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ by

$$\tilde\psi_\gamma(t) \;=\; \frac{-c}{ct+d}\,. \tag{2.23}$$

This cocycle $\tilde\psi$ is parabolic, since $\tilde\psi_T = 0$. It gives the period function $\tilde\psi_S = \frac{-1}{t}$ in $\mathcal{D}^{\omega^0,\mathrm{exc}}_{1,2}$. It is in the subspace of rational functions, hence one calls it a *rational period function*. In §5.2 we will return to this example.

2.5. **Related work.** Much of the work on the relation between automorphic forms and cohomology is done for integral weights at least 2. The association of cocycles to automorphic forms is stated clearly in 1957 by Eichler, [43, §2]. Eichler gives the integral in (2), and notes [43, (17),§2] that for cusp forms the cocycles have the property that we now call parabolic.

The idea can be found earlier in the literature. As pointed out in [41], Poincaré mentions already in 1905 [100, §3] the repeated antiderivative of automorphic forms and polynomials measuring the non-invariance. Also Cohn [32] mentions this relation in the main theorem, for modular forms of weight 4.

Shimura [112] studies the relation between cusp forms and cohomology groups with the aim of obtaining a lattice in the space of cusp form such that the quotient is an abelian variety. He discusses real and integral structures in the cohomology groups.

Since then the relation between automorphic forms and cohomology has been studied in numerous papers, of which we here mention Manin [86].

The use of the space of rational functions for cocycles associated to modular forms originates in Knopp [67]. Kohnen and Zagier [73] used it for period functions on the modular group. In [73] the concept of mixed parabolic cohomology seems to be arising. See also [119].

3. The image of automorphic forms in cohomology

The main goal of this section is to show that

$$\mathbf{r}^\omega_r A_r(\Gamma; v) \;\subset\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})\,. \tag{3.1}$$

This will contribute to the proof of Theorem A (which will be completed in Subsection 10.5). In Subsection 3.6 we will describe, under assumptions on r and v, and based on the truth of Theorem A, the images $\mathbf{r}^\omega_r S_r(\Gamma, v)$ and $\mathbf{r}^\omega_r M_r(\Gamma, v)$. This gives Theorem B.

We start in Subsection 3.1 with a simple lemma, with which we immediately can prove some of the isomorphism in Theorem E on page 18.

3.1. **Mixed parabolic cohomology groups.** To show that $\psi \in Z^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$ is a parabolic cocycle in $Z^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, W)$ for some Γ-module $W \subset \mathcal{D}^{\omega^*}_{v,2-r}$ we have to find for each cusp $\mathfrak{a}$ of Γ an element $h_{\mathfrak{a}} \in W$ such that

$$\psi_{\pi_{\mathfrak{a}}} \;=\; h_{\mathfrak{a}}|_{v,2-r}(\pi_{\mathfrak{a}} - 1)\,. \tag{3.2}$$

The following result gives the position of the singularities of the solutions.

Lemma 3.1. *If $h \in \mathcal{D}^{\omega^*}_{2-r}$ satisfies $\lambda^{-1}\, h|_{2-r}\pi - h \in \mathcal{D}^{\omega}_{2-r}$ for a parabolic element $\pi \in \mathrm{SL}_2(\mathbb{R})$ and $\lambda \in \mathbb{C}^*$, then $\mathsf{BdSing}\, h \subset \{\mathfrak{a}\}$, where $\mathfrak{a}$ is the unique fixed point of π.*

Proof. Each parabolic element $\pi \in \mathrm{SL}_2(\mathbb{R})$ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ or to T^{-1}. With (1.5) we can transform the hypothesis in both cases to $\lambda^{-1}\, h|_{2-r}T - h \in \mathcal{D}^{\omega}_{2-r}$. If h has singularities in $\mathbb{R}$ put them in increasing order: $\xi_1 < \xi_2 < \cdots$. Then $\xi_1 - 1 = T^{-1}\xi_1 \in \mathsf{BdSing}\,(h|_{2-r}T)$, and cannot be canceled by a singularity of h. So a singularity can occur only at ∞, and only at $\mathfrak{a}$ in the original situation. $\square$

Proposition 3.2. *Let $r \in \mathbb{C}$. Then*

$$\begin{aligned}
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0}_{v,2-r}) &= H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r})\,,\\
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) &= H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})\,,\\
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r}) &= H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})\,,\\
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty}_{v,2-r}) &= H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\infty}_{v,2-r})\,.
\end{aligned}$$

Proof. If $\psi \in Z^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r})$ then we have for each cusp $\mathfrak{a}$ an element $h \in \mathcal{D}^{\omega^*}_{v,2-r}$ such that $h|_{v,r}(\pi_{\mathfrak{a}} - 1) = \psi_{\pi_{\mathfrak{a}}}$. This is the situation considered in Lemma 3.1, so $h \in \mathcal{D}^{\omega}_{v,2-r}[\mathfrak{a}]$. Hence $\psi \in Z^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0}_{v,2-r})$. The same argument is valid for the other cases. $\square$

3.2. **The parabolic equation for an Eichler integral.**

Definition 3.3. We call a function F on a subset of $\mathbb{C}$ that is invariant under horizontal translations, *λ-periodic* if it satisfies $F(t+1) \;=\; \lambda\, F(t)$ for all t in its domain.

Example. For an automorphic form $F \in A_r(\Gamma, v)$ and a cusp $\mathfrak{a}$, the relation $F|_{v,r}\pi_{\mathfrak{a}} = v(\pi_{\mathfrak{a}})\, F$ implies that the translated function $F|_r\sigma_{\mathfrak{a}}$ is $v(\pi_{\mathfrak{a}})$-periodic.

Parabolic difference equation. We take an arbitrary holomorphic λ-periodic function E on $\mathfrak{H}$. It has an absolutely convergent Fourier expansion

$$E(z) \;=\; \sum_{n \equiv \alpha \bmod 1} a_n\, e^{2\pi i n z} \tag{3.3}$$

on $\mathfrak{H}$ with $\lambda = e^{2\pi i \alpha}$, $\alpha \in \mathbb{C}$. In the next subsections we aim to find functions h such that

$$\lambda^{-1}\, h(t+1) - h(t) \;=\; \int_{z_0-1}^{z_0} (z-t)^{r-2}\, E(z)\, dz \tag{3.4}$$

at least for $t \in \mathfrak{H}^- \cup \mathbb{R}$, and to get information concerning its behavior near ∞.

3.3. **Asymptotic behavior at infinity.** It will be useful to understand the behavior of $\mathsf{prj}_{2-r}h$ at ∞ for solutions h of (3.4). For functions f on $\mathbb{R}$ we understand in these notes $f(t) \sim \sum_{n\geq k} c_n\, t^{-n}$ to mean $f(t) = \sum_{n=k}^{N-1} c_n\, t^{-n} + \mathrm{O}(t^{-N})$ as $t \to \pm\infty$ for all $N \geq k$. So $f(t) \sim 0$ means $f(t) = \mathrm{O}(t^{-N})$ for all $N \in \mathbb{Z}_{\geq 0}$.

For elements $f \in \mathcal{D}^\infty_{2-r}$ we know that there are coefficients b_n such that

$$(\mathsf{prj}_{2-r}f)(t) \sim \sum_{n\geq 0} b_n\, t^{-n}$$

as t approaches ∞ through $\mathfrak{H}^- \cup \mathbb{R}$. So we have surely this behavior as t approaches ∞ through $\mathbb{R}$.

Lemma 3.4. *Let $r \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$, and suppose that $f \in \mathcal{D}^\omega_{2-r}[\infty]$ is λ-periodic. We consider asymptotic expansions of $(\mathsf{prj}_{2-r}f)(t) = (i-t)^{2-r} f(t)$ of the type*

$$(\mathsf{prj}_{2-r}f)(t) \;\sim\; \sum_{n\geq k} b_n\, t^{-n} \qquad \text{for some } k \in \mathbb{Z}\,. \tag{3.5}$$

i) *If f satisfies (3.5) for $t \uparrow \infty$ as well as for $t \downarrow -\infty$ (with the same coefficients b_n), then*
   a) *f is a constant function if $\lambda = 1$ and $r \in \mathbb{Z}$. In this case $r \geq k+2$.*
   b) *$f = 0$ in all other cases.*

ii) *Let $\varepsilon \in \{1,-1\}$. Suppose that f satisfies (3.5) as $\varepsilon t \uparrow \infty$. Then*
   a) *if $\lambda = 1$ and $r \in \mathbb{Z}_{\geq k+2}$, then f is a constant function;*
   b) *else if $|\lambda| = 1$, then $f = 0$;*
   c) *else $f(t) \sim 0$ as $\varepsilon t \uparrow \infty$.*

Proof. Consider $f \in \mathcal{D}^\omega_{2-r}[\infty]$ that is λ-periodic and has an expansion (3.5) as $t \uparrow \infty$, or $t \downarrow -\infty$, or both. If the expansion is non-zero it has the form $b_n\, t^{-n} + b_{n+1}\, t^{-n-1} + \cdots$ where $b_n \neq 0$. Insertion in

$$(\mathsf{prj}_{2-r}f)(t+1) \;=\; \lambda\Bigl(\frac{1-(i-1)/t}{1-i/t}\Bigr)^{2-r} (\mathsf{prj}_{2-r}f)(t)$$

gives

$$\lambda\, b_n \;=\; b_n\,, \quad \lambda\,(b_{n+1} - (r-2)\,b_n) \;=\; b_{n+1} - n\, b_n\,.$$

This is impossible with $b_n \neq 0$ if $\lambda \neq 1$. If $\lambda = 1$ it is possible if $n = r-2 \geq k$, and has solutions corresponding to a constant function $f(t) = c$, and

$$(\mathsf{prj}_{2-r}f)(t) \;=\; (i-t)^{2-r}\, c\,.$$

This shows that non-zero expansions occur only in the Case a) in Part i).

In this case we set $f_0(t) = f(t) - c$. We set $f_0 = f$ otherwise. Then $f_0 \in \mathcal{D}^\omega_{2-r}[\infty]$ is λ-periodic with expansion $(\mathsf{prj}_{2-r}f_0)(t) \sim 0$. In other words, $(\mathsf{prj}_{2-r}f_0)(t) = \mathrm{O}(t^{-N})$ for any order $N \in \mathbb{Z}_{\geq 0}$, and then the same holds for $f_0(t)$. To show that $f = 0$ in Case b) of i), we notice that, as a λ-periodic function, f_0 has a Fourier expansion and the estimate $f(t) = \mathrm{O}(t^{-N})$ holds for each Fourier term, which is of the form $c_n\, e^{2\pi i n t}$ with $e^{2\pi i n} = \lambda$. For expansion in both directions, $|t| \to \infty$, this implies that all Fourier terms vanish, and hence $f = 0$. This finishes the proof of Part i).

For a one-sided expansion, say $t \to \infty$, there might be Fourier terms that satisfy $e^{2\pi i n t} \sim 0$ as $t \uparrow \infty$, namely if $\operatorname{Im} n < 0$. This possibility and the same possibility as $t \downarrow -\infty$ are excluded by the assumption $|\lambda| = 1$ in Part ii)b). Without this assumption, $f(t) \sim 0$. □

3.4. **Construction of solutions.** We break up the Fourier expansion (3.3) in three parts, according to $\operatorname{Re} n > 0$, $\operatorname{Re} n < 0$ and $\operatorname{Re} n = 0$.

Cuspidal case.

Lemma 3.5. *Suppose that the Fourier expansion* (3.3) *has the form*

$$E(z) \;=\; \sum_{n\equiv\alpha \bmod 1,\, \operatorname{Re} n>0} a_n\, e^{2\pi i n z}\,,$$

with $\alpha \in \mathbb{C}$*,* $\lambda = e^{2\pi i\alpha}$*.*

i) *If* $r \in \mathbb{C} \setminus \mathbb{Z}_{\geq 2}$*, then there is a unique* $h \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$ *satisfying* (3.4).

ii) *If* $r \in \mathbb{Z}_{\geq 2}$ *then* (3.4) *has solutions in* $\mathcal{D}^{\mathrm{pol}}_{2-r}$*.*

a) *If* $\lambda = e^{2\pi i\alpha} \neq 1$*, then there is a unique solution.*

b) *If* $\lambda = 1$*, then the solutions of* (3.4) *in* $\mathcal{D}^{\mathrm{pol}}_{2-r}$ *are unique up to addition of a constant.*

Proof. Lemma 2.5 states that

$$h_0(t) \;:=\; \int_{z_0}^{\infty} (z-t)^{r-2}\, E(z)\, dz \tag{3.6}$$

defines $h_0 \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$. If we take a vertical path of integration, then h_0 is a holomorphic function on $\mathbb{C} \setminus (z_0 + i[0,\infty))$.

Let us consider $t \in \mathbb{C}$ with $\operatorname{Im} t < \operatorname{Im} z_0$. The integral over the closed path sketched in Figure 2 on the right equals zero for all $a > 0$, and due to the exponential decay of E the limit as $a \to \infty$ of the integrals over the sides depending on $a > 0$ exist. Hence we get

$$\int_{z_0-1}^{\infty} (z-t)^{r-2}\, E(z)\, dz = \int_{z_0-1}^{z_0} (z-t)^{r-2}\, E(z)\, dz + h_0(t)\,.$$

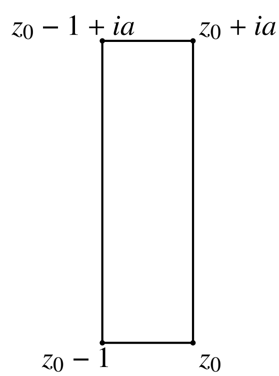

FIGURE 2

Like in Part iii) of Lemma 2.3 this gives

$$\begin{aligned}\lambda^{-1}\, h_0(t+1) \;&=\; \lambda^{-1}\int_{z_0}^{\infty} (z-t-1)^{r-2}\, E(z)\, dz \;=\; \int_{z_0-1}^{\infty} (z-t)^{r-2}\, E(z)\, dz\\ &=\; h_0(t) + \int_{z_0-1}^{z_0} (z-t)^{r-2}\, E(z)\, dz\,.\end{aligned}$$

This relation extends holomorphically to all $t \in \mathbb{C}$ outside the region determined by $\operatorname{Re} z_0 - 1 \le \operatorname{Re} t \le \operatorname{Re} z_0$ and $\operatorname{Im} t \ge \operatorname{Im} z_0$. So h_0 is a solution of (3.4) in $\mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$.

Let h be another solution in $\mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$. Then $p = h - h_0$ is a λ-periodic function in $\mathcal{D}^{\infty}_{2-r}$, and hence $\mathsf{prj}_{2-r} p$ has an expansion as in Lemma 3.4, with $k \ge 0$. If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\ge 2}$ then Part i) a) and Part ii) of Lemma 3.4 implies that $p = 0$ so that we have proved Part i) of this lemma.

To prove Part ii) let $r \in \mathbb{Z}_{\ge 2}$. It is clear from the integral that $h_0 \in \mathcal{D}^{\mathrm{pol}}_{2-r}$ if $r \in \mathbb{Z}_{\ge 2}$. If $\lambda = 1$ it reduces to the case b) in Part i) of Lemma 3.4. So p is a non-zero constant. This handles Part ii)b) of the present lemma. If $\lambda \ne 1$, it reduces to the case a) in Part ii) of Lemma 3.4 so that $p = 0$. This gives Part a) in Part ii) of the present lemma. □

Exponentially increasing part.

Lemma 3.6. *Suppose that the Fourier expansion* (3.3) *has the form*

$$E(z) \;=\; \sum_{n \equiv \alpha \bmod 1,\ \operatorname{Re} n < 0} a_n\, e^{2\pi i n z}\,, \tag{3.7}$$

with $\alpha \in \mathbb{C}$, $\lambda = e^{2\pi i\alpha}$.

i) *Equation* (3.4) *has solutions $h \in \mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$, among which occurs a solution for which $\mathsf{prj}_{2-r} h(t)$ has an asymptotic expansion as $t \uparrow \infty$ of the form $\sum_{n\ge 0} c_n t^{-n}$.*

ii) *Let $|\lambda| = 1$ and $E \ne 0$. For none of these solutions h do we have an asymptotic expansion of the form $\mathsf{prj}_{2-r} h(t) \sim \sum_{n\ge k} q_k\, t^{-n}$ valid for $t \uparrow \infty$ and for $t \downarrow -\infty$ with the same coefficients.*

Proof. We cannot use the integral in (3.6), since E has exponential growth on $\mathfrak{H}$.

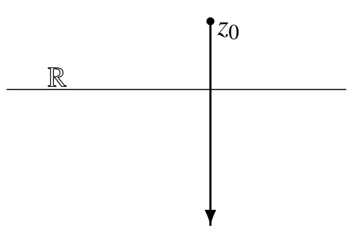

FIGURE 3

The convergence of $E(z)$ in $\mathfrak{H}$ implies good growth for its Fourier coefficients. This growth then implies that $E(z)$ can be defined on $\mathbb{C}$ with exponential decay as $\operatorname{Im} z \downarrow -\infty$. So we use a path of integration as in Figure 3.

In this way we obtain a holomorphic function h_{ri} on the region $\operatorname{Re} t > \operatorname{Re} z_0$ given by

$$h_{\mathrm{ri}}(t) = \int_{z_0 - i[0,\infty)} (z-t)^{r-2} E(z)\, dz\,, \tag{3.8}$$

and satisfying (3.4) for these values of t.

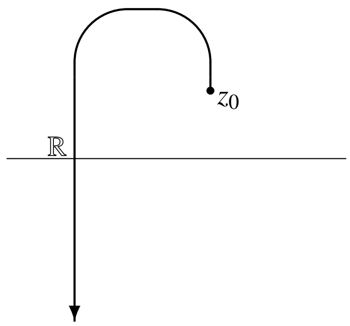

Figure 4

Deforming the integral as in Figure 4 we get the holomorphic continuation of h_{ri} to a larger region. In this way we get the continuation to the region $\mathbb{C} \smallsetminus (z_0 + i[0,\infty))$. By analytic continuation, the extension satisfies (3.4) on the region $\mathbb{C}\smallsetminus(z_0+[-1,0]+i[0,\infty))$. We normalize the factor $(z-t)^{r-2}$ by requiring that $\frac{\pi}{2} \leq \arg(z-t) \leq \frac{3\pi}{2}$ if $\operatorname{Re} t > \operatorname{Re} z_0$ and z is on the path of integration.

We have

$$(\mathsf{prj}_{2-r} h_{\mathrm{ri}})(t) = \int_{z_0} \Bigl(\frac{z-t}{i-t}\Bigr)^{r-2} E(z)\, dz\,, \tag{3.9}$$

over a path of integration starting at z_0 going down to ∞, adapted to t, and normalized by $\arg(\frac{z-t}{i-t}) \to 0$ as $t \uparrow \infty$. We consider the asymptotic behavior of $(\mathsf{prj}_{2-r} h_{\mathrm{ri}})(t)$ as $t \uparrow \infty$ through $\mathbb{R}$. The exponential decay of E as $\operatorname{Im} z \downarrow -\infty$ implies that for a fixed large t in $\mathbb{R}$ the contribution of the integral over $\operatorname{Im} z < -\frac{1}{2}t$ can be estimated by $\mathrm{O}(e^{-\varepsilon t})$ as $t \to \infty$, with $\varepsilon > 0$ depending on E, α in (3.7), and z_0. We insert the Taylor expansion of order N of $(\frac{z-t}{i-t})^{r-2}$ in $\frac{1}{t}$ into the remaining part of the integral, and find an expansion starting at $k = 0$, but only as $t \uparrow \infty$. In this way we obtain the second statement in Part i).

Actually, if we apply the same reasoning to the integral in (3.9) for $t \downarrow -\infty$ we get the same expansion, with the same coefficients. However, that is not an expansion of h_{ri}, but of another solution h_{le}, which we can define in the following way.

An equally sensible choice is the path of integration sketched in Figure 5. Now the path has to be chosen so that t is to the left of it, and below it if $\mathrm{Re}\, t > \mathrm{Re}\, z_0$. This defines another solution $h_{\mathrm{le}} \in \mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$ of (3.4). The normalization of the corresponding integrand for $\mathsf{prj}_{2-r} h_{\mathrm{le}}$ is also by $\arg\big(\frac{z-t}{i-t}\big) \to 0$ as $t \uparrow \infty$.

As indicated above, $h_{\mathrm{le}}(t)$ has an asymptotic expansion as $t \downarrow -\infty$, with the same coefficients as in the expansion of $h_{\mathrm{ri}}(t)$ as $t \uparrow \infty$.

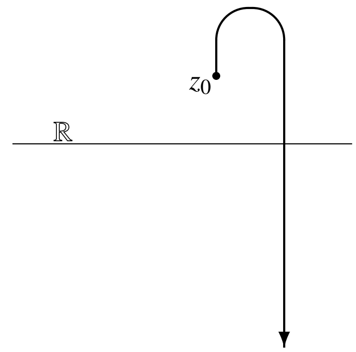

FIGURE 5

Both h_{ri} and h_{le} are solutions of (3.4) for E as in (3.7). We have for $t \in \mathbb{R}$

$$\begin{aligned}&(\mathsf{prj}_{2-r}(h_{\mathrm{ri}}(t) - h_{\mathrm{le}}))(t)\\ &\qquad = \int_D \Big(\frac{z-t}{i-t}\Big)^{r-2} E(z)\, dz\,,\end{aligned}$$

over a path of integration indicated in Figure 6.

This integral is holomorphic in $r \in \mathbb{C}$. For $\mathrm{Re}\, r > 1$ it can be computed by deforming the path of integration to the vertical half-line downward from t. This leads to the following result:

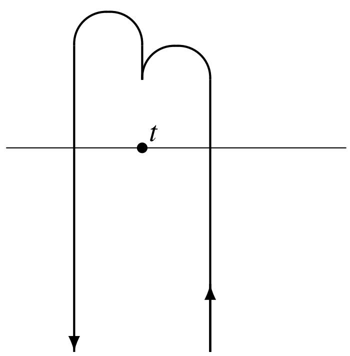

FIGURE 6

(3.10)
$$(\mathsf{prj}_{2-r}(h_{\mathrm{ri}} - h_{\mathrm{le}}))(t) \;=\; \frac{(2\pi)^{2-r}\, e^{\pi i r/2}}{\Gamma(2-r)}\, (i-t)^{r-2} \sum_{n \equiv \alpha \bmod 1,\, \mathrm{Re}\,(n)<0} \frac{a_n}{(-n)^{r-1}}\, e^{2\pi i n t}\,.$$

This difference gives a λ-periodic function $H = h_{\mathrm{ri}} - h_{\mathrm{le}}$. Moreover, the difference is holomorphic in $r \in \mathbb{C}$. So the equality is valid for all $r \in \mathbb{C}$.

Now let $|\lambda| = 1$. Suppose that $h \in \mathcal{D}^{\omega,\mathrm{exc}}_{2-r}$ is a solution of (3.4) with a two-sided asymptotic expansion. We have $h = h_{\mathrm{ri}} + p_r = h_{\mathrm{le}} + p_\ell$ with λ-periodic $p_r, p_\ell \in \mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$.

Suppose that the difference $p_r = h - h_{\mathrm{ri}}$ has an asymptotic expansion as $t \uparrow \infty$. Part ii) of Lemma 3.4 shows that p_r is constant (and zero in most cases). Similarly $p_\ell = h - h_{\mathrm{le}}$ is constant. So $h_{\mathrm{ri}} - h_{\mathrm{le}}$ is constant. However, in (3.10) we see that this implies $h_{\mathrm{ri}} - h_{\mathrm{le}} = 0$ by the assumption $\mathrm{Re}\, n < 0$ in (3.7). Then all a_n vanish and $E = 0$. □

Remaining Fourier term. We are left with the multiples of $e^{2\pi i n z}$ with $\mathrm{Re}\, n = 0$. So $n = i\, \mathrm{Im}\, \alpha \equiv \alpha \bmod 1$.

Lemma 3.7. *Suppose that* $E(z) = e^{-2\pi z n}$ *with* $\operatorname{Re} n = 0$, $n \neq 0$. *Then Equation* (3.4) *has solutions in* $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$.

Proof. We can still find a direction in which $E(z)$ decays exponentially.

In the case $\operatorname{Im} n < 0$ we choose a path as indicated in Figure 7. We can proceed as in the proof of Lemma 3.6.

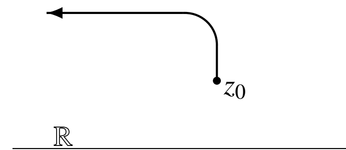

FIGURE 7

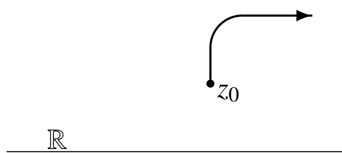

FIGURE 8

For $\operatorname{Im} n > 0$ we use the path in Figure 8. Again,we can proceed as in the proof of Lemma 3.6. □

Lemma 3.8. *Let* $r \in \mathbb{C}$, $\lambda = 1$ *and* $E(z) = 1$.

i) *Equation* (3.4) *has the following solution in* $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$*:*

$$h(t) = \begin{cases} (1-r)^{-1}\,(z_0 - t)^{r-1} & \text{if } r \neq 1\,,\\ -\log(z_0 - t) & \text{if } r = 1\,,\end{cases} \tag{3.11}$$

where we choose in both cases $-\frac{\pi}{2} < \arg(z_0 - t) < \frac{3\pi}{2}$.

ii) *For* $r \neq 1$ *this is the unique solution in* $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$ *for which* $\mathsf{prj}_{2-r} h$ *has an asymptotic expansion valid for* $t \uparrow \infty$ *and for* $t \downarrow -\infty$.

iii) *If* $r = 1$ *there are no solutions in* $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$ *that have a two-sided asymptotic expansion at* ∞ *in the projective model.*

Proof. Part i) can be checked by a computation of the integral in (3.4). For $r \neq 1$ it is seen that $(\mathsf{prj}_{2-r} h)(t)$ has a two-sided asymptotic expansion of the form $(\mathsf{prj}_{2-r} h)(t) \sim \sum_{n \geq -1} c_n t^{-n}$. For $r = 1$ this solution clearly has no such expansion.

Any other solution is of the form $h + p$ with a 1-periodic function p. If it has a two-sided asymptotic expansion it is zero, by Part i) of Lemma 3.4. This gives Part ii) of the present lemma. For Part iii) one can check that no 1-periodic function can produce logarithmic behavior at ∞. □

Example. Only for a constant function E we have given an explicit formula for a solution h. It is possible to express solutions for the other cases in terms of sums of incomplete gamma-functions.

For the powers of the Dedekind eta-function we get for $\operatorname{Re} r > 0$ a solution of the form

$$\begin{aligned} h(t) &= -ie^{\pi i r/2}\,(2\pi)^{1-r} \sum_{k\geq 0} p_k(r)\,(r/12+k)^{1-r} \\ &\quad \cdot e^{2\pi i(12k+r)t/12}\,\Gamma\Big(r-1, 2\pi i(r/12+k)(t-z_0)\Big)\,, \end{aligned} \tag{3.12}$$

with the incomplete gamma-function

$$\Gamma(a,u) \;=\; \int_u^\infty v^{a-1}\, e^{-v}\, dv \;=\; e^{-u} \int_{x=0}^\infty (u+x)^{a-1}\, e^{-x}\, dx\,. \tag{3.13}$$

The incomplete gamma-function is well defined on $\mathbb{C} \smallsetminus (-\infty, 0]$. That suffices for (3.12) if $\operatorname{Re} r > 0$ and $t \in \mathfrak{H}^-$.

If $\operatorname{Re} r \leq 0$, the same formulas can be used for the terms in the Fourier expansion with $\operatorname{Re} \frac{r}{12} + k > 0$. For the remaining terms with $k + \frac{r}{12} \neq 0$ the choices in this subsection lead also to the same expression with incomplete gamma-functions, but now interpreted with a choice of a suitable branch of the multivalued extension. For $k + \frac{r}{12} = 0$ we can use the formula for $r \neq 1$ in Lemma 3.8.

3.5. **Image of automorphic forms in the analytic cohomology.** Now we can take a step towards the proof of Theorem A:

Theorem 3.9. *For all $r \in \mathbb{C}$ and all multiplier systems for the weight r:*

$$\mathbf{r}_r^\omega\, A_r(\Gamma, v) \;\subset\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})\,. \tag{3.14}$$

Proof. As explained in §3.2 we have to solve, for each cusp $\mathfrak{a}$ of Γ, Equation (3.2) with an element of $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$. Lemma 3.1 shows that such solutions always satisfy $\mathsf{BdSing}\, h \subset \{\mathfrak{a}\}$, hence if we have a solution in $\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$ it is in $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$.

By conjugation, the task is equivalent to solving (3.4) with E replaced by $F|_r\sigma_{\mathfrak{a}}$. The Fourier series of E, which is the Fourier expansion of F at the cusp $\mathfrak{a}$, is split up as a sum of two or three terms. The existence of solutions is obtained in the Lemmas 3.5–3.8. □

3.6. **Proof of Theorem B.** Assuming that Theorem A has been proved, we now prove Theorem B. We use the results concerning asymptotic expansions in §3.4. There we have seen that we need $|\lambda| = 1$ to get satisfactory results. Hence we impose the assumptions of real weight and unitary multiplier system, which are the same assumptions as in the Theorem of Knopp and Mawi. See Theorem 2.7.

Proof of Theorem B on the basis of Theorem A. In Proposition 2.6 we saw that the space $\mathbf{r}_r^\omega\, S_r(\Gamma, v)$ is contained in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r})$ for all $r \in \mathbb{C}$. Part i) of the Lemmas 3.5 and 3.8 show that $\mathbf{r}_r^\omega\, M_r(\Gamma, v)$ is contained in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp},\mathrm{exc}}_{v,2-r})$ for $r \in \mathbb{C} \smallsetminus \{1\}$, since for h as in (3.11) the function $\mathsf{prj}_{2-r} h(t)$ has an asymptotic expansion at ∞ starting at t^1 (“$k = -1$”).

Let $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$. Under Theorem A any class in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$ is of the form $\mathbf{r}_r^\omega F$ for some $F \in A_r(\Gamma, v)$. We want to show that F satisfies the following:

	Part i)	Part ii)
	$r \notin \mathbb{Z}_{\geq 2}$	$r \notin \mathbb{Z}_{\geq 1}$
wish:	$F \in S_r(\Gamma, v)$	$F \in M_r(\Gamma, v)$

We consider a cusp $\mathfrak{a}$ of Γ, and put $E = F|_r\sigma_\mathfrak{a}$. The assumptions on F imply that there is $h_\mathfrak{a}$ in $\mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$, respectively $\mathcal{D}^{\omega,\mathrm{smp},\mathrm{exc}}_{2-r}[\infty]$ such that

$$h_\mathfrak{a}|_{2-r}(1 - v(\pi_\mathfrak{a})^{-1}\, T)\,(t) = \int_{z_0-1}^{z_0} (z-t)^{r-2}\, E(z)\, dz\,.$$

we write $E = E_c + E_0 + E_e$ by taking the Fourier terms with $n > 0$, $n = 0$ and $n < 0$, respectively. Note that $|v(\pi_\mathfrak{a})| = 1$, hence the Fourier term orders are real.

We take h_c provided by Lemma 3.5, h_0 by Lemma 3.8, and h_e by Lemma 3.6. Then $h_\mathfrak{a}|_{2-r}\sigma_\mathfrak{a} = h_c + h_0 + h_e + p$, with a $v(\pi_\mathfrak{a})$-periodic element p in $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$. Table 1 gives information on the asymptotic behavior, where we use the definitions of $\mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}$ and $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}$, and Lemmas 3.5, 3.6, and 3.8. (The c_n in the table depend on the function.)

	Part i)	Part ii)
	$r \notin \mathbb{Z}_{\geq 2}$ $h_\mathfrak{a}\vert\sigma_\mathfrak{a} \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$	$r \notin \mathbb{Z}_{\geq 1}$ $h_\mathfrak{a}\vert\sigma_\mathfrak{a} \in \mathcal{D}^{\omega,\mathrm{smp},\mathrm{exc}}_{2-r}[\infty]$
2-sided:	$(\mathsf{prj}_{2-r} h_\mathfrak{a}\vert\sigma_\mathfrak{a})(t) \sim \sum_{n\geq 0} c_n\, t^{-n}$	$(\mathsf{prj}_{2-r} h_\mathfrak{a}\vert\sigma_\mathfrak{a})(t) \sim \sum_{n\geq -1} c_n\, t^{-n}$
2-sided:	$(\mathsf{prj}_{2-r} h_c)(t) \sim \sum_{n\geq 0} c_n\, t^{-n}$	
as $t \uparrow \infty$:	$(\mathsf{prj}_{2-r} h_e)(t) \sim \sum_{n\geq 0} c_n\, t^{-n}$	
$v(\pi_\mathfrak{a}) \neq 1$	$h_0 = 0$	
$v(\pi_\mathfrak{a}) = 1$	$(\mathsf{prj}_{2-r} h_0)(t) \sim \sum_{n\geq -1} c_n t^{-n}$ if $r \neq 1$	
	no asymptotic expansion if $r = 1$	

TABLE 1

Let $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 1}$. Note that the $v(\pi_\mathfrak{a})$-periodic function p has an asymptotic expansion as $t \uparrow \infty$, and is $\mathrm{O}(t)$. Part ii) of Lemma 3.4 implies that either $p = 0$ or p is a non-zero constant and $r \in \mathbb{Z}_{\geq 1}$. Since the latter case is impossible, we deduce that $p = 0$.

We conclude that $h_e = h_\mathfrak{a}|_{2-r}\sigma_\mathfrak{a} - h_c - h_0$ has a two-sided expansion. Part ii) of Lemma 3.6 shows that $h_e = 0$, and hence $E_e = 0$. So $F|_{2-r}\sigma_\mathfrak{a} = E_c$, and hence F behaves like an element of $S_r(\Gamma, v)$ at the cusp $\mathfrak{a}$. Since $\mathfrak{a}$ was chosen arbitrarily, this finishes the proof of both parts under the assumption $r \neq 1$.

For Part i) we have still to consider $r = 1$. If we work modulo functions with an asymptotic expansion in powers of t^{-1} as $t \uparrow \infty$, the $v(\pi_\mathfrak{a})$-periodic function p has to compensate for the possible logarithmic behavior of h_0 given in Lemma 3.8. The logarithmic term is growing as $t \uparrow \infty$ and the periodic function is bounded (since $|v(\pi_\mathfrak{a})| = 1$), so this is impossible, and $h_0 = 0$.

Now we proceed as above, with asymptotic expansions starting at t^0. For h_e this rules out the constant function, and we arrive again at h_e=0, and hence $F|_{2-r}\sigma_\mathfrak{a} = E_c$. □

Remark 3.10. Since we have used the unitarity of the multiplier system v only for $|v(\pi_{\mathfrak{a}})| = 1$, Theorem B is still true under the assumption that $|v(\pi)| = 1$ for all *parabolic* $\pi \in \Gamma$, without such an assumption concerning hyperbolic elements.

3.7. **Related work.** Pribitkin [103, Theorem 1] uses integrals along paths like those in §3.4.

Proposition 3.2 is analogous to [15, Proposition 10.3]. Here we use the explicit integrals in §3.4, since we want to handle complex weights.

4. One-sided averages

In §3.2 we considered the parabolic equation with an Eichler integral as the given function. We now take the right hand side to be more general, and use the *one-sided averages* given by

$$(4.1)\qquad \begin{aligned} (\mathrm{Av}^+_{T,\lambda} g)(t) &:= \sum_{n\geq 0} \lambda^{-n}\, g(t+n)\,, \\ (\mathrm{Av}^-_{T,\lambda} g)(t) &:= -\sum_{n\leq -1} \lambda^{-n}\, g(t+n)\,, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, and where the subscript T refers to $T : t \mapsto t+1$.

4.1. **One-sided averages with absolute convergence.** If one of the series in (4.1) converges absolutely then $h = \mathrm{Av}^{\pm}_{T,\lambda} g$ provides a solution of the equation

$$(4.2)\qquad h(t) - \lambda^{-1}\, h(t+1) \;=\; g(t)\,.$$

Proposition 4.1. *Let $\lambda \in \mathbb{C}^*$, and suppose that g represents an element of $\mathcal{D}^{\omega}_{2-r}$.*

a) *the one-sided average $\mathrm{Av}^{\pm}_{T,\lambda} g$ converges absolutely if one of the following the conditions is satisfied:*

(4.3)

λ	$\lvert\lambda\rvert > 1$	$\lvert\lambda\rvert = 1$ *and* $\mathrm{Re}\, r < 1$	$\lvert\lambda\rvert < 1$
$\mathrm{Av}^+_{T,\lambda}\varphi$	*convergent*	*convergent*	*undecided*
$\mathrm{Av}^-_{T,\lambda}\varphi$	*undecided*	*convergent*	*convergent*

b) *The average defines a holomorphic function $\mathrm{Av}^{\pm}_{T,\lambda} g$ on a region*

$$(4.4)\qquad D^{\pm}_{\varepsilon} \;:=\; \Bigl\{ z \in \mathbb{C} \;:\; y < \varepsilon \text{ or } \pm x > \varepsilon^{-1} \Bigr\},$$

for some $\varepsilon \in (0,1)$.

c) $\mathrm{Av}^{\pm}_{T,\lambda} g$ *satisfy* (4.2).

d) $\mathrm{Av}^{\pm}_{T,\lambda} g$ *represent an element of* $\mathcal{D}^{\omega}_{2-r}[\infty]$.

Remark 4.2. We can interpret the phrase *g represents an element of $\mathcal{D}^{\omega}_{2-r}$* in two ways, and we will have reasons to use both interpretations.

a) g is a holomorphic function on an $\{\infty\}$-excised neighbourhood, and $\mathsf{prj}_{2-r} g$ is holomorphic on a neighbourhood of ∞ in $\mathbb{P}^1_{\mathbb{C}}$;

b) $g \in C^2(\mathbb{C})$ has a holomorphic restriction to an $\{\infty\}$-excised neighbourhood, and $\mathsf{prj}_{2-r} g$ is holomorphic on a neighbourhood of ∞ in $\mathbb{P}^1_{\mathbb{C}}$.

Under the first interpretation, $\mathrm{Av}^{\pm}_{T,\lambda} g$ is a holomorphic function on $D^{\pm}_{\varepsilon}$, where ε depends on the domain of g. In the second interpretation, we have $g \in C^2(\mathbb{C})$ such that $\mathsf{prj}_{2-r} g$ is holomorphic on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, and $\mathrm{Av}^{\pm}_{T,\lambda} g$ is in $C^2(\mathbb{C})$ and holomorphic on $D^{\pm}_{\varepsilon}$.

Proof of proposition 4.1. Representatives of the projective model $\mathsf{prj}_{2-r} \mathcal{D}^{\omega}_{2-r}$ are holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus K$ for some compact set $K \subset \mathfrak{H}$, which is contained in a set of the form $[-\varepsilon^{-1}, \varepsilon^{-1}] \times [i\varepsilon, i\varepsilon^{-1}]$ for some small $\varepsilon \in (0,1)$. To get a representative in the space $\mathcal{D}^{\omega}_{2-r}$ itself, we have to multiply by $(i-t)^{r-2}$. So we work with functions g that are holomorphic on

$$\mathbb{C} \smallsetminus \Bigl([-\varepsilon^{-1}, \varepsilon^{-1}] \times i[\varepsilon, \infty) \Bigr).$$

For t in a compact region V anywhere in $\mathbb{C}$, there is a tail of the series with $z+n$ in the region where g is holomorphic. Moreover, $g(t) = \mathrm{O}\bigl(|t|^{\mathrm{Re}\, r-2}\bigr)$ as $|t| \to \infty$. So the tail converges absolutely on V, and represents a holomorphic function on the interior of V. If $g \in C^2(\mathfrak{H})$, then the remaining terms give a C^2 contribution, and $g \in C^2(\mathbb{C})$. If we take $V \subset D^{\pm}_{\varepsilon}$ the whole series may be taken as the tail, and we get the holomorphy of the one-sided average on $D^{\pm}_{\varepsilon}$.

Note that the form of the set $D^{\pm}_{\varepsilon}$ implies that $\mathrm{Av}^{\pm}_{T,\lambda} g$ represents an element of $\mathcal{D}^{\omega}_{2-r}[\infty]$, but not necessarily of $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$. □

Remark. The relation (4.2) between $\mathrm{Av}^{\pm}_{T,\lambda} g$ and g implies this relation for the elements in $\mathcal{D}^{\omega}_{2-r}[\infty]$ and $\mathcal{D}^{\omega}_{2-r}$ that they represent.

Proposition 4.3. *Let $r \in \mathbb{Z}_{\leq 0}$ and $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R}$. Suppose that g is a representative of an element of $\mathcal{D}^{\omega}_{2-r}$ of the type b) in Remark 4.2.*

a) *Then*

$$\mathrm{Av}_{T,\lambda} g \;:=\; \mathrm{Av}^{+}_{T,\lambda} g - \mathrm{Av}^{-}_{T,\lambda} g \tag{4.5}$$

defines a λ-periodic element of $C^2(\mathbb{C})$.

b) *There is $\varepsilon \in (0,1)$ such that the function $\mathrm{Av}_{T,\lambda} g$ is holomorphic on two regions, with Fourier expansions of the following form:*

$$\mathrm{Av}_{T,\lambda} g\,(z) \;=\; \begin{cases} \sum_{m \equiv \alpha(1),\, m>0} a^{\mathrm{up}}_m \, e^{2\pi i m z} & \text{on } \{ z \in \mathfrak{H} \;:\; y > \varepsilon^{-1} \}, \\ \sum_{m \equiv \alpha(1),\, m<0} a^{\mathrm{down}}_m \, e^{2\pi i m z} & \text{on } \{ z \in \mathbb{C} \;:\; y < \varepsilon \}. \end{cases} \tag{4.6}$$

Proof. The function $(\mathsf{prj}_{2-r} g)(z) = (i-z)^{2-r} g(z)$ represents an element of the projective model of $\mathcal{D}^{\omega}_{2-r}$. Since $r \in \mathbb{Z}_{\leq 0}$ the function g itself is holomorphic on a neighbourhood of ∞ in $\mathbb{P}^1_{\mathbb{C}}$, and has a zero of order at least $2-r$ at ∞. Since $|\lambda| = 1$, both series $\mathrm{Av}^{+}_{T,\lambda} g$ and $\mathrm{Av}^{-}_{T,\lambda} g$ converge absolutely on $\mathbb{C}$, by Proposition 4.1.

These functions are now holomorphic on a set of the form

$$\mathbb{C} \smallsetminus \Bigl((-\infty, \varepsilon^{-1}] \times i[\varepsilon, \varepsilon^{-1}] \Bigr), \text{ respectively } \mathbb{C} \smallsetminus \Bigl((-\varepsilon^{-1}, \infty] \times i[\varepsilon, \varepsilon^{-1}] \Bigr).$$

So $(\mathrm{Av}_{T,\lambda} g)(z) = \sum_{n\equiv\alpha(1)} \lambda^{-n} g(z+n)$ defines a λ-periodic function on $\mathbb{C}$ that is holomorphic on the two regions in the proposition. On both regions the Fourier coefficients are given by integral

$$\int_{\mathrm{Im}\, z=v} e^{-2\pi i m z}\, g(z)\, dz\,,$$

representing the coefficients a_m^{up} if $v > \varepsilon^{-1}$, and the coefficients a_m^{down} if $v < \varepsilon$. The coefficients can differ on both regions. The integral is invariant under changes in v in the corresponding interval. Since $g(z) = \mathrm{O}(|z|^{\mathrm{Re}\, r-2})$ as $|z| \to \infty$ through $\mathbb{C}$, the integral satisfies $a_m = o(e^{2\pi m v})$ for fixed v. So $a_m^{\mathrm{up}} = 0$ for $m \le 0$, and $a_m^{\mathrm{down}} = 0$ for $m \ge 0$. □

In §11 we will use the following result:

Lemma 4.4. *Let $r \in \mathbb{Z}_{\le 0}$. Suppose that g is a representative of an element of $\mathcal{D}^{\omega}_{2-r}$, of the type a) in Remark 4.2. Suppose that h represents an element of $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$ such that $h - h|_{2-r}T$ and g represent the same element of $\mathcal{D}^{\omega}_{2-r}$.*

Then there are 1-periodic $p_+, p_- \in \mathcal{O}(\mathbb{C})$ such that for all sufficiently small $\varepsilon > 0$

$$\begin{aligned} h(z) &= (\mathrm{Av}^{+}_{T,1} g)(z) + p_+(z) \quad \text{if } \mathrm{Im}\,(z) < \varepsilon \text{ or } \mathrm{Re}\,(z) > \varepsilon^{-1}\,, \\ h(z) &= (\mathrm{Av}^{-}_{T,1} g)(z) + p_+(z) \quad \text{if } \mathrm{Im}\,(z) < \varepsilon \text{ or } \mathrm{Re}\,(z) < -\varepsilon^{-1}\,. \end{aligned} \tag{4.7}$$

Proof. Proposition 4.1 shows that we are in the domain of absolute convergence of $\mathrm{Av}^{\pm}_{T,1} g$, and that these averages are holomorphic on a set $D^{\pm}_{\varepsilon}$ with some $\varepsilon \in (0,1)$. Now the weight is an integer, and the factor $(i-z)^{r-2}$ is non-zero and holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i,\infty\}$. The function $z \mapsto g(z+n)$ is holomorphic outside the smaller region

$$[-\varepsilon^{-1} - n, \varepsilon^{-1} - n] \times i[\varepsilon, \varepsilon^{-1}]\,.$$

Hence the averages $\mathrm{Av}^{\pm}_{T,1} g$ are holomorphic on

$$\mathbb{C} \smallsetminus \Bigl((-\infty, \varepsilon^{-1}] \times i[\varepsilon, \varepsilon^{-1}]\Bigr)\,, \text{ respectively } \mathbb{C} \smallsetminus \Bigl([-\varepsilon^{-1}, \infty) \times i[\varepsilon, \varepsilon^{-1}]\Bigr)\,.$$

The function h is holomorphic on an $\{\infty\}$-excised neighbourhood. So after adaptation of $\varepsilon > 0$ on a region

$$\mathbb{C} \smallsetminus [-\varepsilon^{-1}, \varepsilon^{-1}] \times i\,[\varepsilon, \infty)\,.$$

On $0 < \mathrm{Im}\,(z) = y < \varepsilon$ the functions h, $\mathrm{Av}^{+}_{T,1} g$ and $\mathrm{Av}^{-}_{T,1} g$ satisfy the same relation, hence there are 1-periodic p_+ and p_- on this region that satisfy (4.7). The relations between h and the averages extend by holomorphy to the half-plane $y < \varepsilon$ in $\mathbb{C}$, and the 1-periodic functions p_+ and p_- extend holomorphically to $y < \varepsilon$.

The relation $p_+ = h - \mathrm{Av}^{+}_{T,1} g$ extends p_+ to a region

$$\{z \in \mathbb{C} \ :\ \mathrm{Im}\,(z) < \varepsilon\} \cup \{z \in \mathbb{C} \ :\ \mathrm{Re}\,(x) > \varepsilon^{-1}\}\,.$$

Then by 1-periodicity p_+ has a 1-periodic holomorphic extension to $\mathbb{C}$.

The case of p_- and $\mathrm{Av}^{-}_{T,1} g$ goes similarly. □

4.2. **Analytic continuation of one-sided averages.** To obtain the one-sided averages with $|\lambda| = 1$ on representatives of $\mathcal{D}^\omega_{2-r}$ for more values of r, we use that the space of the projective model $\mathsf{prj}_{2-r}\mathcal{D}^\omega_{2-r}$ does not depend on r. The representatives in the projective model are holomorphic functions h on a neighbourhood Ω of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. For a fixed h the function $g_r := \mathsf{prj}^{-1}_{2-r}h$ represents an element of $\mathcal{D}^\omega_{2-r}$ for each $r \in \mathbb{C}$, ie.

$$g_r(t) \;=\; (i-t)^{r-2}\, h(t) \qquad (t \in \Omega)\,. \tag{4.8}$$

In this subsection we work with the interpretation a) in Remark 4.2.

Lemma 4.5. *Let $|\lambda| = 1$, and let g_r be as defined in* (4.8). *Let $\varepsilon \in (0,1)$ be such that $\mathrm{Av}^\pm_{T,\lambda}\, g_r$ is holomorphic on the set $D^\pm_\varepsilon$ in* (4.4) *for* $\mathrm{Re}\, r < 1$.

a) *The function $(r,z) \mapsto (\mathrm{Av}^\pm_{T,\lambda} g_r)(z)$ extends as a holomorphic function on $(\mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}) \times D^\pm_\varepsilon$.*
b) *If $h(\infty) = 0$ then $(r,z) \mapsto (\mathrm{Av}^\pm_{T,\lambda}\, g_r)(z)$ can be extended holomorphically to the slightly larger region $(\mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}) \times D^\pm_\varepsilon$.*

Proof. We use the *Hurwitz-Lerch zeta-function*

$$H(s,a,z) \;=\; \sum_{n\geq 0} e^{2\pi i a n}\, (z+n)^{-s}\,, \tag{4.9}$$

which converges absolutely and is holomorphic in (s,z) for $\mathrm{Im}\, a \leq 0$, $z \in \mathbb{C} \smallsetminus \mathbb{Z}_{\leq 0}$ and $\mathrm{Re}\, s > 1$. Kanemitsu, Katsurada and Yoshimoto [61, Theorem 1*] give the holomorphy of $(s,z) \mapsto H(s,a,z)$ on $(\mathbb{C} \smallsetminus \{1\}) \times \{z \in \mathbb{C} \;:\; \mathrm{Re}\, z > 0\}$ with a first order singularity at $s = 1$ if $a \in \mathbb{Z}$, and no singularity in s at all otherwise. With

$$H(s,a,z) \;=\; \sum_{n=0}^{m-1} e^{2\pi i a n}\, (z+n)^{-s} + H(s,a,z+m)$$

for each $m \in \mathbb{Z}_{\geq 1}$, we obtain holomorphy in $z \in \mathbb{C} \smallsetminus (-\infty, 0]$. (Lagarias and Li [77] study the continuation in three variables. Here we need only the continuation in (s,z).)

The function h in (4.8) is holomorphic on a neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, and hence has a convergent power series expansion on a neighbourhood of ∞:

$$h(z) = \sum_{k=0}^{\infty} \tilde{a}_k\, z^{-k}\,. \tag{4.10}$$

This implies that we have for $z \in D^\pm_\varepsilon$

$$g_r(z) \;=\; \sum_{k=0}^{N-1} a_k(r)\, (z-i)^{r-2-k} + g_{r,N}(z)\,, \tag{4.11}$$

with $g_{r,N}(z) = \mathrm{O}(z^{r-2-N})$ as $z \to \infty$ through $D^\pm_\varepsilon$, uniformly for r in compact sets in $\mathbb{C}$. The $a_k(r)$ are polynomials in r. We take $\arg(z-i) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$. The one-sided averages of $g_{r,N}$ converge absolutely, and provide holomorphic functions in (r,z)

on $\mathrm{Re}\, r < N+1$ and $z \in D_\varepsilon^\pm$. For the remaining finitely many terms we have a sum of

$$a_k(r)\,\lambda^{-n}\,(z+n-i)^{r-2-k}$$
$$= \; a_k(r)\,e^{-2\pi i n\alpha} \begin{cases} (z-i+n)^{r-2-k} & \text{for } n \geq 0 \text{ and } \mathrm{Re}\, z > \varepsilon^{-1}\,, \\ e^{-\pi i(r-2-k)}\,(i-z+|n|)^{r-2-k} & \text{for } n \leq -1 \text{ and } \mathrm{Re}\, z < -\varepsilon^{-1}\,, \end{cases}$$

where $\alpha \in \mathbb{C}$ has been chosen so that $\lambda = e^{2\pi i\alpha}$. In this way we obtain:

$$\begin{aligned} (\mathrm{Av}^+_{T,\lambda} g_r)(z) \;&=\; \sum_{k=0}^{N-1} a_k(r)\,H(k+2-r, -\alpha, z-i) + (\mathrm{Av}^+_{T,\lambda} g_{r,N})(z)\,, \\ (\mathrm{Av}^-_{T,\lambda} g_r)(z) \;&=\; \sum_{k=0}^{N-1} a_k(r)\,\lambda\, e^{\pi i(k-r)}\,H(k+2-r, \alpha, 1+i-z) \\ &\qquad + (\mathrm{Av}^-_{T,\lambda} g_{r,N})(z)\,. \end{aligned} \tag{4.12}$$

The function $(r,z) \mapsto H(k+2-r, -\alpha, z-i)$ is meromorphic on the region $r \in \mathbb{C}$, $z \in i + \mathbb{C} \smallsetminus (-\infty, 0]$, with a singularity at $r = k+1$. The function $(r,z) \mapsto (\mathrm{Av}^+_{T,\lambda} g_{r,N})(z)$ is holomorphic on the region $\mathrm{Re}\, r < 1+N$, $z \in D_\varepsilon^+$. So $(r,z) \mapsto (\mathrm{Av}^+_{T,\lambda} g_r)(z)$ extends meromorphically to the region $\mathrm{Re}\, r < 1+N$, $z \in D^+_\varepsilon$, and its singularities can occur only at $r = a$ with $a \in \{1, \ldots, N\}$. The case of $\mathrm{Av}^-_{T,\lambda} g_r$ goes similarly. □

Proposition 4.6. *Let $r \in \mathbb{C}$, $\lambda \in \mathbb{C}^*$, and let g represent an element of $\mathcal{D}^\omega_{2-r}$.*

i) *There are well-defined one-sided averages $\mathrm{Av}^\pm_{T,\lambda} g$ holomorphic on $D^\pm_\varepsilon$, as in* (4.4)*, for some $\varepsilon \in (0,1)$ depending on g, under the following conditions*

(4.13)

	$\lvert\lambda\rvert > 1$	$\lvert\lambda\rvert = 1$		$\lvert\lambda\rvert < 1$
		$(\mathsf{prj}_{2-r} g)(\infty) = 0$	$(\mathsf{prj}_{2-r} g)(\infty) \neq 0$	
$\mathrm{Av}^+_{T,\lambda} g$	$r \in \mathbb{C}$	$r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$	$r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$	
$\mathrm{Av}^-_{T,\lambda} g$		$r \in \mathbb{Z}_{\geq 2}$	$r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$	$r \in \mathbb{C}$

ii) *These one-sided averages satisfy $\mathrm{Av}^\pm_{T,\lambda} g|_{2-r}(1 - \lambda^{-1}\,T) = g$.*

iii) *If $g = g_r = \mathsf{prj}^{-1}_{2-r} h$ as in* (4.8)*, and $|\lambda|^{\pm 1} \geq 1$, then $r \mapsto \mathrm{Av}^\pm_{T,\lambda} g_r$ is a meromorphic function on $\mathbb{C}$.*

Proof. Each representative g of an element of $\mathcal{D}^\omega_{2-r}$ is of the form $g_r = \mathsf{prj}_{2-r} h$ for some holomorphic function on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_\mathbb{R}$ in $\mathbb{P}^1_\mathbb{C}$. If $|\lambda| \neq 1$, Proposition 4.1 gives the convergence of one of the averages and the relation in Part ii). The convergence is sufficiently quick to have holomorphy in r.

Let $|\lambda| = 1$. Proposition 4.1 gives convergence of both averages for $\mathrm{Re}\, r < 1$, and Lemma 4.5 provides the meromorphic continuation to $\mathbb{C}$, with singularities only in the points indicated in Part i). The relation in Part ii) stays valid by analytic continuation. □

Remark 4.7. If $|\lambda| = 1$, $\lambda \neq 1$, the proof of Lemma 4.5 can be adapted to give holomorphy of $\mathrm{Av}^\pm_{T,\lambda}\, g_r$ in $r \in \mathbb{C}$. We can strengthen the statements in Cases i) and iii) of Proposition 4.6 as well.

Asymptotic behavior. To get the asymptotic behavior of $\mathrm{Av}^{\pm}_{T,\lambda} g_r(t)$ as $\pm\mathrm{Re}\, t \to \infty$, we use the following result:

Proposition 4.8. (Katsurada, [63]) *Let $s, a \in \mathbb{C}$, $\mathrm{Im}\, a \le 0$. There are $b_k(\lambda, s) \in \mathbb{C}$ such that for each $K \in \mathbb{Z}_{\ge 0}$ we have as $|z| \to \infty$ on any region $\delta - \pi \le \arg z \le \pi - \delta$ with $\delta > 0$*

$$H(s, a, \tfrac{1}{2} + z) \;=\; \frac{\varepsilon(\lambda)}{1-s}\, z^{1-s} + \sum_{k=0}^{K-1} b_k(\lambda, s)\, z^{-k-s} + \mathrm{O}(|z|^{-\mathrm{Re}\, s - K})\,, \tag{4.14}$$

with $\lambda = e^{2\pi i a}$, $\varepsilon(\lambda) = 1$ if $\lambda = 1$ and $\varepsilon(\lambda) = 0$ otherwise. The coefficients b_k satisfy

$$\lambda^{-1}\, b_k(\lambda^{-1}, s) \;=\; (-1)^{k+1}\, b_k(\lambda, s)\,. \tag{4.15}$$

The first three coefficients are as follows:

(4.16)

	$\lambda = 1$	$\lambda \neq 1$
$b_0(\lambda, s)$	0	$\frac{1}{1-\lambda}$
$b_1(\lambda, s)$	$-\frac{s}{48}$	$-\frac{s}{4}\frac{1+\lambda}{(1-\lambda)^2}$
$b_2(\lambda, s)$	0	$\frac{s(s+1)(1+6\lambda+\lambda^2)}{48(1-\lambda^3)}$

Proof. This is a direct consequence of [63, Theorem 1], applied with $\alpha = \frac{1}{2}$.
We have $H(s, a, z) = \Phi(a, z, s)$ with Katsurada's Φ. This gives (4.14) with

$$b_k(\lambda, s) \;=\; \frac{(-1)^{k+1}}{(k+1)!}\, B_{k+1}(\tfrac{1}{2}, \lambda)\, (s)_k\,,$$

where the B_k are generalized Bernoulli polynomials, given by

$$\sum_{k \ge 0} B_k(x, y)\, \frac{z^k}{k!} \;=\; \frac{z e^{xz}}{y e^z - 1}\,.$$

Relation (4.15) follows from

$$\frac{z e^{z/2}}{y e^z - 1} \;=\; \frac{z}{y e^{z/2} - e^{-z/2}} \;=\; y^{-1}\, \frac{-z}{y^{-1} e^{-z/2} - e^{z/2}}\,. \qquad \square$$

Proposition 4.9. *Let $r \in \mathbb{C}$, and let g be a representative of an element of $\mathcal{D}^{\omega}_{2-r}$.*

i) *Let $|\lambda| = 1$ and suppose that $r \in \mathbb{C}$ is such that $\mathrm{Av}^{+}_{T,\lambda}\, g$ and $\mathrm{Av}^{-}_{T,\lambda}\, g$ exist.*

a) *There are coefficients c_k depending on λ, r and on the coefficients of the expansion of $\mathrm{prj}_{2-r} g$ at ∞, such that for each $M \in \mathbb{Z}_{\ge 0}$ we have:*

$$(\mathrm{Av}^{\pm}_{T,\lambda}\, g)(t) \;=\; (it)^{r-2} \sum_{k=-1}^{M-1} c_k\, t^{-k} + \mathrm{O}(|t|^{r-2-M}) \tag{4.17}$$

as $|t| \to \infty$ with $\pm\mathrm{Re}\, t \ge 0$, $\mathrm{Im}\, t \le 0$.

b) *If* $g(t) = (it)^{r-2}\,(a_0 + a_1\,t^{-1} + \cdots)$ *near* ∞*, then*

(4.18)

	$\lambda = 1$	$\lambda \neq 1$
c_{-1}	$\frac{a_0}{r-1}$	0
c_0	$\frac{a_1}{r-2}$	$\frac{\lambda\, a_0}{\lambda-1}$
c_1	$\frac{a_2}{r-3} + \frac{(r-2)\,a_0}{48}$	$\frac{\lambda\, a_1}{\lambda-1} + \frac{(r-2)\,\lambda\,(\lambda+1)\,a_0}{4(\lambda-1)^2}$

ii) *Let* $|\lambda| \neq 1$*. If* $\pm$ *is such that* $|\lambda|^{\pm 1} > 1$*, then there are coefficients* c_k *such that for each* $M \in \mathbb{Z}_{\geq 0}$

$$(\mathrm{Av}^{\pm}_{T,\lambda}\, g)(t) \;=\; (it)^{r-2} \sum_{k=0}^{M-1} c_k\, t^{-k} + \mathrm{O}(|t|^{r-2-M}) \tag{4.19}$$

as $|t| \to \infty$ *with* $\pm\mathrm{Re}\, t \geq 0$*,* $\mathrm{Im}\, t \leq 0$*.*

Remark. It is remarkable that if both one-sided averages exist, then the coefficients in both expansions are the same, although $\mathrm{Av}^{+}_{T,\lambda}\, g$ and $\mathrm{Av}^{-}_{T,\lambda}\, g$ have in general no reason to be equal.

