Chapter 5

Geometry Learning in the Early Years: Developing Understanding of Shapes and Space with a Focus on Visualization

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Abstract In this chapter, we address children's geometry learning in the early years with a focus on visualization. We start the chapter with some background information about visualization in mathematics and geometry and its relationship with language and gestures paying special attention to the early years. The next parts of the chapter aim to give insight into how young children solve geometrical activities with emphasis on the uses of visualization in developing understanding of space and shape concepts. In particular, we discuss three different research approaches which investigated young children's development of geometrical thinking when dealing with shapes' transformations, imaginary perspective taking, and space and shape aspects with the use of gestural and verbal acts. Finally, in light of the above, a number of conclusions are drawn about the multiple qualities and uses of visualization in the development of the understanding of shapes and space and the diverse factors that may intervene in early geometry learning which involves the use of visualization.

Keywords Geometry · Visualization · Shape transformation · Imaginary perspective taking · Spatial concepts · Gesture · Early years

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5.1 Introduction

Geometry is an indispensable part of contemporary early childhood curricula and educational programs (e.g., Sarama & Clements, 2009). This mathematical domain includes “the study of spatial relationships of all kinds; relationships that can be found in the three-dimensional space we live in and on any two-dimensional surface in this three-dimensional space. These relationships can be discovered all around us” (Egsgard, 1970, p. 478). This approach to geometry is in line with the view which is adopted in this chapter about early geometry, that it needs to be studied as a subject with a dynamic, spatial, and imaginative character, rather than as a static subject focusing on the naming and sorting of shapes (Moss, Hawes, Naqvi, & Caswell, 2015).

Researching geometry learning and teaching in the early years has received increasing attention in an attempt to promote the importance of geometry in early childhood and to propose new directions (Sinclair & Bruce, 2015). Nevertheless, there is still much to be done to unravel young children’s development in this domain (Dindyal, 2015). This chapter addresses young children’s geometrical thinking with a focus on visualization, which Duval (2014) considers “the first crucial point in geometry learning” (p. 1). Visualization is an inherent component of students’ making sense of aspects of space and shape when learning and thinking in geometry. For example, visualizing geometrical shapes and their manipulation can be a powerful heuristic for solving geometrical problems.

5.2 Visualization

Visualization has been defined and used in various ways in the literature. In this chapter, we adopt the broad meaning provided by Arcavi (2003) for this term.

> Visualization is the ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas, and advancing understandings. (p. 217)

Mental images, external representations, visualization processes, and abilities are major constituents of visualization (Gutiérrez, 1996). A mental image is the basic component in visualization and refers to “any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements” (Gutiérrez, 1996, p. 9). According to Presmeg (1986), mental images (of high school learners) can be classified into five types: (a) concrete images (“picture in the mind”), (b) kinaesthetic images (of physical movements), (c) dynamic images (images in the mind that are moved or transformed), (d) pattern images (visual representations of abstract relationships), and (e) images of formulae (mental images with symbols as they appear, e.g., in textbooks). Presmeg (2006) points out that the categories of mental images may overlap and that the concrete images need to be articulated with analytical thinking to be effectively employed in mathematics.

Mental images are generated and transformed on the basis of the interpretation of external representations or objects and are expressed in verbal or graphical form, so there is an interaction between mental images and external representations (Yakimanskaya, 1991). External representations which may support visual reasoning in mathematics include, for example, pictures, diagrams, drawings, and verbal descriptions standing for mathematical concepts or properties (Gutiérrez, 1996).

In visualization, two key processes of dealing with visual images are performed either mentally or physically: interpreting/transforming information into visual terms (visual processing) and interpreting visual images to produce information (interpretation of figural information) (Bishop, 1983; Gutiérrez, 1996). In performing the above processes, there are various visualization abilities that need to be developed and used, depending on the task and the images concerned. According to Gutiérrez (1996), major abilities are, for example, figure–ground perception (identifying figures out of a complex configuration), perceptual constancy (ability to focus on critical attributes of objects despite changes in other non-critical attributes, e.g., size, orientation), mental rotation, perception of spatial positions (e.g., relate an object to oneself), perception of spatial relationships (e.g., relate several objects to one another or to oneself), visual discrimination (compare objects, images, etc., to one another).

5.3 Visualization in Geometry

Like any other mathematical object, objects in geometry are abstract concepts and can be studied only through their semiotic representations. A geometrical figure is a visual representation constructed with specific tools (e.g., ruler, compass, software). A figure integrates semiotic representations which are produced within three different registers: (a) shapes, (b) magnitudes, and (c) marks and words. In a geometrical task which includes figures, three corresponding types of cognitive activity are required: seeing shapes and recognizing what is seen, measuring magnitudes and comparing, and making inferences from the given properties represented in marks and words (Duval, 2014). Given that figures possess a central role in geometrical tasks and activities, visualization is a key cognitive process of geometrical thinking. Visualization involves the recognition of figural units which can be identified in a configuration of shapes as well as operating with figures (Duval, 1995). In geometry, visualization goes beyond a “spontaneous perceptual way of seeing” a figure (Duval, 2014, p. 11). This visual perception applies to any visual representation of material objects or spatial organization (images, diagrams, plans, etc.) outside mathematics. Perceptual shape recognition is sometimes misleading for the recognition of geometrical properties and, therefore, for the recognition of the geometrical objects represented (object recognition) (Duval, 2014). On the contrary, visualization enables the simultaneous and immediate apprehension of a configuration as a whole by distinguishing at the same time all its figural units, that is, all the elements that can be visually identified in a figure (characterized by their dimensionality, for example, 2D shapes or 1D objects, i.e., lines and segments), and their interrelations and also by recognizing other possible configurations. This is a mathematical way of looking at a figure which enables learners to develop the understanding that a geometrical figure is a representation of the abstract geometrical object and its properties, and not the perceptual object. This understanding, which plays an essential role in reasoning, defining, and problem solving or proving in geometry (Duval, 2014), is difficult for children to grasp, particularly for young children (Dindyal, 2015).

Visualization is the basis for the heuristic use of figures in geometry problems, as it enables the solution of problems without explicit references to properties. This is based on transformations of the 2D figural units into other figural units of the same dimensionality (2D), which Duval (2014) called the "operative apprehension of figures" (p. 15). More specifically, operative apprehension is a form of visual processing on geometrical figures which depends on the various ways of modifying a given figure, including the mereologic way, the optic way, and the place way. The mereologic way refers to the division of the whole given figure into parts and the combination of them in another figure or subfigures (reconfiguration). The optic way is when one makes the figure larger or narrower, or slant, while the place way of modification refers to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. Within the operative apprehension, the given figure becomes a starting point to explore other configurations that stem from the applications of these visual operations (Duval, 1995).

