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Virtual Special Issue - L.E.J. Brouwer after 50 years

Lewis meets Brouwer: Constructive strict implication

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Abstract

C.I. Lewis invented modern modal logic as a theory of "strict implication" $\prec$. Over the classical propositional calculus one can as well work with the unary box connective. Intuitionistically, however, the strict implication has greater expressive power than $\Box$ and allows to make distinctions invisible in the ordinary syntax. In particular, the logic determined by the most popular semantics of intuitionistic K becomes a proper extension of the minimal normal logic of the binary connective. Even an extension of this minimal logic with the "strength" axiom, classically near-trivial, preserves the distinction between the binary and the unary setting. In fact, this distinction has been discovered by the functional programming community in their study of "arrows" as contrasted with "idioms". Our particular focus is on arithmetical interpretations of intuitionistic $\prec$ in terms of *preservativity* in extensions of HA, i.e., Heyting's Arithmetic.

1. Introduction

More is possible in the constructive realm than is dreamt of in classical philosophy. For example, we have nilpotent infinitesimals [92] and the categoricity of weak first-order theories of arithmetic [89,90], this paper Appendix C.4.2. We zoom in on one such possibility: the original modal connective of "strict implication" $\prec$ proposed by C. I. Lewis [70,74], and hence called here the *Lewis arrow*, does not reduce to the unary box $\Box$ over constructive logic. This simple insight opens the doors for a plethora of new intuitionistic modal logics that cannot be understood solely in terms of the box. To the best of our knowledge, this observation was originally made in the area of *preservativity logic* [55,56,127,129] and metatheory of arithmetic provides perhaps the

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most interesting applications of intuitionistic $\prec$. However, one can claim that a similar discovery has been independently made in the study of *functional programming* in computer science (cf. Section 7.1).

We begin in Section 2 by recalling Lewis' invention of strict implication, mostly remembered by historians; these days, modal logic is almost by default taken to be the theory of boxes and diamonds. After sketching how $\prec$ fell into disuse and neglect, we speculate whether removing the law of excluded middle could have saved Lewis' vision of modal logic. This is also a good opportunity to highlight some unexpected analogies between the fates of Brouwer's and Lewis' projects.

In Section 3, we clarify how the intuitionistic distinction between $\phi \prec \psi$ and $\Box(\phi \to \psi)$ is reflected in Kripke semantics. This may well prove the most natural way of introducing this connective for many readers.

In Section 4, we present the minimal deduction system[1] iA and numerous additional principles used in the remainder in the paper. In Section 4.2, we clarify connections between them, i.e., the inclusion relation between corresponding logics.

With the syntactic apparatus ready, we turn in Section 5 to a major motivation for the study of $\prec$: logics of Σ^0_1-preservativity of arithmetical theories as contrasted with more standard logics of provability. In order to provide an umbrella notion for the study of arithmetical interpretations of modal connectives, we begin this section by setting up a general framework for *schematic logics*, which may prove of interest in its own right.

In Section 6, we are finally tying together the semantic setup of Section 3 and the syntactic infrastructure of Section 4 by providing a discussion of completeness and correspondence results. Some of them are well-known, others are new. Having a complete semantics for the logics under consideration allows us in Section 6.3 to complement earlier syntactic derivations (given in Section 4.2) with examples of *non*-derivations.

In Section 7, we are presenting other applications of *strong* arrows and *strong* boxes. In fact, what we call here "strong arrows" turns out to correspond directly to "arrows" in functional programming. We are also briefly discussing connections with logics of guarded (co)recursion and intuitionistic logics of knowledge.

But while intuitionistic $\prec$ can be (re)discovered in areas ranging from computer science to philosophy, in our view arithmetical interpretations are most developed and interesting. Thus, in Section 8 we return to the theme of Section 5 presenting some applications of the logic of preservativity. In Section 8.1 we discuss the application of preservativity to the study of the provability logic of Heyting Arithmetic HA. In Section 8.2, we show that preservativity allows a more satisfactory expression of the failure of *Tertium non Datur*.

The paper has several appendices that offer some supporting material. Appendix A collects basic facts about realizability needed in other sections. In Appendices B and C, we provide some basic insights in Π^0_1-conservativity logics and interpretability logics. These insights strengthen our understanding of preservativity logic both by extending this understanding and by offering a contrast to this understanding. Finally, Appendix D discusses the collapse of $\prec$ in Lewis' first monograph, i.e., *A Survey of Symbolic Logic* [70] from the perspective of our deductive systems.

Of course, we are of the opinion that the reader should carefully study everything we put in the paper. However, we realize that this expectation is not realistic. For this reason, we present several roadmaps through the paper.[2]

[1] It was baptized "iP" by Iemhoff and coauthors [54–56], but this acronym ties $\prec$ too tightly to preservativity.

[2] Note also that reading the electronic version may sometimes prove easier due to omnipresent hyperlinks: apart from all the usually clickable entities (citations or numbers of (sub)sections, footnotes and table- or theorem-like environments

The basic option is to read Sections 2–4 to get the basics of motivational background, the Kripke semantics and an impression of possible reasoning systems.

The reader who wants more solid treatment of Kripke semantics can extend the basic option with Section 6.

The computer science package consists of the basic option and Section 7.

The reader who wants to go somewhat more deeply into the history of the subject can extend the basic option with Appendix D.

○ The reader who wants to understand the basics of arithmetical interpretations can extend the basic option with Section 5.

○○ An extended package for arithmetical interpretations combines ○ with Section 8.

○○○ The full arithmetical package extends ○○ with Appendices A, B and C.

2. The rise and fall of the house of Lewis

2.1. *"The error of philosophers"*

We are reflecting on L.E.J. Brouwer's heritage half a century after his passing. Given his negative views on the rôle of logic and formalisms in mathematics, it seems somewhat paradoxical that these days the name of intuitionism survives mostly in the context of *intuitionistic logic*.[3] One is reminded in this context of what Nietzsche called *the error of philosophers*:

> The philosopher believes that the value of his philosophy lies in the whole, in the structure. Posterity finds it in the stone with which he built and with which, from that time forth, men will build oftener and better—in other words, in the fact that the structure may be destroyed and yet have value as material.[4]

We feel thus excused to focus on propositional logics based on the intuitionistic propositional calculus (IPC). More specifically, our interest lies in an intuitionistic take on a formal language developed by an author nearly perfectly contemporary with Brouwer: Clarence Irving Lewis,[5] the father of modern modal logic. And this time, the reason for this does *not* come from the well-known Gödel(–McKinsey–Tarski) translation of IPC into the system Lewis denoted as $\mathsf{S4}$, which is discussed elsewhere in this collection.

One can also see a certain irony in the fate of Lewis' systems. They were explicitly designed to give an account of "strict implication" $\dashv\!3$. The unary $\Box$ can be introduced using

$$\Box\phi \leftrightarrow (\top \dashv\!3\, \phi). \tag{1}$$

. . .), even most names of logical systems can be clicked upon to retrieve their definition in Tables 4.1 and 4.2. When reading a hardcopy, we advise keeping these Tables handy, perhaps jointly with Fig. 6.2.

[3] A related and better-known paradox is that Brouwer's own name survives in mainstream mathematics mostly in connection with his work on topology, which is confirmed by several contributions in this collection. This despite the fact that he rejected these results on philosophical grounds and was actively involved in topological research only for the period necessary to secure academic recognition and international status. Moreover, it seems a myth that the non-constructive character of his most famous topological publications *turned* Brouwer into an intuitionist. There is ample evidence that while the exact form of his intuitionism evolved somewhat, his philosophical beliefs predate these results. Cf. van Stigt [118] for a detailed discussion of all these points.

[4] *Human, All-Too-Human, Part II*, translated by Paul V. Cohn.

[5] He was born two years later than Brouwer and died two years earlier.

In fact, Lewis designed $\prec$ and $\Box$ as mutually definable,[6] setting

$$\phi \prec \psi := \Box(\phi \to \psi) \tag{2}$$

and over subsequent decades, modal logic in a narrow sense turned into the theory of unary $\Box$ and/or $\Diamond$. In a broader sense, pretty much any *intensional* operator extending the usual supply of connectives can be called a modality. Modalities came to represent not only *necessity*, but also *arithmetical provability*, *knowledge*, *belief*, *obligation*, and various forms of *guarded quantification*: *validity after all possible program executions*, *in all accessible states*, *in all future time instants* or *at every point in an open neighborhood* (the list, of course, is far from being exhaustive). Just like in the case of intuitionistic logic, a wide range of semantics for modalities have been investigated, the most prominent being the Kripke semantics (relational structures), but also topologies, coalgebras, monoidal endofunctors on categories or more recent "possibility semantics".

Thus, Lewis' dissatisfaction with *material* or *extensional* implication and disjunction, expressed first in a short 1912 article [66], has ultimately led to the spectacular success story of modal logic, much like Brouwer's[7] dissatisfaction with non-constructive usage of implication and disjunction has ultimately led to the spectacular success story of intuitionistic logic. And yet, while Lewis did not write much on formal logic after *Symbolic Logic*[8] published in 1932 [74], his occasional remarks do not suggest he would approve of the scattering of his Strict Implication systems into a bewildering galaxy of unimodal calculi. Indeed, he was not only opposed to the very name *modal logic*, but believed that his formalisms is the exact opposite of real "modal" logic, which in his view was . . . the extensional system of *Principia Mathematica*:

> There *is* a logic restricted to indicatives; the truth-value logic most impressively developed in *Principia Mathematica*. But those who adhere to it usually have thought of it—so far as they understood what they were doing—as being the universal logic of propositions which is independent of mode. And when that universal logic was first formulated in exact terms, they failed to recognize it as the only logic which is *independent* of the mode in which propositions are entertained and dubbed it "modal logic". (Cf. [95, p. 203].)

His own belief was that

> the relation of strict implication expresses *precisely that relation which holds when valid deduction is possible* [emphasis ours]. It fails to hold when valid deduction is not possible.

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6 To be precise, in his books Lewis did not use $\Box$ as a primitive. His exact formulation of $\phi \prec \psi$ was $\neg\Diamond(\phi \wedge \neg\psi)$. However, in the classical setting, this one is obviously equivalent to the one given by (2), and the reliance of Lewis' formulation on involutive negation would be a major problem over IPC. See Appendix D for a more detailed examination of the rôle of involutive/classical negation in Lewis' original system.

7 Speaking of Brouwer, note again the parallelism of dates: 1912, the year when Lewis fired his first shots for intensional connectives by publishing *Implication and the Algebra of Logic* [66], is also the year when Brouwer obtained his position at the University of Amsterdam, was elected to the Royal Netherlands Academy of Arts and Sciences, delivered his famous inaugural address *Intuitionism and Formalism* and became liberated to pursue his own program. We refrain here from investigating further analogies, such as the fact that Lewis wrote his 1910 Ph.D. on *The Place of Intuition in Knowledge* (cf. Murphey [95, Ch. 1] for an extended discussion), that he had a solid background in idealism and Kant and that he remained under strong influence of these philosophical positions throughout his career.

8 *Symbolic Logic* was a collaboration between C.I. Lewis and C.H. Langford. The authors, however, made it clear in the preface who wrote and is "ultimately responsible" for which chapter, a practice rather uncommon today. All the passages quoted in this paper come from chapters written by Lewis. As Murray G. Murphey says in his monograph on C.I. Lewis: "*Symbolic Logic* was less a cooperative venue than a coauthored book . . . To what extent each advised the other on their separate chapters is left unclear, but probably there was not much of an attempt to harmonize . . . Langford's theory of propositions, for example, in Chapter IX is clearly not Lewis's theory". [95, p. 183].

> In that sense, the system of Strict Implication may be said to provide that canon and critique of deductive inference which is the desideratum of logical investigation [74, p. 247]

and that

> Strict Implication explains the paradoxes incident to truth-implication. [74, p. 247]

While the failure of Lewis' systems to conquer this intended territory had to do with philosophical prejudices of the following decades, they were also simply less suited for these purposes than Lewis thought. The original system of *A Survey of Symbolic Logic* in 1918 [70]—stemming back to a 1914 paper [68]—was plagued by a number of issues, the most famous one pointed out by Post: the combination of an axiom equivalent to (in an updated notation)

$$(\Box\phi \prec \Box\psi) \prec (\neg\psi \prec \neg\phi)$$

with other axioms and classical negation laws trivialized the modality and collapsed strict implication to material implication [71]. We provide an extended analysis of Lewis' SSL problem in Appendix D; we believe it is an interesting application of the intuitionistic theory of $\prec$ discussed in this paper.[9] In *Symbolic Logic* [74]—more precisely, in its famous Appendix II—Lewis was more cautious, creating several "lines of retreat" (as Parry [100] described it) in the form of $\mathsf{S3}$, $\mathsf{S2}$ and $\mathsf{S1}$. At least on the technical front, this time things went better. Immediate polemics focused on possibility of definability of intensional connectives in extensional systems, but none of the authors involved proposed anything resembling what we much later came to know as the *Standard Translation* of modal logic into predicate logic.[10] There were, however, subtler problems, pointed out in the post-war period by Ruth Barcan Marcus[11]:

> it is plausible to maintain that if strict implication is intended to systematize the familiar concept of deducibility or entailment, then some form of the deduction theorem should hold for it. [10]

She showed [9,10] that $\mathsf{S1}$ to $\mathsf{S3}$ fail this criterion, for several conceivable formulations of the Deduction Theorem. And those which behave somewhat better in this respect, i.e., from $\mathsf{S4}$ upwards are too strong to capture a general notion of strict implication which Lewis would approve of.

In fact, $\mathsf{S4}$ and $\mathsf{S5}$, which we came to count among *normal* systems (unlike $\mathsf{S1}$–$\mathsf{S3}$) and for which the advantage of switching to the unary setting is most obvious, for Lewis himself were foster children he was forced to adopt. As is well-known, it was Oskar Becker[12] [12] who proposed these axioms, even calling one of them the *Brouwersche Axiom*; let us not discuss the adequacy of this name here, but not only does it provide us with another excuse to mention Brouwer in this paper, it has also survived until today in names of systems like KB or KTB. Becker intended to cut the number of non-equivalent modalities in the calculus, a goal which seems rather orthogonal to Lewis' plans:

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[9] Cf. also the discussion by Murphey [95, pp. 101–102] or Parry [100].

[10] Cf., e.g., the attempts of Bronstein&Tarter or Abraham addressed, respectively, by McKinsey and Fitch; see Murphey [95, Ch. 6] for references. It is worth pointing out that Lewis himself [75] dealt with this question in a paper published only posthumously (with Langford as a "nominal" coauthor, see editor's note [86] for a contemporary perspective).

[11] Her earliest papers [9] are signed by her maiden surname, Ruth Barcan, which survives until today in the name of the *Barcan formula*.

[12] Although many developments discussed in this subsection – in particular proposing and justifying $\mathsf{S4}$ axioms with an explicit Brouwerian motivation – had their forerunner in a neglected 1928 paper by Ivan E. Orlov, cf. [11,36].

> Those interested in the merely mathematical properties of such systems of symbolic logic tend to prefer more comprehensive and less strict systems such as S5 and material implication. The interests of logical study would probably be best served by an exactly opposite tendency. [74, p. 502]

Kurt Gödel did review Becker's work [47, p. 216–217] and was familiar with William T. Parry's early analysis of the notion of *analytic* implication based on $\prec$ [47, p. 266–267].[13] This apparently led[14] to his landmark 1933 paper [47, p. 296–303] translating the nascent intuitionistic calculus into what turns out to be a notational variant of S4 formulated with unary box as a primitive. Thus, immediately after *Symbolic Logic* was published, Gödel pretty much doomed the fate of $\prec$ and condemned non-normal systems to at most secondary status: his paper not only provided an independent motivation (in terms of "the intuitionistic logic of Brouwer and Heyting" ...) for the study of extensions of S4 rather than subsystems of S3, but also highlighted the elegance and conciseness of $\Box$-based axiomatizations for these logics.

In short, it appears that regardless of the fact that historical circumstances did not favor Lewis, none of his systems was destined to success or genuinely free of design or conceptual issues. Nevertheless, the idea of providing an implication connective yielding tautologies only when the antecedent is genuinely *relevant* for the consequent proved prescient.[15] In fact, one can easily argue that even the later enterprise of relevance logic would not satisfy Lewis' expectations: he wanted to *supplement* material implication with a strict one, not *replace* it altogether. In this sense, still more recent *resource-aware* formalisms with computer-science motivation where *both* a substructural *and* an intuitionistic/classical implication are present (either as an abbreviation or directly in the signature) like linear logic [2,18,46,122] or the logic of bunched implications BI [98,104,105] seem closer to Lewis' original idea.

2.2. *Could Brouwerian inspiration help Lewis' systems?*

At the time of publication of *Symbolic Logic*, Lewis was both familiar with and open to non-boolean extensional connectives. The chapters he wrote for that monograph deal in detail with n-valued systems of Łukasiewicz.[16] At the same time, he published a paper on *Alternative*

[13] As another small example how modal and intuitionistic inspirations tended to work hand-in-hand for Gödel: his proof that IPC is not characterized by any finite algebra [47, p. 268–271] is presented as an answer to a question posed by Otto Hahn during a discussion following Parry's presentation.

[14] His short review of Becker points out that Becker's attempts to relate modal logic to "the intuitionistic logic of Brouwer and Heyting" and claims that steps taken by Becker to "deal with this problem on a formal plane" are unlikely to succeed; Orlov (cf. Footnote 12) was more insightful, but it does not appear that Gödel was familiar with his paper.

[15] The connection between modal logics and relevance logics has been always actively debated, see, e.g., Mares [85, Ch. 6] for an extended presentation, including a reminder that Ackermann's 1956 paper which "began the study of relevant entailment" took issue with some tautologies valid for Lewis' $\prec$, in particular *ex falso quodlibet*. But in fact the relationship can be traced back at least to 1933, when Parry in his work on analytic implication based on $\prec$ proposed what relevance logicians came to know as the *variable sharing criterion*: much later, Dunn [40] noted that Parry's system is contained in S4 and proposed a "demodalization" of Parry's original system still preserving that criterion. As another connection with Gödel, let us note that his discussion [47, p. 266–267] of the work of Parry suggested a completeness result that was only proved in 1986 by Fine [43]. Moreover, one can push the clock back even beyond Parry and Gödel, to the paper of Orlov (cf. Footnote 12), which seems the first attempt to relate relevance, intuitionistic, and modal principles, including the first axiomatization of what came to be known as the implicative-negative fragment of the relevance logic R [36]. Let us note here the view of van Atten [124] that "logic as Brouwer sees it is a relevance logic", rejecting in particular *ex falso* (absent also in earliest versions of formalizations of intuitionistic logic by Kolmogorov and Glivenko), which subverts the standard understanding of the *BHK interpretation* (cf. Section 7.1).

[16] At the time, Lewis still attributed it to a collaboration between Łukasiewicz and Tarski.

systems of logic [73]. In both these references, he discusses possible definitions of "truth-implications" [74] or "implication-relations" [73] one can entertain in finite, but not necessarily binary matrices. The latter paper also contains a rare (perhaps the only one) reference to Brouwer in his writings:

> [T]he mathematical logician Brouwer has maintained that the law of the Excluded Middle is not a valid principle at all. The issues of so difficult a question could not be discussed here; but let us suggest a point of view at least something like his. ... The law of the Excluded Middle is not writ in the heavens: it but reflects our rather stubborn adherence to the simplest of all possible modes of division, and our predominant interest in concrete objects as opposed to abstract concepts. The reasons for the choice of our logical categories are not themselves reasons of logic any more than the reasons for choosing Cartesian, as against polar or Gaussian coördinates, are themselves principles of mathematics, or the reason for the radix 10 is of the essence of number. [73, p. 505]

Of course, the question of Lewis' own potential take on combining IPC and $\prec\!\!\!\!-\!\!\!3$ remains speculative: it does not seem he was familiar with the work of Kolmogorov, Glivenko and Heyting, turning Brouwer's philosophical insights into a propositional calculus. Nevertheless, let us note two points:

- even the collapse of Lewis' original system [68,70] was caused by classical laws combined with a misguided boolean inspiration, namely the insistence on involutivity of the *strict* negation (cf. Appendix D);
- even when considering classical Kripke frames, the negation-free logic obtained by replacing $\to$ with $⥽$ is a sublogic of the intuitionistic logic [27,28,31,37] (see also Question 4.3.).

Our paper, however, focuses on an even more fundamental advantage of studying the theory of $⥽$ over IPC. Whatever is there to be said about *the universal logic of propositions which is independent of mode* and its extensional basis, defining $⥽$ using (2) is premature in the constructive setting. Furthermore, instances of such a "constructive strict implication" can be executed seen in areas ranging from metatheory of intuitionistic arithmetic to functional programming, often satisfying very different laws to those strict implication was supposed to obey; indeed, sometimes rather meaningless classically. For example,

$$\mathsf{S_a} \quad (\phi \to \psi) \to (\phi ⥽ \psi)$$

holds in numerous logics justified from a computational/Curry–Howard (Section 7), arithmetical (Section 5.4.4) or even philosophical (Section 7.3) point of view.[17]

3. Strict implication in intuitionistic Kripke semantics

It is time to begin a more systematic discussion, starting with the relational interpretation of $⥽$. In this paper, we are concerned with the following propositional languages: $\mathcal{L}_{⥽}$ (with Lewis' arrow), $\mathcal{L}_{\Box}$ (the unimodal one, identified with a fragment of $\mathcal{L}_{⥽}$) and $\mathcal{L}$ (the propositional language of

[17] From a Lewisian point of view, would intuitionistic $\to$ be the "strict" implication and $⥽$ be the "material" implication in such systems?

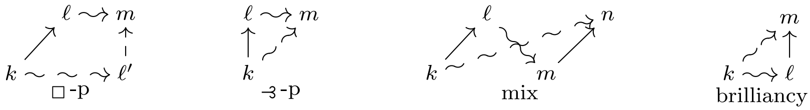


Fig. 3.1. Minimal conditions one can impose on $\Box$-frames and $\strictif$-frames. See Fig. 6.2 for a visual representation of other conditions corresponding to additional axioms.

IPC):

$$\begin{array}{rl} \mathcal{L}_{\strictif} & \phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid (\phi \strictif \phi), \\ \mathcal{L}_{\Box} & \phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid (\Box\phi), \\ \mathcal{L} & \phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi). \end{array}$$

As usual, $\neg\phi$ abbreviates $\phi \to \bot$.

For the sake of clarity, the binding priorities are as follows: unary connectives $\neg$ and $\Box$ bind strongest, next comes $\strictif$, then $\wedge$ and $\vee$, and finally $\to$.

Regarding associativity, it is used tacitly for $\wedge$ and $\vee$, just like commutativity. Regarding $\to$ and $\strictif$, they are commonly assumed to associate to the right, but we will be careful not to overuse this convention, as it can be confusing.