Proof. It suffices to consider large values of M. We take $M > \mathrm{Re}\, r + 1$. If $g(t) = \mathrm{O}(|t|^{r-3-M})$ we have $(\mathrm{Av}^{\pm}_{T,\lambda}\, g)(t) = \mathrm{O}(|t|^{r-2-M})$, which only influences the error term. So the explicit terms in the asymptotic expansion are determined by the part

$$(it)^{r-2} \sum_{j=0}^{M} a_j\, t^{-j}$$

of the expansion of g at ∞. We consider for $0 \leq j \leq M$ functions g_j representing elements of $\mathcal{D}^{\omega}_{2-r}$ for which $g_j(t) = (it)^{r-2}\, t^{-j} + \mathrm{O}(|t|^{r-3-M})$ as $t \to \infty$.

In Part i) we have $|\lambda| = 1$. In (4.11) we took $t - i$ as the variable. Now $t - \frac{1}{2}$ is more convenient. We put $\lambda = e^{2\pi i \alpha}$, and have, modulo terms that can be absorbed into the error term:

$$\begin{aligned}(\mathrm{Av}^{+}_{T,\lambda}\, g_j)(1/2 + t) \;&\equiv\; -e^{\pi i r/2}\, H(2 + j - r, -\alpha, 1/2 + t)\,,\\ (\mathrm{Av}^{-}_{T,\lambda}\, g_j)(1/2 + t) \;&\equiv\; e^{-\pi i r/2}\, (-1)^j\, \lambda\, H(2 + j - r, \alpha, 1/2 - t)\,.\end{aligned}$$

With Proposition 4.8 this gives

$$\begin{aligned}(\mathrm{Av}^{+}_{T,\lambda}\, g_j)(1/2 + t) \;&\equiv\; (it)^{r-2}\Bigl(\frac{\varepsilon(\lambda)}{r - j - 1}\, t^{1-j} + \sum_{k=0}^{M-j} b_k(\lambda^{-1}, 2 + j - r)\, t^{-k-j}\Bigr)\,,\\ (\mathrm{Av}^{-}_{T,\lambda}\, g_j)(1/2 + t) \;&\equiv\; e^{-\pi i r/2}(-1)^j \lambda\, \Bigl(\frac{\varepsilon(\lambda)}{r - 1 - j}\, (-t)^{r-1-j}\\ &\qquad + \sum_{k=0}^{M-j} b_k(\lambda, 2 + j - r)\, (-t)^{-k-j+r-2}\Bigr)\end{aligned}$$

$$= (it)^{r-2} \Big(\frac{\varepsilon(\lambda)}{r-1-j} t^{1-j} - \sum_{k=0}^{M-j} (-1)^k \lambda\, b_k(\lambda, 2+j-r)\, t^{-k-j} \Big).$$

(In the last step we have used that $\lambda = 1$ if $\varepsilon(\lambda) \neq 0$.)

For g with expansion $(it)^{r-2} \sum_{j\geq 0} a_j t^{-j}$ near ∞ this leads to an expansion as in (4.17), with coefficients $c_\ell^\pm$ of the form

$$c_{-1}^\pm = \frac{\varepsilon(\lambda)}{r-1} a_0 ,$$

and for $\ell \geq 0$

$$c_\ell^+ = \frac{\varepsilon(\lambda)}{r-\ell-2} a_{\ell+1} + \sum_{j=0}^{\ell} b_{\ell-j}(\lambda^{-1}, 2+j-r)\, a_j ,$$

$$c_\ell^- = \frac{\varepsilon(\lambda)}{r-\ell-2} a_{\ell+1} - \sum_{j=0}^{\ell} (-1)^{\ell-j} \lambda\, b_{\ell-j}(\lambda, 2+j-r)\, a_j .$$

Relation (4.15) shows that $c_\ell^+ = c_\ell^-$.

In part ii) the factor λ^{-n} takes care of the convergence of the one-sided average. The asymptotic behavior follows directly from the behavior of $g(t)$ near ∞. □

Lemma 4.10. *Let $|\lambda| = 1$.*

i) *Let $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 1}$. The following statements concerning $\varphi \in \mathcal{D}^\omega_{2-r}$ are equivalent:*
   a) *There is a representative g of φ for which $\mathrm{Av}^+_{T,\lambda} g$ and $\mathrm{Av}^-_{T,\lambda} g$ represent the same element of $\mathcal{D}^\omega_{2-r}[\infty]$.*
   b) *There is a function h representing an element of $\mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\infty]$ such that $h|_{2-r}(1-\lambda^{-1} T)$ represents φ.*

   If these statements hold, then $\mathrm{Av}^+_{T,\lambda} g$, $\mathrm{Av}^-_{T,\lambda} g$ and h represent the same element of $\mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\infty]$.

ii) *Let $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$. The following statements concerning $\varphi \in \mathcal{D}^\omega_{2-r}$ are equivalent:*
   a) *There is a representative g of φ such that $\mathsf{prj}_{2-r} g(\infty) = 0$, and for which $\mathrm{Av}^+_{T,\lambda} g$ and $\mathrm{Av}^-_{T,\lambda} g$ represent the same element of $\mathcal{D}^\omega_{2-r}[\infty]$.*
   b) *There is a function h representing an element of $\mathcal{D}^{\omega,\infty}_{2-r}[\infty]$ such that $h|_{2-r}(1-\lambda^{-1} T)$ represents φ.*

   If these statements hold, then $\mathrm{Av}^+_{T,\lambda} g$, $\mathrm{Av}^-_{T,\lambda} g$ and h represent the same element of $\mathcal{D}^{\omega,\infty}_{2-r}[\infty]$.

Proof. Let g be a representative of φ as in one of the statements a). Then $\mathrm{Av}^+_{T,\lambda} g(z) = \mathrm{Av}^-_{T,\lambda} g(z)$ for $y < \varepsilon$ for some $\varepsilon \in (0,1)$. Let us call this function f. It is holomorphic on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{R}$ in $\mathbb{C}$, and Proposition 4.9 shows $(\mathsf{prj}_{2-r} f)(z)$ has an asymptotic expansion as $z \to \infty$ through $\mathfrak{H}^- \cup \mathbb{R}$ required in Definition 1.10 for representatives of elements of $\mathcal{D}^{\omega,\mathrm{smp}}_{2-r}[\infty]$. This gives b) in Part i). If we have the additional condition $(\mathsf{prj}_{2-r} g)(\infty) = 0$, the asymptotic expansion starts at $k = 0$

instead of $k = -1$, and we conclude that f represents an element of $\mathcal{D}^{\omega,\infty}_{2-r}[\infty]$. This concludes the proof of a)⇒b) in both parts.

Let h as b) be given. With any representative g of φ, we have also $\mathrm{Av}^+_{T,\lambda} g$ and $\mathrm{Av}^-_{T,\lambda} g$ in $\mathcal{D}^{\omega}_{2-r}[\infty]$ satisfying the same relation. So $h - \mathrm{Av}^{\pm}_{T,\lambda} g$ is a λ-periodic function on a neighbourhood of $\mathbb{R}$, with a one-sided asymptotic expansion of the type (4.17) as $\pm\mathrm{Re}\, z \to \infty$. Hence this λ-periodic function is zero by Lemma 3.4, and the three functions h, $\mathrm{Av}^+_{T,\lambda} g$ and $\mathrm{Av}^-_{T,\lambda} g$ are the same on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{R}$ in $\mathbb{C}$, and represent the same element of $\mathcal{D}^{\omega}_{2-r}[\infty]$. That gives a) in Part i). For Part ii) we note that the fact that h represents an element of $\mathcal{D}^{\omega,\infty}_{2-r}[\infty]$ implies $(\mathsf{prj}_{2-r} g)(\infty) = 0$. □

4.3. **Parabolic cohomology groups.** With the one-sided averages we can prove some of the isomorphisms in Theorem E on page 18.

Proposition 4.11. *Let $r \in \mathbb{R}$, and let v be a unitary multiplier system.*

i) *If $r \notin \mathbb{Z}_{\geq 2}$ then*

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}) \;=\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty}_{v,2-r})\,.$$

ii) *The codimension of $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{smp},\mathrm{exc}}_{v,r})$ in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,r})$ is finite if $r = 1$, and zero if $r \notin \mathbb{Z}_{\geq 1}$.*

Proof. Let $\psi \in Z^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty}_{v,2-r})$. For cusps $\mathfrak{a}$ in a (finite) set of representatives of the Γ-orbits of cusps we consider $h_{\mathfrak{a}} \in \mathcal{D}^{\omega^0,\infty}_{v,2-r}$ such that $h_{\mathfrak{a}}|_{v,2-r}(1 - \pi_{\mathfrak{a}}) = \psi_{\pi_{\mathfrak{a}}} \in \mathcal{D}^{\omega}_{v,2-r}$. After conjugation, we are in the situation of Part ii)b) of Lemma 4.10 with $\lambda = v(\pi_{\mathfrak{a}})$. Since the conditions on r and λ in that lemma are satisfied, we have $h = \mathrm{Av}^+_{T,\lambda}\psi_{\pi_{\mathfrak{a}}} = \mathrm{Av}^-_{T,\lambda}\psi_{\pi_{\mathfrak{a}}}$ near $\mathfrak{H}^- \cup \mathbb{R}$. Since $\mathrm{Av}^{\pm}_{T,\lambda}\psi_{\pi_{\mathfrak{a}}}$ is holomorphic on $D^+_{\varepsilon} \cup D^-_{\varepsilon}$ for some $\varepsilon > 0$, with $D^{\pm}_{\varepsilon}$ as in (4.4), the function h is holomorphic on a $\{\infty\}$-excised neighbourhood, hence $h \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\infty]$, and $h_{\mathfrak{a}} \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{2-r}[\mathfrak{a}] \subset \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{2-r}$.

The other case goes similarly, except if $r = 1$ and $v(\pi_{\mathfrak{a}}) = 1$. If $\mathsf{prj}_1(\psi_{\pi_{\mathfrak{a}}})(\infty) = 0$ then Proposition 4.9 implies that the starting term of the asymptotic expansion (4.17) satisfies $k \geq 0$, and $h_{\mathfrak{a}}$ is in $\mathcal{D}^{\omega,\infty}_{v,1}[\mathfrak{a}]$, and the same reasoning applies. Since the number of cuspidal orbits is finite, this imposes conditions on the cocycles determining a subspace of finite codimension. □

Proposition 4.12. *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, then*

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r}) \;=\; H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})\,. \tag{4.20}$$

If $r = 1$, then the space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,1}, \mathcal{D}^{\omega^*}_{v,1})$ has finite codimension in the space $H^1(\Gamma; \mathcal{D}^{\omega}_{v,1})$.

Remark. So for all $r \notin \mathbb{Z}_{\geq 2}$ the space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r})$ has finite codimension in $H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$.

We prepare the proof of Proposition 4.12 by a lemma.

Lemma 4.13. *Let $r \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$. Then*

$$\begin{aligned} &\mathcal{D}^\omega_{2-r} \subset \mathcal{D}^\omega_{2-r}[\infty]|_{2-r}(1-\lambda^{-1}T) && \text{if } r \notin \mathbb{Z}_{\geq 1}\,,\\ &\dim\Bigl(\mathcal{D}^\omega_1 \,/\, \bigl(\mathcal{D}^\omega_1 \cap (\mathcal{D}^\omega_1[\infty]|_1(1-\lambda^{-1}T))\bigr)\Bigr) \leq 1 && \text{if } r = 1\,. \end{aligned} \tag{4.21}$$

In the case $r = 1$ an element $\varphi \in \mathcal{D}^\omega_1$ is in $\mathcal{D}^\omega_1[\infty]|_1(1-\lambda^{-1}T)$ if $\lambda \neq 1$ or if $\mathsf{prj}_1\varphi(\infty) = 0$.

Proof. Proposition 4.6 shows that if $r \notin \mathbb{Z}_{\geq 1}$ or if $\lambda \neq 1$, we can use at least one of the one-sided averages to show that $\mathcal{D}^\omega_{2-r}$ is contained in $\mathcal{D}^\omega_{2-r}[\infty]|_{2-r}(1-\lambda^{-1}T)$. If $r = 1$ and $\lambda = 1$ we have to restrict ourselves to a subspace of $\mathcal{D}^\omega_1[\infty]$ of codimension 1. □

Proof of Proposition 4.12. The inclusion ⊂ follows from the definition of parabolic cohomology. To prove the other inclusion we consider a cocycle $\psi \in Z^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$ and need to show that for a representative $\mathfrak{a}$ of each Γ-orbit of cusps there is $h \in \mathcal{D}^{\omega^*}_{v,2-r}$ such that $h|_{v,2-r}(1-\pi_\mathfrak{a}) = \psi_{\pi_\mathfrak{a}}$. By conjugation this can be brought to ∞, into the situation considered in Lemma 4.13. Since there are only finitely many cuspidal orbits, we get for $r = 1$ a subspace of finite codimension. □

4.4. **Related work.** Knopp uses one-sided averages in [66, Part IV], attributing the method to B.A. Taylor (non-published). In [15] the one-sided averages are an important tool, defined in Section 4, and used in Sections 9 and 12.

Part II. **Harmonic functions**

5. Harmonic functions and cohomology

5.1. **The sheaf of harmonic functions.** By associating to open sets $U \subset \mathfrak{H}$ the vector space $\mathcal{H}_r(U)$ of r-harmonic functions on U (as defined in Definition 1.15) we form the *sheaf $\mathcal{H}_r$ of r-harmonic functions* on $\mathfrak{H}$.

The shadow operator ξ_r in (1.28) determines a morphism of sheaves $\mathcal{H}_r \to \mathcal{O}_\mathfrak{H}$, where $\mathcal{O}_\mathfrak{H}$ denotes the sheaf of holomorphic functions on $\mathfrak{H}$, and leads to an exact sequence

$$0 \to \mathcal{O}_\mathfrak{H} \to \mathcal{H}_r \xrightarrow{\xi_r} \mathcal{O}_\mathfrak{H} \to 0\,. \tag{5.1}$$

The maps $\xi_r : \mathcal{H}_r(U) \to \mathcal{O}(U)$ are antilinear for the structure of vector spaces over $\mathbb{C}$. The surjectivity of ξ_r follows from classical properties of the operator $\partial_{\bar z}$. (It suffices to solve locally $\partial_{\bar z} h = \varphi$ for given holomorphic φ. See, eg., Hörmander [57, Theorem 1.2.2].)

Actions. Let $r \in \mathbb{C}$. For each $g \in \mathrm{SL}_2(\mathbb{R})$ the operator $|_r g$ gives bijective linear maps $\mathcal{H}_r(U) \to \mathcal{H}_r(g^{-1}U)$ and $\mathcal{O}_\mathfrak{H}(U) \to \mathcal{O}_\mathfrak{H}(g^{-1}U)$. For sections F of $\mathcal{H}_r$

$$\Delta_r(F|_r g) = (\Delta_r F)|_r g\,, \qquad \xi_r(F|_r g) = (\xi_r F)|_{2-\bar r} g\,. \tag{5.2}$$

If v is a multiplier system for Γ for the weight r, then we have also the actions $|_{v,r}$ of Γ on $\mathcal{H}_r$ and $|_{\bar v,2-\bar r}$ on $\mathcal{O}$. With these actions $\mathcal{H}_r$ and $\mathcal{O}$ are Γ-equivariant sheaves.

5.2. **Harmonic lifts of automorphic forms.** In this subsection we will prove Theorem C.

Example. The image in $H^1(\Gamma(1);\mathcal{D}_{1,2}^{\omega^0,\mathrm{exc}})$ of $\mathbf{r}_0^\omega 1 \in H^1(\Gamma(1);\mathcal{D}_{1,2}^\omega)$ can be represented by the cocycle $\tilde\psi$ in (2.23), given by $\tilde\psi_\gamma(t) = \frac{-c}{ct+d}$. The cocycle $\iota\tilde\psi$ in the functions on $\mathfrak{H}$, obtained with the involution ι in (1.6) can be written as $\gamma \mapsto h|_{1,2}(1-\gamma)$, with the 2-harmonic function $h(z) = \frac{i}{2y}$.

The holomorphic Eisenstein series of weight 2

$$E_2(z) \;=\; 1 - 24 \sum_{n\geq 1} \sigma_1(n)\, e^{2\pi i n z} \tag{5.3}$$

is not a modular form. Adding a multiple of h we get $E_2^* = \frac{6i}{\pi}\, h + E_2$, which is a harmonic modular form in $\mathrm{Harm}_2(\Gamma(1),1)$. The function E_2^* is a 2-harmonic lift of the constant function $\frac{3}{\pi}$. See Definition 1.17. Furthermore, we have

$$\iota\tilde\psi_\gamma \;=\; \frac{\pi}{6i}\, E_2|_{1,2}(\gamma-1)\,.$$

Since E_2 has polynomial growth near the boundary $\mathbb{P}^1_\mathbb{R}$ of $\mathfrak{H}$, we conclude that with $b = \frac{\pi i}{6}\,\iota E_2 \in \mathcal{D}_{1,2}^{-\infty}$ we obtain $b|_{1,2}(\gamma-1) = \tilde\psi_\gamma$. So the class $\mathbf{r}_0^\omega 1$ becomes trivial under the natural map to $H^1(\Gamma(1);\mathcal{D}_{1,2}^{-\infty})$.

Alternative description of cocycles. Generalizing this example, we first use the differential form $\omega_r(F;\cdot,\cdot)$ in (2.1) to describe the cocycle $\psi_F^{z_0}$ in (2) in an alternative way. We recall the involution ι in (1.6).

Lemma 5.1. *Let $r\in\mathbb{C}$ and $F \in A_r(\Gamma, v)$. We put for $t\in\mathfrak{H}^-$:*

$$Q_F(t) \;:=\; \int_{z_0}^{\bar t} \omega_r(F;t,z)\,. \tag{5.4}$$

i) *The function Q_F on $\mathfrak{H}^-$ satisfies*

$$Q_F|_{v,2-r}(\gamma-1) \;=\; \psi_{F,\gamma}^{z_0} \qquad \text{for each } \gamma\in\Gamma. \tag{5.5}$$

ii) *The corresponding function ιQ_F on $\mathfrak{H}$ is $(2-\bar r)$-harmonic, and*

$$\xi_{2-\bar r}\,\iota\, Q_F(z) \;=\; 2^{r-1}\, e^{\pi i (r-1)/2}\, F(z)\,. \tag{5.6}$$

Proof. For Part i) we use Part iii) of Lemma 2.3. For $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$

$$\begin{aligned} Q_F|_{v,2-r}\gamma\,(t) \;&=\; v(\gamma)^{-1}\,(ct+d)^{r-2}\, Q_F(\gamma t)\\ &=\; \int_{z_0}^{\gamma\bar t} \omega_r(F;\cdot,z)|_{v,2-r}\gamma\,(t) \;=\; \int_{\gamma^{-1}z_0}^{\bar t} \omega_r(F;t,z)\,. \end{aligned}$$

For Part ii) we note that the function ιQ_F on $\mathfrak{H}$ satisfies

$$\overline{\iota Q_F(z)} \;=\; \int_{\tau=z_0}^{z} (\tau-\bar z)^{r-2}\, F(\tau)\, d\tau\,,$$

$$\begin{aligned} \xi_{2-\bar r}\, \iota Q_F(z) \;&=\; 2i\, y^{2-r}\, \frac{\partial}{\partial z} \overline{\iota Q_F(z)} \;=\; 2i\, y^{2-r}\, (z-\bar z)^{r-2}\, F(z)\\ &=\; 2^{r-1}\, e^{\pi i (r-1)/2}\, F(z)\,. \end{aligned}$$

Since the image of ιQ_F under the shadow operator is holomorphic, the function ιQ_F is in $\mathcal{H}_{2-\bar r}(\mathfrak{H})$. □

Proof of Theorem C. The theorem states the equivalence of two statements concerning an automorphic form $F \in A_r(\Gamma, v)$. Statement a) means that the cocycle $\psi_F^{z_0}$ is a coboundary in $B^1(\Gamma; \mathcal{D}^{-\omega}_{v,2-r})$. This is equivalent to

a') $\exists \Phi \in \mathcal{O}(\mathfrak{H}^-)\ \forall \gamma \in \Gamma\ :\ \Phi|_{v,2-r}(\gamma - 1) \;=\; \psi^{z_0}_{F,\gamma}$.

Statement b) amounts to the existence of a $(2-\bar r)$-harmonic lift of F, and is equivalent to

b') $\exists H \in \mathrm{Harm}_{2-\bar r}(\Gamma, \bar v)\ :\ \xi_r H = F$.

We relate these two statements by a chain of intermediate equivalent statements s1)–s6).

s1) $\exists \Phi \in \mathcal{O}(\mathfrak{H}^-)\ \forall \gamma \in \Gamma\ :\ \Phi|_{v,2-r}(\gamma-1) \;=\; Q_F|_{v,2-r}(\gamma-1)$.

Relation (5.5) implies the equivalence of a') and s1).

We rewrite s1) as follows:

s2) $\exists \Phi \in \mathcal{O}(\mathfrak{H}^-)\ \forall \gamma \in \Gamma\ :\ (\Phi - Q_F)|_{v,2-r}\gamma = \Phi - Q_F$.

Functions on $\mathfrak{H}^-$ and $\mathfrak{H}$ are related by the involution ι in (1.6), which preserves holomorphy. So s2) is equivalent to the following statement:

s3) $\exists M \in \mathcal{O}(\mathfrak{H})\ \forall \gamma \in \Gamma\ :\ (M - \iota Q_F)|_{\bar v, 2-\bar r}\gamma = M - \iota Q_F$.

The holomorphy of M is equivalent to the vanishing of $\xi_{2-\bar r} M$. Hence s3) is equivalent to the following statement:

s4) There is a function M on $\mathfrak{H}$ such that $\forall \gamma \in \Gamma\ :\ (M - \iota Q_F)|_{\bar v,2-\bar r}\gamma = M - \iota Q_F$ and $\xi_{2-\bar r} M = 0$.

Now we relate functions H and M on $\mathfrak{H}$ by $H = M - \iota Q_F$. With (5.6) this shows that s4) is equivalent to the following statement:

s5) There is a function H on $\mathfrak{H}$ such that $\forall \gamma \in \Gamma\ :\ H|_{\bar v 2-\bar r}\gamma = H$ and $\xi_{2-\bar r} H = -2^{r-1} e^{\pi i (r-1)/2} F$.

The statement that $\xi_r H$ is holomorphic is equivalent to the statement that H is $(2-\bar r)$-harmonic. Hence we get the equivalent statement:

s6) $\exists H \in \mathrm{Harm}_{2-\bar r}(\Gamma, \bar v)\ :\ \xi_{2-\bar r} H = -2^{r-1} e^{\pi i (r-1)/2} F$.

Up to replacing H by a non-zero multiple, statement s6) is equivalent to statement b'). □

Remark. The r-harmonic function Q_F in (5.4) describes the cocycle $\psi_F^{z_0}$ by the relation (5.5). Theorem C relates the existence of a holomorphic function also describing $\psi_F^{z_0}$ to the existence of a r-harmonic lift. One may call such holomorphic functions *automorphic integrals*. In the work of Knopp [66] and others there is the additional requirement that automorphic integrals are invariant under T.

Consequences. Kra's result [75, Theorem 5] is equivalent to the statement that $H^1(\Gamma;\mathcal{D}^{-\omega}_{1,2-r}) = \{0\}$ for even weights r. So we have the following direct consequence of Theorem C.

Corollary 5.2. *Let $r \in 2\mathbb{Z}$ and let v be the trivial multiplier system. Then each automorphic form in $A_r(\Gamma, 1)$ has a harmonic lift in* $\mathrm{Harm}_{2-r}(\Gamma, 1)$.

A bit more work is needed for the following consequence of Theorem C.

Theorem 5.3. *Let v be a unitary multiplier system for the weight $r \in \mathbb{R}$. If each cusp form in $S_r(\Gamma, v)$ has a $(2-r)$-harmonic lift, then each unrestricted holomorphic automorphic form in $A_r(\Gamma, v)$ has a $(2 - r)$-harmonic lift.*

Proof. Comparing our results with the Theorem of Knopp and Mawi [71], reformulated as Theorem 2.7 above, we noted that the diagram (2.10) shows that for real r and unitary v we can decompose $A_r(\Gamma, v) = S_r(\Gamma, v) \oplus X$, where X is the kernel of the composition

$$A_r(\Gamma, v) \xrightarrow{\mathbf{r}_r^\omega} H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r}) \to H^1(\Gamma;\mathcal{D}^{-\infty}_{v,2-r})\,.$$

So all elements of this space X have $(2 - \bar{r})$-harmonic lifts, which are $(2 - r)$-harmonic lifts, since here the weight is real. So if one can lift cusps forms, one can lift all elements of $A_r(\Gamma, v)$. □

5.3. **Related work.** Knopp [66, §V.2] discussed the question how far the module has to be extended before a cocycle attached to an automorphic forms becomes a coboundary.

The relation between harmonic automorphic forms, automorphic integrals and cocycles for the shadow is mentioned by Fay on p. 145 of [46].

Bruinier and Funke [17] explicitly considered the shadow operator and the question whether harmonic lifts exist. Existence of harmonic lifts is often shown with help of real-analytic Poincaré series with exponential growth, introduced by Niebur [93]. For instance, Bringmann and Ono [3] (cusp forms for $\Gamma_0(N)$ weight $\frac{1}{2}$), Bruinier, Ono and Rhoades [18] (integral weights at least 2), Jeon, Kang and Kim [60] (weight $\frac{3}{2}$, exponential growth), Duke, Imamoḡlu, and Tóth [42] (weight 2). The approach in [16] (modular forms of complex weight with at most exponential growth) is similar; it uses no Poincaré series but similar meromorphic families. Bruinier and Funke [17, Corollary 3.8] use Hodge theory for the the existence of r-harmonic lifts, and Bringmann, Kane and Zwegers [5, §3, §5] explain how to employ holomorphic projection for this purpose.

The harmonic lifts are related to "mock automorphic forms". For a given unrestricted holomorphic automorphic form $F \in A_r(\Gamma, v)$ it is relatively easy to write down a harmonic function C such that $\xi_{2-\bar{r}}\, C = F$. The function ιQ_F in (5.4) is an example. Any holomorphic function M such that $M + C$ is a harmonic automorphic form may be called a mock automorphic form. The function E_2 in (5.3) is a well known example. In the last ten years a vast literature on mock automorphic forms has arisen. For an overview we mention [47, 123].

It should be stressed that our Theorem C concerns the existence of harmonic lifts and of automorphic integrals. An enjoyable aspect of the theory is the large number of mock modular forms with a explicit, number-theoretically nice description, often with weights $\frac{1}{2}$ and $\frac{3}{2}$, related to functions on the Jacobi group. See, for instance, [29, 30, 31]). This leads to explicit harmonic lifts, and via Theorem C to the explicit description of cocycles as coboundaries.

6. Boundary germs

To complete the proof of Theorem A we have to show that each cohomology class in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$ is of the form $\mathbf{r}^{\omega}_r F$ for some unrestricted holomorphic automorphic form. To do this, we use the spaces of "analytic boundary germs", in Definition 6.3. This allows us to define, for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, Γ-modules isomorphic to $\mathcal{D}^{\omega}_{v,2-r}$ and $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$, consisting of germs of functions. These germs are sections of a sheaf on the common boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}^-$ and $\mathfrak{H}$. Using these isomorphic modules we will be able, in Section 10, to complete the proof of Theorem A.

6.1. **Three sheaves on the real projective line.**

6.1.1. *The sheaf of real-analytic functions on* $\mathbb{P}^1_{\mathbb{R}}$. Recall that $\mathcal{O}$ denotes the sheaf of holomorphic functions on $\mathbb{P}^1_{\mathbb{C}}$.

Definition 6.1. For each open set $I \subset \mathbb{P}^1_{\mathbb{R}}$ we define the sheaf $\mathcal{V}^{\omega}_{2-r}$ by

$$\mathcal{V}^{\omega}_{2-r}(I) \;:=\; \varinjlim \mathcal{O}(U)\,, \tag{6.1}$$

where U runs over the open neighbourhoods of I in $\mathbb{P}^1_{\mathbb{C}}$. The operator $|^{\mathrm{prj}}_{2-r} g$ in (1.20) gives a linear bijection $\mathcal{V}^{\omega}_{2-r}(I) \to \mathcal{V}^{\omega}_{2-r}(g^{-1}I)$ for each $g \in \mathrm{SL}_2(\mathbb{R})$.

The sections in $\mathcal{V}^{\omega}_{2-r}(I)$ for I open in $\mathbb{P}^1_{\mathbb{R}}$ are holomorphic on some neighbourhood of I in $\mathbb{P}^1_{\mathbb{C}}$, and hence have a real-analytic restriction to I. Conversely, any real-analytic function on I is locally given by a convergent power series, and hence extends as a holomorphic function to some neighbourhood of I. So we can view $\mathcal{V}^{\omega}_{2-r}$ for each r as the sheaf of real-analytic functions on $\mathbb{P}^1_{\mathbb{R}}$, provided with the operators $|^{\mathrm{prj}}_{2-r} g$ with $g \in \mathrm{SL}_2(\mathbb{R})$.

The space of global sections $\mathcal{V}^{\omega}_{2-r}(\mathbb{P}^1_{\mathbb{R}})$ contains a copy of $\mathcal{D}^{\omega}_{2-r}$. Indeed, the map $(\mathsf{prj}_{2-r}\varphi)(t) = (i-t)^{2-r}\,\varphi(t)$ induces the injection

$$\mathsf{prj}_{2-r} : \mathcal{D}^{\omega}_{2-r} \to \mathcal{V}^{\omega}_{2-r}(\mathbb{P}^1_{\mathbb{R}})$$

that intertwines the operators $|_{2-r} g$ and $|^{\mathrm{prj}}_{2-r} g$ for $g \in \mathrm{SL}_2(\mathbb{R})$. It further induces a morphism of Γ-modules $\mathsf{prj}_{2-r} : \mathcal{D}^{\omega}_{v,2-r} \to \mathcal{V}^{\omega}_{v,2-r}(\mathbb{P}^1_{\mathbb{R}})$ and an injective map from $\mathcal{D}^{\omega}_{2-r}[\xi_1, \ldots, \xi_n]$ into $\mathcal{V}^{\omega}_{2-r}\left(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}\right)$.

6.1.2. *The sheaf of harmonic boundary germs.* We recall that $\mathcal{H}_r$ denotes the sheaf of r-harmonic functions on $\mathfrak{H}$.

Definition 6.2. For open $I \subset \mathbb{P}^1_{\mathbb{R}}$

$$\mathcal{B}_r(I) \;:=\; \varinjlim \mathcal{H}_r(U \cap \mathfrak{H})\,, \tag{6.2}$$

where U runs over the open neighbourhoods of I in $\mathbb{P}^1_{\mathbb{C}}$. The induced sheaf $\mathcal{B}_r$ on $\mathbb{P}^1_{\mathbb{R}}$ is called the sheaf of *r-harmonic boundary germs*.

The operator $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$ induces linear bijections $\mathcal{B}_r(I) \to \mathcal{B}_r(g^{-1}I)$ for open $I \subset \mathbb{P}^1_{\mathbb{R}}$.

We identify $\mathcal{H}_r(\mathfrak{H})$ with its image in $\mathcal{B}_r(\mathbb{P}^1_{\mathbb{R}})$.

6.1.3. *The sheaf of analytic boundary germs.* We now turn to the boundary germs that are most useful for the purpose of this paper.

Definition 6.3. Let $r \in \mathbb{C}$.

i) Consider the real-analytic function f_r on $\mathfrak{H} \smallsetminus \{i\}$ given by

$$f_r(z) \;:=\; \frac{2i}{z-i}\Bigl(\frac{\bar z - i}{\bar z - z}\Bigr)^{r-1}\,. \tag{6.3}$$

ii) For open $U \subset \mathbb{P}^1_{\mathbb{C}}$ we define

$$\mathcal{H}^{\mathrm{b}}_r(U) := \{F \in \mathcal{H}_r(U \cap \mathfrak{H}) \;:\; F/f_r \text{ has a real-analytic continuation to } U\}\,. \tag{6.4}$$

iii) For open sets $I \subset \mathbb{P}^1_{\mathbb{R}}$ we define

$$\mathcal{W}^{\omega}_r(I) \;:=\; \varinjlim \mathcal{H}^{\mathrm{b}}_r(U)\,, \tag{6.5}$$

where U runs over the open neighbourhoods of I in $\mathbb{P}^1_{\mathbb{C}}$. This defines a subsheaf $\mathcal{W}^{\omega}_r$ of $\mathcal{B}_r$, called the sheaf of *analytic boundary germs*.

Remark 6.4. The function f_r is analogous to the function $z \mapsto \bigl(\frac{4y}{|z+i|^2}\bigr)^s$ (or to $w \mapsto (1-|w|^2)^s$ in the disk model) in [13], Definition 5.2 and the examples after Equation (5.9).

The function f_r can be written as $cq(z)^{1-r}\,\frac{z+i}{z-i}\,(z+i)^{2-r}$, where $q(z) = \frac{y}{|z+i|^2}$ is a real-valued real-analytic function on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{-i\}$ with $\mathbb{P}^1_{\mathbb{R}}$ as its zero set, and c is some factor in $\mathbb{C}^*$.

The motivation for our choice of f_r is that it is the right choice to make Proposition 6.6 below work.

Lemma 6.5. *Let $r \in \mathbb{C}$. For each $g \in \mathrm{SL}_2(\mathbb{R})$ the operator $|_r g$ induces a linear bijection $\mathcal{W}^{\omega}_r(I) \to \mathcal{W}^{\omega}_r(g^{-1}I)$.*

Proof. Let $g = \bigl(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\bigr) \in \mathrm{SL}_2(\mathbb{R})$. We have

$$(cz+d)^{-r}\frac{f_r(gz)}{f_r(z)} \;=\; (a-ic)^{r-2}\;\frac{z-i}{z-g^{-1}i}\;\Bigl(\frac{\bar z - i}{\bar z - g^{-1}i}\Bigr)^{1-r}\,. \tag{6.6}$$

To see this up to a factor depending on the choice of the arguments is just a computation. Both sides of the equality are real-analytic in $z \in \mathfrak{H} \smallsetminus \{i, g^{-1}i\}$ and in g in the dense open set $G_0 \subset \mathrm{SL}_2(\mathbb{R})$, defined in (1.3). The equality holds for $g = \mathrm{I}$, hence for $g \in G_0$. Elements in $\mathrm{SL}_2(\mathbb{R}) \smallsetminus G_0$ are approached with $c \downarrow 0$, and a and d tending to negative values. The argument conventions §1.1 and Proposition 1.5 are such that both $(cz+d)^{-r}$ and $(a-ic)^{r-2}$ are continuous under this approach.

Suppose that $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ represents a section in $\mathcal{W}_r^\omega(I)$, with $I = U \cap \mathbb{P}^1_{\mathbb{R}}$ and $A = F/f_r$ real-analytic on $U \subset \mathbb{P}^1_{\mathbb{R}}$. Then we have

$$F|_r g(z) = f_r(z)\,(cz+d)^{-r}\,\frac{f_r(gz)}{f_r(z)}\,A(gz)\,,$$

which is of the form f_r times a real-analytic function near $g^{-1}I$. $\square$

Examples. (a) Consider the r-harmonic function

$$F(z) \;=\; y^{1-r} \tag{6.7}$$

on $\mathfrak{H}$. Since

$$A(z) = \frac{F(z)}{f_r(z)} \;=\; \frac{z-i}{2i}\Bigl(\frac{\bar z - i}{-2i}\Bigr)^{1-r}$$

extends as a real-analytic function to $U = \mathbb{C} \smallsetminus \{i\}$, the function F is in $\mathcal{H}_r^{\mathrm{b}}(\mathbb{C} \smallsetminus \{i\})$. It is not in $\mathcal{H}_r^{\mathrm{b}}\bigl(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i\}\bigr)$, since A is not given by a convergent power series in $1/z$ and $1/\bar z$ on a neighbourhood of ∞ in $\mathbb{P}^1_{\mathbb{C}}$. So the function F represents an element of $\mathcal{W}_r^\omega(\mathbb{R})$.

(b) For $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\ge 2}$ and $\mu \in \mathbb{Z}_{\ge 0}$

$$\mathrm{M}_{r,\mu}(z) \;:=\; f_r(z)\Bigl(\frac{z-i}{z+i}\Bigr)^{\mu+1}\,{}_2F_1\Bigl(1+\mu, 1-r; 2-r; \frac{4y}{|z+i|^2}\Bigr)\,. \tag{6.8}$$

At this moment we only state that $\mathrm{M}_{r,\mu}$ is r-harmonic on $\mathfrak{H} \smallsetminus \{i\}$, and postpone giving arguments for this statement till §7. The function $z \mapsto \frac{4y}{|z+i|^2} = 1 - \Bigl|\frac{z-i}{z+i}\Bigr|^2$ is real-analytic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{-i\}$, with value 0 on $\mathbb{P}^1_{\mathbb{R}}$. Since the hypergeometric function is holomorphic on the unit disk in $\mathbb{C}$, this implies that $\Bigl(\frac{z-i}{z+i}\Bigr)^{-\mu-1}\mathrm{M}_{r,\mu}(z)$ is real-analytic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i, -i\}$, and hence $\mathrm{M}_{r,\mu}$ is in $\mathcal{H}_r^{\mathrm{b}}\bigl(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i,-i\}\bigr)$, and represents an element of $\mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$.

(c) Let $\operatorname{Re} r > 0$. Then $\eta(z)^{2r} = e^{2r \log \eta(z)}$ is a cusp form of weight r for the modular group $\Gamma(1)$. The function

$$\Phi(z) \;=\; \int_0^{i\infty} \eta^{2r}(\tau)\,\frac{2i}{z-\tau}\Bigl(\frac{\bar z - \tau}{\bar z - z}\Bigr)^{r-1}\,d\tau\,, \tag{6.9}$$

defines an r-harmonic function on $\mathfrak{H} \smallsetminus i(0,\infty)$, if we take the path of integration along the geodesic from 0 to ∞. (To check the harmonicity one may apply ξ_r; this gives a holomorphic function.) Deforming the path of integration leads to other domains. Such a change in the function does not change the r-harmonic boundary germ in $\mathcal{W}_r^\omega(\mathbb{R})$ it represents.

6.2. **Relation between the sheaves of harmonic boundary and analytic boundary germs.** The sheaf $\mathcal{W}_r^\omega$ is related to the simpler sheaf $\mathcal{V}_{2-r}^\omega$ by the important *restriction morphism* that we will define now.

Proposition 6.6. *Let $r \in \mathbb{C}$. There is a unique morphism of sheaves $\rho_r^{\mathrm{prj}} : \mathcal{W}_r^\omega \to \mathcal{V}_{2-r}^\omega$ with the following property: If $f \in \mathcal{W}_r^\omega(I)$ for an open set $I \subset \mathbb{P}^1_\mathbb{R}$ is represented by $F \in \mathcal{H}_r^{\mathrm{b}}(U)$, and $\rho_r^{\mathrm{prj}} f \in \mathcal{V}_{2-r}^\omega(I)$ is represented by $\varphi \in \mathcal{O}(U_0)$ for some open neighbourhood U_0 of I in $\mathbb{P}^1_\mathbb{C}$, then the real-analytic function F/f_r on U is related to φ by*

$$(F/f_r)(t) \;=\; \varphi(t) \qquad \text{for } t \in I\,.$$

This is called the restriction morphism *and is compatible with the actions of* $\mathrm{SL}_2(\mathbb{R})$*:*

$$\rho_r^{\mathrm{prj}}(f|_r g) = (\rho_r^{\mathrm{prj}} f)|_{2-r}^{\mathrm{prj}} g \qquad \text{for } f \in \mathcal{W}_r^\omega(I) \text{ and } g \in \mathrm{SL}_2(\mathbb{R})\,. \tag{6.10}$$

Proof. Let $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ for some neighbourhood U of I in $\mathbb{P}^1_\mathbb{C}$. Then $A := F/f_r$ on $U \cap \mathfrak{H}$ extends as a real-analytic function to U. If we replace F by another representative $F_1 \in \mathcal{H}_r^{\mathrm{b}}(U_1)$ of the same element of $\mathcal{W}_r^\omega(I)$, then F_1 and F have the same restriction to $U_2 \cap \mathfrak{H}$ for a connected neighbourhood $U_2 \subset U \cap U_1$ of I in $\mathbb{P}^1_\mathbb{C}$. Since U_2 is connected, the functions A and A_1 extend uniquely to U_2, and hence to $I \subset U_2$. We thus obtain a well-defined function on I which has further a holomorphic extension to some neighbourhood U_0 of I in $\mathbb{P}^1_\mathbb{C}$ since it is real-analytic on I. Hence it represents an element of $\mathcal{V}_{2-r}^\omega(I)$ that is uniquely determined by the element $f \in \mathcal{W}_r^\omega(I)$ represented by F.

This defines $\rho_r^{\mathrm{prj}} : \mathcal{W}_r^\omega(I) \to \mathcal{V}_{2-r}^\omega(I)$. We have compatibility with the restriction maps associated to $I_1 \subset I$, and hence obtain a morphism of sheaves.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. From relation (6.6) we see that the map $\mathcal{W}_r^\omega(I) \to \mathcal{W}_r^\omega(g^{-1}I)$ determined by $F \mapsto F|_r g$ sends F/f_r to the real-analytic function

$$\begin{aligned}\Bigl(\frac{F|_r g}{f_r}\Bigr)(z) \;&=\; \frac{(cz+d)^{-r}\,F(gz)}{f_r(z)} \;=\; (cz+d)^{-r}\,\frac{f_r(gz)}{f_r(z)}\,(F/f_r)(gz)\\ &=\; (a-ic)^{r-2}\,\frac{z-i}{z-g^{-1}i}\Bigl(\frac{\bar z - i}{\bar z - g^{-1} i}\Bigr)^{1-r}\,(F/f_r)(gz)\,.\end{aligned}$$

For $z = t \in I$, this equals $((F/f_r)|_{2-r}^{\mathrm{prj}} g)(t)$ by (1.20). Thus, we obtain (6.10). □

Illustration. Let I be an interval in $\mathbb{P}^1_\mathbb{R}$, and let U be an open neighbourhood of I in $\mathbb{P}^1_\mathbb{C}$. For representatives F of $f \in \mathcal{W}_r^\omega(I)$ a representative φ of the image $\rho_r^{\mathrm{prj}} f \in \mathcal{V}_{2-r}^\omega$ is obtained by a sequence of extensions and restrictions. See Figure 9.

Examples. (a) The restriction of $F(z) = y^{1-r}$ in (6.7) is

$$(\rho_r^{\mathrm{prj}} F)(t) = \frac{t-i}{2i}\Bigl(\frac{t-i}{-2i}\Bigr)^{1-r} = -\Bigl(\frac{t-i}{-2i}\Bigr)^{2-r}\,.$$

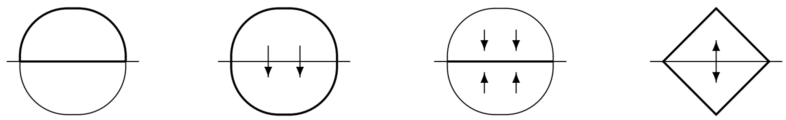

FIGURE 9. ρ_r^{prj} as a sequence of extensions and restrictions.

(b) For $\mathrm{M}_{r,\mu}$ in (6.8) we use that $\frac{4y}{|z+i|^2} = 0$ on $\mathbb{P}^1_{\mathbb{R}}$ and that ${}_2F_1(\cdot,\cdot;\cdot;0) = 1$ to obtain

$$(\rho_r^{\mathrm{prj}} \mathrm{M}_{r,\mu})(t) = \Bigl(\frac{t-i}{t+i}\Bigr)^{\mu+1} . \tag{6.11}$$

6.3. **Kernel function for the map from automorphic forms to boundary germ cohomology.**

Proposition 6.7. *For $r \in \mathbb{C}$ let K_r be the function on $(\mathfrak{H} \times \mathfrak{H}) \smallsetminus (diagonal)$ given by:*

$$K_r(z;\tau) \ := \ \frac{2i}{z-\tau} \Bigl(\frac{\bar z - \tau}{\bar z - z}\Bigr)^{r-1} . \tag{6.12}$$

For each $z \in \mathfrak{H}$ the function $K_r(z;\cdot)$ is holomorphic on $\mathfrak{H} \smallsetminus \{z\}$, and for each $\tau \in \mathfrak{H}$ the function $K_r(\cdot;\tau)$ is r-harmonic on $\mathfrak{H} \smallsetminus \{\tau\}$ and represent an element of $\mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$ with restriction

$$(\rho_r^{\mathrm{prj}} K_r(\cdot;\tau))(t) \ = \ \Bigl(\frac{\tau - t}{i-t}\Bigr)^{r-2} . \tag{6.13}$$

For each $g \in \mathrm{SL}_2(\mathbb{R})$ it satisfies

$$K_r(\cdot;\cdot)\,|_r g \otimes |_{2-r} g \ = \ K_r . \tag{6.14}$$

Remark. In (6.14) we use $K_r(\cdot;\cdot)|_r g \otimes |_{2-r} g(z,\tau) = (cz+d)^{-r}\,(c\tau+d)^{r-2}\,K_r(gz;g\tau)$ for $g = \bigl(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\bigr)$.

Proof. The shadow operator, defined in (1.28), gives

$$(\xi_r K_r(\cdot;\tau))(z) \ = \ (\bar r - 1)\Bigl(\frac{z - \bar\tau}{2i}\Bigr)^{\bar r - 2} . \tag{6.15}$$

The result is holomorphic in z, hence $z \mapsto K_r(z;\tau)$ is r-harmonic.

The quotient

$$K_r(z;\tau)/f_r(z) \ = \ \frac{i-z}{\tau - z}\Bigl(\frac{\tau - \bar z}{i - \bar z}\Bigr)^{r-1} . \tag{6.16}$$

extends real-analytically in z across $\mathbb{P}^1_{\mathbb{R}}$, to $U_\tau = \mathbb{P}^1_{\mathbb{C}} \smallsetminus (\{\tau\} \cup p)$, where p is a path in $\mathfrak{H}^-$ from $-i$ to $\bar\tau$. So $K_r(\cdot;\tau) \in \mathcal{H}_r^{\mathrm{b}}(U_\tau)$ represents an analytic boundary germ on $\mathbb{P}^1_{\mathbb{R}}$. On $\mathbb{P}^1_{\mathbb{R}}$ the values of z and $\bar z$ coincide, and we find the restriction in (6.13).

For the equivariance in (6.14) we check by a computation similar to the computation in the proof of Lemma 6.5 that

$$(cz+d)^{-r}\,(c\tau+d)^{r-2}\,K_r(gz;g\tau) \;=\; K_r(z;\tau)\,. \qquad \square$$

Remark. The restriction $\rho_r^{\mathrm{prj}} K_r(\cdot;\tau)$ in (6.13) gives a function in $\mathsf{prj}_{2-r}\mathcal{D}^{\omega}_{2-r}$. Since it is convenient to work with $\mathcal{D}^{\omega}_{2-r}$ itself, we introduce the following operator:

Definition 6.8. We set

$$\rho_r \;:=\; \mathsf{prj}_{2-r}^{-1}\,\rho_r^{\mathrm{prj}} \tag{6.17}$$

So $(\rho_r f)(t) \;=\; (i-t)^{r-2}\,(\rho_r^{\mathrm{prj}} f)(t)$, and we find

$$(\rho_r K_r(\cdot;\tau))(t) \;=\; (\tau-t)^{r-2}\,. \tag{6.18}$$

This shows that the kernel $K_r(\cdot;\cdot)$ is analogous to the kernel function $(z,t)\mapsto(z-t)^{r-2}$ in the Eichler integral. For fixed $\tau\in\mathfrak{H}$ the representative

$$z\mapsto K_r(z;\tau)$$

of an element of $\mathcal{W}^{\omega}_r(\mathbb{P}^1_{\mathbb{R}})$ is sent by the restriction map to the representative

$$t\mapsto (z-t)^{r-2}$$

of an element of $\mathcal{D}^{\omega}_{2-r}$.

Definition 6.9. Let $F\in A_r(\Gamma,v)$. We put for $z_0\in\mathfrak{H}$ and $\gamma\in\Gamma$:

$$c^{z_0}_{F,\gamma}(z) \;:=\; \int_{\gamma^{-1}z_0}^{z_0} K_r(z;\tau)\,F(\tau)\,d\tau\,. \tag{6.19}$$

Proposition 6.10. *Let $r\in\mathbb{C}$, $z_0\in\mathfrak{H}$, and $F\in A_r(\Gamma,v)$.*

i) *The map $\gamma\mapsto c^{z_0}_{F,\gamma}$ defines a cocycle $c^{z_0}_F\in Z^1(\Gamma;\mathcal{W}^{\omega}_r(\mathbb{P}^1_{\mathbb{R}}))$.*
ii) *The cohomology class $\mathbf{q}^{\omega}_r F \;:=\; [c^{z_0}_F]$ in $H^1(\Gamma;\mathcal{W}^{\omega}_{v,r}(\mathbb{P}^1_{\mathbb{R}}))$ does not depend on the base point $z_0\in\mathfrak{H}$.*
iii) *With the natural map $H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r})\to H^1(\Gamma;\mathcal{V}^{\omega}_{2-r}(\mathbb{P}^1_{\mathbb{R}}))$ corresponding to $\mathsf{prj}_{2-r}:\mathcal{D}^{\omega}_{v,2-r}\to\mathcal{V}^{\omega}_{v,2-r}$, the following diagram commutes:*

$$\begin{tikzcd} A_r(\Gamma,v) \arrow[r, "\mathbf{r}^{\omega}_r"] \arrow[drr, "\mathbf{q}^{\omega}_r"] & H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r}) \arrow[r, "\mathsf{prj}_{2-r}"] & H^1(\Gamma;\mathcal{V}^{\omega}_{v,2-r}(\mathbb{P}^1_{\mathbb{R}})) \\ & & H^1(\Gamma;\mathcal{W}^{\omega}_{v,r}(\mathbb{P}^1_{\mathbb{R}})) \arrow[u, "\rho^{\mathrm{prj}}_r"'] \end{tikzcd} \tag{6.20}$$

Proof. Proposition 6.7 shows that the differential form $K_r(z;\tau)\,F(\tau)\,d\tau$ has properties analogous to those of $\omega_r(F;t,\tau)$ in §2.1. The proof of Parts i) and ii) goes along the same lines as the proof of Proposition 2.4. For Part iii) use (6.13). $\square$

Thus, we see that the map $\mathbf{r}^{\omega}_r$ to cohomology in Theorem A is connected to the map $\mathbf{q}^{\omega}_r$ to boundary germ cohomology by the restriction map $\rho^{\mathrm{prj}} r$. However, in Theorem A the basic module is $\mathcal{D}^{\omega}_{v,2-r}$ and not the larger module $\mathcal{V}^{\omega}_{2-r}(\mathbb{P}^1_{\mathbb{R}})$. We need to study the boundary germs more closely, in order to identify inside $\mathcal{W}^{\omega}_{v,r}(\mathbb{P}^1_{\mathbb{R}})$ a smaller module that can play the role of $\mathcal{D}^{\omega}_{v,2-r}$.

6.4. **Local study of the sheaf of analytic boundary germs.**

6.4.1. *Positive integral weights.* At many places in this section positive integral weights require separate treatment. A reader wishing to avoid these complications may want to concentrate on the general case of weights in $\mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$.

The next definition will turn out to be relevant for weights in $\mathbb{Z}_{\geq 1}$ only.

Definition 6.11. For U open in $\mathbb{P}^1_{\mathbb{C}}$ let

$$\begin{aligned} \mathcal{H}^{\mathrm{h}}_r(U) \;:=\; \{F \in \mathcal{O}(U \cap \mathfrak{H}) \cap \mathcal{H}^{\mathrm{b}}_r(U) \;:\; \\ F \text{ has a holomorphic extension to } U\}\,. \end{aligned} \tag{6.21}$$

Lemma 6.12. *Let $r \in \mathbb{Z}_{\geq 1}$. For open sets $U \subset \mathbb{P}^1_{\mathbb{R}}$ such that $U \cap \mathbb{P}^1_{\mathbb{R}} \neq \emptyset$ and $-i \notin U$ the restriction to $U \cap \mathfrak{H}$ of $F \in \mathcal{O}(U)$ is in $\mathcal{H}^{\mathrm{h}}_r(U)$ if one of the following conditions is satisfied:*

a) *$U \subset \mathbb{C}$,*
b) *$\infty \in U$ and F has at ∞ a zero of order at least r.*

Proof. F is holomorphic on $U \cap \mathfrak{H}$, hence r-harmonic on $U \cap \mathfrak{H}$. For $z \in U \cap \mathfrak{H}$:

$$F(z)/f_r(z) \;=\; \frac{1}{2i}\, F(z)\,(z-i)\,(\bar z - i)^{1-r}\,(\bar z - z)^{r-1} \tag{6.22}$$

$$\;=\; \frac{1}{2i}\,(z^r F(z))\,(1-i/z)\,(1-i/\bar z)^{1-r}\,(1/2 - 1/\bar z)^{r-1}\,. \tag{6.23}$$

Equality (6.22) shows that F/f_r is real-analytic on $U \smallsetminus \{\infty, -i\} = U \smallsetminus \{\infty\}$. If $\infty \in U$ then (6.23) shows that it is also real-analytic on some neighbourhood of ∞. □

6.4.2. *Local structure.* We return to the sheaves $\mathcal{V}^{\omega}_{2-r}$ and $\mathcal{W}^{\omega}_r$, in Definitions 6.1 and 6.5.

The sections of $\mathcal{V}^{\omega}_{r-2}$ are holomorphic on neighbourhoods of open sets $I \subset \mathbb{P}^1_{\mathbb{R}}$, and are locally at $x \in \mathbb{R}$ given by a power series expansion in $z - x$ converging on some open disk with center x. At ∞ we have a power series in z^{-1}. The real-analytic functions $A = F/f_r$ corresponding to representatives of sections of $\mathcal{W}^{\omega}_r$ are also given by a power series near $x \in I$, now in two variables, $z - x$ and $\bar z - x$, which also converges on a disk around x. At ∞ we have a power series expansion in $1/z$ and $1/\bar z$.

With the operators $|_r g$ for $g \in \mathrm{SL}_2(\mathbb{R})$ we can construct isomorphisms between the stalks of $\mathcal{W}^{\omega}_r$. So for a local study it suffices to work with a disk around 0. A problem is that the points i and $-i$ play a special role in the function f_r. Hence it is better not to use arbitrary elements of $\mathrm{SL}_2(\mathbb{R})$ to transport points of $\mathbb{P}^1_{\mathbb{R}}$ to 0, but to use $k(\vartheta) = \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right) \in \mathrm{SO}(2)$.

We denote disks around 0 by

$$D_p \;=\; \{z \in \mathbb{C} \;:\; |z| < p\}\,, \tag{6.24}$$

where we take $p \in (0,1)$ to have $\pm i \notin D_p$. All points of $\mathbb{P}^1_{\mathbb{R}}$ are uniquely of the form $k(\vartheta)\,0 = \tan\vartheta$ with $\vartheta \in \mathbb{R} \bmod \pi\mathbb{Z}$. All $k\,D_p \subset \mathbb{P}^1_{\mathbb{C}}$ with $k \in \mathrm{SO}(2)$ do not contain $\pm i$; in general they are Euclidean disks in $\mathbb{C}$. The sets $k\,D_p$ are invariant under complex conjugation.

Proposition 6.13. *Suppose that the set $U \subset \mathbb{P}^1_{\mathbb{C}}$ is of the form $U = kD_p$ with $k \in \mathrm{SO}(2)$, $0 < p < 1$.*

i) Restriction. *Let $r \in \mathbb{C}$. If $F \in \mathcal{H}^b_r(U)$ then the restriction $\rho^{\mathrm{prj}}_r F$ extends as a holomorphic function on U.*

ii) a) *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ then $\mathcal{H}^{\mathrm{h}}_r(U) = \{0\}$.*
   b) *If $r = 1$, then $\mathcal{H}^{\mathrm{b}}_1(U) = \mathcal{H}^{\mathrm{h}}_1(U)$.*
   c) *If $r \in \mathbb{Z}_{\geq 1}$, then $\mathcal{H}^{\mathrm{h}}_r(U) \subset \mathcal{H}^{\mathrm{b}}_r(U)$.*

iii) a) *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, then the restriction map $\rho^{\mathrm{prj}}_r : \mathcal{H}^b_r(U) \to \mathcal{O}(U)$ is bijective.*
   b) *If $r \in \mathbb{Z}_{\geq 2}$ then the following sequence is exact:*

$$0 \to \mathcal{H}^{\mathrm{h}}_r(U) \to \mathcal{H}^{\mathrm{b}}_r(U) \xrightarrow{\rho^{\mathrm{prj}}_r} \mathsf{prj}_{2-r}\mathcal{D}^{\mathrm{pol}}_{2-r} \to 0\,,$$

   where the last space has to be interpreted as the space of functions on U that extend to $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i\}$ as elements of the projective model of $\mathcal{D}^{\mathrm{pol}}_{2-r}$.

iv) Shadow operator. *If $F \in \mathcal{H}^b_r(U)$, then the holomorphic function $\xi_r F \in \mathcal{O}(U \cap \mathfrak{H})$, defined in* (1.28)*, extends holomorphically to U and satisfies*

$$(\xi_r F)(z) \;=\; (\bar r - 1)\Bigl(\frac{z+i}{2i}\Bigr)^{\bar r - 2}\,\overline{\rho^{\mathrm{prj}}_r F(\bar z)} \qquad (z \in U \cap \mathfrak{H})\,. \tag{6.25}$$

Remarks. (a) Part i) shows that for small disks U the restriction of an element of $\mathcal{H}^{\mathrm{b}}_r(U)$ is represented by a function defined on the whole disk U, not just on some unspecified neighbourhood of $U \cap \mathbb{P}^1_{\mathbb{R}}$.
(b) The shadow operator and the restriction morphism were defined in different ways. Part iv) shows that they are related.

Proof. We start with proving the statements for $U = D_p$, and will at the end derive the general case.

Part i), $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. First we consider the case when $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. We'll show that for any $F \in \mathcal{H}^{\mathrm{b}}_r(D_p) \subset \mathcal{H}_r(D_p \cap \mathfrak{H})$ (a representative of) $\rho^{\mathrm{prj}}_r F$ is holomorphically continuable to D_p.