The above theoretical considerations were focused on visualization of plane shapes. Considering that geometry refers to a model of space (Soury-Lavergne & Maschietto, 2015), spatial visualization, and reasoning is prevalent in this mathematical domain and it underlies most geometrical thinking (Dindyal, 2015; Van den Heuvel-Panhuizen & Buys, 2008). This spatial interpretation of geometry is the natural way in which young children encounter geometry. In line with this view, Freudenthal (1973) was one of the pioneers who argued strongly for starting with spatial geometry in the early years of education: "Geometry is grasping space… that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it." (p. 403) Spatial reasoning and visualization abilities develop through children's activities. For example, when playing hide-and-seek, children try to hide in a place in which they will be invisible to the child that is looking for them. Therefore, they try to imagine or to reason what the other child will and will not be able to see from different points of view.

Learning to mentally take a particular point of view and to see a model of it is an important aspect of spatial reasoning in the domain of geometry that is strongly highlighted by the NCTM Principles and Standards of School Mathematics (2000) in the K − 2 grades and also in the TAL teaching/learning trajectory for the first and second year of preschool (K1 and K2) (Van den Heuvel-Panhuizen & Buys, 2008).

5.4 Language, Gestures, and Visualization in Geometry

Geometrical figures alone are not sufficient for using properties to solve geometrical problems. Language production, either implicit or explicit, has an essential role and needs to be coordinated with visualization in doing geometry (Duval, 2014). According to Duval (2014) in geometry, three types of language production are necessary: (a) definitions of objects and properties, which are visualized through geometrical figures or configurations and theorems; (b) descriptions of the production of figures and configurations; and (c) inferences from given properties resulting in justifications or proofs. Among the three types of language production, definitions and descriptions are most closely associated with visualization (and this relationship is based on figural units rather than on whole figures).

Besides visual abilities, verbal skills need to be addressed and developed in the teaching and learning of geometry (Hoffer, 1981), beginning in early years of education (Dindyal, 2015). In geometrical activities (involving either 2D or 3D shapes), young children often share their experiences and reflections with others and thus develop their abilities to describe visual images (e.g., geometrical figures or configurations), spatial concepts, relations, and reasoning (Van den Heuvel-Panhuizen & Buys, 2008). By using language related to geometrical activities, children learn how to use geometry language which can support spatial visualization and reasoning. Developing children's geometry language means that they develop their knowledge and understanding not only of geometrical terms for shapes—"condensations of definitions" (Duval, 2014, p. 17)—but also of naming and describing actions and transformations that are performed with shapes, figures, and other objects, such as rotating, moving, and identifying their position—descriptions of configuration productions (Duval, 2014). It is to be noted that a broad perspective is taken here when using the term language, since apart from words, an indispensable component of children's communication about space and shapes is gesture (e.g., Elia, Gagatsis, & Van den Heuvel-Panhuizen, 2014).

The study of gestures in mathematical thinking, learning, and teaching has gained increased attention in mathematics education research in the last few years. The connection of gesture and visualization, acknowledged in the work of Presmeg (2006, 2014), examined the teachers' and students' visual thinking in high school mathematics classes. The use of gesture by a teacher or a student was considered a strong indication of visual imagery in mathematics learning and teaching. Gestures can serve as a dynamic representational tool of various mathematical ideas through which people can get a deeper level of consciousness of their meaning. The embodied character of gestures may facilitate the process of reaching abstract concepts through the visual modality. Moreover, as a consequence of this process, people can communicate mathematical concepts more easily (Nemirovsky & Ferrara, 2009). Thus, the roles of gestures in mathematical visualization are an issue of major importance. Further systematic investigation is still needed however (Presmeg, 2006; 2014), and this is particularly true for visualization in geometry, a domain in which the study of gestures has received limited attention, especially in the early years (Elia, Gagatsis, & Van den Heuvel-Panhuizen, 2014).

Gestures and spatial thinking are closely linked to one another, with gestures playing a significant role in cognitive processing (Alibali, 2005) and in conveying spatial information (McNeill, 1992) such as location and movement. The visuospatial nature of gesture makes it suitable for capturing spatial information. Gestures represent spatial properties and action-based characteristics of concepts (Krauss, Chen, & Gottesman, 2000). They assist speakers in activating mental images and in maintaining these spatial representations in working memory (Alibali, 2005). At the same time, using gestures to express spatial properties may support the activation of related mental representations of the concepts also in verbal form (Krauss, Chen, & Gottesman, 2000). As a result, visualization is used and manifested in diverse ways in geometry.

The next section of the chapter aims to give further insight into how young children solve geometrical activities, paying attention to the uses of visualization by children in developing understanding of space and shape concepts. In particular, we discuss three different research approaches drawn on in our studies in which visualization is used in different types of shape and spatial tasks, contexts, and with different tools and resources. Since mental images can be perceived only through external support (e.g., verbal, graphical) (Sutherland, 1995), in order to identify children's mental images and visualization abilities in these studies, we interpreted the actions produced or the outcomes of these actions that were the result of children's activity with mental images in response to the geometrical tasks.

5.5 A Dynamic Approach to Plane Geometry: Optic Transformation of Geometrical Figures

Identifying, describing, and classifying two-dimensional shapes have been the focus of a major body of research on geometry learning in mathematics education literature in the last years. Yet these aspects cover only a part of children's thinking in plane geometry. This focus probably stems from the strong and lasting influence of the Van Hiele (1985) model on mathematics education research at all age levels (Sinclair & Bruce, 2015). Taking a different perspective, Castelnuovo (1972) points out that children do not easily observe figures when they are steady, but rather when they move or vary in a continuous manner. Owens (1999) suggests that students visualize movement as they make connections between shapes. An example is constructing a square that becomes a rectangle as it gets thinner. Such intuitive thinking, which involves a continuous variation process, is called "dynamic intuition" (Castelnuovo, 1972). According to Presmeg (1986), it is a kind of dynamic visualization, as it involves the creation of dynamic mental images for specific geometrical figures.

Following this dynamic approach to geometry learning, Gagatsis, Sriraman, Elia, and Modestou (2006) explored young children's geometrical thinking by focusing on the dynamic use of drawings. Specifically, the study investigated the strategies children use in the optic transformation of 2D shapes (e.g., increasing or decreasing the size of figures) (Duval, 1995) and the relationship of these strategies with children's age and shape identification abilities.