We begin with recalling the basic setup of intuitionistic Kripke frames for $\mathcal{L}_{\Box}$.[18] They come equipped with two accessibility relations. One of them, which we will denote by $\preceq$, is a partial ordering[19] interpreting intuitionistic implication:

$$k \Vdash \phi \to \psi \text{ if, for all } \ell \succeq k, \text{ if } \ell \Vdash \phi, \text{ then } \ell \Vdash \psi. \tag{3}$$

This forces the denotation of $\to$ to be $\preceq$-*persistent* or, as some authors say, "monotone" or "upward-closed". It is enough to impose (3) and require $\preceq$-persistence of atoms to ensure persistence for all $\mathcal{L}$-formulas. The other accessibility relation $\sqsubset$ is the modal one. There are two choices one can make to ensure $\preceq$-persistence for $\Box$:

> One is to modify the satisfaction clauses. This might be a reasonable thing to do, for one might wish to use the partial order to give a more intuitionistic reading of the modalities. The other remedy is to impose conditions on models that ensure that the monotonicity lemma does hold. [112, §3.3]

In fact, in a unimodal language the difference between these two strategies is not essential; it becomes more consequential when a single accessibility relation is used to interpret, for example, both $\Box$ and $\Diamond$ (see [112, §3.3] for a discussion and more references). Still, most references choose the latter one, i.e., keeping the same reading of $\Box$ as in the classical case and imposing conditions on the interaction of $\preceq$ and $\sqsubset$ to ensure persistence (see Fig. 3.1).

Božić and Došen [26] have established that in the presence of unary $\Box$ with semantics defined by

$$k \Vdash \Box\phi \text{ if, for all } \ell \sqsupset k, \ell \Vdash \psi$$

persistence is equivalent to the condition

[18] As far as $\mathcal{L}_{\Box}$ is concerned, our discussion largely follows Litak [81]. The reader is referred there for more details and references.

[19] In fact, it is essential only that the relation is a preorder (i.e., a reflexive and transitive relation), but such a generalization brings no tangible benefits from the point of view of expressivity, definability and completeness of propositional logics.

□-p if $k \preceq \ell \sqsubset m$, then, for some ℓ', we have $k \sqsubset \ell' \preceq m$

(i.e., $\preceq \cdot \sqsubset \subseteq \sqsubset \cdot \preceq$, where " $\cdot$ " denotes relational composition). However, most references require tighter interaction. On certain occasions, like in Goldblatt [48], one sees a strengthening to

⥽-p if $k \preceq \ell \sqsubset m$, then $k \sqsubset m$ (i.e., $\preceq \cdot \sqsubset \subseteq \sqsubset$).

But the most common one (see, e.g., [117,140,141]) is the still stronger

mix if $k \preceq \ell \sqsubset m \preceq n$, then $k \sqsubset n$ (i.e., $\preceq \cdot \sqsubset \cdot \preceq \subseteq \sqsubset$).

This condition naturally obtains in a canonical model construction à la Stone and Jónsson–Tarski for prime filters of (reducts of) Heyting algebras with normal □ [17,26,61,117]. Moreover, mix is "mostly harmless" for □: it can be obtained from □-p by adding the requirement that for any ℓ, the set of its $\sqsubset$-successors is $\preceq$-upward closed, that is,

brilliancy if $k \sqsubset \ell \preceq m$, then $k \sqsubset m$ (i.e., $\sqsubset \cdot \preceq \subseteq \sqsubset$).

The name, to the best of our knowledge, has been proposed by Iemhoff [53–56], another one being *strongly condensed* [26]. As noted in standard references [26,48], not only brilliancy can*not* be defined using □, but any model satisfying □-p can be made brilliant *without changing the satisfaction relation for □-formulas* in a straightforward way: by replacing $\sqsubset$ by its composition with $\preceq$.

Consider now the Lewisian strict implication $\phi \mathrel{⥽} \psi$. Here is the natural satisfaction clause in this semantics, directly transferring the classical one:

$$k \Vdash \phi \mathrel{⥽} \psi \text{ if, for all } \ell \sqsupset k, \text{ if } \ell \Vdash \phi, \text{ then } \ell \Vdash \psi. \tag{4}$$

The first consequence of such an enrichment of the language is that □-p becomes too weak to ensure persistence. Let us state this formally, defining for this purpose a somewhat too general notion:

Definition 3.1. A *preframe* is a triple $\mathcal{F} := \langle W, \preceq, \sqsubset \rangle$, where $\preceq$ is a partial order, and $\sqsubset$ is a binary relation. A *premodel* based on $\mathcal{F}$ is $\mathcal{K} := \langle \mathcal{F}, V \rangle$, where V is a *valuation* mapping propositional variables to $\preceq$-upward closed sets. The *forcing relation* $\mathcal{K}, k \Vdash \phi$ is defined in the standard way for the intuitionistic connectives and using Eq. (4) for ⥽. □

It can be easily shown (see, e.g., [56,144]) that *the condition equivalent to persistence becomes precisely* ⥽-p, that is:

Fact 3.2. *For a preframe $\mathcal{K} := \langle W, \preceq, \sqsubset \rangle$, ⥽-p above corresponds to the following condition: for any two sets U, V upward closed wrt $\preceq$, the set*

$$U \mathrel{⥽} V := \{k \in W \mid \forall \ell \sqsupset k,\ \textit{if } \ell \in U,\ \textit{then } \ell \in V\}$$

is upward closed wrt $\preceq$.

We will thus take ⥽-p to be the minimal condition in what follows.

Definition 3.3. A (⥽-)*frame* is a preframe satisfying ⥽-p. □

We can define in a standard way what it means for a formula to be *valid* or *refuted* in a class of models.

As we have already suggested, for $\mathcal{L}_{\dashv}$ the brilliancy condition does not remain "mostly harmless" in the sense described above for $\mathcal{L}_{\Box}$:

Fact 3.4 (*[144]*). *The following conditions are equivalent for a $\dashv$-frame:*

- *validity of* $(\phi \wedge \psi) \dashv \chi \to \phi \dashv (\psi \to \chi)$*;*
- *validity of* $\psi \dashv \chi \to \top \dashv (\psi \to \chi)$*;*
- *validity of brilliancy.*

One easily sees the converse implication

$$\phi \dashv (\psi \to \chi) \to (\phi \wedge \psi) \dashv \chi$$

and, consequently, its special instance (where ϕ is equal to $\top$)

$$\Box(\psi \to \chi) \to \psi \dashv \chi$$

to be valid on any $\dashv$-frame; Lemma 4.1 for a syntactic derivation.

Let us take stock. In order to restore definability of $\dashv$ in terms of $\Box$, i.e., validity of (2) above, one needs to impose the brilliancy condition. In general, $\Box(\phi \to \psi)$ implies $\phi \dashv \psi$, but not necessarily the other way around. Of course, in classical Kripke frames, $\preceq$ is a discrete order, which trivializes all conditions discussed above and all distinctions between them. As we will see in Corollary 4.8, the boolean deconstruction of $\dashv$ can be also derived syntactically. We will return to Kripke semantics in Section 6.

4. Axiomatizations

4.1. *A fistful of logics*

In this section, we present a Hilbert-style study of $\mathcal{L}_{\dashv}$-logics. Discussion of arithmetically oriented principles was originated by Visser [125–127,129] and developed further by Iemhoff and coauthors [54–56], who also studied the basic theory of $\dashv$-frames. IPC and CPC denote, respectively, the intuitionistic propositional calculus and its classical counterpart.

4.1.1. *Logics in* $\mathcal{L}_{\Box}$

Before we start discussing $\dashv$-logics in Section 4.1.2, Table 4.1 presents some axioms involving only $\Box$, which is a definable connective in $\mathcal{L}_{\dashv}$.

- The axioms of i-GL$_{\Box}$ (intuitionistic Löb logic) and c-GL$_{\Box}$ (classical Löb logic) are well-known. The logic c-GL$_{\Box}$ is arithmetically complete for all classical Σ^0_1-sound theories extending Elementary Arithmetic EA. The logic i-GL$_{\Box}$ is arithmetically valid in all arithmetical theories extending i-EA. We discuss these matters further in Section 5.3.
- The principle Lei is known as *Leivant's Principle*. The principle is, in a sense, a shadow of the disjunction property. The disjunction property of an arithmetical theory T cannot be verified in T itself. Leivant's Principle is arithmetically valid in a substantial class of arithmetical theories that includes Heyting Arithmetic HA. We discuss Leivant's Principle in Section 5.3.
- S$_{\Box}$ axiomatizes *strong modalities* (cf. Section 7), but arises also in some arithmetically motivated logics (Section 5.4.4). *Strong Löb logic* is obtained by adding S$_{\Box}$ to i-GL$_{\Box}$—or, alternatively, by using SL$_{\Box}$ instead of L$_{\Box}$ as an axiom.

Table 4.1
List of principles for $\Box$. Here, the names of systems in the right column refer to languages restricted to connectives appearing in the axiomatization, i.e., not involving $\prec\!\!\!\!\!\prec$. Later in the text, we will also use some of these principles as axioms over iA, i.e., the minimal "normal" system for $\prec\!\!\!\!\!\prec$ (cf. Table 4.2), where $\Box$ is a defined connective.

$\mathsf{N}_\Box$	$\vdash \phi \quad \Rightarrow \vdash \Box\phi$
$\mathsf{K}_\Box$	$\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$
$\mathsf{4}_\Box$	$\Box\phi \to \Box\Box\phi$
$\mathsf{C4}_\Box$	$\Box\Box\phi \to \Box\phi$
$\mathsf{L}_\Box$	$\Box(\Box\phi \to \phi) \to \Box\phi$
$\mathsf{S}_\Box$	$\phi \to \Box\phi$
$\mathsf{SL}_\Box$	$(\Box\phi \to \phi) \to \phi$
Lei	$\Box(\phi \vee \psi) \to \Box(\phi \vee \Box\psi)$
$\mathsf{CB}_\Box$	$\Box\phi \to (\psi \to \phi) \vee \psi$
$\mathsf{CB}'_\Box$	$\Box(\psi \to \phi) \to (\psi \to \phi) \vee \psi$
$\mathsf{Lin}_\Box$	$\Box(\phi \to \psi) \vee \Box(\psi \to \phi)$
peirce	$((\phi \to \psi) \to \phi) \to \phi$
em	$\phi \vee \neg\phi$

CPC	$:= \mathsf{IPC} + \mathsf{peirce}$
$\mathsf{i\text{-}K}_\Box$	$:= \mathsf{IPC} + \mathsf{N}_\Box + \mathsf{K}_\Box$
$\mathsf{i\text{-}GL}_\Box$	$:= \mathsf{i\text{-}K}_\Box + \mathsf{L}_\Box$
$\mathsf{c\text{-}GL}_\Box$	$:= \mathsf{CPC} + \mathsf{i\text{-}GL}_\Box$
$\mathsf{i\text{-}S}_\Box$	$:= \mathsf{i\text{-}K}_\Box + \mathsf{S}_\Box$
$\mathsf{i\text{-}SL}_\Box$	$:= \mathsf{i\text{-}K}_\Box + \mathsf{SL}_\Box$
$\mathsf{i\text{-}PLL}_\Box$	$:= \mathsf{i\text{-}S}_\Box + \mathsf{C4}_\Box$
$\mathsf{i\text{-}mHC}_\Box$	$:= \mathsf{i\text{-}S}_\Box + \mathsf{CB}_\Box$
$\mathsf{i\text{-}KM}_\Box$	$:= \mathsf{i\text{-}SL}_\Box + \mathsf{CB}_\Box$
$\mathsf{i\text{-}KM.lin}_\Box$	$:= \mathsf{i\text{-}KM}_\Box + \mathsf{Lin}_\Box$

$\mathsf{4}_\Box$ is known to be derivable in $\mathsf{i\text{-}GL}_\Box$.
In the provability logic literature:

- $\mathsf{N}_\Box$ is known as L1,
- $\mathsf{K}_\Box$ is known as L2,
- $\mathsf{4}_\Box$ is known as L3,
- $\mathsf{L}_\Box$ is known as L4.

- The principle $\mathsf{C4}_\Box$ classically corresponds to a semantic condition known as *density* (cf. Fig. 6.2). From another point of view, this axiom arises naturally in the Curry–Howard logic of *monads* (Section 7). It is a typical "non-Löb-like" axiom: in combination with $\mathsf{L}_\Box$, we could derive $\Box\bot$.
- $\mathsf{CB}_\Box$ comes from the intuitionistic system $\mathsf{i\text{-}KM}_\Box$ of Kuznetsov and Muravitsky and its later weakening to $\mathsf{i\text{-}mHC}_\Box$ by Esakia and the Tbilisi group (see [81] for more information and references); its equivalent variant $\mathsf{CB}'_\Box$ (Lemma 4.16a) was discussed [126] in connection with PA^* (Section 5.4.4). In our setting, it is interesting to contrast it with CB_a in Table 4.2 (Fig. 6.2 and Example 6.11). See also Section 8.2 for the arithmetical perspective on the contrast between $\mathsf{i\text{-}mHC}_\Box$ and $\mathsf{i\text{-}mHC}_\mathsf{a}$.
  The name CB used here comes from Litak [81], where it was used to suggest the Cantor–Bendixson derivative.

- $\mathsf{Lin}_\Box$ is a typical axiom valid on total orders. In Fact 4.18 and Example 6.12, we compare and contrast this axiom with its $\dashv$-counterpart.

4.1.2. Logics in $\mathcal{L}_\dashv$

Table 4.2 displays potential axioms for $\dashv$ central for this paper. Most of them come with an explicit arithmetical interpretation. Typically, the "primed" variants of axioms will be their equivalent reformulations (Section 4.2).

- Iemhoff [54–56] identified system iA as the logic of all (finite) frames satisfying the $\dashv$-p condition; this and other completeness results are discussed in Section 6. However, Di is an axiom which is not exactly trivial from an arithmetical point of view. It does hold in the preservativity logic of Heyting Arithmetic but it fails in the preservativity logic of Peano Arithmetic (Section 5.4 and Appendix B). The non-triviality of Di, and the potential interest in a disjunction-free system (Section 7) are the reasons why we isolated iA^- as a subsystem.
- The principles $\mathsf{4_a}$, $\mathsf{W_a}$, $\mathsf{M_a}$ are arithmetically valid for the preservativity interpretation of $\dashv$. This means that they are in the logic $\mathsf{i\text{-}PreL}^-$ which is arithmetically valid in all arithmetical theories we consider in this paper (Section 5.4). The principle $\mathsf{L_a}$, weaker than $\mathsf{W_a}$, is mainly of technical interest.
- If we interpret $\phi \dashv \psi$ as $\neg \psi \rhd \neg \phi$, then the principle $\mathsf{P_a}$ is the distinctive principle of the interpretability logic of finitely axiomatized extensions of EA^+ aka $\mathrm{I}\Delta_0 + \mathsf{Supexp}$. The modality $\rhd$ stands for interpretability over a theory. This modality is explained in Appendix C.[20] The specific result mentioned here is discussed in detail in Appendix C.3.
- $\mathsf{S_a}$ and $\mathsf{S'_a}$ are $\dashv$-variants axiomatize the same logic as $\mathsf{S}_\Box$ (Lemma 4.10). In general, this is rarely the case with $\dashv$-generalizations of $\Box$-axioms; often the $\dashv$-version is stronger, but $\mathsf{Lin_a}$ illustrates such a rule is not universal.
- We have already seen Box in Section 3 above; its equivalence with Box' and Box'' is established in Lemma 4.4. The conjunction of this axiom with BL, derivable in iA (Lemma 4.1c), collapses $\dashv$. Note that $\mathsf{CB_a}$ makes Box derivable (Lemma 4.16), unlike $\mathsf{CB}_\Box$ (Example 6.11).
- The last group of $\dashv$-principles—i.e., $\mathsf{App_a}$, $\mathsf{C4_a}$ and Hug—which should be contrasted with $\mathsf{C4}_\Box$, will play a prominent rôle in Section 7.1 on monads, idioms and arrows in functional programming. For similar reasons as $\mathsf{C4}_\Box$, they are of drastically "anti-Löb" character, a fact made explicit by their semantic correspondents displayed in Fig. 6.2 in Section 6. It is worth mentioning that $\mathsf{App_a}$ was in fact adopted by Lewis as an axiom even in his weakest system $\mathsf{S1}$, cf. Remark 7.3.

4.2. An armful of derivations

In this subsection we put the Hilbert-systems proposed above to actual use. We begin with a discussion of minimal axiom systems, with and without $\mathsf{K_a}$ or Di. Later on, we move to those inspired by concrete applications. We are not giving the details of these derivations here; some are available in existing references (and we give references in several cases), some are left for the reader as an exercise, and some will be published in future work [82].

[20] On a side note, some CS readers may be familiar with the use of triangle-like notation like $\rhd$ for *unary* modalities in the context of guarded (co)recursion discussed in Section 7.2. The tradition of using such notation for *binary* operators and connectives such as arithmetical interpretability is much longer and we believe this convention to be more natural.

Table 4.2
List of principles for $\dashv$ and logics considered in this paper.

$\mathsf{N_a}$	$\vdash \phi \to \psi \;\Rightarrow \vdash \phi \dashv \psi$
Tr	$\phi \dashv \psi \to \psi \dashv \chi \to \phi \dashv \chi$
$\mathsf{K_a}$	$\phi \dashv \psi \to \phi \dashv \chi \to \phi \dashv (\psi \wedge \chi)$
$\mathsf{K_a'}$	$\phi \dashv \psi \to (\phi \wedge \chi) \dashv (\psi \wedge \chi)$
$\mathsf{K_a''}$	$\phi \dashv \psi \to \phi \dashv (\psi \to \chi) \to \phi \dashv \chi$
$\mathsf{K_a'''}$	$\phi \dashv (\psi \to \chi) \to (\phi \wedge \psi) \dashv \chi$
BL	$\Box(\phi \to \psi) \to \phi \dashv \psi$
LB	$\phi \dashv \psi \to \Box\phi \to \Box\psi$
Di	$\phi \dashv \chi \to \psi \dashv \chi \to (\phi \vee \psi) \dashv \chi$
$\mathsf{Di'}$	$\phi \dashv \psi \to (\phi \vee \chi) \dashv (\psi \vee \chi)$
$\mathsf{P_a}$	$\phi \dashv \psi \to \Box(\phi \dashv \psi)$
$\mathsf{4_a}$	$\phi \dashv \Box\phi$
$\mathsf{L_a}$	$(\Box\phi \to \phi) \dashv \phi$
$\mathsf{W_a}$	$(\phi \wedge \Box\psi) \dashv \psi \to \phi \dashv \psi$
$\mathsf{W_a'}$	$\phi \dashv \psi \to (\Box\psi \to \phi) \dashv \psi$
$\mathsf{M_a}$	$\phi \dashv \psi \to (\Box\chi \to \phi) \dashv (\Box\chi \to \psi)$
$\mathsf{M_a'}$	$(\phi \wedge \Box\chi) \dashv \psi \to \phi \dashv (\Box\chi \to \psi)$
$\mathsf{S_a}$	$(\phi \to \psi) \to \phi \dashv \psi$
$\mathsf{S_a'}$	$\phi \dashv \psi \to \phi \to \Box\psi$
Box	$\phi \dashv \psi \to \Box(\phi \to \psi)$
$\mathsf{Box'}$	$(\chi \wedge \phi) \dashv \psi \to \chi \dashv (\phi \to \psi)$
$\mathsf{Box''}$	$\phi \dashv \psi \to (\chi \to \phi) \dashv (\chi \to \psi)$
$\mathsf{CB_a}$	$\phi \dashv \psi \to (\phi \to \psi) \vee \phi$
$\mathsf{Lin_a}$	$\phi \dashv \psi \vee \psi \dashv \phi$
$\mathsf{App_a}$	$(\phi \wedge (\phi \dashv \psi)) \dashv \psi$
$\mathsf{C4_a}$	$\Box\phi \dashv \phi$
Hug	$(\phi \to \Box\psi) \to (\phi \dashv \psi)$

Everywhere below, when we write i-$\mathcal{X}^?$, the superscript "?" can be either "−" or nothing, depending whether or not Di is used.

$$\begin{aligned} \mathsf{iA_0} &:= \mathsf{IPC} + \mathsf{N_a} + \mathsf{Tr}, \\ \mathsf{iA}^- &:= \mathsf{iA_0} + \mathsf{K_a}, \\ \mathsf{iA} &:= \mathsf{iA}^- + \mathsf{Di}, \\ \mathsf{i\text{-}GL_a}^? &:= \mathsf{iA}^? + \mathsf{L_a}, \\ \mathsf{i\text{-}GW_a}^? &:= \mathsf{iA}^? + \mathsf{W_a}, \\ \mathsf{i\text{-}PreL}^? &:= \mathsf{i\text{-}GW_a}^? + \mathsf{M_a}. \end{aligned}$$

For each logic i-$\mathcal{X}^?$, i-S$\mathcal{X}^?$ denotes its extension with $\mathsf{S}_\Box$, in particular

$\mathsf{i\text{-}SA} := \mathsf{iA} + \mathsf{S}_\Box$.

Set also:

$\mathsf{i\text{-}PLAA} := \mathsf{i\text{-}SA} + \mathsf{C4_a}$,

$\mathsf{i\text{-}mHC_a} := \mathsf{i\text{-}SA} + \mathsf{CB_a}$,

$\mathsf{i\text{-}KM_a} := \mathsf{i\text{-}mHC_a} + \mathsf{L}_\Box$,

$\mathsf{i\text{-}KM.lin_a} := \mathsf{i\text{-}KM_a} + \mathsf{Lin_a}$,

For each logic i-$\mathcal{X}$, i-Box§ denotes its extension with Box, e.g.,

$\mathsf{i\text{-}BoxA} := \mathsf{iA} + \mathsf{Box}$,

$\mathsf{i\text{-}BoxGL_a} := \mathsf{i\text{-}GL_a} + \mathsf{Box}$.

Note that $\mathsf{i\text{-}BoxGL_a}$ is just a notational variant of $\mathsf{i\text{-}GL}_\Box$. Note also that notation i-Box$\mathcal{X}^-$ would be redundant, Lemma 4.4c. A fortiori, the same applies to extensions of $\mathsf{i\text{-}mHC_a}$ by Lemma 4.16c. Similarly, $\mathsf{i\text{-}PLAA}^-$ would be redundant by Lemma 4.17g. In all these systems, Di can be derived from the remaining axioms. Furthermore, as we will show in Lemma 4.19, $\mathsf{i\text{-}KM.lin_a}$ and $\mathsf{i\text{-}KM.lin}_\Box$ are notational variants of the same system.

For a calculus $\mathcal{X}$ defined by a list of axioms and rules, write $\mathcal{X} \vdash \phi$ to denote deducibility from all substitution instances of axioms/rules in $\mathcal{X}$ plus Modus Ponens. Whenever we have that for any ϕ, $\mathcal{Y} \vdash \phi$ implies $\mathcal{X} \vdash \phi$, we write $\mathcal{X} \vdash \mathcal{Y}$. For $\mathcal{X} \vdash \phi \to \psi$, $\mathsf{iA}_0 \vdash \phi \to \psi$, $\mathsf{iA}^- \vdash \phi \to \psi$ or $\mathsf{iA} \vdash \phi \to \psi$ (see Table 4.2 below), write, respectively, $\phi \vdash_{\mathcal{X}} \psi$, $\phi \vdash_0 \psi$, $\phi \vdash_- \psi$ and $\phi \vdash_\vee \psi$. In other words, we use $\vdash_-$ (and $\vdash_{\mathcal{X}-}$) for derivability without instances of non-IPC schemes involving disjunction (Di or equivalently Di') and $\vdash_0$ for a still more restrictive case when deduction relies on monotonicity only. Correspondingly, interderivability (equivalence) is denoted using, respectively $\dashv\vdash$, $\dashv\vdash_{\mathcal{X}}$, $\dashv\vdash_0$, $\dashv\vdash_-$ and $\dashv\vdash_\vee$. Also, let us abbreviate $\mathcal{X} \vdash \phi \dashv\!\!3\, \psi$, $\mathsf{iA}_0 \vdash \phi \dashv\!\!3\, \psi$, $\mathsf{iA}^- \vdash \phi \dashv\!\!3\, \psi$ or $\mathsf{iA} \vdash \phi \dashv\!\!3\, \psi$ as, respectively, $\phi \models\!\!3_{\mathcal{X}}\, \psi$, $\phi \models\!\!3_0\, \psi$, $\phi \models\!\!3_-\, \psi$ and $\phi \models\!\!3_\vee\, \psi$. Note that even the weakest of these relations, i.e., $\models\!\!3_0$ is transitive and contains $\vdash_0$; in fact, this is precisely essence of the minimal deduction system iA_0. Finally, for deductions in $\Box$-only language, using $\mathsf{i} - \mathsf{K}_\Box$ as the minimal system, one can use similar conventions as above with $\Box$ as subscript (e.g, "$\vdash_\Box$" and "$\dashv\vdash_\Box$").