We note that we have $F(z) = y^{1-r}\, B(z)$ where

$$B(z) = \frac{2i}{z-i}\Bigl(\frac{\bar z - i}{-2i}\Bigr)^{r-1}\,\frac{F(z)}{f_r(z)}.$$

Since $F \in \mathcal{H}^b_r(D_p)$, B is real-analytic on D_p and there are coefficients $b_{n,m}$ such that

$$B(z) \;=\; \sum_{n,m\geq 0} b_{n,m} z^n \bar z^m \tag{6.26}$$

converges absolutely on a disk D_{p_1} where $0 < p_1 \le p$. We define

$$b(z) \;:=\; 2iy^r \partial_{\bar z}\big(y^{1-r}B(z)\big) \;=\; 2i\,y\,B_{\bar z}(z) + (r-1)\,B(z) \qquad \text{on } D_p\,.$$

The condition of r-harmonicity on F is equivalent to the antiholomorphicity of b. On D_{p_1} we have, by (6.26), the expansion

$$\begin{aligned} b(z) \;&=\; \sum_{m\ge0} (r-1-m)\,b_{0,m}\,\bar z^m \\ &\quad + \sum_{m\ge0,\,n\ge1} \Big((m+1)\,b_{n-1,m+1} + (r-1-m)\,b_{n,m}\Big) z^n \bar z^m\,. \end{aligned} \tag{6.27}$$

The antiholomorphy implies that for each $n \ge 1$ the coefficient of $z^n\bar z^m$ has to vanish. If $r \notin \mathbb{Z}_{\ge1}$ this gives the relation

$$b_{n,m} \;=\; \frac{(1-r)_m}{m!}\, b_{n+m,0} \tag{6.28}$$

(with the Pochhammer symbol $(a)_m = \prod_{j=0}^{m-1}(a+j)$), and

$$b(z) \;=\; -\sum_{m\ge0} \frac{(1-r)_{m+1}}{m!}\, b_{m,0}\,\bar z^m \qquad (z \in D_{p_1})\,. \tag{6.29}$$

This is the power series of an antiholomorphic function on D_p, hence it converges absolutely on D_p.

Now, if ϕ is a representative of $\rho_r^{\mathrm{prj}} F$, then, for $t \in (-p,p)$,

$$\begin{aligned} \varphi(t) \;=\; -\Big(\frac{t-i}{-2i}\Big)^{2-r} B(t) \;&=\; -\Big(\frac{t-i}{-2i}\Big)^{2-r} \sum_{n,m\ge0} \frac{(1-r)_m}{m!}\, b_{n+m,0}\, t^{n+m} \\ &=\; -\Big(\frac{t-i}{-2i}\Big)^{2-r} \sum_{\ell\ge0} \frac{(2-r)_\ell}{\ell!}\, b_{\ell,0} t^\ell\,. \end{aligned} \tag{6.30}$$

Here we use the well-known formula $\sum_{m=0}^{\ell} \frac{(1-r)_m}{m!} = \frac{(2-r)_\ell}{\ell!}$. A comparison of the absolutely convergent series (6.29) and (6.30) shows that (6.30) also converges absolutely on D_p. This implies that the restriction gives a map $\mathcal{H}_r^b(D_p) \to \mathcal{O}(D_p)$.

Part i), $r \in \mathbb{Z}_{\ge1}$. Secondly we show that in the case $r \in \mathbb{Z}_{\ge1}$ the same conclusion holds. In this case (6.28) is valid for $m \le r-1$. For $m \ge r$ we get successively $b_{n,m} = 0$. Since the corresponding Pochhammer symbols vanish, expansion (6.29) stays valid. We get the same estimate for $b_{n+m,0}$ and arrive at

$$\varphi(t) \;=\; -\Big(\frac{t-i}{-2i}\Big)^{2-r} \begin{cases} \sum_{\ell=0}^{r-2} (1-r)_\ell\, (\ell!)^{-1}\, b_{\ell,0}\, t^\ell & \text{if } r \ge 2\,, \\ \sum_{n=0}^{\infty} b_{n,0}\,(n!)^{-1}\, t^n & \text{if } r = 1\,, \end{cases} \tag{6.31}$$

on D_p. This completes the proof of Part i) for $U = D_p$.

Part ii) a) and c). Let $F \in \mathcal{O}(U)$, and suppose that its restriction to $U \cap \mathfrak{H}$ is in $\mathcal{H}_r(U)$, then

$$F(z)/f_r(z) \;=\; \frac{z-i}{2i}\Big(\frac{\bar z - i}{-2i}\Big) F(z)\, y^{r-1}\,.$$

If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, then the presence of the factor y^{1-r} shows that this can be real-analytic near 0 only if $F = 0$. This implies Part a). If $r \in \mathbb{Z}_{\geq 2}$, all factors are real-analytic. This implies Part c).

Part ii) b). If $r = 1$, all $b_{n,m}$ with $m \geq 1$ vanish. Hence $B(z)$ and $F(z) = y^{1-1}\, B(z)$ are holomorphic. This gives Part ii)b).

Part iii) a), surjectivity. In the case when $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, take $\varphi \in \mathcal{O}(D_p)$, which is represented by an absolutely convergent power series

$$\varphi(t) \;=\; \sum_{\ell \geq 0} a_\ell\, t^\ell\,. \tag{6.32}$$

Hence $a_\ell = \mathrm{O}(c^{-\ell})$ for all $c \in (0, p)$. We put

$$b_{n,m} \;=\; \frac{(1-r)_m\,(n+m)!}{m!\,(2-r)_{n+m}}\, a_{n+m}\,, \tag{6.33}$$

and define B by (6.26) with these coefficients $b_{n,m}$. The factor $\frac{(1-r)_m\,(n+m)!}{m!\,(2-r)_{n+m}}$ has polynomial growth in m and n. We arrive at absolute convergence of the power series (6.26) on $|z| < c$ for all $c < p$. Hence B is real-analytic on D_p. The structure of the $b_{n,m}$ shows that $b(z) = 2i\, yB_{\bar z} + (r-1)\, B$ is antiholomorphic, hence $F := y^{1-r}\, B$ on $D_p \cap \mathfrak{H}$ is in $\mathcal{H}^b_r(D_p)$ and $\rho^{\mathrm{prj}}_r F(t) = -\big(\frac{t-i}{-2i}\big)^{r-2}\, \varphi(t)$. This shows the surjectivity if $r \notin \mathbb{Z}_{\geq 2}$.

Part iii) b), surjectivity. In the case of $r \in \mathbb{Z}_{\geq 2}$, Equation (6.31) shows that the restriction map has the projective model of $\mathcal{D}^{\mathrm{pol}}_{2-r}$ as its image, since we can freely choose the $b_{\ell,0}$ with $\ell \leq r-2$. This gives the surjectivity in the exact sequence in Part iii)b).

Part iii) a),b) injectivity. We suppose that $\rho^{\mathrm{prj}}_r F = 0$ for $F \in \mathcal{H}^b_r(D_p)$. Then $\varphi = 0$, which by (6.30) implies $F = 0$ if $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$. Thus we have the injectivity in Part iii)a), which completes the proof of iii)a).

For $r \in \mathbb{Z}_{\geq 2}$ we have $b_{\ell,0} = 0$ for $\ell \leq r-2$. So

$$B(z) \;=\; \sum_{\ell \geq r-1} b_{\ell,0}\, z^\ell \sum_{m=0}^{r-1} \frac{(1-r)_m}{m!} (\bar z/z)^m \;=\; \sum_{\ell \geq r-1} b_{\ell,0} z^{\ell+1+1-r} (2iy)^{r-1}\,.$$

So the kernel consists of the functions F on $D_p \cap \mathfrak{H}$ with expansion

$$(2i)^{r-1} \sum_{\ell \geq 0} b_{\ell+r-1,0}\, z^\ell\,,$$

which are just the restrictions to $U \cap \mathfrak{H}$ of the holomorphic functions on D_p. This completes also the proof of Part iii)b).

Part iv). We find by a direct computation for $z \in U \cap \mathfrak{H}$

$$(\xi_r F)(z) \;=\; \frac{4}{\bar z + i} \Big(\frac{z+i}{2i}\Big)^{\bar r - 1} \Big(\frac{1-\bar r}{2i}\, \frac{\bar z + i}{z+i}\, \overline{A(z)} + y\, \overline{A_{\bar z}(z)}\Big),$$

where $A(z) = F(z)/f_r(z)$, analytically continued. This shows that $\xi_r F$ extends as a real-analytic function to U. We know that it is holomorphic on $U \cap \mathfrak{H}$, hence on U. It is determined by its values for $x \in U \cap \mathbb{R}$:

$$\begin{aligned}(\xi_r F)(x) &= -2i\Bigl(\frac{x+i}{2i}\Bigr)^{\bar r-2}\Bigl(\frac{1-\bar r}{2i}\,\overline{A(x)}+0\Bigr)\\ &= (\bar r-1)\Bigl(\frac{x+i}{2i}\Bigr)^{\bar r-2}\,\overline{A(x)}\,.\end{aligned}$$

On $U \cap \mathbb{R}$ we have $A = \rho_r^{\mathrm{prj}} F$, hence

$$(\xi_r F)(x) = (\bar r-1)\Bigl(\frac{x+i}{2i}\Bigr)^{\bar r-2}\,\overline{(\rho_r^{\mathrm{prj}} F)(x)}\,,$$

which extends to an equality of holomorphic functions on U, which is (6.25).

Shifted disks. To prove the proposition for kD_p with general $k = k(\vartheta)$ ($\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$) we note that the bijective operator $|^{\mathrm{prj}}_{2-r}k : O(D_p) \to O(k^{-1}D_p)$ preserves holomorphy. This together with the bijective operator $|_r k : \mathcal{H}^{\mathrm{b}}_r(D_p) \to \mathcal{H}^{\mathrm{b}}_r(k^{-1}D_p)$, and Relation (6.10) imply Parts i), ii) and iii).

To prove (iv) for general kD_p, we first apply (iv) to $F|_r k \in \mathcal{H}^{\mathrm{b}}_r(D_p)$ to get, for $z \in D_p \cap \mathbb{C}$:

$$(\xi_r(F|_r k))(z) = (\bar r-1)\Bigl(\frac{z+i}{2i}\Bigr)^{\bar r-2}\,\overline{\rho_r^{\mathrm{prj}}(F|_r k)(\bar z)}. \tag{6.34}$$

Upon an application of (1.30) and (6.10), this becomes, with $k = \begin{pmatrix} a & -c\\ c & a\end{pmatrix}$:

$$\begin{aligned}(\xi_r F)(kz) &= (cz+a)^{2-\bar r}\,(\bar r-1)\Bigl(\frac{z+i}{2i}\Bigr)^{\bar r-2}\,\overline{\rho_r^{\mathrm{prj}}(F|_r k)(\bar z)}\\ &= (cz+a)^{2-\bar r}\,(\bar r-1)\Bigl(\frac{z+i}{2i}\Bigr)^{\bar r-2}\,\overline{(\rho_r^{\mathrm{prj}} F)|^{\mathrm{prj}}_{2-r}k\;(z)}\\ &= (cz+a)^{2-\bar r}\,(\bar r-1)\Bigl(\frac{z+i}{2i}\Bigr)^{\bar r-2}\,(a+ic)^{\bar r-2}\\ &\qquad\cdot\overline{\Bigl(\frac{\bar z-i}{\bar z-k^{-1}i}\Bigr)^{2-r}}\;\overline{(\rho_r^{\mathrm{prj}} F)(k\bar z)}\\ &= (\bar r-1)\Bigl(\frac{kz+i}{2i}\Bigr)^{\bar r-2}\;\overline{(\rho_r^{\mathrm{prj}} F)(k\bar z)}\,.\end{aligned} \tag{6.35}$$

We used that $k^{-1}i = i$, and that $(a+ic)\,(z+i) = (cz+a)(kz+i)$. Since the conjugate of a holomorphic function on $\mathfrak{H}^-$ is holomorphic on $\mathfrak{H}$, this proves the statement. $\square$

Proposition 6.13 gives a rather precise description of the local relation between the sheaves $\mathcal{V}^{\omega}_{2-r}$ and $\mathcal{W}^{\omega}_r$. The next theorem ties this together to a global statement, which will turn out to be crucial in Sections 8 and 10.

Theorem 6.14. i) *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ the morphism of sheaves $\rho_r^{\mathrm{prj}} : \mathcal{W}^{\omega}_r \to \mathcal{V}^{\omega}_{2-r}$ is an isomorphism.*

ii) *For $r \in \mathbb{Z}_{\geq 1}$ we define the subsheaf* ${}^{\mathrm{h}}\mathcal{W}_r^\omega$ *of* $\mathcal{W}_r^\omega$ *by*

$$
{}^{\mathrm{h}}\mathcal{W}_r^\omega(I) \;:=\; \varinjlim \mathcal{H}_r^{\mathrm{h}}(U)\,, \tag{6.36}
$$

where U runs over the open neighbourhoods in $\mathbb{P}^1_{\mathbb{C}}$ of open sets I in $\mathbb{P}^1_{\mathbb{R}}$.

a) *If $r = 1$,* ${}^{\mathrm{h}}\mathcal{W}_1^\omega = \mathcal{W}_1^\omega$.

b) *If $r \in \mathbb{Z}_{\geq 2}$, the following sequence is exact:*

$$
0 \to {}^{\mathrm{h}}\mathcal{W}_r^\omega \to \mathcal{W}_r^\omega \xrightarrow{\rho_r} \mathcal{D}^{\mathrm{pol}}_{2-r} \to 0\,. \tag{6.37}
$$

The space $\mathcal{D}^{\mathrm{pol}}_{2-r}$ is interpreted as a constant sheaf on $\mathbb{P}^1_{\mathbb{R}}$.

Proof. Proposition 6.13 gives the corresponding statements on sets U near all points of $\mathbb{P}^1_{\mathbb{R}}$, giving all statements in the theorem on the level of stalks. □

6.5. **Related work.** In [15] the analytic boundary germs form the essential tool to prove the surjectivity of the map from Maass forms of weight zero to cohomology considered there. This gave the motivation to study these boundary germs for themselves, in the paper [13]. In the introduction of [13] ("Further remarks", p. 111) it is indicated that the boundary germs have been studied in the much wider context of general symmetric spaces.

One finds the isomorphism analogous to the isomorphism in Part i) of Theorem 6.14 in [13, §5.2]. There the isomorphism is approached in two ways: by power series expansions and by integrals. In the proof of Proposition 6.13 we have used only power series.

7. Polar harmonic functions

The subject of this section may seem slightly outside the line of thought of the previous sections. It has its interest on itself, and it provides more examples of r-harmonic function that do or do not represent analytic boundary germs. The main reason to discuss it is in the case $r \in \mathbb{Z}_{\geq 2}$. Though Theorem 6.14 leads directly to spaces of analytic boundary germs isomorphic to the spaces $\mathcal{D}^\omega_{2-r}$ for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, the situation is less clear for $r \in \mathbb{Z}_{\geq 2}$. With polar harmonic functions we will arrive in §8.1 at a satisfactory definition for all $r \in \mathbb{C}$.

7.1. **Polar expansion.** The map $z \mapsto w(z) := \frac{z-i}{z+i}$ with inverse $w \mapsto z(w) := i\frac{1+w}{1-w}$ gives a bijection between the upper half-plane $\mathfrak{H}$ and the unit disk in $\mathbb{C}$. We write a continuous function F on $\mathfrak{H}$ in the form $F(z) = \left(\frac{2i}{z+i}\right)^r P(w(z))$. This has the advantage that the transformation $F \mapsto F|_r \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$ with $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ corresponds to sending P to the function $w \mapsto e^{ir\vartheta}\, P(e^{2i\vartheta} w)$.

We put

$$
\begin{aligned}
F(\mu; z) \;&:=\; \Bigl(\frac{2i}{z+i}\Bigr)^r \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{-2i\mu\vartheta} P(e^{2i\vartheta} w)\, d\vartheta \\
&=\; \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{-i(2\mu+r)\vartheta} \Bigl(F|_r \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}\Bigr)(z)\, d\vartheta \qquad (\mu \in \mathbb{Z})\,.
\end{aligned} \tag{7.1}
$$

In the first expression we see a coefficient of the Fourier expansion of the function $\vartheta \mapsto P(e^{2i\vartheta} w)$. Thus we have a convergent representation

$$F(z) = \sum_{\mu\in\mathbb{Z}} F(\mu; z)\,, \tag{7.2}$$

the *polar expansion*. If we do not work on the whole of $\mathfrak{H}$, but on an annulus $c_1 < \left|\frac{z-i}{z+i}\right| < c_2$, we can proceed similarly.

We use this in particular for r-harmonic functions F. From the second expression in (7.1) we see that $F(\mu; \cdot)$ is r-harmonic, since the operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$ preserve r-harmonicity and we can exchange the order of differentiation and integration. The terms $F(\mu; \cdot)$ can be written in the form $F(\mu, z) = \bigl(2i/(z+i)\bigr)^r (w/\bar{w})^\mu\, p_\mu(|w|^2)$, for some function p_μ on $[0, \infty)$. With some computations one can obtain an ordinary differential equation the p_μ, which turns out to be related to the hypergeometric differential equation, with a two-dimensional solution space. This leads to the following r-harmonic functions, all depending holomorphically on r in a large subset of $\mathbb{C}$.

$$\mathrm{P}_{r,\mu}(z) = \Bigl(\frac{2i}{z+i}\Bigr)^r w^\mu = \Bigl(\frac{2i}{z+i}\Bigr)^r \Bigl(\frac{z-i}{z+i}\Bigr)^\mu, \quad \mu \in \mathbb{Z}\,, \tag{7.3}$$

$$\mathrm{M}_{r,\mu}(z) = \begin{cases} f_r(z) \left(\frac{z-i}{z+i}\right)^{\mu+1} {}_2F_1\Bigl(1+\mu, 1-r; 2-r; \frac{4y}{|z+i|^2}\Bigr) & \text{if } \mu \geq 0\,, \\ f_r(z)\, \frac{z-i}{z+i} \left(\frac{\bar{z}+i}{\bar{z}-i}\right)^{-\mu} {}_2F_1\Bigl(1-\mu-r, 1; 2-r; \frac{4y}{|z+i|^2}\Bigr) & \text{if } \mu \leq 0\,, \end{cases} \tag{7.4}$$

$$\mathrm{H}_{r,\mu}(z) = f_r(z)\, \frac{z-i}{z+i} \Bigl(\frac{\bar{z}+i}{\bar{z}-i}\Bigr)^{-\mu} {}_2F_1\Bigl(1-\mu-r, 1; 1-\mu; \Bigl|\frac{z-i}{z+i}\Bigr|^2\Bigr), \quad \mu \leq -1. \tag{7.5}$$

The function $\mathrm{P}_{r,\mu}$ is holomorphic, hence r-harmonic. Checking the r-harmonicity of $\mathrm{M}_{r,\mu}$ and $\mathrm{H}_{r,\mu}$ requires work, for which there are several approaches:

a) Carry out the computation for the differential equation for p_μ, transform it to a hypergeometric differential equation, and check that the hypergeometric functions in (7.4) and (7.5) are solutions.
b) Check by a direct computation (for instance with formula manipulation software like Mathematica) that the shadow operator sends the functions to the holomorphic functions indicated in Table 2, thus establishing r-harmonicity.
c) Transform the problem to the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, and use the remarks in §A.1.5 in the Appendix.

Most of these facts follow directly from the formulas, and the properties of the hypergeometric function. We note the following:

- The factor $\frac{4y}{|z+i|^2}$ is real-analytic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{-i\}$ with zero set $\mathbb{P}^1_{\mathbb{R}}$. It has values between 0 and 1 on $\mathfrak{H}$, reaching the value 1 only at $z = i$. Since the hypergeometric functions are holomorphic on $\mathbb{C} \smallsetminus [1, \infty)$, the definition shows that $\mathrm{M}_{r,\mu} \in \mathcal{H}^{\mathrm{b}}_r(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i, -i\})$. To investigate the behavior of $\mathrm{M}_{r,\mu}$ at i we note that its shadow has a singularity at $z = i$ if $\mu \geq 1$, so $\mathrm{M}_{r,\mu}$ cannot be real-analytic at i for $\mu \geq 0$.

$f =$	$\mathrm{P}_{r,\mu}$	$\mathrm{M}_{r,\mu}$	$\mathrm{H}_{r,\mu}$
$f \in$	$\mathcal{H}_r(\mathfrak{H})$ $(\mu \geq 0)$ $\mathcal{H}_r(\mathfrak{H} \smallsetminus \{i\})$ $(\mu < 0)$	$\mathcal{H}_r(\mathfrak{H} \smallsetminus \{i\})$	$\mathcal{H}_r(\mathfrak{H})$ $(\mu \leq -1)$
$\xi_r f$	0	$(\bar r - 1)\left(\frac{2i}{z+i}\right)^{2-\bar r}\left(\frac{z-i}{z+i}\right)^{-\mu-1}$	$-\mu\left(\frac{2i}{z+i}\right)^{2-\bar r}\left(\frac{z-i}{z+i}\right)^{-\mu-1}$
f reprs. elt. of $\mathcal{W}_r^\omega(\mathbb{P}^1_\mathbb{R})$	if $r \in \mathbb{Z}_{\geq 1}$	if $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ or if $r \in \mathbb{Z}_{\geq 2}$ and $1 - r \leq \mu \leq -1$	
$\rho_r^{\mathrm{prj}} f$	0 $(r \in \mathbb{Z}_{\geq 1})$	$\left(\frac{t-i}{t+i}\right)^{\mu+1}$	

TABLE 2. Properties of polar r-harmonic functions.

- The functions $\mathrm{P}_{r,\mu}$ and $\mathrm{M}_{r,\mu}$ are linearly independent for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$.
- The Kummer relation [45, §2.9, (33)] implies

$$\mathrm{H}_{r,\mu} = \frac{\mu}{1-r}\mathrm{M}_{r,\mu} + \frac{|\mu|!}{(1-r)_{|\mu|}}\,\mathrm{P}_{r,\mu} \quad (\mu \leq -1)\,. \tag{7.6}$$

  From the singularity of $\mathrm{P}_{r,\mu}$ at i and the fact that $\mathrm{H}_{r,\mu} \in \mathcal{H}_r(\mathfrak{H})$ we see that $\mathrm{M}_{r,\mu}$ has a singularity at i for $\mu \leq -1$ as well.
- If $r_0 \in \mathbb{Z}_{\geq 2}$ the meromorphic function $r \mapsto \mathrm{M}_{r,\mu}$ has in general a first order singularity at $r = r_0$ with a non-zero multiple of $\mathrm{P}_{r_0,\mu}$ as the residue. However, if $1 - r_0 \leq \mu \leq -1$ it turns out to be holomorphic at $r = r_0$. So under these conditions $\mathrm{M}_{r_0,\mu}$ is well defined.

Proposition 7.1. $\mathcal{H}_r(\mathfrak{H}) \cap \mathcal{W}_r^\omega(\mathbb{P}^1_\mathbb{R}) = \{0\}$ *for* $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$.

Proof. Let $F \in \mathcal{H}_r(\mathfrak{H}) \cap \mathcal{W}_r^\omega(\mathbb{P}^1_\mathbb{R})$. With (6.6) and the second line of (7.1) we have

$$F(\mu; z) = \int_{-\pi/2}^{\pi/2} A(\vartheta, z)\, d\vartheta\,,$$

with a function A that is real-analytic in (ϑ, z) with ϑ in a neighbourhood of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in $\mathbb{C}$, and z in a neighbourhood of $\mathbb{P}^1_\mathbb{R}$ in $\mathbb{P}^1_\mathbb{C}$. So all terms $F(\mu; z)$ in the polar expansion of F also represent elements of $\mathcal{W}_r^\omega(\mathbb{P}^1_\mathbb{R})$. Consulting Table 2 we conclude for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ that $F(\mu; \cdot)$ is a multiple of $\mathrm{M}_{r,\mu}$.

On the other hand, as mentioned above, since F is r-harmonic in $\mathfrak{H}$, so is $F(\mu; z)$ for each μ. Again from table 2 we see that $F(\mu, \cdot)$ should be a multiple of $\mathrm{P}_{r,\mu}$ for $\mu \geq 0$ and a multiple of $\mathrm{H}_{r,\mu}$ if $\mu \leq -1$.

Hence all terms in the polar expansion of F vanish, and $F = 0$. □

7.2. **Polar expansion of the kernel function.** The kernel function $K_r(z; \tau) = \frac{2i}{z-\tau}\left(\frac{\bar z-\tau}{\bar z-z}\right)^{r-1}$ in §6.3 gives, for a fixed $\tau \in \mathfrak{H}$, rise to two polar expansions in z, on the disk $\left|\frac{z-i}{z+i}\right| < \left|\frac{\tau-i}{\tau+i}\right|$ and on the annulus $1 > \left|\frac{z-i}{z+i}\right| > \left|\frac{\tau-i}{\tau+i}\right|$.

Proposition 7.2. i) *Consider z,τ satisfying $\left|\frac{z-i}{z+i}\right| > \left|\frac{\tau-i}{\tau+i}\right|$.*

a) *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, then*

$$K_r(z;\tau) \;=\; \sum_{\mu\leq -1} \frac{(2-r)_{-\mu-1}}{(-\mu-1)!}\, \mathrm{P}_{2-r,-\mu-1}(\tau)\, \mathrm{M}_{r,\mu}(z)\,. \tag{7.7}$$

b) *If $r \in \mathbb{Z}_{\geq 2}$, then*

$$K_r(z;\tau) \;=\; \sum_{\mu=1-r}^{-1} (-1)^{-\mu-1} \binom{r-2}{-\mu-1} \mathrm{P}_{2-r,-\mu-1}(\tau)\, \mathrm{M}_{r,\mu}(z) + p_r(z;\tau)\,, \tag{7.8}$$

with

$$p_r(z;\tau) \;:=\; \frac{2i}{z-\tau}\Bigl(\frac{\tau-i}{z-i}\Bigr)^{r-1}\,. \tag{7.9}$$

ii) *Consider z,τ satisfying $\left|\frac{z-i}{z+i}\right| < \left|\frac{\tau-i}{\tau+i}\right|$. For all $r \in \mathbb{C}$:*

$$K_r(z;\tau) \;=\; -\sum_{\mu\leq -1} \frac{(1-r)_{-\mu}}{(-\mu)!}\, \mathrm{P}_{2-r,-\mu-1}(\tau)\, \mathrm{H}_{r,\mu}(z) - \sum_{\mu\geq 0} \mathrm{P}_{2-r,-\mu-1}(\tau)\, \mathrm{P}_{r,\mu}(z)\,. \tag{7.10}$$

iii) *For $r \in \mathbb{Z}_{\geq 2}$:*

$$\begin{aligned} y^{1-r} \;&=\; \sum_{\mu=1-r}^{1} \frac{(1-r)_{-\mu}}{(-\mu)!}\, \mathrm{H}_{r,\mu}(z) + \Bigl(\frac{2i}{z+i}\Bigr)^{r-1} \\ &=\; -\sum_{\mu=1-r}^{-1} \frac{(2-r)_{-\mu-1}}{(-\mu-1)!}\, \mathrm{M}_{r,\mu}(z) + \Bigl(\frac{2i}{z-i}\Bigr)^{r-1}\,. \end{aligned} \tag{7.11}$$

Proof. We use the coordinates $w = \frac{z-i}{z+i}$ and $\xi = \frac{\tau-i}{\tau+i}$, and put

$$X_+ \;:=\; \bigl\{(z,\tau)\in \mathfrak{H}^2 \;:\; |\xi| < |w| < 1\bigr\}\,, X_- \;:=\; \bigl\{(z,\tau)\in \mathfrak{H}^2 \;:\; |w| < |\xi| < 1\bigr\}\,.$$

General expansion. The function $K_r(\cdot;\cdot)$ has polar expansions on both regions, that have the following form on $X_\pm$:

$$K_r(z;\tau) \;=\; \sum_{\mu\in\mathbb{Z}} A^\pm_\mu(r,\tau)\, F^\pm_\mu(r,z)\,.$$

We consider this first for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$. The fact that $K_r(\cdot;\tau)$ on X_+ represents an element of $\mathcal{W}^\omega_r(\mathbb{P}^1_\mathbb{R})$ implies that we can take $F^+_\mu = \mathrm{M}_{r,\mu}$ and where the fact that $K_r(\cdot;\tau)$ has no singularity on X_- implies that we can take $F^-_\mu = \mathrm{H}_{r,\mu}$ for $\mu \leq -1$ and $F^-_\mu = \mathrm{P}_{r,\mu}$ for $\mu \geq 0$.

The invariance relation

$$K_r(\cdot;\cdot)|_r g \otimes |_{2-r} g \;=\; K_r \qquad \text{for each } g \in \mathrm{SL}_2(\mathbb{R}) \tag{7.12}$$

in (6.14), applied with $g = k(\vartheta)$ for small ϑ implies that $A^\pm_\mu(r,\tau)$ transforms under $|_r k(\vartheta)$ by $e^{-i(r+2\mu)\vartheta}$, hence it has the form $A^\pm_\mu(r,\tau) = d^\pm_\mu(r)\, \mathrm{P}_{2-r,-\mu-2}(\tau)$, for some quantity $d^\pm_\mu(r)$.

For a given $z \in \mathfrak{H}$ the function $K_r(z;\cdot)$ has only a singularity in the upper half-plane at $\tau = z$. Since $\mathrm{P}_{2-r,-\mu-1}(\tau)$ has a singularity at $\tau = i$ if $-\mu-1 \leq 0$, we have $d^+_\mu(r) = 0$ for $\mu \geq 0$. Thus, we have the following:

$$\begin{aligned} \text{on } X_+:\quad K_r(z;\tau) &= \sum_{\mu\leq -1} d^+_\mu(r)\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{M}_{r,\mu}(z)\,,\\ \text{on } X_-:\quad K_r(z;\tau) &= \sum_{\mu\leq -1} d^-_\mu(r)\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{H}_{r,\mu}(z)\\ &\quad + \sum_{\mu\geq 0} d^-_\mu(r)\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{P}_{r,\mu}(z)\,. \end{aligned} \tag{7.13}$$

To both sides of both equations we apply the shadow operator ξ_r. With (6.15), Table 2, and the fact that $\mathrm{P}_{r,\mu}$ is holomorphic, we get

$$\begin{aligned} &(\bar r - 1)\Bigl(\frac{z-\bar\tau}{2i}\Bigr)^{\bar r-2}\\ &\quad = \sum_{\mu\leq -1}\overline{\mathrm{P}_{2-r,-\mu-1}(\tau)}\Bigl(\frac{2i}{z+i}\Bigr)^{2-\bar r}\Bigl(\frac{z-i}{z+i}\Bigr)^{-\mu-1}\cdot\begin{cases}\overline{d^+_\mu(r)}\,(\bar r-1) & \text{on } X_+\,,\\ \overline{d^-_\mu(r)}\,(-\mu) & \text{on } X_-\,.\end{cases} \end{aligned}$$

We consider this for $(z,\tau)\in X_+$ with ξ near 0, and for $(z,\tau)\in X_-$ with w near 0. Then we can rewrite the left hand side of the equation so that the main factor is

$$\begin{aligned} (\bar r-1)\,(1-w\bar\xi)^{\bar r-2} &= (\bar r-1)\sum_{a\geq 0}\frac{(2-\bar r)_a}{a!}\,w^a\,\bar\xi^a\\ &= (\bar r-1)\sum_{\mu\leq -1}\frac{(2-\bar r)_{-\mu-1}}{(-\mu-1)!}\Bigl(\frac{\bar\tau+i}{\bar\tau-i}\Bigr)^{-\mu-1}\Bigl(\frac{z-i}{z+i}\Bigr)^{-\mu-1}\,. \end{aligned}$$

So for $\mu\leq -1$ we have

$$d^+_\mu(r) = \frac{(2-r)_{-\mu-1}}{(-\mu-1)!}\,,\qquad d^-_\mu(r) = -\frac{(1-r)_{-\mu}}{(-\mu)!}\,. \tag{7.14}$$

Part i)a) is now clear.

Part i)b). The integrals in (7.1) with $F(z) = K_r(z;\tau)$ show that each term $A^\pm_\mu(r,\tau)\,F^\pm_\mu(r,z)$ in the polar expansion is holomorphic in r. So we can handle the case $r_0\in\mathbb{Z}_{\geq 2}$ by a limit argument. For $1-r_0\leq\mu\leq -1$ with $r_0\in\mathbb{Z}_{\geq 2}$, the function $\mathrm{M}_{r,\mu}$ has a holomorphic extension to $r = r_0$ (remarks to Table 2). This gives the sum in (7.8). With (7.6) the terms with $\mu\leq -r_0$ can be written as

$$\frac{(1-r)\,(1-r)_{-\mu-1}}{\mu\,(-\mu-1)!}\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{H}_{r,\mu}(z) + \mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{P}_{r,\mu}(z)\,.$$

The first of these terms has limit 0 as $r\to r_0$. The second term leads to a series with $p_r(z;\tau)$ as its sum.

Part ii). On X_- we have

$$K_r(z;\tau) = -\sum_{\mu\leq 0}\frac{(1-r)_{-\mu}}{(-\mu)!}\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{H}_{r,\mu}(z) + \sum_{\mu\geq 0} d^-_\mu(r)\,\mathrm{P}_{2-r,-\mu-1}(\tau)\,\mathrm{P}_{r,\mu}(z)\,,$$

with still unknown $d_\mu^-(r)$ for $\mu \geq 0$. In the coordinates w and ξ this becomes:

$$\begin{aligned}
&\frac{(1-w)^r\,(1-\xi)^{2-r}\,(1-\bar w\xi)^{r-1}}{(w-\xi)\,(1-|w|^2)^{r-1}}\\
(7.15)\qquad &= -\sum_{\mu\leq -1} \frac{(1-r)_{-\mu}}{(-\mu)!}\,(1-\xi)^{2-r}\,\xi^{-\mu-1}\,\mathrm{H}_{r,\mu}(z)\\
&+\sum_{\mu\geq 0} d_\mu^-(r)\,(1-\xi)^{2-r}\,\xi^{-\mu-1}\,(1-w)^r\,w^\mu\,.
\end{aligned}$$

We divide by $(1-\xi)^{2-r}$. The remaining quantity on the left has the following expansion:

$$\frac{(1-w)^r}{(1-|w|^2)^{r-1}}\sum_{a,b\geq 0}(-1)w^a\xi^{-a-1}\binom{r-1}{b}(-1)^b\bar w^b\xi^b\,.$$

Let $n \geq 1$. The coefficient of ξ^{-n} in this expansion is

$$-\sum_{b\geq 0} w^{b+n-1}\binom{r-1}{b}(-\bar w)^b \;=\; -w^{n-1}\,(1-|w|^2)^{r-1}\,.$$

Hence $d_\mu^{-1}(r) = -1$ for $\mu \geq 0$. By holomorphic extension from $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ to $r \in \mathbb{C}$, this gives Part ii).

Part iii). Let $r \in \mathbb{Z}_{\geq 2}$. The equality (7.15) divided by $(1-\xi)^{2-r}$ becomes

$$\frac{(1-w)^r\,(1-\bar w\xi)^{r-1}}{(w-\xi)\,(1-|w|^2)^{r-1}} \;=\; -\sum_{\mu=1-r}^{-1}\frac{(1-r)_{-\mu}}{(-\mu)!}\,\xi^{-\mu-1}\,\mathrm{H}_{r,\mu}(z) - (1-w)^r\xi^{-1}\,(1-w/\xi)^{-1}\,.$$

Now we let ξ tend to 1. We obtain:

$$-(1-w)^{r-1}\,(1-\bar w)^{r-1}\,(1-|w|^2)^{1-r} \;=\; -\sum_{\mu=1-r}^{-1}\frac{(1-r)_{-\mu}}{(-\mu)!}\,\mathrm{H}_{r,\mu}(z) - (1-w)^{r-1}\,.$$

In terms of the coordinate z this is

$$-y^{1-r} \;=\; -\sum_{\mu=1-r}^{-1}\frac{(1-r)_{-\mu}}{(-\mu)!}\,\mathrm{H}_{r,\mu}(z) - \Big(\frac{2i}{z+i}\Big)^{r-1}\,,$$

which gives the first expression for y^{1-r} in Part iii). We use (7.6) to obtain the second expression. □

7.3. **Related work.** The polar expansion generalizes the power series expansion in $w = \frac{z-i}{z+i}$ for holomorphic functions on the upper half-plane. When dealing with r-harmonic functions a straightforward generalization leads to the functions in Table 2. Proposition 7.2 is analogous to [13, Proposition 3.3].

Part III. Cohomology with values in analytic boundary germs

We turn to the proof of the surjectivity in Theorem A and the proof of Theorem D, by relating cohomology with values in $\mathcal{D}^{\omega}_{v,2-r}$ to cohomology in modules $\mathcal{E}^{\omega}_{v,r} \subset \mathcal{W}^{\omega}_{v,r}(\mathbb{P}^1_{\mathbb{R}})$. Section 8 gives the definition of these modules.

We use a description of cohomology that turned out to be useful in the analogous result for Maass forms, in [15]. This description of cohomology is based on a tesselation of the upper half-plane. See Section 9.

Theorem 10.18 describes the relation between holomorphic automorphic form and boundary germ cohomology. This theorem immediately implies the surjectivity in Theorem A. For the weights in $\mathbb{Z}_{\geq 2}$ work has to be done, in Section 11, to prove Theorem D.

8. Highest weight spaces of analytic boundary germs

This section serves to define the modules $\mathcal{E}^{\omega}_{v,r}$ to take the place of the modules $\mathcal{D}^{\omega}_{v,2-r}$.

8.1. Definition of highest weight space.

The cases $r \in \mathbb{Z}_{\geq 2}$, and $r \in \mathbb{C} \setminus \mathbb{Z}_{\geq 2}$ are dealt with separately.

8.1.1. *Weight in* $\mathbb{C} \setminus \mathbb{Z}_{\geq 2}$. Part i) of Theorem 6.14 points the way how to treat this case. It states that $\rho^{\mathrm{prj}}_r : \mathcal{W}^{\omega}_r(\mathbb{P}^1_{\mathbb{R}}) \to \mathcal{V}^{\omega}_r(\mathbb{P}^1_{\mathbb{R}})$ is bijective. For $\varphi \in \mathcal{D}^{\omega}_{2-r}$

$$\rho_r^{-1}\varphi \;=\; (\rho^{\mathrm{prj}}_r)^{-1}\mathsf{prj}_{2-r}\varphi \;\in\; \mathcal{W}^{\omega}_r(\mathbb{P}^1_{\mathbb{R}})\,, \tag{8.1}$$

where we use that $(\rho_r f)(t) = (i-t)^{r-2}\,(\rho^{\mathrm{prj}}_r f)(t)$. See (6.17). For $\varphi \in \mathcal{D}^{\omega}_{2-r}[\xi_1,\ldots,\xi_n]$ we can proceed similarly to get $\rho_r^{-1}\varphi \in \mathcal{W}^{\omega}_r[\xi_1,\ldots,\xi_n]$.

Definition 8.1. For $r \in \mathbb{C} \setminus \mathbb{Z}_{\geq 2}$ we define

$$\mathcal{E}^{\omega}_r \;:=\; \rho_r^{-1}\mathcal{D}^{\omega}_{2-r}, \quad \mathcal{E}^{\omega,\mathrm{exc}}_r[\xi_1,\ldots,\xi_n] \;:=\; \rho_r^{-1}\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\xi_1,\ldots,\xi_n] \tag{8.2}$$

for each finite set $\{\xi_1,\ldots,\xi_n\} \subset \mathbb{P}^1_{\mathbb{R}}$.

Weight 1. The case $r = 1$ is special. The restriction morphism is given by $(\rho^{\mathrm{prj}}_1 F)(t) = \frac{1}{2i}\,(t-i)\,F(t)$, and hence $(\rho_1 F)(t) = \frac{i}{2}\,F(t)$. This gives the following equalities:

$$\mathcal{E}^{\omega}_1 \;=\; \mathcal{D}^{\omega}_1\,, \qquad \mathcal{E}^{\omega,\mathrm{exc}}_1[\xi_1,\ldots,\xi_n] = \mathcal{D}^{\omega,\mathrm{exc}}_1[\xi_1,\,\ldots,\xi_n]\,. \tag{8.3}$$

Characterization with series. The projective model $\mathsf{prj}_{2-r}\mathcal{D}^{\omega}_{2-r}$ consists of the holomorphic functions on some neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{C}$. So it consists of the functions

$$t \mapsto \sum_{\mu\leq 0} c_\mu \Big(\frac{t-i}{t+i}\Big)^{\mu}$$

with coefficients that satisfy $c_\mu = \mathrm{O}(e^{-a|\mu|})$ as $|\mu| \to \infty$, for some $a > 0$ depending on the domain of the function. Table 2 in §7.1 gives $\big(\frac{t-i}{t+i}\big)^\mu = \rho_r^{\mathrm{prj}} M_{r,\mu-1}$. Hence we have for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$

$$\mathcal{E}_r^\omega \;=\; \Big\{ \sum_{\mu \in \mathbb{Z}_{\leq -1}} c_\mu \,\mathrm{M}_{r,\mu} \;:\; c_\mu = \mathrm{O}(e^{-a|\mu|}) \text{ for some } a > 0 \Big\}. \tag{8.4}$$

Highest weight spaces. We call $\mathcal{D}_{2-r}^\omega$ and $\mathcal{E}_r^\omega$ *highest weight spaces*. The use of this terminology is explained in §A.2.1 in the Appendix.

8.1.2. *Case $r \in \mathbb{Z}_{\geq 2}$.* We note that representatives of elements of $\mathcal{D}_r^\omega$ are holomorphic functions on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ that have at ∞ a zero of order at least r, since (1.19) shows that the elements of $\mathcal{D}_r^\omega$ are of the form $t \mapsto (i-t)^{-r} \cdot (\text{holo. at } \infty)$. These functions represent also elements of $\mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$.

Definition 8.2. For $r \in \mathbb{Z}_{\geq 2}$ we define:

$$\begin{aligned} \mathcal{E}_r^\omega \;&:=\; \mathcal{D}_r^\omega + \sum_{\mu=1-r}^{-1} \mathbb{C}\,\mathrm{M}_{r,\mu}\,, \\ \mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n] \;&:=\; \mathcal{D}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n] + \sum_{\mu=1-r}^{-1} \mathbb{C}\,\mathrm{M}_{r,\mu}\,, \end{aligned} \tag{8.5}$$

for finite sets $\{\xi_1,\ldots,\xi_n\} \subset \mathbb{P}^1_{\mathbb{R}}$.

Remark. This defines $\mathcal{E}_r^\omega$ as a subspace of $\mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$, and $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]$ as a subspace of $\mathcal{W}_r^\omega\big(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1,\ldots,\xi_n\}\big)$.

Comparison with weight 1. If we apply the formulas in (8.5) with $r = 1$, the sum over μ is empty, and we get back (8.3).

Characterization with series. The elements of the projective model $\mathsf{prj}_{2-r}\mathcal{D}_r^\omega$ are the functions of the form $h(t) = \sum_{\mu\leq 0} d_\mu \big(\frac{t-i}{t+i}\big)^\mu$ with $d_\mu = \mathrm{O}(e^{-a|\mu|})$ for some $a > 0$. In view of (1.19) and (7.3) $f = \mathsf{prj}_r^{-1} h$ has an expansion of the form

$$(z-i)^{-r} \sum_{\mu \leq 0} d_\mu \Big(\frac{t-i}{t+i}\Big)^\mu \;=\; \sum_{\mu \leq -r} c_\mu \,\mathrm{P}_{r,\mu}(z)\,,$$

with the c_μ satisfying the same estimate. This leads to

$$\mathcal{E}_r^\omega \;=\; \Big\{ \sum_{\mu\leq -r} c_\mu \,\mathrm{P}_{r,\mu}(z) + \sum_{\mu=1-r}^{-1} c_\mu \,\mathrm{M}_{r,\mu} \;:\; c_\mu = \mathrm{O}(e^{-a|\mu|}) \text{ for some } a > 0 \Big\}, \tag{8.6}$$

which is similar to (8.4).

In the following result we use the subsheaf ${}^{\mathrm{h}}\mathcal{W}_r^\omega$, defined in (6.36). Its sections are represented by holomorphic functions, contained in the kernel of the restriction morphism ρ_r.

Proposition 8.3. *Let $r \in \mathbb{Z}_{\geq 2}$.*

i) *$\mathcal{D}_r^\omega$ is a subspace of ${}^{\mathrm{h}}\mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$ invariant under the operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$, and $\mathcal{D}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]|_r g = \mathcal{D}_r^{\omega,\mathrm{exc}}[g^{-1}\xi_1 \ldots, g^{-1}\xi_n]$ for all $g \in \mathrm{SL}_2(\mathbb{R})$.*

ii) *$\mathcal{E}_r^\omega$ is a subspace of $\mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$ invariant under the operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$, and $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]|_r g = \mathcal{E}_r^{\omega,\mathrm{exc}}[g^{-1}\xi_1 \ldots, g^{-1}\xi_n]$ for all $g \in \mathrm{SL}_2(\mathbb{R})$.*

iii) *The following sequences are exact:*

$$
\begin{aligned}
&0 \to \mathcal{D}_r^\omega \to \mathcal{E}_r^\omega \xrightarrow{\rho_r} \mathcal{D}_{2-r}^{\mathrm{pol}} \to 0\,,\\
&0 \to \mathcal{D}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n] \to \mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n] \xrightarrow{\rho_r} \mathcal{D}_{2-r}^{\mathrm{pol}} \to 0\,.
\end{aligned}
\tag{8.7}
$$

Proof. For Part i) we use the definitions in §1.6, applied with r instead of $2-r$. In particular, elements of $\mathcal{D}_r^\omega$ are represented by holomorphic functions with at ∞ a zero of order at least r. In that way we see that all elements of $\mathcal{D}_r^\omega$ are sections in ${}^{\mathrm{h}}\mathcal{W}_r^\omega$, and similarly for $\mathcal{D}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$.

For Part ii) there remains to show that $\mathrm{M}_{r,\mu}|_r g \in \mathcal{E}_r^\omega$. Relation (7.8) in Proposition 7.2 expresses $K_r(\cdot;\tau)$ as a linear combination of the $M_{r,\mu}$ in $\mathcal{E}_r^\omega$ and an explicit kernel $p_r(\cdot;\tau)$. Since $P_{2-r,-\mu-1}(\tau)$ is essentially equal to $\left(\frac{\tau-i}{\tau+i}\right)^{-\mu-1}$, we can invert the relation, and express each $\mathrm{M}_{r,\mu}$ with $1-r \leq \mu \leq -1$ as a linear combination of $K_r(\cdot;\tau_i) - p_r(\cdot;\tau_i)$ for $r-1$ elements $\tau_i \in \mathfrak{H}$. The invariance relation (7.12) implies that $K_r(\cdot;\tau_i)|_r g$ is a multiple of $K_r(\cdot; g^{-1}\tau_i)$, which is in $\mathcal{E}_r^\omega$ by an application of (7.8). The contribution of $p_r(\cdot;\tau_i)$ is in $\mathcal{D}_r^\omega$, which is invariant under the operators $|_r g$.

The exactness of the sequences in Part iii) follows directly from the fact that ρ_r vanishes on $\mathcal{D}_r^\omega$ and the relations $(\rho_r \mathrm{M}_{r,\mu})(t) = (i-t)^{r-2}\left(\frac{t-i}{t+i}\right)^{\mu+1}$, with (6.11) and (8.1). □

8.2. **General properties of highest weight spaces of analytic boundary germs.** In the previous subsection we have chosen spaces $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$ of boundary germs for all finite subsets $\{\xi_1, \ldots, \xi_n\}$ of $\mathbb{P}^1_{\mathbb{R}}$, and the space $\mathcal{E}_r^\omega$, which we call also $\mathcal{E}_r^{\omega,\mathrm{exc}}[]$. The following proposition lists properties that these system have in common for all $r \in \mathbb{C}$. In Section 10 we shall work on the basis of these properties.

Proposition 8.4. *The systems of spaces in Definitions 8.1 and 8.2 have the following properties:*

i) *$\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n] \subset \mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\})$ consists of boundary germs represented by functions in $\mathcal{H}_r^{\mathrm{b}}(U)$ where U is open in $\mathbb{P}^1_{\mathbb{C}}$ so that $U \cup \mathfrak{H}^-$ is a $\{\xi_1, \ldots, \xi_n\}$-excised neighbourhood.*

ii) a) *If $\{\xi_1, \ldots, \xi_n\} \subset \{\eta_1, \ldots, \eta_m\}$, then*

$$\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n] \subset \mathcal{E}_r^{\omega,\mathrm{exc}}[\eta_1, \ldots, \eta_m]\,.$$

b) *If $\{\xi_1, \ldots, \xi_n\} \cap \{\eta_1, \ldots, \eta_m\} = \emptyset$, then*

$$\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n] \cap \mathcal{E}_r^{\omega,\mathrm{exc}}[\eta_1, \ldots, \eta_m] = \mathcal{E}_r^\omega\,.$$

With the inclusion relation ii)a) we define

$$\mathcal{E}_r^{\omega^*,\mathrm{exc}} := \varinjlim \mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]\,, \tag{8.8}$$

where $\{\xi_1,\ldots,\xi_n\}$ runs over the finite subsets of $\mathbb{P}^1_{\mathbb{R}}$.

iii) $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]|_r g = \mathcal{E}_r^{\omega,\mathrm{exc}}[g^{-1}\xi_1,\ldots,g^{-1}\xi_n]$ *for each* $g \in \mathrm{SL}_2(\mathbb{R})$.

iv) *The function* $z \mapsto \int_{z_1}^{z_2} K_r(z;\tau)\, f(\tau)\, d\tau$ *represents an element of* $\mathcal{E}_r^{\omega}$ *for all* $z_1, z_2 \in \mathfrak{H}$ *and each holomorphic function* f *on* $\mathfrak{H}$.

v) *If* $F \in \mathcal{H}_r(\mathfrak{H})$ *represents an element of* $\mathcal{E}_r^{\omega}$, *then* $F = 0$.

vi) *If* $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ *represents an element of* $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]$ *then its shadow* $\xi_r F \in \mathcal{O}(U \cap \mathfrak{H})$ *extends holomorphically to* $\mathfrak{H}$.

vii) *Let* $\lambda \in \mathbb{C}^*$. *Suppose that* $f \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$ *has a representative* F *that satisfies:*

   a) $F \in \mathcal{H}_r(U \cap \mathfrak{H})$ *for some neighbourhood* U *of* $\mathbb{P}^1_{\mathbb{R}}$ *in* $\mathbb{P}^1_{\mathbb{C}}$,

   b) *the function* $z \mapsto \lambda^{-1} F(z+1) - F(z)$ *on* $\mathfrak{H} \cap U \cap T^{-1}U$ *represents an element of* $\mathcal{E}_r^{\omega}$,

then $f = p + g$ *with an element* $g \in \mathcal{E}_r^{\omega}$ *and a* λ*-periodic element* $p \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$.

Remarks. (a) In Property a) it is not always possible to choose the representative so that the set U contains $\mathfrak{H}^-$. Moreover, the property does not state that all functions in $\mathcal{H}_r^{\mathrm{b}}(U)$ with U as indicated represent elements of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]$.

(b) Condition a) in Part vii) is strong. In general representatives of an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$ are r-harmonic only on an $\{\infty\}$-excised neighbourhood.

Proof. We consider the various parts of the theorem, often separately for the general case $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ and the special case $r \in \mathbb{Z}_{\geq 2}$.

a. *Part i).* Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$. An element $F \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]$ is determined by $h = \rho_r^{\mathrm{prj}} F$ in the projective model of $\mathcal{D}_{2-r}^{\omega,\mathrm{exc}}[\xi_1,\ldots,\xi_n]$. So h is holomorphic on a $\{\xi_1,\ldots,\xi_n\}$-excised neighbourhood U_0. Each point $\xi \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1,\ldots,\xi_n\}$ is of the form $k_\xi \cdot 0$ with $k \in \mathrm{SO}(2)$. We choose $p(\xi) \in (0,1)$ such that $k_\xi\ D_{p(\xi)} \subset U_0$. Then Part iii)a) of Proposition 6.13 implies that $\rho_r^{\mathrm{prj}} \mathcal{H}_r^{\mathrm{b}}(k_\xi\, D_{p(\xi)}) = \mathcal{O}(k_\xi\, D_{p(\xi)})$. So $F \in \mathcal{H}_r^{\mathrm{b}}(U)$, with

$$U := \bigcup_{\xi \in \mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1,\cdots,\xi_n\}} k_\xi\, D_{p(\xi)}\,.$$

However, not for all choices of the p_ξ the set $U \cup \mathfrak{H}^-$ is an excised neighbourhood.

We return to the choice of the $p_\xi \in (0,1)$. By conjugation by an element of $\mathrm{SO}(2)$ we arrange that all ξ_j are in $\mathbb{R}$.

We recall that near each of the points ξ_j a $\{\xi_1, \ldots, \xi_n\}$-excised neighbourhood looks like a full neighbourhood of ξ_j minus the sector between two geodesic half-lines with end-point ξ_j.

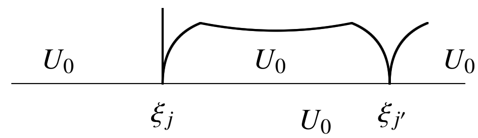

FIGURE 10

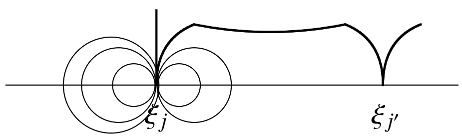

FIGURE 11

Since $\xi_j \in \mathbb{R}$ those geodesic half-lines are parts of euclidean circles with their center on $\mathbb{R}$, or vertical euclidean lines. This implies that there is a small $\varepsilon > 0$ such that for all $\xi \in \mathbb{R}$ with $0 < |\xi - \xi_j| < \varepsilon$ the open euclidean disk around ξ with radius $|\xi - \xi_j|$ is contained in U_0. If ε is sufficiently small these euclidean disks are of the form $k_{\xi'}\, D_{p(\xi')}$, with in general $\xi' \neq \xi$.

In this way we can choose for all ξ sufficiently near to ξ_j the value $p_\xi \in (0,1)$ in such a way that ξ_j is in the closure of $k_\xi\, D_{p(\xi)}$. We see that U is near ξ_j a full neighbourhood of ξ_j minus the sectors between two geodesics half-lines at ξ_j in the upper and the lower half-plane. So $\mathfrak{H}^- \cup U$ is an excised neighbourhood provided we take the p_ξ appropriately.

b. *Part i) for $r \in \mathbb{Z}_{\geq 2}$.* Elements of $\mathcal{D}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$ are already represented by functions of the desired form. The functions $\mathrm{M}_{r,\mu}$ are in $\mathcal{H}_r^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i, -i\})$.

c. *Part ii).* Immediate from the corresponding property of $\mathcal{D}_p^{\omega,\mathrm{exc}}$, with $p \in \{r, 2-r\}$.

d. *Part iii).* Immediate from Part i) of Proposition 1.14 and (6.10) if $r \notin \mathbb{Z}_{\geq 2}$, and from Proposition 8.3, Part ii), if $r \in \mathbb{Z}_{\geq 2}$.

e. *Part iv) for $r \notin \mathbb{Z}_{\geq 2}$.* Integration over τ in a compact set in $\mathfrak{H}$ preserves the property that $K_r(\cdot;\tau)$ represents an element of $\mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$, and commutes with taking the restriction. Applying ρ_r gives the integral $\int_{z_1}^{z_2} (z-\tau)^{r-2}\, F(\tau)\, d\tau$, which has a value in $\mathcal{D}^{\omega}_{2-r}$.

f. *Part iv) for $r \in \mathbb{Z}_{\geq 2}$.* Equation (7.8) in Proposition 7.2 expresses $K_r(\cdot;\tau)$ as a linear combination of the $\mathrm{M}_{r,\mu}$ in $\mathcal{E}_r^{\omega}$ and an explicit kernel $p_r(\cdot;\tau)$. In the integral of the terms with $\mathrm{M}_{r,\mu}$ only the coefficient depends on τ. Hence the result is a multiple of $\mathrm{M}_{r,\mu}$. The kernel $p_r(\cdot;\tau)$ is in $\mathcal{D}_r^{\omega}$ by the description in Part i) of Proposition 7.2, and it stays there under integration with respect to τ.

g. *Part v) for $r \notin \mathbb{Z}_{\geq 1}$.* See Proposition 7.1.

h. *Part v) for $r = 1$.* Let $F \in \mathcal{H}_1(\mathfrak{H})$ represent an element of $\mathcal{E}_1^{\omega}(\mathbb{P}^1_{\mathbb{R}})$. So $\rho_1 F(z) = \frac{z-i}{2i}\, F(z)$ is holomorphic on a neighbourhood U of $\mathfrak{H}^- \cap \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. Then F itself is holomorphic on $\mathfrak{H} \cap U \smallsetminus \{i\}$, hence F is holomorphic on $\mathfrak{H}$ since it is already real-analytic. Thus, $F \in \mathcal{O}(\mathbb{P}^1_{\mathbb{C}})$ with a zero at ∞, hence $F = 0$.

i. *Part v) for $r \in \mathbb{Z}_{\geq 2}$.* For $r \in \mathbb{Z}_{\geq 2}$, we consider $F = F_0 + \sum_{\mu=1-r}^{-1} c_\mu \, \mathrm{M}_{r,\mu} \in \mathcal{H}_r(\mathfrak{H})$ with F_0 representing an element of $\mathcal{D}_r^\omega$ and $c_\mu \in \mathbb{C}$. Since the $\mathrm{M}_{r,\mu}$ are r-harmonic on $\mathfrak{H} \smallsetminus \{i\}$ the function F_0 is holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{i\}$ with at ∞ a zero of order at least r.

To investigate the singularity of F_0 at i we use Kummer relation (7.6), which relates $\mathrm{M}_{r,\mu}$, $\mathrm{H}_{r,\mu}$ and $\mathrm{P}_{r,\mu}$. Since $\mathrm{H}_{r,\mu}$ is r-harmonic on $\mathfrak{H}$, the singularity at i of $\sum_{\mu=1-r}^{-1} c_\mu \mathrm{M}_{r,\mu}$ is the same as that of

$$\begin{aligned} F_1(z) \;&=\; \sum_{\mu=1-r}^{-1} \frac{(-\mu-1)!}{(2-r)_{-\mu-1}}\, c_\mu \, \mathrm{P}_{r,\mu} \\ &=\; \sum_{\mu=1-r}^{-1} \frac{(-\mu-1)!}{(2-r)_{-\mu-1}}\, c_\mu \Big(\frac{2i}{z+i}\Big)^r \Big(\frac{z-i}{z+i}\Big)^\mu . \end{aligned} \tag{8.9}$$

This leads to a holomorphic function $F_0 + F_1$ on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{-i\}$, with at $z = -i$ a pole of order at most $r-1$. At ∞ the function F_0 has a zero of order at least r. The same holds for F_1 by the factor $(z+i)^{-r}$ in (8.9). So the number of zeros is larger than the number of poles, and we conclude that $F_0 + F_1 = 0$. However, $F_1 = -F_0$ has to be in $\mathcal{D}_r^\omega$, in particular, it has to be holomorphic at $z = -i$. Inspection of (8.9) shows that successively $c_{1-r}, c_{2-r}, \ldots$ have to vanish. So $F = 0$.

j. *Part vi).* The representative F is defined on $U \cap \mathfrak{H}$, where the open set $U \subset \mathbb{P}^1_{\mathbb{R}}$ contains $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}$. Part iv) of Proposition 6.13 implies that there is an open set $U_1 \subset U$, still containing $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}$ (obtained as the union of sets $k\ D_p$) such that on U_1 the shadow $\xi_r F(z)$ is a holomorphic multiple of $a(z) = \overline{(\rho_r^{\mathrm{prj}} F)(\bar z)}$. So the domain of a is some neighbourhood U_2 of $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}$ in $\mathbb{P}^1_{\mathbb{C}}$. Since $\rho_r F \in \mathcal{D}^{\omega^*}_{v,2-r}$ the functions $\rho_r F$ and $\rho_r^{\mathrm{prj}} F$ are holomorphic on $\mathfrak{H}^-$. Hence the domain of the holomorphic function a contains $\overline{\mathfrak{H}^-} = \mathfrak{H}$.

k. *Part vii).* See §8.5. □

8.3. **Splitting of harmonic boundary germs, Green's form.** We discuss now a splitting of the space of global sections of r-harmonic boundary germs. We shall use this to prove Part vii) in Proposition 8.4 in the case that $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. To obtain the splitting we use the Green's form for harmonic functions and the resolvent kernel.