The participants of the study were 291 children ranging from four to eight years of age. Children were asked to draw a stairway of triangles, squares, and rectangles, respectively, each one bigger than the preceding one, and then a stairway for each shape, each one smaller than the preceding one. To assess children's shape identification ability, children were asked to identify and color the squares, rectangles, and triangles among other figures. Three models of action/transformation strategies were observed in children's responses to the transformation tasks:

(a) Conservation of shape by increasing or decreasing both dimensions of a plane figure at the same time (Figs. 5.1 and 5.2). This was named as T-strategy (modification of Two-shape dimensions).
(b) Differentiating mainly one dimension of the figures: In the case of triangles, this dimension was the altitude to the base; that is why children sometimes produced isosceles triangles although their paths started with an equilateral one (most common case) (Fig. 5.3). In the case of rectangles, it was usually the longer side (Fig. 5.4); sometimes, a square occurred among the rectangles in a very natural way (Fig. 5.4). In the case of squares, children produced rectangles (Fig. 5.5). This was called O-strategy (modification of One-shape dimension).
(c) Producing a defective series of irregular figures. This was called N-strategy.

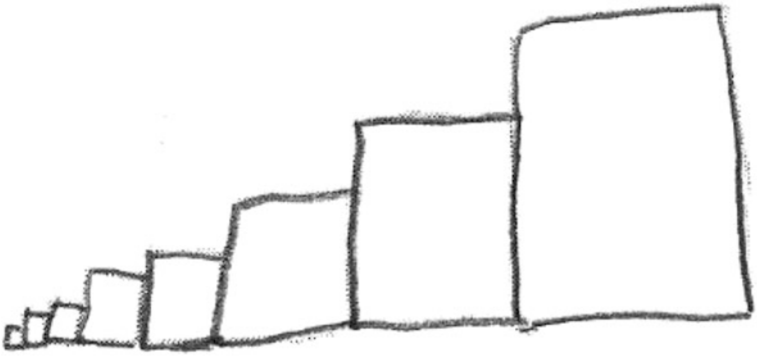


Fig. 5.1 Increasing both dimensions (T-strategy), from Gagatsis, Sriraman, Elia, & Modestou (2006, p. 34), with permission from Nordic Studies in Mathematics Education

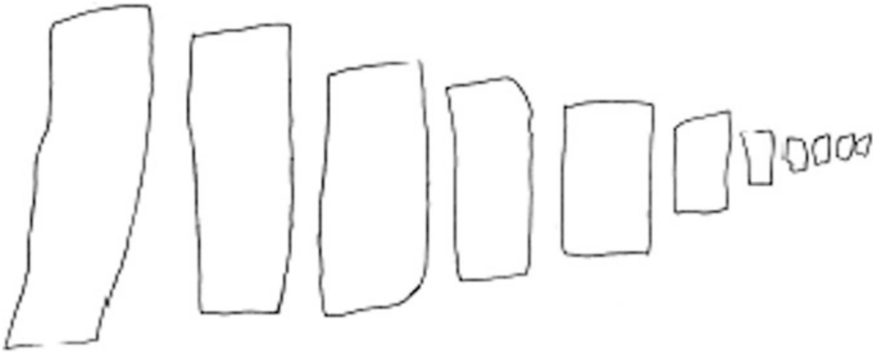


Fig. 5.2 Decreasing both dimensions (T-strategy), from Gagatsis et al. (2006, p. 34), with permission from Nordic Studies in Mathematics Education

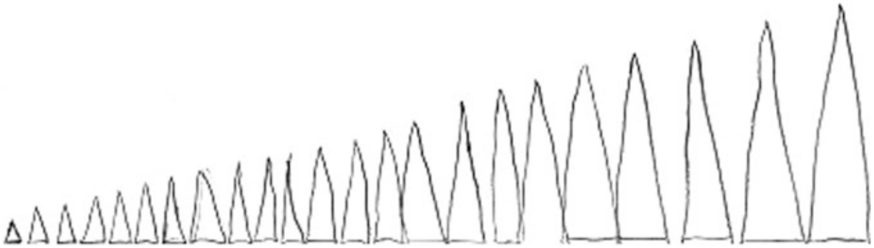


Fig. 5.3 Increasing only one dimension (O-strategy), from Gagatsis et al. (2006, p. 35), with permission from Nordic Studies in Mathematics Education

Fig. 5.4 Squares in the series of rectangles, from Gagatsis et al. (2006, p. 35), with permission from Nordic Studies in Mathematics Education

Fig. 5.5 Rectangles in the series of squares, from Gagatsis et al. (2006, p. 35), with permission from Nordic Studies in Mathematics Education

The findings of the study and specifically the results of the hierarchical similarity analysis (Gras, Suzuki, Guillet, & Spagnolo, 2008) showed that the children tended to use each strategy in a consistent way across the different figures and types of optic modification (increasing or decreasing size) in their attempt to solve the tasks. This kind of behavior may be attributed to children's different ways of looking at geometrical figures.

The application of O-strategy by many children to different tasks signifies a global apprehension of the figure with partial analysis and use of its figural units (Duval, 2014). Children following the O-strategy were not confined to keeping the prototypical form of the initial shape of the pathway. They were more inclined on the one hand to focus on the attributes they regarded as critical for the geometrical figure and keep them unchanged (e.g., number of sides, parallel or vertical sides), and on the other hand, to give less attention to the attributes, they considered as non-critical and alter them (e.g., ratio of dimensions). While in the paths of rectangle and triangle this alteration did not lead to a change in the type of figures (they remained rectangles or triangles), in the path of squares this alteration resulted in a change into (non-square) rectangles. This deficit might stem from children's lack of understanding of the relationship between squares and rectangles, a possible consequence of either prior teaching (Sarama & Clements, 2009) or not being taught geometrical shapes. Nevertheless, the capacity that these children demonstrated in visualizing dynamically the optic transformation of geometrical figures cannot be ignored. O-strategy can be seen as an indication of children's initial steps in the development of the understanding that a geometrical figure is a representation of an object and not the perceptual object itself which has to keep its initial form.

The use of T-strategy may be a result of children's attempt not to change the holistic appearance of the figure. It is likely that in dealing with the optic transformation tasks these children mainly used visual perception. This is a simple way of looking at a figure as for any image of material objects, which hinders the recognition of the geometrical objects represented (object recognition) (Duval, 2014). A further explanation, related to the former one, is the possibility that prototypical images of shapes were central to children's reasoning. These limited views of the particular shapes may have reinforced children's tendency to keep the prototypical form of the initial shape constant in the series they produced, because, otherwise, different forms of shapes may have occurred, not as "good" as the first one (Mesquita, 1998).