4.2.1. Axiomatizations for minimal systems

Lemma 4.1.

a. *The principles* $\mathsf{K_a}$, $\mathsf{K_a'}$, $\mathsf{K_a''}$ *and* $\mathsf{K_a'''}$ *are equivalent over* iA_0.
b. *The principles* Di *and* Di' *are equivalent over* iA_0.
c. $\mathsf{iA}_0 \vdash \mathsf{BL}$ *and* $\mathsf{iA}_0 \vdash \mathsf{LB}$.

As noted in existing references (cf., e.g., [54, Chapter 3] or [56, Theorem 2.5]), there is some freedom in the choice of minimal rules:

Fact 4.2. $\mathsf{iA}^- \dashv\vdash \mathsf{IPC} + \mathsf{N}_\Box + \mathsf{K}_\Box + \mathsf{Tr} + \mathsf{K_a}$.

Open Question 4.3. Even in the absence of intuitionistic $\preceq$, the negation-free logic obtained by replacing $\to$ with $\dashv\!\!3$ is a subintuitionistic logic [31,37,27,28]. Is there a good a presentation of the minimal logic of $\dashv\!\!3$ in terms of *fusion* or *dovetailing/fibring* of IPC and this minimal subintuitionistic logic, rather analogous to the logic of bunched implications BI [99,106,107]? Note that the analogy with BI is limited, e.g., both local and global consequence relations of $\dashv\!\!3$ in the absence of $\to$ cease to be *protoalgebraic* [27,28]. ⊞

4.2.2. Collapsing and decomposing $\dashv\!\!3$

In Fact 3.4, we have observed that there are two syntactically similar conditions one can use to enforce brilliancy. Now we can prove syntactically their equivalence, which explains why we used the seemingly weaker one as Box:

Lemma 4.4.

a. $\mathsf{i\text{-}BoxA}^- \dashv\vdash \mathsf{iA} + \mathsf{Box}'$, *i.e.,* Box *and* Box' *are equivalent over* iA^-.
b. $\mathsf{i\text{-}BoxA}^- \dashv\vdash \mathsf{iA} + \mathsf{Box}''$, *i.e.,* Box *and* Box'' *are equivalent over* iA^-.
c. $\mathsf{i\text{-}BoxA}^- \vdash \mathsf{Di}$ *and consequently* $\mathsf{i\text{-}Box}\mathcal{X} \dashv\vdash \mathsf{i\text{-}Box}\mathcal{X}^-$ *for any* $\mathcal{X}$.

Remark 4.5. We presented one possible way to translate a $\dashv\!\!3$-logic i-$\mathcal{X}$ into a $\Box$-logic, to wit to take $\phi \dashv\!\!3\, \psi$ as an abbreviation for $\Box(\phi \to \psi)$. This translation relates i-$\mathcal{X}$ to its extension $\mathsf{i\text{-}Box}\mathcal{X}$, which is term-equivalent to a $\Box$-logic. Another way, studied in detail by Iemhoff and coauthors [54–56], takes the validity of LB as a starting point and translates $\phi \dashv\!\!\!3\, \psi$ as $\Box\phi \to \Box\psi$. A third interpretation of $\dashv\!\!\!3$ in terms of $\Box$ relating i-PLL$_\Box$ and i-PLAA is discussed in Remark 7.2; it builds on a i-PLAA-decomposition of $\dashv\!\!\!3$ in terms of $\to$ provided by Lemma 4.17f. For more on reductions of $\dashv\!\!\!3$ to unary modalities see [82]. $\Box$

We have suggested that the degeneration of $\dashv\!\!\!3$ in the presence of classical laws can be derived syntactically. In fact, this can be obtained as a consequence of an equivalence derivable over the intuitionistic base but, atypically, using disjunction with its Di axiom in an essential way:

Lemma 4.6. *We have:* $\psi \dashv\!\!\!3\, \chi \dashv\vdash_\vee (\psi \vee \neg\psi) \dashv\!\!\!3\, (\psi \to \chi)$.

Nevertheless, as $(\psi \vee \neg\psi) \dashv\!\!\!3\, (\psi \to \chi)$ is parametric in the antecedent of strict implication, it does not seem a satisfying reduction of $\dashv\!\!\!3$ to $\to$. Let us also note in passing that if one adds $\dashv\!\!\!3$ to Johansson's minimal logic instead of IPC, even this transformation does not work anymore. Moreover, there is no one-variable formula $\phi(p)$ in the disjunction-free fragment of the intuitionistic signature s.t. $p \dashv\!\!\!3\, q \dashv\vdash_- \phi(p) \dashv\!\!\!3\, (p \to q)$ and CPC $\vdash \phi(p)$, cf. Example 6.10.

Open Question 4.7. In general, we stick to extensions of $\mathcal{L}$, but let us make a digression concerning a language without all standard connectives. Suppose we define $[\phi]\psi$ as $(\phi \vee \neg\phi) \dashv\!\!\!3\, \psi$. As we saw above, $\phi \dashv\!\!\!3\, \psi$ is equivalent with $[\phi](\phi \to \psi)$. Is there an elegant axiomatization for the minimal fragment of the language with $[\cdot](\cdot)$? It seems richer than the disjunction-free fragment of $\mathcal{L}_{\dashv\!\!\!3}$. $\boxplus$

Corollary 4.8. iA + em $\vdash$ i-BoxA.

Remark 4.9. This is one of very few places in this section where we need full iA rather than iA$^-$, i.e., where Di is used in an essential way. This happens for a very good reason: it is not possible to derive i-BoxA from iA$^-$ + em. One can see this, e.g., by considering the interpretation of $\phi \dashv\!\!\!3\, \psi$ as $\Box\phi \to \Box\psi$. $\Box$

In Appendix B we will explain that the logic ILM of Π^0_1-conservativity and interpretability corresponds to c-PreL := i-PreL$^-$ + em. This provides a proof that even c-PreL does not extend i-BoxA. The proof may use either the arithmetical interpretation or the Veltman semantics used for ILM.

We will discuss collapsing and decomposing further in a later paper [82]; see also remarks preceding Theorem 6.6.

4.2.3. Derivations between arithmetical principles

We turn our attention to derivations between principles of central importance, especially from the point of view of arithmetical interpretations.

Lemma 4.10. *We have:*

a. i-SA$^-$ $\dashv\vdash$ iA$^-$+ S$_a$, *i.e., over* iA$^-$, *the principles* S$_\Box$ *and* S$_a$ *are equivalent.*
b. i-SA$^-$ $\dashv\vdash$ iA$^-$+ S$'_a$, *i.e., over* iA$^-$, *the principles* S$_\Box$ *and* S$'_a$ *are equivalent.*
c. *In the presence of* S$_a$, N$_a$ *is derivable using just Modus Ponens.*

Hence, axiomatizations of "strength" in terms of $\Box$ and in terms of $\prec$ yield the same logic over iA^-. As we are going to see below, this is a relatively rare phenomenon. Still, many well-known modal derivations can be easily translated into the $\prec$-setting, e.g., a derivation of $\mathsf{4}$ from the Löb axiom:

Lemma 4.11. $\mathsf{i\text{-}GL}_\mathsf{a}^- \dashv\vdash \mathsf{iA}^- + \mathsf{L}_\Box + \mathsf{4}_\mathsf{a}$. *It follows that, over* $\mathsf{iA}^- + \mathsf{4}_\mathsf{a}$, *the principles* $\mathsf{L}_\Box$ *and* L_a *are interderivable.*

Lemma 4.12.

a. $\mathsf{iA}^- + \mathsf{4}_\mathsf{a} \vdash \mathsf{4}_\Box$.
b. $\mathsf{i\text{-}SA}^- \vdash \mathsf{4}_\mathsf{a}$.
c. $\mathsf{i\text{-}SA}^- \vdash \mathsf{P}_\mathsf{a}$.
d. $\mathsf{i\text{-}SA}^- + \mathsf{L}_\Box \vdash \mathsf{i\text{-}GL}_\mathsf{a}^-$.

Lemma 4.13 (*[56, Cor. 2.6 and 2.7]*). W_a *and* W'_a *are equivalent over* iA_0^-. *Similarly,* M_a *and* M'_a *are equivalent over* iA_0^-.

Lemma 4.14. *We have:*

a. $\mathsf{i\text{-}GW}_\mathsf{a}^- \vdash \mathsf{i\text{-}GL}_\mathsf{a}^-$.
b. $\mathsf{i\text{-}GL}_\mathsf{a}^- + \mathsf{M}_\mathsf{a} \vdash \mathsf{W}_\mathsf{a}$. *In other words,* $\mathsf{i\text{-}PreL}^- \dashv\vdash \mathsf{i\text{-}GL}_\mathsf{a}^- + \mathsf{M}_\mathsf{a}$.
c. $\mathsf{i\text{-}GL}_\mathsf{a}^- + \mathsf{P}_\mathsf{a} \vdash \mathsf{W}_\mathsf{a}$.
d. $\mathsf{iA}^- + \mathsf{M}_\mathsf{a} \vdash (\Box\phi \prec \psi) \to \Box(\Box\phi \to \psi)$.

Examples 6.7–6.9 illustrate that clauses (a) and (b) cannot be reversed.

Lemma 4.15. $\mathsf{iA} + \mathsf{4}_\mathsf{a} \vdash \mathsf{Lei}$.

This implies that the logics $\mathsf{i\text{-}GL}_\mathsf{a}$, $\mathsf{i\text{-}GW}_\mathsf{a}$ and $\mathsf{i\text{-}PreL}$ are not conservative over $\mathsf{i\text{-}GL}_\Box$. Both $\mathsf{i\text{-}GL}_\mathsf{a}$ and $\mathsf{i\text{-}GW}_\mathsf{a}$ are conservative over $\mathsf{i\text{-}GL}_\Box + \mathsf{Lei}$ [56].

4.2.4. More derivations

Derivations discussed in the remainder of this section are mostly of importance in Section 7, although, e.g., Lemma 4.16a will be also relevant in Section 5.4.4:

Lemma 4.16. *We have:*

a. $\mathsf{i\text{-}K}_\Box^- + \mathsf{CB}_\Box \dashv\vdash_\Box \mathsf{i\text{-}K}_\Box^- + \mathsf{CB}'_\Box$
b. $\mathsf{iA}^- + \mathsf{CB}_\mathsf{a} \vdash \mathsf{CB}_\Box$.
c. $\mathsf{i\text{-}mHC}_\mathsf{a}^- \vdash \mathsf{Box}$ *and consequently* $\mathsf{i\text{-}mHC}_\mathsf{a}^- \vdash \mathsf{Di}$.
d. $\mathsf{i\text{-}mHC}_\mathsf{a} \vdash \mathsf{M}_\mathsf{a}$.
e. $\mathsf{i\text{-}KM}_\mathsf{a} \vdash \mathsf{W}_\mathsf{a}$.

Clause (c) implies that the notation "$\mathsf{i\text{-}mHC}_\mathsf{a}^-$" is redundant. Example 6.11 illustrates that clause (b) cannot be reversed.

Lemma 4.17. *We have:*

a. $\mathsf{iA}^- + \mathsf{C4}_\mathsf{a} \vdash \mathsf{C4}_\Box$.

b. $\mathsf{iA}^- + \mathsf{App_a} \vdash \mathsf{C4_a}$.
c. $\mathsf{iA}^- + \mathsf{Hug} \vdash \mathsf{C4_a}$.
d. $\mathsf{i\text{-}PLAA}^- \vdash \mathsf{App_a}$.
e. $\mathsf{i\text{-}PLAA}^- \vdash \mathsf{Hug}$.
f. $\phi \dashv\!\!\!3\ \psi \dashv\vdash_{\mathsf{i-PLAA}^-} \phi \to \Box\psi$
g. $\mathsf{i\text{-}PLAA}^- \vdash \mathsf{Di}$.

Example 6.13 illustrates irreversibility of several clauses in this lemma. So far, principles involving $\dashv$ tended to be stronger than their relatives formulated in $\mathcal{L}_\Box$. It is indeed quite often but not always the case. For example, in the case of "semi-linearity" axioms, the situation is reversed:

Fact 4.18. *We have:*

- $\mathsf{iA}^- + \mathsf{Lin}_\Box \vdash \mathsf{Lin_a}$.
- $\mathsf{i\text{-}BoxA} + \mathsf{Lin_a} \vdash \mathsf{Lin}_\Box$, *in particular* $\mathsf{i\text{-}mHC_a} + \mathsf{Lin_a} \vdash \mathsf{Lin}_\Box$.

It is hard to make $\mathsf{Lin_a}$ and $\mathsf{Lin}_\Box$ coincide in the absence of Box, cf. Example 6.12. Nevertheless, here is an important exception, used in Section 7.2:

Lemma 4.19. *We have:*

a. $(\Box\phi \to \Box\psi) \to \Box(\phi \to \psi) \vdash_0 \mathsf{Box}$.
b. $\mathsf{i\text{-}KM.lin}_\Box \vdash (\Box\phi \to \Box\psi) \to \Box(\phi \to \psi)$.
c. $\mathsf{i\text{-}KM.lin}_\Box \dashv\vdash \mathsf{i\text{-}KM.lin_a}$. *That is, not only both systems are notational variants of each other, but the* Box *axiom can be derived from* $\mathsf{i\text{-}KM.lin}_\Box$.

5. Arithmetical interpretations: provability and preservativity ○

In Section 6 we continue the discussion of the modal side of our calculi. But now, we cannot postpone any further the presentation of our original motivation for studying constructive $\dashv$ and a number of its axiom systems in terms of Σ^0_1-*preservativity* for an arithmetical theory T:

- $A \dashv_T B$ if, for all Σ^0_1-sentences S, if $T \vdash S \to A$, then $T \vdash S \to B$.

In order to provide a framework for such interpretations of modal connectives, we introduce the notion of a *schematic logic*. This notion can be given a very general treatment. However, for the purposes of this paper, we will restrict ourselves to the case of arithmetical theories, studying propositional logics of theories (Section 5.2), provability logics (Section 5.3) and our true target: preservativity logics (Section 5.4). For an instructive contrast, we provide some extra information about logics for Π^0_1-conservativity and interpretability in Appendices B and C.

5.1. Schematic logics

An arithmetical theory T is, for the purposes of this paper, an extension of i-EA, the intuitionistic version of Elementary Arithmetic, in the arithmetical language.[21] We demand that the axiom set of T is given by a $\Delta_0(\mathsf{exp})$-formula.

[21] The classical theory EA is $\mathrm{I}\Delta_0 + \mathsf{Exp}$. This theory consists of the basic axioms for zero, successor, addition, multiplication plus Δ_0-induction plus the axiom that states that exponentiation is total. The theory i-EA is the same theory only with constructive logic as underlying logic. The theory proves the decidability of $\Delta_0(\mathsf{exp})$-formulas. Some basic information about constructive arithmetic can be found in [38,121,123].

Let $\mathcal{L}_{\circledcirc_0,\ldots,\circledcirc_{k-1}}$ be the language extending $\mathcal{L}$ with operators $\circledcirc_0, \ldots, \circledcirc_{k-1}$, where $\circledcirc_i$ has arity n_i. Let a function F be given that assigns to every $\circledcirc_i$ an arithmetical formula $A(v_0, \ldots, v_{n_i-1})$, where all free variables are among the variables shown. We write $\circledcirc_{i,F}(B_0, \ldots, B_{n_i-1})$ for $F(\circledcirc_i)(\underline{\ulcorner B_0 \urcorner}, \ldots, \underline{\ulcorner B_{n_i-1} \urcorner})$. Here $\underline{\ulcorner C \urcorner}$ is the numeral of the Gödel number of C. Suppose f is a mapping from the propositional atoms to arithmetical sentences. We define $(\phi)^f_F$ as follows:

- $(p)^f_F := f(p)$
- $(\cdot)^f_F$ commutes with the propositional connectives
- $(\circledcirc_i(\phi_0, \ldots, \phi_{n_i-1}))^f_F := \circledcirc_{i,F}((\phi_0)^f_F, \ldots, (\phi_{n_i-1})^f_F)$.

Let T be an arithmetical theory. We say that a modal formula in our given signature is T-*valid* w.r.t. F if, for all assignments f of arithmetical sentences to the propositional atoms, we have $T \vdash (\phi)^f_F$. We write $\Lambda_{T,F}$ for the set of modal formulas that are T-valid w.r.t. F. Of course, we will focus exclusively on "natural" F yielding well-behaved $\Lambda_{T,F}$ with interesting properties.

5.2. *Propositional logics of a theory*

Let us first consider the case where our finite set of modal operators is empty. If T is consistent and classical, then $\Lambda_T := \Lambda_{T,\emptyset}$ is, trivially, precisely CPC and if T is Heyting Arithmetic (HA), then Λ_T has the *de Jongh property*: $\Lambda_T = \mathsf{IPC}$.

There are theories for which Λ_T is an intermediate logic strictly between IPC and CPC. De Jongh, Verbrugge & Visser [34] show that whenever Θ is an intermediate logic with the finite frame property (cf. Section 6) and U is the result of extending HA with all axioms of Θ as schemes, $\Lambda_U = \Theta$.

For some theories like *Markov's Arithmetic* $\mathsf{MA} = \mathsf{HA} + \mathsf{MP} + \mathsf{ECT}_0$, where MP is the *Markov's Principle* ([123, 4.5, p. 203], [121, 1.11.5, p. 93]):

$$(\forall x(Ax \vee \neg Ax) \wedge \neg\neg\exists x\, Ax) \to \exists x\, Ax$$

and ECT_0 is the *Extended Church's Thesis* (cf. Appendix A), the characterization of the set of valid principles is an open problem connected to the question of the propositional logic of realizability. See e.g. [102, § 13]. For more on intuitionistic schematic logics see [5,34,102,113,131].

5.3. *Provability logic*

Next we consider the extension of propositional logic with a unary modal operator $\Box$. It allows numerous interesting arithmetical interpretations, but at this point we focus on the interpretation of $\Box$ as provability. Consider any arithmetical theory T. We assume that T comes equipped with a $\Delta_0(\mathsf{exp})$-predicate α_T that gives the codes of its axiom set. Let provability in T be arithmetized by prov_T. We note that T really occurs in the guise of α_T. We set $\mathsf{F}_{0,T}(\Box) := \mathsf{prov}_\mathsf{T}(v_0)$. Let $\Lambda^*_T := \Lambda_{T,\mathsf{F}_{0,T}}$. Intuitionistic Löb's Logic $\mathsf{i\text{-}GL}_\Box$ is contained in all Λ^*_T, where T is an arithmetical theory in the sense of this paper. This insight is due to Löb [83].[22]

[22] Three remarks are in order. The fact that Löb's Principle follows from Löb's work was noted by Leon Henkin who was the referee of Löb's paper. Secondly, Löb's proof of Löb's Principle is fully constructive and goes through even in constructive versions of S^1_2. Thirdly, Kripke's proof of Löb's Principle from the Second Incompleteness Theorem is not constructive—even if we give the Second Incompleteness Theorem the form: if a theory proves its own consistency then it is inconsistent.

Remark 5.1. In many treatments of intuitionistic modal logic the interdefinability of $\Box$ and $\Diamond$ fails and $\Box$ and $\Diamond$ both are treated as primitive operations. This is not so in the context of provability logic for constructive theories and its extensions. Here the connective $\Diamond$ is always defined as $\neg\Box\neg$. Thus, $\Box$, which signals the existence of a proof, is the positive notion and $\Diamond$ is the negative less informative notion. We note that $\neg\Diamond\neg$ is equivalent to $\neg\neg\Box\neg\neg$ which, in the context of theories like HA, is certainly weaker than $\Box$. □

Remark 5.2. One of the first global insights into schematic theories is due to George Gargov [44]: they inherit the disjunction property from the underlying arithmetic theory. Thus, if an extension of i-EA has the disjunction property, then so has its provability logic.[23] □

The theory $\mathsf{c\text{-}GL}_{\Box}$ is obtained by extending $\mathsf{i\text{-}GL}_{\Box}$ with classical logic. If T is a Σ^1_0-sound classical theory, then $\Lambda^*_T = \mathsf{c\text{-}GL}_{\Box}$. This insight is due to Solovay [115]. In contrast, the logic $\mathsf{i\text{-}GL}_{\Box}$ is not complete for HA. The system for preservativity logic $\mathsf{i\text{-}PreL} + \mathsf{V}$ discussed in Section 5.4 derives many more arithmetically valid principles for the provability logic of HA underivable in $\mathsf{i\text{-}GL}_{\Box}$, e.g.,

- $\Box\neg\neg\Box\phi \to \Box\Box\phi$.
- $\Box(\neg\neg\Box\phi \to \Box\phi) \to \Box\Box\phi$
- $\Box(\phi \vee \psi) \to \Box(\phi \vee \Box\psi)$. ($\mathsf{Lei}$).

We note that the first principle is a consequence of classical $\mathsf{c\text{-}GL}_{\Box}$, but the second and third are not. This illustrates that $\Lambda^*_{(\cdot)}$ is not monotonic. To make this understandable, the reader may note that we both change the theory and the interpretation of the modal operator.

Note that substituting $\Box\bot$ for ϕ and $\neg\Box\bot$ for ψ in Lei yields

$$\begin{aligned} \vdash \Box(\Box\bot \vee \neg\Box\bot) &\to \Box(\Box\bot \vee \Box\neg\Box\bot) \\ &\to \Box(\Box\bot \vee \Box\bot) \\ &\to \Box\Box\bot. \end{aligned}$$

Hence, adding Lei to $\mathsf{c\text{-}GL}_{\Box}$ yields $\Box\Box\bot$, i.e., Leivant's principle is 'weakly inconsistent' with classical logic over $\mathsf{i\text{-}GL}_{\Box}$.

We write $\Lambda \boxplus \Lambda'$ for the closure of $\Lambda \cup \Lambda'$ under modus ponens. Our insight above yields: $\mathsf{c\text{-}GL}_{\Box} + \Box\Box\bot \subseteq \Lambda^*_{\mathsf{HA}} \boxplus \Lambda^*_{\mathsf{PA}}$. In fact, by Theorem 5.3, we have: $\Lambda^*_{\mathsf{HA}} \boxplus \Lambda^*_{\mathsf{PA}} = \mathsf{c\text{-}GL}_{\Box} + \Box\Box\bot$.

Theorem 5.3 (*Silly Upperbound*)**.** *We have:*

$$(\Box\Box\bot \to \neg\neg\Box\bot) \notin \Lambda^*_T \quad \textit{iff} \quad \Lambda^*_T \subseteq \mathsf{c\text{-}GL}_{\Box} + \Box\Box\bot.$$

Our proof presupposes knowledge of the proof of Solovay's Theorem. The proof can be skipped since nothing but the Silly Upperbound rests on it.