Theorem 8.5. *If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ then*

$$\mathcal{B}_r(\mathbb{P}^1_{\mathbb{R}}) \;=\; \mathcal{H}_r(\mathfrak{H}) \;\oplus\; \mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}}) . \tag{8.10}$$

Since we have already Proposition 7.1, we need only to show that $\mathcal{B}_r(\mathbb{P}^1_{\mathbb{R}}) = \mathcal{H}_r(\mathfrak{H}) + \mathcal{W}_r^\omega(\mathbb{P}^1_{\mathbb{R}})$.

Resolvent kernel. We put

$$Q_r(z_1, z_2) \;=\; \mathrm{M}_{r,0}\Big(\frac{z_2 - \mathrm{Re}\, z_1}{\mathrm{Im}\, z_1}\Big), \tag{8.11}$$

with the r-harmonic function $\mathrm{M}_{r,0} \in \mathcal{H}_r(\mathfrak{H} \smallsetminus \{i\})$ in (7.4). So $Q_r(z_1, z_2)$ is defined on $\mathfrak{H} \times \mathfrak{H} \smallsetminus$ (diagonal). It is called the *free space resolvent kernel* . It is a special case of the resolvent kernel that inverts the differential operator $\Delta_r - \lambda$ on suitable functions. See, eg., [78, §3, Chap. XIV].

The following properties can be checked by a computation, but are more easily seen on the universal covering group, as we explain in §A.1.6.

$$\Delta_r Q_r(z_1, \cdot) \;=\; 0\,, \tag{8.12}$$

$$4y^2 \partial_z \partial_{\bar z} Q_r(\cdot, z_2) + 2iry\, \partial_{\bar z} Q_r(\cdot, z_2) + r\, Q_r(\cdot, z_2) \;=\; 0\,, \tag{8.13}$$

$$\text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \quad (cz_1 + d)^r\, (cz_2 + d)^{-r}\, Q_r(g\, z_1, g\, z_2) \;=\; Q_r(z_1, z_2)\,. \tag{8.14}$$

The r-harmonic function $z_2 \mapsto Q_r(z_1, z_2)$ represents an element of $\mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$.

Green's form. For $f_1, f_2 \in C^\infty(U)$, with $U \subset \mathfrak{H}$, we define the *Green's form*

$$[f_1, f_2]_r \;=\; \Bigl(\partial_z f_1 + \frac{r}{z - \bar z} f_1\Bigr) f_2\, dz + f_1 (\partial_{\bar z} f_2)\, d\bar z\,. \tag{8.15}$$

This is a 1-form on U, which satisfies

$$[f_1|_r g, f_2|_{-r} g]_r \;=\; [f_1, f_2]_r \circ g \quad \text{on } g^{-1}U \qquad \text{for } g \in \mathrm{SL}_2(\mathbb{R})\,. \tag{8.16}$$

If f_1 is r-harmonic on U and if f_2 satisfies the differential equation in (8.13) on U, then $[f_1, f_2]_r$ is a closed differential form on U. (These results can be checked by some computations.)

Cauchy-like integral formula.

Proposition 8.6. *Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. Let U be an open set in $\mathfrak{H}$, and let C be a positively oriented simple closed curve in U such that the region V enclosed by C is contained in U. Then for each $F \in \mathcal{H}_r(U)$:*

$$\int_C [F, Q_r(\cdot, z')]_r \;=\; \begin{cases} 2\pi i(1 - r) F(z') & \text{if } z' \in V\,, \\ 0 & \text{if } z' \in \mathfrak{H} \smallsetminus (C \cup V)\,. \end{cases}$$

Proof. In this result the kernel Q_r and the Green's form are combined to give for r-harmonic functions $F \in \mathcal{H}_r(\mathfrak{H})$ the closed differential form

$$[F, Q_r(\cdot, z')]_r(z)$$

on $\mathfrak{H} \times \mathfrak{H} \smallsetminus$ (diagonal). It satisfies for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$

$$(cz' + d)^{-r} [F, Q_r(\cdot, gz')]_r(gz) \;=\; [F|_r g, Q_r(\cdot, z')]_r(z)\,. \tag{8.17}$$

Hence it suffices to establish the relation for $z' = i$. The proof proceeds along the same lines as the proof of [13, Theorem 3.1].

We have

$$\begin{aligned} Q_r(z,i) &= \frac{2iy}{i-\bar z}\Bigl(\frac{z+i}{2i}\Bigr)^{r-1} {}_2F_1\Bigl(1,1-r;2-r;\frac{4y}{|z+i|^2}\Bigr)\\ &= 2(r-1)\,\log|z-i| + \mathrm{O}(1) \qquad \text{as } z\to i\,, \qquad (8.18)\\ \partial_{\bar z} Q_r(z,i) &= \frac{r-1}{\bar z + i} + \mathrm{O}(\log|z-i|) \qquad \text{as } z \to i\,. \end{aligned}$$

The integral of the term with dz in $[F, Q_r(\cdot,i)]$ over a circle around i with radius ε is $\mathrm{O}(\varepsilon\,\log\varepsilon) = o(1)$ as $\varepsilon\downarrow 0$. The other term gives

$$\int_{\varphi=0}^{2\pi} (F(i)+\mathrm{O}(\varepsilon))\Bigl(\frac{r-1}{\varepsilon\, e^{-i\varphi}} + \mathrm{O}(\log\varepsilon)\Bigr)(-i\varepsilon\, e^{-i\varphi})\,d\varphi \;=\; 2\pi i(1-r)\,F(i) + o(1)\,. \quad \square$$

We first illustrate a possible use of Proposition 8.6 in an example, and will after that complete the proof of Theorem 8.5.

Let $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq 1}$.In the situation sketched in Figure 12, the integral

$$\frac{1}{2\pi i(1-r)}\int_C [F, Q_r(\cdot, z')]$$

represents a function of z' on the regions inside and outside the simple positively oriented closed path C. According to Proposition 8.6 the resulting function inside C is equal to F, and outside C we get the zero function.

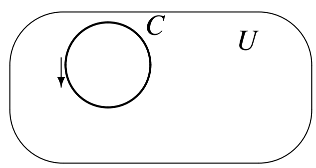

FIGURE 12

The situation is different if we let C run around a hole in U. Now the integral defines an r-harmonic function F_i on the region inside C (including the hole), and a function F_o outside C.

Since the differential form is closed, we can deform the path of integration inside U, thus obtaining extensions of F_i and F_o to overlapping regions inside U. On the intersection of the domains Proposition 8.6 implies $F_i - F_o = F$.

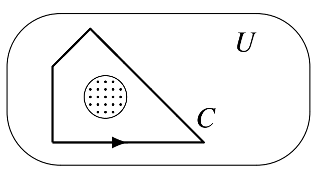

FIGURE 13

Completion of the proof of Theorem 8.5. Let F represent an element of $\mathcal{B}_r(\mathbb{P}^1_{\mathbb{R}})$. So $F\in\mathcal{H}_r(U)$ for any open $U\subset\mathfrak{H}$ that contains a region of the form $1-\varepsilon < \left|\frac{z-i}{z+i}\right| < 1$. The disk $\left|\frac{z-i}{z+i}\right|\leq 1-\varepsilon$ will play the role of the hole in Figure 13.

For a positively oriented simple closed curve C in U we have two r-harmonic functions:

$$\begin{aligned} F_i(z') &= \frac{1}{2\pi i(1-r)} \int_C [F, Q_r(\cdot, z')]_r \quad \text{for } z' \in \mathfrak{H} \text{ inside } C\,, \\ F_o(z') &= \frac{1}{2\pi i(1-r)} \int_C [F, Q_r(\cdot, z')]_r \quad \text{for } z' \in \mathfrak{H} \text{ outside } C\,. \end{aligned} \tag{8.19}$$

By moving the path closer and closer to the boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}$ we obtain that $F_i \in \mathcal{H}_r(\mathfrak{H})$. Further F_o is r-harmonic on a region $U' \subset U$ that contains the intersection with $\mathfrak{H}$ of a neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The function $Q_r(z, \cdot)$ represents an element of $\mathcal{W}^\omega_r(\mathbb{P}^1_{\mathbb{R}})$ for each $z \in \mathfrak{H}$. This property is preserved under integration. Hence F_o represents an element of $\mathcal{W}^\omega_r(\mathbb{P}^1_{\mathbb{R}})$.

If z' is in the intersection of the domains of F_i and F_o, then we apply the integral representation with different paths, and get $F(z') = F_i(z') - F_o(z')$ by Proposition 8.6. This gives the desired decomposition.

Together with Proposition 7.1, this implies the theorem. □

8.4. **Periodic harmonic functions and boundary germs.** In Definition 3.3 we introduced the concept of λ-periodic functions, for $\lambda \in \mathbb{C}^*$. We use this terminology also for boundary germs satisfying $f|_p T = \lambda f$, with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (The transformation does not depend on the weight p.

Lemma 8.7. *Put*

$$F_{r,n}(z) \;:=\; e^{2\pi i n z}\, y^{1-r}\, {}_1F_1(1-r; 2-r; 4\pi n y) \qquad (r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2},\; n \in \mathbb{C})\,. \tag{8.20}$$

(For $r = 1$ we have $F_{1,n}(z) = e^{2\pi i n z}$.)

i) *For $r \in \mathbb{Z}_{\geq 2}$ the λ-periodic elements $\varphi \in \mathcal{D}^{\mathrm{pol}}_{2-r}$ form the one-dimensional subspace of constant functions if $\lambda = 1$, and are zero otherwise.*

ii) *If $F \in \mathcal{H}^{\mathrm{b}}_r(U)$ represents a λ-periodic element of $\mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$ then it has an r-harmonic extension as an element of $\mathcal{H}_r(\mathfrak{H})$, and is given by a Fourier expansion*

$$F(z) \;=\; \begin{cases} \sum_{n \equiv \alpha(1)} c_n\, F_{r,n}(z) & \text{if } r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}\,, \\ \sum_{n \equiv \alpha(1)} c_n\, e^{2\pi i n z} + a_0\, y^{1-r} & \text{if } r \in \mathbb{Z}_{\geq 2}\,, \end{cases} \tag{8.21}$$

where the coefficients c_n satisfy $c_n = \mathrm{O}(e^{-b|\mathrm{Re}\, n|})$ as $|\mathrm{Re}\, n| \to \infty$, and where $a_0 \in \mathbb{C}$ is equal to zero unless $\lambda = 1$.

iii) *Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. If $F \in \mathcal{H}_r(\mathfrak{H})$ represents a λ-periodic element of $\mathcal{W}^\omega_r(\mathbb{R})$, then $F \in \mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$.* (Hence the statements in Part ii) apply to F.)

Proof. Since $\mathcal{D}^{\mathrm{pol}}_{2-r}$ consists of polynomials of degree at most $r-2$, Part i) is immediately clear.

For Parts ii) and iii) we consider first a neighbourhood U of $\mathbb{R}$ in $\mathbb{C}$ and $F \in \mathcal{H}^{\mathrm{b}}_r(U)$ that represents an element of $\mathcal{W}^\omega_r(\mathbb{R})$. Since F is λ-periodic the set U contains a strip $-\varepsilon < \mathrm{Im}\, z < \varepsilon$ for some $\varepsilon > 0$.

The λ-periodic, r-harmonic function F on $U \cap \mathfrak{H}$ is given by a Fourier expansion

$$F(z) \;=\; \sum_{n \equiv \alpha(1)} e^{2\pi i n x}\, f_n(y)$$

that is absolutely convergent on $\mathfrak{H}$. Since the operator Δ_r defining r-harmonicity commutes with translation $z \mapsto z + u$ with $u \in \mathbb{R}$, all Fourier terms have the form $e^{2\pi i n x}\, f_n(y)$ and are also r-harmonic. Hence they are in a two-dimensional solution space.

We use that the condition $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ is inherited by the Fourier terms. The multiples of $F_{r,n}$ are in $\mathcal{H}_r^{\mathrm{b}}(U)$ for some neighbourhood U of $\mathbb{R}$ in $\mathbb{C}$. If $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, a linearly independent element of the solution space is the holomorphic function $z \mapsto e^{2\pi i n z}$, which does not represent an analytic boundary germ. Thus we get for $r \notin \mathbb{Z}_{\geq 1}$ a Fourier expansion of the form indicated in (8.21).

For $r \in \mathbb{Z}_{\geq 1}$ we consider the functions $F(z) = e^{2\pi i n z}\, g(y)$ for which $\xi_r F$ is holomorphic. In all cases we can take $g(y)$ constant, and obtain the multiples of $e^{2\pi i n z}$.

If g is not constant, the condition that $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ leads to $g(y) = y^{1-r}\, a(y)$ with a real-analytic function a on a neighbourhood of $y = 0$ in $\mathbb{R}$. We find that we should have

$$\frac{r-1}{2i}\, a(y) + y\, a'(y) \;=\; c\, e^{4\pi n y}$$

with some $c \in \mathbb{C}$. If $r = 1$ this is possible only with $c = 0$, and then $a'(y) = 0$. So we do not get more than indicated in (8.21).

For $r \in \mathbb{Z}_{\geq 2}$ we take the restriction:

$$\rho_r^{\mathrm{prj}}\big(e^{2\pi i n z}\, y^{1-r}\, a(y)\big)(t) \;=\; -(-2i)^{r-2}\, (t-i)^{2-r}\, e^{2\pi i n t}\, a(0)\,.$$

This should be a λ-periodic element of $\mathcal{D}_{2-r}^{\mathrm{pol}}$, which can be non-zero only if $\lambda = 1$, and then is a constant function. This leads to the term $a_0\, y^{1-r}$ in (8.21).

For Part ii) we suppose that F represents an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$. This is an assumption in Part ii). Part i) in Proposition 8.4 implies that $U \cap \mathfrak{H}$ contains a region of the form

$$\{z \in \mathfrak{H} \;:\; |\mathrm{Re}\, z| > \varepsilon^{-1}\} \cup \{z \in \mathfrak{H} \;:\; \mathrm{Im}\, z < \varepsilon\}$$

for some $\varepsilon > 0$. The relation $F(z+1) = \lambda\, F(z)$ allows us to find a real-analytic continuation of F to all of $\mathfrak{H}$. So under the assumptions of Part ii) we have $U = \mathbb{C}$. In Part iii) it is given that U contains $\mathfrak{H}$. So now we know only that U contains all $z \in \mathbb{C}$ with $\mathrm{Im}\, z > -\varepsilon$ for some $\varepsilon > 0$.

In both parts we have the expansion (8.21) for all $z \in \mathfrak{H}$. This leads to information concerning the coefficients.

For $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, we quote from [115, §4.1.1] the asymptotic behavior of the confluent hypergeometric series:

$${}_1F_1(1-r; 2-r; t) \;\sim\; \begin{cases} (1-r)\, t^{-1}\, e^{t} & \text{as } \mathrm{Re}\, t \to \infty\,, \\ \Gamma(2-r)\, (-t)^{r-1} & \text{as } \mathrm{Re}\, t \to -\infty\,. \end{cases}$$

The absolute convergence of the Fourier expansion of F implies the estimate of the coefficients c_n.

This gives, for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ the growth of the coefficients, and finishes the proof of Part ii) for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ Moreover, dividing by y^{1-r} we get a Fourier expansion converging on all of $\mathbb{C}$, and representing a real-analytic function on $\mathbb{C}$. That implies that $F(z)/f_r(z)$ is real-analytic on $\mathbb{C}$, hence $F \in \mathcal{H}_r^{\mathrm{b}}(\mathbb{C})$, which shows that F represents an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$, by Part i) of Proposition 8.4. This gives Part iii).

We are left with Part ii) for $r \in \mathbb{Z}_{\geq 1}$. For $r = 1$ we have $(\rho_1^{\mathrm{prj}} F)(z) = \frac{z-i}{2i} F(z)$ in $\mathcal{D}_1^{\omega,\mathrm{exc}}[\infty]$. Hence F has a holomorphic extension to $\mathbb{C}$. This extension is still given by a convergent Fourier expansion, which should be the expansion $\sum_n c_n F_{1,n}(z) = \sum_n c_n e^{2\pi i n z}$. This convergence on $\mathbb{C}$ implies the estimate of the coefficients.

Finally, if $r \in \mathbb{Z}_{\geq 2}$, then the term $\sum_n c_n e^{2\pi i n z}$ is holomorphic, and hence is in $\mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$. Again, we get convergence on all of $\mathbb{C}$. This ends the proof of Part ii). □

8.5. **Completion of the proof of Proposition 8.4.**

Proof of Part vii) for $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. The function $F \in \mathcal{H}_r(U \cap \mathfrak{H})$ represents an r-harmonic boundary germ $f \in \mathcal{B}_r(\mathbb{P}^1_{\mathbb{R}})$. According to Theorem 8.5 we have a unique decomposition $f = P + g$, with $P \in \mathcal{H}_r(\mathfrak{H})$ identified with the boundary germ it represents, and $g \in \mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$ with representative $G = F - P$ in $\mathcal{H}_r(U \cap \mathfrak{H})$. Since G represent an element of $\mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$ it is an element of $\mathcal{H}_r^{\mathrm{b}}(U_1)$ for some neighbourhood $U_1 \subset U$ of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$.

Now

$$f|_r(\lambda^{-1}T - 1) \;=\; P|_r(\lambda^{-1}T - 1) + g|_r(\lambda^{-1}T - 1)\,.$$

The left hand side is in $\mathcal{E}_r^{\omega} \subset \mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}})$ by condition b) in the assumption. So the direct sum in (8.10) shows that $\lambda^{-1} P|_r T = P$.

Since $P = F - G$ represents an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty] + \mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}}) \subset \mathcal{W}_r^{\omega}(\mathbb{R})$ we can apply Part iii) of Lemma 8.7 to P and conclude that $P \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$. Then $G = F - P$ represents an element

$$\begin{aligned} g \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty] \cap \mathcal{W}_r^{\omega}(\mathbb{P}^1_{\mathbb{R}}) \;&=\; (\rho_r)^{-1}\Big(\mathcal{D}_{2-r}^{\omega,\mathrm{exc}}[\infty] \cap \mathsf{prj}_{2-r}^{-1} \mathcal{V}_{2-r}^{\omega}(\mathbb{P}^1_{\mathbb{R}})\Big) \\ &=\; \rho_r^{-1} \mathcal{D}_{2-r}^{\omega} \;=\; \mathcal{E}_r^{\omega}\,. \end{aligned}$$

□

The proof for $r \in \mathbb{Z}_{\geq 1}$ requires some preparation.

Lemma 8.8. *Let $r \in \mathbb{Z}_{\geq 1}$. Suppose that F representing an element of $\mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$ satisfies the conditions in Part vii) in Proposition 8.4. Then there is a λ-periodic function $P \in \mathcal{O}(\mathbb{C})$ such that $G = F - P$ is holomorphic on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, and satisfies $G(\infty) = 0$.*

Proof. Condition a) in Part vii) of Proposition 8.4 tells us that F is r-harmonic on $\mathfrak{H} \smallsetminus K$ for some compact set $K \subset \mathfrak{H}$. Since we have the additional information that F represents an element of $\mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$ it is holomorphic on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{R}$ in $\mathbb{C}$. Hence $F \in \mathcal{O}(\mathbb{C} \smallsetminus K)$.

We define a holomorphic function P on $\mathbb{C}$ by

$$P(z) \;=\; \frac{1}{2\pi i}\int_{|\tau|=c} F(\tau)\,\frac{d\tau}{\tau-z}\,, \tag{8.22}$$

where c is chosen larger than $|z|$, and in such a way that K is enclosed by the path of integration. The function $z\mapsto \lambda^{-1}\,F(z{+}1)-F(z)$ is holomorphic on $\mathbb{C}\smallsetminus\bigl(K\cup T^{-1}K\bigr)$. It represents an element of $\mathcal{E}^\omega_r$ by Assumption b), hence it represents an element of $\mathcal{D}^\omega_r$, and is holomorphic on $\mathbb{P}^1_\mathbb{C}\smallsetminus\bigl(K\cup T^{-1}K\bigr)$, with at ∞ a zero of order at least r. Since $r\geq 1$, Cauchy's theorem implies that

$$\frac{1}{2\pi i}\int_{|\tau|=c}\Bigl(\lambda^{-1}\,F(\tau+1)-F(\tau)\Bigr)\frac{d\tau}{\tau-z} \;=\; 0\,,$$

for all sufficiently large c. This implies that $\lambda^{-1}P(z+1)=P(z)$ for all $z\in\mathbb{C}$.

Take $G=F-P$. With (8.22) we find for all sufficiently large c and $|z|<c<c_1$

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\tau|=c} G(\tau)\,\frac{d\tau}{\tau-z} \;&=\; P(z)-\frac{1}{2\pi i}\int_{|\tau_1|=c_1}\frac{1}{2\pi i}\int_{|\tau|=c}\frac{d\tau}{(\tau_1-\tau)\,(\tau-z)}\,F(\tau_1)\,d\tau_1\\ &=\; P(z)-P(z)\;=\;0\,.\end{aligned}$$

Insertion of the Laurent expansion $G(\tau)=\sum_{k\in\mathbb{Z}} b_k\,\tau^k$ of G at ∞ into the integral shows that $b_k=0$ for $k\geq 0$. So the function G is holomorphic on a neighbourhood of ∞ with a zero at ∞. □

Proof of Part vii) for $r=1$. We have $\mathcal{E}^\omega_1=\mathcal{D}^\omega_1$ and $\mathcal{E}^{\omega,\mathrm{exc}}_1[\infty]=\mathcal{D}^{\omega,\mathrm{exc}}_1[\infty]$; see (8.3). We apply Lemma 8.8, and use that a first order zero at ∞ suffices to conclude that G represents an element of $\mathcal{D}^\omega_1$. □

Proof of Part vii) for $r\in\mathbb{Z}_{\geq 2}$. We have $H\in\mathcal{H}_r(\mathfrak{H}\smallsetminus K)$ with a compact set $K\subset\mathfrak{H}$, for which we can arrange that $i\in K$.

We write $F=F_0+m$, with F_0 representing an element of $\mathcal{D}^{\omega,\mathrm{exc}}_r[\infty]$ and $m=\sum_{\mu=1-r}^{-1} c_\mu\,\mathrm{M}_{r,\mu}\in\mathcal{E}^\omega_r$, with the c_μ in $\mathbb{C}$. Then F_0 is r-harmonic on $\mathfrak{H}\smallsetminus K$, and is holomorphic on an $\{\infty\}$-excised neighbourhood. It also satisfies Condition b) in Part vii) of Proposition 8.4, so we can apply Lemma 8.8. This gives $F_0=G+P$, with a λ-periodic holomorphic function P on $\mathbb{C}$, and G holomorphic on a neighbourhood of $\mathfrak{H}^-\cup\mathbb{P}^1_\mathbb{R}$ in $\mathbb{P}^1_\mathbb{C}$ with a zero at ∞. For G to represent an element of $\mathcal{D}^\omega_r$ we would need a zero at ∞ of order at least r.

The function G shares with F_0 the property that $z\mapsto\lambda^{-1}\,G(z+1)-G(z)$ represents an element of $\mathcal{E}^\omega_r$, even an element of $\mathcal{D}^\omega_r$ by holomorphy. Insertion of this property in the power series of G at ∞ shows that the zero of G at ∞ has order at least $r-1$.

If $\lambda=1$, we cannot reach order r. In this case we write $G=G_0+c_0\,R_r$, where

$$R_r(z)\;=\;\Bigl(\frac{2i}{z-i}\Bigr)^{r-1},$$

and where c_0 is chosen such that G_0 has a zero at ∞ of order at least r. So G_0 represents an element of $\mathcal{D}^\omega_r$.

Part iii) of Proposition 7.2 shows that $R_r(z) = y^{1-r} + m_1$, with $m_1 \in \mathcal{E}_r^\omega$. Working modulo elements of $\mathcal{E}_r^\omega$ we have

$$F \;=\; F_0 + m \;\equiv\; G + P \;=\; G_0 + c_0\, R_r + P \;\equiv\; c_0\, y^{1-r} + c_0\, m_1 + P \;\equiv\; c_0\, y^{1-r} + P\,.$$

The function $c_0\, y^{1-r} + P$ represents a λ-periodic element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$. □

8.6. **Related work.** In [15] the analytic cohomology groups have values in the space $\mathcal{V}_s^\omega(\mathbb{P}^1_\mathbb{R})$ which is isomorphic to the space $\mathcal{W}_s^\omega(\mathbb{P}^1_\mathbb{R})$ by the results in [13, §5.2]. In the present context we work with the subspace $\mathcal{D}^\omega_{v,2-r}$ of $\mathsf{prj}_{2-r}^{-1}\mathcal{V}^\omega_{v,2-r}(\mathbb{P}^1_\mathbb{R})$, and we have to do more work to determine a submodule of analytic boundary germs related to $\mathcal{D}^\omega_{v,2-r}$.

For weights $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ we have the isomorphism in Part i) of Theorem 6.14, which points the way to the definition of $\mathcal{E}^\omega_{v,r}$. The power series approach in that theorem is more complicated than that in [13, Proposition 5.6]. For weights $r \in \mathbb{Z}_{\geq 1}$ we defined $\mathcal{E}^\omega_{v,r}$ so that it satisfies the properties in Proposition 8.4.

Property vii) in Proposition 8.4 is similar to [15, Lemma 9.23]. It will be essential for the proof of Theorem 10.4 in §10. The proof of this property in the case $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$ follows the proof in [15]. It uses the boundary germ splitting in Theorem 8.5 which is similar to [13, Proposition 5.3]. The resolvent kernel Q_r in (8.11), the Green's form in (8.15) and the Cauchy-like result Proposition 8.6 have their examples in (3.8), (3.13), Theorem 3.1 in [13], respectively.

To prove property vii) in Proposition 8.4 for positive integral weights we had to find other methods, which were inspired by the use of hyperfunctions and the Poisson transformation (§2.2 and §3.3 in [13]). In these notes we avoid the explicit use of hyperfunctions and the Poisson transformation.

9. Tesselation and cohomology

Up till now we worked with the standard description of group cohomology, recalled in §1.4. For the boundary germ cohomology we turn to the description of cohomology that turned out to be useful in [15]. We use the concepts and notations of those notes, and do not repeat a complete discussion. We invite the reader to have a quick look at [15, §6.1–3], where the approach is explained for cocompact discrete groups, and then to consult [15, §11] for the case of groups with cusps.

9.1. **Tesselations of the upper half-plane.** The tesselations that we use are called "of type **Fd**" in [15]. They are based on the choice of a suitable fundamental domain for $\Gamma\backslash\mathfrak{H}$.

Tesselation for the modular group. With the standard choice of the fundamental domain $\mathfrak{F}$ for $\Gamma(1)\backslash\mathfrak{H}$, a part of the tesselation looks as in Figure 14. The tesselation $\mathcal{T}$ is obtained by taking all $\Gamma(1)$-translates of the fundamental domain $\mathfrak{F}$ divided in a cuspidal triangle V_∞ and a compact part $\mathfrak{F}_Y$. The set of faces is $X_2^{\mathcal{T}} = \{\gamma^{-1}V_\infty, \gamma^{-1}\mathfrak{F}_Y \;:\; \gamma \in \Gamma(1)\}$. In the boundary $\partial_2\mathfrak{F}$ of the fundamental domain there are oriented edges e_∞ from a point $P_\infty = \frac{1}{2} + iY$ (with some $Y > 1$) to the cusp ∞, and compact edges e_1 from $e^{\pi i/3}$ to P_∞ and e_2 from i to $e^{\pi i/3}$. There is

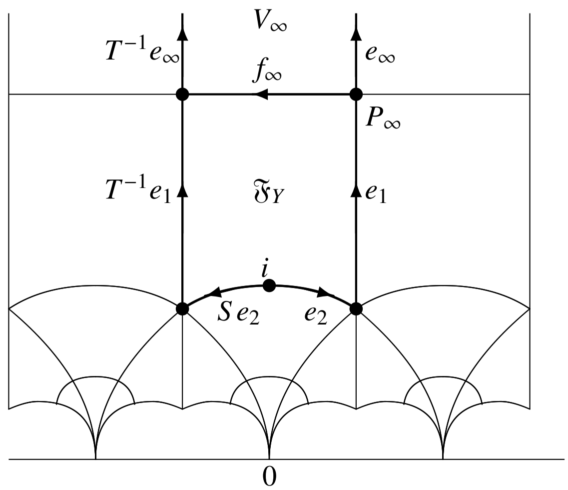

FIGURE 14. Sketch of a tesselation for the modular group, based on the standard fundamental domain.

also the horizontal edge f_∞ from P_∞ to $T^{-1}P_\infty$. These four edges generate a set $X_1^{\mathcal{T}}$ of oriented edges freely over $\overline{\Gamma(1)} = \{\pm 1\}\backslash\Gamma(1)$. If $e \in X_1^{\mathcal{T}}$ then the same edge with the opposite orientation is written as $-e$. We follow the convention used in [15] to include in $X_1^{\mathcal{T}}$ only one of the two oriented edges corresponding to a given unoriented edge. The points i, $e^{\pi i/3}$, P_∞ of $\mathfrak{H}$ together with the cusp ∞ generate over $\Gamma(1)$ the set $X_0^{\mathcal{T}}$ of vertices, but not freely over $\overline{\Gamma(1)}$, since i and $e^{\pi i/3}$ are fixed by subgroups of $\overline{\Gamma(1)}$ of orders 2 and 3 respectively. The subgroup of $\overline{\Gamma(1)}$ fixing P_∞ consists only of 1, and the group $\Gamma(1)_\infty$ fixing ∞ is infinite.

We define the subsets $X_i^{\mathcal{T},Y}$ consisting of all elements that are compact in $\mathfrak{H}$. So $X_0^{\mathcal{T},Y}$ is generated by i, $e^{\pi i/3}$, and P_∞; $X_1^{\mathcal{T},Y}$ by e_1, e_2 and f_∞; and $X_2^{\mathcal{T},Y}$ by $\mathfrak{F}_Y$.

General groups. In general, the fundamental domain $\mathfrak{F}$ is chosen in such a way that its closure in $\mathfrak{H} \cup \mathbb{P}^1_{\mathbb{R}}$ contains only one cusp of Γ from each Γ-orbit of cusps. The fundamental domain is the union of a compact part $\mathfrak{F}_Y$ and a number of cuspidal triangles $V_\mathfrak{a}$, for the cusps $\mathfrak{a}$ in the closure of $\mathfrak{F}$. Each $V_\mathfrak{a}$ has vertices $\mathfrak{a}$, $P_\mathfrak{a}$ and $\pi_\mathfrak{a}^{-1}P_\mathfrak{a}$, and a boundary consisting of edges $e_\mathfrak{a} \in X_1^{\mathcal{T}}$ from $P_\mathfrak{a}$ to $\mathfrak{a}$, $\pi_\mathfrak{a}^{-1}e_\mathfrak{a}$, and $f_\mathfrak{a} \in X_1^{\mathcal{T},Y}$ from $P_\mathfrak{a}$ to $\pi_\mathfrak{a}^{-1}P_\mathfrak{a}$. So each of these cuspidal triangles looks the same as the triangle V_∞ for the modular group.

9.2. **Resolutions based on a tesselation.** The tesselation $\mathcal{T}$ gives rise to Γ-modules $F_i^{\mathcal{T}} := \mathbb{C}[X_i^{\mathcal{T}}] \supset F_i^{\mathcal{T},Y} := \mathbb{C}[X_i^{\mathcal{T},Y}]$, which are considered as *right* modules, by $(x)|\gamma = (\gamma^{-1}x)$. There are the obvious boundary operators $\partial_i : \mathbb{C}[X_i^{\mathcal{T}}] \to \mathbb{C}[X_{i-1}^{\mathcal{T}}]$ that satisfy $\partial_i\mathbb{C}[X_i^{\mathcal{T},Y}] \subset \mathbb{C}[X_{i-1}^{\mathcal{T},Y}]$.

For the modular group:

$$\partial_2(V_\infty) \;=\; (e_\infty)|(1-T)-(f_\infty)\,, \quad \partial_2(\mathfrak{F}_Y) \;=\; (e_1)|(1-T)+(e_2)|(1-S)+(f_\infty)\,.$$

This leads to complexes $(F_\cdot^{\mathcal{T}}) \supset (F_\cdot^{\mathcal{T},Y})$ of Γ-modules. It turns out ([15, §11.2]) that for right Γ-modules V that are vector spaces over $\mathbb{C}$ the cohomology of the resulting complex $\hom_{\mathbb{C}[\Gamma]}(F_\cdot^{\mathcal{T},Y},V)$ is canonically isomorphic to the group cohomology $H^\cdot(\Gamma;V)$. In working with this description of cohomology it is often useful to identify a $\mathbb{C}[\Gamma]$-homomorphism $F_i^{\mathcal{T}} = \mathbb{C}[X_i^{\mathcal{T}}] \to V$ with the corresponding map $c : X_i^{\mathcal{T}} \to V$, which satisfies $c(\gamma^{-1}x) = c(x)|\gamma$ for all $\gamma \in \Gamma$, $x \in X_i^{\mathcal{T}}$.

We use the complex $(F_\cdot^{\mathcal{T}})$ to describe the mixed parabolic cohomology. The mixed parabolic cochains are defined by

$$C^i(F_\cdot^{\mathcal{T}};V,W) \;=\; \Bigl\{c : X_i^{\mathcal{T}} \to W \;:\; c(x) \in V \text{ if } x \in X_i^{\mathcal{T},Y}\,, \qquad c(\gamma^{-1}x) = c(x)|\gamma \text{ for all } \gamma \in \Gamma\Bigr\}\,. \tag{9.1}$$

A derivation can be defined by $d^i c(x) \;=\; (-1)^{i+1}\,c(\partial_{i+1}x)$ for $x \in X_i^{\mathcal{T}}$. We often write d instead of d^i.

The space $Z^i(F_\cdot^{\mathcal{T}};V,W)$ of mixed parabolic cocycles is defined as the kernel of $d^i : C^i(F_\cdot^{\mathcal{T}};V,W) \to C^{i+1}(F_\cdot^{\mathcal{T}};V,W)$ and the subspace of mixed parabolic coboundaries $B^i(F_\cdot^{\mathcal{T}};V,W)$ as $d^{i-1}C^{i-1}(F_\cdot^{\mathcal{T}};V,W)$ if $i \geq 1$ and as the zero subspace if $i = 0$. Then the cohomology groups of the complex,

$$Z^i(F_\cdot^{\mathcal{T}};V,W)\;/\;B^i(F_\cdot^{\mathcal{T}};V,W)\,, \tag{9.2}$$

are for $i = 1$ isomorphic to the mixed parabolic cohomology groups $H^1_{\mathrm{pb}}(\Gamma;V,W)$ in Definition 1.3. In [15, §11.3] the mixed parabolic cohomology groups $H^i_{\mathrm{pb}}(\Gamma;V,W)$ are defined as the spaces in (9.2) for all i.

In particular for $i = 1$ we have the following commutative diagram for $\mathbb{C}[\Gamma]$-modules $V \subset W$:

$$\begin{tikzcd}
Z^1(\Gamma;V)/B^1(\Gamma;V) & Z^1(F_\cdot^{\mathcal{T},Y};V)/B^1(F_\cdot^{\mathcal{T},Y};V) \arrow[l, "\cong"'] \\
Z^1(\Gamma;V,W)/B^1(\Gamma;V) \arrow[u, hook] & Z^1(F_\cdot^{\mathcal{T}};V,W)/B^1(F_\cdot^{\mathcal{T}};V,W) \arrow[l, "\cong"'] \arrow[u, hook]
\end{tikzcd}$$

The isomorphic spaces in the top row give two isomorphic descriptions of $H^1(\Gamma;V)$, and the two spaces in the bottom row of $H^1_{\mathrm{pb}}(\Gamma;V,W)$.

The conditions on the tesselations are such that the action of $\bar\Gamma = \{\pm 1\}\backslash\Gamma$ on $X_1^{\mathcal{T}}$ and $X_2^{\mathcal{T}}$ is free on finitely many elements. So for $i \geq 1$ the cochains $c \in C^i(F_\cdot^{\mathcal{T}};V,W)$ are determined by their values on a $\mathbb{C}[\Gamma]$-basis of $\mathbb{C}[X_i^{\mathcal{T}}]$. For the modular group, $c \in C^1(F_\cdot^{\mathcal{T}};V,W)$ is completely determined by $c(e_1)$, $c(e_2)$ and $c(f_\infty)$ in V, and $c(e_\infty) \in W$, and $b \in C^2(F_\cdot^{\mathcal{T}};V,W)$ is determined by $c(\mathfrak{F}_Y) \in V$ and $c(V_\infty) \in W$. For $i = 0$ there are in general no bases over $\mathbb{C}[\bar\Gamma]$. The fact that each cusp is fixed by an infinite subgroup of $\Gamma(1)$ makes the difference between parabolic cohomology and standard group cohomology. Points of $X_0^{\mathcal{T},Y}$ may be fixed by non-trivial finite subgroups of $\bar\Gamma$. As long as we work with Γ-modules that are vector spaces over $\mathbb{C}$ this is not important. For $\overline{\Gamma(1)}$ it suffices if we can divide by 2 and 3 in the modules that we use.

For a cocycle $c \in Z^1(F^{\mathcal{T}}_{\cdot}; V, W)$ the value $c(p)$ on a cycle $p \in \mathbb{Z}[X_1^{\mathcal{T}}]$ corresponding to a path from P_1 to P_2 (both in $X_0^{\mathcal{T}}$) does not depend on the choice of the path along edges in $X_1^{\mathcal{T}}$, only on the end-points P_1 and P_2. So we can write $c(p) = c(P_1, P_2)$, and view c as a function on $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}}$. In general $c(p) \in W$. It satisfies

$$\begin{aligned} c(P_1,P_3) &= c(P_1,P_2)+c(P_2,P_3) && \text{for } P_j \in X_0^{\mathcal{T}}\,,\\ c(\gamma^{-1}P_1,\gamma^{-1}P_2) &= c(P_1,P_2)|\gamma && \text{for } \gamma\in\Gamma\,,\ P_j \in X_0^{\mathcal{T}}\,. \end{aligned} \tag{9.3}$$

If both P_1 and P_2 are in $X_0^{\mathcal{T},Y}$, then the path can be chosen in $\mathbb{Z}[X_0^{\mathcal{T},Y}]$, and hence $c(p) \in V$.

Now choose a base point $P_0 \in X_0^{\mathcal{T},Y}$. Then $\psi_\gamma = c(\gamma^{-1}P_0, P_0)$ is in V for each $\gamma \in \Gamma$. It turns out to define a group cocycle $\psi \in Z^1(\Gamma; V)$. It is even a mixed parabolic group cocycle in $Z^1_{\mathrm{pb}}(\Gamma; V, W)$. Let us check this in the situation of the modular group, with the tesselation discussed above. Then

$$\begin{aligned} \psi_T &= c(T^{-1}P_0,P_0) = c(T^{-1}P_0,T^{-1}P_\infty)+c(T^{-1}P_\infty,T^{-1}\infty)\\ &\quad + c(\infty,P_\infty)+c(P_\infty,P_0) = \bigl(-c(e_\infty)+c(P_\infty,P_0)\bigr)\,\Big|\,(1-T) \in W|(1-T)\,. \end{aligned}$$

This computation shows that the presence of ∞ as a vertex of the tesselation forces parabolicity of the cocycle. (We use $|$ to denote the action of Γ on the $F_i^{\mathcal{T}} = \mathbb{C}[X_i^{\mathcal{T}}]$, as well as in the modules V and W.)

On can check that this association $c \mapsto \psi$ sends coboundaries to coboundaries and that taking a different base point P_0 does not change the cohomology class. The map $c \mapsto \psi$ is an easy way to describe the canonical isomorphism between the description of cohomology with a tesselation and the standard description of group cohomology.

9.3. **Cocycles attached to automorphic forms.** To describe the linear maps $\mathbf{r}_r^\omega : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$ and $\mathbf{q}_r^\omega : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{E}^\omega_{v,r})$ in Theorem A and Proposition 6.10 in the approach to cohomology based on a tesselation $\mathcal{T}$ we use for an unrestricted holomorphic automorphic form $F \in A_r(\Gamma, v)$ the cocycles $\psi_F \in Z^1(F^{\mathcal{T},Y}_{\cdot}; \mathcal{D}^\omega_{v,2-r})$ and $c_F \in Z^1(F^{\mathcal{T},Y}_{\cdot}; \mathcal{E}^\omega_{v,r})$ given on edges $x \in X_1^{\mathcal{T},Y}$ by

$$\psi_F(x;t) := \int_{\tau\in x} \omega_r(F;t,\tau) = \int_{\tau\in x} (\tau-t)^{r-2}\,F(\tau)\,d\tau\,, \tag{9.4}$$

$$c_F(x;z) := \int_{\tau\in x} K_r(z;\tau)\,F(\tau)\,d\tau = \int_{\tau\in x} \frac{2i}{z-\tau}\Bigl(\frac{\bar z-\tau}{\bar z - z}\Bigr)^{r-1} F(\tau)\,d\tau\,. \tag{9.5}$$

The orientation of the edge x determines the direction of the integration. We use the boundary germ cohomology in the next section, and hence will work with the cocycle c_F. Property iv) in Proposition 8.4 implies that c_F has values in $\mathcal{E}^\omega_r$.

Notations. Let $\mathcal{E}_r^{\omega^0,\mathrm{exc}} = \varinjlim \mathcal{E}_r^{\omega,\mathrm{exc}}[\mathfrak{a}_1,\ldots,\mathfrak{a}_n]$, where $\{\mathfrak{a}_1,\ldots,\mathfrak{a}_n\}$ runs over the finite subsets of cusps of Γ.

The space $\mathcal{E}_r^{\omega^*,\mathrm{exc}}$, defined in Proposition 8.4, is invariant under the operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$, but these operators act in $\mathcal{E}_r^{\omega^0,\mathrm{exc}}$ only if g maps cusps to cusps. By $\mathcal{E}_{v,r}^{\omega}$, $\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}}$, and $\mathcal{E}_{v,r}^{\omega^*,\mathrm{exc}}$ we denote the Γ-modules for the action $|_{v,r}$ on the corresponding spaces.

Image of $\mathbf{q}_r^\omega$. For edges in $X_1^{\mathcal{T}} \smallsetminus X_1^{\mathcal{T},Y}$ the integration does not make sense, unless F happens to be a cusp form. To extend c_F to $X_1^{\mathcal{T}}$ we need to define $c_F(e_\mathfrak{a})$ for each cusp $\mathfrak{a}$ of Γ such that $c_F(\partial_2 V_\mathfrak{a}) = 0$, in the notation of §9.1.

For weights $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ and the highest weight spaces in Definition 8.1, Theorem 3.9 implies $\mathbf{q}_r^\omega A_r(\Gamma, v) \subset H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}_{v,r}^{\omega}, \mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}})$. For $r \in \mathbb{Z}_{\geq 2}$ we will see in §11.1 that not all automorphic forms give rise to mixed parabolic cocycles with values in the analytic boundary germs.

9.4. **Derivatives of *L*-functions.** In the introduction we mentioned that derivatives of L-functions can be related to cocycles. We illustrate this here by an example.

Let f be a newform of weight 2 for $\Gamma = \Gamma_0(N)$ such that $L_f(1) = 0$ (under the assumption that f is even for the Fricke involution). Set

$$u(z) = \log(\eta(z)\eta(Nz)), \qquad z \in \mathfrak{H}.$$

Then, as shown in [51],

$$L_f'(1) = \frac{1}{\pi}\int_0^\infty f(iy)u(iy)dy. \tag{9.6}$$

This integral, though reminiscent of a period integral, has an integrand that is far from Γ-invariant and thus does not give a cocycle. To address this problem, we first note that, by the defining formula of $u(z)$, the RHS of (9.6) equals the value at $r = 0$ of the derivative

$$\frac{d}{dr}\left(\frac{1}{\pi}\int_0^\infty f(iy)\,(\eta(iy)\eta(iNy))^r\,dy\right)\bigg|_{r=0}.$$

This integral is still not Γ-invariant but now it can be formulated in terms of cocycles considered in these notes.

Set

$$f_r(z) = f(z)(\eta(z)\eta(Nz))^r\,.$$

This is a cusp form of weight $2+r$ for Γ depending holomorphically on r on a neighbourhood of 0 in $\mathbb{C}$. The corresponding multiplier system w_r is also holomorphic in r.

We refine the tesselation in Figure 14 so that the geodesic from 0 to ∞ is a sum of edges, forming a path $p \in \mathbb{Z}[X_1^{\mathcal{T}}]$. (Then the faces V_∞ and $\mathfrak{F}_Y$ are each divided into two faces.) We have for any automorphic form F the value

$$\psi_F(p;t) = \int_{\tau\in p} \omega_r(F;t,\tau)\,.$$

Applying this to f_r defined above, we obtain $\psi_{f_r}(p;\cdot) \in \mathcal{D}^{\omega,\infty}_{w,-r}[0,\infty]$, since f_r is a cusp form. In particular

$$\psi_{f_r}(p;0) \;=\; i\,e^{\pi i r/2} \int_0^\infty f(iy)\,e^{r\,u(iy)}\,y^r\,dy\,. \tag{9.7}$$

With the change of variables $y \mapsto 1/Ny$, using the invariance of f_r under the Fricke involution, this can be seen to be equal to

$$-i\,N^{-r/2}\,e^{\pi i r} \int_0^\infty f(iy)\,e^{r\,u(iy)}\,dy\,.$$

With Goldfeld's result we obtain the following relation between the cocycle ψ_{f_r} and the L-function:

$$\begin{aligned} \psi_{f_r}(p;0) \;&=\; -i\,L_f(1) + r\Big(\frac{\pi i}{2}\,\log N\,L_f(1) - \pi^2\,L_f(1) - \pi i\,L_f'(1)\Big) \\ &\qquad + \mathrm{O}(r^2) \quad (r\to 0) \\ &=\; -\pi i r\,L_f'(1) + \mathrm{O}(r^2) \quad (r\to 0)\,. \end{aligned} \tag{9.8}$$

9.5. **Related work.** The general approach to group cohomology via an arbitrary projective resolution is well known. See for instance, in Brown [8], Chap. III, §1, for the definition, and Chap. I, §5, for the standard complex. Also more topologically oriented complexes are well known; see for instance [8, §4, Chap. 1]. In [15] the tesselations of the upper half-plane based on a fundamental domain of the discrete group in question turned out to be useful.

10. Boundary germ cohomology and automorphic forms

10.1. **Spaces of global representatives for highest weight spaces.** Property i) in Proposition 8.4 shows that elements of $\mathcal{E}^\omega_r$ are represented by elements of $\mathcal{H}^{\mathrm{b}}_r(U)$, hence by r-harmonic functions on $U\cap\mathfrak{H}$. Property v) shows that if $F\in\mathcal{H}^{\mathrm{b}}_r(U)$ is non-zero, then $U\supset\mathfrak{H}$ is impossible. For the cohomological manipulations in this section it is desirable to have spaces of representatives that are defined on $\mathfrak{H}$. If non-zero, these functions cannot be r-harmonic everywhere on $\mathfrak{H}$.

Definition 10.1. We define the spaces $\mathcal{G}^\omega_r$, $\mathcal{G}^{\omega^*}_r$ and $\mathcal{G}^{\omega^0}_r$ of functions on $\mathfrak{H}$:

$$\begin{aligned} \mathcal{G}^\omega_r \;:=\; \Big\{F\in C^2(\mathfrak{H}) \;:\; &\text{there exists an open neighbourhood } U \\ &\text{of } \mathbb{P}^1_{\mathbb{R}} \text{ in } \mathbb{P}^1_{\mathbb{C}} \text{ such that } F|_{U\cap\mathfrak{H}} \text{ is in } \mathcal{H}^{\mathrm{b}}_r(U) \\ &\text{and represents an element of } \mathcal{E}^\omega_r\Big\}, \end{aligned} \tag{10.1}$$

$$\begin{aligned} \mathcal{G}^{\omega^*,\mathrm{exc}}_r \;:=\; \Big\{F\in C^2(\mathfrak{H}) \;:\; &\text{there exists an excised neighbourhood } U \\ &\text{such that } F|_{U\cap\mathfrak{H}} \text{ is in } \mathcal{H}^{\mathrm{b}}_r(U) \text{ and represents} \\ &\text{an element of } \mathcal{E}^{\omega^*,\mathrm{exc}}_r\Big\}, \end{aligned} \tag{10.2}$$

$$\begin{aligned}(10.3)\qquad \mathcal{G}_r^{\omega^0,\mathrm{exc}} := \Big\{F \in C^2(\mathfrak{H}) \ :\ &\text{there exists an excised neighbourhood } U\\ &\text{with excised set consisting of cusps, such that } F|_{U\cap\mathfrak{H}}\\ &\text{is in } \mathcal{H}_r^{\mathrm{b}}(U) \text{ and represents an element of } \mathcal{E}_r^{\omega^0,\mathrm{exc}}\Big\}.\end{aligned}$$

The operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$ act in $\mathcal{G}_r^{\omega}$ and $\mathcal{G}_r^{\omega^*,\mathrm{exc}}$, and in $\mathcal{G}_r^{\omega^0,\mathrm{exc}}$ if $g \in \Gamma$. By $\mathcal{G}_{v,r}^{\omega}$, $\mathcal{G}_{v,r}^{\omega^*,\mathrm{exc}}$, $\mathcal{G}_{v,r}^{\omega^0,\mathrm{exc}}$ we denote the corresponding Γ-modules with the action $|_{v,r}$.

Remarks. (a) This definition formalizes for $\mathcal{E}_r^{\omega}$ what we did informally for $\mathcal{D}_{2-r}^{\omega}$ in Remark 4.2, b).

(b) While, by Part v) of Proposition 8.4,

$$(10.4)\qquad \mathcal{H}_r(\mathfrak{H}) \cap \mathcal{G}_r^{\omega} = \{0\},$$

the space $\mathcal{H}_r(\mathfrak{H}) \cap \mathcal{G}_r^{\omega^*,\mathrm{exc}}$ contains non-zero elements, for instance the functions $F_{r,n}$ in (8.20).

Definition 10.2. We define $\mathcal{N}_r^{\omega}$, $\mathcal{N}_r^{\omega^*,\mathrm{exc}}$, or $\mathcal{N}_r^{\omega^0,\mathrm{exc}}$ as the kernels of the natural maps $\mathcal{G}_r^{\omega} \to \mathcal{E}_r^{\omega}$, $\mathcal{G}_r^{\omega^*,\mathrm{exc}} \to \mathcal{E}_r^{\omega^*,\mathrm{exc}}$, or $\mathcal{G}_r^{\omega^0,\mathrm{exc}} \to \mathcal{E}_r^{\omega^0,\mathrm{exc}}$ which assign to F the boundary germ represented by it.

Proposition 10.3. i) *$\mathcal{N}_r^{\omega}$, and $\mathcal{N}_r^{\omega^*,\mathrm{exc}}$ are invariant under the operators $|_r g$ with $g \in \mathrm{SL}_2(\mathbb{R})$, and the action $|_{v,r}$ makes $\mathcal{N}_r^{\omega^0,\mathrm{exc}}$ into a Γ-module $\mathcal{N}_{v,r}^{\omega^0,\mathrm{exc}}$.*

ii) *The space $\mathcal{N}_r^{\omega}$ is the space $C_c^2(\mathfrak{H})$ of the twice differentiable compactly supported functions on $\mathfrak{H}$, and $\mathcal{N}_r^{\omega^*,\mathrm{exc}}$, respectively $\mathcal{N}_r^{\omega^0,\mathrm{exc}}$, is the space of the twice differentiable functions on $\mathfrak{H}$ with support contained in a set $\mathfrak{H} \smallsetminus U$ where U is an excised neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$, which in the case of $\mathcal{N}_r^{\omega^0,\mathrm{exc}}$ has an excised set consisting of cusps.*

iii) *The diagram of Γ-equivariant maps*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}_{v,r}^{\omega} \arrow[r] \arrow[d, hook] & \mathcal{G}_{v,r}^{\omega} \arrow[r] \arrow[d, hook] & \mathcal{E}_{v,r}^{\omega} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathcal{N}_{v,r}^{\omega^*,\mathrm{exc}} \arrow[r] & \mathcal{G}_{v,r}^{\omega^*,\mathrm{exc}} \arrow[r] & \mathcal{E}_{v,r}^{\omega^*,\mathrm{exc}} \arrow[r] & 0 \end{tikzcd}$$

commutes. The rows are exact sequences.

Proof. Part i) follows directly from Definition 10.2.

For Part ii), suppose that F is in the kernel of $\mathcal{G}_r^{\omega} \to \mathcal{E}_r^{\omega}$, $\mathcal{G}_r^{\omega^*,\mathrm{exc}} \to \mathcal{E}_r^{\omega^*,\mathrm{exc}}$, or $\mathcal{G}_r^{\omega^0,\mathrm{exc}} \to \mathcal{E}_r^{\omega^0,\mathrm{exc}}$, then $F = 0$ on a set $U \cap \mathfrak{H}$ with U a neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, or $U \cap \mathfrak{H} = U_0 \cap \mathfrak{H}$ for an excised neighbourhood U_0 of $\mathbb{P}^1_{\mathbb{R}}$. In the former case $\mathfrak{H} \smallsetminus U$ is relatively compact in $\mathfrak{H}$, hence F has compact support. In the latter case F is zero on an excised neighbourhood intersected with $\mathfrak{H}$.

For the exactness in Part iii) we need to prove the surjectivity of the linear maps $\mathcal{G}_r^{\omega} \to \mathcal{E}_r^{\omega}$ and $\mathcal{G}_r^{\omega^*,\mathrm{exc}} \to \mathcal{E}_r^{\omega^*,\mathrm{exc}}$. The commutativity of the diagram is clear.

We start with a representative $F \in \mathcal{H}_r^b(U)$ of an element of $\mathcal{E}_r^\omega$, respectively $\mathcal{E}_r^{\omega^*,\mathrm{exc}}$, where U is a neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, respectively contained in an excised neighbourhood U_0 of $\mathbb{P}^1_{\mathbb{R}}$ such that $U \cap \mathfrak{H} = U_0 \cap \mathfrak{H}$. We take smaller sets $U_1 \subset U_2 \subset U$ such that U_2 is a neighbourhood of the closure of U_1 and U is a neighbourhood of the closure of U_2, and consider a cut-off function $\psi \in C^2(\mathfrak{H})$ equal to 1 on U_1 and equal to 0 on $\mathfrak{H} \smallsetminus U_2$. Then $z \mapsto \psi(z)\, F(z)$, extended by 0, is an element of $\mathcal{G}_r^\omega$, respectively $\mathcal{G}_r^{\omega^*,\mathrm{exc}}$, representing the same boundary germ as F. □

Lemma 10.4. *Let $\lambda \in \mathbb{C}^*$. If $h \in \mathcal{E}_r^{\omega^*,\mathrm{exc}}$ and $\lambda^{-1} h|_r T - h \in \mathcal{E}_r^\omega$, then $h \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$.*

Proof. In the same way as for Lemma 3.1. □

Definition 10.5.
- i) For $f \in \mathcal{E}_r^{\omega^*,\mathrm{exc}}$ we denote by $\mathsf{BdSing}\, f$ the minimal set $\{\xi_1, \ldots, \xi_n\}$ such that $f \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$, and call it the set of *boundary singularities* of f. For $F \in \mathcal{H}_r^{\mathrm{b}}(U)$ we denote by $\mathsf{BdSing}\, F$ the set $\mathsf{BdSing}\, f$ for the boundary germ f represented by F.
- ii) For any twice differentiable function F on $\mathfrak{H}$ we denote by $\mathsf{Sing}_r\, F$ the complement of the maximal open set in $\mathfrak{H}$ on which $\Delta_r F = 0$, and call it the set of *singularities* of F.
- iii) Analogously we define $\mathcal{G}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$ as the set of $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$ representing an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\xi_1, \ldots, \xi_n]$, and $\mathsf{BdSing}\,_r F$ for $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$ as the set of boundary singularities of the element of $\mathcal{E}_r^{\omega^*,\mathrm{exc}}$ that F represents.

Remarks. (a) $\mathsf{BdSing}\, F \subset \mathbb{P}^1_{\mathbb{R}}$ and $\mathsf{Sing}_r\, F \subset \mathfrak{H}$ for each $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$.

(b) For elements of $\mathcal{D}_{2-r}^{\omega^*}$ we dealt only with boundary singularities, and often called them singularities. For $\mathcal{E}_r^{\omega^*,\mathrm{exc}}$ and $\mathcal{G}_r^{\omega^*,\mathrm{exc}}$ it is important to distinguish both types of singularities.

(c) The properties in Proposition 8.4 imply properties of sets of boundary singularities. For instance

$$\mathsf{BdSing}\,(f|_r g) \;=\; g^{-1}\, \mathsf{BdSing}\, f \qquad \text{for } g \in \mathrm{SL}_2(\mathbb{R})\,. \tag{10.5}$$

(d) If $F \in \mathcal{G}_r^\omega$ then $\mathsf{Sing}_r\, F$ is a compact subset of $\mathfrak{H}$, and if $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$ then $\mathsf{Sing}_r\, F$ is contained in an excised neighbourhood.

λ-periodic elements.

Definition 10.6. For $\lambda \in \mathbb{C}^*$, put $\mathcal{I}_r(\lambda) := \{ f \in \mathcal{E}_r^{\omega^*,\mathrm{exc}} \;:\; f|_r T = \lambda f\}$.

Lemma 10.7. *Let $\lambda \in \mathbb{C}^*$.*
- i) *Each element of $\mathcal{I}_r(\lambda)$ is represented by a unique λ-periodic function in $\mathcal{H}_r(\mathfrak{H}) \cap \mathcal{G}_r^{\omega^*,\mathrm{exc}}$.*
- ii) *If $F \in \mathcal{N}_r^{\omega^*,\mathrm{exc}}$ is λ-periodic, then $F = 0$.*

Proof. Let $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$ represent an element of $\mathcal{I}_r(\lambda)$. Then it represents an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$ by Lemma 10.4.

From Part i) in Proposition 8.4 we see that $F \in \mathcal{H}_r(U \cap \mathfrak{H})$ for an excised neighbourhood U with excised set $\{\infty\}$, and that $\lambda^{-1}F(z+1) = F(z)$ for all $z \in \mathfrak{H} \cap U \cap T^{-1}U$.

This implies that F can be analytically extended to give a λ-periodic element of $\mathcal{H}_r(\mathfrak{H})$. Since this analytic extension is determined by its values on a strip $0 < \operatorname{Im} z < \varepsilon$ it is unique.