The construction of a defective series of irregular figures (N-approach), which was also applied consistently by the children, suggests that these children had not yet developed the ability to discriminate visually between shapes and could not even focus on the visual form of a figure irrespective of changes in size. Furthermore, it is possible that the children encountered difficulties in interpreting the verbal information in the task instructions as visual terms needed to form images for the geometrical figures and their optic modification.

Children's approaches toward the tasks varied with respect to their age. O-strategy was used mostly by seven- to eight-year-old children, while N-approach was used primarily by four- to six-year-old children. T-strategy was used mostly by children of six to seven years of age in some figures and four- to six-year-old children in other figures. Children's age, combined with the strategy they used, offered further support to the above interpretations of both children's ways and difficulties in looking at and making sense of geometrical figures.

Children's ability to recognize triangles, squares, and rectangles was found to connect only marginally with their responses to the transformation tasks. Thus, it can be concluded that shape identification requires different types of abilities from shape construction and transformation, processes that involve dynamic visualization processes.

5.6 Preschoolers' Imaginary Perspective Taking

The perception of spatial relationships, that is, whether a person can relate spatially several objects to one another or to oneself, is a major visualization and spatial reasoning ability (Gutiérrez, 1996). This ability is pertinent to mentally taking a particular point of view, an important aspect of spatial visualization that is often found in children's everyday activities (e.g., game "hide-and-seek"). To further explore the development of spatial visualization abilities in early childhood, Van den Heuvel-Panhuizen, Elia and Robitzsch (2015) set up a study focusing on the performance of preschoolers in mentally representing a viewpoint different from one's own, namely "imaginary perspective taking" (IPT). A major concern of the study was also to find out whether there were cross-cultural patterns in the IPT performances by children in different countries.

The ability of IPT can be divided into subcomponents. Flavell, Abrahams, Croft, & Flavell (1981) proposed and validated a distinction for these subcomponents into two abilities of perspective taking. Both were tested through tasks with cards placed between the experimenter and the child, in which the child had to take the experimenter's perspective. The so-called Level 1 competence concerns the visibility of objects. It implies the ability to deduce which objects are visible or not from the other viewpoint. To determine whether an object is visible, a possible strategy is to imagine oneself in the other position, projecting an observer's line of sight and verifying whether the target object meets with this line (Yaniv & Shatz, 1990). However, Michelon and Zacks (2006) suggested that this Level 1 competence might also require a line-of-sight tracing, that is, an imaginary process that acts as if an actual line is drawn between the other observer and the target object. Thus, while the former strategy in Level 1 competence involves the use of a dynamic mental image, the latter strategy is based on the creation of a concrete image (Presmeg, 1986).

The Level 2 competence concerns the appearance of object. It implies the ability to indicate how an object looks when it is seen from a different viewpoint. This competence requires a child to deal with multiple aspects of the visual appearance of an object, including features such as size, shape, and location, and to understand that these features differ when an object is seen from different perspectives

(Pillow & Flavell, 1986). Level 2 competence therefore principally involves the use of dynamic mental imagery (Presmeg, 1986), as it requires applying specific knowledge about how changes in the observer–object relationship influence aspects of the appearance. Flavell et al. (1981) found that both IPT competences are acquired by children as young as five years of age. Specifically, three-year-olds performed well on Level 1 tasks but had difficulties with Level 2 tasks, even after a brief training. Usually, this Level 2 competence is attained at about four or five years of age (Pillow & Flavell, 1986).

In Van den Heuvel-Panhuizen, Elia, & Robitzsch's study (2015), a survey was carried out in the Netherlands and in Cyprus. Table 5.1 shows the sample composition of the preschoolers that participated in the study. Their performance in IPT was assessed by administering two test booklets, each with pictorial paper-and-pencil items about imagining visibility (IPT type 1) and about imagining appearance (IPT type 2). For example, the Tower item (Fig. 5.6) is used to measure IPT type 1. In this item, the children were asked what the girl who stands on top of the tower sees. The Mouse item (Fig. 5.7) measures IPT type 2. Here, the children had to determine how the mouse looks from above.

The study confirmed previous studies' findings (e.g., Flavell, Abrahams Everett, Croft, & Flavell, 1981) that development of the IPT type 1 competence (visibility) precedes the IPT type 2 competence (appearance). Preschoolers in the Netherlands and Cyprus answered on average, respectively, 70 and 55% of the visibility items correctly, 40 and 30% of the appearance items correctly. For the visibility items, these percentages are more or less in agreement with Flavell et al. (1981), but not for the appearance items. These findings are confined by the nature of the study's items, which included drawings representing the objects, and the environment in which the objects (and sometimes also the observer) were situated (2D representations). In Flavell's studies, the perspective taking tasks were situated in concrete situations, mostly with physical objects (3D displays). These findings imply that the creation and use of dynamic mental imagery with high cognitive demands in IPT

Table 5.1 Sample composition

Child characteristic	Group	Number of children			
		NL (N = 334)		Cyprus (N = 304)	
Preschool year	K1	123		86	
	K2	211		218	
Gender	Boys	176		141	
	Girls	158		163	
Age		Age			
		M	SD	*M*	SD
	K1	4.67	0.38	4.67	0.28
	K2	5.69	0.37	5.61	0.32
	K1 + K2	5.32	0.62	5.35	0.53