Proof. "$\to$" Suppose $\phi \in \Lambda^*_T$ and $\mathsf{c\text{-}GL}_{\Box} + \Box\Box\bot \nvdash \phi$. Then there is a counter Kripke model of depth 2 to ϕ, say with nodes $0, \ldots, n-1$ and root 0. We have $i \sqsubset j$ iff $i = 0$ and $j > 0$. Let T^+ be T plus $\Box_T\Box_T\bot$ plus *sentential excluded third*. We work in T^+. We define a Solovay function in the usual way:

[23] Interestingly, Gargov's argument itself uses classical logic.

- $h0 := 0$
- $h(p+1) := \begin{cases} i & \text{if } h(p) \sqsubset i \text{ and } \mathsf{proof}_T(p, \ulcorner \ell \neq i \urcorner) \\ h(p) & \text{otherwise} \end{cases}$
  Here ℓ is the limit of h.

We note that, since $\Box_T \Box_T \bot$ we have $\Box_T \bigvee_{0<j<n} \exists x\, hx = j$. This tells us that inside the box, we can indeed prove that the limit exists. Moreover we have excluded third for sentences of the form $\ell = i$. Outside the box we can also prove the existence of the limit by sentential excluded third. Using these two observations we can execute the usual Solovay argument. This gives us $\Box_T \bot$ and we may conclude that $T \vdash \Box_T \Box_T \bot \to \neg\neg \Box_T \bot$.
"$\leftarrow$" Suppose $(\Box\Box\bot \to \neg\neg\Box\bot) \notin \Lambda^*_T$ and $\Lambda^*_T \subseteq \mathsf{c\text{-}GL}_\Box + \Box\Box\bot$. Then it would follow that $\mathsf{c\text{-}GL}_\Box \vdash \Box\Box\bot \to \Box\bot$. Quod non. □

Note that $(\Box\Box\bot \to \neg\neg\Box\bot) \in \Lambda^*_T$ if T is one of HA, $\mathsf{HA} + \mathsf{MP}$, $\mathsf{HA} + \mathsf{ECT}_0$. The situation is different for $T = \mathsf{HA}^*$ (cf. Section 5.4.4).

We formulate the main question of constructive provability logic.

Open Question 5.4. What is the provability logic of HA? Is it decidable? We note that the logic is *prima facie* Π^0_2. ⊞

The basic information about classical provability logic can be found in [6,24,25,50,58,78,114, 119]. For information about intuitionistic provability logic, see e.g. [5,53,54,129,135].

5.4. Preservativity logic

As stated above, Σ^0_1-preservativity [55,56,127,129,132] for a theory T is defined as follows:

- $A \dashv_T B$ if, for all Σ^0_1-sentences S, if $T \vdash S \to A$, then $T \vdash S \to B$.

In contrast to Π^0_1-conservativity and interpretability (see Appendices B and C), defining Σ^0_1-preservativity does not require an inter-theory notion $T \dashv U$.

We give a characterization of Σ^0_1-preservativity that is analogous to the Orey–Hájek characterization for interpretability over PA. Suppose T extends HA. We write $\Box_{T,n}$ for the arithmetization of provability from the axioms of T with Gödel number $\leq n$. The theory T is, HA-verifiably, essentially reflexive: for all n and A, we have $T \vdash \Box_{T,n} A \to A$. Here we allow parameters in the formulation of the reflection principle.[24]

Theorem 5.5. *Suppose T is an extension of HA. Then, $A \dashv_T B$ iff, for all n, $T \vdash \Box_{T,n} A \to B$. This result is verifiable in i-EA.*

Proof. "$\to$" Suppose $A \dashv_T B$. It follows that (a) if $T \vdash \Box_{T,n} A \to A$, then (b) $T \vdash \Box_{T,n} A \to B$. Now note that (a) follows from essential reflexivity.

[24] This result is folklore. We could not locate a fully worked-out proof in the literature. Some ingredients can be found in [121, Part I, §5], but the treatment of these ingredients contains some gaps. The proof looks as follows. The theory HA verifies cut-elimination for predicate logic. Consider any n. Reason in T. Suppose $\Box_{T,n} A$. Let p be a cut-free witness of $\Box_{T,n} A$. All formulas occurring in p will have complexity $\leq m$, for some standard m. Here our complexity measure is *depth of logical connectives*. We can develop a partial satisfaction predicate for formulas of complexity $\leq m$ that HA-verifiably satisfies the commutation conditions. The standard axioms of T that have Gödel number $\leq n$ are true (in the sense of our satisfaction predicate), since the Tarski bi-conditionals are derivable. By induction, we can show that all m-derivations from true axioms yield true conclusions. So, *a fortiori*, we have A.

"$\leftarrow$" Suppose (c) for all n, $T \vdash \Box_{T,n} A \to B$ and (d) $T \vdash S \to A$. From (d), we have, for some m, that $T \vdash \Box_{T,m}(S \to A)$. We choose m so large that the finite axiomatization of i-EA has Gödel number $\leq m$. By i-EA verifiable Σ^0_1-completeness of extensions of i-EA, $T \vdash S \to \Box_{T,m} A$. Hence, by (c), $T \vdash S \to B$.

The verifiability in i-EA can be seen by inspection of the above proof. $\Box$

5.4.1. i-PreL$^-$ and i-PreL

We note that $\top \dashv_{\mathsf{HA}} A$ is i-EA-provably equivalent to $\Box_{\mathsf{HA}} A$. This means that in our study of Σ^0_1-preservativity logic of arithmetical theories, we can treat $\Box$ as a defined connective and focus on $\mathcal{L}_{\dashv}$. Let $\mathsf{F}_{1,T}(\dashv) := \mathsf{p}_T(v_0, v_1)$, where p_T is a good arithmetization of the relation $A \dashv_T B$. We define $\Lambda^\circ_T := \Lambda_{T,\mathsf{F}_{1,T}}$.

The principles of i-PreL$^-$ are arithmetically valid for all arithmetical theories in our sense, to wit all $\Delta_0(\mathsf{exp})$-axiomatized theories in the arithmetical language extending i-EA. However, i-PreL $=$ i-PreL$^-$ $+$ Di is valid for a more restricted group. We need the notion of *closure under q-realizability*: $T \vdash A$ implies there is an n such that $T \vdash \underline{n} \cdot \varepsilon \downarrow \wedge \underline{n} \cdot \varepsilon \widetilde{q} A$ (cf. Appendix A for notation). A sufficient condition for Di to hold is that T is not only closed under q-realizability, but it also verifies this fact.

Theorem 5.6. *Suppose T is T-provably closed under q-realizability. Then Di is arithmetically valid in T, i.e., Di is in Λ°_T.*

Proof. We write $\dashv$ for $\dashv_T$ and $\Box$ for $\Box_T$.

Suppose T is T-provably closed under q-realizability. We reason in T. Suppose (a) $A \dashv C$ and (b) $B \dashv C$. Suppose $\Box(S \to (A \vee B))$, where S is a Σ^0_1-sentence. By q-realizability and the fact that Σ^0_1-sentences are self-realizing, we can find a recursive index e of a 0-ary, 0, 1-valued function, such that (c) $\Box(S \to e \cdot \varepsilon \downarrow)$, (d) $\Box((e \cdot \varepsilon = 0 \wedge S) \to A)$ and (e) $\Box((e \cdot \varepsilon = 1 \wedge S) \to B)$. From (a) and (d), we get: (f) $\Box((e \cdot \varepsilon = 0 \wedge S) \to C)$. From (b) and (e) we get (g) $\Box((e \cdot \varepsilon = 1 \wedge S) \to C)$. From (c), (f) and (g), we obtain $\Box(S \to C)$. $\Box$

The following salient theories T are T-provably (even i-EA-provably) closed under q-realizability: HA, HA $+$ MP, HA $+$ ECT_0, MA $=$ HA $+$ MP $+$ ECT_0 and HA* (see Section 5.4.4), hence i-PreL is arithmetically valid in them.

Open Question 5.7. It would be interesting to have a more perspicuous condition for the satisfaction of Di than closure under q-realizability.

Moreover, in many cases we can also prove Di using the de Jongh translation. Are there separating examples where either q-realizability works and the de Jongh translation does not or where the de Jongh translation works but q-realizability does not? ⊞

5.4.2. The preservativity logic of HA

The logic i-PreL is incomplete for HA [55,129]. Define:

- $(\chi)(\sigma) := \sigma$ for $\sigma ::= \top \mid \bot \mid (\top \dashv \phi) \mid (\sigma \vee \sigma)$, where ϕ ranges over the full language.
- $(\chi)(\phi \wedge \psi) := ((\chi)(\phi) \wedge (\chi)(\psi))$,
- $(\chi)(\phi) := (\chi \to \phi)$ in all other cases.

The following principle is arithmetically valid over HA.

V For $\chi := \bigwedge_{i<n}(\phi_i \to \psi_i)$, we have:
$\vdash (\chi \to (\phi_n \vee \phi_{n+1})) \dashv \bigvee_{j<n+2}(\chi)(\phi_j)$.

An example of a consequence of V is as follows. Consider any non-modal propositional formula $\phi(p)$ with at most p free. Suppose that $\phi(p)$ is not constructively valid. Then, the principle $\phi(\Box\psi) \dashv (\Box\psi \vee \neg\Box\psi)$ is arithmetically valid over HA.

Remark 5.8. We have the following salient result about the admissible rules of HA. Suppose ϕ and ψ are non-modal propositional formulas. Define:

- $\phi \;\; \vdash\!\!\!\sim_{\mathsf{HA}} \psi$ if for all arithmetical substitutions σ we have:
  $\mathsf{HA} \vdash \sigma(\phi) \;\; \Rightarrow \;\; \mathsf{HA} \vdash \sigma(\psi)$.

The following are equivalent:
(i) $\phi \;\; \vdash\!\!\!\sim_{\mathsf{HA}} \psi$, (ii) i-PreL + V $\vdash \phi \dashv \psi$, (iii) i-PreL + V $\vdash \Box\phi \to \Box\psi$.
See [55] in combination with [129]. □

Is i-PreL + V the preservativity logic of HA? We do not think so. The second author has discovered a valid scheme that does not appear to be derivable from i-PreL + V. To save space, we postpone a detailed discussion to future work.

Open Question 5.9. Here is a list of more open problems.

I. Is i-PreL$^-$ the preservativity logic of all extensions of i-EA?
In other words, is i-PreL$^-$ the intersection of all Λ°_T, where T is an arithmetical extension of i-EA?

II. Is i-PreL$^-$ the preservativity logic of all extensions of HA?

III. Is there an extension T of i-EA such that $\Lambda^\circ_T =$ i-PreL$^-$?

IV. Is there an extension T of HA such that $\Lambda^\circ_T =$ i-PreL$^-$?

V. Is there an extension T of i-EA such that $\Lambda^\circ_T =$ i-PreL?

VI. Is there an extension T of HA such that $\Lambda^\circ_T =$ i-PreL?

VII. What is the preservativity logic of HA?

VIII. What is the preservativity logic of HA + MP?

IX. What is the preservativity logic of HA + ECT_0?

The questions VII, VIII, IX are obviously quite difficult. [a] As far as we know nobody has seriously worked on questions I–VI. ⊞

[a] They could be easier than the question what the provability logic of HA, HA + MP or HA + ECT_0 is. Sometimes theories in a richer language are easier to manage.

5.4.3. The preservativity logic of classical theories

We know a lot about the preservativity logic of classical theories, since $\dashv_T$ can be intertranslated with Π^0_1-conservativity $\blacktriangleright_T$ in the classical case. As a consequence we can translate what we know about the logic of Π^0_1-conservativity to a result about preservativity logic. Let c-PreL := i-PreL^- + em.

Theorem 5.10. *Suppose that T is Σ^0_1-sound classical theory that extends $\mathrm{I}\Pi_1^- + \mathsf{Exp}$. Then, the preservativity logic of T is precisely* c-PreL.

This result is a translation of Theorem 12 of [14], which is a strengthening of the main result of [49], the latter in turn being an adaptation of [108] and [16]. For details see Appendix B.

5.4.4. HA^* *and* PA^*

The Completeness Principle for a theory T is defined as

$$\mathsf{CP}_T \quad A \to \Box_T A.$$

Here A is allowed to contain parameters. Consider any theory T such that T is $\mathsf{HA} + \mathsf{CP}_T$. Such a theory is easily constructed by the Fixed Point Lemma. One can show that, if HA verifies that T is $\mathsf{HA} + \mathsf{CP}_T$, then T is unique modulo provable equivalence. Thus, the following definition is justified: HA^* is the unique theory such that, HA-verifiably, HA^* is $\mathsf{HA} + \mathsf{CP}_{\mathsf{HA}^*}$. The theory HA^* was introduced and studied in [126].

We have a second way of access to HA^* via a variant of Gödel's translation of IPC in $\mathsf{S4}$. We define:

- $A^{\mathsf{g}} := A$ if A is atomic.
- $(\cdot)^{\mathsf{g}}$ commutes with $\wedge$, $\vee$ and $\exists$.
- $(B \to C)^{\mathsf{g}} := ((B^{\mathsf{g}} \to C^{\mathsf{g}}) \wedge \Box_{\mathsf{HA}}(B^{\mathsf{g}} \to C^{\mathsf{g}}))$.
- $(\forall x\, B)^{\mathsf{g}} := (\forall x\, B^{\mathsf{g}} \wedge \Box_{\mathsf{HA}} \forall x\, B^{\mathsf{g}})$.

We have $\mathsf{HA}^* \vdash A$ iff $\mathsf{HA} \vdash A^{\mathsf{g}}$. Using the translation $(\cdot)^{\mathsf{g}}$ on can show that HA^* is conservative over HA with respect to formulas that have only Σ_1-formulas as antecedents of implications.

The theory HA^* is the theory in which the incompleteness phenomena lie most closely to the logical surface. We have the strong form of Löb's Principle $\mathsf{HA}^* \vdash (\Box_{\mathsf{HA}^*} A \to A) \to A$. Note that $\mathsf{HA}^* \vdash \neg\neg\, \Box_{\mathsf{HA}^*} \bot$ is a special case. We are inclined to read this principle as: inconsistency can never be excluded.

If we extend PA to $U = \mathsf{PA} + \mathsf{CP}_U$, we end up with the inglorious $U \vdash \Box_U \bot$. However, HA^* is conservative over HA for a wide class of formulas. So, the Completeness Principle is an example of a kind of extension that makes no real sense in the classical case.

The theory HA^* can be used to provide easy proofs of the independence of KLS (Kreisel–Lacombe–Schoenfield) and MS (Myhill–Shepherdson) from HA [126], simplifying the original ones by Beeson [13] while preserving their basic idea.

De Jongh and Visser showed that every prime recursively enumerable Heyting algebra on finitely many generators can be embedded in the Heyting algebra of HA^*. See [35]. Their proof is an adaptation of a proof by Shavrukov [109] in the simplified form due to Zambella [143] concerning the embeddability of Magari algebras in the Magari algebra of Peano Arithmetic.

A consequence of the De Jongh–Visser result is the fact that the admissible propositional rules for HA^* are precisely the derivable rules. In contrast, the admissible propositional rules for HA are the same as the admissible rules for IPC: this is the maximal set of admissible rules that is possible for a theory with the de Jongh property. Thus among theories with the de Jongh property both the minimal possible set of admissible rules and the maximal one are exemplified. See also [131].

We want to show that HA^* is HA-verifiably closed under q-realizability. The easiest route is via the notion of self-q-realizability. A formula $A(\vec{x})$ (with all free variables shown) is *self-q-realizing* if there is a number s^A such that $\mathsf{HA} \vdash A(\vec{x}) \to (\underline{s^A} \cdot (\vec{x}))\widetilde{q}A(\vec{x})$, cf. Appendix A for notation.

A substantial class of i-EA-verifiably self-q-realizing formulas is the class of *auto-q formulas* given as follows. Let S range over all Σ^0_1-formulas, let A range over all formulas and let v range over all variables:

- $B ::= S \mid (B \wedge B) \mid \forall v\, B \mid (A \to B)$.

We note that the class of auto-q formulas substantially extends the almost negative formulas that are self-r-realizing.

The instances of the completeness scheme have the form $\forall \vec{x}\,(A(\vec{x}) \to S(\vec{x}))$, where S is Σ^0_1. Thus, these instances are auto-q. It follows that HA^* is HA-verifiably closed under q-realizability. Thus, $\mathsf{i\text{-}PreL}^* := \mathsf{iA} + \mathsf{SL}_\Box + \mathsf{M_a}$ is contained in the preservativity logic of HA^*, to wit $\Lambda^\circ_{\mathsf{HA}^*}$. There are examples of valid principles that are most probably not in $\mathsf{i\text{-}PreL}^*$. We do not know whether this has any traces in the provability logic of HA^*. As will be explained in Remark C.3, there is a certain analogy between $\mathsf{HA} + \mathsf{CT}_0!$ and HA^*.

We turn to the theory PA^*, axiomatized by the set α of all sentences A such that $\mathsf{PA} \vdash A^{\mathsf{g}}$. One can easily show that α is closed under deduction and that PA^* satisfies $\mathsf{CP}_{\mathsf{HA}^*}$.[25] The theory PA^* verifies the Trace Principle :

$$\mathsf{TP} \quad \Box_{\mathsf{PA}^*}\forall x\,(Ax \to Bx) \to (\exists x\, Ax \vee \forall x\,(Ax \to Bx)).$$

This principle is equivalent to

$$\Box_{\mathsf{PA}^*}\forall x\, Bx \to (\exists x\, Ax \vee \forall x\,(Ax \to Bx)).$$

The presence of the trace principle has as a modal consequence the principle

$$\mathsf{CB}_\Box \quad \Box\phi \to ((\psi \to \phi) \vee \psi).$$

In [126], it is shown that the logic $\mathsf{i\text{-}KM}_\Box$ is precisely the provability logic of PA^*.[26] We remind the reader that:

$$\mathsf{i\text{-}KM}_\Box = \mathsf{i\text{-}SL}_\Box + \mathsf{CB}_\Box = \mathsf{i\text{-}GL}_\Box + \mathsf{S}_\Box + \mathsf{CB}_\Box.$$

The preservativity logic of PA^* contains $\mathsf{i\text{-}PreL}^-$ and $\mathsf{S_a}$, but neither Di nor $\mathsf{CB_a}$ (Section 8.2).

6. Kripke completeness and correspondence

Apart from being our original motivation to study $\dashv$, the arithmetical interpretation can occasionally complement the deductive systems proposed in Section 4 by providing a route to *disprove* certain judgements of the form $\mathcal{X} \vdash \phi$, i.e., to show *non*-derivability from suitable sets of axioms (namely, those valid in some arithmetical interpretations):

Example 6.1. Interpreting $\phi \dashv \psi$ as $\Box(\phi \to \psi)$ over HA yields $\mathsf{iA} + \mathsf{L}_\Box + \mathsf{M_a}$. This interpretation refutes $\mathsf{4_a}$, $\mathsf{L_a}$ and *a fortiori* $\mathsf{W_a}$. It follows that $\mathsf{4_a}$ is really needed in Lemma 4.14b above to derive $\mathsf{W_a}$. □

[25] We have demanded that the axiom set of a theory is $\Delta_0(\mathsf{exp})$. The axioms of PA^* do not satisfy this demand. So, the official axiom set should be a suitable $\Delta_0(\mathsf{exp})$-set manufactured from α using a version of Craig's trick.

[26] In [126] the equivalent form $\mathsf{CB}'_\Box$ is used, cf. Lemma 4.16.

To disprove more such judgements, we need to return to relational insights of Section 3 and provide Kripke completeness and correspondence results. Most of this section is based on work we will discuss in a parallel publication [82].

6.1. Notions of completeness

Given a logic i-$\mathcal{X}$, set $\mathsf{Fram}(\text{i-}\mathcal{X}) := \{\mathcal{F} \mid \text{for any } V, \langle \mathcal{F}, V\rangle \Vdash \text{i-}\mathcal{X}\}$. Say that i-$\mathcal{X}$ is (*weakly*) *complete for* (or *with respect to*) a class of frames K if it is

- *sound* wrt K, i.e., $K \subseteq \mathsf{Fram}(\text{i-}\mathcal{X})$ and
- any α s.t i-$\mathcal{X} \nvdash \alpha$ can be refuted in a model based on a frame from K.

We say that a condition (which may be expressed in a natural language or in a formalized metalanguage like first- or second-order logic) *corresponds* to a given $\multimap$-logic i-$\mathcal{X}$ if it defines precisely $\mathsf{Fram}(\text{i-}\mathcal{X})$. In particular, when a condition is a correspondent of $\mathsf{iA} + \phi$, we say it *corresponds* to ϕ and correspondingly (pun unintended) use notation $\mathsf{Fram}(\phi)$. A logic i-$\mathcal{X}$ can be complete for much smaller a class than $\mathsf{Fram}(\text{i-}\mathcal{X})$ but if it is complete for *some* class of frames, it is also complete for $\mathsf{Fram}(\text{i-}\mathcal{X})$; we can thus take this as a definition what it means to be (weakly) complete without additional qualifications. Incomplete logics, i.e., those which have some non-theorems which cannot be refuted in $\mathsf{Fram}(\text{i-}\mathcal{X})$, are sometimes even encountered among those with an arithmetical interpretation, c.f. systems known as GLB and GLP [24,51,57], though most "naturally" defined logics tend to be complete.

Remark 6.2. Let us recall an important difference between completeness and correspondence when it comes to combinations (conjunctions) of axioms. Clearly, $\mathsf{Fram}(\bigwedge \Gamma) = \bigcap_{\gamma \in \Gamma} \mathsf{Fram}(\gamma)$, so whenever α is a correspondent of ϕ and β is a correspondent of ψ, $\alpha \wedge \beta$ is a correspondent of $\phi \wedge \psi$. Nothing like this needs to hold for completeness, even for a finitely axiomatizable logic. Completeness of $\mathsf{iA} + \phi$ and $\mathsf{iA} + \psi$ for frames defined, respectively, by α and β does not automatically imply that $\mathsf{iA} + \phi + \psi$ is complete for $\alpha \wedge \beta$—or, indeed, for any class of frames whatsoever. This is why in Fig. 6.2, Theorems 6.4 and 6.6 we do not mention *correspondence* conditions for logics axiomatized by conjunctions/combinations of axioms, but *completeness* results for such logics need to be stated explicitly. □

The notion of completeness can be refined further in two orthogonal directions. One of them is the *finite model property* (*fmp*, also known as the *finite frame property*) which simply means completeness wrt a class of *finite* frames. While the fmp is a much stronger property than weak completeness, it is still rather standard among most "natural" logics. It is not quite the case, however, with another refinement of interest: the notion of *strong* completeness, i.e., completeness for deductions from infinite sets of premises. This notion can be defined in two different ways using either

- the relation $\Gamma \vdash_{\text{i-}\mathcal{X}} \phi$ defined as "ϕ is deducible from Γ using all theorems of i-$\mathcal{X}$ and Modus Ponens" or
- the relation $\Gamma \vdash^{\multimap}_{\text{i-}\mathcal{X}} \phi$ defined as "ϕ is deducible from Γ using all theorems of i-$\mathcal{X}$, Modus Ponens *and the rule* $\mathsf{N_a}$".