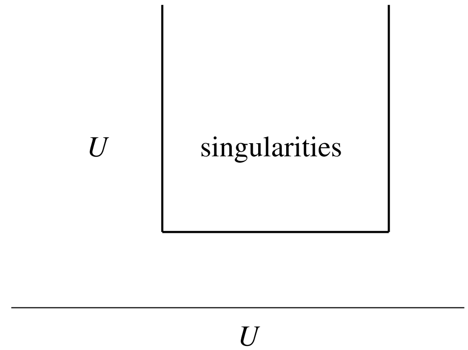

Figure 15

If the λ-periodic function F represents an element of $\mathcal{N}_r^{\omega^*,\mathrm{exc}}$ then F is zero on U, hence the extension is zero. □

Lemma 10.8. *Let $\mathfrak{a}$ be a cusp of Γ. Denote by $\Gamma_{\mathfrak{a}}$ the subgroup of Γ fixing $\mathfrak{a}$. Then the following sequence is exact:*

$$0 \to \left(\mathcal{N}_{v,r}^{\omega^*,\mathrm{exc}}\right)^{\Gamma_{\mathfrak{a}}} \to \left(\mathcal{G}_{v,r}^{\omega^*,\mathrm{exc}}\right)^{\Gamma_{\mathfrak{a}}} \to \left(\mathcal{E}_{v,r}^{\omega^*,\mathrm{exc}}\right)^{\Gamma_{\mathfrak{a}}} \to 0\,.$$

Proof. The group $\Gamma_{\mathfrak{a}}$ is generated by $\pi_{\mathfrak{a}} = g_{\mathfrak{a}} T g_{\mathfrak{a}}^{-1}$. By conjugation we can reduce the statement of the lemma to the exactness of the sequence one obtains if one takes in the sequence

$$0 \to \mathcal{N}_r^{\omega^*,\mathrm{exc}} \to \mathcal{G}_r^{\omega^*,\mathrm{exc}} \to \mathcal{E}_r^{\omega^*,\mathrm{exc}} \to 0$$

the kernel of the operator $|_r(\lambda^{-1}T - 1)$ with $\lambda = v(\pi_{\mathfrak{a}})$. This does not necessarily produce an exact sequence, but here we get by Lemma 10.7 the sequence

$$0 \to 0 \to \mathcal{I}_r(\lambda) \to \mathcal{I}_r(\lambda) \to 0\,,$$

which is exact. (We identify $\mathcal{I}_r(\lambda)$ with the space of the harmonic representatives in Part i) of Lemma 10.7.) □

Lemma 10.9. *Let $\lambda \in \mathbb{C}^*$. Suppose that the function F on $\mathfrak{H}$ satisfies:*

a) $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$,
b) $\mathsf{Sing}_r F$ *is a compact subset of* $\mathfrak{H}$,
c) $z \mapsto \lambda^{-1} F(z+1) - F(z)$ *is an element of* $\mathcal{G}_r^{\omega}$.

Then $F = P + G$ with $P \in \mathcal{I}_r(\lambda)$ and $G \in \mathcal{G}_r^{\omega}$.

Proof. The open set $\mathfrak{H} \smallsetminus \mathsf{Sing}_r F$ is of the form $U \cap \mathfrak{H}$ with U a neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The restriction f of F to $U \cap \mathfrak{H}$ represents an element of $\mathcal{E}_r^{\omega^*,\mathrm{exc}}$. Assumptions a) and c) imply that f represents an element of $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$, by Lemma 10.4. Part vii) in Proposition 8.4 implies the existence of $p \in \mathcal{I}_r(\lambda)$ such that $g = f - p \in \mathcal{E}_r^{\omega}$. Taking P as the global representative of p in Part i) of Lemma 10.7 we get a representative $G := F - P$ of g in $\mathcal{G}_r^{\omega}$. □

10.2. **From parabolic cocycles to automorphic forms.** Now we start with a mixed parabolic cocycle and construct a corresponding holomorphic automorphic form.

Proposition 10.10. i) *If $c \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ there is $u([c],\cdot) \in A_r(\Gamma, v)$ that depends only on the cohomology class of c, and $[c] \mapsto u([c],\cdot)$ defines a linear map*

$$\alpha_r : H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \to A_r(\Gamma, v)\,. \tag{10.6}$$

ii) *Let $F \in A_r(\Gamma, v)$ such that $\mathbf{q}^{\omega}_r F \in H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$, then $u([c_F],\cdot) = F$, with c_F as in* (9.5).

Proof. The proof is almost identical to that of [15, Proposition 12.2]. Table 3 compares the analogous quantities. Instead of repeating the proof, we give below

holomorphic forms	**Maass forms**
Γ-module $\mathcal{E}^{\omega}_{v,r}$	Γ-module $\mathcal{W}^{\omega}_{s}$
Γ-module $\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}$	Γ-module $\mathcal{W}^{\omega^*,\mathrm{exc}}_{s}$
cocycle c	cocycle ψ
cochain $\tilde{c}$	cochain $\tilde{\psi}$
$\mathcal{G}^{\omega}_{v,r},\ \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}$	$\mathcal{G}^{\omega}_{s},\ \mathcal{G}^{\omega^*,\mathrm{exc}}_{s}$
$\mathcal{N}^{\omega}_{v,r},\ \mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r}$	$\mathcal{N}^{\omega}_{s},\ \mathcal{N}^{\omega^*,\mathrm{exc}}_{s}$
$u([c],\cdot)$	u_{ψ}

TABLE 3. Correspondence between the quantities in the proof here, for holomorphic automorphic forms, and the quantities in the proof of [15, Proposition 12.2], for Maass forms of weight 0 and more general invariant eigenfunctions. Here we work with boundary singularities restricted to the cusps, whereas in [15] the singularities were general at first, and had to be reduced to singularities in cusps by an additional step.

a discussion of the main ideas in the context of the modular group. There is one complication, which is not present in [15]. We handle it in Lemma 10.11. □

Lift of the cocycle. Let $c \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ be given. Its values $c(x)$ on $x \in X^{\mathcal{T}}_1$ are boundary germs in $\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,e}$. See the right column in the diagram in Proposition 10.3. We want to lift c to a cochain $\tilde{c} \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^{\omega}_{v,r}, \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$, which involves the central column in the diagram. For each x in the $\mathbb{C}[\Gamma(1)]$-basis $\{e_1, e_2, f_\infty\}$ of $F^{\mathcal{T},Y}_1$ we can, according to Proposition 10.3 choose a representative $\tilde{c}(x) \in \mathcal{G}^{\omega}_{v,r}$ of $c(x) \in \mathcal{E}^{\omega}_{v,r}$. For $c(e_\infty)$ we can choose a representative

$\tilde c(e_\infty) \in \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}$. Since $\tilde c(e_\infty)|_{v,r}(1-T)$ represents $c(e_\infty)|_{v,r}(1-T) = c(f_\infty) \in \mathcal{E}^\omega_{v,r}$, we have $\mathsf{BdSing}\,\tilde c(e_\infty) \subset \{\infty\}$ by Lemma 10.4. So $\tilde c$ is determined by

$$\begin{aligned} \tilde c(e_1),\ \tilde c(e_2),\ \tilde c(f_\infty) \in \mathcal{G}^\omega_{v,r}\,,&\ \text{ representatives of } c(e_1),\ c(e_2),\ c(f_\infty)\,,\\ \tilde c(e_\infty) \in \mathcal{G}^{\omega,\mathrm{exc}}_{v,r}[\infty]\,,&\ \text{ representative of } c(e_\infty)\,. \end{aligned} \tag{10.7}$$

For each $x \in \{e_1, e_2, f_\infty\}$ the set $\mathsf{Sing}_r\,\tilde c(x)$ is compact. So we can find $R > 0$ such that for each of these three edges the set $\mathsf{Sing}_r\,\tilde c(x)$ is contained in the R-neighbourhood (for the hyperbolic distance) of x. Furthermore $\mathsf{Sing}_r\,\tilde c(e_\infty)$ is contained in the complement of an excised neighbourhood with excised set $\{\infty\}$, hence

$$\mathsf{Sing}_r\,\tilde c(e_\infty) \subset \{z \in \mathfrak{H} \;:\; |\mathrm{Re}\, z| \leq \varepsilon^{-1}\,, \mathrm{Im}\, z > \varepsilon\}\,,$$

for some $\varepsilon > 0$.

Since $\tilde c$ is given on a basis of $F^{\mathcal{T}}_1$, we can extend it $\mathbb{C}[\Gamma(1)]$-linearly, and obtain a cochain $\tilde c \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^\omega_{v,r}, \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$. There is no reason for the lift $\tilde c$ to be a cocycle. For any $y \in X^{\mathcal{T}}_1$ and any $\gamma \in \Gamma(1)$ we have $\mathsf{Sing}_r\,\tilde c(\gamma^{-1}y) = \gamma^{-1}\,\mathsf{Sing}_r\,\tilde c(y)$. So for $y \in X^{\mathcal{T},Y}_1$ the set $\mathsf{Sing}_r\,\tilde c(y)$ is contained in the R-neighbourhood of x. Similarly, $\mathsf{Sing}_r\,\tilde c(\gamma^{-1}e_\infty)$ is contained in

$$\{\gamma^{-1}z \;:\; |\mathrm{Re}\, z| \leq \varepsilon^{-1}\,, \mathrm{Im}\, z > \varepsilon\}\,. \tag{10.8}$$

This means that the singularities of any $\tilde c(y)$ cannot be "too far" from the edge $y \in X^{\mathcal{T}}_1$.

Construction. We start the construction of an automorphic form. First we work on a connected set $Z \subset \mathfrak{H}$ that is contained in finitely many $\Gamma(1)$-translates of the standard fundamental domain $\mathfrak{F}$. We choose a closed path $C \in \mathbb{Z}[X^{\mathcal{T}}_1]$ encircling Z once in positive direction. Since Z may contain a translate of $\mathfrak{F}$ this path may have to go through cusps, as illustrated in Figure 16. We can take the cycle C far away

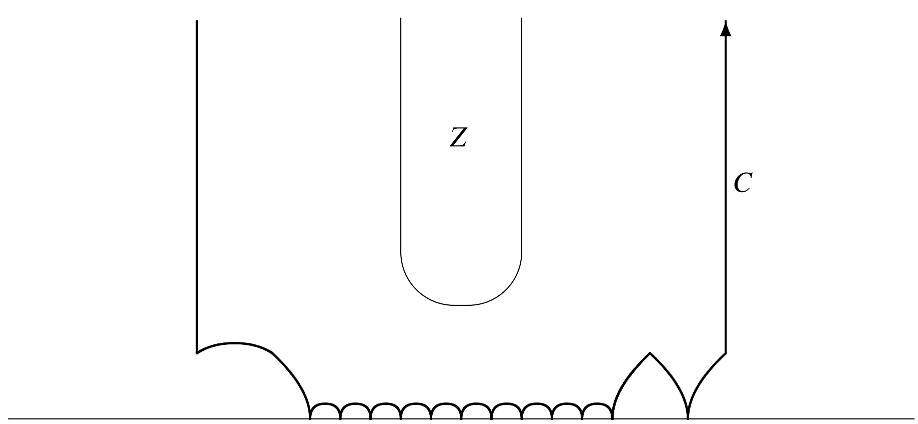

FIGURE 16

from Z, such that $\mathsf{Sing}_r\,\tilde c(x) \cap Z = \emptyset$ for all x occurring in the path C. We choose

C such that for all edges x occurring in C the R-neighbourhoods of edges in $X_1^{\mathcal{T},Y}$ and the sets in (10.8) do not intersect Z. So for each edge x occurring in C the set Z is in the region on which $\tilde{c}(x)$ is a representative of $c(x)$.

We define for $z \in Z$:[1]

$$u(C;z) := \frac{1}{4\pi}\,\tilde{c}(C)\,(z)\,. \tag{10.9}$$

So $u(C;z)$ is the sum of contributions $\frac{\pm 1}{4\pi}\tilde{c}(x)(z)$ with $x \in X_1^{\mathcal{T}}$ occurring in C. Since C is far away from Z the function $u(C;\cdot)$ is r-harmonic on (the interior of) Z.

Independence of choices. The next step is to get rid of the choice of the lift $\tilde{c}$ and of the choice of c in its cohomology class. This can be done in exactly the same way as in [15, §7.1 and §12.2]. The main reasoning is given in [15, §7.1] for the cocompact case. There it is explained that the definition does not depend on the choice of the cycle C, provided it is far enough from Z. For a given $\gamma \in \Gamma$ we can take C such that both C and $\gamma^{-1}C$ can be used in (10.9). Then the Γ-equivariance of $\tilde{c}$ implies that $u(C;\cdot)|_{v,r}\gamma = u(C;\cdot)$ on the intersection $Z \cap \gamma^{-1}Z$ for $\gamma \in \Gamma$. The function is r-harmonic on the interior of Z, by the same argument as in [15, §7.1]. The independence on C allows us to enlarge the set Z, thus ending up with an element of $\mathrm{Harm}_r(\Gamma, v)$, which we now can call $u(\tilde{c};\cdot)$.

By the reasoning in [15, §7.1] the r-harmonic automorphic form that we obtained is independent of the choice of the lift $\tilde{c}$ of c, and of the choice of c in its cohomology class. So we may now denote it by $u([c];\cdot)$.

Remaining questions. There are two questions left: (1) Is $u([c];\cdot)$ a *holomorphic* automorphic form? (2) If $\mathbf{q}_r^\omega F$ happens to be in $H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$, what is then the relation between F and $u(\mathbf{q}_r^\omega F;\cdot)$?

Question (1) does not arise in [15]. The following lemma treats it, for general cofinite Γ with cusps.

Lemma 10.11. *Let $\tilde{c} \in C^1(F^{\mathcal{T}}_{\cdot};\mathcal{G}^\omega_{v,r},\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$ be a lift of $c \in Z^1(F^{\mathcal{T}}_{\cdot};\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$. Suppose that $C = \sum_j \varepsilon_j\, x_j \in \mathbb{Z}[X_1^{\mathcal{T}}]$ (finite sum, with $\varepsilon_j = \pm 1$, $x_j \in X_1^{\mathcal{T}}$) is a cycle encircling an open set $Z \in \mathfrak{H}$ once in positive direction, so that for each x_j the set Z is contained in the set where $\tilde{c}(x_j)$ represents $c(x_j)$. Then $\tilde{c}(C)$ is holomorphic on Z.*

Proof. For each x_j the function $\tilde{c}(x_j)$ represents an element of $\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}$ on some set U_j as in Property i) in Proposition 8.4, and $Z \subset U_j$. By Property vi) we know that $\xi_r\tilde{c}(x_j)$ has a holomorphic extension $h_j \in \mathcal{O}(\mathfrak{H})$. Since C is a closed cycle, we have $c(C) = 0$ in $\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}$. So $\tilde{c}(C) = 0$ on an excised neighbourhood V on which $\tilde{c}(C)$ represents $c(C)$. This neighbourhood is contained in the intersection of the U_j, but will in general not contain Z. See Figure 17.

[1]The factor $\frac{1}{4\pi}$ differs from the factor $\frac{1}{\pi i}$ in [15]. This is caused by a difference in the normalization of $K_r(\cdot;\cdot)$ in (6.12), and $q_s(\cdot,\cdot)$ in [15, (1.4)].

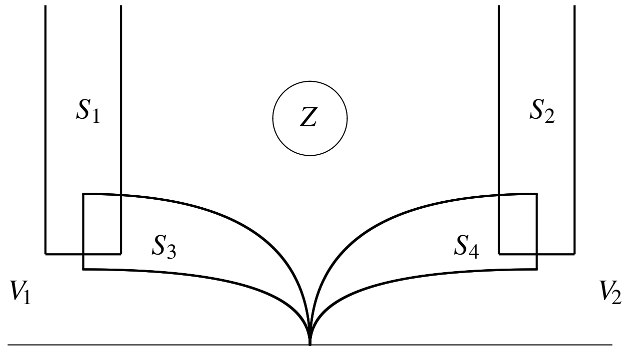

FIGURE 17. Illustration for the proof of Lemma 10.11. We take $C = \sum_{j=1}^{4} \varepsilon_j x_j$. The singularities of $\tilde{c}(x_j)$ are contained in S_j, and we can take $U_j = \mathfrak{H} \smallsetminus S_j$. The union $V_1 \cup V_2$ is an excised neighbourhood, on which $\tilde{c}(C)$ represents $c(C)$.

We now know the following:

$$\begin{array}{llll}
\forall_j: & \tilde{c}(x_j) &= c(x_j) & \text{on } U_j \supset Z\,,\\
\forall_j: & h_j &\in \mathcal{O}(\mathfrak{H})\,, & \\
\forall_j: & \xi_r\,\tilde{c}(x_j) &= h_j & \text{on } U_j \supset Z\,,\\
& \tilde{c}(C) &= c(C) \;=\; 0 & \text{on } V\,,\\
& \xi_r\tilde{c}(C) &= \sum_j \varepsilon_j\,\xi_r\tilde{c}(x_j) \;=\; 0 & \text{on } V\\
& \sum_j \varepsilon_j\,h_j &= \sum_j \varepsilon_j\,\xi_r\tilde{c}(x_j) \;=\; 0 & \text{on } \mathfrak{H} \text{ (continuation)}\\
& \xi_r\tilde{c}(C) &= \sum_j \varepsilon_j\,\xi_r\tilde{c}(x_j) \;=\; \sum_j \varepsilon_j\,h_j = 0 & \text{on } Z\\
& u([c];\cdot) &= \frac{1}{4\pi}\tilde{c}(C) \text{ is holomorphic on } Z\,. \quad \square &
\end{array}$$

This lemma implies directly that $u([c];\cdot) \in A_r(\Gamma, v)$, thus completing the proof of Part i) of Proposition 10.10.

The other remaining question concerns Part ii), to which we apply Cauchy's formula:

Lemma 10.12. *Suppose that* $[c] = \mathbf{q}_r^{\omega} F \in H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ *for some automorphic form* $F \in A_r(\Gamma, v)$. *Then* $u([c];\cdot) = F$.

Proof. By analytic continuation it suffices to show the equality on some non-empty open set. Let us take Z open and relatively compact in the interior of the compact face $\mathfrak{F}_Y$ of the tesselation contained in the fundamental domain.

For z in the interior of $\mathfrak{F}_Y$ we have

$$c(\partial_2\mathfrak{F}_Y)(z) \;=\; \int_{\partial_2\mathfrak{F}_Y} \frac{2i}{z-\tau}\Bigl(\frac{\bar z-\tau}{\bar z - z}\Bigr)^{r-1} F(\tau)\,d\tau\,, \tag{10.10}$$

as follows from (9.5). The factor $\Bigl(\frac{\bar z-\tau}{\bar z-z}\Bigr)^{r-1}$ is holomorphic as a function of τ. So the value of the integral is $4\pi\, F(z)$ for z in the interior of $\mathfrak{F}_Y$, in particular for $z\in Z$. The hyperbolic distance of Z to $\partial_2\mathfrak{F}_Y$ is larger than some $\varepsilon>0$. We can choose the lift $\tilde c$ of c such that for each $x\in X_1^{\mathcal{T},Y}$ the singularities of $\tilde c(x)$ are contained in the ε-neighbourhood of x. Then $\tilde c(\partial_2\mathfrak{F}_Y)$ is equal to $c(\partial_2\mathfrak{F}_Y)$ on Z. □

Averages. An alternative to (10.9) is the description of $u([c];\cdot)$ as an infinite sum, which is a kind of Poincaré series.

Definition 10.13. Let f be a continuous function on $\mathfrak{H}$ with support contained in finitely many Γ-translates of a fundamental domain of $\Gamma\backslash\mathfrak{H}$. We define the Γ-*average* of f by

$$(\mathrm{Av}_{\Gamma,v,r} f)(z) \;:=\; \sum_{\gamma\in\{\pm1\}\backslash\Gamma} (f|_{v,r}\gamma)\,(z)\,. \tag{10.11}$$

Remarks. (a) We have $|_{v,r}(-\gamma) = |_{v,r}\gamma$, so it makes sense to sum over $\{\pm1\}\backslash\Gamma$.

(b) Under the support condition in the definition the sum is locally finite and defines a continuous function that is invariant for the action $|_{v,r}$ of Γ.

(c) To use the average to describe $u([c],\cdot)$, we start with the exact sequence

$$\to H^1_{\mathrm{pb}}(\Gamma;\mathcal{G}^\omega_{v,r},\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^2_{\mathrm{pb}}(\Gamma;\mathcal{N}^\omega_{v,r},\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,w}) \to \tag{10.12}$$

The exactness follows from [15, Proposition 11.9]. To see that the conditions of that theorem are satisfied, we use the diagram in Part iii) of Proposition 10.3 and Lemma 10.8.

Let $\tilde c\in C^1(F^{\mathcal{T}}_{\cdot};\mathcal{G}^\omega_{v,r},\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$ be a lift of $c\in Z^1(F^{\mathcal{T}}_{\cdot};\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$. The exact sequence (10.12) shows that $d\tilde c\in H^2_{\mathrm{pb}}(\Gamma;\mathcal{N}^\omega_{v,r},\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r})$. We apply $d\tilde c$ to the fundamental domain $\mathfrak{F}$ of $\Gamma\backslash\mathfrak{H}$ underlying the tesselation $\mathcal{T}$. So $d\tilde c(\mathfrak{F}) = d\tilde c(\mathfrak{F}_Y)+d\tilde c(V_\infty)$ in the case of $\Gamma=\Gamma(1)$, and in general $d\tilde c(\mathfrak{F}) = d\tilde c(\mathfrak{F}_Y)+\sum_{\mathfrak{a}} d\tilde c(V_{\mathfrak{a}})$, where $\mathfrak{a}$ runs over the cusps in the closure of the fundamental domain $\mathfrak{F}$. This implies that $d\tilde c(\mathfrak{F})\in\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r}$, hence we can apply $\mathrm{Av}_{\Gamma,v,r}$ to it.

Proposition 10.14. *With the notations of Proposition 10.10:*

$$u([c],z) \;=\; \frac{1}{4\pi}(\mathrm{Av}_{\Gamma,v,r} d\tilde c(\mathfrak{F}))(z) \;=\; \frac{1}{4\pi}\sum_{\gamma\in\{\pm1\}\backslash\Gamma} (d\tilde c(\mathfrak{F}))\,\Big|_{v,r}\,\gamma\,(z)\,. \tag{10.13}$$

Proof. The proof follows the approach to Propositions 7.1 and (12.5) in [15]. □

Remark. On first sight it may seem amazing that the sum of translates of the non-analytic function $d\tilde c(\mathfrak{F})$ is a holomorphic function. See the discussion after [15, Proposition 7.1].

10.3. **Injectivity.** Proposition 10.10 gives us a linear map α_r from mixed parabolic cohomology that is left inverse to $\mathbf{q}_r^\omega$. It might have a non-zero kernel.

Proposition 10.15. *The linear map α_r in* (10.6) *is injective.*

Proof. The proof is based on the exact sequence (10.12) and the average in (10.13):

$$\begin{tikzcd} H^1_{\mathrm{pb}}(\Gamma;\mathcal{G}^\omega_{v,r},\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r] & H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r, "\delta"] \arrow[d, "\alpha_r"] & H^2_{\mathrm{pb}}(\Gamma;\mathcal{N}^\omega_{v,r},\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[d, "{[b]\mapsto \mathrm{Av}_{\Gamma,v,r}b(\mathfrak{F})}"'] \\ & A_r(\Gamma,v) \arrow[r, hook] & C^2(\mathfrak{H})^\Gamma_{v,r} \end{tikzcd} \tag{10.14}$$

The vertical map on the right is given by associating to the cohomology class $[b]$ the average $\mathrm{Av}_{\Gamma,v,r}b(\mathfrak{F})$. By $C^2(\mathfrak{H})_{v,r}$ we mean the space $C^2(\mathfrak{H})$ provided with the action $|_{v,r}$ of Γ. The map α_r is the composition of the connecting homomorphism δ and the vertical map. Failure of injectivity might be caused by δ and by the average.

Lemma 10.16 below implies that the vertical map cannot contribute to the kernel of α_r. That leaves us with the connection homomorphism δ. Lemma 10.17 below gives the vanishing of $H^1_{\mathrm{pb}}(\Gamma;\mathcal{G}^\omega_{v,w},\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$, and hence the injectivity of δ. □

Lemma 10.16. *Let $c\in Z^1(F^{\mathcal{T}}_{\cdot};\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ and let $\tilde c$ be a lift of c as in* (10.7)*. If $\mathrm{Av}_{\Gamma,v,r}d\tilde c(\mathfrak{F})=\sum_{\gamma\in\{\pm1\}\backslash\Gamma} d\tilde c(\mathfrak{F})|_{v,r}\gamma=0$ then $d\tilde c\in B^2(F^{\mathcal{T}}_{\cdot};\mathcal{N}^\omega_{v,r},\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r})$.*

Proof. The proof is analogous to that of [15, Lemma 12.6]. Here we discuss it in the modular case $\Gamma=\Gamma(1)$.

A cocycle $b\in Z^2(F^{\mathcal{T}}_{\cdot};\mathcal{N}^\omega_{v,r},\mathcal{N}^{\omega^0,\mathrm{exc}}_{v,r})$ is determined by its values on the faces (V_∞) and $(\mathfrak{F}_Y)$. The freedom that we have within a cohomology class is to add to $(b(V_\infty),b(\mathfrak{F}_Y))$ elements of three forms: (1) $(u,-u)$, with $u\in\mathcal{N}^\omega_{v,r}$ (related to the edge f_∞), (2) $(t|_{v,r}(1-T),0)$ with $t\in\mathcal{N}^{\omega^*,\mathrm{exc}}_{v,r}$ (related to the edge e_∞), (3) $(0,w|_{v,r}(1-\gamma))$ with $w\in\mathcal{N}^\omega_{v,r}$, $\gamma\in\Gamma$ (related to the edges e_1 and e_2). So $b(\mathfrak{F})=b(V_\infty)+b(\mathfrak{F}_Y)$ is determined by b up to addition of an element of $\mathcal{N}^{\omega^*,\mathrm{exc}}_{v,r}|_{v,r}(1-T)+\sum_{\gamma\in\Gamma}\mathcal{N}^\omega_{v,r}|_{v,r}(1-\gamma)$.

The first consequence of this description is that $\mathrm{Av}_{\Gamma,v,r}b(\mathfrak{F})$ does not depend on the choice of b in its cohomology class.

Now we consider $b=d\tilde c$ as in the lemma. The element $b(\mathfrak{F}_Y)$ is in $\mathcal{N}^\omega_{v,r}\subset C^2_c(\mathfrak{H})$. So there is $q>Y$ such that the support of $b(\mathfrak{F}_Y)$ does not intersect the region

$$\bigcup_{\gamma\in\Gamma}\Bigl\{\gamma z\;:\;\mathrm{Im}\,z\geq q\Bigr\}.$$

Further, $b(V_\infty)=\tilde c(e_\infty)|_{v,r}(1-T)-\tilde c(f_\infty)$ represents the zero element of $\mathcal{E}^{\omega,\mathrm{exc}}_{v,r}[\infty]$. Hence $b(V_\infty)$ has support in a set of the form $\{z\in\mathfrak{H}\;:\;\mathrm{Im}\,z>\varepsilon,\ |\mathrm{Re}\,z|\leq\varepsilon^{-1}\}$ for some $\varepsilon>0$. We deal with C^2-functions, and hence we can split off from $b(V_\infty)$ an element $u\in C^2_c(\mathfrak{H})=\mathcal{N}^\omega_{v,r}$ and move it to $b(\mathfrak{F}_Y)$, by the freedom indicated above. In this way we arrange that $b(V_\infty)$ has support in the set $\{z\in\mathfrak{H}\;:\;\mathrm{Im}\,z>q-1\,,\ |\mathrm{Re}\,z|\leq\varepsilon^{-1}\}$.

We take a partition of unity α on $\mathbb{R}$: $\alpha \in C_c^2(\mathbb{R})$ such that $\sum_{n\in\mathbb{Z}} \alpha(x+n) = 1$ for all $x \in \mathbb{R}$. We take $\beta \in C^2(0,\infty)$ such that $\beta(y) = 1$ for $y \geq q+\delta$ with $\delta > 0$ and $\beta(y) = 0$ for $y \leq q$, and put $\chi(z) = \alpha(\operatorname{Re} z)\beta(\operatorname{Im} z)$. So $\sum_n \chi(z+n) = 1$ for all z with $\operatorname{Im} z \geq q+\delta$.

The element $b_1 \in C^2(F_\cdot^{\mathcal{T}}; \mathcal{N}_{v,r}^{\omega}, \mathcal{N}_{v,r}^{\omega^*,\mathrm{exc}})$ determined by $b_1(\mathfrak{F}_Y) = 0$ and

$$b_1(V_\infty)(z) \;=\; \sum_{n\in\mathbb{Z}} \big(b(V_\infty)\chi(\cdot+n)\big)\Big|_{v,r} (1-T^{-n})(z)$$

is a coboundary. (Note that the terms in the sum vanish for all but finitely many n.) We define $\hat{b} = b - b_1$, which is in the same cohomology class as b. For $\operatorname{Im} z \geq q+\delta$

$$\hat{b}(V_\infty)(z) \;=\; b(V_\infty)(z) - \sum_n \Big(b(V_\infty)(z)\cdot\chi(z+n) - b(V_\infty)(z-n)\cdot\chi(z)\Big)$$

$$= \alpha(\operatorname{Re} z)\beta(\operatorname{Im} z) \sum_{\gamma\in\{\pm1\}\backslash\Gamma(1)_\infty} b(V_\infty)|_{v,r}\gamma\,(z) \tag{10.15}$$

Now we use the assumption that $\mathrm{Av}_{\Gamma,v,r} b(\mathfrak{F}) = 0$. From our knowledge of the support $b(V_\infty)$ we conclude that $\big(\mathrm{Av}_{\Gamma,v,r} b(\mathfrak{F})\big)(z) = \big(\mathrm{Av}_{\Gamma,v,r} b(V_\infty)\big)(z)$ if $\operatorname{Im} z \geq q+\delta$. Furthermore, for $\operatorname{Im} z \geq q+\delta$ the expression in (10.15) is equal to $\alpha(x)$ $\big(\mathrm{Av}_{\Gamma,v,r} b(V_\infty)\big)(z)$, since since for $\gamma \notin \Gamma(1)_\infty$, the support of $b(V_\infty)|\gamma$ does not intersect the support of $b(V_\infty)$. for So $\hat{b}(V_\infty)$ vanishes on this domain, hence it has compact support. So we can move $\hat{b}(V_\infty)$ to $\hat{b}(\mathfrak{F}_Y)$.

We are left with a cocycle $\hat{b}$ given by $\hat{b}(V_\infty) = 0$ and $\hat{b}(\mathfrak{F}_y)$ with support not intersecting the region

$$\bigcup_{\gamma\in\Gamma} \Big\{\gamma z \;:\; \operatorname{Im} z \geq q+\delta\Big\}.$$

We take a $\Gamma(1)$-partition of unity $\psi \in C^\infty(\mathfrak{H})$ with support of ψ contained in the union of finitely many $\Gamma(1)$-translates of $\mathfrak{F}$. So $\sum_{\gamma\in\{\pm1\}\backslash\Gamma} \psi(\gamma z) = 1$ for all $z \in \mathfrak{H}$, and the sum is a finite sum for all z. We write $f = \hat{b}(\mathfrak{F}) = \hat{b}(\mathfrak{F}_Y)$, and know by the assumption that $\mathrm{Av}_{\Gamma,v,r} f = 0$. For $z \in \mathfrak{H}$:

$$\begin{aligned} f(z) \;&=\; f(z) - \psi(z)(\mathrm{Av}_{\Gamma,v,r} f)(z) \;=\; \sum_{\gamma\in\{\pm1\}\backslash\Gamma} \big(f(z)\,\psi(\gamma^{-1}z) - \psi(z)\,f|_{v,r}\gamma\,(z)\big) \\ &=\; \sum_{\gamma\in\{\pm1\}\backslash\Gamma} \big(f\cdot(\psi\circ\gamma^{-1})\big)|_{v,r}(1-\gamma)\,. \end{aligned}$$

For almost all γ the intersection of the supports of f and $\psi\circ\gamma^{-1}$ have empty intersections. So the sum is finite, and $\hat{b}$ is a coboundary. □

Lemma 10.17. $H^1_{\mathrm{pb}}(\Gamma; \mathcal{G}_{v,r}^{\omega}, \mathcal{G}_{v,r}^{\omega^0,\mathrm{exc}}) = \{0\}$.

Proof. Similar to the proof of [15, Proposition 12.5], to which we refer for the full proof. Table 4 gives a list of corresponding notations and concepts.

Let $c \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_{v,r}^{\omega}, \mathcal{G}_{v,r}^{\omega^0,\mathrm{exc}})$ be given. This cocycle induces a map $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}} \to \mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}}$ which we also indicate by c. It has the properties in (9.3). The aim is to

holomorphic forms	**Maass forms wt.** 0
Γ-module $\mathcal{G}^{\omega}_{v,r}$	Γ-module $\mathcal{G}^{\omega}_{s}$
Γ-module $\mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}$	Γ-module $\mathcal{G}^{\omega^*,\mathrm{exc}}_{s}$
cocycle c	cocycle ψ
$c(\xi,\xi')$	$\psi_{\xi,\xi'}$

TABLE 4. Correspondence with [15, Proposition 12.5].

show that it is a coboundary. To do that is suffices to show that the group cocycle $\gamma \mapsto c(\gamma^{-1}P_0, P_0)$ is a coboundary for one base point $P_0 \in X_0^{\mathcal{T}}$.

(a) There exists $R > 0$ such that $\mathsf{Sing}_r\, c(x) \subset N_R(x)$ for all edges $x \in X_1^{\mathcal{T}}$. The set $N_R(x)$ is an R-neighbourhood of x for the hyperbolic metric if x is an edge in $X_1^{\mathcal{T},Y}$, and a more general neighbourhood defined in [15, (12.2)] if x is an edge going to a cusp.

(b) We prove that $c(\mathfrak{a},\mathfrak{b}) \in \mathcal{H}_r(\mathfrak{H})$ for any two cusps $\mathfrak{a},\mathfrak{b}$.

Suppose that $z \in \mathsf{Sing}_r\, c(\mathfrak{a},\mathfrak{b})$. The value of $c(\mathfrak{a},\mathfrak{b})$ is the value $c(p)$ for any path in $\mathbb{Z}[X_1^{\mathcal{T}}]$ from $\mathfrak{a}$ to $\mathfrak{b}$. We can move the path p away from z in such a way that z is not in $N_R(x)$, in (a), for any of the edges x occurring in p. So $\mathsf{Sing}_r\, c(\mathfrak{a},\mathfrak{b}) = \emptyset$.

(c) By breaking up a path from $\mathfrak{a}$ to $\mathfrak{b}$ at a point $P \in X_0^{\mathcal{T},Y} = X_0^{\mathcal{T}} \cap \mathfrak{H}$ it can be shown that $\mathsf{Sing}_r\, c(P,\mathfrak{a})$ is a compact subset of $\mathfrak{H}$ for any path $\mathbb{Z}[X_1^{\mathcal{T}}]$ from P to $\mathfrak{a} \in C$.

(d) Now Lemma 10.9 can be applied to the conjugated element $F = c(P,\mathfrak{a})|_r \sigma_{\mathfrak{a}}^{-1}$. We note that $F \in \mathcal{G}_r^{\omega^*,\mathrm{exc}}$, for Condition a), that its singularities are contained in a compact set, for Condition b), and that

$$F|_{v,r}(1-\pi_{\mathfrak{a}}) \;=\; c(P,\mathfrak{a})|_{v,r}(1-\pi_{\mathfrak{a}}) \;=\; c(P,\pi_{\mathfrak{a}}^{-1}P) \in \mathcal{G}^{\omega}_{v,r}$$

implies Condition c). The conclusion is that $F = Q_{\mathfrak{a}} + G$, with $Q_{\mathfrak{a}} \in \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r}$ satisfying $Q_{\mathfrak{a}}|_{v,r}\pi_{\mathfrak{a}} = P$, and $G \in \mathcal{G}^{\omega}_{v,r}$ representing an element of $\mathcal{E}^{\omega}_r$. Then use Lemma 10.7 to see that $Q_{\mathfrak{a}} \in \mathcal{H}_r(\mathfrak{H})$.

(e) Such an element $Q_{\mathfrak{a}}$ exists for all cusps $\mathfrak{a}$, and for $\mathfrak{b} = \gamma^{-1}\mathfrak{a}$ we have $Q_{\mathfrak{b}} = Q_{\mathfrak{a}}|_{v,r}\gamma$.

(f) The transformation properties of the $Q_{\mathfrak{a}}$ allow us to define another cocycle $\hat{c}$ in the same class as c by taking for $x, y \in X_0^{\mathcal{T}}$:

$$\hat{c}(x,y) \;:=\; c(x,y) + \begin{Bmatrix} Q_x & \text{if } x \in X_0^{\mathcal{T}} \smallsetminus X_0^{\mathcal{T},Y} \\ 0 & \text{if } x \in X_0^{\mathcal{T},Y} \end{Bmatrix} + \begin{Bmatrix} -Q_y & \text{if } y \in X_0^{\mathcal{T}} \smallsetminus X_0^{\mathcal{T},Y}, \\ 0 & \text{if } y \in X_0^{\mathcal{T},Y}. \end{Bmatrix}$$

It has the property that $\hat{c}(\mathfrak{a},\mathfrak{b}) \in \mathcal{H}_r(\mathfrak{H}) \cap \mathcal{G}^{\omega}_r$ for all cusps $\mathfrak{a},\mathfrak{b}$. Part v) of Proposition 8.4 implies that $\hat{c}(\mathfrak{a},\mathfrak{b}) = 0$ for all cusps. Taking a cusp as the base point P_0, we see that the cohomology class of the cocycle $\hat{c}$, and hence of the original cocycle c, is zero. $\square$

10.4. **From analytic boundary germ cohomology to automorphic forms.** We have obtained two linear maps, $\mathbf{q}_r^\omega$ (Proposition 6.10) and α_r (Proposition 10.10):

$$\begin{tikzcd} A_r(\Gamma,v) \arrow[r, "\mathbf{q}_r^\omega"] & H^1(\Gamma;\mathcal{E}_{v,r}^\omega) \\ & H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}_{v,r}^\omega,\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}}) \arrow[ul, "\alpha_r"] \end{tikzcd} \tag{10.16}$$

We recall that $H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}_{v,r}^\omega,\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}}) \subset H^1(\Gamma;\mathcal{E}_{v,r}^\omega)$. The following theorem shows the relation between these maps.

Theorem 10.18. *Let Γ be a cofinite discrete group of $\mathrm{SL}_2(\mathbb{R})$ with cusps. Let $r\in\mathbb{C}$ and let v be a corresponding multiplier system.*

i) *Both linear maps $\mathbf{q}_r^\omega$ and α_r in (10.16) are injective.*
ii) *Define*

$$A_r^{\mathcal{E}}(\Gamma,v) \;:=\; \big(\mathbf{q}_r^\omega\big)^{-1} H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}_{v,r}^\omega,\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}})\,. \tag{10.17}$$

Then the restriction of $\mathbf{q}_r^\omega$ to $A_r^{\mathcal{E}}(\Gamma,v)$ and the restriction of α_r to the image $\mathbf{q}_r^\omega A_r^{\mathcal{E}}(\Gamma,v) \subset H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}_{v,r}^\omega,\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}})$ are inverse to each other.

$$\begin{tikzcd} A_r^{\mathcal{E}}(\Gamma,v) \arrow[r, shift left, "\mathbf{q}_r^\omega"] & \mathbf{q}_r^\omega A_r^{\mathcal{E}}(\Gamma,v) \arrow[l, shift left, "\alpha_r"] \end{tikzcd}$$

Proof. Proposition 10.15 gives the injectivity of α_r. Suppose that $F\in\ker\mathbf{q}_r^\omega$. (Then $F\in A_r^{\mathcal{E}}(\Gamma,v)$ by the definition in (10.17).) Proposition 10.10 ii) shows that $\alpha_r\mathbf{q}_r^\omega F = F$, hence $F=\alpha_r\,0=0$. This shows that $\mathbf{q}_r^\omega$ is injective. This gives Part i).

Part ii) of Proposition 10.10 shows that $\alpha_r\circ\mathbf{q}_r^\omega$ is the identity on $A_r^{\mathcal{E}}(\Gamma,v)$. Since $\mathbf{q}_r^\omega : A_r^{\mathcal{E}}(\Gamma,v)\to\mathbf{q}_r^\omega A_r^{\mathcal{E}}(\Gamma,v)$ is surjective Part ii) follows. □

10.5. **Completion of the proof of Theorem A for general weights.** We consider $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq 2}$. Proposition 2.4 shows that $\mathbf{r}_r^\omega : A_r(\Gamma,v)\to H^1(\Gamma;\mathcal{D}_{r,2-r}^\omega)$ is a well-defined linear map. Theorem 3.9 shows that the image is contained in $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}_{v,2-r}^\omega,\mathcal{D}_{v,2-r}^{\omega^0,\mathrm{exc}})$. We have the following relations:

$$\begin{tikzcd} A_r(\Gamma;v) \arrow[r, "\mathbf{r}_r^\omega"] & H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}_{v,2-r}^\omega,\mathcal{D}_{v,2-r}^{\omega^0,\mathrm{exc}}) \arrow[r, hook] & H^1(\Gamma;\mathcal{D}_{v,2-r}^\omega) \\ A_r^{\mathcal{E}}(\Gamma,v) \arrow[u, hook] \arrow[r, "\mathbf{q}_r^\omega"] & H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}_{v,r}^\omega,\mathcal{E}_{v,r}^{\omega^0,\mathrm{exc}}) \arrow[u, "\cong"', "\rho_r"] & \end{tikzcd}$$

Definition 8.1 of the highest weight spaces of boundary germs $\mathcal{E}_r^*$ as isomorphic to the corresponding highest weight spaces $\mathcal{D}_{2-r}^*$ induces an isomorphism in cohomology. Theorem 3.9 implies that $\mathbf{r}_r^\omega A_r(\Gamma,v)\subset H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}_{v,2-r}^\omega,\mathcal{D}_{v,2-r}^{\omega^0,\mathrm{exc}})$, and hence

$\mathbf{q}_r^\omega A_r(\Gamma,v) \subset H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,2-r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,2-r})$ by Proposition 6.10. So $A_r^{\mathcal{E}}(\Gamma,v) = A_r(\Gamma,v)$. Theorem 10.18 then gives the inverse α_r of $\mathbf{q}_r^\omega$:

$$\begin{tikzcd}
A_r(\Gamma;v) \arrow[r, "\mathbf{r}_r^\omega"] & H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^\omega_{v,2-r},\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \arrow[r, hook] & H^1(\Gamma;\mathcal{D}^\omega_{v,2-r}) \\
A_r(\Gamma,v) \arrow[u, "="'] \arrow[r, shift left, "\mathbf{q}_r^\omega"] & H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[u, "\cong"', "\rho_r"] \arrow[l, shift left, "\alpha_r"] &
\end{tikzcd}$$

10.6. **Related work.** As indicated at several places in this section, we followed closely the approach of [15], §7 and §12.2–3.

11. Automorphic forms of integral weights at least 2 and analytic boundary germ cohomology

In this section we will prove Theorem D, which concerns automorphic forms with weight $r \in \mathbb{Z}_{\geq 2}$ and analytic boundary germ cohomology.

Throughout this section we only treat the case of weight $r \in \mathbb{Z}_{\geq 2}$.

11.1. **Image of automorphic forms in mixed parabolic cohomology.** The linear map $\mathbf{q}_r^\omega : A_r(\Gamma,v) \to H^1(\Gamma;\mathcal{W}^\omega_{v,r}(\mathbb{P}^1_{\mathbb{R}}))$ in Proposition 6.10 has image in $H^1(\Gamma;\mathcal{E}^\omega_r)$, by Property iv) in Proposition 8.4.

Definition 11.1. For all $r \in \mathbb{C}$ we define

$$A_r^0(\Gamma,v) := \left\{F \in A_r(\Gamma,v) \;:\; a_0(\mathfrak{a},F) = 0 \text{ for all cusps } \mathfrak{a} \text{ with } v(\pi_{\mathfrak{a}}) = 1\right\}. \tag{11.1}$$

See (1.14) for the Fourier coefficients $a_n(\mathfrak{a},F)$ at the cusp $\mathfrak{a}$.

The idea is to allow automorphic forms with large growth at the cusps, but not to allow constant terms in the Fourier expansion.

Proposition 11.2. *Let $r \in \mathbb{Z}_{\geq 2}$. For each $F \in A_r(\Gamma,v)$ the following statements are equivalent:*

a) $\mathbf{q}_r^\omega F \in H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$
b) $F \in A_r^0(\Gamma,v)$.

We will base the proof on the following lemma:

Lemma 11.3. *Let $r \in \mathbb{Z}_{\geq 2}$, $z_0 \in \mathfrak{H}$, and $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{C}$. Suppose that the holomorphic function E on $\mathfrak{H}$ is given by the Fourier expansion*

$$E(\tau) = \sum_{n\equiv\alpha(1)} a_n\, e^{2\pi i n\tau}\,.$$

Then there exists $h \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$ such that

$$\lambda^{-1}\, h(z+1) - h(z) \;=\; \int_{\tau=z_0-1}^{z_0} K_r(z;\tau)\, E(\tau)\, d\tau\,, \tag{11.2}$$

if and only if $\lambda \neq 1$ or $a_0 = 0$.

Remark. If $\lambda \neq 1$, then n in the Fourier expansion does not run over the integers, and a_0 is not defined.

Proof. This is a situation similar to that in §3.4. We can split up the Fourier expansion of E. For the cuspidal part

$$E_c(\tau) \;=\; \sum_{n\equiv\alpha(1),\,\mathrm{Re}\,n>0} a_n\, e^{2\pi i n\tau}$$

we can use

$$h_c(z) \;=\; \int_{\tau=z_0}^{\infty} K_r(z;\tau)\, E_c(\tau)\, d\tau \tag{11.3}$$

for $z \in \mathfrak{H} \smallsetminus (z_0 + i[0,\infty))$. (In this way we avoid the singularity at $\tau = z$. See (6.12).)

Expression (7.8) gives, for those τ that have smaller hyperbolic distance to i than z, an expression for $K_r(z;\tau)$ in terms of a linear combination of $\mathrm{M}_{r,\mu}$ with $1 - r \le \mu \le -1$ and an explicit expression $p_r(z;\tau)$. The factors of the $\mathrm{M}_{r,\mu}$ are holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus \{-i\}$, and $p_r(z;\tau)$ is meromorphic on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, with singularity in $\mathfrak{H}\times\mathfrak{H}$ given by $\frac{1}{z-\tau}$. By analytic continuation (7.8) is valid for all (z,τ) of interest in (11.3).

On insertion of (7.8) the integrals of the term with $\mathrm{M}_{r,\mu}(z)$ yield, by the exponential decay of $E_c(\tau)$, a multiple of $\mathrm{M}_{r,\mu}$, hence in $\mathcal{E}^{\omega}_r$. The term with $p_r(z;\tau)$ gives

$$\int_{\tau=z_0}^{\infty} \frac{2i}{z-\tau}\, \frac{(\tau-i)^{r-1}}{(z-i)^{r-1}}\, E_c(\tau)\, d\tau\,.$$

It yields a holomorphic function on $\mathbb{C} \smallsetminus (z_0 + i[0,\infty))$, hence the result is an element of $\mathcal{D}^{\omega,\mathrm{exc}}_r[\infty]$. Together with the multiples of $\mathrm{M}_{r,\mu}$ we obtain an element of $\mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$ with the desired property.

We proceed similarly with the contribution $E_\infty(\tau)$ of the part of the Fourier series with $\mathrm{Re}\,n < 0$. The path of integration is replaced by the path used in the proof of Lemma 3.6. If $|\lambda| \neq 1$ we can take $\alpha \in i\mathbb{R}$, $\alpha \neq 0$, and use the paths as in the proof of Lemma 3.7. This gives a function h_∞ satisfying

$$\lambda^{-1}\, h_\infty(z+1) - h_\infty(z) \;=\; \int_{\tau=z_0-1}^{z_0} K_r(z;\tau)\, E_\infty(\tau)\, d\tau\,.$$

There remains the case of a constant Fourier term, present only if $\lambda = 1$. We look for $h \in \mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$ such that

$$h(z+1) - h(z) \;=\; \int_{\tau=z_0-1}^{z_0} K_r(z;\tau)\, d\tau\,,$$

which maps under the restriction map to a relation for $\varphi = \rho_r h \in \mathcal{D}^{\mathrm{pol}}_{2-r}$:

$$\varphi(t+1) - \varphi(t) \;=\; \int_{\tau=z_0-1}^{z_0} (\tau - t)^{r-2}\, d\tau \;=\; \frac{(z_0-t)^{r-1} - (z_0-t-1)^{r-1}}{r-1}\,.$$

(We have used (6.13) and (1.19).) The right hand side is a polynomial in t with $(-t)^{r-2}$ as the term of highest degree in t. Any polynomial solution φ is a polynomial with degree $r-1$ in t, and hence is not in $\mathcal{D}^{\mathrm{pol}}_{2-r}$. □

Remarks. (a) The function $h = h_c + h_\infty + h_0 \in \mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$ constructed in the proof has $\mathsf{Sing}_r\, h \subset z_0 + i[0,\infty)$.

(b) If $\lambda = 1$ the constant term can be handled by $\int_{z_0-1}^{z_0} K_r(z;\tau)\, d\tau = h_0(z+1) - h_0(z)$, with

$$h_0(z) \;=\; -2i\,\log(z-z_0) + 2i\log y - 2i\sum_{l=1}^{r-1}\binom{r-1}{l}\frac{(z-z_0)^l}{l\,(\bar z - z)^l}\,. \tag{11.4}$$

We note that although h_0 is an r-harmonic function, it does not represent an analytic boundary germ: $h_0 \notin \mathcal{W}^\omega_r(\mathbb{R})$.

Proof of Proposition 11.2. Let $z_0 \in \mathfrak{H}$, and consider the cocycle $c^{z_0}_F$ in (6.19), which represents the cohomology class $\mathbf{q}^\omega_r F$ of $F \in A_r(\Gamma, v)$. The following statements are equivalent:

- $\mathbf{q}^\omega_r F \in H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$.
- For each cusp $\mathfrak{a}$ there exists $h \in \mathcal{E}^{\omega,\mathrm{exc}}_r[\infty]$ that satisfies the relation (11.2) with $\lambda = v(\pi_\mathfrak{a})$, $E = F|_r\sigma_\mathfrak{a}$, and z_0 replaced by $\sigma_\mathfrak{a}^{-1} z_0$.

This gives the proposition. □

Relation (10.17) in Theorem 10.18 defines a subspace $A^{\mathcal{E}}_r(\Gamma, v) \subset A_r(\Gamma, v)$. We state a direct consequence of Proposition 11.2:

Corollary 11.4. *Let $r \in \mathbb{Z}_{\geq 2}$. Then $A^{\mathcal{E}}_r(\Gamma, v) = A^0_r(\Gamma, v)$.*

Next we would like to know that $\mathbf{q}^\omega_r A^0_r(\Gamma, v)$ is equal to $H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$. This requires quite some work, carried out in the following subsection.

11.2. **Image of mixed parabolic cohomology classes in automorphic forms.**

Proposition 11.5. *Let $r \in \mathbb{Z}_{\geq 2}$. The linear map $\alpha_r : H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \to A_r(\Gamma, v)$ in Proposition 10.10 has image in $A^0_r(\Gamma, v)$.*

Proof. Let a cohomology class $[c] \in H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ be given. In the proof of Proposition 10.10 the image $u = \alpha_r([c])$ is constructed in (10.9) as $u(z) := u(C; z) = \frac{1}{4\pi}\tilde c(C)(z)$, where the cochain $\tilde c \in C^1(F^{\mathcal{T}}_{\cdot}; \mathcal{G}^\omega_r, \mathcal{G}^{\omega^0,\mathrm{exc}}_{v,r})$ is a lift of $c \in Z^1(F^{\mathcal{T}}_{\cdot}; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$, and where $C \in \mathbb{Z}[X^{\mathcal{T}}_1]$ is a path around z adapted to $\tilde c$. We use a tesselation $\mathcal{T}$ and the corresponding resolution $(F^{\mathcal{T}}_i) = (\mathbb{Z}[X^{\mathcal{T}}_i])$ as discussed in §9.

We want to show that for each cusp the Fourier term of order zero of the automorphic form vanish. It suffices to do this for one representative $\mathfrak{a}$ of each Γ-orbit of cusps for which $v(\pi_\mathfrak{a}) = 1$. (If $v(\pi_\mathfrak{a}) \neq 1$ there is no Fourier term of order zero at the cusp $\mathfrak{a}$.) This can be handled for each such cusp separately. By conjugation we can assume that $\mathfrak{a} = \infty$ and $\pi_{\mathfrak{a}} = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

After the conjugation, the cuspidal sector V_∞ looks exactly like that for the modular group, in Figure 14, §9.1. The sector V_∞ is bounded by edges e_∞ from $P_\infty = \frac{1}{2} + iY$ (for some $Y > 0$) to ∞, $T^{-1}e_\infty$ from $T^{-1}\infty$ to ∞, and f_∞ from P_∞ to $T^{-1}P_\infty$. By holomorphy it suffices to consider the Fourier term of order 0 high up in the cuspidal sector.

The cocycle c satisfies $c(f_\infty) \in \mathcal{E}_r^\omega$, $c(e_\infty) \in \mathcal{E}_r^{\omega^*,\mathrm{exc}}$, and $c(e_\infty)|_r(1-T) = c(f_\infty)$. By Lemma 10.4 this implies $c(e_\infty) \in \mathcal{E}_r^{\omega,\mathrm{exc}}[\infty]$. We change the cocycle within its cohomology class. Definition 8.2 shows that $\mathcal{E}_r^{\omega,\mathrm{exc}}[\infty] = \mathcal{D}_r^{\omega,\mathrm{exc}}[\infty] + \mathcal{E}_r^\omega$. So there is $k \in \mathcal{E}_r^\omega$ such that $c(e_\infty) - k \in \mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$. We define $f \in C^0(F_\cdot^{\mathcal{T}}; \mathcal{E}_r^\omega, \mathcal{E}_r^{\omega^0,\mathrm{exc}})$ by taking $f(\gamma^{-1}P_\infty) = k|_r\gamma$ for all $\gamma \in \Gamma$, and $f = 0$ on all other Γ-orbits in $X_0^{\mathcal{T}}$. Then $c_1 = c - df$ is in the same cohomology class as c, and satisfies $c_1(e_\infty) \in \mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$. Replacing c by c_1, we can assume that the cocycle c now satisfies $c(e_\infty) \in \mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$ and then automatically $c(f_\infty) \in \mathcal{D}_r^\omega$ by the cocycle relation.

In the construction of $u(C; z)$ in (10.9) we started with z in a given set Z and showed that there are suitable cycles around it. Here we will take a special cycle C and choose a region Z encircled by it, high up in $\mathfrak{H}$.

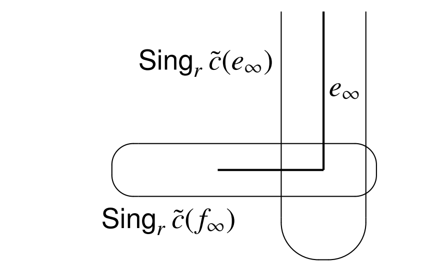

FIGURE 18

Let $\tilde{c} \in C^1(F_\cdot^{\mathcal{T}}; \mathcal{G}_{v,r}^\omega, \mathcal{G}_{v,r}^{\omega^0,\mathrm{exc}})$ be a lift of c. Our choice of the lift $\tilde{c}(f_\infty)$ of $c(f_\infty)$ in the proof of Proposition 10.10 implies that $\mathsf{Sing}_r\, \tilde{c}(f_\infty)$ is compact in $\mathfrak{H}$. The lift $\tilde{c}(e_\infty)$ was chosen so that $\mathfrak{H} \smallsetminus \mathsf{Sing}_r\, \tilde{c}(e_\infty)$ is an $\{\infty\}$-excised neighbourhood. The regions where $\tilde{c}(e_\infty)$ and $\tilde{c}(f_\infty)$ are not holomorphic may be large, and cover the sector V_∞. We have drawn the edges $x = e_\infty, f_\infty$ inside the singular set $\mathsf{Sing}_r\, \tilde{c}(x)$. For $\tilde{c}(f_\infty)$ this is a consequence of the choice, and for $\tilde{c}(e_\infty)$ we can arrange the choice so that $e_\infty \subset \mathsf{Sing}_r\, \tilde{c}(e_\infty)$.

We would like to enclose the set Z on which to study the function u by the boundary $\partial_2 V_\infty = e_\infty - T^{-1}e_\infty - f_\infty$. However, the corresponding sets of singularities may very well overlap, leaving no space for a region Z. Instead of this, we take the union of a number of translates $T^{-n}V_\infty$.

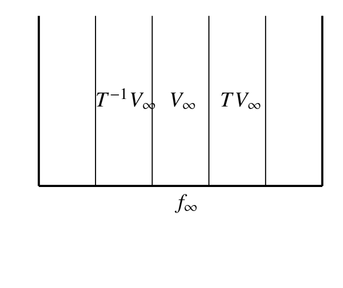

FIGURE 19

We take $k \in \mathbb{Z}_{\geq 1}$ large, so that there is a region of width at least 2 between $\mathsf{Sing}_r\, \tilde{c}(T^{-k}e_\infty) = T^{-k}\mathsf{Sing}_r\, \tilde{c}(e_\infty)$ and $\mathsf{Sing}_r\, \tilde{c}(T^k e_\infty) = T^k \mathsf{Sing}_r\, \tilde{c}(e_\infty)$. We put $g_k = \sum_{n=-k}^{k-1} T^{-n} f_\infty$. This leads to the situation in Figure 20. There is a connected

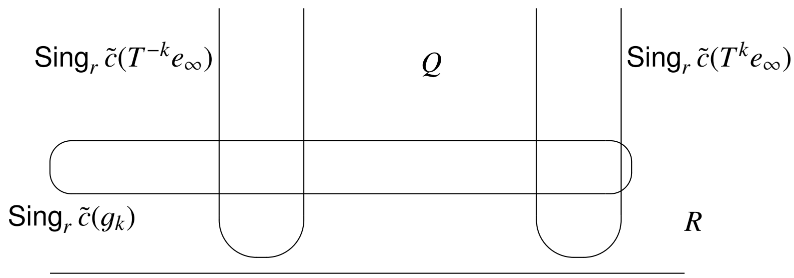

FIGURE 20. Illustration of regions of non-holomorphy.

region Q high up in the upper half-plane of width at least 2 on which $\tilde c(T^{\pm k}e_\infty)$ and $\tilde c(g_k)$ are holomorphic. The region Q is disjoint from the connected region R in the complement of the three singular sets that has $\mathbb{R}$ in its boundary.

We consider the cycle $C = (T^k e_\infty) - (T^{-k}e_\infty) - (f_\infty^{(k)}) \in \mathbb{Z}[X_1^{\mathcal{T}}]$, with k as fixed above. It encircles Q once, so $4\pi\, u(z) = \tilde c(C)\,(z)$ for $z \in Q$. Furthermore, on the $\{\infty\}$-excised neighbourhood R the function $\tilde c(C)$ is zero, since it represents $c(C)$, which is zero by the cocycle relation.

The element $h = \tilde c(e_\infty) \in \mathcal{G}_{v,r}^{\omega^0,\mathrm{exc}}$ represents an element of $\mathcal{D}_r^{\omega,\mathrm{exc}}[\infty]$ and $g = \tilde c(f_\infty) \in \mathcal{G}_{v,r}^{\omega}$ represents an element of $\mathcal{D}_r^{\omega}$. Furthermore the multiplier system v is trivial on $\pi_{\mathfrak a}$, and we have conjugated $\mathfrak a$ to ∞. So we can apply Lemma 4.4. The averages $\mathrm{Av}_{T,1}^{+} g$ and $\mathrm{Av}_{T,1}^{-} g$ are functions in $C^2(\mathfrak{H})$ that are holomorphic on the region $0 < \mathrm{Im}\,(z) < \varepsilon$ and on $\pm\mathrm{Re}\,(z) > \varepsilon^{-1}$ for some sufficiently small positive ε. Lemma 4.4 gives us 1-periodic p_+ and p_- such that $h = \mathrm{Av}_{T,1}^{\pm} g + p_\pm$, on regions as indicated in the lemma.