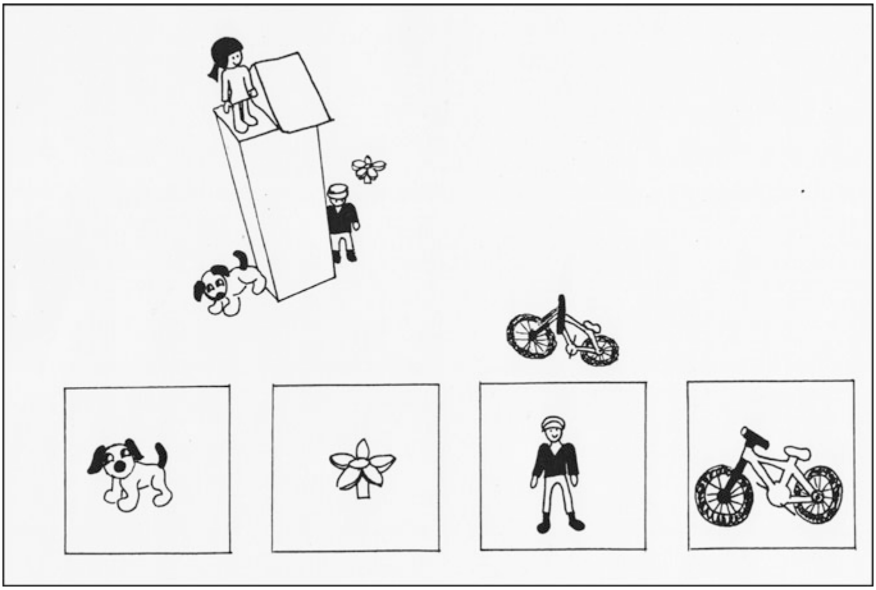


Fig. 5.6 Tower item

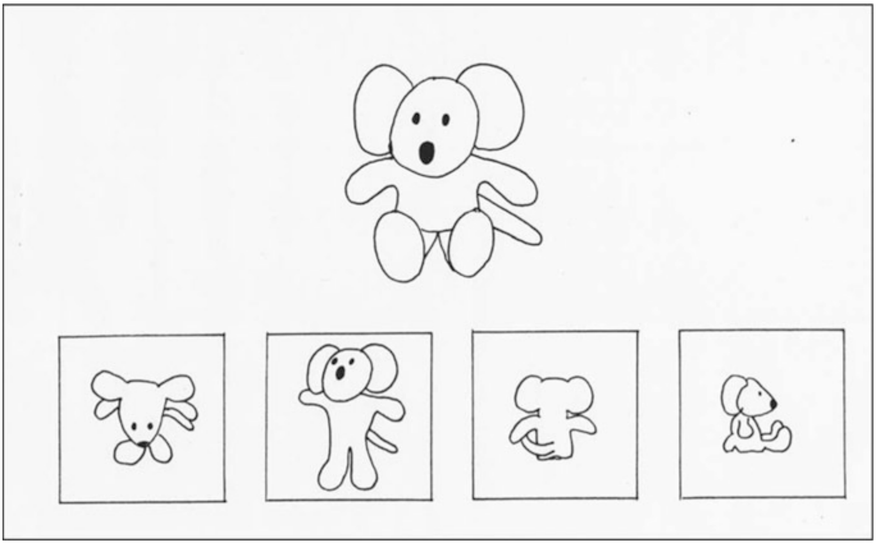


Fig. 5.7 Mouse item

(IPT type 2 competence) are more difficult when it is based upon pictures rather than on concrete objects. This is not the case when visualization in IPT includes processes of lower cognitive demands (IPT type 1 competence), as the children performed well in both pictorial and concrete situations.

Using regression analysis on the data showed that in both countries, the children's mathematics ability (based on scores on a test developed by the Central Institute for Test Development, *Cito*, for children in the Netherlands and on teachers' perceptions of their students' level in mathematics for children in Cyprus) was significantly positively related to IPT performance (the Netherlands: $B = 0.06$, $\mathrm{SE} = 0.01$, $t = 7.05$, $p < 0.001$, $\eta^2 = 0.036$; Cyprus: $B = 0.04$, $\mathrm{SE} = 0.01$, $t = 4.02$, $p < 0.001$, $\eta^2 = 0.020$). For the Netherlands' sample, the older children (in K2) significantly outperformed those in K1 ($B = 0.04$, $\mathrm{SE} = 0.02$, $t = 2.01$, $p = 0.044$, $\eta^2 = 0.001$), while in Cyprus, the group year was not found to be a significant predictor of the IPT scores. Results also showed that in both countries there was no significant effect of gender on the IPT scores.

Although the children in the Netherlands and Cyprus may have grown up in a culturally different environment (southern versus northern Europe), the findings in the two samples generally were quite similar. In fact, the main striking difference was that in the Netherlands the children performed higher on both IPT types than those in Cyprus. An explanation could be found in the implementation of the preschool curriculum. Even if in both countries spatial reasoning is a part of the mathematics program, this does not mean that the topic is adequately implemented by the teachers of both countries. Another explanation for the performance difference might be that the children in Cyprus are less familiar than the Netherlands' children with taking a class-administered paper-and-pencil test.

5.7 Making Connections Between Space and Shape Aspects and Their Verbal Representations: The Role of Gestures

McNeill (1992) suggests that "[s]peech and gesture are elements of a single integrated process of utterance formation" (p. 35). However, in contrast to speech, which can be decomposed into parts with isolated meanings, gesture is immediate and represents an image which depends on the whole. Thus, gestures are strongly connected with visualization and can be regarded as a strong indication of this mode of thinking (see also Presmeg, 2006). This view is adopted in a study we conducted to investigate the role of gestures in manipulating and communicating spatial concepts and concepts of shape at preschool level (Elia et al., 2014). In this study, we examined a five-year-old child while interacting with her teacher in the context of a geometrical activity, based on a task which involves a major visualization ability (Gutiérrez, 1996), that is, the understanding and operating on relationships between various positions in space (Sarama & Clements, 2009). Moreover, the task required semiotic transformations, that is, conversions between spatial representations and verbal descriptions.

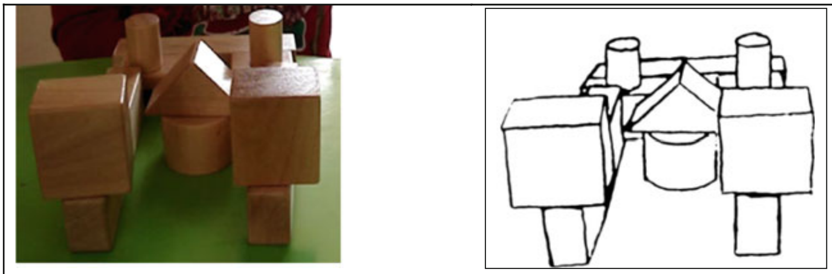
The activity had the form of a game for two players, one of whom was the child's teacher. During the activity, the child and the teacher sat opposite each other with a screen-divider to hide each other's work. The activity included three parts. In Part 1 and Part 3, the child freely created a construction with wooden blocks (Fig. 5.8) and then described the structure, step-by-step. The teacher built the construction using blocks from the child's verbal directions. In Part 2 of the activity, the child and the teacher switched roles, and during her verbal description, the teacher produced gestures.

In Fig. 5.8, we present the child's construction in Part 3 of the activity. The figure includes also the description of her construction and her gestural production which was then analyzed to unravel the role of gestures in using and communicating spatial and shape-related ideas by the child.

The child was found to use gestures throughout the whole part of the activity in which she acted as a describer. This finding provides evidence for the strong interrelations between geometrical thinking and gestures shown in previous studies. Gesture production provided support to internal spatial visualization (Chu & Kita, 2011) of various geometry aspects that the child had to describe at the same time, such as shape, size, location, and orientation of blocks.