A given $\multimap$-logic i-$\mathcal{X}$ is then

- *strongly locally complete* if whenever $\Gamma \nvdash_{\text{i-}\mathcal{X}} \phi$, there exists $\mathcal{F} \in \mathsf{Fram}(\text{i-}\mathcal{X})$, a valuation V and a point k in $\mathcal{F}$ s.t. $\mathcal{F}, V, k \Vdash \Gamma$ and $\mathcal{F}, V, k \nVdash \phi$.

- *strongly globally complete* if whenever $\Gamma \not\vdash^{\dashv}_{\text{i-}\mathcal{X}} \phi$, there exists $\mathcal{F} \in \mathsf{Fram}(\text{i-}\mathcal{X})$, a valuation V s.t. $\mathcal{F}, V \Vdash \Gamma$ yet for some point k in $\mathcal{F}$, $\mathcal{F}, V, k \nVdash \phi$.

As discovered by Frank Wolter [139], these two notions coincide for Kripke semantics of ordinary modal logics. While Wolter was not working with extensions of iA, his reasoning extends to our setting:

Theorem 6.3. *A* $\dashv$*-logic* i-$\mathcal{X}$ *is strongly locally complete iff it is strongly globally complete.*

Strong completeness is typically achieved as a corollary of stronger results, such as *canonicity*, which in turn, as first observed by Fine [42] (see also Gehrke et al. [45] for a general treatment), can be obtained as a corollary of *elementarity*: that is, being complete wrt a *first-order definable* class of frames. It is not hard to see intuitively the reason for this connection: for a (weakly) complete logic at least, a failure of strong completeness implies a failure of *compactness* of the Kripke consequence relation, whereas being elementarily definable guarantees compactness of this relation. A suitable notion of canonicity for $\dashv$-logics has been proposed and studied in the literature [54,55,144]; in fact, clauses regarding strong completeness in Theorem 6.4 are corollaries of such canonicity results.

6.2. *Completeness and correspondence results*

Fig. 6.2 lists various completeness/correspondence conditions for $\mathcal{L}_{\dashv}$-principles. Löb-like axioms tend to have counterparts which are not of first-order character, but numerous others can in fact be expressed in first-order logic.

Let us turn these claims into proper theorems. First, let us summarize results which are available in the existing literature, or can be relatively easily derived:

Theorem 6.4.

a. iA *is strongly complete (wrt the class of all frames) and enjoys the finite model property* [54, Prop. 4.1.1], [55, Prop. 7], [144, Th. 2.1.10].
b. i-BoxA $=$ iA $+$ Box *corresponds to the class of brilliant frames, is strongly complete and enjoys the finite model property.*
c. i-SA $=$ iA $+$ $\mathsf{S}_{\Box}$ *corresponds to the class of strong frames, is strongly complete and enjoys the finite model property.*
d. iA $+$ $4_{\Box}$ *corresponds to the class of semi-transitive frames, is strongly complete and enjoys the finite model property.*
e. iA $+$ 4_{a} *corresponds to the class of gathering frames, is strongly complete and enjoys the finite model property* [54, Prop. 4.2.1], [55, Prop. 8].
f. iA $+$ $\mathsf{L}_{\Box}$ *corresponds to the class of Noetherian semi-transitive frames and enjoys the finite model property* [54, Prop. 4.3.2], [144, Th. 2.2.7].
g. i-GL$_{\mathsf{a}}$ $=$ iA $+$ $\mathsf{L}_{\Box}$ $+$ 4_{a} *(cf. Lemma 4.11) corresponds to the class of Noetherian gathering frames* [55, Lem. 9], [56, Lem. 3.10] *and enjoys the finite model property.*
h. i-GW$_{\mathsf{a}}$ $=$ iA $+$ W_{a} *corresponds to the "supergathering" property of Fig. 6.2 on the class of finite frames* [144, Lem. 3.5.1], [56, Th. 3.31].
i. iA $+$ M_{a} *corresponds to the class of Montagna frames of Fig. 6.2, is strongly complete* [55, Prop. 11] *and enjoys the finite model property* [56, Lem. 3.21], [144, Th. 3.3.5].

Box	brilliant	$k \sqsubset \ell \preceq m \Rightarrow k \sqsubset m$	$\begin{tikzcd} & m \\ k \arrow[r, rightsquigarrow] \arrow[ur, rightsquigarrow] & \ell \arrow[u] \end{tikzcd}$
$4_\Box$	semi-transitive	$k \sqsubset \ell \sqsubset m \Rightarrow$ $\exists x. k \sqsubset x \preceq m$	$\begin{tikzcd} x \arrow[r, dashed] & m \\ k \arrow[u, rightsquigarrow] \arrow[r, rightsquigarrow] & \ell \arrow[u, rightsquigarrow] \end{tikzcd}$
4_a	gathering	$k \sqsubset \ell \sqsubset m \Rightarrow \ell \preceq m$	$\begin{tikzcd} k \arrow[r, rightsquigarrow] & \ell \arrow[r, rightsquigarrow, bend left] \arrow[r, dashed, bend right] & m \end{tikzcd}$
$L_\Box$	$\sqsubset$-Noetherian (conversely well-founded) and semi-transitive		
W_a	supergathering	on finite frames: $k \sqsubset \ell \sqsubset m \Rightarrow$ $\exists x \sqsupset k.(\ell \prec x \preceq m)$	$\begin{tikzcd} & x \arrow[r, dashed] & m \\ k \arrow[ur, rightsquigarrow] \arrow[rr, rightsquigarrow] & & \ell \arrow[u, rightsquigarrow] \arrow[ul, dashed, "\neq"] \end{tikzcd}$
M_a	Montagna	$k \sqsubset \ell \preceq m \Rightarrow \exists x \sqsupset k.(\ell \preceq x \preceq m \;\&\; x{\uparrow}_{\sqsubset\preceq} \subseteq m{\uparrow}_{\sqsubset\preceq})$	
$S_\Box$	strong	$k \sqsubset \ell \Rightarrow k \preceq \ell$	$\begin{tikzcd} k \arrow[r, rightsquigarrow, bend left] \arrow[r, dashed, bend right] & \ell \end{tikzcd}$
CB_a	$\sqsubset$-dominated	$k \prec \ell \Rightarrow k \sqsubset \ell$	$\begin{tikzcd} k \arrow[r, rightsquigarrow, dashed, bend left] \arrow[r, bend right, "\neq"'] & \ell \end{tikzcd}$
$CB_\Box$	weakly $\sqsubset$-dominated	$k \prec \ell \Rightarrow \exists m \sqsupset k. m \preceq \ell$	$\begin{tikzcd} & \ell \\ k \arrow[ur, "\neq"] \arrow[r, rightsquigarrow, dashed] & m \arrow[u, dashed] \end{tikzcd}$
Lin_a	weakly semi-linear	$k \sqsubset \ell \;\&\; k \sqsubset m \Rightarrow (m \preceq \ell \text{ OR } \ell \preceq m)$	
$Lin_\Box$	strongly semi-linear	$k \sqsubset \ell \preceq \ell' \;\&\; k \sqsubset m \preceq m' \Rightarrow (m' \preceq \ell' \text{ OR } \ell' \preceq m')$	
$C4_\Box$	semi-dense	$k \sqsubset \ell \Rightarrow \exists x \preceq \ell. \exists y \sqsupset k. y \sqsubset x$	$\begin{tikzcd} \ell & x \arrow[l, dashed] \\ k \arrow[u, rightsquigarrow] \arrow[r, rightsquigarrow, dashed] & y \arrow[u, rightsquigarrow, dashed] \end{tikzcd}$
$C4_a$	pre-reflexive	$k \sqsubset \ell \Rightarrow \exists x \sqsupset \ell. x \preceq \ell$	$\begin{tikzcd} k \arrow[r, rightsquigarrow] & \ell \arrow[r, rightsquigarrow, dashed, bend left] & x \arrow[l, dashed, bend left] \end{tikzcd}$
Hug	semi-nucleic	$k \sqsubset \ell \Rightarrow \exists m \succeq k.$ $\exists m' \sqsupset m.\; \ell \preceq m. m' \preceq \ell$	$\begin{tikzcd} & m \arrow[r, rightsquigarrow, dashed] & {} \arrow[dl, dashed] \\ k \arrow[ur, dashed] \arrow[r, rightsquigarrow] & \ell \arrow[u, dashed] & \end{tikzcd}$
App_a	almost reflexive	$k \sqsubset \ell \Rightarrow \ell \sqsubset \ell$	$\begin{tikzcd} k \arrow[r, rightsquigarrow] & \ell \arrow[loop right, rightsquigarrow, dashed] \end{tikzcd}$

Fig. 6.2. Correspondence conditions. In this figure, and elsewhere in this paper, $\rightsquigarrow$ stands for $\sqsubset$ and $\rightarrow$ stands for $\preceq$. Some names of principles are taken from Iemhoff and coauthors [55,56,144], others come from our work to be published separately [82]. and subset $X \subseteq W$, set $X{\uparrow}_R := \{y \in W \mid \exists x \in X. x R y\}$; in particular, write $x{\uparrow}_R$ for $\{x\}{\uparrow}_R$.

We could not find in the literature an explicit statement of the finite model property of i-PreL = i-GW_a + M_a. Moreover, an astute reader probably noticed that we do not claim strong completeness for all logics appearing in the statement of this theorem. The reason is obvious: it is very well-known that variants of the Löb axiom clash with strong completeness and, a fortiori, with canonicity. Boolos and Sambin [25] credit Fine and Rautenberg with this observation, which can be now found in any standard monograph on modal logic. This can be extended in several directions, e.g., to logics with weaker axioms (cf. Amerbauer [3]) or to failure of broader notions of strong completeness [79]; see Litak [80, § 3] for more on both counts. In the context of logics for (relative) interpretability (cf. Appendix C), problems with canonicity and strong completeness have been pointed out, e.g., by de Jongh and Veltman [33]. Let us adapt such arguments to our setting:

Theorem 6.5. i-$\mathcal{X}$ *is not strongly complete whenever*

- *it is contained between* $\mathsf{iA} + \mathsf{L}_\Box$ *and* $\mathsf{c\text{-}GL}_\Box + \mathsf{Lin}_\Box$ *or*
- *it is contained between* $\mathsf{iA} + \mathsf{L}_\Box$ *and* $\mathsf{i\text{-}KM.lin_a}$.

In particular, $\mathsf{i\text{-}GL_a}$, $\mathsf{i\text{-}GW_a}$, $\mathsf{i\text{-}PreL}$ *or* $\mathsf{i\text{-}KM_a}$ *fail to be strongly complete.*

Proof Sketch. We can work in the standard modal language containing just $\Box$ rather $\dashv$ (in fact, $\Box$ and $\to$ are the only connectives really used). We can also use the freedom offered by Theorem 6.3 and choose to disprove global completeness. Consider now $\Gamma := \{\Box p_{i+1} \to p_i \mid i \in \omega\}$ and note that in any model where Γ is globally satisfied but p_0 fails, there must exists an infinite $\sqsubset$-ascending chain, which allows us to refute Noetherianity, hence refuting $\mathsf{L}_\Box$.

However, taking $\sqsubset$ to be an ordered sum of ω with its copy with reverse ordering ω^*, $\preceq$ to be either (for the first clause) discrete or (for the second clause) the reflexive version of $\sqsubset$ and setting $V(p_i) := (i+1)\Uparrow_\preceq$ produces a model where Γ is globally valid, p_0 fails and all theorems of i-$\mathcal{X}$ hold under V (despite being refutable in the underlying frame). $\Box$

Theorem 6.4 above does not cover correspondence and completeness claims for all axioms and frame conditions displayed in Fig. 6.2, especially those not directly related to preservativity and provability principles. As it turns out, there is a technique of transferring generic results available for (bi)modal logics over CPC into the intuitionistic setting. For $\Box$-logics, it has been developed in a series of papers by Wolter and Zakharyaschev [140,141]. We are going to present details of generalization of this technique to $\dashv$-logics in a separate paper [82]. For now, let us just list some consequences regarding strong completeness and canonicity (we leave the finite model property out of the picture here):

Theorem 6.6 (*[82]*).

a. $\mathsf{iA} + \mathsf{CB_a}$ *correspond to the class of* $\sqsubset$*-dominated frames of Fig. 6.2 and is strongly complete.*
b. $\mathsf{iA} + \mathsf{CB}_\Box$ *correspond to the class of weakly* $\sqsubset$*-dominated frames of Fig. 6.2 and is strongly complete.*
c. $\mathsf{i\text{-}mHC_a}$ *is strongly complete (wrt the class of strong* $\sqsubset$*-dominated frames).*
d. $\mathsf{iA} + \mathsf{Lin_a}$ *correspond to the class of weakly semilinear frames of Fig. 6.2 and is strongly complete.*
e. $\mathsf{iA} + \mathsf{Lin}_\Box$ *correspond to the class of strongly semilinear frames of Fig. 6.2 and is strongly complete.*
f. $\mathsf{iA} + \mathsf{C4}_\Box$ *correspond to the class of semi-dense frames of Fig. 6.2 and is strongly complete.*
g. $\mathsf{iA} + \mathsf{C4_a}$ *correspond to the class of pre-reflexive frames of Fig. 6.2 and is strongly complete.*

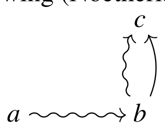
h. iA + $\mathsf{App_a}$ *correspond to the class of almost reflexive frames of* Fig. 6.2 *and is strongly complete.*
i. i-PLAA *is strongly complete (wrt the class of strong almost reflexive frames).*

6.3. Non-derivations

Having a developed semantics, we are now in a position to provide more examples of *non*-derivations between formulas and *non*-containments between logics.

Example 6.7. Consider the formula $\phi_0 := \Box\bot \strictif \bot \to \Box\bot$. It is easy to see that this formula is in the closed fragment of $\mathsf{i\text{-}GW_a}$. This means that ϕ_0 is variable-free and provable in $\mathsf{i\text{-}GW_a}$. We show that ϕ_0 is not in the closed fragment of $\mathsf{i\text{-}GL_a}$.

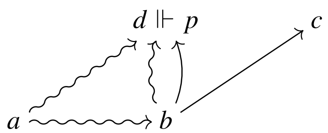
By Theorem 6.4g, $\mathsf{i\text{-}GL_a}$ is determined by Noetherian gathering frames. Consider the following (Noetherian gathering) model:



All points are $\sqsubset$-irreflexive. We set $b \preceq c$, $a \sqsubset b \sqsubset c$ and the valuation is empty, i.e., $V(a) = V(b) = V(c) = \emptyset$. Note that $\mathsf{CB_a}$ holds in this model.

Clearly, $a \Vdash \Box\bot \strictif \bot$, but $a \nVdash \Box\bot$. □

Example 6.8. Consider the formula $\phi_1 := \Box\bot \strictif p \to \Box(\Box\bot \to p)$. Lemma 4.14d implies that this formula is provable in i-PreL. We prove that $\mathsf{i\text{-}GW_a} \nvdash \phi_1$ by considering the following model satisfying the condition for finite frames for $\mathsf{i\text{-}GW_a}$ as stated in Theorem 6.4h (and Fig. 6.2):



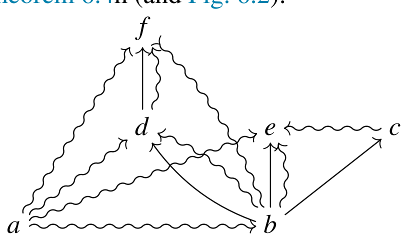
All points are $\sqsubset$-irreflexive. We set $b{\uparrow_\preceq} \supseteq \{c, d\}$, $a \sqsubset b$, $b \sqsubset d$, $a \sqsubset d$ and the valuation is $V(d) = p$ and empty otherwise.

It is now easy to see that $a \Vdash \Box\bot \strictif p$, but $a \nVdash \Box(\Box\bot \to p)$. □

Example 6.9. We can improve Example 6.8 by providing a separating closed formula. Consider the formula

$$\phi_2 := \Box\Box\bot \strictif \neg\neg\Box\bot \to \Box(\Box\Box\bot \to \neg\neg\Box\bot).$$

Again, Lemma 4.14d implies that this formula is provable in i-PreL. We prove that $\mathsf{i\text{-}GW_a} \nvdash \phi_2$ by considering the following model satisfying the condition for finite frames for $\mathsf{i\text{-}GW_a}$ as stated in Theorem 6.4h (and Fig. 6.2):



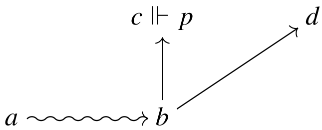
All points are $\sqsubset$-irreflexive. As usual, we do not draw the transitive and reflexive closure of $\to$ (which, recall, stands for the poset order $\preceq$). The valuation is empty (and irrelevant anyway).

It is now easy to see that $a \Vdash \Box\Box\bot \strictif \neg\neg\Box\bot$, but $a \nVdash \Box(\Box\Box\bot \to \neg\neg\Box\bot)$. □

Example 6.10. Recall that following Lemma 4.6, we noted that in the disjunction-free setting, there is no one-variable formula $\phi(p)$ s.t. $p \mathrel{-\!\!3} q \dashv\vdash_{-} \phi(p) \mathrel{-\!\!3} (p \to q)$ and $\mathsf{CPC} \vdash \phi(p)$. This follows from the fact that

$$\phi_3 := p \mathrel{-\!\!3} q \to (\neg\neg p \to p) \mathrel{-\!\!3} (p \to q)$$

is *not* a theorem of iA:

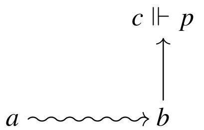


All points are $\sqsubset$-irreflexive. We set q to be false everywhere. It is easy to check the antecedent of ϕ_3 holds at a, but the consequent fails. It is worth noting that i-PreL holds in this model.

We can complement this observation by another one: it is not possible to improve the iA-equivalence of Lemma 4.6 by taking a one-variable intuitionistic formula stronger that $p \vee \neg p$ as the antecedent of $\mathrel{-\!\!3}$ replacing em, as

$$\phi_4 := p \mathrel{-\!\!3} q \to (\neg\neg p \vee \neg p) \mathrel{-\!\!3} (p \to q)$$

is not a theorem of iA either:

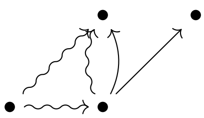


All points are $\sqsubset$-irreflexive. We set q to be false everywhere. Again the antecedent of ϕ_4 holds at a, but the consequent fails; moreover, i-PreL holds in this model. □

Example 6.11. Here are diagrams illustrating that $\mathsf{CB_a}$ is not a theorem of $\mathsf{i\text{-}mHC_\Box}$; that is, strong frames which are only weakly $\sqsubset$-dominated. We use the convention that $\circ$ stand for a $\sqsubset$-reflexive loop and $\bullet$ for lack thereof.

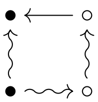
Arithmetical interpretation provides another interesting way of distinguishing between $\mathsf{CB_\Box}$ and $\mathsf{CB_a}$: Section 5.4.4 noted that $\mathsf{CB_\Box}$ is in the preservativity logic of PA^*, whereas as stated in Theorem 8.10, $\mathsf{CB_a}$ does not belong to this system (the only problem is that neither does Di). □

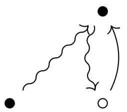
Example 6.12. So far, we were seeing examples showing that principles for $\mathrel{-\!\!3}$ are often properly stronger than their relatives formulated in terms of $\Box$ only. Recall that when introducing Fact 4.18, we indicated it is not always the case, as witnessed by semi-linearity axioms. Here is a simple frame for $\mathsf{i\text{-}GW_a} + \mathsf{Lin_a}$ where $\mathsf{Lin_\Box}$ fails (for both claims one can use Theorem 6.6 and Fig. 6.2, but they are straightforward to verify anyway). We are following the same conventions regarding $\sqsubset$-reflexive and $\sqsubset$-irreflexive points as in the preceding example:



□

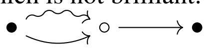
Example 6.13. In order to separate $\mathsf{C4}_{\Box}$, $\mathsf{C4}_{\mathsf{a}}$ and $\mathsf{App}_{\mathsf{a}}$, we provide an example of a semi-dense frame which is not pre-reflexive (on the left) and a pre-reflexive one which are not almost reflexive (on the right):





Again, even without using completeness results of Theorem 6.6, one can easily verify everything by hand (including finding suitable valuations). □

Example 6.14. In Section 7.1, we will use the fact that $\mathsf{i\text{-}PLAA}$ does not contain Box. As made clear by Theorem 6.6 and Fig. 6.2, for this purpose we need a strong almost reflexive frame which is not brilliant:



□

7. Strength: arrows, monads, idioms and guards

We have already seen that arithmetical interpretation of modalities provides good motivation for studying intuitionistic logics with strict implication, including those with the strength axiom. This is a very good motivation indeed, but by no means the only one. Such formalisms have continuously reappeared in several recent lines of research, especially in theoretical computer science.

7.1. Notions of computation and arrows

Surprisingly, the functional programming community discovered a variant of constructive strict implication at roughly the same time as it appeared in the context of preservativity. More specifically, "(classical) arrows" in the terminology of John Hughes [52] (see also [77]) are in our terminology *strong* Lewis arrows. Interestingly enough, their *unary* cousins knows as "idioms" or "applicative functors" [88] were discovered *later* in this community, though a special subclass of applicative functors – to wit, *monads* corresponding to $\mathsf{i\text{-}PLL}_{\Box}$ modalities [15,41,60]—has been enjoying continuous attention since the seminal paper of Moggi [94]. A particularly convenient basis for our discussion contrasting arrows, idioms and monads is provided by Lindley et al. [77], which we take as the main reference for this subsection.

The connection between intuitionistic logics and functional programming is provided by the *Curry–Howard correspondence*, also known as the *Curry–Howard isomorphism* or *proposition-as-types paradigm* (cf. [116]).[27] While the details are outside of the scope of this paper, the shortest outline is that

- (intuitionistic) formulas correspond to *types*,
- logical connectives correspond to type operators/constructors,

[27] As pointed out by Sørensen and Urzyczyn, "The *Brouwer - Heyting - Kolmogorov - Schönfinkel - Curry - Meredith - Kleene - Feys - Gödel - Läuchli - Kreisel - Tait - Lawvere - Howard - de Bruijn - Scott - Martin-Löf - Girard - Reynolds - Stenlund - Constable - Coquand - Huet - . . . - isomorphism* might be a more appropriate name, still not including all the contributors". [116, p. viii] Indeed, the Curry–Howard isomorphism provides the most commonly accepted specification of the Brouwer–Heyting–Kolmogorov interpretation of intuitionistic connectives. We could thus only half-jokingly argue that this subsection is yet another place in our paper where Lewis meets Brouwer.

- logical axioms correspond to *inhabited* types and hence deciding theoremhood corresponds to the type inhabitation problem,
- logical proofs – e.g., in a variant of a natural deduction system or in a Hilbert-style system – are encoded by *proof terms* – in a variant of lambda calculus or of combinatory logic – understood as a (functional) programming language and hence
- proof *normalization* corresponds to reduction of these terms, understood as representing *computation*.

In particular, ordinary intuitionistic implication $\phi \to \psi$ corresponds to forming the *function space* of programs (proofs) which take data from (proofs for) ϕ as their input and produce members of (proofs for) ψ as their output. The introduction rule for $\to$ corresponds to λ-abstraction and its elimination rule (i.e., ordinary Modus Ponens) corresponds to function application.