Shifting this with $T^{\pm k}$ we get

$$\begin{aligned} \tilde c(T^k e_\infty) &= \tilde c(e_\infty)|_r T^{-k} = (\mathrm{Av}_{T,1}^{-} g)|_r T^{-k} + p_-\,, \\ \tilde c(T^{-k} e_\infty) &= \tilde c(e_\infty)|_r T^{k} = (\mathrm{Av}_{T,1}^{+} g)|T^{k} + p_+\,, \end{aligned} \tag{11.5}$$

first on regions as indicated in the lemma, and then by analytic continuation to regions containing Q and a strip $0 < \mathrm{Im}\,(z) < \varepsilon$.

For z in Q or near $\mathbb{R}$ we have

$$\begin{aligned} \tilde c(C)(z) &= \tilde c(T^k e_\infty)(z) - \tilde c(T^{-k}e_\infty)(z) - \tilde c(g_k)(z) \\ &= (\mathrm{Av}_{T,1}^{-} g)|_r T^{-k}(z) + p_-(z) - (\mathrm{Av}_{T,1}^{+} g)|_r T^{k}\,(z) - p_+(z) - \sum_{n=-k}^{k-1} g|_r T^n\,(z) \\ &= -\sum_{n\le -1} g|_r T^{n-k}\,(z) - \sum_{n\ge 0} g|_r T^{n+k}\,(z) - \sum_{n=-k}^{k-1} g|_r T^n\,(z) + p_-(z) - p_+(z) \\ &= -(\mathrm{Av}_{T,1} g)\,(z) + p_-(z) - p_+(z)\,, \end{aligned}$$

with $\mathrm{Av}_{T,1}$ as defined in (4.5).

Next we apply Proposition 4.3 to $g = \tilde c(f_\infty)$. The 1-periodic function $\mathrm{Av}_{T,1}g(z) = \sum_{n\in\mathbb{Z}} g(z+n)$ on $\mathfrak{H}$ is holomorphic on a region of the form $0 < \operatorname{Im} z < \varepsilon$ and on a region $y > \varepsilon^{-1}$ for some $\varepsilon \in (0,1)$. We denote the holomorphic function on the upper region by $\mathrm{Av}^{\uparrow}_{T,1}\tilde c(f_\infty)$, and the holomorphic function on the lower region by $\mathrm{Av}^{\downarrow}_{T,1}\tilde c(f_\infty)$. Proposition 4.3 states that $\mathrm{Av}^{\uparrow}_{T,1}\tilde c(f_\infty)$ has a Fourier expansion with terms of positive order only, and $\mathrm{Av}^{\downarrow}_{T,1}\tilde c(f_\infty)$ a Fourier expansion with only terms of negative order.

The domain of $\mathrm{Av}^{\downarrow}_{T,1} g$ is contained in the region R. There we find

$$0 \;=\; \tilde c(C)(z) \;=\; p_-(z) - p_+(z) - \bigl(\mathrm{Av}^{\downarrow}_{T,1} g(z)\,.$$

So all Fourier terms of $p_- - p_+$ of order $n \geq 0$ vanish. This holds on $\mathfrak{H}$, since $p_- - p_+$ is holomorphic and 1-periodic on $\mathfrak{H}$.

If $z \in Q$ then z is in the domain of $\mathrm{Av}^{\uparrow}_{T,1} g$, and

$$4\pi\, u(C;z) \;=\; \tilde c(C)(z) \;=\; p_-(z) - p_+(z) - \mathrm{Av}^{\uparrow}_{T,1} g\,.$$

The function $(\mathrm{Av}^{\uparrow}_{T,1} g)(z)$ is given by a Fourier expansions with terms of positive order. The term $p_- - p_+$ has a Fourier expansion with terms of negative order. Hence the Fourier coefficient of u at ∞ of order 0 vanishes. $\square$

Corollary 11.6. *Let $r \in \mathbb{Z}_{\geq 2}$. Then $H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) = \mathbf{q}^\omega_r\, A^0_r(\Gamma, v)$.*

Proof. By Proposition 11.5 we have $\alpha_r H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \subset A^0_r(\Gamma, v)$.

A given class $[c] \in H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ has image $\alpha_r[c] \in A^0_r(\Gamma, v)$, and hence $\mathbf{q}^\omega_r \alpha_r[c] \in \mathbf{q}^\omega_r A^0_r(\Gamma, v)$. Part ii) of Theorem 10.18 implies that $\alpha_r \mathbf{q}^\omega_r \alpha_r[c] = \alpha_r[c]$, and then $\mathbf{q}^\omega_r \alpha_r[c] = [c]$, by the injectivity of α_r in Part i) of that theorem. This proves that $H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \subset \mathbf{q}^\omega_r\, A^0_r(\Gamma, v)$.

The other inclusion follows from $(\mathbf{q}^\omega_r)^{-1} H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) = A^0_r(\Gamma, v)$ (Proposition 11.2). $\square$

11.3. **Exact sequences for mixed parabolic cohomology groups.**

Proposition 11.7. *Let $r \in \mathbb{Z}_{\geq 2}$. We put $K_{1,2} := \mathcal{D}^{\mathrm{pol}}_{1,0} \cong \mathbb{C}$ (trivial representation), and $K_{v,r} := \{0\}$ if $r \in \mathbb{Z}_{\geq 3}$ or $v \neq 1$.*

The rows in the following commuting diagram are exact. (We have suppressed Γ from the notation.)

$$\begin{tikzcd}
K_{v,r} \arrow[r] \arrow[d, equal] & H^1(\mathcal{D}^\omega_{v,r}) \arrow[r] & H^1(\mathcal{E}^\omega_{v,r}) \arrow[r, "\rho_r"] & H^1(\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[r] & 0 \\
K_{v,r} \arrow[r] & H^1_{\mathrm{pb}}(\mathcal{D}^\omega_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r] \arrow[u, hook] & H^1_{\mathrm{pb}}(\mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r, "\rho_r"] \arrow[u, hook] & H^1_{\mathrm{pb}}(\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[u, hook] &
\end{tikzcd} \tag{11.6}$$

Proof. We use the following commuting diagram of Γ-modules with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{D}^{\omega}_{v,r} \arrow[r] \arrow[d, hook] & \mathcal{E}^{\omega}_{v,r} \arrow[r, "\rho_r"] \arrow[d, hook] & \mathcal{D}^{\mathrm{pol}}_{v,2-r} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r} \arrow[r] & \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r} \arrow[r, "\rho_r"] & \mathcal{D}^{\mathrm{pol}}_{v,2-r} \arrow[r] & 0
\end{tikzcd}
\tag{11.7}
$$

See Part iii) of Proposition 8.3.

The upper row in (11.6) is part of the corresponding long exact sequence in group cohomology. Since Γ has cusps, all groups $H^2(\Gamma; V)$ are zero. (See, *e.g.*, [15, §11.2].) The Γ-invariants of $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$ are zero, unless $r = 2$ and $v = 1$, when $\mathcal{D}^{\mathrm{pol}}_{1,2-2}$ is the trivial representation. This gives the exactness of the upper row.

To use [15, Proposition 11.9] for the lower row, we need also exactness of

$$0 \to (\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r})^{\Gamma_\mathfrak{a}} \to (\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})^{\Gamma_\mathfrak{a}} \xrightarrow{\rho_r} (\mathcal{D}^{\mathrm{pol}}_{v,2-r})^{\Gamma_\mathfrak{a}} \to 0$$

for each cusp $\mathfrak{a}$ of Γ. Most of the exactness follows from Part iii) in Proposition 8.3. For the surjectivity of ρ_r we conjugate $\mathfrak{a}$ to ∞, and use the Fourier expansion in Part ii) of Lemma 8.7. The restriction ρ_r sends the holomorphic contributions to zero. Only if $v(\pi_\mathfrak{a}) = 1$ there may be a multiple of y^{1-r}; and we note that $\rho_r y^{1-r} = -(2i)^{r-2}$ spans the T-invariants in $\mathcal{D}^{\mathrm{pol}}_{2-r}$. If $v(\pi_\mathfrak{a}) \neq 1$ there is no multiple of y^{1-r}, and there are no non-zero elements in $\mathcal{D}^{\mathrm{pol}}_{2-r}$ on which T acts as multiplication by $v(\pi_\mathfrak{a})$.

Proposition 11.9 in [15] gives a long exact sequence of the corresponding mixed parabolic cohomology groups, of which the lower row is a part. We use [15, (11.11)], which tells us that $H^0_{\mathrm{pb}}(\Gamma; V)$ is the space of invariants V^Γ. Hence we get $K_{v,r}$ on the left. □

Remark. In the second line of diagram (11.6) we have not written a terminating → 0. We did not succeed in proving this directly, for instance by showing that

$$H^2_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^2_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$$

is injective. For unitary multiplier system the surjectivity of

$$\rho_r : H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r})$$

is known to hold, by classical results, as we will discuss in §11.5.

11.4. **Automorphic forms and analytic boundary germ cohomology.** We proceed under the assumption $r \in \mathbb{Z}_{\geq 2}$. In the diagram

$$
\begin{tikzcd}
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^{\omega}_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[d, shift left, "\alpha_r\ \cong"] \\
A_{2-r}(\Gamma, v) \arrow[u, "\mathbf{r}^{\omega}_{2-r}", "\cong"'] & A^0_r(\Gamma, v) \arrow[u, shift left, "\mathbf{q}^{\omega}_r"]
\end{tikzcd}
\tag{11.8}
$$

we use Theorem A in the weight $2-r \in \mathbb{Z}_{\le 0}$ to get the isomorphism $\mathbf{r}^\omega_r$ on the left. The isomorphisms α_r and $\mathbf{q}^\omega_r$ on the right follow from Theorem 10.18, Corollaries 11.4 and 11.6. The horizontal arrow denotes the natural map associated to the inclusions $\mathcal{D}^\omega_{v,r} \subset \mathcal{E}^\omega_{v,r}$ and $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r} \subset \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}$. The following results makes this into a commutative diagram:

Lemma 11.8. *Let $r \in \mathbb{Z}_{\ge 2}$. Let $c_r = \frac{i}{2}\frac{1}{(r-1)!}$. Let $K_{v,r}$ be as defined in Proposition 11.7.*

The following diagram commutes and has exact rows:

$$\begin{tikzcd} K_{v,r} \arrow[r] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[r, "\mathrm{id}"] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[d, shift left, "\alpha_r\ \cong"] \\ K_{v,r} \arrow[u, "\times(-i/2)"] \arrow[r] & A_{2-r}(\Gamma, v) \arrow[u, "\cong\ \mathbf{r}^\omega_{2-r}"'] \arrow[r, "c_r\,\partial_z^{r-1}"] & A^0_r(\Gamma, v) \arrow[u, shift left, "\mathbf{q}^\omega_r"] \end{tikzcd} \tag{11.9}$$

By id we indicate the homomorphism induced by the inclusions $\mathcal{D}^*_{v,r} \to \mathcal{E}^*_{v,r}$.

Proof. Bol's equality $\partial_z^{r-1}(F|_{2-r}g) = F^{(r-1)}|_r g$ for $g \in \mathrm{SL}_2(\mathbb{R})$, which appears in [6, §8], implies that $c_r\,\partial_z^{r-1}$ determines a map $A_{2-r}(\Gamma, v) \to A_r(\Gamma, v)$. Since the constant functions are the sole polynomials that can be automorphic forms, the kernel is $K_{v,r}$. So the lower row is exact. Proposition 11.7 gives the exactness of the upper row.

For the commutativity of the left rectangle we assume $r = 2$ and $v = 1$. The map $\mathbf{r}^\omega_{2-r}$ sends the constant function 1 to the class represented by the cocycle $\gamma \mapsto \psi^{z_0}_{1,\gamma}(t) = \frac{1}{t-z_0} - \frac{1}{t-\gamma^{-1}z_0}$, for an arbitrary base point $z_0 \in \mathfrak{H}$. The constant function $-\frac{i}{2} \in K_{1,2} \cong (\mathcal{D}^{\mathrm{pol}}_{1,0})^\Gamma$ has a lift $t \mapsto -\frac{i}{2}K_2(t; z_0)$ in $\mathcal{E}^\omega_{1,2}$, with the kernel function K_2 defined in (6.12). So the connecting homomorphism sends $\psi^{z_0}_1$ to the cocycle χ determined by $\chi_\gamma = -\frac{i}{2}K_2(\cdot; z_0)|_{1,2}(1-\gamma)$. The kernel function K_r has the invariance property $K_r(\cdot;\cdot)|_r g \otimes |_{2-r} g = K_r(\cdot;\cdot)$, in (6.14). For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$:

$$\chi_\gamma(t) \;=\; -\frac{i}{2}K_2(t; z_0) + \frac{i}{2}(a - cz_0)^0\, K_2(t; \gamma^{-1}z_0) \;=\; \frac{1}{t-z_0} - \frac{1}{t-\gamma^{-1}z_0} \;=\; \psi^{z_0}_{1,\gamma}(t)\,.$$

For the commutativity of the second rectangle we start with $F \in A_{r-2}(\Gamma, v)$ and compute its image under the composition $\alpha_r \circ \mathrm{id} \circ \mathbf{r}^\omega_{r-2}$. We use the description of cohomology with a tesselation as discussed in §9. The cocycle c representing $\mathbf{r}^\omega_r F$ is determined by

$$c_F(x; z) \;=\; \int_x (z-t)^{-r}\, F(z)\, dz\,,$$

where $x \subset \mathfrak{H}$ is an oriented edge in $X^{\mathcal{T},Y}_1$.

The function $c_F(x;\cdot)$ is defined on $\mathbb{C} \smallsetminus x$, and represents an element of $\mathcal{D}^\omega_{v,2}$. Since $\mathcal{D}^\omega_{v,r} \subset \mathcal{E}^\omega_{v,r}$, the same cocycle represents $\mathrm{id}(\mathbf{r}^\omega_r F)$. The image of $\mathrm{id}(\mathbf{q}^\omega_r F)$ under the map α_r in Proposition 10.10 is an automorphic form $u \in A_r(\Gamma, v)$. By analytic continuation it is determined by its value on the interior $\mathring{\mathfrak{F}}_Y$ of the face $\mathfrak{F}_Y \in X^{\mathcal{T},Y}_2$. (It is important to use a face that is completely contained in $\mathfrak{H}$; otherwise $c(x)$ need not be given by the integral above for all edges x in the boundary of $\partial_2\mathfrak{F}_Y$.) We apply Proposition 10.15. It gives, for $z \in \mathring{\mathfrak{F}}_Y$

$$\begin{aligned} u(z) &= \frac{1}{4\pi}\, c_F(\partial_2\mathfrak{F}_Y)(z) = \frac{1}{4\pi}\sum_{x\in\partial_2\mathfrak{F}_Y} c_F(x;z) = \frac{1}{4\pi}\sum_{x\in\partial_2\mathfrak{F}_Y}\int_x (\tau-z)^{-r}\,F(\tau)\,d\tau \\ &= \frac{1}{4\pi}\int_{\partial_2\mathfrak{F}_Y}(\tau-z)^{-r}\,F(\tau)\,d\tau = \frac{2\pi i}{4\pi}\frac{1}{(r-1)!}\,F^{(r-1)}(z) = \bigl(c_r\,\partial_z^{r-1}F\bigr)(z)\,. \qquad \square \end{aligned}$$

Lemma 11.9. *Let $r\in\mathbb{Z}_{\geq 2}$. The map $\mathbf{q}_r^\omega : A_r(\Gamma,v)\to H^1(\Gamma;\mathcal{E}^\omega_{v,r})$ is injective and the following diagram commutes:*

(11.10)
$$\begin{tikzcd} H^1(\Gamma;\mathcal{E}^\omega_{v,r}) \arrow[rr, "\rho_r"] & & H^1(\Gamma;\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \\ H^1_{\mathrm{pb}}(\Gamma;\mathcal{E}^\omega_{v,r},\mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[u, hook] & & \\ A^0_r(\Gamma,v) \arrow[u, "\mathbf{q}_r^\omega", "\cong"'] \arrow[r, hook] & A_r(\Gamma,v) \arrow[uul, hook, "\mathbf{q}_r^\omega"'] \arrow[uur, "\mathbf{r}_r^\omega"'] & \end{tikzcd}$$

Proof. Suppose that $F\in A_r(\Gamma,v)$ satisfies $\mathbf{q}_r^\omega F = 0$. Then F is in the space $A_r^{\mathcal{E}}(\Gamma,v)$ in (10.17), and then $F = 0$ by the injectivity in Part i) of Theorem 10.18.

The commutativity of the left triangle is a direct consequence of the definitions. For the right triangle we start with the commutativity of the diagram in Proposition 6.10, where we can replace $\mathcal{D}^\omega_{v,2-r}$ by $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$, since $r\in\mathbb{Z}_{\geq 2}$.

$$\begin{tikzcd} H^1(\Gamma;\mathcal{W}^\omega_{v,r}(\mathbb{P}^1_\mathbb{R})) \arrow[rr, "\rho_r^{\mathrm{prj}}"] & & H^1(\Gamma;\mathcal{V}^\omega_{v,2-r}(\mathbb{P}^1_\mathbb{R})) \\ & H^1(\Gamma;\mathcal{E}^\omega_{v,r}) \arrow[ul] \arrow[r, "\rho_r"] & H^1(\Gamma;\mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[u, "\mathsf{prj}_{2-r}"'] \\ & A_r(\Gamma,v) \arrow[uul, "\mathbf{q}_r^\omega"] \arrow[u, "\mathbf{q}_r^\omega"'] \arrow[ur, "\mathbf{r}_r^\omega"'] & \end{tikzcd}$$

We have chosen the spaces $\mathcal{E}^*_{v,r}$ in such a way the the image of $\mathbf{q}_r^\omega$ is in $H^1(\Gamma;\mathcal{E}^\omega_{v,r})$. From $\rho_r = \mathsf{prj}_{2-r}^{-1}\,\rho_r^{\mathrm{prj}}$ (Definition 6.17) and Part iii) of Proposition 8.3 it follows that $\rho_r^{\mathrm{prj}}\,\mathbf{q}_r^\omega = \mathbf{r}_r^\omega$. $\square$

Recapitulation of the proof of Theorem D. The commutativity of various parts of the diagram in (5) in Theorem D follows from Proposition 11.7 and the Lemmas 11.8 and 11.9.

The exactness of the top row and the second row, in Part ii), are given by Proposition 11.7, which gives also the information in Part iii) of the theorem.

The injectivity of $\mathbf{q}_r^\omega : A_r(\Gamma,v)\to H^1(\Gamma;\mathcal{E}^\omega_{v,r})$ is shown in Lemma 11.9, the injectivity of the vertical maps between cohomology groups follows directly from the definition of (mixed) parabolic cohomology. The bijectivity of $\mathbf{r}^\omega_{2-r}$ is given by Theorem A for weights not in $\mathbb{Z}_{\ge 2}$, and the injectivity of $\mathbf{q}_r^\omega : A_r^0(\Gamma, v) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r})$ is a consequence of Theorem 10.18 and Corollary 11.4. □

11.5. **Comparison with classical results.** In the following part of diagram (5) in Theorem D

$$\begin{tikzcd} H^1(\Gamma; \mathcal{E}^\omega_{v,r}) \arrow[r, "\rho_r"] & H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[r] & 0 \\ H^1_{\mathrm{pb}}(\Gamma; \mathcal{E}^\omega_{v,r}, \mathcal{E}^{\omega^0,\mathrm{exc}}_{v,r}) \arrow[u, hook] \arrow[r, "\rho_r"] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[u, hook] & \end{tikzcd} \tag{11.11}$$

the absence of an arrow $\to 0$ in the second row is remarkable. The surjectivity of ρ_r in the top row is a consequence of the general fact that $H^2(\Gamma; V) = \{0\}$ for any Γ-module for groups Γ with cusps. See, eg., [15, §11/2]. In the long exact sequence corresponding to the diagram in (11.7) there is a sequel

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) \to H^2_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r}) \to H^2_{\mathrm{pb}}(\Gamma; \mathcal{D}^\omega_{v,r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,r})$$

that may be non-zero. It would be interesting to see that in general the second row in (11.11) is surjective.

We review some classical results, under the assumption that the multiplier system v for weight $r \in \mathbb{Z}_{\ge 2}$ is unitary.

The elements of $\mathcal{D}^{\mathrm{pol}}_{2-r}$ are polynomial functions, and hence are holomorphic on $\mathbb{C}$. This space of polynomials is invariant under the involution ι in (1.6). The action is changed (unless v is real-valued): $\iota : \mathcal{D}^{\mathrm{pol}}_{v,2-r} \leftrightarrow \mathcal{D}^{\mathrm{pol}}_{\bar v,2-r}$. This induces involutions

$$\begin{tikzcd} H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[r, "\iota"] & H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{\bar v,2-r}) \\ H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) \arrow[u, hook] \arrow[r, "\iota"] & H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\mathrm{pol}}_{\bar v,2-r}) \arrow[u, hook] \end{tikzcd} \tag{11.12}$$

The linear map $\mathbf{r}_r^\omega : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r})$ has an antilinear counterpart $\iota\mathbf{r}_r^\omega : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{\bar v,2-r})$, in which $\iota\mathbf{r}_r^\omega F$ is represented by

$$\gamma \mapsto \int_{z=\gamma^{-1}z_0}^{z_0} \overline{F(z)}\,(\bar z - t)^{r-2}\,d\bar z\,.$$

We now look at the classical theory in [58], where Theorem 1 gives

$$H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,2-r}) = \mathbf{r}_r^\omega M_r(\Gamma, v) \oplus \iota\mathbf{r}_{\bar r}^\omega S_r(\Gamma, \bar v)\,. \tag{11.13}$$

The restriction of $\mathbf{r}_r^\omega$ to $M_r(\Gamma, v)$ is a multiple of the map β in [58] and [66, §1.3]. It is described by $(r-1)$-fold integration. The construction of $(\iota\mathbf{r}_{\bar r}^\omega)f$ for $f \in S_r(\Gamma, \bar v)$ is carried out by forming $g^* \in A_r(\Gamma, v)$, and then forming, with the "supplementary function", (a multiple of) $\mathbf{r}_r^\omega g^*$ with the property that $\mathbf{r}_r^\omega g^*$ is a multiple of $\iota\mathbf{r}_{\bar r}^\omega f$. (The resulting antilinear map $S_r(\Gamma, \bar v) \to H^1(\Gamma; \mathcal{D}^{\mathrm{pol}}_{v,w-r})$ is called α

in [58]. In particular, $\mathbf{r}_r^\omega g$ is a parabolic class, in $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\mathrm{pol}}_{v,2-r})$. The computations in §3.4, especially Lemma 3.8, show that $g\in A^0_r(\Gamma,v)$. With Theorem 1 in [58], we conclude that $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\mathrm{pol}}_{v,2-r})$ is contained in $\mathbf{r}_r^\omega\, A^0_r(\Gamma,v)$. The diagram in Theorem D implies that $H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\mathrm{pol}}_{v,2-r}) = \mathbf{r}_r^\omega\, A^0_r(\Gamma,v)$. So indeed, the classical theory gives us the missing surjectivity, for unitary multiplier systems.

We note that in [66] the map α is constructed in a different way, with automorphic integrals of Niebur [94]. For the purpose of this subsection the supplementary functions used in [58] are more useful.

Remark. Knopp, Lehner and Raji [70] [72] [105, 106] have studied cohomology classes associated to *generalized modular forms* for which the multiplier systems need to satisfy $|v(\pi)| = 1$ only for parabolic $\pi\in\Gamma$.

11.6. **Related work.** In this section we connected the classical results concerning the relation between automorphic forms and Eichler cohomology to the boundary germ cohomology in Theorem 10.18.

Part IV. Miscellaneous

We have proved Theorems A–D in the introduction, and some of the isomorphisms in Theorem E in §1.7. In Sections 12 and 13 we complete the proof of Theorem E. In Section 14 we discuss quantum automorphic forms and their relation to cohomology. We close this part with Section 15, which gives further remarks on the literature.

12. Isomorphisms between parabolic cohomology groups

12.1. **Invariants under hyperbolic and parabolic elements.** For Γ-modules $V\subset W$ there is a natural map $H^1_{\mathrm{pb}}(\Gamma;V,W)\to H^1_{\mathrm{pb}}(\Gamma;W)$, which turns out to be an isomorphism in several cases under consideration. It takes quite some work to sort this out. As a first step, we consider for parabolic and hyperbolic elements $\gamma\in\Gamma$ the spaces $V^\gamma = \{v\in V \;:\; v|\gamma = v\}$ of invariants in the Γ-modules V under consideration.

Parabolic elements. Lemma 3.1 implies that for a parabolic $\pi\in\Gamma$ we have $(\mathcal{D}^{\omega^*}_{r,2-r})^\pi \subset \mathcal{D}^{\omega}_{v,2-r}[\mathfrak{a}]$, where $\mathfrak{a}$ is the cusp fixed by π.

Lemma 12.1. *Let $r\in\mathbb{C}$, and let $\pi\in\Gamma$ be parabolic. We denote $\lambda = v(\pi)$.*

a) *The dimensions of various spaces of invariants are as follows:*

	$r\notin\mathbb{Z}_{\geq1}$ *or* $\lambda\neq1$	$r=1$ *and* $\lambda=1$	$r\in\mathbb{Z}_{\geq2}$ *and* $\lambda=1$
$\dim(\mathcal{D}^{\omega^*}_{v,2-r})^\pi$	∞	∞	∞
$\dim(\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})^\pi$	∞	∞	∞
$\dim(\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})^\pi$	0	1	1
$\dim(\mathcal{D}^{\omega^*,\infty}_{v,2-r})^\pi$	0	0	1

b) *In all cases* $(\mathcal{D}^{\omega^*,\infty}_{v,2-r})^\pi = (\mathcal{D}^{\omega}_{v,2-r})^\pi$*, and* $(\mathcal{D}^{\omega}_{v,2-r})^\pi = (\mathcal{D}^{\mathrm{pol}}_{v,2-r})^\pi$ *if* $r \in \mathbb{Z}_{\geq 2}$.

Proof. Going over to π^{-1} if necessary, the element π is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. After conjugation we find that invariance amounts to $\varphi(t+1) = \lambda\,\varphi(t)$, with $\lambda = v(\pi) \in \mathbb{C}^*$. This has solutions given by $\sum_{n\equiv\alpha(1)} a_n\, e^{2\pi i n t}$ with $e^{2\pi i\alpha} = \lambda$. For $\mathcal{D}^{\omega^*}_{v,2-r}$ we need convergence on a half-plane $\mathrm{Im}\, t < \varepsilon$ for some $\varepsilon > 0$. For $\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$ the λ-periodicity of φ implies that φ extends holomorphically to all of $\mathbb{C}$, and hence we need convergence on all of $\mathbb{C}$. In both cases in Part i) we get an infinite-dimensional space of invariants.

In the other parts there is a condition at ∞, which implies that $(\mathsf{prj}_{2-r}\varphi)(t) := (i-t)^{2-r}\,\varphi(t)$ has an asymptotic expansion of the form $(\mathsf{prj}_{2-r}\varphi)(t) \sim \sum_{\ell\geq k} c_\ell\, t^{-\ell}$, valid as $t \in \mathfrak{H}^-$ approaches ∞. For $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ we have $k = -1$, and for $\mathcal{D}^{\omega^*,\infty}_{v,2-r}$ and its submodules, $k = 0$.

So if $\varphi \neq 0$ the expansion starts with $d_n\, t^{r-2-n} + d_{n+1}\, t^{r-3-n} + \cdots$, where $d_n \neq 0$ and $n \geq k$. We insert this into the invariance relation. If $\lambda \neq 1$, the starting term shows that $d_n = 0$. So for $\lambda \neq 1$ no invariants exist in $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ and smaller modules.

If $\lambda = 1$ then we find from the second term that $d_n\,(r-2-n) = 0$. So for an invariant the expansion should start at $n = r-2$. Since $n \geq k$, this leads to $r \in \mathbb{Z}_{\geq 1}$ for $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$, and $r \in \mathbb{Z}_{\geq 2}$ for the smaller modules. Thus we have $\varphi(t) = d_{r-2} + d_{r-1}\, t^{-1} + \cdots$. There is indeed an easy invariant under these conditions, namely the constant function $\varphi_{\mathrm{cst}}(t) = 1$. It is in each of the modules. Then $\varphi - d_{r-2}\varphi_{\mathrm{cst}}$ is 1-periodic and $\mathrm{O}(t^{-1})$, hence zero. So the dimension of the space of invariants equals 1. □

Remark. For $\pi = T$ and $v(T) = 1$, the proof shows that the invariants are the constant functions. For other parabolic elements $\pi = gTg^{-1}$ the spaces of invariants in Parts ii)–iv) of the lemma have dimension 1, but need not consist of constant functions.

Hyperbolic elements and closed geodesics. An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is *hyperbolic* if $a+d > 2$. A hyperbolic element γ of $\mathrm{SL}_2(\mathbb{R})$ has exactly two invariant points in $\mathbb{P}^1_{\mathbb{C}}$, situated on $\mathbb{P}^1_{\mathbb{R}}$, say ξ and ξ'. On the geodesic in $\mathfrak{H}$ connecting ξ and ξ' the action of γ on the points of the geodesic amounts to a shift over a fixed distance for the hyperbolic metric, which we call $\ell(\gamma)$. We note that $\ell(\gamma^n) = |n|\,\ell(\gamma)$ for $n \in \mathbb{Z}$. The image in $\Gamma\backslash\mathfrak{H}$ of that invariant geodesic is a so-called closed geodesic, with length $\ell(\gamma)$.

A *hyperbolic subgroup* H of Γ is a subgroup generated by a hyperbolic γ and -1. Such a hyperbolic generator γ is a *primitive hyperbolic element* of Γ. The inverse γ^{-1} is the other primitive hyperbolic element in H. We can conjugate a hyperbolic element γ in $\mathrm{SL}_2(\mathbb{R})$ to $\begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2} \end{pmatrix}$ with $p = e^{\ell(\gamma)} > 1$. This element has ∞ as attracting fixed point, and 0 as repelling fixed point.

Lemma 12.2. *Let* $\lambda \in \mathbb{C}^*$*, and let* $\gamma \in \mathrm{SL}_2(\mathbb{R})$ *be hyperbolic. If* $f \in \mathcal{D}^{\omega^*}_{2-r}$ *satisfies* $f|_{2-r}\gamma = f$*, then* $f \in \mathcal{D}^{\omega}_{2-r}[\xi,\xi']$*, where* ξ *and* ξ' *are the fixed points of* γ*.*

Proof. Analogous to the proof of Lemma 3.1. □

To formulate the following result it is convenient to introduce for a hyperbolic $\gamma \in \Gamma$ the quantity $\kappa := \kappa_{v,2-r}(\gamma) \in \mathbb{C}$ that is uniquely determined by

$$e^{\kappa\,\ell(\gamma)} \;=\; v(\gamma)\, e^{(r/2-1)\,\ell(\gamma)}\,, \quad \text{and} \quad -\frac{\pi}{\ell(\gamma)} < \operatorname{Im}\kappa \le \frac{\pi}{\ell(\gamma)}\,, \tag{12.1}$$

where $\ell(\gamma)$ is the length of the periodic geodesic corresponding to γ.

Lemma 12.3. *Let $r \in \mathbb{C}$, and let γ be a hyperbolic element of Γ, corresponding to a closed geodesic in $\Gamma\backslash\mathfrak{H}$ with length $\ell(\gamma)$.*

a) *With $\kappa = \kappa_{v,2-r}(\gamma)$ as in* (12.1) *the dimensions of various spaces of invariants are as follows:*

	$r \notin \mathbb{Z}$ *or* $\kappa \le -2$ *or* $\kappa \ge r$	$r \in \mathbb{Z}_{\ge 0}$ *and* $\kappa \in \{-1, r-1\}$	$\kappa, r \in \mathbb{Z}$ *and* $0 \le \kappa \le r-2$
$\dim(\mathcal{D}^{\omega^*}_{v,2-r})^\gamma$	∞	∞	∞
$\dim(\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})^\gamma$	∞	∞	∞
$\dim(\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})^\gamma$	0	1	1
$\dim(\mathcal{D}^{\omega^*,\infty}_{v,2-r})^\gamma$	0	0	1

b) *In all cases $(\mathcal{D}^{\omega}_{v,2-r})^\gamma = (\mathcal{D}^{\omega^*,\infty}_{v,2-r})^\gamma$, and $(\mathcal{D}^{\omega}_{v,2-r})^\gamma = (\mathcal{D}^{\mathrm{pol}}_{v,2-r})^\gamma$ if $r \in \mathbb{Z}_{\ge 2}$.*

Proof. We conjugate γ in $\mathrm{SL}_2(\mathbb{R})$ to $\begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2}\end{pmatrix}$, where $p = e^{\ell(\gamma)}$, which leaves fixed 0 and ∞, and the geodesic between them. This leads to the equation

$$p^{1-r/2}\,\varphi(pt) \;=\; v(\gamma)\,\varphi(t)\,. \tag{12.2}$$

By examining the Fourier expansion of $\psi(x) := \varphi(e^x)$, the solutions in $\mathcal{D}^{\omega^*}_{v,2-r}$ can be obtained as

$$\sum_{n\in\mathbb{Z}} d_n\,(it)^{\alpha+2\pi i n/\ell(\gamma)}\,, \tag{12.3}$$

convergent for at least $-\frac{\pi}{2} < \arg(it) < \frac{\pi}{2}$, and α in the set

$$\begin{aligned} E_{v,2-r}(\gamma) \;&=\; \Bigl\{\frac{r}{2} - 1 + \frac{\log v(\gamma) + 2\pi i m}{\ell(\gamma)} \;:\; m \in \mathbb{Z}\Bigr\} \\ &=\; \Bigl\{\kappa_{v,2-r}(\gamma) + \frac{2\pi i m}{\ell(\gamma)} \;:\; m \in \mathbb{Z}\Bigr\}. \end{aligned} \tag{12.4}$$

With the standard choice of the argument, $(it)^\alpha$ is well defined on $\mathbb{C} \smallsetminus i[0,\infty)$.

By Lemma 12.2, a function φ representing an element of $(\mathcal{D}^{\omega^*}_{v,2-r})^\gamma$ should extend holomorphically to neighbourhoods of $(0,\infty)$ and $(-\infty,0)$ in $\mathbb{C}$. The γ-invariance implies that φ should be holomorphic on a Γ-invariant domain. So φ should be holomorphic on at least a region $-\pi-\varepsilon < \arg t < \varepsilon$ for some $\varepsilon > 0$. Hence the series in (12.3) represents an element of $(\mathcal{D}^{\omega^*}_{v,2-r})^\gamma$ precisely if it converges on a region $-\frac{\pi}{2} - \varepsilon < \arg(it) < \frac{\pi}{2} + \varepsilon$ with some $\varepsilon > 0$. To get a holomorphic function on an excised neighbourhood with excised set $\{0,\infty\}$, we need to pick coefficients such that we have convergence for $-\pi < \arg(it) < \pi$. In this way we obtain a complete description of the, infinite-dimensional, spaces $(\mathcal{D}^{\omega^*}_{v,2-r})^\gamma$ and $(\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})^\gamma$ in the first two lines in the table in Part a).

For the smaller modules there should be asymptotic expansions at 0 and ∞. Let $k=-1$for $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ and $k=0$ for $\mathcal{D}^{\omega^*,\infty}_{v,2-r}$. In the expansion at zero, $d_n=0$ for $n\neq 0$, and α should be in $E_{v,2-r}(\gamma)\cap\mathbb{Z}_{\geq k}$. The function $t\mapsto t^{r-2}\,\varphi(-1/t)$ should also have an expansion with terms t^m with $m\geq k$. Hence we have the further restriction $r-2-\alpha\in\mathbb{Z}_{\geq k}$. So the exponents $\alpha\in E_{v,2-r}(\gamma)$ should satisfy

$$\alpha\in E_{v,2-r}(\gamma)\cap\mathbb{Z}\cap(r-2+\mathbb{Z})\quad\text{ and } k\leq\alpha\leq r-2-k\,.$$

So we should have $r\in\mathbb{Z}$. The condition on $\mathrm{Im}\,\kappa_{v,2-r}(\gamma)$ in (12.1) implies that $\alpha=\kappa_{v,2-r}(\gamma)\in\mathbb{Z}$ and $n=0$. The remaining condition gives $k\leq\kappa_{v,2-r}(\gamma)\leq r-2-k$. This gives the third and fourth line in the table. This completes the proof of Part a).

Moreover, if $r\in\mathbb{Z}_{\geq 2}$, any invariant $t\mapsto t^\kappa$ that is in $\mathcal{D}^{\omega^*,\infty}_{v,2-r}$ is in $\mathcal{D}^{\mathrm{pol}}_{v,2-r}$. This gives Part b). □

Remark. The characterization depends on the primitive hyperbolic element γ. The element γ^{-1} is primitive hyperbolic as well, and

$$\kappa_{v,2-r}(\gamma^{-1})\;\equiv\;r-2-\kappa_{v,2-r}(\gamma)\bmod 2\pi i/\ell(\gamma)\,. \tag{12.5}$$

The transition $x\mapsto r-2-x$ maps the set $E_{v,2-r}(\gamma)$ in (12.4) into $E_{v,2-r}(\gamma^{-1})$.

Lemma 12.4. *Suppose that both fixed points ξ and ξ' of the hyperbolic element $\gamma\in\Gamma$ are in $\mathbb{R}$ and satisfy $\xi<\xi'$. If r and κ in the previous lemma are integral, and $-1\leq\kappa\leq r-1$, then the γ-invariants in $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ are spanned by the rational function*

$$t\mapsto(t-\xi')^{r-2-\kappa}(t-\xi)^\kappa\,.$$

Proof. We use

$$g\;=\;|\xi-\xi'|^{-1/2}\begin{pmatrix}\xi' & \xi\\ 1 & 1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$$

to transform the geodesic $i(0,\infty)$ into the geodesic from ξ to ξ'. We work with $t\in\mathfrak{H}^-$, and denote by $\doteq$ that we ignore non-zero factors that do not depend on t:

$$(it)^\kappa|_{2-r}g^{-1}\,(t)\;\doteq\;(-t+\xi')^{r-2}\Bigl(\frac{t-\xi}{-t+\xi'}\Bigr)^\kappa\;\doteq\;(t-\xi')^{r-2-\kappa}\,(t-\xi)^\kappa\,.\qquad\square$$

12.2. **Modules of singularities.** In the sequel we will deal with two Γ-modules $V\subset W$, where $V=\mathcal{D}^{\omega}_{v,2-r}$, and W is one of the following larger modules:

$$(a):\;\mathcal{D}^{\omega^*}_{v,2-r}\,,\quad(b):\;\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}\,,\quad(c):\;\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}\,,\quad(d):\;\mathcal{D}^{\omega^*,\infty}_{v,2-r}\,. \tag{12.6}$$

Definition 12.5. In each of the cases in (12.6) we consider the quotient module

$$\mathcal{S}\;:=\;W/V\,, \tag{12.7}$$

which we call the *module of singularities*. We write $\mathcal{S}^{\omega^*}_{v,2-r}$, $\mathcal{S}^{\omega^*,\mathrm{exc}}_{v,2-r}$, ..., if we want to indicate the case under consideration explicitly.

Definition 12.6. For $\xi \in \mathbb{P}^1_{\mathbb{R}}$ we put $\mathcal{S}_\xi := W[\xi]/V \subset \mathcal{S}$, where $W[\xi]$ consists of the elements $f \in W$ with $\mathsf{BdSing}\, f \subset \{\xi\}$.

Remarks. (a) The space $\mathcal{S}_\xi$ is a subspace of $\mathcal{S}$, not the stalk of a sheaf.

(b) The direct sum $\bigoplus_{\xi\in\mathbb{P}^1_{\mathbb{R}}} \mathcal{S}_\xi$ is a submodule of $\mathcal{S}$.

Definition 12.7. We say that the module $\mathcal{S} = W/V$ has *separation of singularities* if

$$\mathcal{S} = \bigoplus_{\xi\in\mathbb{P}^1_{\mathbb{R}}} \mathcal{S}_\xi\,. \tag{12.8}$$

Proposition 12.8. *For all cases in* (12.6) *the module* $\mathcal{S}$ *has separation of singularities.*

Proof. In [15, Proposition 13.1] this is shown for the sheaves used in that paper: $V = \mathcal{V}^\omega_s$, the sheaf of analytic functions with action of $\mathrm{PSL}_2(\mathbb{R})$ specified by the spectral parameter s and W a subsheaf of $\mathcal{V}^{\omega^*,\mathrm{exc}}_s$. It is based on the result of complex function theory that if Ω_1 and Ω_2 are open sets in $\mathbb{C}$ any holomorphic function f on $\Omega_1 \cap \Omega_2$ can be written as $f = f_1 - f_2$ with $f_1 \in \mathcal{O}(\Omega_1)$, $f_2 \in \mathcal{O}(\Omega_2)$. See, e.g., [57, Proposition 1.4.5].

This shows that if, for some open set containing $\mathfrak{H}^-$ and $\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1, \ldots, \xi_n\}$, $f \in \mathcal{O}(U)$ represents an element of $\mathcal{D}^{\omega^*}_{v,2-r}$, then we can take a neighbourhood $U_1 \supset U$ of $\mathfrak{H}^- \cup (\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_1\})$, and $U_2 \supset U$ a neighbourhood of $\mathfrak{H}^- \cup (\mathbb{P}^1_{\mathbb{R}} \smallsetminus \{\xi_2, \ldots, \xi_n\})$. Then f_2 represents an element of $\mathcal{D}^\omega_{v,2-r}[\xi_1]$ and f_1 an element of $\mathcal{D}^\omega_{v,2-r}[\xi_2, \ldots, \xi_n]$. Successively we can write each element of $\mathcal{D}^\omega_{v,2-r}[\xi_1, \ldots, \xi_n]$ non-uniquely as the sum of elements in the spaces $\mathcal{D}^\omega_{v,2-r}[\xi_j]$. This shows that $\mathcal{S}^{\omega^*}_{v,2-r}$ has separation of singularities.

For the other spaces $\mathcal{S} = W/V$ we have $W \subset \mathcal{D}^{\omega^*}_{v,2-r}$. All these subspaces are defined by conditions on the singularities of a local nature, based on the properties of a representative at each ξ in the set of singularities separately. Addition of an element for which ξ is not in the set of boundary singularities does not influence the condition at ξ. So separation of singularities is inherited from $\mathcal{S}^{\omega^*}_{v,2-r}$. □

Lemma 12.9. *Let* $r \in \mathbb{C}$.

i) *The space of invariants* $\mathcal{S}^\Gamma$ *is zero.*
ii) *Let* $\gamma \in \Gamma$ *be hyperbolic, with fixed points* ξ *and* ξ'. *By* $\ell(\gamma)$ *we denote the length of the associated geodesic. Then the dimensions of the spaces of*

invariants are as follows.

	$v(\gamma) \neq e^{-r\ell(\gamma)/2}$	$v(\gamma) = e^{-r\ell(\gamma)/2}$
$\dim\bigl((\mathcal{S}^{\omega^*}_{v,2-r})_\xi\bigr)^\gamma$	∞	∞
$\dim\bigl((\mathcal{S}^{\omega^*,\mathrm{exc}}_{v,2-r})_\xi\bigr)^\gamma$	∞	∞
$\dim\bigl((\mathcal{S}^{\omega^*,\mathrm{smp}}_{v,2-r})_\xi\bigr)^\gamma$	0	1
$\dim\bigl((\mathcal{S}^{\omega^*,\infty}_{v,2-r})_\xi\bigr)^\gamma$	0	0

Proof. For Part i) we note that if $f \in W$ represents an element of $\mathcal{S}^\Gamma$, then $\mathsf{Sing}_r f$ is a Γ-invariant subset of $\mathbb{P}^1_\mathbb{R}$. All orbits in $\mathbb{P}^1_\mathbb{R}$ of the cofinite discrete group Γ are infinite. However, elements of $\mathcal{D}^{\omega^*}_{v,2-r}$ have only finitely many singularities.

In Part ii) we denote $V = \mathcal{D}^{\omega}_{v,2-r}$, and take for W one of the modules $\mathcal{D}^{\omega^*}_{v,2-r}$, $\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$, $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$, and $\mathcal{D}^{\omega^*,\infty}_{v,2-r}$. There is an injective map $W^\gamma/V^\gamma \to \mathcal{S}^\gamma$. The image is contained in $\mathcal{S}_\xi \oplus \mathcal{S}_{\xi'}$. With separation of singularities, we can split each element of $\mathcal{S}^\gamma$ as a component in $(\mathcal{S}_\xi)^\gamma$ and a component in $(\mathcal{S}_{\xi'})^\gamma$.

We conjugate γ in $\mathrm{SL}_2(\mathbb{R})$ to $\begin{pmatrix} p^{1/2} & 0 \\ 0 & p^{-1/2}\end{pmatrix}$ with $p = e^{\ell(\gamma)}$. The invariants $t \mapsto (it)^{\kappa+\frac{2\pi i n}{\ell(\gamma)}}$ in the proof of Lemma 12.3 have a singularity at 0, except possibly in the case $n = 0$. This leads to the first two lines in the table.

Now let $W = \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ or $W = \mathcal{D}^{\omega^*,\infty}_{v,2-r}$. The component in $(\mathcal{S}_0)^\gamma$ of the image $f + V$ in $\mathcal{S}$ is invariant if and only if $0 \in \mathsf{BdSing}\, f$ and $f|_{v,2-r}(\gamma - 1) \in V$. Let $f(t) \sim \sum_{n \geq k} c_m\, t^m$ be the asymptotic expansion at 0, with $k = -1$ for $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$, and $k = 0$ for $\mathcal{D}^{\omega^*,\infty}_{v,2-r}$.

If $k = -1$ the term $c_{-1}t^{-1}$ can be non-zero if $p^{-r/2} = v(\gamma)$. Then $f(t) = t^{-1}$ leads to a non-zero element of $(\mathcal{S}_0)^\gamma$.

If $m_0 \geq 0$ a term with $p^{-r/2+1+m_0} = v(\gamma)$ leads to an invariant in W^γ which is also in V^γ, so not to a non-zero element of $(\mathcal{S}_0)^\gamma$. The remaining asymptotic series $\sum_{m\geq 0,\, m\neq m_0} c_m\, t^m$ for $\varphi \in W[\xi]$, leads to an asymptotic series

$$\sum_{m\geq 0,\, m\neq m_0} c_m \left(v(\gamma)^{-1} p^{-r/2+1+m} - 1 \right) t^m$$

for $\varphi|_{v,2-r}(\gamma - 1)$. For an invariant in $\mathcal{S}$ this last series should be convergent on a neighbourhood of 0 in $\mathbb{C}$. But since p^m is exponentially increasing, then also $\sum_{m\geq 0,\, m\neq m_0} c_m\, t^m$ is convergent on that neighbourhood, and hence is in V. □

12.3. **Mixed parabolic cohomology and parabolic cohomology.** For Γ-modules $V \subset W$ as in (12.6) there is a natural map $H^1_{\mathrm{pb}}(\Gamma; V, W) \to H^1_{\mathrm{pb}}(\Gamma; W)$. We'll show that it is injective, and investigate its surjectivity.

Lemma 12.10. *Let $V = \mathcal{D}^{\omega}_{v,2-r} \subset W$, where W is one of the modules $\mathcal{D}^{\omega^*,\mathrm{cond}}_{v,2-r}$ in* (12.6)*, or one of the corresponding modules $\mathcal{D}^{\omega^0,\mathrm{cond}}_{v,2-r}$. The following sequence* *is exact:*

$$0 \to H^1_{\mathrm{pb}}(\Gamma; V, W) \to H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S})\,. \tag{12.9}$$

Proof. The exact sequence of Γ-modules $0 \to V \to W \to \mathcal{S} \to 0$ induces a long exact sequence in mixed parabolic cohomology. This is discussed in [15] at the end of §11. We use the following part of the long exact sequence:

$$H^0(\Gamma; \mathcal{S}) \to H^1_{\mathrm{pb}}(\Gamma; V, W) \to H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S}) \tag{12.10}$$

Part i) of Lemma 12.9 leads to the desired sequence. □

The lemma shows that $H^1_{\mathrm{pb}}(\Gamma; V, W) \to H^1_{\mathrm{pb}}(\Gamma; W)$ is injective. It is surjective if the image of $H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S})$ is zero.

Definition 12.11. For each Γ-orbit $x \subset \mathbb{P}^1_{\mathbb{R}}$ put

$$\mathcal{S}\{x\} := \bigoplus_{\xi \in x} \mathcal{S}_\xi\,. \tag{12.11}$$

For each orbit $x \in \Gamma\backslash\mathbb{P}^1_{\mathbb{R}}$ the space $\mathcal{S}\{x\}$ is a Γ-module. Since $\mathcal{S}$ has separation of singularities we have

$$\mathcal{S} = \bigoplus_{x \in \Gamma\backslash\mathbb{P}^1_{\mathbb{R}}} \mathcal{S}\{x\}\,. \tag{12.12}$$

To investigate the image of $H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S})$ we can investigate separately the images of $H^1(\Gamma; W) \to H^1(\Gamma; \mathcal{S}\{x\})$. The following statement is analogous to [15, Proposition 13.4]:

Proposition 12.12. *Let x be a Γ-orbit in $\mathbb{P}^1_{\mathbb{R}}$. The natural map*

$$H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S}\{x\})$$

is the zero map in each of the following cases:

a) *the stabilizers Γ_ξ of the elements $\xi \in x$ are equal to $\{1, -1\}$,*
b) *the orbit x consists of cusps of Γ,*
c) *the stabilizers Γ_ξ of the elements $\xi \in x$ contain hyperbolic elements, with the additional condition that for all $\gamma \in \Gamma_\xi$ the space of invariants $\mathcal{S}^\gamma_\xi$ is zero.*

Remarks. (a) This result shows an important difference between hyperbolic elements of Γ and other elements. If the Condition c) is not satisfied it opens the way to construct cocycles that do not come from automorphic forms via the injection $\mathbf{r}^\omega_r$ and the natural map from mixed parabolic cohomology to parabolic cohomology.

(b) In Case c) we need to check that $\mathcal{S}^\gamma_\xi = \{0\}$ only for one generator γ of Γ_ξ.

(c) We do not repeat the proof, since it is completely analogous to the proof of [15, Proposition 13.4]. We explain the main steps.

The proof uses the description of cohomology based on a tesselation $\mathcal{T}$ of the upper half-plane, as discussed in §9. We quote two lemmas from [15] before sketching the proof of Proposition 12.12.

Lemma 12.13. *For each cocycle $c_1 \in Z^1(F_\cdot^{\mathcal{T}}; W)$ there is $c \in Z^1(F_\cdot^{\mathcal{T}}; W)$ in the same cohomology class with the properties that $c(e) = 0$ for all edges $e \in X_1^{\mathcal{T}}$ that occur in the boundary of any cuspidal triangle.*

We recall that each cusp $\mathfrak{a}$ of Γ occurs as vertex of infinitely many faces $\pi_\mathfrak{a}^{-n} V_\mathfrak{a} \in X_2^{\mathcal{T}}$, $n \in \mathbb{Z}$, $\pm\pi_\mathfrak{a}$ generators of the stabilizer of $\mathfrak{a}$. These $\pi_\mathfrak{a}^{-n} V_\mathfrak{a}$ are cuspidal triangles. The edges $\pi_\mathfrak{a}^{-n} f_\mathfrak{a}$ form a *horocycle* in $\mathfrak{H}$. If the cusp $\mathfrak{a}$ is in $\mathbb{R}$ this horocycle is a euclidean circle.

The lemma says that we can arrange that c vanishes on all edges $\pi_\mathfrak{a}^{-n} e_\mathfrak{a}$ and $\pi_\mathfrak{a}^{-n} f_\mathfrak{a}$.

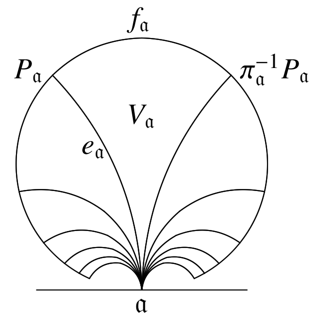

FIGURE 21

Proof. See the proof of [15, Lemma 13.2]. □

Let $x \in \Gamma\backslash\mathbb{P}^1_\mathbb{R}$. If a cohomology class in $H^1_{\mathrm{pb}}(\Gamma; W)$ is given by a cocycle in $Z^1(F_\cdot^{\mathcal{T}}; W)$ as in Lemma 12.13 its image $c \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{S}\{x\})$ vanishes on all edges in $X_1^{\mathcal{T}} \smallsetminus X_1^{\mathcal{T},Y}$, so it is in fact a cocycle on $F_1^{\mathcal{T},Y}$. Therefore c represents a class in $H^1(\Gamma; \mathcal{S}\{x\})$. Anyhow, c is a cocycle that vanishes on all edges $\gamma^{-1} f_\mathfrak{a}$ and $\gamma^{-1} e_\mathfrak{a}$ with $\gamma \in \Gamma$ and $\mathfrak{a} \in \mathfrak{F}^{\mathrm{cu}}$ (the intersection of $\mathbb{P}^1_\mathbb{R}$ with the closure of the fundamental domain $\mathfrak{F}$ underlying the tesselation $\mathcal{T}$).

For any edge $e \in X_1^{\mathcal{T}}$ we denote by $c(e)_\xi$ the component of $c(e)$ in $\mathcal{S}_\xi$ in the decomposition $\mathcal{S}\{x\} = \bigoplus_{\xi \in x} \mathcal{S}_\xi$. We put, for the fixed cocycle c

$$D(\xi) := \{e \in X_1^{\mathcal{T}} : c(e)_\xi \neq 0\}. \tag{12.13}$$

Lemma 12.14. *For each $\xi \in x$, $x \in \Gamma\backslash\mathbb{P}^1_\mathbb{R}$, the set $D(\xi)$ consists of finitely many Γ_ξ-orbits.*

Proof. See the proof of [15, Lemma 13.5]. □

Sketch of the proof of Proposition 12.12. To the cocycle $c \in Z^1(F_\cdot^{\mathcal{T}}; \mathcal{S}\{x\})$ is associated a function $X_0^{\mathcal{T},Y} \times X_0^{\mathcal{T},Y} \to \mathcal{S}\{x\}$, also denoted c. The value $c(P, Q)$ is determined by the value of c on any path in $\mathbb{Z}[X_1^{\mathcal{T}}]$ from P to Q.

Let $\mathfrak{a}$ be a cusp of Γ. If we can show that $c(\gamma^{-1} P_\mathfrak{a}, P_\mathfrak{a}) = 0$ for all $\gamma \in \Gamma$, then the group cocycle $\psi_\gamma = c(\gamma^{-1} P_\mathfrak{a}, P_\mathfrak{a})$ vanishes, and hence the cohomology class of c is trivial.

The proof in [15, §13.1] considers the three cases given in Proposition 12.12 separately. In all cases it is argued for a given $\xi \in x$, that there is a path in $\mathbb{Z}[X_1^{\mathcal{T},Y}]$ from $\gamma^{-1} P_\mathfrak{a}$ to $P_\mathfrak{a}$ that does not contain edges in $D(\xi)$. This gives $c(\gamma^{-1} P_\mathfrak{a}, P_\mathfrak{a})_\xi = 0$, and leads to $[c] = 0$ in $H^1(\Gamma; \mathcal{S}\{x\})$.

Case a) in Proposition 12.12 is easiest, since in this case $D(\xi)$ is a finite set of edges, which is easily avoided.

In Case b) the orbit x consists of cusps, and we take $\mathfrak{a} \in x$. Now the set $D(\xi)$ may be infinite. Let $\gamma \in \Gamma$ be fixed. If $\xi \notin \{\mathfrak{a}, \gamma^{-1}\mathfrak{a}\}$ it is shown that there is a path from

$\gamma^{-1}P_{\mathfrak{a}}$ to $P_{\mathfrak{a}}$ avoiding $D(\xi)$. Then the observation that $c(\gamma^{-1}P_{\mathfrak{a}}, P_{\mathfrak{a}}) \in \mathcal{S}_{\gamma^{-1}\mathfrak{a}} \oplus \mathcal{S}_{\mathfrak{a}}$ is the basis for an argument showing that $\gamma \mapsto c(\gamma^{-1}P_{\mathfrak{a}}, P_{\mathfrak{a}})$ is a coboundary.

In Case c) the set $D(\xi)$ may be a barrier that makes it impossible to find a suitable path between $\gamma^{-1}P_{\mathfrak{a}}$ and $P_{\mathfrak{a}}$ if they are on opposite sides of the barrier. If this happens the cocycle relation can be used to show that $c(\gamma^{-1}P_{\mathfrak{a}}, P_{\mathfrak{a}})_\xi$ is in $\mathcal{S}^{\gamma}_{\xi}$ for a generator γ of Γ_ξ. Under the additional condition in Part c) in Proposition 12.12, this invariant is zero. □

Theorem 12.15. *Let v be a multiplier system for the weight $r \in \mathbb{C}$ on the cofinite discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ with cusps. The natural map*

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, W) \to H^1_{\mathrm{pb}}(\Gamma; W)$$

is an isomorphism for each of the following Γ-modules W:

i) *W is one of the Γ-modules $\mathcal{D}^{\omega^0}_{v,2-r}$, $\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}$, $\mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r}$, or $\mathcal{D}^{\omega^0,\infty}_{v,2-r}$.*

ii) *$W = \mathcal{D}^{\omega^*,\infty}_{v,2-r}$.*

iii) *$W = \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$, under the additional condition that $v(\gamma) \neq e^{-r\ell(\gamma)/2}$ for all primitive hyperbolic elements $\gamma \in \Gamma$.* (By $\ell(\gamma)$ we denote the length of the associated closed geodesic in $\Gamma\backslash\mathfrak{H}$.)

Proof. We use Proposition 12.12 to show that $H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S})$ is the zero map. Then the exact sequence in (12.9) gives the desired bijectivity.

For the spaces W in Part i) we have $W/V = \bigoplus_{\mathfrak{a}\ \mathrm{cusp}} \mathcal{S}_{\mathfrak{a}}$, and need only Case b) in Proposition 12.12. For Parts ii) and iii) we have to take into account all cases in Proposition 12.12, and need the vanishing of $\mathcal{S}^{\gamma}_{\xi}$ for all hyperbolic $\gamma \in \Gamma$ that leave fixed ξ. Part ii) of Lemma 12.9 shows that this is the case for $W = \mathcal{D}^{\omega^*,\infty}_{v,2-r}$, and also for $\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ provided $e^{-r\ell(\gamma)/2} \neq v(\gamma)$. □

Missing case. Missing in Theorem 12.15 is the module $W = \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$. That case is discussed in Proposition 13.5.

12.4. **Related work.** We followed closely the approach in [15, §13.1].

13. Cocycles and singularities

There are several natural maps between cohomology groups that we did not yet handle in the previous sections. Theorem E in §1.7 states explicitly some maps that are not isomorphisms. In this section we prove those statements by constructing cocycles with the appropriate properties.

13.1. **Cohomology with singularities in hyperbolic fixed points.** In the exceptional case in Part iii) of Theorem 12.15 we want to show that the injective map

$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, W) \to H^1_{\mathrm{pb}}(\Gamma; W)$$

is not surjective for $W = \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$, or for $W = \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ under the additional condition $v(\gamma) = e^{-r\ell(\gamma)/2}$ for at least one primitive hyperbolic element of Γ.

We use the description of cohomology based on a tesselation $\mathcal{T}$ of the upper half-plane, as discussed in §9.

Notations. We work with a hyperbolic subgroup H of Γ. So H is generated by a primitive hyperbolic element δ, and -1. All elements of H leave fixed the repelling fixed point ζ_1 and the attracting fixed point ζ_2 of δ. The elements of H leave invariant the geodesic between ζ_1 and ζ_2. The image of this geodesic in $\Gamma\backslash\mathfrak{H}$ is a closed geodesic, whose length we indicate by $\ell(\delta)$.