The analysis of the child's description indicated that various space and shape aspects were visualized through gestures. Moreover, different aspects of geometrical content were more likely to stimulate the use of specific types of gestures by the child based on McNeill's (1992) classification. Specifically, when the child described the shape of a block, e.g., a cylinder (named as circle by the child, lines 31–33, see Fig. 5.9), the orientation of a block, e.g., horizontal direction of a parallelepiped (lines 29–30, see Fig. 5.10) and topological relations of proximity or separation, e.g., shapes that were attached or not (lines 6–8, see Fig. 5.11), she tended to produce iconic gestures (McNeill, 1992) which depicted the geometry aspects involved. Interestingly, in some cases when explaining the placement of blocks, the child produced iconic gestures to represent mental images of kinesthetic character (Presmeg, 1986). Specifically, these iconic gestures visualized the movement of placing the objects in their current location (lines 11–12, see Fig. 5.12). In other cases, when the child explained the location of the blocks in her construction (e.g., in front), she used deictic gestures (McNeill, 1992), indicating the position in which the blocks were placed (lines 26–27, see Fig. 5.13). In summary, iconic gestures served multiple functions in the child's geometrical thinking and were used more often relatively to deictic gestures which were rather mono-functional.

The child's gestures were found not only to represent visually the geometrical information (e.g., naming of shapes) given by her verbal expressions, thus reinforcing the meaning of speech (e.g., lines 31–33, Fig. 5.9), but also to visually complement, enrich, and specify her verbal descriptions, particularly when her verbal utterances were unclear, general, or incomplete. Some space and shape



Child: Take two long shapes. [Stretches out one hand vertically to her body and forms a straight line in the air by moving her hand with the palm open near her chest].
Two.
Teacher: How shall I put them?
Child: Like this. [Moves her hands away from one another with one palm opposite the other]
Not like this. [Puts the palms of her hands together]
Take two squares.
Teacher: Where shall I put them?
Child: On [Moves her hand downwards pretending to hold the block and place it on another block] the ….. [Stretches out her hands vertically to her body and forms a straight line in the air by moving her hands with the pointing fingers stretched near her chest] but not like this [takes a parallelepiped with dimensions of similar length and turns it widthwise], like this [turns the parallelepiped lengthwise].
Then take two [Shows two and then three fingers] small … [Moves her pointing finger to form a semicircle], two small [Moves her pointing finger to form a semicircle], two small bridges.
Then put them attached [Joins the fingers of her hands] to the long shapes [Stretches out one hand vertically to her body and forms a straight line in the air by moving her hand with the palm open near her chest], in front of them, not here [Points with both her hands behind the parallelepipeds], but here [points with both her hands in front of the parallelepipeds, close to her].
Then take another long shape.
Put it in front of the bridges [Opens her hands to form a flat surface and joins her fingers in front of the bridges, close to her].
Take two circles [Makes a round line vertically in the air with her pointing finger], but small ones [Moves her hands close to her face and forms fists].
Put them on the bridges [Moves both her hands downwards pretending to hold the blocks and place them on other blocks].

Fig. 5.8 Construction made by the child, in Part 3 of the activity, followed by the child's description of her construction, from Elia et al. (2014, p. 745)

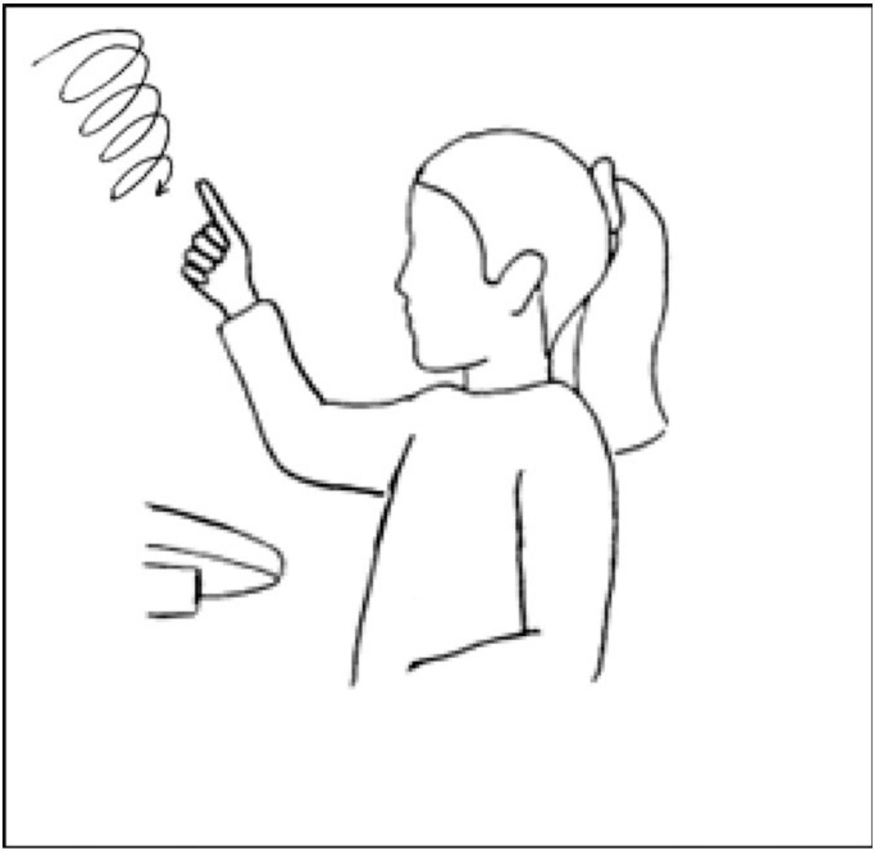


Fig. 5.9 Iconic gesture for the shape of cylinder (lines 31–33), from Elia et al. (2014, p. 745)

aspects of the construction were manifested mainly by the child's gestures instead of her words. For example, the orientation of a shape on the plane was a spatial aspect that was never expressed verbally by the child during the activity, but started to become explicit thanks to the iconic gestures she produced. Even when the child verbally explained the location of a block (e.g., in front of another block), she simultaneously used a gesture to illustrate the orientation of that block, that is, whether it was horizontally or vertically positioned (lines 29–30, see Fig. 5.10). These iconic gestures seemed to be essential and valuable in visually representing the orientation of a shape and other geometrical concepts that were complex for the child. It can be claimed that the child's visualization through gestures, together with her verbalization through speech, were harmonically coordinated to successfully accomplish the given description task.