Nevertheless, one may ask: are "computations" exactly co-extensional with "members of function space"? In the words of Ross Paterson

> Many programs and libraries involve components that are function-like, in that they take inputs and produce outputs, but are not simple functions from inputs to outputs... [S]uch "notions of computation" defin[e] a common interface, called "arrows". [101, p. 201]

What are the laws such a notion of computation is supposed to satisfy? The inhabitation laws of the calculus of "classic arrows" [77, Fig. 4] in a disjunction-free language are given by the following axioms:[28]

$$
\begin{array}{ll}
\mathsf{S_a} & (\phi \to \psi) \to \phi \dashv \psi, \\
\mathsf{Tr} & \phi \dashv \psi \to \psi \dashv \chi \to \phi \dashv \chi, \\
\mathsf{K'_a} & \phi \dashv \psi \to (\phi \wedge \chi) \dashv (\psi \wedge \chi).
\end{array}
$$

Thanks to Lemmas 4.1 and 4.10, we know it is just an axiomatization for $\mathsf{i\text{-}SA^-}$!

Open Question 7.1. As Lindley et al. [78] work in a type theory without the co-product operator (i.e., the Curry–Howard counterpart of disjunction), the issue of validity of Di simply does not arise. Nevertheless, given the problematic status of Di in preservativity logics of some theories (cf. Open Question 5.7), it seems a valid question whether Di should be a law imposed on all notions of computation—and if not, how to characterize those where it holds. It is an inhabited type for both *arrows with apply* (*monads*) and *static arrows* (*idioms*), as follows from the discussion below and, correspondingly, Lemmas 4.17g and 4.16c. ⊞

What is the status of the Box law then (or any of its equivalent forms)? As it turns out, the Curry–Howard interpretation provides another rationale for considering (strong) Lewis arrows not determined by an unary $\Box$. Lindley et al. [77] call arrows satisfying Box *static arrows* and show that such arrows correspond to the "idioms" or "applicative functors" of McBride and Paterson [88]. Indeed, the inhabitation laws of the calculus for idioms [77, Fig. 3] are exactly those of $\mathsf{i\text{-}S}_\Box$. This is, however, only a special subclass of computations encoded by arrows: namely those computations "in which commands are *oblivious* to input" [77]. Lindley [76] rephrases this claim to the effect that idioms are distinguished by their static approach to *data flow*.

[28] Lindley et al. [77] call these axioms arr, $>>>$ and first, respectively. They also use $\rightsquigarrow$ in place of $\dashv$.

However, as said above, just a special subclass of applicative functors is by far the most important from a programming point of view: that of *(strong) monads*. This subclass of idioms whose type system satisfies in addition the inhabitation law corresponding to $\mathsf{C4}_\Box$ (and, obviously, a number of equalities between proof terms, which are not of concern to us here) provides the most popular framework for *effectful computations*. In other words, the Curry–Howard counterpart (the logic of type inhabitation) of the calculus for (strong) monads proposed by Moggi under the name of *computational metalanguage* [94] is $\mathsf{i\text{-}PLL}_\Box$: *propositional lax logic* [15,41,60].

Monads can be shown [52,77] to be in 1–1 correspondence with *higher-order arrows* or *classical arrows with apply*. To wit, these are arrows satisfying the law:

$$\mathsf{App_a} \quad (\phi \wedge (\phi \dashv\!\!3\, \psi)) \dashv\!\!3\, \psi.$$

Thus, by Lemma 4.17, the logic of type inhabitation for this subclass of arrows is precisely $\mathsf{i\text{-}PLAA}$ (*propositional logic of arrows with apply*). Lindley et al. present a two-context natural deduction system for both $\mathsf{i\text{-}SA}^-$ and $\mathsf{i\text{-}PLAA}$, whose proof-term assignment is based on a distinction between *terms* and *commands* and argue that higher-order arrows are "*promiscuous* (in the broader sense of *undiscriminating*)", as the "apply" construct corresponding to $\mathsf{App_a}$ bridges this distinction carefully maintained in the calculus for $\mathsf{i\text{-}SA}^-$ (which can be thus called *meticulous*). Another perspective is offered by Lindley [76]: higher-order arrows are distinguished by their dynamic approach not only to *data flow*, but also to *control flow*.

Remark 7.2. The correspondence between monads and arrows with apply should *not* be conflated with the one between idioms (whose logic of type inhabitation is $\mathsf{i\text{-}SA}^-$) and static arrows, whose logic of type inhabitation is $\mathsf{i\text{-}BoxSA}$: i.e., a system where $\phi \dashv\!\!3\, \psi$ is definable as $\Box(\phi \to \psi)$. In contrast, Box is obviously not valid in $\mathsf{i\text{-}PLAA}$ (cf. Example 6.14) and the $\Box$-only fragment of $\mathsf{i\text{-}PLAA} + \mathsf{Box}$ is a $\Box$-logic stronger than $\mathsf{i\text{-}PLL}_\Box$; e.g., we have that

$$\mathsf{i\text{-}PLAA} + \mathsf{Box} \vdash \Box(\Box\phi \to \phi)$$

and one can easily check that $\mathsf{i\text{-}PLL}_\Box \nvdash \Box(\Box\phi \to \phi)$. Instead, $\mathsf{i\text{-}PLAA}$ is embedded into $\mathsf{i\text{-}PLL}_\Box$ by interpreting $\phi \dashv\!\!3\, \psi$ as $\phi \to \Box\psi$, cf. [77, § 6]. In fact, we can derive this fact syntactically from Lemma 4.17f above! □

Remark 7.3. To finish this subsection on another theme from Lewis, note that *Symbolic Logic* [74] had this to say about $\mathsf{App_a}$ (appearing therein as postulate $\mathsf{11.7}$ in the main text and in the famous Appendix II as $\mathsf{B7}$):

> It might be supposed that this principle would be implicit in any set of assumptions for a calculus of deductive inference. As a matter of fact, $\mathsf{11.7}$ cannot be deduced from other postulates. [74, p. 125]

The last sentence[29] is pertinent indeed: $\mathsf{App_a}$ is the only axiom of the smallest system Lewis was interested in, i.e., $\mathsf{S1}$, which is not a theorem of iA^-! □

[29] It is proved later on p. 495 of [74] using a matrix proposed by Parry.

7.2. Modalities for guarded (co)recursion

Another area of recent computer science where strong intuitionistic modalities have found numerous applications is the study of guarded (co)recursion: as an important tool to ensure *productivity* in (co)programming with *coinductive types* [8,23,29,62–64,93] and, on the metalevel, in semantic reasoning about programs involving *higher-order store* or a combination of *impredicative quantification* with *recursive types* [20–22,39,59,111,120].

The logics of type inhabitation of these systems are mostly extensions of $\mathsf{i\text{-}SL}_\Box$, involving either first- or higher-order quantifiers (corresponding to dependent, polymorphic or impredicative types) or additional entities like *clock variables* [8,22,23,93], or (a constructive analogue of) the universal modality [29]. Nakano [96,97] proposed using the axioms of $\mathsf{i\text{-}SL}_\Box$ for *approximation modality* crediting Sambin–de Jongh-style results on elimination of fixpoints as one of his motivations (see [81, § 3] for a detailed discussion of this point); more recent discussion of Nakano-style systems can be found in Abel and Vezzosi [1] and Severi [107]. The idea of using such modalities also in the metalanguage for reasoning about *semantics* of programs has been popularized by Appel et al. [4], who were nevertheless working with the axiom $\mathsf{L}_\Box$ rather than $\mathsf{SL}_\Box$ seen in most later references.

As the above overview makes clear, the area has grown too large to allow an adequate summary in this paper. See [81] for more information and [91] for an overview of *models of guarded (co)recursion*, i.e., from our point of view, categorical models for proof systems for fragments of such logics. Our question here is whether the Lewis arrow naturally occurs in this context.

In fact, starting from the original paper of Nakano [96] and even more so in references like Abel and Vezzosi [1], the introduction/elimination/inference rules governing the behavior of such an "approximation" or "delay" modality are often formulated *combining* $\Box$ and $\to$. This point is perhaps most explicitly addressed by Clouston and Goré [30], a reference highly relevant from our point of view, as it does use $\prec\!\!3$ (denoted therein as $\twoheadrightarrow$), claiming moreover:

> The main technical novelty of our sequent calculus is that we leverage the fact that the intuitionistic accessibility relation is the reflexive closure of the modal relation, by decomposing implication into a static (classical) component and a dynamic 'irreflexive implication' $\prec\!\!3$ that looks forward along the modal relation. In fact, this irreflexive implication obviates the need for $\Box$ entirely, as $\Box\phi$ is easily seen to be equivalent to $\top \prec\!\!3\ \phi$. Semantically, the converse of this applies also, as $\phi \prec\!\!3\ \psi$ is semantically equivalent to $\Box(\phi \to \psi)$, but the $\prec\!\!3$ connective is a necessary part of our calculus. We maintain $\Box$ as a first-class connective in deference to the computer science applications and logic traditions from which we draw, but we note that formulae of the form $\Box(\phi \to \psi)$ are common in the literature—see Nakano's $(\to E)$ rule [96], and even more directly the $\circledast$ constructor of [19]. We therefore suspect that treating $\prec\!\!3$ as a first-class connective could be a conceptually fruitful side-benefit of this work ([30], in a notation adjusted to this paper).

Clouston and Goré [30] provide a sequent calculus for a logic called here $\mathsf{i\text{-}KM.lin_a}$. The focus on this logic is motivated by Litak's observation [81] that $\mathsf{i\text{-}KM.lin}_\Box$ is the propositional fragment of the Mitchell–Bènabou logic of the *topos of trees* proposed as a model of guarded (co)recursion by Birkedal and coauthors [20] and used ever since [29,93,107].

Let us note here that Lemma 4.19 implies that *any* semantics for $\mathsf{i\text{-}KM.lin_a}$ must make Box valid: in other words, $\mathsf{i\text{-}KM.lin_a}$ can be just seen as another syntactic presentation of $\mathsf{i\text{-}KM.lin}_\Box$. However, Lemma 4.19 requires all the axioms of $\mathsf{i\text{-}KM.lin}_\Box$ and when studying broader classes of models of guarded (co)recursion [91], more flexibility in adding $\prec\!\!3$ is possible.

Open Question 7.4 Are there natural applications of $\strictif$-logics not including the Box axiom in terms of guarded (co)recursion? And, more broadly, do arithmetically relevant principles discussed in this paper have a computational interpretation? $\boxplus$

Let us add that, while Gentzen-style systems are not our main interest here, the above quote from Clouston and Goré [30] hints at another motivation for studying constructive $\strictif$. Namely, even in the setups which make it a definable connective, it can still prove a more convenient primitive from a proof-theoretical point of view than $\Box$ is.

7.3. Intuitionistic epistemic logic

Finally, let us briefly mention yet another recent area of research where strong intuitionistic modalities made a surprising appearance: in the work of Artemov and Protopopescu on intuitionistic epistemic logic [7], presented also in this collection.[30] These authors work with unary $\Box$ and call $\mathsf{S}_\Box$ the principle of "co-reflection". The minimal system denoted by these authors as IEL^- corresponds to $\mathsf{i\text{-}S}_\Box$ in our notation, their IEL is obtained by adding[31] $\neg\Box\bot$ and IEL^+ arises by adding $\mathsf{C4}_\Box$—i.e., is an extension of $\mathsf{i\text{-}PLL}_\Box$.

A proof-theoretic justification for these systems is presented in terms of the Brouwer–Heyting–Kolmogorov interpretation. This seems to provide a natural connection with references discussed in Section 7.1—but, curiously, none of them seems to be mentioned by Artemov and Protopopescu, neither the extensive literature on $\mathsf{i\text{-}PLL}_\Box$, nor the rôle of $\mathsf{i\text{-}S}_\Box$ as the logic of applicative functors (idioms, prenuclei …). We leave an epistemic interpretation of strong arrows and extensions of $\mathsf{i\text{-}SA}$ as a promising subject for future study.

8. Applications of preservativity

Having briefly overviewed other motivations for studying constructive $\strictif$, let us return to our main one. Preservativity has many applications. A number of these applications can be found in [127] and [129]. We describe one of the main results of those papers in Section 8.1. In Section 8.2, we show how one can capture the invalidity of the law of excluded middle in terms of preservativity. We illustrate how this result imposes a constraint on possible preservativity logics of theories.

8.1. NNIL

The NNIL-formulas (*No Nestings of Implications to the Left* [127,129]) are defined as follows:

- $\phi ::= \bot \mid \top \mid p \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (p \to \phi)$.

It is easy to see that there are only finitely many nonequivalent NNIL-formulas on finitely many variables. Let $\vec{p}$ be the propositional variables of ϕ and define $\phi^\star$ as the disjunction of representatives of all IPC-equivalence classes of NNIL-formulas ψ in the variables $\vec{p}$ such that $\mathsf{IPC} \vdash \psi \to \phi$. Using the Interpolation Theorem, we see that, for any NNIL-formula χ, we have $\mathsf{IPC} \vdash \chi \to \phi$ if and only if $\mathsf{IPC} \vdash \chi \to \phi^\star$. So, $\phi^\star$ is the best NNIL-approximation from below of ϕ. In more fancy terms, $(\cdot)^\star$ is the right adjoint of the embedding functor of the

[30] For other approaches to intuitionistic epistemic logic cf. also Williamson [138] or Proietti [103] and for a more dynamic take, see Kurz and Palmigiano [65].

[31] Litak [81] denotes $\neg\Box\bot$ as (nv)—*non-verum*.

preorder category of the NNIL-formulas into the preorder category of all propositional formulas, both preorders being IPC-provable implication.

Theorem 8.1 (*[127,129]*). *For any function f from the propositional variables to Σ^0_1-sentences, $\phi^f \dashv_{\mathsf{HA}} (\phi^\star)^f$. Hence, if $\mathsf{HA} \vdash \phi^f$, then $\mathsf{HA} \vdash (\phi^\star)^f$.*

The original aim of [127] was to show: if $\mathsf{HA} \vdash \phi^f$, then $\mathsf{HA} \vdash (\phi^\star)^f$. However, it turned out that the inductive assumption requires the stronger statement involving preservativity. Thus, preservativity was discovered as a tool for induction loading.

Theorem 8.1 can be reformulated in terms of admissible consequence. We define:

- $\phi \mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1} \psi$ if for any Σ^0_1-substitution f, $\mathsf{HA} \vdash \psi^f$ whenever $\mathsf{HA} \vdash \phi^f$.

Thus, $\phi \mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1} \psi$ means that ϕ/ψ is an admissible rule for Σ^0_1-substitutions over HA. Theorem 8.1 now simply says: $\phi \mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1} \phi^\star$. It is optimal in the sense that, whenever $\phi \mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1} \psi$, we have $\mathsf{IPC} \vdash \phi^\star \to \psi$ [129]. Thus,

$$\phi \mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1} \psi \text{ iff } \phi^\star \vdash_{\mathsf{IPC}} \psi.$$

If we view $\mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1}$ and $\vdash_{\mathsf{IPC}}$ are pre-ordering categories, this says that $(\cdot)^\star$ is the left adjoint of the embedding functor of $\mathrel{\mid\!\sim}_{\mathsf{HA},\Sigma^0_1}$ in $\vdash_{\mathsf{IPC}}$.

The NNIL-formulas play an important rôle in: the characterization of the provability logic of HA for Σ^0_1-substitutions by Ardeshir and Mojtahedi [5], the study of infon logic [32] and several other contexts [106,137,142].

8.2. *On the falsity of Tertium non Datur*

In intuitionistic propositional logic, we have the principle $\neg\neg(\phi \vee \neg\phi)$. As a consequence, there is no direct logical expression of the constructive insight of the invalidity of the law of excluded middle.[32] The connective $(\cdot) \dashv \bot$ is a weaker form of negation, say $\sim$. Can we have, provably in i-EA, that $\sim_{\mathsf{HA}}(A \vee \neg A)$, for some suitable A?

We will show that, for a wide range of theories T, we can indeed find such a sentence A, including T being HA, HA + MP or HA + ECT_0, HA^*. We write:

- $T \leq U$ if i-EA verifies that T is a subtheory of U.

Suppose i-EA verifies Di for U, i.e. suppose that Di is in $\Lambda_{\mathsf{F}_{1,U},\text{i-EA}}$. We note that over i-EA we have $(\Box_U \bot \vee \neg\Box_U\bot) \dashv_U \Box_U\bot$. This is in the desired direction since we can consider $\Box_U\bot$ as a weak form of falsity. However, we cannot get the desired result as long as we stay with Σ^0_1-sentences.

Theorem 8.2. *Consider any consistent theory U. There is, verifiably in* i-EA + $\Diamond_U\top$, *no Σ_1-sentence S such that $\sim_U(S \vee \neg S)$.*

Proof. We work in i-EA + $\Diamond_U\top$. Consider a Σ_1-sentence S. Suppose we have $\sim_U (S \vee \neg S)$. It follows that $(S \to (S \vee \neg S)) \dashv_U (S \to \bot)$. Thus, $\top \dashv_U \neg S$, $\neg S \dashv_U (S \vee \neg S)$ and $(S \vee \neg S) \dashv_U \bot$. Ergo, $\Box_U\bot$. Quod non. $\Box$

[32] We can consistently add $\neg\forall x\,(A(x) \vee \neg A(x))$ to constructive arithmetic for certain A. E.g., HA plus a weak version of Church's Thesis (cf. Appendix A) proves $\neg\forall x\,(x \cdot x{\downarrow} \vee x \cdot x{\uparrow})$.

To prepare the construction of the promised sentence, we first consider theories V with $\mathsf{HA} \leq V$. Recall that $\Box_{V,x} A$ stands for (arithmetized) provability from the axioms of V with Gödel number $\leq x$.

- *Feferman provability* for V is defined by: $\triangle_V A := \exists x\, (\Box_{V,x} A \wedge \Diamond_{V,x} \top)$.

We have:[33]

Fe1 $V \vdash A \;\;\Rightarrow\;\; V \vdash \triangle_V A$.
Fe2 i-EA $\vdash \triangle_V(A \to B) \to (\triangle_V A \to \triangle_V B)$.
Fe3 i-EA $\vdash S \to \triangle_V S$, for Σ^0_1-sentences S.
We note that it follows that i-EA $\vdash \Box_V B \to \triangle_V \Box_V B$.
Fe4 i-EA $\vdash \triangle_V B \to \Box_V B$.
Fe5 i-EA $\vdash \Diamond_V \top \to (\triangle_V A \leftrightarrow \Box_V A)$.
Fe6 i-EA $\vdash \triangledown_V \top$, where $\triangledown$ is $\neg\triangle\neg$.

We note that classically Fe4 follows form Fe5.

Shavrukov [110] provides a complete axiomatization for the bimodal logic of ordinary provability and Feferman provability for PA.

Open Question 8.3 Shavrukov employs a different interpretation of $\Box_{\mathsf{PA},x}$, to wit provability in $\mathrm{I}\Sigma_x$. It would be interesting to find a better analogue of the version of the Feferman predicate employed by Shavrukov for the case of (extensions of) HA. Moreover, the principles given above provide a part of the principles given by Shavrukov for the classical case. We do not get all Shavrukov's principles in the constructive case. It would be interesting to study how close we can get to his system. ⊞

Supposing that V is consistent, we cannot get that, for all A, we have $V \vdash \triangle_V A \to \triangle_V \triangle_V A$. Otherwise, we could reproduce the reasoning for Gödel's Second Incompleteness Theorem. This leads immediately to a contradiction with Fe6.

We remind the reader that the theory V is V-verifiably *essentially reflexive*. This means that both truly and V-provably, we have: for all n and all A, we have $V \vdash \Box_{V,\underline{n}} A \to A$.[34]

Theorem 8.4. *Suppose* HA $\leq T$. *We have* i-EA $\vdash A \dashv_V \triangle_V A$.

Proof. Reason in i-EA. Consider any x. We have, by essential reflexivity,

$$\Box_V(\Box_{V,x} A \to (\Box_{V,x} A \wedge \Diamond_{V,x} \top)).$$

Hence, $\Box_V(\Box_{V,x} A \to \triangle_V A)$. Ergo, by Theorem 5.5, $A \dashv_V \triangle_V A$. □

Consider the Gödel sentence G_V of Feferman provability for V. We have

$$\text{i-EA} \vdash G_V \leftrightarrow \neg \triangle_V G_V.$$

Whenever the intended theory is clear from the context, we write G for G_V.

[33] We do not present the principles for triangle as a schematic logic. This is because of the occurrence of a variable over Σ^0_1-sentences. We would need a many-sorted propositional theory. Of course this is perfectly doable. We just did not develop it in this paper.

[34] We have this even for formulas A, when we employ the usual convention for free variables under the box.

Theorem 8.5. *Suppose* $\mathsf{HA} \leq V$. *Then,* $\mathsf{i\text{-}EA} \vdash G \dashv_V \bot$ *and* $\mathsf{i\text{-}EA} \vdash \neg G \dashv_V \bot$.

Proof. We reason in i-EA.

We have $G \dashv_V \triangle_V G$, and hence, $G \dashv_V \neg G$. Since also $G \dashv_V G$, it follows that $G \dashv_V \bot$.

We have, $\neg G \dashv_V \triangle_V G$ and $\neg G \dashv_V \triangle_V \neg G$. Hence, $\neg G \dashv_V \triangle_V \bot$. So, by Fe6, we have $\neg G \dashv_V \bot$. □

Theorem 8.6. *Suppose* $\mathsf{HA} \leq T$ *and that* T_0 *verifies* Di *for* T, *i.e. that* Di *is in* $\Lambda_{\mathsf{F}_{1,U},T_0}$. *Then, we have* $T_0 \vdash \sim_T (G_T \vee \neg G_T)$.

This follows immediately from Theorem 8.5. The reason why T_0 appears in the formulation is that we want the result both for $T_0 = \mathsf{i\text{-}EA}$ and for $T_0 = T$.

Open Question 8.7 Can we extend Theorem 8.6 to cases where $\mathsf{HA} \leq T$ fails? ⊞

We can now show that the preservativity logic of PA^* does not contain Di and $\mathsf{CB_a}$. We first prove a purely modal result that delivers both cases. We can achieve it in two ways.

Theorem 8.8.

A. $\mathsf{i\text{-}GW_a} + \mathsf{CB}_\Box \vdash (p \dashv \bot \wedge \neg p \dashv \bot) \to \Box\bot$.
B. $\mathsf{i\text{-}GL_a} + \mathsf{CB}_\Box \vdash \boxdot(p \dashv \bot \wedge \neg p \dashv \bot) \to \Box\bot$.

Proof. (A): We reason in $\mathsf{i\text{-}GW_a} + \mathsf{CB}_\Box + (p \dashv \bot \wedge \neg p \dashv \bot)$. By Di, we have (a) $(p \vee \neg p) \dashv \bot$. On the other hand, we have, by $\mathsf{CB}_\Box$, that $\Box\bot \to (p \vee \neg p)$. By $\mathsf{N_a}$, we have (b) $\Box\bot \dashv (p \vee \neg p)$. Combining (a) and (b), we find $\Box\bot \dashv \bot$ and, hence, by $\mathsf{W_a}$, we obtain $\Box\bot$.