Lemma 13.1. *Let $\mathcal{T}$ be a Γ-invariant tesselation of $\mathfrak{H}$. Let $\delta \in \mathsf{H}$ and $\zeta_1, \zeta_2 \in \mathbb{P}^1_{\mathbb{R}}$ as indicated above.*

There is a path p from ζ_1 to ζ_2 in $\mathfrak{H}$ with the following properties:

a) *p is an oriented C^1-curve in $\mathfrak{H} \cup \mathbb{P}^1_{\mathbb{R}}$, with respect to the structure of $\mathbb{P}^1_{\mathbb{C}}$ as a C^1-variety.*
b) *p has no self-intersection, and intersects $\mathbb{P}^1_{\mathbb{R}}$ only in the end-points ζ_1 and ζ_2.*
c) *p does not go through points of $X_0^{\mathcal{T},Y} = X_0^{\mathcal{T}} \cap \mathfrak{H}$.*
d) *p intersects each edge $e \in X_1^{\mathcal{T}}$ transversely, at most a finite number of times.*
e) *For each edge $e \in X_1^{\mathcal{T}}$ there are only finitely many Γ-translates $\gamma^{-1}p$ that intersect e.*
f) *$\delta^{-1}p = p$.*

Remark. All Γ-translates $\gamma^{-1}p$ form C^1-paths in $\mathfrak{H}$ from $\gamma^{-1}\zeta_1$ to $\gamma^{-1}\zeta_2$ with properties b)–e), and $(\gamma^{-1}\delta\gamma)^{-1}\,\gamma^{-1}p = \gamma^{-1}p$.

Proof. Intuitively, we may start with the geodesic from ζ_1 to ζ_2 and deform it to satisfy the conditions.

More precisely, we take a point P_0 in the interior of a face of the tesselation, and take a C^1-path p_0 from P_0 to δP_0, taking care to arrive in δP_0 with the same derivative as δp_0 departs from δP_0. If p_0 goes through a vertex or has a non-transversal intersection with an edge, or coincides with an edge, we deform it locally. In this way we arrange that p_0 intersects finitely many edges once, transversally. Near P_0 and δP_0 we have not changed p_0. Taking the union $\bigcup_{m\in\mathbb{Z}} \delta^m p_0$ and closing it in $\mathbb{P}^1_{\mathbb{C}}$, we get a C^1-path p satisfying Properties a)-d), and f).

The compact path p_0 intersects only finitely many Γ-translates of the fundamental domain $\mathfrak{F}$ on which the tesselation $\mathcal{T}$ is built. A given edge e is contained in the closure of one Γ-translate of $\mathfrak{F}$. So there are only finitely many $\gamma \in \Gamma$ such that e intersects $\gamma^{-1}p_0$. This implies that the path satisfies Property e) as well. □

Definition 13.2. Let p be a path as in Lemma 13.1, let $\gamma \in \Gamma$, and let $x \in \pm X_1^{\mathcal{T}}$ be an oriented edge.

i) For each point intersection point $P \in x \cap \gamma^{-1}p$ we define $\epsilon_P(x, \gamma^{-1}p) \in \{\pm 1\}$ depending on the orientation as indicated in Figure 22.

$\epsilon_P(x, \gamma^{-1}p) = 1$
$\gamma^{-1}p$ crosses x from right to left

$\epsilon_P(x, \gamma^{-1}p) = -1$
$\gamma^{-1}p$ crosses x from left to right

FIGURE 22. Choice of $\epsilon_P(x, \gamma^{-1}p)$.

ii) We put

$$\epsilon(x, \gamma^{-1}p) \;:=\; \sum_{P \in x \cap \gamma^{-1}p} \epsilon_P(x, \gamma^{-1}p)\,. \tag{13.1}$$

iii) We extend $x \mapsto \epsilon(x, \gamma^{-1}p)$ to a $\mathbb{C}$-linear map $\mathbb{C}[X_1^{\mathcal{T}}] \to \mathbb{C}$.

Remarks. (a) If x and $\gamma^{-1}p$ have no intersection, then the sum in (13.1) is empty, hence $\epsilon(x, \gamma^{-1}p) = 0$.

(b) Like in [15] we use the convention that $X_1^{\mathcal{T}}$ consists of oriented edges of the tesselation, and that if $e \in X_1^{\mathcal{T}}$, then the edge $-e$ with the opposite orientation is not in $X_1^{\mathcal{T}}$.

(c) Property d) in Lemma 13.1 implies that the total number of crossing of x and $\gamma^{-1}p$ is finite. So $\epsilon(x, \gamma^{-1}p)$ in Part ii) is well defined. It counts the number of crossings from right to left minus the number of crossings from left to right.

(d) The definition of ϵ is arranged in such a way that for each oriented edge x occurring in the boundary $\partial_2 V$ of a face $V \in X_2^{\mathcal{T}}$ the quantity $\epsilon(x, \gamma^{-1}p)$ counts the number of times that $\gamma^{-1}p$ enters the face V through the edge x minus the number of times it leaves V through x. This gives $\epsilon(\partial_2 V, \gamma^{-1}p) = 0$ for all faces $V \in X_2^{\mathcal{T}}$.

(e) We have $\epsilon_P(-x, \gamma^{-1}p) = -\epsilon_P(x, \gamma^{-1}p)$ and $\epsilon(-x, \gamma^{-1}p) = -\epsilon(x, \gamma^{-1}p)$. Hence the $\mathbb{C}$-linear extension in Part iii) is possible.

(f) For an oriented path $q \in \mathbb{Z}[X_1^{\mathcal{T}}]$ we can view $\epsilon(q, \gamma^{-1}p)$ as the number of times q crosses $\gamma^{-1}p$ where $\gamma^{-1}p$ goes from right to left, with respect to the orientation of q, minus the number of times q crosses $\gamma^{-1}p$ where $\gamma^{-1}p$ goes from left to right.

(g) The function ϵ is Γ-invariant:

$$\epsilon(\beta^{-1}x, \beta^{-1}\gamma^{-1}p) \;=\; \epsilon(x, \gamma^{-1}p) \qquad \text{for all } \beta \in \Gamma\,. \tag{13.2}$$

Proposition 13.3. *Let* H *be a hyperbolic subgroup of* Γ*, and let* p *be a path as in Lemma 13.1 between the fixed points* ζ_1 *and* ζ_2 *of* H*. Let* W *be a* Γ*-module.*

For each $a \in W^{\mathsf{H}}$ we put

$$c(p,a;x) := \sum_{\gamma\in\mathsf{H}\backslash\Gamma} \epsilon(x,\gamma^{-1}p)\; a|\gamma \qquad \textit{for } x \in F_1^{\mathcal{T}} = \mathbb{C}[X_1^{\mathcal{T}}]\,. \tag{13.3}$$

a) *This defines a cocycle $c(p,a;\cdot) \in Z^1(F_\cdot^{\mathcal{T}};W)$.*
b) *If p_1 and p_2 are paths as in Lemma 13.1 with the same initial point ζ_1 and the same final point ζ_2, then $c(p_1,a;\cdot)$ and $c(p_2,a;\cdot)$ are in the same cohomology class in $H^1_{\mathrm{pb}}(\Gamma;W)$.*

Remark. Without the counting function $\epsilon(\cdot,\cdot)$ the values $c(p,a;x)$ of the cocycle $c(p,a;\cdot)$ are hyperbolic Poincaré series. Hence we call the sums in (13.3) *signed hyperbolic Poincaré series*.

Proof. The terms in the sum are invariant under $\gamma \mapsto \delta\gamma$ with $\delta \in \mathsf{H}$. It is a finite sum by Property e) in Lemma 13.1. So $c(p,a;x)$ is well-defined. In Remark (c) after Definition 13.2 we have noted that $\epsilon(\partial_2 V,\gamma^{-1}p) = 0$ for each $V \in X_2^{\mathcal{T}}$. This gives the cocycle property. With (13.2) we have for $\beta \in \Gamma$

$$\begin{aligned} c(p,a;\beta^{-1}x) &= \sum_{\gamma\in\mathsf{H}\backslash\Gamma} \epsilon(\beta^{-1}x,\gamma^{-1}p)\; a|\gamma \;=\; \sum_{\gamma\in\mathsf{H}\backslash\Gamma} \epsilon(\beta^{-1}x,(\gamma\beta)^{-1}p)\; a|\gamma\beta \\ &= \sum_{\gamma\in\mathsf{H}\backslash\Gamma} \epsilon(x,\gamma^{-1}p)\; a|\gamma\beta \;=\; c(p,a;x)|\beta\,. \end{aligned}$$

This gives the $\mathbb{C}[\Gamma]$-linearity of $x \mapsto c(p,a;x)$, and ends the proof of Part i).

To prove Part b) we consider the function $c(p,a;\cdot,\cdot)$ on $X_0^{\mathcal{T}} \times X_0^{\mathcal{T}}$ given by $c(p,a;Q_1,Q_2) = c(p,a;q)$ independent of the choice of the path $q \in \mathbb{Z}[X_1^{\mathcal{T}}]$ from Q_1 to Q_2. The cohomology class of $c(p,a;\cdot)$ is determined by the group cocycle $\gamma \mapsto c(p,a;\gamma^{-1}Q_0,Q_0)$ for any base point $Q_0 \in X_0^{\mathcal{T}}$. See the final paragraphs of §9.2. We will show that for cusps $\mathfrak{a}$ and $\mathfrak{b}$ the value of $c(p,a;\mathfrak{a},\mathfrak{b})$ depends only on the position of $\mathfrak{a}$ and $\mathfrak{b}$ in relation to ζ_1 and ζ_2, and not on the actual path p from ζ_1 to ζ_2. With a cusp as the base point Q_0 this gives Part b).

The points ζ_j divide $\mathbb{P}^1_{\mathbb{R}}$ into two *cyclic intervals* $(\zeta_1,\zeta_2)_{\mathrm{cycl}}$ and $(\zeta_2,\zeta_1)_{\mathrm{cycl}}$ for the cyclic order on $\mathbb{P}^1_{\mathbb{R}}$. See Figure 23.

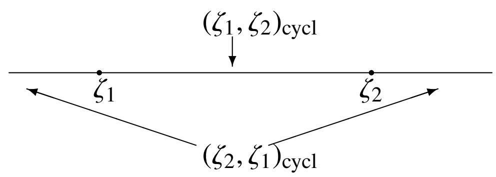

FIGURE 23

For cusps $\mathfrak{a}$ and $\mathfrak{b}$ we choose a path $q_{\mathfrak{a},\mathfrak{b}} \in \mathbb{Z}[X_1^{\mathcal{T}}]$ from $\mathfrak{a}$ to $\mathfrak{b}$. By Remark (f) after Definition 13.2, the values of $\epsilon(q_{\mathfrak{a},\mathfrak{b}},\gamma^{-1}p)$ are zero if $\mathfrak{a}$ and $\mathfrak{b}$ are not separated in $\mathbb{P}^1_{\mathbb{R}}$ by the points $\gamma^{-1}\zeta_1$ and $\gamma^{-1}\zeta_2$. Table 5 gives the values of $\epsilon(q_{\mathfrak{a},\mathfrak{b}},\gamma^{-1}p)$ for $\gamma \in \Gamma$ and the cusps $\mathfrak{a}$ and $\mathfrak{b}$ for the fixed path p. See also Figure 24. This implies

$\epsilon(q_{\mathfrak{a},\mathfrak{b}},\gamma^{-1}p)$	$\mathfrak{b}\in(\gamma^{-1}\zeta_1,\gamma^{-1}\zeta_2)_{\mathrm{cycl}}$	$\mathfrak{b}\in(\gamma^{-1}\zeta_2,\gamma^{-1}\zeta_1)_{\mathrm{cycl}}$
$\mathfrak{a}\in(\gamma^{-1}\zeta_1,\gamma^{-1}\zeta_2)_{\mathrm{cycl}}$	0	−1
$\mathfrak{a}\in(\gamma^{-1}\zeta_2,\gamma^{-1}\zeta_1)_{\mathrm{cycl}}$	1	0

TABLE 5

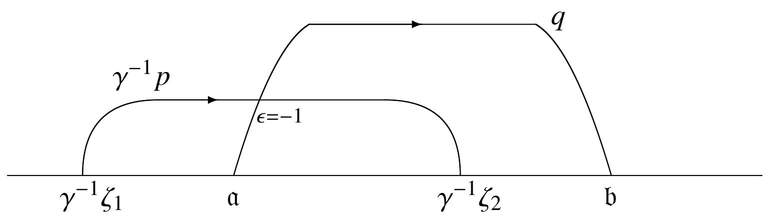

FIGURE 24. Illustration of case $\mathfrak{a}\in(\gamma^{-1}\zeta_1,\gamma^{-1}\zeta_2)_{\mathrm{cycl}}$ and $\mathfrak{b}\in(\gamma^{-1}\zeta_2,\gamma^{-1}\zeta_1)_{\mathrm{cycl}}$ in Table 5. The path $\gamma^{-1}p$ crosses q from left to right.

that $c(p,a;\mathfrak{a},\mathfrak{b})$ only depends on the position of the cusp $\mathfrak{a}$ and $\mathfrak{b}$ in relation to ζ_1 and ζ_2, not on the actual path p. □

Remarks. (a) The cocycle is Γ-equivariant in the following way:

$$c(p,a;\cdot) = c(\gamma^{-1}p, a|\gamma;\cdot) \qquad \text{for all } \gamma\in\Gamma\,. \tag{13.4}$$

(b) The cocycle $c(p,a;\cdot)$ depends linearly on $a\in W^{\mathsf{H}}$; *ie*, for all $\lambda_1,\lambda_2\in\mathbb{C}$

$$c(p,\lambda_1 a_1+\lambda_2 a_2;\cdot) = \lambda_1\, c(p,a_1;\cdot)+\lambda_2\, c(p,a_2;\cdot)\,.$$

(c) The construction is canonical for a morphism of Γ-modules $W\to W_1$: If $a\in W^{\mathsf{H}}$ is mapped to $b\in W_1^{\mathsf{H}}$, then

$$c(p,a;\cdot)\mapsto c(p,b;\cdot) \quad \text{under the natural map } Z^1(F_\cdot^{\mathcal{T}};W)\to Z^1(F_\cdot^{\mathcal{T}};W_1)\,.$$

Geodesics with elliptic fixed points. The geodesic from $\frac12-\frac12\sqrt5$ to $\frac12+\frac12\sqrt5$ induces a closed geodesic on $\Gamma(1)\backslash\mathfrak{H}$. The corresponding hyperbolic subgroup H of $\Gamma(1)$ can be generated by $D=\begin{pmatrix}2&1\\1&1\end{pmatrix}$ and $-\mathrm{I}$. This geodesic passes through the point $i\in\mathfrak{H}$, which is fixed by the elliptic element $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\in\Gamma(1)$. It induces an element $\pm S$ in $\overline{\Gamma(1)}$ of order two, so i is an elliptic point of $\Gamma(1)$ of order 2. All points $D^n\, i$, with $n\in\mathbb{Z}$, are elliptic points of $\Gamma(1)$ of order 2, fixed by $D^nS\,D^{-n}\in\Gamma(1)$.

In general, a geodesic of Γ may go through elliptic fixed points of Γ of order 2 in $\bar\Gamma$. Then there are elliptic elements of order two in $\sigma\in\Gamma$ such that $\sigma\gamma\sigma=\gamma^{-1}$ for all $\gamma\in\mathsf{H}$. The action of σ interchanges the two fixed points of H. The element σ normalizes H, but is not an element of H. Conversely, each $\sigma\in\Gamma\smallsetminus\mathsf{H}$ such that $\sigma\mathsf{H}\sigma^{-1}=\mathsf{H}$, is elliptic with a fixed point of order 2 on the geodesic.

Lemma 13.4. *Let $V = \mathcal{D}^{\omega}_{v,2-r}$, and either $W = \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$ or $W = \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$. Denote $\mathcal{S} = W/V$. For each hyperbolic subgroup $\mathsf{H} \subset \Gamma$ there is a linear map*

$$\Psi_{\mathsf{H}} : W^{\mathsf{H}} \to H^1_{\mathrm{pb}}(\Gamma; W) \tag{13.5}$$

with the following properties:

i) *The image of the composition $\tilde\Psi_{\mathsf{H}} : W^{\mathsf{H}} \xrightarrow{\Psi_{\mathsf{H}}} H^1_{\mathrm{pb}}(\Gamma; W) \to H^1(\Gamma; \mathcal{S})$ is contained in the Γ-invariant summand $H^1(\Gamma; \mathcal{S}\{\Gamma\,\zeta_1\} + \mathcal{S}\{\Gamma\,\zeta_2\})$ of $H^1(\Gamma; \mathcal{S})$, where ζ_1 and ζ_2 are the fixed points of H.*
ii) *The kernel of $\tilde\Psi_{\mathsf{H}}$ is the space*
$$V^{\mathsf{H}} + \Bigl\{ a \in W^{\mathsf{H}} \;:\; a|_{v,2-r}\sigma = a \text{ for some } \sigma \in \Gamma \smallsetminus \mathsf{H} \text{ normalizing } \mathsf{H} \Bigr\}.$$

Remarks. (a) The summands $\mathcal{S}\{\Gamma\,\zeta_1\}$ and $\mathcal{S}\{\Gamma\,\zeta_2\}$ of $\mathcal{S}$ either coincide or have intersection $\{0\}$.

(b) Any $\sigma \in \Gamma \smallsetminus \mathsf{H}$ normalizing H is elliptic of order two. (If $\gamma \in \Gamma$ normalizes H and fixes ζ_1 and ζ_2, then it is hyperbolic, and hence in H. If it interchanges the ζ_j it has order two, and hence is elliptic.)

(c) The second term in the description of $\ker \tilde\Psi_{\mathsf{H}}$ in Part ii) is zero if there are no elliptic elements normalizing H.

Proof. We use a path p from ζ_1 to ζ_2 as in Lemma 13.1. For $a \in W^{\mathsf{H}}$ we define $\Psi_H(a)$ as the cohomology class of $c(p, a; \cdot)$ in Proposition 13.3. This gives a linear map, and the construction and Lemma 12.2 show that the cocycle $c(p, a; \cdot)$ has values with singularities in the Γ-orbits of $\mathsf{BdSing}\, a \subset \{\zeta_1, \zeta_2\}$. This gives Part i).

Suppose first that there is an elliptic $\sigma \in \Gamma$ normalizing H. In the sum over $\gamma \in \mathsf{H}\backslash\Gamma$ in the definition of $c(p, a; \cdot)$ in (13.3) we combine the summands γ and $\sigma\gamma$. Since $\sigma^{-1}p$ is p with the opposite orientation, we have $\epsilon(x, (\sigma\gamma)^{-1}p) = \epsilon(\gamma^{-1}(-p)) = -\epsilon(x, \gamma^{-1}p)$. The two corresponding terms in the sum in (13.3) give

$$\epsilon(x, \gamma^{-1}p)\,(a|_{v,2-r}\gamma - a|_{v,2-r}\sigma\gamma) \;=\; \epsilon(x, \gamma^{-1}p)\; a|_{v,2-r}(1-\sigma)\gamma\,.$$

So if $a \in W^{\mathsf{H}}$ satisfies $a|_{v,2-r}\sigma = a$, then the cocycle $c(p, a; \cdot)$ is zero, so $a \in \ker \Psi_{\mathsf{H}} \subset \ker \tilde\Psi_{\mathsf{H}}$.

If $a \in V^{\mathsf{H}}$, then $c(p, a; \cdot)$ has values in V, hence the image cocycle in $\mathcal{S}$ vanishes. This establishes the inclusion $\supset$ in Part ii), for the case that an elliptic σ normalizing H exists and for the other case.

To show the other inclusion, suppose that $a \in W^{\mathsf{H}}$ is in $\ker \tilde\Psi_{\mathsf{H}}$. If there are elliptic $\sigma \in \Gamma \smallsetminus \mathsf{H}$ normalizing H, then $a|_{v,2-r}\sigma = \pm a$. If $a|_{v,2-r}\sigma = a$ then a is in the right hand side in Part ii) and and we are done. If $a|_{v,2-r}\sigma = -a$ then we will show that $a \in V^{\mathsf{H}}$.

Since a is H-invariant, $\mathsf{BdSing}\, a \subset \{\zeta_1, \zeta_2\}$ by Lemma 12.2. So the image $\tilde a$ of a in $\mathcal{S}$ is in $\mathcal{S}_{\zeta_1} \oplus \mathcal{S}_{\zeta_2}$. Since the class of $c(p, \tilde a; \cdot)$ is zero, this cocycle is a coboundary, and there exists $\tilde f \in C^0(X_0^{\mathcal{T}}; \mathcal{S}\{\zeta_1, \zeta_2\})$ such that $c(p, \tilde a; \cdot) = d\tilde f$, with the notation $\mathcal{S}\{\zeta_1, \zeta_2\} = \mathcal{S}\{\zeta_1\} + \mathcal{S}\{\zeta_2\}$.

By Γ-equivariance, $f(\mathfrak{a}) = f(\mathfrak{a})|_{v,2-r}\pi_{\mathfrak{a}}$ for all cusps $\mathfrak{a}$. So $\mathsf{BdSing}\, f(\mathfrak{a}) \subset \{\mathfrak{a}\}$, by Lemma 3.1. Since ζ_1 and ζ_2 are no cusps, we have $\tilde{f}(\mathfrak{a}) = 0$ for all cusps.

Let $q_{\mathfrak{a},\mathfrak{b}}$ be a path in $\mathbb{Z}[X_1^{\mathcal{T}}]$ from a cusp $\mathfrak{a} \in (\zeta_2, \zeta_1)_{\mathrm{cycl}}$ to a cusp $\mathfrak{b} \in (\zeta_1, \zeta_2)_{\mathrm{cycl}}$. If no elliptic $\sigma \in \Gamma$ normalizing H exists, then $\epsilon(q_{\mathfrak{a},\mathfrak{b}}, p) = 1$. The contribution to $c(p, a; q_{\mathfrak{a},\mathfrak{b}})$ with singularities in $\{\zeta_1, \zeta_2\}$ is given by a. So the component of $c(p, \tilde{a}; q)$ in $\mathcal{S}_{\zeta_1} \oplus \mathcal{S}_{\zeta_2}$ is equal to the component of $\tilde{a}$ in $\mathcal{S}_{\zeta_1} \oplus \mathcal{S}_{\zeta_2}$ in the decomposition (12.8). On the other hand $c(p, \tilde{a}; q) = \tilde{f}(\mathfrak{a}) - \tilde{f}(\mathfrak{b}) = 0$. Since $\mathsf{BdSing}\, a \subset \{\zeta_1, \zeta_2\}$, this implies that $\tilde{a} = 0$, hence $a \in V$. But $a \in W^{\mathsf{H}}$, so $a \in V^{\mathsf{H}}$.

If $\sigma \in \Gamma \smallsetminus \mathsf{H}$ normalizes H some changes in this reasoning are needed, since we have also $\epsilon(q_{\mathfrak{a},\mathfrak{b}}, \sigma^{-1}p) = -1$. Now the contribution to $c(p, a; q_{\mathfrak{a},\mathfrak{b}})$ with singularities in $\{\zeta_1, \zeta_2\}$ is $a - a|_{v,2-r}\sigma$, and the component of $c(p, \tilde{a}; q)$ in $\mathcal{S}_{\zeta_1} \oplus \mathcal{S}_{\zeta_2}$ is equal to $2\tilde{a}$. We can finish the proof by the same argument as in the other case. $\square$

Theorem 12.15 asserts that the canonical map $H^1_{\mathrm{pb}}(\Gamma; V, W) \to H^1_{\mathrm{pb}}(\Gamma; W)$ is an isomorphism if $V = \mathcal{D}^{\omega}_{v,2-r}$ and W is one of a list of larger modules, each contained in $\mathcal{D}^{\omega^*}_{v,2-r}$. Now we focus on the following two cases, for which Theorem 12.15 does not give information:

a) $V = \mathcal{D}^{\omega}_{v,2-r}$, $W = \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$.

b) $V = \mathcal{D}^{\omega}_{v,2-r}$, $W = \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}$, and there are primitive hyperbolic elements $\gamma \in \Gamma$ for which $v(\gamma) = e^{-r\ell(\gamma)/2}$.

The following result gives information concerning Case a), and partial information concerning Case b).

Proposition 13.5. *Let $r \in \mathbb{C}$.*

i) *The natural map*
$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$$

1) *is injective,*

2) *and its image in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ has infinite codimension.*

ii) *Suppose that the set*
$$P = \Big\{\gamma \in \Gamma \;:\; \gamma \text{ is primitive hyperbolic, and } v(\gamma) = e^{-r\ell(\gamma)/2}\Big\}$$
is non-empty. (Recall that $\ell(\gamma)$ is the length of the closed geodesic associated to γ.)

1) *The natural map*
$$H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}) \tag{13.6}$$
is injective.

2) *It is not surjective if $r \in \mathbb{Z}_{\geq 0}$ and for some $\gamma \in P$ one of the following two conditions is satisfied:*

a) *There are no $\sigma \in \Gamma$ such that $\sigma\gamma\sigma^{-1} = \gamma^{-1}$.*

b) *$r = 0$, $v(\gamma) = 1$, and there exist $\sigma \in \Gamma$ such that $\sigma\gamma\sigma^{-1} = \gamma^{-1}$, and $v(\sigma) = 1$.*

Remark. If in Part ii)2) none of the conditions a) and b) holds, we do not know whether the map in (13.6) is surjective.

Proof. The *injectivity* in Parts i)1) and ii)1) follows from the exact sequence (12.9).

Part i)2). Let H be a hyperbolic subgroup of Γ, with primitive hyperbolic generator γ. We consider two cases:

- Suppose that there are no elliptic $\sigma \in \Gamma$ normalizing H. Lemma 12.9 implies that $W^{\mathsf{H}}/V^{\mathsf{H}}$, with $W = \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ and $V = (\mathcal{D}^{\omega}_{v,2-r})^{\mathsf{H}}$, has infinite dimension. Lemma 13.4 and the exact sequence (12.9) imply that its image in $H^1(\Gamma; \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ has infinite dimension.
- If there exists $\sigma \in \Gamma \smallsetminus \mathsf{H}$ normalizing H, we need an subspace of
$$W^{\mathsf{H}} \;/\; (V^{\mathsf{H}} + \ker(\sigma - 1))$$
of infinite dimension. If $f \in W^{\mathsf{H}} = (\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})^{\mathsf{H}}$, then $f|_{v,2-r}\sigma \in (\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})^{\mathsf{H}}$ too (because of $\sigma\gamma\sigma^{-1} = \gamma^{-1}$). Then $a := f - f|_{v,2-r}\sigma$ satisfies $a|_{v,2-r}\sigma = -a$. By Part ii) of Lemma 12.9 there are infinitely many linearly independent such f not in $V = \mathcal{D}^{\omega}_{v,2-r}$.

Furthermore, Lemma 13.4 shows that different hyperbolic subgroups of Γ lead to cohomology classes with values in different summands of $\mathcal{S}$, which is an other source of infinite dimensionality.

Part ii)2). We show non-surjectivity of the map by producing cocycles that have non-zero image in $H^1(\Gamma; \mathcal{S})$. See Lemma 12.10. Lemma 13.4 provides us with cocycles. More precisely, consider $\gamma \in P$. To apply Lemma 13.4 we need $a \in (\mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})^{\gamma}$ that is not in $(\mathcal{D}^{\omega}_{v,2-r})^{\gamma}$. Lemma 12.3 shows that there is a one-dimensional space with such elements, occurring for $r \in \mathbb{Z}_{\geq 0}$ and $\kappa \in \{-1, r-1\}$, with κ as indicated in that lemma. That gives $e^{\ell(\gamma)(\kappa+1)} = 1$, hence $\kappa = -1$. Under Condition a) in Part ii)2) we conclude that there is a class in $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})$ with non-trivial image in $H^1(\Gamma; \mathcal{S}^{\omega^*,\mathrm{smp}}_{v,2-r})$.

Under Condition b) we have $\pm 1 = v(\gamma) = e^{-r\ell(\gamma)/2}$. Since $r \in \mathbb{Z}_{\geq 0}$ this is possible only for $r = 0$ and $v(\gamma) = 1$. Conjugation as above brings us to the situation $a(t) \doteq (it)^{-1}$. Hence $a|_{v,2-0}\sigma = -v(\sigma)\, a$. So we need the value $v(\sigma) = 1$ of the two possible values ± 1 to complete the proof with Lemma 13.4. □

13.2. **Mixed parabolic cohomology and condition at cusps.**

Proposition 13.6. *Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$. The space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ has infinite codimension in the space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r})$.*

We prepare the proof of this proposition in two lemmas, one of geometric nature, like Proposition 13.3, the other an infinite codimension result.

In the lemma with a geometric flavor, we work with a tesselation as discussed in §9.1, based on a fundamental domain $\mathfrak{F}$ of $\Gamma\backslash\mathfrak{H}$, which is split up in a compact set $\mathfrak{F}_Y$, and cuspidal triangles $V_{\mathfrak{b}}$ where $\mathfrak{b}$ runs over a set of representatives of the Γ-orbits of cusps. The edge $f_{\mathfrak{b}}$ is the intersection of the boundaries of $\mathfrak{F}_Y$ and $V_{\mathfrak{b}}$.

Lemma 13.7. *Let $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$, let $\mathfrak{a}$ be a cusp of Γ, and let $\delta \in \Gamma \smallsetminus \Gamma_{\mathfrak{a}}$. Let $a \in \mathcal{D}^{\omega}_{v,2-r}$ such that*

$$a|_{v,2-r}(1-\delta^{-1}) \in \mathcal{D}^{\omega^*}_{v,2-r} \Big|_{v,2-r} (1-\pi_{\mathfrak{a}})\,.$$

a) *There exists a $\mathbb{C}$-linear map $a \mapsto c(a;\cdot)$ from $\mathcal{D}^{\omega}_{v,2-r}$ to $Z^1(F^{\mathcal{T},Y}_{\cdot};\mathcal{D}^{\omega}_{v,2-r})$ such that $c(a;f_{\mathfrak{a}}) = a|_{v,2-r}(1-\delta^{-1})$, and if there are cusps $\mathfrak{b}$ not in the orbit $\Gamma\,\mathfrak{a}$ then $c(a;f_{\mathfrak{b}}) = 0$.*

b) *The $\mathbb{C}[\Gamma]$-equivariant linear map $c(a;\cdot) : F^{\mathcal{T},Y}_1 \to \mathcal{D}^{\omega}_{v,2-r}$ has a $\mathbb{C}[\Gamma]$-equivariant linear extension $\tilde{c}(a;\cdot) : F^{\mathcal{T}}_1 \to \mathcal{D}^{\omega^*}_{v,2-r}$ such that*

$$\tilde{c}(a;\cdot) \in Z^1(F^{\mathcal{T}}_{\cdot};\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^*}_{v,2-r})\,.$$

c) *The cohomology class $[\tilde{c}(a;\cdot)] \in H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^*}_{v,2-r})$ satisfies*

$$[\tilde{c}(a;\cdot)] \in H^1_{\mathrm{pb}}(\Gamma;\mathcal{D}^{\omega}_{v,2-r},\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}) \Longleftrightarrow a|_{v,2-r}(1-\delta^{-1}) \in \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r} \Big|_{v,2-r} (1-\pi_{\mathfrak{a}})\,.$$

Remark. In the simplest situation, we apply the lemma with a choice of $\delta \in \Gamma$ such that the fundamental domain $\delta^{-1}\mathfrak{F}$ is a neighbour of the fundamental domain $\mathfrak{F}$, and has common edges with it. Since $\delta \notin \Gamma_{\mathfrak{a}}$ the edges $e_{\mathfrak{a}}$ and $\pi_{\mathfrak{a}}^{-1}e_{\mathfrak{a}}$ in $\partial_2\mathfrak{F}$ that go to $\mathfrak{a}$ are not edges of $\delta^{-1}\mathfrak{F}$. A general choice of $\delta \in \Gamma \smallsetminus \Gamma_{\mathfrak{a}}$ leads to fundamental domains $\mathfrak{F}$ and $\delta^{-1}\mathfrak{F}$ that are far apart. We can connect them by a finite corridor of fundamental domains $\gamma_j^{-1}\mathfrak{F}$ such that $\gamma_{j-1}^{-1}\mathfrak{F}$ and $\gamma_j^{-1}\mathfrak{F}$ have a common edge.

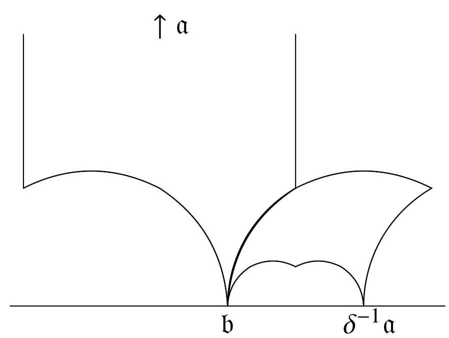

FIGURE 25

Proof. To construct a cocycle $c(a;\cdot)$ with the desired properties we adapt the geometric approach in §13.1 to the present needs.

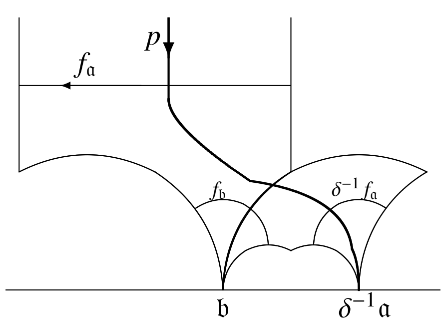

FIGURE 26

We take a C^1 path from $\mathfrak{a}$ to $\delta^{-1}\mathfrak{a}$ not going through vertices in $X^{\mathcal{T}}_0$, except the initial and final points $\mathfrak{a}$ and $\delta^{-1}\mathfrak{a}$, passing through the interior of $V_{\mathfrak{a}}$, leaving it through a point of $f_{\mathfrak{a}}$, then going on through the interior of $\mathfrak{H}_Y = \bigcup_{\gamma\in\Gamma}\gamma^{-1}\mathfrak{F}_Y$, crossing edges in $X^{\mathcal{T},Y}_1$ transversally, entering $\delta^{-1}V_{\mathfrak{a}}$ via a point of $\delta^{-1}f_{\mathfrak{a}}$ and going through the interior of $\delta^{-1}V_{\mathfrak{a}}$ to $\delta^{-1}\mathfrak{a}$.

We can choose the path p in such a way that it has many of the properties in Lemma 13.1, namely a)–d).

In b) we replace ζ_1 and ζ_2 by $\mathfrak{a}$ and $\delta^{-1}\mathfrak{a}$. In d) we have intersections only with the edges $f_\mathfrak{a}$, $\delta^{-1}f_\mathfrak{a}$, and a finite number of intermediate edges in $X_1^{\mathcal{T},Y}$. Since p runs through finitely many translates of $\mathfrak{F}$, Property e) is also satisfied. Moreover, all edges $e_\mathfrak{b}$ to cusps $\mathfrak{b}$ do not intersect p. Property f) does not apply here.

We define $\epsilon(x,\gamma^{-1}p)$ for $x \in X_1^{\mathcal{T}}$ and $\gamma \in \Gamma$ as in Definition 13.2, and next define $c_0(a;\cdot) \in C^1(F_\cdot^{\mathcal{T}}; \mathcal{D}^\omega_{v,2-r})$ by

$$c_0(a;x) := \sum_{\gamma\in\{\pm1\}\backslash\Gamma} \epsilon(x,\gamma^{-1}p)\, a|\gamma \qquad \text{for } x \in F_1^{\mathcal{T}}\,. \tag{13.7}$$

The Γ-equivariance is clear from the equivariance of ϵ, however $c_0(a;\cdot)$ need not be a cocycle. Indeed, on the one hand, $c_0(a;e_\mathfrak{a}) = c_0(a;\pi_\mathfrak{a}^{-1}e_\mathfrak{a}) = 0$, since p does not intersect any Γ-translate of $e_\mathfrak{a}$ in an interior point. On the other hand, $c_0(a;f_\mathfrak{a}) = a|(1-\delta)$ may very well be non-zero. However, we still have

$$c_0(a;\partial_2\gamma^{-1}\mathfrak{F}_Y) = 0 \qquad \text{for all } \gamma\in\Gamma\,.$$

So the restriction $c(a;\cdot)$ of $c_0a;\cdot)$ to $F_1^{\mathcal{T},Y} = \mathbb{C}[X_1^{\mathcal{T},Y}]$ is in $Z^1(F_\cdot^{\mathcal{T},Y};\mathcal{D}^\omega_{v,2-r})$.

The path p intersects $f_\mathfrak{a}$ with $\epsilon(f_\mathfrak{a},p)=1$ and $\delta^{-1}f_\mathfrak{a}$ with $\epsilon(\delta^{-1}f_\mathfrak{a},p)=-1$, and no other Γ-translates of edges $f_{\mathfrak{b}'}$ with $\mathfrak{b}'$ a cusp of Γ. So no path $\gamma^{-1}p$ with $\gamma\in\Gamma$ intersects $f_\mathfrak{b}$ with $\mathfrak{b}\neq\mathfrak{a}$ in the closure of $\mathfrak{F}$ in $\mathbb{P}^1_\mathbb{C}$. For $f_\mathfrak{a}$ we find

$$c(a;f_\mathfrak{a}) \;=\; \epsilon(f_\mathfrak{a},p)\,a + \epsilon(f_\mathfrak{a},\delta\,p)\,a|\delta^{-1} \;=\; a|_{v,2-r}(1-\delta^{-1})\,.$$

So $c(a;\cdot)$ satisfies the requirements in Part a) of the lemma.

Part b) asks for defining $\tilde c(a;e_\mathfrak{b})$ for the cusps $\mathfrak{b}$ in the closure of $\mathfrak{F}$. For $\mathfrak{b}\neq\mathfrak{a}$ this is easy: We have $c(a;f_\mathfrak{b})=0$, and define $\tilde c(a;e_\mathfrak{b})=0$ to have $\tilde c(a;\partial_2V_\mathfrak{b})=0$. The assumptions on a in the lemma show that there exists $h\in\mathcal{D}^\omega_{v,2-r}[\mathfrak{a}]$ such that $h|_{v,2-r}(1-\pi_\mathfrak{a}) = a|_{v,2-r}(1-\delta^{-1}) = c(a;f_\mathfrak{a})$. By taking $\tilde c(a;e_\mathfrak{a}) = h$ we have $\tilde c(a;\partial_2V_\mathfrak{a})=0$. By Γ-equivariance we use this to define a cocycle $\tilde c(a;\cdot)\in Z^1(F_\cdot^{\mathcal{T}};\mathcal{D}^\omega_{v,2-r},\mathcal{D}^{\omega^*}_{v,2-r})$ that coincides with c on $F_1^{\mathcal{T},Y}$.

The implication $\Leftarrow$ in Part c) is a direct consequence of the definition of c. For the implication $\Rightarrow$ we suppose that there exists $f\in C^0(F_\cdot^{\mathcal{T}};\mathcal{D}^\omega_{v,2-r},\mathcal{D}^{\omega^*}_{v,2-r})$ such that $\tilde c(a;\cdot)-(df)(\cdot)\in Z^1(F_\cdot^{\mathcal{T}};\mathcal{D}^\omega_{v,2-r},\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$. The Γ-equivariance of f implies that $f(\mathfrak{a})|_{v,2-r}\pi_\mathfrak{a} = f(\mathfrak{a})$. Denote $k=\tilde c(a;e_\mathfrak{a})-df(e_\mathfrak{a})$; so $k\in\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}$. Then

$$\begin{aligned}
k|_{v,2-r}(1-\pi_\mathfrak{a}) \;&=\; \tilde c(a;e_\mathfrak{a})|_{v,2-r}(1-\pi_\mathfrak{a}) - \big(f(P_\mathfrak{a})-f(\mathfrak{a})\big)|_{v,2-r}(1-\pi_\mathfrak{a})\\
&=\; h|_{v,2-r}(1-\pi_\mathfrak{a}) - f(P_\mathfrak{a})|_{v,2-r}(1-\pi_\mathfrak{a}) + 0\\
&\in\; c(a;f_\mathfrak{a}) + \mathcal{D}^\omega_{v,2-r}|_{v,2-r}(1-\pi_\mathfrak{a}) \qquad (\text{since } P_\mathfrak{a}\in X_0^{\mathcal{T},Y})\\
&=\; a|_{v,2-r}(1-\delta^{-1}) + \mathcal{D}^\omega_{v,2-r}|_{v,2-r}(1-\pi_\mathfrak{a})\,.
\end{aligned}$$

Hence $a|_{v,2-r}(1-\delta^{-1})\in\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}(1-\pi_\mathfrak{a})$. □

Lemma 13.8. *Let $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq2}$, $\lambda,\mu\in\mathbb{C}^*$, and $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$ with $c>0$. Then the space*

$$\Big(\mathcal{D}^\omega_{2-r}\,\Big|_{2-r}\,(1-\mu\,\gamma^{-1})\Big)\cap\Big(\mathcal{D}^{\omega^*,\mathrm{exc}}_{2-r}\,\Big|_{2-r}\,(1-\lambda^{-1}T)\Big) \tag{13.8}$$

has infinite codimension in the space

$$\left(\mathcal{D}^{\omega}_{2-r}\,\Big|_{2-r}\,(1-\mu\,\gamma^{-1})\right)\cap\left(\mathcal{D}^{\omega^*}_{2-r}\,\Big|_{2-r}\,|(1-\lambda^{-1}T)\right). \tag{13.9}$$

Proof. This may be compared with Lemma 4.13, which implies, with Lemma 3.1, that for $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq 2}$ the space

$$\mathcal{D}^{\omega}_{2-r}\cap\left(\mathcal{D}^{\omega^*}_{2-r}|_{2-r}(1-\lambda^{-1}T)\right)$$

has finite codimension in the space $\mathcal{D}^{\omega}_{2-r}$. So we have to show that imposing the condition "exc" and applying $|_{2-r}(1-\mu\gamma^{-1})$ makes an infinite-dimensional difference. We do this by giving an infinite-dimensional space

$$R\subset\left(\mathcal{D}^{\omega}_{2-r}\,\Big|_{2-r}\,(1-\mu\,\gamma^{-1})\right)\cap\left(\mathcal{D}^{\omega^*}_{2-r}\,\Big|_{2-r}\,(1-\lambda^{-1}T)\right),$$

for which we then show that it has zero intersection with

$$\mathcal{D}^{\omega^*,\mathrm{exc}}_{2-r}|_{2-r}(1-\lambda^{-1}T)\,.$$

We take $z_0\in\mathfrak{H}$, on which we will impose some restrictions later on, and put

$$\begin{aligned}R\;=\;\Big\{\varphi|_{2-r}(1-\mu\,\gamma^{-1})\in\mathcal{D}^{\omega}_{2-r}\;:\;&\varphi(t)=(i-t)^{r-2}\,p(t)\text{ where }p\text{ is a rational}\\&\text{function on }\mathbb{P}^1_{\mathbb{C}}\text{, such that }p(\infty)=p(\gamma^{-1}\infty)=0\text{, and}\\&p\text{ has a singularity at }t=z_0\text{, and nowhere else in }\mathbb{P}^1_{\mathbb{C}}\Big\}.\end{aligned}$$

Since the order of the singularity of p at $t=z_0$ is not prescribed, this space has infinite dimension. There should be a zero at at least two points in $\mathbb{P}^1_{\mathbb{C}}$, so any non-zero p has a singularity at $t=z_0$ of order at least 2. The factor $(i-t)^{r-2}$ may give φ a boundary singularity at $t=\infty$. This factor has no influence on the singularities of φ at $t=z_0$ and $t=\gamma\,z_0$.

The singularities of φ in $\mathbb{P}^1_{\mathbb{C}}$ occur at z_0, from p, and on the line $i[1,\infty]$, from the factor $(i-t)^{r-2}$. The singularities of $\varphi|_{2r}\gamma^{-1}(t)=(a-cz)^{r-2}\,\varphi(\gamma^{-1}t)$ are contained in the union of $a/c+i[0,\infty]$ and γ applied to the singularities of φ.

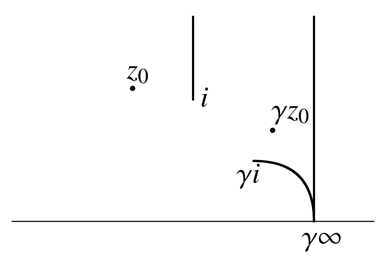

FIGURE 27

We choose z_0 such that the set $z_0+\mathbb{Z}$ does not contain points of $i[1,\infty]\cup\gamma(i[1,\infty)]\cup(\gamma\infty+i[0,\infty])\cup\{\gamma\,z_0\}$.

Let $f=\varphi|_{2-r}(1-\mu\,\gamma^{-1})\in R$. We have $\mathsf{prj}_{2-r}\varphi(t)=p(t)$, hence $(\mathsf{prj}_{2-r}\varphi)(\infty)=0$; and also $(\mathsf{prj}_{2-r}(\varphi|_{2-r}\gamma^{-1}))(\infty)=(p|^{\mathrm{prj}}_{2-r}\gamma^{-1})(\infty)=0$. (See (1.20).) Using a one-sided average (Proposition 4.6) we find $h\in\mathcal{D}^{\omega}_{2-r}[\infty]$ such that

$$h(t)-\lambda^{-1}h(t+1)\;=\;f(t)\;=\;\varphi(t)-\mu(\varphi|_{2-r}\gamma^{-1})(t)\,, \tag{13.10}$$

at least for $t\in\mathfrak{H}^-$. So $f\in\mathcal{D}^{\omega}_{2-r}[\infty]|_{2-r}(1-\lambda^{-1}T)$, and obviously also in $\mathcal{D}^{\omega}_{2-r}|_{2-r}$ $(1-\mu^{-1}\gamma)$. We have to show that if $p\neq 0$, then none of the solutions of (13.10) can be in $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$.

If a solution h of (13.10) were in $\mathcal{D}^{\omega,\mathrm{exc}}_{2-r}[\infty]$, then it extends holomorphically to an $\{\infty\}$-excised neighbourhood. So h can have singularities only inside a strip $|\mathrm{Re}\, z| \le N$ for some $N > 0$. In particular h can have singularities at $z_0 + n$ only for a finite number of $n \in \mathbb{Z}$.

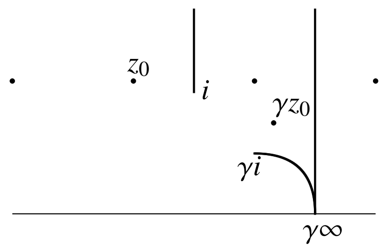

FIGURE 28

The right hand side in Relation (13.10) has singularities at $z_0 + n$ only if $n = 0$. So the maximal $n \ge 0$ such that $z_0 + n$ is a singularity of h cannot be larger than 0, since otherwise there would be a singularity $z_0 + n + 1$ as well. Similarly, the minimum value of $n \le 0$ such that h is singular at $z_0 + n$ is also 0. However, a singularity of h only at z_0 is also impossible, since f is holomorphic at $z_0 \pm 1$.

So h cannot have a singularity at any point of $z_0 + \mathbb{Z}$. The choice of z_0 shows that then φ has no singularity at z_0, in contradiction with $p \neq 0$. □

Proof of Proposition 13.6. We have to show that

$$\dim\Bigl(H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r}) \,/\, H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})\Bigr) \;=\; \infty\,.$$

We choose a cusp $\mathfrak{a}$ of Γ and $\delta \in \Gamma \smallsetminus \Gamma_{\mathfrak{a}}$, and apply Parts b) and c) of Lemma 13.7. The map $a \mapsto [\tilde{c}(a;\cdot)]$ induces a linear map

$$\begin{aligned}\Bigl(\mathcal{D}^{\omega}_{v,2-r}\bigr|_{v,2-r}(1-\delta^{-1})\Bigr) \cap \Bigl(\mathcal{D}^{\omega^*}_{v,2-r}\bigr|_{v,2-r}(1-\pi_{\mathfrak{a}})\Bigr)&\\ \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r}) \,/\, H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})\,,&\end{aligned}$$

with kernel

$$\Bigl(\mathcal{D}^{\omega}_{v,2-r}\bigr|_{v,2-r}(1-\delta^{-1})\Bigr) \cap \Bigl(\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}\bigr|_{v,2-r}(1-\pi_{\mathfrak{a}})\Bigr).$$

So it suffices to show that this kernel has infinite codimension in

$$\Bigl(\mathcal{D}^{\omega}_{v,2-r}\bigr|_{v,2-r}(1-\delta^{-1})\Bigr) \cap \Bigl(\mathcal{D}^{\omega^*}_{v,2-r}\bigr|_{v,2-r}(1-\pi_{\mathfrak{a}})\Bigr).$$

Conjugating $\mathfrak{a}$ to ∞ and δ to γ, we arrive at a statement handled in Lemma 13.8, with λ and μ determined by $v(\pi_{\mathfrak{a}})$ and $v(\delta)$. □

13.3. **Recapitulation of the proof of Theorem E.** *Part i)* concerns the case $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\ge 2}$. We have to show

a) $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) = H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}) \cong H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$.

b) $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r})$ has infinite codimension in $H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$.

c) $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \to H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})$ is injective with an image of infinite codimension.

In the following diagram we indicate where we have carried out the various steps. (To save space we suppress Γ in the notation.) For Parts i)a) and i)b) we have:

$$
\begin{tikzcd}
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \arrow[r, equal, "\text{Prop. 3.2}"] \arrow[d, "\cong"', "\text{Thm. 12.15}"] & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}) \arrow[d, hook, "\text{inf. codim.}"', "\text{Prop. 13.6}"] \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*}_{v,2-r}) \arrow[d, hook, "\text{fin. codim.}"', "\text{Prop. 4.12}"] \\
 & H^1(\mathcal{D}^{\omega}_{v,2-r})
\end{tikzcd}
\tag{13.11}
$$

Part i)c) follows from the following commuting diagram:

$$
\begin{tikzcd}
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \arrow[r, equal, "\text{Prop. 3.2}"] \arrow[d, "\cong"', "\text{Thm. 12.15}"] & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r}) \arrow[d, hook, "\text{inf. cod.}"', "\text{Prop. 13.5}"] \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \arrow[r] & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^*,\mathrm{exc}}_{v,2-r})
\end{tikzcd}
\tag{13.12}
$$

Part ii) of the theorem states the following identities and isomorphisms:

$$
\begin{aligned}
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}) \;&=\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty}_{v,2-r}) \;=\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\infty}_{v,2-r})\\
&\cong\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\infty}_{v,2-r}) \;\cong\; H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\infty}_{v,2-r})\,.
\end{aligned}
$$

It follows from the diagram

$$
\begin{tikzcd}
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty,\mathrm{exc}}_{v,2-r}) \arrow[d, equal, "\text{Prop. 4.11}"] & \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\infty}_{v,2-r}) \arrow[r, equal, "\text{Prop. 3.2}"] \arrow[d, "\cong"', "\text{Thm. 12.15}"] & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\infty}_{v,2-r}) \arrow[d, "\cong"', "\text{Thm. 12.15}"] \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^0,\infty}_{v,2-r}) & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^*,\infty}_{v,2-r})
\end{tikzcd}
\tag{13.13}
$$

Only for the first equality we need $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$. For all other steps $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 2}$ suffices.

Part iii) states for $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 1}$:

a) The image $\mathbf{r}^{\omega}_r M_r(\Gamma, v) = H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp},\mathrm{exc}}_{v,2-r})$ is equal to

$$
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r})\,, \qquad H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})
$$

and canonically isomorphic to $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r})$.

b) The space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp,exc}}_{v,2-r})$ is canonically isomorphic to the space $H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r})$ if $v(\gamma) \neq e^{-r\ell(\gamma)/2}$ for all primitive hyperbolic elements $\gamma \in \Gamma$, where $\ell(\gamma)$ is the hyperbolic length of the closed geodesic associated to γ

Part iii)a) follows from the diagram

$$\begin{tikzcd}
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp,exc}}_{v,2-r}) \arrow[d, equal, "\text{Prop. 4.11, ii)}"'] & \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r}) \arrow[r, equal, "\text{Prop. 3.2}"] \arrow[d, "\cong"', "\text{Thm. 12.15}"] & H^1_{\mathrm{pb}}(\mathcal{D}^{\omega}_{v,2-r}, \mathcal{D}^{\omega^*,\mathrm{smp}}_{v,2-r}) \\
H^1_{\mathrm{pb}}(\mathcal{D}^{\omega^0,\mathrm{smp}}_{v,2-r}) &
\end{tikzcd} \tag{13.14}$$

The condition that r is real is needed only for the first step. Part iii)b) follows also from Theorem 12.15 under a condition on hyperbolic elements.

13.4. **Related work.** The constructions in this section arose from a generalization of the examples in Propositions 13.7 and 14.3 in [15]. The paths p in Lemma 13.1 and in the proof of Lemma 13.7 represent cycles in homology. It is conceivable that they can be related to the computations of Ash [1], who computes the parabolic cohomology with values in the rational functions by computing first homology groups. We have not succeeded in making this relation explicit.

14. Quantum automorphic forms

Theorem E implies that $\mathbf{r}^{\omega}_r : A_r(\Gamma, v) \to H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$ is far from surjective. Quantum automorphic forms may be put, for weights $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, on the place of the question mark in the diagram

$$\begin{tikzcd}
A_r(\Gamma, v) \arrow[r, dotted, no head] \arrow[d, "\mathbf{r}^{\omega}_r"', "\cong"] & ? \arrow[d, dotted, no head] \\
H^1_{\mathrm{pb}}(\Gamma; \mathcal{D}^{\omega}_{v,2-r}; \mathcal{D}^{\omega^0,\mathrm{exc}}_{v,2-r}) \arrow[r, hook] & H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})
\end{tikzcd} \tag{14.1}$$

This is similar to the role of quantum Maass forms in [15, §14.4].

14.1. **Quantum modular forms.** Zagier [124] gives examples of *quantum modular forms* as functions on $\mathbb{Q}$ that have a modular transformation behavior modulo a smooth function on $\mathbb{R}$.

Example: Powers of the Dedekind eta-function. We attach a quantum modular form to η^{2r} with $\mathrm{Re}\, r > 0$.

The cusps of $\Gamma(1)$ form one orbit, $\mathbb{P}^1_{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. For each cusp $\mathfrak{a} \in \mathbb{Q}$ the function

$$h_{\mathfrak{a}}(t) := \int_{z_0}^{\mathfrak{a}} \omega_r(\eta^{2r}; t, z) = \int_{z_0}^{\mathfrak{a}} \eta^{2r}(z)\,(z-t)^{r-2}\,dz \tag{14.2}$$

is well defined for $t \in \mathfrak{H}^- \cup \mathbb{R}$.

Let $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $\mathfrak{a}, \delta^{-1}\mathfrak{a} \in \mathbb{Q}$. Then

$$\begin{aligned} v_r(\delta)^{-1}\,(ct+d)^{r-2}\,h_{\delta\mathfrak{a}}(\delta t) - h_{\mathfrak{a}}(t) &= \left(\int_{\delta^{-1}z_0}^{\mathfrak{a}} - \int_{z_0}^{\mathfrak{a}}\right)\omega_r(\eta^{2r}; t, z) \\ &= \psi^{z_0}_{\eta^{2r},\delta}(t)\,, \end{aligned} \tag{14.3}$$

by Lemma 2.3. All terms in this relation are in the space $\mathcal{D}^{\infty}_{2-r}$ of smooth vectors, hence

$$p(\mathfrak{a}) := h_{\mathfrak{a}}(\mathfrak{a}) \qquad (\mathfrak{a} \in \mathbb{Q}) \tag{14.4}$$

is well defined, and satisfies

$$p|_{v_r,2-r}(\delta - 1)\,(\mathfrak{a}) = \psi^{z_0}_{\eta^{2r},\delta}(\mathfrak{a}) \qquad (\mathfrak{a}, \delta\mathfrak{a} \in \mathbb{Q})\,. \tag{14.5}$$

The function $p : \mathbb{Q} \to \mathbb{C}$ has no reason to have a continuous extension to $\mathbb{R}$. However, $p|_{v_r,2-r}(\delta - 1)$ is the restriction of a real-analytic function on $\mathbb{R}$. The function p is an example of a quantum modular form.

Strong quantum modular forms. Since $h_{\mathfrak{a}}$ as indicated above is an element of $\mathcal{D}^{\infty}_{2-r}$, we have an asymptotic series $h_{\mathfrak{a}}(t) \sim P(\mathfrak{a}, t) := \sum_{n\geq 0} c_n(\mathfrak{a})\,(t-\mathfrak{a})^n$, approximating $h_{\mathfrak{a}}(t)$ as $t \to \mathfrak{a}$ through $\mathfrak{H}^- \cup \mathbb{R}$. For $\delta \in \Gamma$ as above we have from (14.3):

$$v_r(\delta)^{-1}\,(ct+d)^{r-2}\,P(\delta\mathfrak{a}, \delta t) - P(\mathfrak{a}, t) \sim \psi^{z_0}_{\eta^{2r},\delta}(t) \tag{14.6}$$

as $t \to \mathfrak{a}$ through $\mathfrak{H}^- \cup \mathbb{R}$. This means that P is a *strong quantum modular form* in the sense of Zagier [124].

Constant function. Now we take $r = 0$, hence $\eta^0 = 1 \in M_0(\Gamma(1), 1)$. It seems sensible to take now

$$h_{\mathfrak{a}}(t) = \frac{1}{t-\mathfrak{a}}\,. \tag{14.7}$$

Now we cannot substitute $t = \mathfrak{a}$. However, with δ as above

$$v_0(\delta)^{-1}\,(ct+d)^{0-2}\,h_{\delta\mathfrak{a}}(\delta t) - h_{\mathfrak{a}}(t) = \frac{-c}{ct+d} = \tilde{\psi}_\delta(t)\,, \tag{14.8}$$

with the cocycle $\tilde{\psi} \in Z^1(\Gamma(1); \mathcal{D}^{\omega^*,\mathrm{exc}}_{1,2})$ in (2.23). So $P(\mathfrak{a}, t) = \frac{1}{t-\mathfrak{a}}$ can be viewed as a strong quantum automorphic form if we allow asymptotic series of the form $\sum_{n\geq -1} c_n(\mathfrak{a})\,(t-\mathfrak{a})^n$.

14.2. **Quantum automorphic forms.** For general cofinite discrete groups Γ we define quantum automorphic forms as simply as possible for our purpose. The example of the constant function shows that we need to use series starting at order -1.

It turns out that we get satisfactory results in the context of these notes if we use expansion starting at order -1, namely, $P(\mathfrak{a},t) := c_{-1}\,(t-\mathfrak{a})^{-1} + c_0 \cdots$, and leaving implicit the terms $c_1, c_2, \ldots$.

Definition 14.1. By C we denote the set of cusps of Γ. By a *system of expansions* p on C we mean a map assigning to all except finitely many points $\mathfrak{a} \in C \cap \mathbb{R}$ an expression

$$p(\mathfrak{a},t) \;=\; c_{-1}(\mathfrak{a})\,(t-\mathfrak{a})^{-1} + c_0(\mathfrak{a}) + (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!]\,,$$

where $\mathbb{C}[\![t-\mathfrak{a}]\!]$ is the ring of formal power series in $t-\mathfrak{a}$. Two such systems p and p_1 are equivalent if $p(\mathfrak{a},t) \equiv p_1(\mathfrak{a},t) \bmod (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!]$ for all but finitely many $\mathfrak{a} \in C \cap \mathbb{R}$. By $\mathcal{R}$ we denote the linear space of equivalence classes of systems of expansions.

If $t \mapsto \varphi(t)$ is real-analytic on a neighbourhood of $\mathfrak{a}$ in $\mathbb{R}$, then multiplication by $\varphi(t)$ is well defined for elements of $(t-\mathfrak{a})^{-1}\mathbb{C}[\![t-\mathfrak{a}]\!] \bmod (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!]$.