The child was found to take the teacher's gestures as a visual model in describing her construction after observing the teacher's corresponding verbal and gestural description. In one case, she mimicked and extended the teacher's gesture

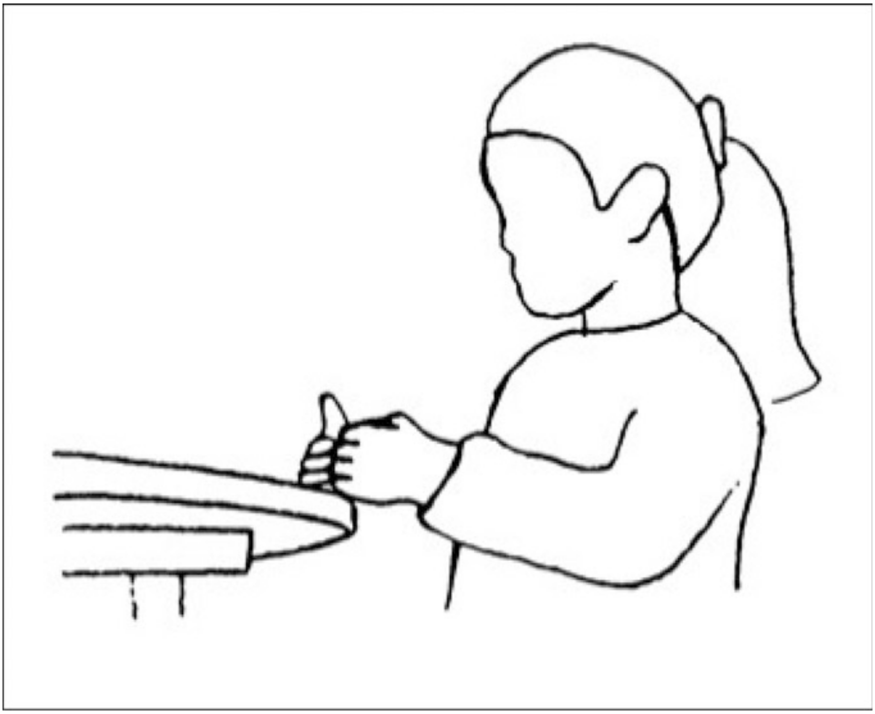


Fig. 5.10 Child's iconic gesture for the orientation of the parallelepiped in front of the blocks named "bridges" (lines 29–30), from Elia et al. (2014, p. 747)

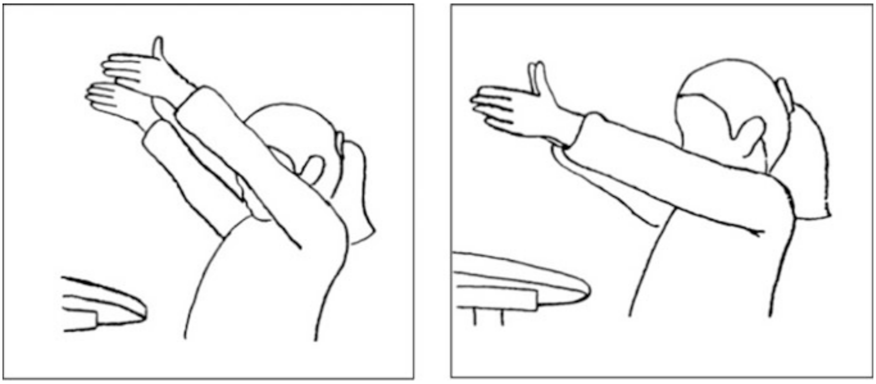


Fig. 5.11 Child's gesture supplementing the verbal expressions "like this" (left) and "not like this" (right) about the positions of two parallelepipeds (lines 6–8), from Elia et al. (2014, p. 751)

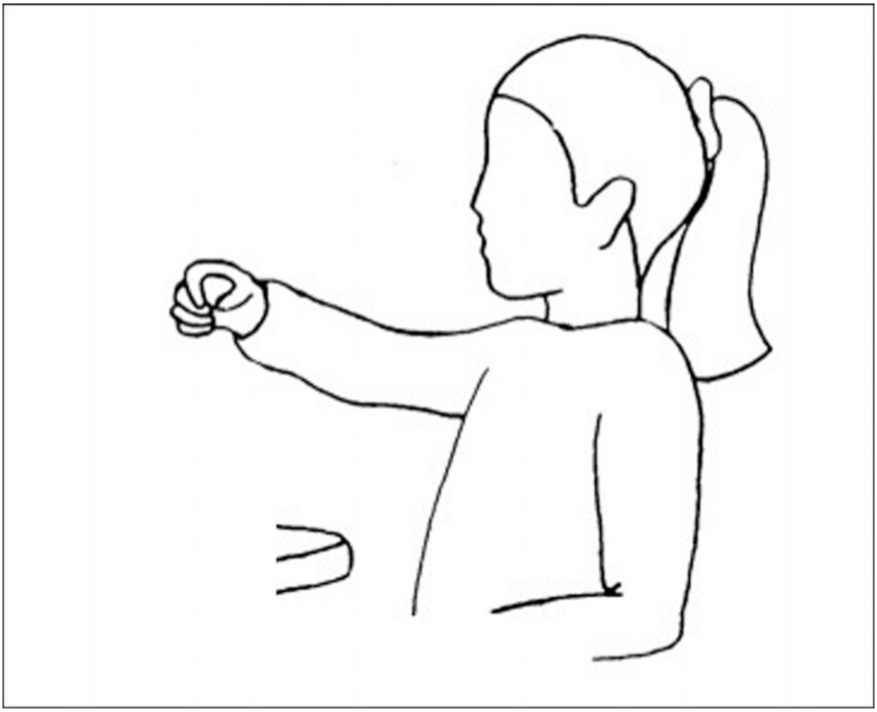


Fig. 5.12 Child's iconic gesture to represent the movement of placing the block on another block (lines 11–12)

that depicted the spatial relation of proximity between two blocks (i.e., placing the palms of her hands together) by using the same gesture and also by adding a contrast to it in a similar situation. Prior to the teacher's description, the child did not make any reference (either verbal or visual) to the proximity between blocks. Specifically, after the teacher's description, the child used a gesture that represented the relative position (separation) of two blocks in her construction, and then a gesture to show how this spatial relation was opposed to the image of two attached blocks (counterexample) (lines 6–8, see Fig. 5.11), that had been represented previously by the teacher's gesture. This change in the child's verbal and gestural acts provides evidence for the contribution of the teacher's expression of the spatial relation of proximity through gestural and speech production on the child's visualization of the particular concept. Furthermore, the contrast added by the child indicates that she internalized and creatively used the meaning of the gesture she observed and then produced visually in her own description. This finding indicates the positive influence on the child's learning of these concepts from the teacher's gestures and verbal expressions representing her mental images of spatial concepts.