(B): We reason in $\mathsf{i\text{-}GL_a} + \mathsf{CB}_\Box + \boxdot(p \dashv \bot \wedge \neg p \dashv \bot)$. By Di, we have (a) $(p \vee \neg p) \dashv \bot$. The principle $\mathsf{CB}_\Box$ gives us $\Box\bot \to (p \vee \neg p)$. It follows, by $\mathsf{N_a}$, that $\Box\Box\bot \to \Box(p \vee \neg p)$. Ergo, we have $\Box\Box\bot \to \Box\bot$. We now apply the extended Löb's Rule, using that our assumption $\boxdot(p \dashv \bot \wedge \neg p \dashv \bot)$ is self-necessitating, to conclude that $\Box\bot$. □

As an immediate consequence of Theorems 8.5, 8.6 and 8.8, we have:

Theorem 8.9. *Suppose* $\mathsf{HA} \leq T$ *and* T *is* Σ^0_1*-sound. Then, we cannot have both* Di *and* $\mathsf{CB}_\Box$ *in* Λ°_T.

Theorem 8.10. *Neither* Di *nor* $\mathsf{CB_a}$ *are in* $\Lambda^\circ_{\mathsf{PA}^*}$.

Proof. Since PA^* is Σ^0_1-sound and validates $\mathsf{CB}_\Box$, by Theorem 8.9, it cannot validate Di. Suppose now PA^* validates $\mathsf{CB_a}$. Then $\Lambda^\circ_{\mathsf{PA}^*}$ extends $\mathsf{i\text{-}mHC_a}^- = \mathsf{iA}^- + \mathsf{S}_\Box + \mathsf{CB_a}$. It follows, by Lemma 4.16c, that $\Lambda^\circ_{\mathsf{PA}^*}$ contains Di. Quod non, as we just saw. □

Another salient consequence of Theorems 8.5, 8.6 and 8.8 is the following result.

Theorem 8.11. *For no* $T \geq \mathsf{HA}$, *we have:* $\Lambda^\circ_T = \mathsf{i\text{-}PreL} + \mathsf{CB}_\Box$.

Proof. Suppose $\mathsf{HA} \leq T$. Clearly, if $(\mathsf{i\text{-}PreL} + \mathsf{CB}_\Box) \subseteq \Lambda^\circ_T$, it follows that $T \vdash \Box_T\bot$. But then $\Box\bot \in \Lambda^\circ_T$. On the other hand, by a simple Kripke model argument, we can show that $\mathsf{i\text{-}PreL} + \mathsf{CB}_\Box \nvdash \Box\bot$. □

Thus, not every extension of $\mathsf{i\text{-}PreL}^-$ can be obtained as the preservativity logic of a $T \geq \mathsf{HA}$. We finish this subsection by giving a better condition under which $\mathsf{CB_a}$ cannot be in the preservativity logic of a theory. This condition will again imply that $\mathsf{CB_a}$ is not in $\Lambda^\circ_{\mathsf{PA}^*}$.

Theorem 8.12. *Suppose* $\mathsf{HA} \leq T$, T *has the disjunction property and* T *is consistent. Then,* Λ°_T *does not contain* $\mathsf{CB_a}$.

Proof. Suppose $\mathsf{HA} \leq T$, T has the disjunction property and T is consistent. Moreover, suppose Λ°_T contains $\mathsf{CB_a}$. We will derive a contradiction.

Let $G := G_T$. Since $T \vdash G \dashv\!\!\exists\ \bot$, it follows, by $\mathsf{CB_a}$, that $T \vdash G \vee \neg G$. Hence, by the disjunction property, we find $T \vdash G$ or $T \vdash \neg G$. Hence $\top \dashv\!\!\exists_T\ G$ and $\top \dashv\!\!\exists_T \neg G$. Ergo, $T \vdash \bot$. □

We note that PA trivially satisfies $\mathsf{CB_a}$. Moreover, $\mathsf{HA} \leq \mathsf{PA}$ and PA is (hopefully) consistent. However, PA does not have the disjunction property.

9. Conclusions

We are not nearly done, but our space is running out: if we did not stop now, we would have to turn this paper into a monograph. We hope to have convinced the reader that constructive $\dashv\!\!\exists$ provides a fascinating subject of research wherever it appears—be it computer science, philosophy or, especially, metatheory of arithmetic. This last context is particularly rife in challenges, despite decades of diligent research in the area. Let us highlight again several lists of unsolved problems regarding arithmetical interpretations: Open Questions 5.4., 5.7., 5.9., 8.3, 8.7 and (in Appendix C) C.4 and C.11.

This, however, is not the only area where interesting open questions abound. As a simple example, consider the study of axiomatization and proof systems for various fragments of $\mathcal{L}_{\dashv\exists}$ (e.g., Open Questions 4.3. and 4.7.). Moreover, we have only briefly touched on the question of computational significance of $\dashv\!\!\exists$. Extending category-, proof- and type-theoretic frameworks for "strong arrows" in computer science (Section 7 and references therein) and providing Curry–Howard/computational interpretations of different axioms in Table 4.2 (cf. in particular Open Questions 7.1. and 7.4) would seem a natural research direction.

A century after the publication of Lewis' first papers on $\dashv\!\!\exists$ and the *Survey*, the full potential of the strict implication connective still remains to be exploited. It could have been otherwise if Lewis followed his evident interest in non-boolean logics (cf. Section 2.2). Another decision which in hindsight proved premature was to insist on principles like $\mathsf{App_a}$ in even the weakest variant of his system (cf. Remark 7.3), which effectively rules out some of the most fruitful provability-motivated applications of $\dashv\!\!\exists$. With these conceptual blocks out of the way and having the advantage of an additional century worth of research on constructive logic, we have no excuse not to carry the torch further.

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Appendix A. A recap of realizability

We need Kleene's T-predicate: $\mathsf{T}(e, x, p)$ means p is a halting computation for the partial recursive function with index e on input x. We write $\mathsf{U}(p) = y$ for: the result of computation p is y. We employ the usual assumptions that for at most one p we have $\mathsf{T}(e, x, p)$ and that $\mathsf{U}(p) \leq p$. Define:

- $e \cdot x = y$ if $\exists p\, (\mathsf{T}(e, x, p) \wedge \mathsf{U}(p) = y)$.
- $p : (e \cdot x = y)$ if $\mathsf{T}(e, x, p) \wedge \mathsf{U}(p) = y$.
- $e \cdot^z x = y$ if $\exists p \leq z \;\; p : (e \cdot x = y)$ or $(\forall q \leq z\, \neg \mathsf{T}(e, x, q) \wedge y = 0)$.
- $e \cdot x \downarrow$ for (the partial recursive function with index) e being defined on x and $e \cdot x \uparrow$ otherwise.

Sometimes we will need Kleene application for functions of several arguments. In such cases, we will write $x \cdot (\vec{y}\,)$. The tuple $(\vec{y}\,)$ is tacitly identified with a number, in particular we use ε for the (code of the) empty sequence.

We have several variants of the (intuitionistic) Church's Thesis:

CT_0 $\forall x\, \exists y\, Axy \to \exists e\, \forall x\, \exists y\, (e \cdot x = y \wedge Axy)$. This is the standard arithmetical form of the Thesis, with only numerical quantifiers appearing (modulo a version of the choice principle), rather than an universally quantified function symbol [121, 1.11.7,p.95], [123, 4.3,p.193].

$\mathsf{CT}_0!$ $\forall x\, \exists! y\, Axy \to \exists e\, \forall x\, \exists y\, (e \cdot x = y \wedge Axy)$. This slightly weakened form will play a central rôle in Appendix C.4.2, where more references are provided.

ECT_0 is the *extended Church's Thesis* [123, 4.4,p.199], [121, 3.2.14,p.195]:

$$\forall x\, (Bx \to \exists y\, Axy) \to \exists e\, \forall x\, (Bx \to \exists y\, (e \cdot x = y \wedge Axy))$$

where B ranges over *almost negative* formulas:

- $B ::= S \mid (B \wedge B) \mid (B \to B) \mid \forall v\, B$

and S ranges over Σ^0_1-formulas. Almost negative formulas will play an important rôle in Appendix C.4.3.

From Section 5.4.1 on, we have been using the notion of *q-realizability* [121, § 3.2.3, p. 189], a variant of the usual Kleene realizability:

$$\begin{aligned}
x\widetilde{q}A &:= A \qquad (A \text{ atomic}) \\
x\widetilde{q}(A \wedge B) &:= (j_1x)\widetilde{q}A \wedge (j_2x)\widetilde{q}B \\
x\widetilde{q}(A \vee B) &:= (j_1x = 0 \to (j_2x)\widetilde{q}A) \vee (j_1x \neq 0 \to (j_2x)\widetilde{q}B) \\
x\widetilde{q}(A \to B) &:= (A \to B) \wedge \forall v(v\widetilde{q}A \to \exists z(x \cdot v = z \wedge z\widetilde{q}B)) \\
x\widetilde{q}(\exists v A v) &:= (j_2x)\widetilde{q}A(j_1x) \\
x\widetilde{q}(\forall v A v) &:= \forall v(Av \wedge \exists z(x \cdot v = z \wedge z\widetilde{q}Av))
\end{aligned}$$

where j_1, j_2 are the inverses of a chosen pairing function. Note that, unlike Troelstra [121, § 3.2.3], we choose to plug additional conjuncts into clauses for $\to$ and $\forall$, rather than for $\vee$ and $\exists$.

Apart for Troelstra [121] and Troelstra and van Dalen [123], another reference on realizability in HA we recommend is Dragalin [38].

Appendix B. $\varPi^0_1$-conservativity

In this appendix, we discuss both classical and constructive interpretability logic.

An arithmetical theory U is $\varPi^0_1$-conservative over a theory T or $T \blacktriangleright U$ if, for all $\varPi^0_1$-sentences P, we have, if $U \vdash P$, then $T \vdash P$.[35,36] We write $A \blacktriangleright_T B$ for $(T + A) \blacktriangleright (T + B)$.

We expand the language of propositional logic with the unary $\Box$ and the binary $\rhd$. Consider any theory T. We set $\mathsf{F}_{2,T}(\Box) := \mathsf{prov}_T(v_0)$ and $\mathsf{F}_{2,T}(\rhd) := \mathsf{picon}_T(v_0, v_1)$. *Par abus de langage*, we write $\blacktriangleright_T$ for $\rhd_{\mathsf{F}_{2,T}}$, thus introducing an innocent ambiguity. We write $\Lambda^{\bullet}_T$ for $\Lambda_{T,\mathsf{F}_{2,T}}$.
We note that a $\varPi^0_1$-sentence is constructively equivalent to the negation of $\varSigma^0_1$ sentence. This implies that $A \to P$ is equivalent to $\neg\neg A \to P$. Thus, we find that $\neg\neg A$ and A are mutually $\varPi^0_1$-conservative over T. This means that $\Box_T$ can only be defined from $\blacktriangleright_T$ for theories in which $\Box_T A$ and $\Box_T \neg\neg A$ are provably equivalent for all A. Hence, in general provability cannot be defined from $\varPi^0_1$-conservativity over constructive theories.

B.1. The classical case

Consider a classical theory T. We have T-verifiably that $A \dashv_T B$ iff $\neg B \blacktriangleright_T \neg A$, and that $A \blacktriangleright_T B$ iff $\neg B \dashv_T \neg A$. Thus, over T, $\varSigma^0_1$-preservativity and $\varPi^0_1$-conservativity are intertranslatable. This tells us that the $\varSigma^0_1$-preservativity logic of T can be found via a transformation of the $\varPi^0_1$-conservativity logic of T.

The logic ILM consists of $\mathsf{c\text{-}GL}_{\Box}$ plus the following principles.

J1	$\Box(\phi \to \psi) \to \phi \rhd \psi$	BL	$\Box(\phi \to \psi) \to \phi \dashv \psi$
J2	$(\phi \rhd \psi \wedge \psi \rhd \chi) \to \phi \rhd \chi$	Tr	$(\phi \dashv \psi \wedge \psi \dashv \chi) \to \phi \dashv \chi$
J3	$(\phi \rhd \chi \wedge \psi \rhd \chi) \to (\phi \vee \psi) \rhd \chi$	$\mathsf{K_a}$	$(\phi \dashv \psi \wedge \phi \dashv \chi) \to \phi \dashv (\psi \wedge \chi)$
J4	$\phi \rhd \psi \to (\Diamond\phi \to \Diamond\psi)$	LB	$\phi \dashv \psi \to (\Box\phi \to \Box\psi)$
J5	$\Diamond\phi \rhd \phi$	$\mathsf{4_a}$	$\phi \dashv \Box\phi$
M	$\phi \rhd \psi \to (\phi \wedge \Box\chi) \rhd (\psi \wedge \Box\chi)$	$\mathsf{M_a}$	$\phi \dashv \psi \to (\Box\chi \to \phi) \dashv (\Box\chi \to \psi)$

The list of principles for preservativity given above is equivalent to $\mathsf{c\text{-}PreL} := \mathsf{i\text{-}PreL}^- + \mathsf{em}$. Lemma 4.1, Fact 4.2, Lemmas 4.11 and 4.14.

Theorem 12 of [14] yields that the $\varPi^0_1$-conservativity logic of T is ILM whenever T is an extension of $\mathrm{I}\varPi^-_1 + \mathsf{Exp}$. This class of theories contains such salient theories as $\mathrm{I}\varSigma_1$ and PA.

Thus, we have justified Theorem 5.10, which tells us that $\Lambda^{\circ}_T = \mathsf{c\text{-}PreL}$ if T is a $\varSigma^0_1$-sound extension of $\mathrm{I}\varPi^-_1 + \mathsf{Exp}$.
We note that the principle corresponding to $\mathsf{L_a}$ would have been:

(†) $(\phi \blacktriangleright \psi \wedge \phi \blacktriangleright \chi) \to \phi \blacktriangleright (\psi \wedge \chi)$.

[35] The use of the notation $\blacktriangleright$ is just local in this paper. Often one uses $\rhd_{\varPi^0_1}$.

[36] Reflection of the general case, where we also consider non-arithmetical theories, reveals that $\varPi_1$-conservativity is 'really' a relation between interpretations of a basic arithmetical theory in various theories.

Let T be a Σ_0^1-sound theory with $\mathsf{PA} \leq T$. Consider the sentence $G := G_T$ from Section 8.2. Suppose T satisfies (†). We have, in T, both $\top \blacktriangleright_T G$ and $\top \blacktriangleright_T \neg G$. It follows that we have $\top \blacktriangleright_T \perp$, i.e. $\Box_T \perp$. However, this contradicts Σ_0^1-soundness.

B.2. *The constructive case*

In this subsection we zoom in on the case of HA. Here the situation for Π_1^0-conservativity is quite different. We still have, HA-verifiably, $A \blacktriangleright_{\mathsf{HA}} B$ iff $\neg B \dashv_{\mathsf{HA}} \neg A$. However, we do not have the equivalence of $A \dashv_{\mathsf{HA}} B$ and $\neg B \blacktriangleright_{\mathsf{HA}} \neg A$. The equivalence fails in both directions.

We have $(\neg\neg \Box_{\mathsf{HA}}\perp \to \Box_{\mathsf{HA}}\perp) \dashv \Box_{\mathsf{HA}}\perp$ [129], but we do not have $\neg \Box_{\mathsf{HA}}\perp \blacktriangleright_{\mathsf{HA}} \neg(\neg\neg \Box_{\mathsf{HA}}\perp \to \Box_{\mathsf{HA}}\perp)$, as this is equivalent to $\Box_{\mathsf{HA}}\neg\neg \Box_{\mathsf{HA}}\perp$.

In the other direction, trivially, we do have $\neg(\Box_{\mathsf{HA}}\perp \vee \neg \Box_{\mathsf{HA}}) \blacktriangleright_{\mathsf{HA}} \perp$. But as shown in Section 5.3, $\top \dashv (\Box_{\mathsf{HA}}\perp \vee \neg \Box_{\mathsf{HA}}\perp)$ fails.

It is easily seen that the logic $\Lambda^{\bullet}_{\mathsf{HA}}$ contains i-ILM, the theory axiomatized by i-GL$_{\Box}$ + J1–5 + M. However, it contains more. As noted above, we have the principle $\vdash \neg\neg\phi \rhd \phi$.

Appendix C. Interpretability

In this appendix, we discuss both classical and constructive interpretability logic.

C.1. *Basics*

NB: The definitions of this subsection work for all theories in finite signature. So in this subsection the theory need not be arithmetical and the axiom set can be just any set of axioms regardless of the complexity.

As is well known, purely relational signatures can simulate signatures with terms via a term-unraveling procedure. Thus, we can justify defining interpretations only for relational languages. A *one-dimensional translation* τ between relational signatures Ξ and Θ provides a domain formula $\delta_\tau(v_0)$ of signature Θ and assigns to each n-ary Ξ-predicate a Θ-formula $P_\tau(v_0, \ldots, v_{n-1})$. Here the variables of δ_τ and P_τ are among those shown. We define a translation $A \mapsto A^\tau$ from Ξ-formulas to Θ-formulas as follows:

- $P^\tau(x_0, \ldots, x_{n-1}) := P_\tau(x_0, \ldots, x_{n-1})$ (in case an x_i is not free for v_i in $P_\tau(v_0, \ldots, v_{n-1})$, we employ the mechanism of renaming bound variables.)
- $(\cdot)^\tau$ commutes with the propositional connectives.
- $(\forall x\, B)^\tau := \forall x\, (\delta_\tau(x) \to B^\tau)$, $(\exists x\, B)^\tau := \exists x\, (\delta_\tau(x) \wedge B^\tau)$.

Nota bene: we also allow identity to be translated to a different formula.

We can define the more complex notion of *many-dimensional translation with parameters*. In the many-dimensional case a sequence of objects of the interpreting theory represents an object in the interpreted theory. In the case with parameters allow a sequence of extra free variables, the parameters, to occur in the domain formula and in the translations of the predicate symbols.

Suppose T has signature Θ and U has signature Ξ. We define:

- An interpretation $K : U \to T$ is a triple $\langle U, \tau, T\rangle$, such that, for all Ξ-sentences A, if $U \vdash A$, then $T \vdash A^\tau$.

- $T \rhd U$ if there is an interpretation $K : U \to T$.
- $A \rhd_T B$ if $(T + A) \rhd (T + B)$.

If we allow parameters, we add a parameter-domain α_K to the specification of K. We demand that $K : U \to T$ iff, T proves that α_K is non-empty and that, for all Ξ-sentences A, if $U \vdash A$, then $T \vdash \forall \vec{w}\, (\alpha_K(\vec{w}) \to A^{\tau,\vec{w}})$.

We write δ_K for δ_{τ_K} and P_K for P_{τ_K}. For more information about the definition of an interpretation, see e.g. [133] and [136].

In the case of extensions of i-EA as the interpreting theory one can show that, for our purposes, allowing many-dimensional interpretations makes no difference. We can eliminate the higher dimensions using Cantor pairing. In case we have extensions of PA as the interpreting theory, allowing parameters makes no difference. We can eliminate parameters using the Orey–Hájek Characterization that guarantees an interpretation without parameters whenever there is an interpretation.

In case we are not considering extensions of PA, it is in most cases unknown whether allowing parameters has an effect on the interpretability logic.

If the interpreting theory is an extension of PA we can always eliminate domain relativization and we can always replace an interpretation by an identity preserving equivalent. In case the interpreted theory has PA-provably infinitely many arguments, we even can do both at the same time.

If the interpreting theory is classical and does define one element in the interpreted theory, we can eliminate the domain relativization by setting all elements outside the original domain equal to the definable element. If we allow parameters we can eliminate the domain relativization always as long as the interpreting theory is classical.

C.2. Interpretability logic introduced

The relation $\rhd_T$ can be arithmetized, say by int_T. We expand the language of propositional logic with the unary $\Box$ and the binary $\rhd$. Consider any theory T with a $\Delta_0(\mathsf{exp})$-axiomatization. We set $\mathsf{F}_{3,T}(\Box) := \mathsf{prov}_T(v_0)$ and $\mathsf{F}_{3,T}(\rhd) := \mathsf{int}_T(v_0, v_1)$. *Par abus de langage*, we write $\rhd_T$ for $\rhd_{\mathsf{F}_{3,T}}$, thus introducing an innocent ambiguity. We write $\widetilde{\Lambda}_T$ for $\Lambda_{T,\mathsf{F}_{3,T}}$.

In the classical case $\Box_T A$ is equivalent to $\neg A \rhd_T \bot$. Thus, classically, we also have the option to expand only with $\rhd$ and treat $\Box$ as a defined symbol. This equivalence can fail intuitionistically. One can see this, e.g., by taking $T := \mathsf{HA}$ and $A := (\Box_{\mathsf{HA}}\bot \vee \neg \Box_{\mathsf{HA}}\bot)$. At present it is unknown whether $\Box_{\mathsf{HA}} A$ is HA-provably equivalent to $\top \rhd_{\mathsf{HA}} A$, so we cannot exclude that there would be a definition of the $\Box$ in terms of interpretability over HA.

C.3. Classical interpretability logic

Over PA arithmetic interpretability and Π^0_1-conservativity coincide. Thus, the $\widetilde{\Lambda}_{\mathsf{PA}} = \mathsf{ILM}$. The arithmetical completeness of ILM for interpretability over PA was proven by Berarducci [16] and Shavrukov [108] proved that this result also holds for all Σ^1_0-sound extensions of PA.[37] The reader is referred to [6,58,130] for more information about classical interpretability logic.

[37] If we leave, for a moment, the context of arithmetical theories, we can say that the result holds for all classical essentially reflexive sequential theories (with respect to some interpretation of arithmetic).

We know two further arithmetically complete interpretability logics. The first is ILP. This is the logic of Σ^1_0-sound finitely axiomatized extensions of EA^+, also known as $\mathrm{I}\Delta_0 + \mathsf{Supexp}$. If we take the contraposed preservativity-style version of ILP, we obtain the logic $\mathsf{i\text{-}GW}^-_{\mathsf{a}} + \mathsf{P_a} + \mathsf{em}$ [128].

C.4. *Constructive interpretability logic*

In this subsection we treat constructive interpretability logic with the interpretability logic of HA as our main focus. We need some preliminary material to get the discussion off the ground.

C.4.1. *i-Isomorphism*

The materials of the subsubsection work for any theories of finite signature.

We will need the notion of *i-isomorphism* between interpretations. Two interpretations $K, M : U \to T$ are *i*-isomorphic if there is an *i-isomorphism* G between K and M. A T-formula G is an *i-isomorphism between* K *and* M if the theory T verifies that 'G is a bijection between δ_K and δ_M that preserves the predicate symbols of U'. For example if P is unary, we ask: $T \vdash \forall u \forall v\, ((\delta_K(u) \wedge \delta_M(v) \wedge G(u, v)) \to (P^K(u) \leftrightarrow P^M(v)))$.
Let T be any extension of HA. Suppose $K : \mathsf{i\text{-}EA} \to T$. We also have the identical interpretation $\mathcal{E} : \mathsf{i\text{-}EA} \to T$ that translates all predicate symbols to themselves. E.g. $\mathsf{A}_{\mathcal{E}}(v_0, v_1, v_2) := \mathsf{A}(v_0, v_1, v_2)$, where A is the relation representing addition. Then, by a special case of the Dedekind–Pudlák Theorem, there exists a formula F such that T proves that F is an initial embedding of $\mathcal{E}$ in K. Now it is easy to see that $\mathcal{E}$ is i-isomorphic to K iff T proves that F is surjective. Thus, there is a single fixed statement, say C_K, that expresses that $\mathcal{E}$ is i-isomorphic to K.[38]

C.4.2. $\mathsf{CT}_0!$

In this subsection, we present some basic facts about $\mathsf{CT}_0!$ (cf. Appendix A), which we will use to derive a new interpretability principle over HA.
The theorem below is proven in [134]. For completeness' sake, we repeat the proof here. The proof is an adaptation of the proof of Tennenbaum's Theorem. Such proofs were used before to prove the categoricity of $\mathsf{i\text{-}EA}$ in constructive meta-theories under the assumption of Church's Thesis and Markov's Principle. By taking some extra care we can avoid the assumption of Markov's Principle.