Definition 14.2. The action $|_{v,2-r}$ of Γ on $\mathcal{R}$ is induced by

$$(p|_{v,2-r}\gamma)(\mathfrak{a},t) \;:=\; v(\gamma)^{-1}\,(ct+d)^{r-2}\,p(\gamma\mathfrak{a},\gamma t) \tag{14.9}$$

for all $\mathfrak{a} \in C \cap \mathbb{R}$ and $\gamma = \left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right) \in \Gamma$ for which $p(\mathfrak{a},\cdot)$ and $p(\gamma\mathfrak{a},\cdot)$ are defined. If $r \notin \mathbb{Z}$ we define $(ct+d)^{r-2}$ by the argument convention (1.2) for $t \in \mathfrak{H}^-$.

Remarks. (a) The operations in both parts of the definition preserve the equivalence between systems of expansions. We will mostly identify an equivalence class with a representative of it.

(b) The inclusion $\mathcal{D}^{\omega}_{v,2-r} \to \mathcal{R}$ given by $\varphi \mapsto p_\varphi$, where

$$p_\varphi(\mathfrak{a},t) \;=\; \varphi(\mathfrak{a}) + (t-\mathfrak{a})\mathbb{C}[[t-\mathfrak{a}]] \qquad \text{for all } \mathfrak{a} \in C \cap \mathbb{R}\,, \tag{14.10}$$

is equivariant for the actions $|_{v,2-r}$ of Γ on $\mathcal{D}^{\omega}_{v,2-r}$ and $\mathcal{R}$.

Definition 14.3. Let $r \in \mathbb{C}$ and let v be a multiplier system for the weight r.

a) By $\mathcal{R}_{v,2-r}$ we denote $\mathcal{R}$ provided with the action $|_{v,2-r}$ of Γ.
b) We define the Γ-module $\mathcal{Q}_{v,2-r} := \mathcal{R}_{v,2-r} \,/\, \mathcal{D}^{\omega}_{v,2-r}$.
c) We define the space ${}^{q}\!A_{2-r}(\Gamma,v)$ of *quantum automorphic forms* of weight $2-r$ with multiplier system v as a quotient of Γ-invariants:

$${}^{q}\!A_{2-r}(\Gamma,v) \;:=\; \mathcal{Q}^{\Gamma}_{v,2-r} \,/\, \mathcal{R}^{\Gamma}_{v,2-r}\,. \tag{14.11}$$

Remarks. (a) So we have an exact sequence of Γ-modules

$$0 \to \mathcal{D}^{\omega}_{v,2-r} \to \mathcal{R}_{v,2-r} \to \mathcal{Q}_{v,2-r} \to 0\,,$$

with the associated long exact sequence

$$0 \to (\mathcal{D}^{\omega}_{v,2-r})^{\Gamma} \to \mathcal{R}^{\Gamma}_{v,2-r} \to \mathcal{Q}^{\Gamma}_{v,2-r} \to H^1(\Gamma;\mathcal{D}^{\omega}_{v,2-r}) \to \cdots$$

We choose to define quantum automorphic forms as the quotient $\mathcal{Q}^\Gamma_{v,2-r}/\mathcal{R}^\Gamma_{v,2-r}$, which can automatically mapped into $H^1(\Gamma,\mathcal{D}^\omega_{v,2-r})$ injectively.

In this way a quantum automorphic form is a function defined on almost all of $C\cap\mathbb{R}$ that has automorphic transformation behavior modulo functions that are analytic on $\mathbb{R}$ minus finitely many points. Further we work modulo functions on $C\cap\mathbb{R}$ that are exactly automorphic.

(b) We leave it to the reader to explore the examples of Zagier [124]. The purpose of our definition is not to cover all those examples. We are content to define quantum automorphic forms in such a way that they fill the hole in diagram 14.1.

14.3. **Quantum automorphic forms, cohomology, and automorphic forms.**

Proposition 14.4. *Let v be a multiplier system on Γ for the weight $r\in\mathbb{C}$.*

a) *There is an injective natural map*

$$ {}^q\mathcal{C} : {}^qA_{2-r}(\Gamma,v) \to H^1(\Gamma;\mathcal{D}^\omega_{v,2-r})\,. \tag{14.12}$$

b) *If $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq 1}$, then ${}^q\mathcal{C}$ is surjective.*

Proof. Injectivity, Part a). Definition 14.3 implies that the sequence

$$0\to\mathcal{D}^\omega_{v,2-r}\to\mathcal{R}_{v,2-r}\to\mathcal{Q}_{v,2-r}\to 0 \tag{14.13}$$

is exact. The part

$$\mathcal{R}^\Gamma_{v,2-r}\to\mathcal{Q}^\Gamma_{v,2-r}\to H^1(\Gamma;\mathcal{D}^\omega_{v,2-r})$$

of the corresponding long exact sequence in group cohomology shows that the connecting homomorphism induces an injective linear map

$${}^qA_{2-r}(\Gamma,v)\to H^1(\Gamma;\mathcal{D}^\omega_{v,2-r})\,,$$

which we call ${}^q\mathcal{C}$. It sends a quantum automorphic form represented by $p\in\mathcal{R}_{v,2-r}$ to the class of the cocycle $\gamma\mapsto p|_{v,2-r}(\gamma-1)$.

Surjectivity, Part b). Let $r\in\mathbb{C}\smallsetminus\mathbb{Z}_{\geq 1}$, and let $\lambda\in\mathbb{C}^*$. Proposition 4.6 shows that for each $f\in\mathcal{D}^\omega_{2-r}$ at least one of the one-sided averages $\mathrm{Av}^+_{T,\lambda}f$ and $\mathrm{Av}^-_{T,\lambda}f$ exists in $\mathcal{D}^{\omega^*}_{2-r}$, and that $(\mathrm{Av}^\pm_{T,\lambda}f)|_{2-r}(1-\lambda^{-1}T)=f$. We use $\mathrm{Av}^+_{T,\lambda}f$ if $|\lambda|\geq 1$ and $\mathrm{Av}^-_{T,\lambda}f$ if $|\lambda|<1$. Furthermore, by Proposition 4.9, there is an asymptotic formula $(\mathrm{Av}^\pm_{T,\lambda}f)(t)=(it)^{r-2}\,(c_{-1}t+c_0+\mathrm{O}(t^{-1}))$ as $t\to\pm\infty$ through $\mathbb{R}$, with coefficients c_{-1} and c_0 determined by f. By conjugation, we define for parabolic $\pi=\sigma T\sigma^{-1}$, with $\sigma\in\mathrm{SL}_2(\mathbb{R})$, $\xi=\sigma\infty\neq\infty$ and $f\in\mathcal{D}^\omega_{2-r}$:

$$\mathrm{Av}^\pm_{\pi,\lambda}f \;:=\; (\mathrm{Av}^\pm_{T,\lambda}(f|_{2-r}\sigma))|_{2-r}\sigma^{-1}\,. \tag{14.14}$$

It satisfies

$$(\mathrm{Av}^\pm_{\pi,\lambda}f)|_{2-r}(1-\lambda^{-1}\pi) \;=\; f\,. \tag{14.15}$$

With the transformation $|_{2-r}\sigma^{-1}$ the asymptotic behavior of $\mathrm{Av}^\pm_{T,\lambda}(f|_{2-r}\sigma)$ at ∞ leads to an asymptotic formula of the form

$$(\mathrm{Av}^\pm_{\pi,\lambda}f)(t) \;=\; c_{-1}\,(t-\xi)^{-1}+c_0+\mathrm{O}(t-\xi)\,, \tag{14.16}$$

where $t \uparrow \xi$ for $\mathrm{Av}^+_{\pi,\lambda}$ and $t \downarrow \xi$ for $\mathrm{Av}^-_{\pi,\lambda}$. The constants c_{-1} and c_0 differ from the constants at ∞. The definition of $\mathrm{Av}^\pm_{\pi,\,d} f$ depends on the choice of σ such that $\xi = \sigma\infty$. For cusps $\mathfrak{a} \in \mathcal{C} \cap \mathbb{R}$ we use $\sigma_\mathfrak{a}$ as in §1.3.

After this preparation we consider a cohomology class in $H^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$, represented by the cocycle $\psi \in Z^1(\Gamma; \mathcal{D}^\omega_{v,2-r})$. For $\mathfrak{a} \in \mathcal{C} \cap \mathbb{R}$ we put

$$p(\mathfrak{a}, t) \;:=\; -(\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}})(t) + (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!]\,, \tag{14.17}$$

where we choose $\pm$ so that the average exists.

Let $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $\mathfrak{a}, \delta\mathfrak{a} \in \mathbb{R}$. Then

$$(\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}})|_{v,2-r}(\pi_\mathfrak{a} - 1) \;=\; -\psi_{\pi_\mathfrak{a}}.$$

Since $\pi_{\delta\mathfrak{a}} \;=\; \delta\pi_\mathfrak{a}\delta^{-1}$, we have $\psi_{\pi_{\delta\mathfrak{a}}} \;=\; \psi_{\pi_\mathfrak{a}}|_{v,2-r}\delta^{-1} + \psi_\delta|_{v,2-r}(\pi_\mathfrak{a}-1)\delta^{-1}$. Therefore,

$$\begin{aligned}
&v(\delta)^{-1}\,(ct+d)^{r-2}\,(\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})(\delta t) \;=\; (\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta\,(t)\\
&(\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta(\pi_\mathfrak{a}-1) \;=\; (\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}(\pi_{\delta\mathfrak{a}}-1)\delta\\
&\quad=\; -\psi_{\delta\mathfrak{a}}|_{v,2-r}\delta \;=\; -\psi_{\pi_\mathfrak{a}} - \psi_\delta|_{v,2-r}(\pi_\mathfrak{a}-1).
\end{aligned}$$

Therefore,

$$\Big((\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta - (\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}})\Big)|_{v,2-r}(\pi_\mathfrak{a}-1) \;=\; -\psi_\delta|_{v,2-r}(\pi_\mathfrak{a}-1).$$

From (14.16) we know that $(\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}})\,(t)$ has a one-sided expansion at $\mathfrak{a}$, and $(\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})\,(t)$ at $\delta^{-1}\mathfrak{a}$. The transformation formula for $|_{v,2-r}\delta$ shows that then $((\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta)\,(t)$ has a similar expansion at $\mathfrak{a}$. The function ψ_δ is holomorphic at $\mathfrak{a}$.

So the function $f := (\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta - (\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}}) + \psi_\delta$ has a one-sided asymptotic expansion at $\mathfrak{a}$ starting at a multiple of $(t-\mathfrak{a})^{-1}$ and is invariant under $|_{v,2-r}\pi_\mathfrak{a}$. Conjugating this to ∞ and applying Part ii) of Lemma 3.4 we conclude that $f(t) \sim 0$ as t approaches $\mathfrak{a}$ from one direction. (We use that $r \notin \mathbb{Z}_{\geq 1}$.)

So we have the following equality of asymptotic expansions

$$(\mathrm{Av}^\pm_{\pi_{\delta\mathfrak{a}}, v(\pi_{\delta\mathfrak{a}})} \psi_{\pi_{\delta\mathfrak{a}}})|_{v,2-r}\delta\;(t) - (\mathrm{Av}^\pm_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}})(t) \sim -\psi_\delta(t)$$

as t approaches $\mathfrak{a}$ through $\mathbb{R}$ from the left or the right depending on $\pm$. We conclude that

$$(p|_{v,2-r}\delta)(\mathfrak{a}, t) - p(\mathfrak{a}, t) \;=\; \psi_\delta(t) + (t-\mathfrak{a})\,\mathbb{C}[[t-\mathfrak{a}]]\,. \tag{14.18}$$

So $p|_{v,2-r}(\delta - 1) = \psi_\delta$ in $\mathcal{R}$, and $\mathcal{C}(p) = [\psi]$. $\square$

Remark. For weight $r = 1$, Part i) of Proposition 4.6 implies (after conjugating ∞ to $\mathfrak{a} \in \mathcal{C} \cap \mathbb{R}$) that $\mathrm{Av}_{\pi_\mathfrak{a}, v(\pi_\mathfrak{a})} \psi_{\pi_\mathfrak{a}}$ is defined if $\psi_{\pi_\mathfrak{a}}(\mathfrak{a}) = 0$. Since there are finitely many Γ-orbits of cusps, the construction in proof of surjectivity of $\mathcal{C}$ goes through for a subspace of $H^1(\Gamma; \mathcal{D}^\omega_{v,1})$ of finite codimension.

Proposition 14.5. *Let v be a multiplier system on Γ for the weight $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$. There is an injective linear map $Q : A_r(\Gamma, v) \to {}^{q}\!A_{2-r}(\Gamma, v)$ such that the following diagram commutes:*

$$\begin{tikzcd} A_r(\Gamma, v) \arrow[rr, hook, "\mathbf{r}_r^{\omega}"] \arrow[dr, "Q"'] & & H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r}) \\ & {}^{q}\!A_{2-r}(\Gamma, v) \arrow[ur, "\cong", "{}^{q}\mathcal{C}"'] & \end{tikzcd}$$

Proof. Theorem A implies that $\mathbf{r}_r^{\omega}$ is injective. Since ${}^{q}\mathcal{C}$ is bijective by Proposition 14.4 the map ${}^{q}\mathcal{C}$ is invertible, so $Q = {}^{q}\mathcal{C}^{-1} \circ \mathbf{r}_r^{\omega}$. □

Remark. This result shows that for $r \in \mathbb{C} \smallsetminus \in \mathbb{Z}_{\geq 1}$ each class in $H^1(\Gamma; \mathcal{D}^{\omega}_{v,2-r})$ is the image of an object with automorphic flavor.

Back to the examples. We now discuss the examples of quantum modular forms in 14.1 after Definition 14.3.

For the *powers of the Dedekind eta-function* with $\operatorname{Re} r > 0$ we gave in (14.2)

$$p(\mathfrak{a}) = h_{\mathfrak{a}}(\mathfrak{a}) \qquad \text{with } h_{\mathfrak{a}}(t) = \int_{z_0}^{\mathfrak{a}} \eta^{2r}(z)\,(z-t)^{r-2}\,dz\,. \tag{14.19}$$

On the other hand, if $r \in \mathbb{C} \smallsetminus \mathbb{Z}_{\geq 1}$, then $Q(\eta^{2r}) = {}^{q}\mathcal{C}^{-1}(\mathbf{r}_r^{\omega}(\eta^{2r}))$ in Proposition 14.5 can be given by

$$q(\mathfrak{a}, t) \;=\; -\big(\mathrm{Av}^{\pm}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r}, \pi_{\mathfrak{a}}}\big)(t) + (t-\mathfrak{a})\,\mathbb{C}[[t-\mathfrak{a}]]\,, \tag{14.20}$$

according to the construction in (14.17), where $\pm$ has to be chosen so that the one-sided average exists.

First take $r \in (0,\infty) \smallsetminus \mathbb{Z}_{\geq 1}$. Then $h_{\mathfrak{a}} \in \mathcal{D}^{\omega,\infty,\mathrm{exc}}_{v_r,2-r}$ satisfies $h_{\mathfrak{a}}|_{v,2-r}(\pi_{\mathfrak{a}} - 1) = \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}}$ by Lemma 2.5. By Part ii) of Lemma 4.10 (also conjugated to $\mathfrak{a}$) we have $h_{\mathfrak{a}} = \mathrm{Av}^{\pm}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}}$ for both choices of $\pm$. So $p(\mathfrak{a}) \equiv q(\mathfrak{a}, t) \bmod (t-\mathfrak{a})\,\mathbb{C}[[t-\mathfrak{a}]]$ in this case.

For $\operatorname{Re} r > 0$, $r \in \mathbb{C} \smallsetminus \mathbb{R}$, we consider only the case that $\operatorname{Im} r > 0$; the other case goes similarly. We note that $v_r(\pi_{\mathfrak{a}}) = v_r(T) = e^{\pi i r/6}$ for all cusps $\mathfrak{a}$. We use $\mathrm{Av}^{-}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})}$, and the asymptotic of $(\mathrm{Av}^{-}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}})(t)$ as $t \downarrow \mathfrak{a}$. Since $h_{\mathfrak{a}}$ and $-\mathrm{Av}^{-}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}}$ satisfy the same equation, we have $h_{\mathfrak{a}} = -\mathrm{Av}^{-}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}} + P$ with a$v_r(\pi_{\mathfrak{a}})$-periodic function P. The asymptotic behavior as $t \downarrow \mathfrak{a}$ shows that $P(t)$ has to be $\mathrm{O}(t-\mathfrak{a})$ as $t \downarrow \mathfrak{a}$. So $h_{\mathfrak{a}}$ and $-\mathrm{Av}^{-}_{\pi_{\mathfrak{a}}, v_r(\pi_{\mathfrak{a}})} \psi^{z_0}_{\eta^{2r},\pi_{\mathfrak{a}}}$ determine the same element of $\mathcal{R}_{v_r,2-r}$.

For the *constant function* $\eta^0 = 1$ we used $h_{\mathfrak{a}}(t) = (t-\mathfrak{a})^{-1}$. It leads in (14.8) to a cocycle with values in $\mathcal{D}^{\omega^0,\mathrm{exc}}_{1,2}$, not in $\mathcal{D}^{\omega}_{1,2}$. So it does not represent $Q(1) \in {}^{q}\!A_2(\Gamma(1), 1)$.

For an explicit computation, we write $\mathfrak{a} \in \mathbb{Q}$ in the form $\sigma_{\mathfrak{a}}\infty$, with $\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, and hence $\mathfrak{a} = \frac{a}{c}$. Then

$$\pi_{\mathfrak{a}}^n = \begin{pmatrix} 1 - nac & na^2 \\ -nc^2 & 1 + nac \end{pmatrix}.$$

So we have, with $\pi = \pi_{\mathfrak{a}} = \sigma_{\mathfrak{a}} T \sigma_{\mathfrak{a}}^{-1}$:

$$\begin{aligned} \mathrm{Av}^+_{\pi,1} \psi^{z_0}_{1,\pi}(t) &= \sum_{n\geq 0} \psi^{z_0}_{1,\pi} \Big|_{1,2} \pi^n(t) = \sum_{n\geq 0} \int_{\pi^{-1}z_0}^{z_0} \omega_0(1;\cdot,z)|_{1,2}\pi^n(t) \\ &= \sum_{n\geq 0} \int_{\pi^{-n-1}z_0}^{\pi^{-n}z_0} \frac{dz}{(z-t)^2}. \end{aligned}$$

We have $\lim_{n\to\infty} \pi_{\mathfrak{a}}^{-n} z_0 = a/c = \mathfrak{a}$. Hence

$$-\mathrm{Av}^+_{\pi,1}\psi^{z_0}_{1,\pi}(t) = -\int_{\mathfrak{a}}^{z_0} \frac{dz}{(z-t)^2} = \frac{1}{t-\mathfrak{a}} + \frac{1}{z_0 - t}.$$

This modification of the function $h_{\mathfrak{a}}$ in (14.7) leads to the element of $\mathcal{R}$ given by

$$p(\mathfrak{a},t) = \frac{1}{t-\mathfrak{a}} + (z_0 - \mathfrak{a})^{-1} + (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!],$$

which satisfies $p|_{1,2}(\delta-1)(\mathfrak{a},t) \equiv \psi^{z_0}_{1,\delta}(t) \bmod (t-\mathfrak{a})\,\mathbb{C}[\![t-\mathfrak{a}]\!]$.

Dependence on the parameters. The family of modular forms $r \mapsto \eta^{2r}$ depends holomorphically on r. This suggest to look for quantum modular forms given by

$$p(\mathfrak{a},t) = \frac{\alpha(\mathfrak{a},r)}{t-\mathfrak{a}} + \beta(\mathfrak{a},r) + (t-\mathfrak{a})\,\mathbb{C}[[t-\mathfrak{a}]],$$

where $r \mapsto \alpha(\mathfrak{a},r)$ and $r \mapsto \beta(\mathfrak{a},r)$ are at least continuous on $[0,\infty)$. This is impossible (proof left to the reader). It is a phenomenon similar to the asymptotic expansion in (4.14), where the coefficient in the leading term is discontinuous in λ.

14.4. **Related work.** The concept of quantum automorphic is due to Zagier. His paper [124] gives beautiful explicit examples of quantum modular forms. Zagier mentioned the concept long before the appearance of [124]. The paper [11] was written during the preparation of [15], to fill a hole in a diagram analogous to (14.1).

15. Remarks on the literature

Like we mentioned in §2.5, an indication of what we now call the Eichler integral is present in a paper of Poincaré in 1905, [100]. Eichler's definition in [43] is based on Bol's equality $\partial_\tau^{r-1}(F|_{2-r}\gamma) = F^{(r-1)}|_r\gamma$, which appears in [6, §8]. In [32] Cohn indicated this approach for weight 4. The paper [112] of Shimura has a different atmosphere; it stresses cohomology with values in a $\mathbb{Z}$-module. In the following years Gunning, Knopp, Lehner and others studied the relation between automorphic forms and cohomology: [53, 54, 44, 33, 80, 56, 81, 107, 108, 52]. Kra [75, 76] started the study of cohomology of kleinian groups. Here the cohomology group is not generated by Eichler integrals. We have not included in the list of references all papers on the cohomology of kleinian groups.

Manin [86] discussed arithmetical questions. For a cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$ of even weight the ratio between the even periods are in the field generated by the Fourier coefficients of the cusp form; for the ratios of the odd periods the same holds. The cocycles are present in the background, for instance in the period relations. So apart from the Fourier coefficients there are two, possibly transcendental, numbers involved in the coefficients of the period polynomials. The arithmetic of the period polynomials, associated with values of L-functions at integral points in the critical strip, are an important area of study in connection with the cocycles attached to automorphic forms. It goes further than the central idea in these notes, which is establishing the relation between automorphic forms and cohomology. Therefore we have not tried to include all papers in this area in the list of references. We mention the concept "modular symbol"; see [86, 113]. We mention also Haberland's paper [55], and [62, 48, 49, 118]. In [122] Zagier describes rather explicitly how to reconstruct a cuspidal Hecke eigenform from its period polynomial.

The step from weights in $\mathbb{Z}_{\geq 2}$ to general real weights was done by Knopp in his paper [66]. For general real weights one needs a multiplier system, which Knopp assumes to be unitary. This definition leads to a map from cusps forms to cohomology classes with values in the highest weight module $\mathcal{D}^{-\infty}_{v,2-r}$, characterized by the condition of polynomial growth. Knopp's cocycle integral also occurs in the paper [94] of Niebur. In the proofs in [66] Knopp uses the construction of "supplementary series", from [65]. It is nice to see that with hindsight we can view the resulting functions as mock automorphic forms. See for instance Pribitkin [101], [102].

The isomorphism between the space of cusp forms and the cohomology group was completed for all weights in 2010 by Knopp and Mawi, [71]. In [21] the isomorphism of Knopp is combined with multiplication by non-zero cusp forms.

Knopp, [67], started the study of rational period functions and gave examples. He showed in [68] that the singularities can occur only in the rational points 0, ∞, and in points in real-quadratic fields (which are hyperbolic fixed points of $\Gamma(1)$), and Choie[22] showed the existence of rational period functions with singularities in any real quadratic irrationals. Several authors expanded the theory, [89, 73, 1, 22, 23, 24, 25, 26, 56, 95, 27, 110, 48, 41]. We expect that the approach in Sections 12 and 13 can be applied to cohomology with values in the module of rational functions.

In [69] Knopp and Mason start the study of "generalized modular forms", which are vector-valued automorphic forms with at most exponential growth at the cusps for the modular group $\mathrm{SL}_2(\mathbb{Z})$ with real weight and matrix-valued multiplier systems that need not be unitary. The papers [70, 72, 105, 106] deal with the cohomology classes associated to these automorphic forms.

The Γ-behavior of automorphic forms can be formulated as the vanishing of $F|_{v,r}(\gamma - 1)$ for all $\gamma \in \Gamma$. This has been generalized to the condition that

$$F|_{v,r}(\gamma_1 - 1)\,(\gamma_2 - 1)\cdots(\gamma_q - 1) \;=\; 0 \qquad \text{for all } \gamma_1, \ldots, \gamma_q \in \Gamma\,,$$

leading to "higher order automorphic forms", for which Deitmar, [36, 38], has studied cohomological questions. See also Diamantis and O'Sullivan [40], Sim [114]. Cohomological techniques have also been used in the context of higher-order forms by Taylor [116]. See further [14].

In [2] Bringmann, Guerzhoy, Kane and Ono consider period polynomials for r-harmonic modular forms with negative even weights. Bringmann, Diamantis and Raum [4] extended the construction to account for non-critical values of L-functions.

The condition of holomorphy can be completely removed from the definition of automorphic forms, and replaced by a second order differential equation. Formulated in terms of functions on the universal covering group $\tilde{G}$ this is the eigenvalue equation for the Casimir operator. This leads to the so-called "Maass forms" and their generalizations. For Maass forms of weight 0 the relation between automorphic forms and cohomology has been studied by many authors. Lewis, [82], gave a bijection between even Maass cusp forms and spaces of holomorphic functions on $\mathbb{C} \smallsetminus (-\infty, 0]$ that satisfy a functional equation similar to the equation satisfied by period function for the modular group $\mathrm{PSL}_2(\mathbb{Z})$. In the papers [83] and especially [84] this is further discussed for the modular group. Mühlenbruch, [91] extended this to real weights. See also [92]. Martin [87] uses similar methods in the context of holomorphic modular forms of weight 1. A relation between the period functions of Lewis and the hyperfunctions associated to Maass forms was explored in [10], the ideas in which were expanded in [34, 35, 37].

Another, rather unexpected, relation is with eigenfunctions of the transfer operator introduced by Mayer, [88], in connection with the Selberg zeta-function. Transfer operators are a concept from mathematical physics, applied by Mayer to the geodesic flow on the quotient $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}$. The eigenfunctions of the transfer operator with eigenvalue 1 are, after a simple transformation, identical with Lewis's period functions. So the eigenfunctions of the transfer operator are related to cohomology classes. See [83], [84, Chap. IV, §3], and [121] for a further discussion. In [12] this relation with cohomology is used to relate eigenfunctions of two transfer operators. See also [90, 97, 98, 99]. As far as we see, the use of a transfer operator is less suitable in the present context, since the space $\mathcal{D}^{\omega}_{2-r}$ is not the space of global sections of a sheaf on $\mathbb{P}^1_{\mathbb{R}}$.

The aim of the paper [15] is to explore the relation between Maass forms of weight zero and cohomology more completely, for all cofinite discrete group. For cocompact discrete groups rather complete results were available, even in the context of automorphic forms on more general symmetric spaces, in the work of Bunke and Olbrich, [19, 20]. For groups with cusps a reasonably complete description was obtained with use of three ideas: (1) use of mixed parabolic cohomology groups;

(2) work with boundary germs as coefficient module; (3) description of the mixed parabolic cohomology groups with resolutions based on a suitable tesselation of the upper half-plane. In the present notes we tried to apply these ideas in the context of holomorphic automorphic forms.

Appendix A. Universal covering group and representations

The discussion in this appendix is not really essential for these notes, but several definitions and arguments become more natural if we relate them to the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

A.1. **Universal covering group.** The *universal covering group* $\tilde G$ of $\mathrm{SL}_2(\mathbb{R})$ is a simply connected Lie group that is locally isomorphic to the Lie group $\mathrm{SL}_2(\mathbb{R})$.

We can describe $\tilde G$ with help of the *Iwasawa decomposition* of $\mathrm{SL}_2(\mathbb{R})$, which writes each $g \in \mathrm{SL}_2(\mathbb{R})$ uniquely as

$$g = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix},$$

with $z = x + iy \in \mathfrak{H}$ and $\vartheta \in \mathbb{R} \bmod 2\pi\mathbb{Z}$. As an analytic variety, $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathfrak{H} \times (\mathbb{R}/2\pi\mathbb{Z})$. A simply connected analytic variety that covers $\mathfrak{H} \times (\mathbb{R}/2\pi\mathbb{Z})$ is $\mathfrak{H}\times\mathbb{R}$, with the natural map $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. We denote its points as (z,ϑ), with $z \in \mathfrak{H}$, $\vartheta \in \mathbb{R}$. It is possible to define a group structure on $\mathfrak{H}\times\mathbb{R}$ such that the projection map $\mathfrak{H}\times\mathbb{R} \to \mathfrak{H}\times(\mathbb{R}/2\pi\mathbb{Z})$ is an real-analytic group homomorphism. The resulting group with underlying space $\mathfrak{H}\times\mathbb{R}$ is the universal covering group $\tilde G$ of $\mathrm{SL}_2(\mathbb{R})$, with projection homomorphism

$$\mathrm{pr} : \tilde G \to \mathrm{SL}_2(\mathbb{R})\,. \tag{A.1}$$

Here we do not describe the group structure of $\tilde G$ explicitly. (See, eg., [9, §2.2.1].) We mention that there is a group homomorphism $\tilde k : \mathbb{R} \to \tilde G$, given by $\tilde k(\vartheta) = (i,\vartheta)$. It covers the isomorphism $\mathbb{R}/2\pi\mathbb{Z} \to \mathrm{SO}(2)$ given by $\vartheta \mapsto \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right)$. We note that $\{\tilde k(2\pi n) : n \in \mathbb{Z}\}$ is the kernel of $\mathrm{pr} : \tilde G \to \mathrm{SL}_2(\mathbb{R})$, and that $\tilde Z := \{\tilde k(\pi n) : n \in \mathbb{Z}\}$ is the center of $\tilde G$.

The most important aspect of the group structure is the lift $g \mapsto \tilde g$ from $\mathrm{SL}_2(\mathbb{R})$ to $\tilde G$, given by

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} := \Bigl(\frac{ai+b}{ci+d}, -\arg(ci+d)\Bigr) \qquad \text{with } -\pi < \arg(cz+d) \le \pi\,. \tag{A.2}$$

It takes a preimage for the covering map pr. It satisfies

$$\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z,\vartheta) = \Bigl(\frac{az+b}{cz+d}, \vartheta - \arg(cz+d)\Bigr). \tag{A.3}$$

This map is continuous on the open dense subset $G_0 \subset \mathrm{SL}_2(\mathbb{R})$, in (1.3). We have

$$(\tilde g)^{-1} \;=\; \widetilde{g^{-1}} \qquad \text{for } g \in G_0\,,$$

$$\text{(A.4)} \qquad \widetilde{gpg^{-1}} \;=\; \tilde g \tilde p (\tilde g)^{-1} \quad \text{for } g \in G_0,\ p = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix},\ x+iy \in \mathfrak{H}\,.$$

All elements of $\tilde G$ can be, non-uniquely, written as a product $\tilde g\, k(\pi n)$, with $g \in G_0$, $n \in \mathbb{Z}$.

A.1.1. *Weight functions and actions by right and left translation.*

Definition A.1. A function $f : \tilde G \to \mathbb{C}$ has *weight* $r \in \mathbb{C}$ if $f(z,\vartheta) = f(z,0)\, e^{ir\vartheta}$.

A function f on $\tilde G$ with weight r is determined by its values on $(z, 0)$, with $z \in \mathfrak{H}$. We define a corresponding function $R_r f$ on $\mathfrak{H}$ by

$$\text{(A.5)} \qquad (R_r f)(z) \;:=\; y^{-r/2} f(z,0)\,, \qquad \text{hence } f(z,\vartheta) \;=\; y^{r/2}\,(R_r f)(z)\, e^{ir\vartheta}\,.$$

Left translation. The group $\tilde G$ has a right action in the space of functions $\tilde G \to \mathbb{C}$ given by *left translation*

$$\text{(A.6)} \qquad \text{for } g \in \tilde G : f \mapsto f|g\,,\ \text{given by } (f|g)(g_1) \;=\; f(gg_1)\,.$$

We also use the notation $L_g f = f|g$.

The action by left translation preserves the weight. Moreover, we have

$$\text{(A.7)} \qquad \big(R_r(f|\tilde g)\big)(z) \;=\; (cz+d)^{-r}\,(R_r f)(z) \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\,.$$

Thus, we see that the operators $|_r g$ in (1.1) correspond naturally to the representation of $\tilde G$ by left translation in the functions on $\tilde G$ of weight r. The argument convention for $\arg(cz+d)$ for $z \in \mathfrak{H}$ in (1.2) is coupled to the choice of the argument in (A.2). Then the convention for $z \in \mathfrak{H}^-$ is determined by the wish to have relation (1.7). Since $g \mapsto \tilde g$ is not a group homomorphism, the operators $|_r g$ do not form a representation of $\mathrm{SL}_2(\mathbb{R})$.

Right translation. There is also the left action of $\tilde G$ on the functions on $\tilde G$ by *right translation*:

$$\text{(A.8)} \qquad (R_g f)(g_1) \;:=\; f(g_1 g)\,.$$

It commutes with left translations. It does not preserve the weight.

A.1.2. *Discrete subgroup.* For a cofinite discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ we define

$$\text{(A.9)} \qquad \tilde\Gamma \;:=\; \{g \in \tilde G \;:\; \mathrm{pr}\, g \in \Gamma\}\,.$$

It is a discrete subgroup of $\tilde G$. It contains the center $\tilde Z$.

Any character $\chi : \tilde\Gamma \to \mathbb{C}^*$ of $\tilde\Gamma$ induces a *central character* of $\tilde Z$ which is determined by $\chi(\tilde k(\pi))$, which we can write as $\chi(\tilde k(\pi)) = e^{\pi i r}$ with $r \in \mathbb{C} \bmod 2\pi\mathbb{Z}$. The map $v_\chi : \Gamma \to \mathbb{C}^*$ given by

$$\text{(A.10)} \qquad v_\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \chi\Bigl(\widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\Bigr) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is a multiplier system on Γ for the weight r. One can check that all multiplier systems on Γ arise in this way.

The representation $|_{v_\chi,r}$ of $\bar\Gamma$ on the functions on $\mathfrak{H}$, in (1.10), corresponds to the representation $\chi^{-1} \otimes L$ of $\tilde\Gamma$ on the functions of weight r on $\tilde G$. For these functions the generator $\tilde k(\pi)$ of $\tilde Z$ acts as multiplication by $\chi(\tilde k(\pi))^{-1} e^{\pi i r} = 1$. So indeed, $\chi^{-1} \otimes L$ is a representation of $\tilde\Gamma/\tilde Z \cong \bar\Gamma$.

The invariants of the representation $\chi^{-1} \otimes L$ in the functions of weight r correspond to the space of all functions on $\mathfrak{H}$ with (Γ, v)-automorphic transformation behavior of weight r. For automorphic forms one requires also that the functions are eigenfunctions of a differential operator. These differential operators can be described with the Lie algebra. (See §A.1.3.)

Modular group. The modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ is covered by $\widetilde{\Gamma(1)} \subset \tilde G$. The generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ can be lifted to give generators $t = (i+1, 0)$ and $s = (i, -\pi/2)$ of $\widetilde{\Gamma(1)}$, with relations generated by $ts^2 = s^2 t$ and $tstst = s$.

All characters of $\widetilde{\Gamma(1)}$ are of the form $\chi_r : \widetilde{\Gamma(1)} \to \mathbb{C}^*$ with $r \in \mathbb{C}/12\pi\mathbb{Z}$ given by

$$\text{(A.11)} \qquad \chi_r(t) = e^{\pi i/6}, \qquad \chi_r(s) = e^{-\pi i r/2},$$

corresponding to the multiplier system v_r in (2.12).

A.1.3. *Lie algebra.* The real Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ is

$$\text{(A.12)} \qquad \mathfrak{g}_r := \Bigl\{ g \in M_2(\mathbb{R}) : \operatorname{Trace} g = 0 \Bigr\}.$$

A basis is $\mathbf{W} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{V} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For each $\mathbf{X} \in \mathfrak{g}_r$ the exponential $\exp \mathbf{X} = \sum_{n \ge 0} \frac{1}{n!} \mathbf{X}^n$ is an element of $\mathrm{SL}_2(\mathbb{R})$. For small values of $t \in \mathbb{R}$ we have $\exp(t\mathbf{X}) \in G_0$; the lift $t \mapsto (\exp t\mathbf{X})^\sim$ extends to a group homomorphism $\mathbb{R} \to \tilde G$. This leads to differential operators on $\tilde G$:

$$\text{(A.13)} \quad (L_{\mathbf{X}} f)(g) := \frac{d}{dt} f\bigl((\exp t\mathbf{X})^\sim g\bigr)\Big|_{t=0}, \quad (R_{\mathbf{X}} f)(g) := \frac{d}{dt} f\bigl(g (\exp t\mathbf{X})^\sim\bigr)\Big|_{t=0}.$$

This can be extended to a linear map $\mathbf{X} \mapsto L_{\mathbf{X}}$ from the complexified Lie algebra $\mathfrak{g} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_r$ to the first order right-invariant differential operators on $\tilde G$. Similarly we have a linear map $\mathbf{X} \mapsto R_{\mathbf{X}}$ from $\mathfrak{g}$ to the first order left-invariant differential operators on $\tilde G$. So the operators $R_{\mathbf{X}}$ leave invariant the space of invariants for the representation $\chi^{-1} \otimes L$ in $C^\infty(\tilde G)$, and the operators $L_{\mathbf{X}}$ leave invariant the space of differentiable functions with a given weight.

The relation with the Lie product $[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$ is

$$\text{(A.14)} \qquad R_{\mathbf{X}} R_{\mathbf{Y}} - R_{\mathbf{Y}} R_{\mathbf{X}} = R_{[\mathbf{X},\mathbf{Y}]}, \quad L_{\mathbf{X}} L_{\mathbf{Y}} - L_{\mathbf{Y}} L_{\mathbf{X}} = -L_{[\mathbf{X},\mathbf{Y}]}.$$

We also write $\mathbf{X}f$ instead of $R_{\mathbf{X}}f$. For the basis $\mathbf{W}$, $\mathbf{E}^+ = \mathbf{H} + i\mathbf{V}$, $\mathbf{E}^- = \mathbf{H} - i\mathbf{V}$ of $\mathfrak{g}$ we have in the coordinates $(z, \vartheta) \in \tilde{G}$:

$$\begin{aligned} \mathbf{W} &= \partial_\vartheta\,, \\ \mathbf{E}^+ &= e^{2i\vartheta}(2iy\,\partial_x + 2y\,\partial_y - i\partial_\vartheta)\,, \quad \mathbf{E}^- = e^{-2i\vartheta}(-2iy\,\partial_x + 2y\,\partial_y + i\partial_\vartheta)\,. \end{aligned} \tag{A.15}$$

The Lie algebra $\mathfrak{g}$ can be embedded in the *universal enveloping algebra* $\mathcal{U}$, generated by all products of elements of $\mathfrak{g}$, with the relations $\mathbf{XY} - \mathbf{YX} = [\mathbf{X}, \mathbf{Y}]$ for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{g}$. The maps $\mathbf{X} \mapsto R_{\mathbf{X}}$ and $\mathbf{X} \mapsto L_{\mathbf{X}}$ can be extended to $\mathcal{U}$, and describe the ring of all left-invariant, respectively right-invariant, differential operators on $\tilde{G}$. The center of $\mathcal{U}$ is a polynomial algebra in one variable, for which we can take

$$\omega := -\frac{1}{4}\mathbf{E}^-\mathbf{E}^+ + \frac{1}{4}\mathbf{W}^2 + \frac{i}{2}\mathbf{W} = -\frac{1}{4}\mathbf{E}^+\mathbf{E}^- + \frac{1}{4}\mathbf{W}^2 - \frac{i}{2}\mathbf{W}\,. \tag{A.16}$$

It gives rise to the following bi-invariant differential operator on $\tilde{G}$:

$$L_\omega = R_\omega = e^{-2i\vartheta}(-2iy\,\partial_x + 2y\,\partial_y + i\partial_\vartheta)\,, \tag{A.17}$$

called the *Casimir operator*.

A.1.4. *Automorphic forms on $\tilde{G}$.* One may define an automorphic form on $\tilde{G}$ with character χ as a function $f : \tilde{G} \to \mathbb{C}$ with transformation behavior $f(\gamma g) = \chi(\gamma)\,f(g)$ for all $g \in \tilde{G}$, $\gamma \in \tilde{\Gamma}$, that is an eigenfunction of R_ω and $R_{\mathbf{W}}$. With this definition, an automorphic form has a weight $r \in \mathbb{C}$, determined by $R_{\mathbf{W}}f = irf$, and an eigenvalue $\lambda \in \mathbb{C}$, determined by $R_\omega f = \lambda f$.

There are several interesting sets of values for (λ, r). If one wants to do spectral theory, it is convenient to take $r \in \mathbb{R}$. Then square integrability of the automorphic forms restricts λ to a subset of $\mathbb{R}$ containing the interval $(1/4, \infty)$.

The automorphic forms considered in [15] correspond to $r = 0$ and $\lambda = s(1 - s)$ with $0 < \operatorname{Re} s < 1$.

The differential operator Δ_r in (1.27) corresponds under R_r in (A.6) to $R_\omega - \frac{r}{2}(1 - \frac{r}{2})$. If f has weight r, then $\mathbf{E}^- f$ has weight $r - 2$. With (A.5) we have

$$R_{r-2}(\mathbf{E}^- f) = -4iy^2\,\partial_{\bar z} R_r f\,\Big(= \overline{2y^{r-2}\,\xi_r F}\Big). \tag{A.18}$$

So the condition of holomorphy corresponds to being in the kernel of $\mathbf{E}^-$. Then (A.17) implies that $\lambda = \frac{r}{2}(1 - \frac{r}{2})$. For the same eigenvalue there are more eigenfunctions of the Casimir operator than there are in the kernel of $\mathbf{E}^-$. They correspond to the larger space of r-harmonic automorphic forms.

A.1.5. *Polar functions.* The polar r-harmonic functions $\mathrm{P}_{r,\mu}$, $\mathrm{M}_{r,\mu}$, and $\mathrm{H}_{r,\mu}$ in §7.1 are specializations of functions in [9, §4.2]. Fourier terms $F(\mu, \cdot)$ transforming according to $F(\mu, \cdot)|_r \left(\begin{smallmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{smallmatrix}\right) = e^{i(r+2\mu)\vartheta}\,F(\mu, \cdot)$ for small values of ϑ are of the form $R_r f(\mu, \cdot)$, as in (A.5), where $f(\mu, \cdot) : \tilde{G} \to \mathbb{C}$ satisfies

$$f(\mu, \tilde{k}(\eta) g \tilde{k}(\psi)) = e^{ir(\eta+\psi)+2i\mu\eta}\,f(\mu, g)\,, \quad R_\omega f(\mu, g) = \frac{r}{2}\Big(1 - \frac{r}{2}\Big) f(\mu, g)\,. \tag{A.19}$$

Such a function can be written as

$$\tilde{k}(\eta)\,(it,0)\,\tilde{k}(\psi) \mapsto e^{2i\mu\eta+ir(\eta+\psi)}\,\Big(\frac{u}{u+1}\Big)^{\mu/2}\,(u+1)^{-r/2}\,h_\mu\Big(\frac{1}{u+1}\Big), \tag{A.20}$$

with $t \geq 1$, $u = \big((t^{1/2}+t^{-1/2})/2\big)^2$, where h_μ satisfies the differential equation in [9, §4.2.6]. In Table 6 we summarize the relation between the variables in [9] and

[9]		here	[9]		here
n	$=$	$r+2\mu$	l	$=$	r
u	$=$	$\frac{(t+1)^2}{4t}$	s	$=$	$\frac{r-1}{2}$
p	$=$	$\frac12\lvert\mu\rvert$	ε	$=$	$\operatorname{Sign}\mu$
u	$=$	$\frac{\lvert z-i\rvert^2}{4y}$	$e^{2i\eta}$	$=$	$\frac{z-i}{z+i}\,\frac{\lvert z+i\rvert}{\lvert z-i\rvert}$
$e^{2i(\eta+\psi)}$	$=$	$\frac{2i}{z+i}\,\frac{\lvert z+i\rvert}{2}$			

TABLE 6. Relations for the computation in §A.1.5

here. The solutions in [9, 4.2.6 and 4.2.9] give:

$$\begin{aligned}
\mu(n,s;(it,0)) &= \Big(\frac{u}{u+1}\Big)^{\mu/2}\,(u+1)^{-r/2},\\
\mu(n,-s;(it,0)) &= \Big(\frac{u}{u+1}\Big)^{\mu/2}\,(u+1)^{r/2-1}\,{}_2F_1\Big(1+\mu,1-r;2-r;\frac{1}{u+1}\Big),\\
&\quad\text{if } \mu\leq 0\ :\\
\omega(n,s;(it,0)) &= \Big(\frac{u}{u+1}\Big)^{-\mu/2}\,(u+1)^{-r/2}\,{}_2F_1\Big(|\mu|,r;1+|\mu|;\frac{u}{u+1}\Big).
\end{aligned} \tag{A.21}$$

We write $(z,0) = \tilde{k}(\eta)\,(it,0)\,\tilde{k}(\psi)$, and have to multiply with $y^{-r/2}\,e^{ir(\eta+\psi)+2i\eta\mu}$ to get the corresponding function $F(\mu,\cdot)$. Table 6 shows also that the functions in (A.21) correspond to $\mathrm{P}_{r,\mu}$, $\mathrm{M}_{r,\mu}$, and $\mathrm{H}_{r,\mu}$, respectively. This requires some computations and, for $\mathrm{M}_{r,\mu}$ with $\mu\leq 0$, use of a Kummer relation (Relation (2), [45, §2.9]).

A.1.6. *Resolvent kernel.* Let m_r denote the function on $\tilde{G}$ such that $R_r m_r = \mathrm{M}_{r,0}$. So $m_r(\tilde{k}(\eta)g\tilde{k}(\psi)) = e^{ir(\eta+\psi)}\,m_r(g)$. The kernel function Q_r in (8.11) corresponds to the function $q_r(g_1,g_2) := m_r(g_1^{-1}g_2)$, which satisfies $q_r(g_1\tilde{k}(\vartheta_1), g_2\tilde{k}(\vartheta_2)) = e^{ir(\vartheta_2,\vartheta_1)}\,q_r(g_1,g_2)$. So it has weight $-r$ in g_1 and weight r in g_2, and we should have $Q_r(z_1,z_2) = y_1^{r/2}y_2^{-r/2}\,q_r(z_1,0),(z_2,0))$, which is indeed the case:

$$\begin{aligned}
y_1^{r/2}y_2^{-r/2}\,q_r(z_1,0),(z_2,0)) &= (y_1/y_2)^{r/2}\,m_r\Big(\begin{pmatrix} y_1^{-1/2} & -x_1y_1^{-1/2}\\ 0 & y_1^{1/2}\end{pmatrix}^{\sim}(z_2,0)\Big)\\
&= (y_1/y_2)^{r/2}\,m_r((z_2-x_1)/y_1,0)\\
&= (y_1/y_2)^{r/2}(y_2/y_1)^{r/2}\,M_r(z_2-x_1)/y_1) \;=\; M_r(z_2-x_1)/y_1).
\end{aligned}$$

Since $q_r(gg_1,gg_2) = q_r(g_1,g_2)$ for all $g\in\tilde{G}$, this immediately implies the invariance relation (8.14).

For the differential equations we use that in weight r the Casimir operator corresponds to $\Delta_r + \frac{r}{2}(1-\frac{r}{2})$. Since ω is left-invariant, we have

$$R_\omega q_r(g_1,\cdot) \;=\; \frac{r}{2}(1-\frac{r}{2})\, q_r(g_1,\cdot)\,.$$

This corresponds to (8.12).

The Casimir operator commutes with $g\mapsto g^{-1}$ and with right translations, so $\omega q_r(\cdot,g_2) = \frac{r}{2}(1-\frac{r}{2})\, q_r(\cdot,g_2)$. Since Q_r has weight $-r$ in the first variable, we have

$$\Bigl(\Delta_{-r} + \frac{-r}{2}\,(1+\frac{r}{2})\Bigr) Q_r(\cdot,z_2) \;=\; \frac{r}{2}\,(1-\frac{r}{2})\, Q_r(\cdot,g_2)\,,$$

which is (8.13).

A.2. **Principal series.** *Induced representation.* The set $\tilde P := \{(z,m\pi)\in\tilde G \;:\; z\in\mathfrak{H},\; m\in\mathbb{Z}\}$ is a subgroup of $\tilde G$. The principal series representations of $\tilde G$ are obtained by induction from the characters, which can be written as

$$\chi_{s,r} : (z,m\pi)\mapsto y^s\, e^{-m\pi i r}\,, \tag{A.22}$$

with $s\in\mathbb{C}$, $r\in\mathbb{C} \mod 2\mathbb{Z}$. This leads to the space $\mathcal{V}^\omega[s,r]$ consisting of the real-analytic functions $\tilde G\to\mathbb{C}$ that satisfy $f(gp) = \chi_{s,r}(p)^{-1} f(g)$ with $p\in\tilde P$, $g\in\tilde G$. The action of $\tilde G$ by left translation makes $\mathcal{V}^\omega[s,r]$ into a representation of $\tilde G$. The collection $\{\mathcal{V}^\omega[s,r] \;:\; s\in\mathbb{C},\; r\in\mathbb{R}/2\mathbb{Z}\}$ is called the *principal series* of representations of $\tilde G$, depending on the *spectral parameter* $s\in\mathbb{C}$ and the *central character* $\tilde k(m\pi)\mapsto e^{-\pi i m r}$. The superscript ω indicates that, for the moment, we consider analytic vectors.

The classes of $\tilde G/\tilde P$ can be parametrized as $\tilde k(\vartheta)\tilde P$ with $\vartheta\in\mathbb{R} \mod \pi\mathbb{Z}$. We can describe the elements of $\mathcal{V}^\omega[s,r]$ as functions $f:\mathbb{R}\to\mathbb{C}$ that satisfy $f(\vartheta+\pi) = e^{\pi i r}\, f(\vartheta)$. With some work one can explicitly describe $f\mapsto f\,\Big|\,\widetilde{\begin{pmatrix} a&b\\ c&d\end{pmatrix}}$ in terms of analytic functions of ϑ depending on a, b, c, and d.

This is not a practical way to work with principal series representations. We choose $p\in r+2\mathbb{Z}$ and relate f as above to φ on $\mathbb{P}^1_{\mathbb{R}}$ by $\varphi(-\cot\vartheta) = e^{-ip\vartheta}\, f(\vartheta)$. This leads to a realization of the principal series $\mathcal{V}^\omega[s,r]$ in the real-analytic functions on $\mathbb{P}^1_{\mathbb{R}}$. We denote this realization by $\mathcal{V}^\omega(s,p)$, and call it a *projective model* of $\mathcal{V}^\omega[s,r]$. The model depends on the choice of $p\equiv r \mod 2$. If one carries out the computations one arrives at the following description of the action

$$\begin{aligned}\varphi|^{\mathrm{prj}}_{s,p}\tilde k(\pi)\,(t) \;&=\; e^{\pi i r}\,\varphi(t) \qquad \text{(independent of } p\equiv r \bmod 2)\,,\\ \varphi|^{\mathrm{prj}}_{s,p}\tilde g\,(t) \;&=\; (a+ic)^{-s-p/2}\,(a-ic)^{-s+p/2}\\ &\qquad \Bigl(\frac{t-i}{t-g^{-1}i}\Bigr)^{s-p/2}\Bigl(\frac{t+i}{t-g^{-1}(-i)}\Bigr)^{s+p/2}\varphi\Bigl(\tfrac{at+b}{ct+d}\Bigr)\,,\end{aligned} \tag{A.23}$$

for $g = \begin{pmatrix} a&b\\ c&d\end{pmatrix}\in G_0\subset \mathrm{SL}_2(\mathbb{R})$, as defined in (1.3).

Remarks. (a) To check the consistency of the two lines in (A.23) we note that if $g = \begin{pmatrix} a&b\\ c&d\end{pmatrix}\in G_0$ tends to $\begin{pmatrix} -1&0\\ 0&-1\end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$ with $\pm c\downarrow 0$, then $\tilde g = \tilde p(gi)\,\tilde k(-\arg(ci+d))$

tends to $\tilde{k}(\mp\pi)$, which gives a factor $e^{\mp\pi i r}$ in the first line. Further $\arg(a+ic) \to \pm\pi$ and $\arg(a-ic) \to \mp\pi$. Hence

$$(a+ic)^{-s-p/2}\,(a-ic)^{-s+p/2} \to e^{\pm\pi i(-s-p/2)\mp\pi i(-s+p/2)} = e^{\mp\pi i p} = e^{\mp\pi i r}\,.$$

(b) The description in (A.23) is complicated. The factor $\bigl(\frac{t-i}{t-g^{-1}i}\bigr)^{s-p/2}$ is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ minus a path in $\mathfrak{H}$ from i to $g^{-1}i$, and similarly the factor $\bigl(\frac{t+i}{t-g^{-1}(-i)}\bigr)^{s+p/2}$ is holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ minus a path in $\mathfrak{H}^-$ from $-i$ to $g^{-1}(-i)$. So if φ is a real-analytic function on $\mathbb{P}^1_{\mathbb{R}}$, then $\varphi|^{\mathrm{prj}}_{s,p}\tilde{g}$ is also real-analytic on $\mathbb{P}^1_{\mathbb{R}}$.

(c) Any real-analytic function on $\mathbb{P}^1_{\mathbb{R}}$ is the restriction of a holomorphic function on some neighbourhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. We can view $\mathcal{V}^{\omega}(s,p)$ as a space on holomorphic functions on some neighbourhood U_φ of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The action $|^{\mathrm{prj}}_{s,p}$ preserves this space.

(d) We do not have one projective model of $\mathcal{V}^{\omega}[s,r]$, but infinitely many. Multiplication by the function $t \mapsto \bigl(\frac{t-i}{t+i}\bigr)^{\ell}$, with $\ell \in \mathbb{Z}$, gives an isomorphism

$$\mathcal{V}^{\omega}(s,r+2\ell) \longrightarrow \mathcal{V}^{\omega}(s,r)\,. \tag{A.24}$$

(e) The action $|^{\mathrm{prj}}_{s,p}$ leaves invariant other spaces of functions on $\mathbb{P}^1_{\mathbb{R}}$, for instance the C^∞-functions. This leads to the space $\mathcal{V}^{\infty}(s,p)$ of smooth vectors in the principal series representation. The discussion in [13, §2] of the space distribution vectors and hyperfunction vectors can be applied here, leading to $\mathcal{V}^{-\infty}(s,p)$ and $\mathcal{V}^{-\omega}(s,p)$.

(f) All elements of $\mathcal{V}^{\omega}(s,p)$ can be represented as a sum

$$\sum_{\mu\in\mathbb{Z}} c_\mu \Bigl(\frac{t-i}{t+i}\Bigr)^{\mu}\,, \tag{A.25}$$

with $c_\mu = \mathrm{O}(e^{-a|\mu|})$ for some $a > 0$. For the larger spaces $\mathcal{V}^{x}(s,p)$ with $x = \infty, -\infty, -\omega$, there are similar descriptions, like in [15, (2.18)], each with a condition on the growth of the coefficients c_μ.

A.2.1. *Highest weight subspaces.* For general combinations of $s,p \in \mathbb{C}$ the $\tilde{G}$-module $\mathcal{V}^{\omega}(s,p)$ is irreducible. (Reducibility has to be understood as the existence of a *closed* non-trivial invariant subspace, for the topology on $\mathcal{V}^{\omega}(s,p)$ that in the projective model is induced by the collection of supremum norms on the neighbourhoods U of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$.) Reducibility occurs if $2s \equiv p$ or $2s \equiv -p$ modulo 2. For our purpose we consider $2s \equiv -p \bmod 2$. In view of the isomorphism in (A.24) we can look at the case $(s,p) = (1-\frac{r}{2}, r-2)$. In that case the action in (A.23) is given by

$$\begin{aligned} \varphi|^{\mathrm{prj}}_{1-r/2,r-2}\tilde{k}(\pi)\,(t) &= e^{\pi i r}\,\varphi(t)\,,\\ \varphi|^{\mathrm{prj}}_{1-r/2,r-2}\tilde{g}\,(t) &= (a-ic)^{r-2}\Bigl(\frac{t-i}{t-g^{-1}i}\Bigr)^{2-r}\varphi\Bigl(\tfrac{at+b}{ct+d}\Bigr)\,. \end{aligned} \tag{A.26}$$

The factor $\Big((t-i)/(t-g^{-1}i)\Big)^{2-r}$ has singularities only on a path in $\mathfrak{H}$ from i to $g^{-1}i$. Hence $\mathcal{V}^\omega\left(1-\frac{r}{2}, r-2\right)$ contains as an invariant subspace the vectors represented by a holomorphic function on a neighbourhood of $\mathfrak{H}^- \cap \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. That is just the projective model $\mathsf{prj}_{2-r}\mathcal{D}^\omega_{2-r}$. Moreover, a comparison of (A.26) with (1.20) shows that $|^{\mathrm{prj}}_{1-r/2,r-2}\tilde g$ is the same as the operator $|^{\mathrm{prj}}_{2-r}g$. In this way, the space $\mathcal{D}^\omega_{2-r}$ can be viewed as an invariant subspace of $\mathcal{V}^\omega\left(1-\frac{r}{2}, r-2\right)$.

In the representation (A.25) the subspace $\mathsf{prj}_{2-r}\mathcal{D}^\omega_{2-r} \subset \mathcal{V}^\omega(1-\frac{r}{2}, r-2)$ is characterized by $c_\mu = 0$ for $\mu > 0$. Then the sum represents a a holomorphic function on a neighbourhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$.

The function $t \mapsto \left(\frac{t-i}{t+i}\right)^\mu$ is an eigenfunction of $\tilde k(\vartheta)$ with eigenvalue $e^{\pi i(r+2\mu)}$. One calls $r+2\mu$ the weight. In $\mathsf{prj}_{2-r}\mathcal{D}^\omega_{2-r}$ only weights $r+2\mu$ with $\mu \leq 0$ occur, hence the name *highest weight* subspace.

We may proceed similarly with the larger representations spaces $\mathcal{V}^\infty(1-\frac{r}{2}, r-2)$, $\mathcal{V}^{-\infty}(1-\frac{r}{2}, r-2)$, and $\mathcal{V}^{-\omega}(1-\frac{r}{2}, r-2)$, to obtain descriptions of the projective models of $\mathcal{D}^x_{v,2-r}$ with $x = \infty, -\infty, -\omega$.

A.3. **Related work.** The idea to view automorphic forms as functions on a Lie group is well-known, and has led to wide generalizations. We have not tried to find the first place where this idea appears in the literature. To handle automorphic forms of non-integral weight one has to use a central extension of the Lie group $\mathrm{SL}_2(\mathbb{R})$. For half-integral weights one needs a double cover, the *metaplectic group*. See, *e.g.*, Gelbart‘s treatment [50]. For general complex weights we need the universal covering group $\tilde G$. See Selberg [111], and Roelcke [109, §4].

Covering groups are often described with a 2-cocycle on $\mathrm{SL}_2(\mathbb{R})$ with values in the center, $\mathbb{Z}/2\mathbb{Z}$ for the metaplectic group, $\tilde Z \cong \mathbb{Z}$ for $\tilde G$. This cocycle turns up naturally in the description of multiplier systems, even if one does not use the language of Lie groups. Petersson gives it in [96, (11)], and Roelcke in [109, (1.7)]. We feel more comfortable with the description of $\tilde G$ as the space $\mathfrak{H} \times \mathbb{R}$ provided with an analytic group structure. This keeps the 2-cocycle hidden in the properties of the lift $g \mapsto \tilde g$.

For all semisimple Lie groups the principal series of representations is important. In [64, Chap II] one finds examples. For the universal covering group $\tilde G$ of $\mathrm{SL}_2(\mathbb{R})$ it was developed by Pukánski [104], since he needed it for function theory on $\tilde G$. Chapter VII of [64] discusses the construction of principal series representations as an induced representation.

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