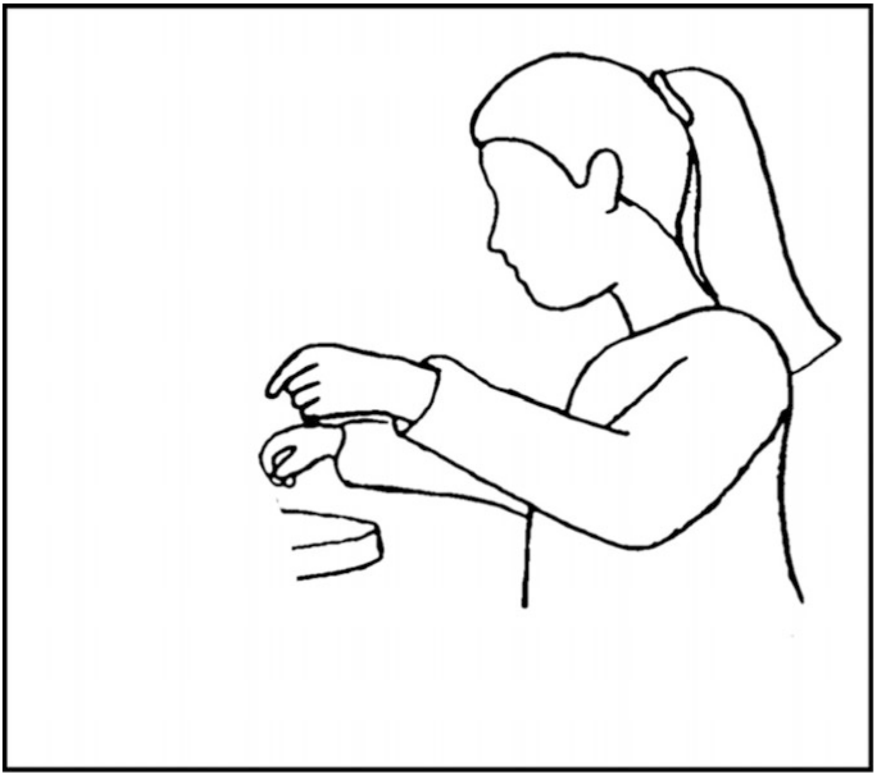


Fig. 5.13 Child's deictic gesture for the position of two "bridges" in front of the two parallelepipeds (lines 26–27)

5.8 Concluding Remarks

The work described above investigated geometry learning in the early years in three different aspects of the development of understanding of shapes and space: transformation of 2D geometrical figures, IPT, and spatial concepts and 3D shapes. Despite the different approaches that were used in this work, all three studies provided evidence for the essential role of visualization in the development of geometrical thinking. This work also revealed how young children use visualization in various types of geometrical activity including either plane or spatial geometry, indicating the multiple qualities and uses of visualization and visual skills in early geometry learning.

Based on the above, it is suggested that geometry learning in the early years cannot be examined without taking into account children's visual reasoning. However, the investigation of visual imagery is a rather difficult task, as a researcher cannot be sure that his/her interpretation about what another person has in his/her mind is accurate (Presmeg, 2014). This investigation is even more challenging in research which focuses on early childhood, as children are still developing their mathematics communication abilities (Van Oers, 2013). Nevertheless, the tasks that were used in the three studies, open dynamic transformation tasks of geometrical figures in the first study, paper-and-pencil pictorial perspective taking tasks with 2D representations of real situations in the second study, and a task involving a child–teacher–child interaction while building a spatial construction and describing it, were found to be successful in this endeavor. They enabled us to identify the use of visualization in the ways children were making sense of space and shape concepts, by means of their own dynamic drawings, selection of given illustrations, or gestures and speech. This shows that selecting or developing appropriate tasks which evoke children's geometrical thinking in meaningful ways, not confined to the simple perceptual apprehension of figures but to the stimulation of deeper spatial insights, is crucial in investigating (and also in developing) visualization in early geometry learning.

This focus on children's visualization processes was found to make "visible" implicit aspects of children's geometrical reasoning. Thus, finding ways to support children in making explicit their visual images and processes in geometry could be useful in assessing children's geometrical thinking. For example, the various strategies children use in dynamic transformation tasks of geometrical figures may enable the teacher to recognize whether children at an early age conceptualize geometrical figures as representations of objects or simple drawings and thus provide the appropriate support. Moreover, giving attention both to the verbal behavior and to the gestural production of children, which reveals the mental images they have constructed for geometrical concepts, in whole classroom interactions, in peer interactions, and in teacher–child interactions, allows the teachers to gain a better understanding of children's learning processes and outcomes and thus make their teaching better match the children's needs.

Our research findings provide evidence for the diversity of factors that may intervene in geometry learning which involves the use of visualization. Task-related characteristics is a major category of these factors, which include, for example, the competence level required by the task, the type of representation (2D or 3D), the type of geometrical figures that are involved, and other cognitive demands. Another category of such factors refer to children's characteristics, including, for example, children's mathematics ability, age, preschool year, and culture. Finally, our work indicates that a number of factors related to teachers and teaching might have an influential role in young children's geometry learning and visualization of space and shapes. These include teachers' gestural production and speech in their interactions with the children, with the teachers' iconic gestures of geometry aspects and corresponding words having the strongest influence on children, as well as the emphasis given by the teachers on spatial reasoning in the teaching of geometry.

The role of the teacher in early geometry learning and in the development of visual reasoning in this domain and determining what is necessary to fulfill this role is an issue that needs further research. Specifically, future studies with more children, longer observations, and a variety of geometrical problem-solving tasks need to be conducted before deriving the characteristics of teaching from a gestural and verbal perspective which can be beneficial for learning in early geometry and the development of visualization abilities. Furthermore, future studies should include in-depth analyses of possible differences in the educational and cultural environment of children which might be associated with the considerable variability found in IPT in children. It would be worthwhile also to explore systematically how the type of presentation of geometrical material to children, including 3D situations, 2D representations (such as work sheets or drawings in picture books), and conversions between 2D representations and 3D situations, influences children's IPT performance. Moreover, it would be theoretically interesting and practically important to investigate the impact of teaching that encourages "visual dynamic intuition" in transformation tasks of geometrical figures on young children's understanding of shape properties and characteristics, as well as of the interconnectivity and hierarchical commonalities and differences among shapes, such as rectangles and squares. Finally, it would be interesting to study the three visualization approaches described in this chapter further, with a particular focus on how they can be connected to contribute to the development of geometrical concepts in early childhood.

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