Theorem C.1. *The theory* $\mathsf{i\text{-}EA}$ *verifies the following. Suppose* T *extends* $\mathsf{HA} + \mathsf{CT}_0!$ *and* $K : T \rhd \mathsf{i\text{-}EA}$*. Then,* $T \vdash \mathsf{C}_K$.

Proof. We give the proof for the case without parameters. We need minor modifications to add parameters.
Suppose T extends $\mathsf{HA} + \mathsf{CT}_0!$ and $K : T \rhd \mathsf{i\text{-}EA}$. We note that $\mathsf{i\text{-}EA}$ proves that $\lambda e \lambda x.(e \cdot^z x)$ is total. Let $\mathsf{sig}(x) = 1$ if $x > 0$ and $\mathsf{sig}(x) = 0$ if $x = 0$. Let F be the initial embedding of $\mathcal{E}$ in K.

[38] We need minor modifications of the formulation in case we have parameters.

We work in T. Fix an element z of δ_K. We define the operation $*$ as follows.

- $e{*}^z x = y$ if $\exists e' \, \exists x' \, \exists y' \, (F(e, e') \wedge F(x, x') \wedge F(y, y') \wedge (\mathsf{sig}(e' \cdot^z x') = y')^K)$.

It is easy to see that $H_z := \lambda e.(1 - e{*}^z e)$ is a total 0,1-valued function. By $\mathsf{CT}_0!$, there is a recursive function that computes H_z, say with index h. Let $p : (h \cdot h = i)$. Suppose $F(h, h')$ and $F(i, i')$ and $F(p, p')$. We have $H_z(h) = i$ and, hence, $h{*}^z h = 1 - i$. This means that $(\mathsf{sig}(h' \cdot^z h') = 1 - i')^K$. On the other hand, since F is an initial embedding, we find $(p' : (h' \cdot h' = i'))^K$.
We reason inside K. In case $p' \leq z$, we have that $h' \cdot h' = h' \cdot^z h'$. Hence, $i' = \mathsf{sig}(1 - i')$. Quod non. Hence $z < p'$. We leave K.
Since F is an initial embedding, we can find a $z^* < p$ such that $F(z^*, z)$. Since z was an arbitrary element of δ_K, we may conclude that F is surjective. □

It follows that the interpretability logic of extensions T of $\mathsf{HA} + \mathsf{CT}_0!$ contains the following principle:

- $\phi \rhd \psi \to \Box(\phi \to \psi)$.

Remark C.2. The Tarski biconditionals TB for the arithmetical language are all sentences of the form $\mathsf{True}(\ulcorner A \urcorner) \leftrightarrow A$. It is clear that every arithmetical theory locally interprets itself plus TB. In the classical case it follows that $\mathsf{PA} \rhd (\mathsf{PA} + \mathsf{TB})$. However, we cannot have $\mathsf{HA} \rhd (\mathsf{HA} + \mathsf{TB})$. If we had $\mathsf{HA} \rhd (\mathsf{HA} + \mathsf{TB})$, then we would have $K : (\mathsf{HA} + \mathsf{CT}_0!) \rhd (\mathsf{HA} + \mathsf{TB})$, for some K. However, since the reduct of K to the arithmetical language is i-isomorphic to $\mathcal{E}$, this would enable us to define truth for the arithmetical language in $\mathsf{HA} + \mathsf{CT}_0!$. By Tarski's Theorem on the undefinability of truth, we would find that $\mathsf{HA} + \mathsf{CT}_0!$ is inconsistent. Quod non. □

For some further information about $\mathsf{CT}_0!$, see [99].

Remark C.3. With respect to interpretability, there is a certain analogy between $\mathsf{HA} + \mathsf{CT}_0!$ and HA^*.

In [134], the following result is proved. Let τ be translation from the arithmetical language to itself. Consider the theory $T := \mathsf{HA}^* + (\text{i-}\mathsf{EA})^\tau$. Clearly, τ carries an interpretation of i-EA in T. Let F_τ be the standard embedding of the T-numbers into the τ-numbers. We have:

$$\mathsf{HA}^* + (\text{i-}\mathsf{EA})^\tau \vdash \forall y \, (\delta_\tau(y) \to (\exists x \, \mathsf{F}_\tau(x, y) \vee \Box_{\mathsf{HA}} \bot)).$$ [39]

It is easy to see that we cannot generally eliminate the $\Box_{\mathsf{HA}}\bot$ from the result since $\mathsf{PA} + \Box_{\mathsf{PA}}\bot$ is an extension of HA^*. The theory $\mathsf{PA} + \Box_{\mathsf{PA}}\bot$ has many non-trivial interpretations of i-EA. It has not been explored whether the result described here throws any shadows on the interpretability logic of HA. □

C.4.3. The interpretability logic of HA

The interpretability logic of HA has not yet been studied. It seems to us that there are some good reasons for this neglect, the first being that the more basic problem of the provability logic of HA is still wide open. Unlike the case of the logic of Σ^0_1-preservativity, there are no indications that the study of the logic of interpretability will help in the study of provability logic.

[39] In case τ has parameters a slight adaptation of the formulation is needed.

Interpretability itself is intuitionistically significant, e.g., the usual translations of elementary syntax in arithmetic work equally well classically and intuitionistically. But – and here is our second reason – the usefulness of interpretations to compare arithmetical theories is much diminished. For example, the $\neg\neg$-translation does not commute with disjunction, and, thus, fails to carry an interpretation. The demand of commutation with disjunction and existential quantification is much more restrictive intuitionistically than classically.

Still, studying the differences between the interpretability logic of HA and that of PA highlights how the classical principles depend on the chosen logic. Also, the relevant methods are quite interesting. Finally, a good friend makes an appearance here: Tennenbaum's Theorem plays a significant rôle.

Which of the axioms of ILM remain in the interpretability logic of HA? The principles of $\mathsf{i}\text{-}\mathsf{GL}_{\Box}$ and the principles J1,2,4 and M are valid over HA. However, J5 fails since, e.g., its instance $\Diamond\Box\bot \rhd \Box\bot$ fails.[40] The status of J3 is unknown. We note that the classical argument for J3 does yield following weakened version.

- $(\phi \rhd \chi \wedge \psi \rhd \chi) \to ((\phi \vee \psi) \wedge \neg(\phi \wedge \psi)) \rhd \chi$.

We define the modal Σ^0_1-formulas as follows:

- $\sigma ::= \top \mid \bot \mid \Box\phi \mid (\sigma \vee \sigma)$.

The following valid principle was noted by Lev Beklemishev in conversation.

- $(\sigma \rhd \chi \wedge \sigma' \rhd \chi) \to (\sigma \vee \sigma') \rhd \chi$, with σ and σ' being modal Σ^0_1.

Open Question C.4 Let $A_0 := \forall S \in \Sigma^0_1 (\mathsf{True}_{\Sigma^0_1} S \vee \neg\,\mathsf{True}_{\Sigma^0_1} S)$ and $A_1 := \forall S \in \Sigma^0_1 (\Box_{\mathsf{HA}} S \to \mathsf{True}_{\Sigma^0_1} S)$. As $\Diamond_{\mathsf{HA}} A_1$ implies, by the Double Negation Translation, $\Diamond_{\mathsf{PA}} A_1$, we have i-$\mathsf{EA}$-verifiably $(A_0 \wedge \Diamond_{\mathsf{HA}} A_1) \rhd_{\mathsf{HA}} A_1$. We can do then the Henkin construction for $\mathsf{PA} + A_1$ using the decidability for Σ^0_1-sentences. We also have trivially $A_1 \rhd_{\mathsf{HA}} A_1$. But do we have:

$$((A_0 \wedge \Diamond_{\mathsf{HA}} A_1) \vee A_1) \rhd_{\mathsf{HA}} A_1\ ?$$

Similarly, we have for any B that $(A_0 \wedge \Diamond_{\mathsf{HA}} \top) \rhd_{\mathsf{HA}} (B \vee \neg\, B)$. But do we have for all B that

$$((A_0 \wedge \Diamond_{\mathsf{HA}} \top) \vee B) \rhd_{\mathsf{HA}} (B \vee \neg\, B)\ ? \qquad \boxplus$$

Is the classically invalid principle $\vdash (\phi \rhd \psi \wedge \phi \rhd \chi) \to \phi \rhd (\psi \wedge \chi)$ still invalid over HA? We do not know that for $\phi = \top$. However, the usual construction of Orey sentences for PA can be adapted to give a sentence O such that $A \rhd_{\mathsf{HA}} O$ and $A \rhd_{\mathsf{HA}} \neg\, O$, where A is the universal closure of an instance of *Tertium non Datur* that is sufficient to make the classical argument work.

Theorem C.1 throws a shadow downward on HA. We need to define the Γ_0-*formulas* to describe it. Let S range over Σ^0_1-formulas and let A range over almost negative formulas, as defined in

[40] The fact that $\Diamond\Box\bot \rhd \Box\bot$ is not valid for HA follows, for example, from Theorem C.8 in combination with what we already know about the provability logic of HA.

Appendix A:

- $B ::= S \mid (B \wedge B) \mid (B \vee B) \mid (A \to B) \mid \forall x\, B \mid \exists x\, B$.

Anne Troelstra shows in [121, §3.6.6] that $\mathsf{HA} + \mathsf{ECT}_0$ is Γ_0-conservative over HA. *A fortiori*, $\mathsf{HA} + \mathsf{CT}_0!$ is Γ_0-conservative over HA. Inspection of the proof shows that this fact is verifiable in $\mathsf{i\text{-}EA}$. We have:

Theorem C.5. *The theory* $\mathsf{i\text{-}EA}$ *verifies the following. Suppose* C *is in* Γ_0. *We have: if* $\bigwedge_{i<n}(A_i \rhd_{\mathsf{HA}} B_i)$ *and* $\mathsf{HA} \vdash \bigwedge_{i<n}(A_i \to B_i) \to C$, *then* $\mathsf{HA} \vdash C$.

Proof. Suppose C is in Γ_0 and $\bigwedge_{i<n}(A_i \rhd_{\mathsf{HA}} B_i)$ and $\mathsf{HA} \vdash \bigwedge_{i<n}(A_i \to B_i) \to C$. It follows that $\mathsf{HA}+\mathsf{CT}_0! \vdash \bigwedge_{i<n}(A_i \to B_i)$ and $\mathsf{HA}+\mathsf{CT}_0! \vdash \bigwedge_{i<n}(A_i \to B_i) \to C$. Hence $\mathsf{HA}+\mathsf{CT}_0! \vdash C$. Since C is in Γ_0, it follows that $\mathsf{HA} \vdash C$. □

Corollary C.6. *The theory* $\mathsf{i\text{-}EA}$ *verifies the following. Suppose* A *is almost negative and* B *is in* Γ_0. *Suppose further that* $A \rhd_{\mathsf{HA}} B$. *Then,* $\mathsf{HA} \vdash A \to B$.

Corollary C.7. *The theory* $\mathsf{i\text{-}EA}$ *verifies the following: if* $\top \rhd_{\mathsf{HA}} O$ *and* $\top \rhd_{\mathsf{HA}} \neg O$, *then* $\mathsf{HA} \vdash \bot$. *Thus, if* HA *is consistent, it has no Orey-sentences.*

We give counterparts of the above classes in the modal language, beginning with the almost negative ones.[41] Let ϕ range over all formulas and

- $\sigma ::= \bot \mid \top \mid \Box\phi \mid (\sigma \vee \sigma)$
- $\psi ::= \sigma \mid (\psi \wedge \psi) \mid (\psi \to \psi)$.

We define the Γ_0-formulas of the bi-modal language as follows. Let ϕ range over all formulas and let ψ range over the almost negative formulas.

- $\chi ::= \bot \mid \top \mid \Box\phi \mid (\phi \rhd \phi) \mid (\chi \wedge \chi) \mid (\chi \vee \chi) \mid (\psi \to \chi)$.

Theorem C.8. *Let* χ *be in* Γ_0. *The following principle is in the interpretability logic of* HA*:*

$$\Big(\bigwedge_{i<n}(\phi_i \rhd \psi_i) \wedge \Box\Big(\bigwedge_{i<n}(\phi_i \to \psi_i) \to \chi\Big)\Big) \to \Box\chi.$$

Example C.9. The principle $\vdash (\neg\neg\Box\bot \to \Box\bot) \rhd \neg\neg\Box\bot \to \Box\neg\neg\Box\bot$ is valid over HA. Since HA is HA-verifiably closed under the primitive recursive Markov's Rule, it follows that $\vdash ((\neg\neg\Box\bot \to \Box\bot) \rhd \neg\neg\Box\bot) \to \Box\Box\bot$ is valid over HA. □

Remark C.10. We note that the seemingly stronger principle

$$\Big(\bigwedge_{i<n}(\phi_i \rhd \psi_i) \wedge \bigwedge_{i<n}(\phi_i \to \psi_i) \rhd \chi\Big) \to \Box\chi,$$

in fact follows from Theorem C.8. □

[41] By the Orey–Hájek characterization, $A \rhd_{\mathsf{PA}} B$ is a Π^0_2-relation. (It was shown to be complete Π^0_2 independently by Per Lindström and Robert Solovay.) No such reduction is known for the relation $A \rhd_{\mathsf{HA}} B$. This relation is *prima facie* Σ^0_3 and might, for all we know, be Σ^0_3-hard. We note that Π^0_2 is almost negative but Σ^0_3 is not. So we cannot take $\phi \rhd \psi$ as a modal almost negative formula. This does not exclude that further insight might allow us to include it at a later stage.

Open Question C.11 Is there an interpretation of i-EA in HA that is not i-isomorphic to $\mathcal{E}$?

There are many strengthenings of our question. We can demand HA-verifiability of the self-interpretation. We could ask whether there is an A such that $\top \rhd_{\mathsf{HA}} A$, but $\mathsf{HA} \nvdash A$. Etcetera.

If one combines the proof of Theorem C.1 with q-realizability, one obtains the following. In case the domain and the parameter domain of an interpretation K of i-EA in HA are auto-q[a], then K is i-isomorphic with $\mathcal{E}$. Thus, a non-trivial interpretation i-EA in HA should either have a sufficiently complex domain or a sufficiently complex parameter domain.[b] Note also that $\top \rhd_{\mathsf{HA}} A$ both implies that $\Box_{\mathsf{HA}+\mathsf{CT}_0!}A$ and that $\top \rhd_{\mathsf{PA}} A$, which puts some severe constraints on the possible A. ⊞

[a] See Section 5.4.4 for the notion of *auto-q*.

[b] We apologize for the classical reasoning. However, since the relevant predicates are decidable, it can be constructively justified.

C.4.4. Interpretability and Π^0_1-conservativity

We have seen that interpretability and Π^0_1-conservativity coincide over PA. Over other classical theories, interpretability and Π_1-conservativity part ways. For example, they come apart over Primitive Recursive Arithmetic PRA: we have $\top \blacktriangleright_{\mathsf{PRA}} \mathrm{I}\Sigma_1$, but not $\top \rhd_{\mathsf{PRA}} \mathrm{I}\Sigma_1$.

Over HA, interpretability and Π_1-conservativity likewise separate ways. We still find that, HA-verifiably, $\rhd_{\mathsf{HA}}$ is a sub-relation of $\blacktriangleright_{\mathsf{HA}}$. However, for example, we have $\Diamond\Box_{\mathsf{HA}}\bot \blacktriangleright_{\mathsf{HA}} \Box_{\mathsf{HA}}\bot$, but not $\Diamond\Box_{\mathsf{HA}}\bot \rhd_{\mathsf{HA}} \Box_{\mathsf{HA}}\bot$. Also, $\neg\neg\,\Box_{\mathsf{HA}}\bot \blacktriangleright_{\mathsf{HA}} \Box_{\mathsf{HA}}\bot$, but not $\neg\neg\,\Box_{\mathsf{HA}}\bot \rhd_{\mathsf{HA}} \Box_{\mathsf{HA}}\bot$.[42]

Appendix D. The problem of the "Survey"

We are returning here to the issue briefly mentioned in the main text: the collapse of $\multimap$ to $\to$ in Lewis' original system [68,70] discovered by Post and addressed by Lewis in a subsequent note [71]. This episode is instructive in illustrating how Lewis' own thinking about $\multimap$ was often sabotaged by a combination of several factors, including:

- an insistence on boolean laws for "material" connectives, including in particular classical, involutive laws for negation;
- especially in the 1910's, a certain carelessness in accepting deductive laws for "intensional" connectives, especially those involving contraposition.

The second problem was pretty much admitted by Lewis himself:

> In developing the system, I had worked for a month to avoid this principle, which later turned out to be false. Then, finding no reason to think it false, I sacrificed economy and put it in ([72], via [95, p. 92]).

In hindsight, these problems are unsurprising, especially given the publication date of the *Survey*. Not only were non-boolean systems in the prenatal stage, but also semantics of propositional logics was poorly understood at the time. *Symbolic Logic* published in 1932 was already in a much better position, mostly thanks to efforts of Mordechaj Wajsberg and William Parry, who

[42] These results follow from Theorem C.8 in combination with facts about provability logic.

provided several crucial algebraic (counter)models used in Appendix II to establish independence results for axioms between S1 and S5. No such assistance was available to Lewis when writing the earlier *Survey* and consequently, when deciding whether or not to adopt a specific axiom for $\prec$, he would mostly rely just on his philosophical intuitions, much like other authors in that period.[43]

From our point of view, it is of particular interest to isolate the actual rôle played by classical logic with its involutive negation, the axiom Box and redefinition of $\Box$ as (1) in the collapse of the system of the *Survey*.

The problematic axiom is the converse of the one which was latter baptized A8 in Appendix II to *Symbolic Logic* :

A8 $(\phi \prec \psi) \prec (\neg\Diamond\psi \prec \neg\Diamond\phi)$.

In the *Survey*, this axiom was postulated as a strict *bi*-implication, i.e.,

$$(\phi \prec \psi) \succ\!\prec (\neg\Diamond\psi \prec \neg\Diamond\phi).$$

In our setting, with $\Box$ rather than $\Diamond$ as the primitive and with ϕ not being equivalent to $\neg\neg\phi$, the missing half can be rendered as

2.21 $(\Box\psi \prec \Box\phi) \prec (\neg\phi \prec \neg\psi)$,

"2.21" being Lewis' name for this axiom [71]. Of course, there are other conceivable variants, for example:

2.21′ $(\Box\neg\phi \prec \Box\neg\psi) \prec (\psi \prec \phi)$.

As it turns out, however, 2.21 is exactly what we need to reproduce Post's derivation over iA (together with a sub-boolean axiom Auxp introduced below).

To present further details, let us also recall that Lewis uses Modus Ponens for $\prec$, i.e., $\frac{\phi \quad \phi\prec\psi}{\psi}$ as the main inference rule. This in itself is telling: in iA, ϕ jointly with $\phi \prec \psi$ entails only $\Box\psi$. The rule $\frac{\Box\phi}{\phi}$ is *admissible*, but not *derivable*, unless one postulates as an axiom explicitly $\Box\phi \to \phi$, something that Lewis' insistence on formulating all the axioms with $\prec$ as the principal connective prevented him from doing; $\Box\phi \prec \phi$ is not quite the same thing.[44]

In the setup with Modus Ponens for $\prec$ as the central rule and $\succ\!\prec$ as the "real" equivalence or identity, instead of deducing $\phi \leftrightarrow \Box\phi$ in the extension of our iA with Lewis' axioms we need to show both $\Box\phi \prec \phi$ (which is already a theorem for Lewis, cf. the discussion of $\mathsf{App_a}$ and Remark 7.3 above) and

[43] Cf. in this respect his remark [71]: "Mr. Post's example which demonstrates the falsity of 2.21 is not here reproduced, since it involves the use of a diagram and would require considerable explanation". A "diagram" is presumably a finite matrix/algebra (which could indicate a largely overlooked inspiration Lewis' work provided for Post in developing non-classical logical "matrices", a.k.a. algebras or truth-tables!). In Appendix II to *Symbolic Logic*, the counterexamples of Parry and Wajsberg were called "groups". It is worth mentioning that early Lewis' papers tended to have titles like *Implication and the Algebra of Logic* [66], *A New Algebra of Implications and Some Consequences* [67], *The Matrix Algebra for Implications* [68] or *A Too Brief Set of Postulates for the Algebra of Logic* [69], but this should not mislead us: the word "algebra" (or "matrix") is not taken here in the sense of modern model theory or universal algebra.

[44] One can see here yet another instance of Lewis' peculiar paradox, pointed out by Ruth Barcan Marcus: despite his insistence that "the relation of strict implication expresses precisely that relation which holds when valid deduction is possible" and that "the system of Strict Implication may be said to provide that canon and critique of deductive inference" [74, p. 247], his own systems tend to run into problems with the relationship between $\to$, $\prec$, entailment and deducibility (relevance logicians would point it out too, cf. Footnote 15, but their own systems have their own share of similar problems).

4_a $\phi \strictif \Box\phi$,

deriving 4_a in turn requiring only finding another theorem χ s.t. $\chi \strictif (\phi \strictif \Box\phi)$ is also a theorem; in other words, to derive still weaker

$\Box 4_a$ $\Box(\phi \strictif \Box\phi)$.

This in turn can be done if one has both 2.21 and a law which is a mild consequence of excluded middle, namely

Auxp $(\neg\neg p \strictif \neg(\Box p \to \Box\neg p)) \strictif (p \strictif \Box p)$.

Note that to derive Auxp, it is enough to have as an axiom scheme, e.g.,

$$(\neg\neg\Box p \wedge \neg\Box\neg p) \to \Box p;$$

this is why we call Auxp a mild consequence of boolean laws.

Note also that in presence of 2.21, we have that

Auxp2 $\Box(\Box p \to \Box\neg p) \strictif \Box\neg p$.

To get this formula, substitute $\bot$ for ϕ and p for ψ in 2.21, use BL and the fact that $\Box p \to \Box\neg p \dashv\vdash_{\_} \Box p \to \Box\bot$. Now we can redo in our setting the Post derivation as quoted by Lewis. Substituting $\Box p \to \Box\neg p$ for ψ and $\neg p$ for ϕ in 2.21 yields

$$((\Box p \to \Box\neg p) \strictif \Box\neg p) \strictif (\neg\neg p \strictif \neg(\Box p \to \Box\neg p)).$$

The antecedent of this strict implication is precisely Auxp2 and the consequent is the antecedent of Auxp.

Remark D.1. Of course, there are simpler ways of collapsing the system of the *Survey* when full boolean logic and all Lewis axioms are assumed. Note that using classical logic and Box (which is an axiom for Lewis, and as we established in Corollary 4.8 can anyway be derived in $\mathsf{iA} + \mathsf{CPC}$), we can replace 2.21 with

$$\Box(\Box\psi \to \Box\phi) \to \Box(\psi \to \phi).$$

Classically, this axiom in turn can be replaced with

$$\Diamond\psi \to \Diamond(\Box\psi \wedge \Diamond\psi).$$

Now, if $\Box\phi \to \phi$ (i.e., reflexivity) is also an axiom or a theorem (which, as shown above, should be indeed the case in a modern representation of Lewis' original system, with $\frac{\Box\phi}{\phi}$ as an admissible or derivable rule), we can derive $\Box\phi \leftrightarrow \phi$, trivializing the modal operator. □

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