Doing Without Nature

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Abstract. We show that every indeterministic n-agent choice model M^i can be transformed into a deterministic n-agent choice model M^d, such that M^i is a bounded morphic image of M^d. This generalizes an earlier result from Van Benthem and Pacuit [16] about finite two-player choice models. It further strengthens the link between STIT logic and game theory, because deterministic choice models correspond in a straightforward way to normal game forms, and choice models are generally used to interpret STIT logic.

1 Introduction

At least since [9], it has become clear that there are strong links between game theory and the model theory of STIT logic. In this paper, we focus on the relation between normal game forms and what we call *choice models*.

A normal game *form* is a strategic game without players' preferences.[1] Choice models will be defined in Sect. 2; they form a specific class of Kripke models for a purely agentive STIT logic, i.e. a logic of (individual and collective) agency that contains no temporal operators. Choice models are e.g. used in [8] to study the complexity of STIT logic for groups. They closely resemble the *choice structures* from [10], the *STIT choice scenarios* from [16], and the *choice Kripke models* from [5].

Normal game forms and choice models are both used to represent the actions of (group) agents, thus providing a basis for the analysis of rational (individual and collective) interaction. The notion of *effectivity for outcomes* is central in both, where an outcome can be thought of as a (non-empty) set of possible worlds. Roughly speaking, an agent is effective for a given outcome in the model if and only if it can ensure that outcome by some action, regardless of what the other agents do.[2]

[1] See e.g. [12] for a solid introduction to the theory of strategic games.
[2] This notion is usually referred to as "α-effectivity" in game theory. We provide a formal definition of it in Sect. 2.

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For reasons of space, we cannot provide the full background and history of STIT logic and its relation to game theory in this paper. We refer to [1,9] for general introductions to STIT logic. See [8] for a discussion of the axiomatization and complexity of group STIT interpreted over choice models. [10] appears to be the first paper that explicitly deals with the relation between strategic game forms and STIT. Horty's [9] work in deontic logic, however, was already strongly inspired by the link between game theory and STIT theory. Publications that are directly relevant to this paper are [5,13,14,16].

To explain the aim of this paper, let us first focus on models with only two agents. Like normal game forms, choice models for two agents can be represented using matrices, where the rows represent choices of agent 1 and columns represent choices of agent 2. Figure 1 is a simple example. Each cell[3] in such a matrix represents a combination of actions of each agent – also known as *action profile* – and the corresponding outcome. For instance, if agent 1 chooses row 1 and agent 2 chooses column 2, then there is a unique outcome, viz. c. Note that the action profile (row 1, column 1) allows for two possible worlds, viz. a and b. In this model, agent 1 is i.a. effective for $\{a, b, c\}$ and $\{d, e\}$, whereas agent 2 is effective for $\{a, b, d\}$ and $\{c, e\}$.[4]

a,b	c
d	e

Fig. 1. A choice model for two agents.

Say two models M_1 and M_2 – whether normal game forms or choice models – are *equivalent* if and only if for any given outcome X and every agent j it holds that j is effective for X in M_1 if and only if j is effective for X in M_2. Every normal game form can be translated into an equivalent choice model, where the action profiles in the former correspond to the worlds in the latter. This was first observed by Tamminga in [13, Sect. 3.1]; we will recall the details in Sect. 2. As is shown in [14], the inverse translation works for a specific class of choice models. This class is characterized by the condition known as *determinism*: each choice of the grand coalition, i.e. each action profile, singles out exactly one world. Note that in the above example, this condition is not satisfied: if agent 1 chooses row 1 and agent 2 chooses column 1, then either a or b may result.

A common motivation for determinism is that we can get it "for free" just by moving to a three-agent model, letting "nature" or "the environment" play the role of the third agent (see e.g. [9, p. 91] and [16, p. 300] where this point is made). In other words, nature is an agent that makes its own choices, and in combination

[3] In the two-agent case, cells correspond to the "innermost squares" in the matrix. See Sect. 2 for the general definition of cells in a choice model.

[4] The notion of effectivity is monotonic: whenever an agent is effective for X, it is also effective for every superset of X. In the current example, this means that agent 1 is e.g. also effective for $\{a, b, c, d\}$.

with the choices of the two "real" agents, this determines the outcome. Applying this idea to the example from Fig. 1 yields the model depicted in Fig. 2, where nature gets to choose between the left and right matrix. In this new choice model, agents 1 and 2 are just as effective as they were in the original model given by Fig. 1, but the choices by the group of all agents (including nature) always determine a singleton outcome.

a	c
d	e

b	c
d	e

Fig. 2. A deterministic choice structure for three agents.

Leaving more philosophical issues aside, one may wonder whether this technical trick is really necessary, mathematically speaking. This question will be answered in the present paper: we show that one can indeed do without nature, as long as one does not consider the effectivity of the grand coalition. We prove this by generalizing a proof method from [16], which is discussed in Sect. 3. We generalize this method, first, to an arbitrary finite number of agents and all groups of such agents except the grand coalition (Sect. 4) and second, to infinite models (Sect. 5). We finish with a summary and some questions for future work (Sect. 6).

2 Preliminaries: Group STIT

The notion of effectivity can be made exact and studied formally, using a well-known STIT logic for group agents. In this section, we introduce the formal language of this logic. After that, we give two different semantics for this logic, using choice models and normal game models, and discuss the relation between these types of semantics.

2.1 The Language of Group STIT

Throughout this paper, we assume a fixed, finite set $N = \{1, \ldots, n\} \subset \mathbb{N}$ of agents. We let j range over members of N and we let G range over non-empty subgroups of N. $\mathfrak{P} = \{p_1, p_2, \ldots\}$ is a (countable) set of propositional variables. The formal language $\mathfrak{L}$ is given by the following Backus-Naur form, where p ranges over $\mathfrak{P}$:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid [G]\varphi \mid \Box\varphi$$

$\mathfrak{L}^{-N}$ denotes the fragment of $\mathfrak{L}$ without the operator $[N]$. Parentheses, brackets, and braces are omitted if the omission does not give rise to ambiguities. The operators $\vee$, $\rightarrow$, $\Diamond$, and $\langle G \rangle$ abbreviate the standard constructions.

A formula $[G]\varphi$ expresses that "the group G sees to it that φ is the case", or alternatively, "given G's choice, φ is necessary".[5] The formula $\Box\varphi$ expresses that "φ is settled true", or equivalently, "whatever the agents choose, φ is the case". In modal logic terminology, $\Box$ corresponds to the universal (or global) modality, since it quantifies over all the worlds in a given model.[6]

2.2 Choice Models

Choice models consist of, on the one hand, a set of possible worlds W, and on the other hand, for every agent $j \in N$, a partition of W that represents the choices of j in the model. The only restriction to these partitions is that they satisfy a specific frame condition, known as *independence of agents*. This condition expresses that no group of agents $G \subseteq N \setminus \{j\}$ can render any of the choices that are available to agent j impossible. In other words, if the agent j has a certain choice, then it can make this choice regardless of what all the other agents do.

To stay in line with standard modal logic terminology, the choices of each agent j will be represented by an equivalence relation $\sim_j$ on W.

Definition 1. *A* choice frame F *is a tuple* $\langle W, \langle \sim_j \rangle_{j \in N} \rangle$, *where* W *is a non-empty set (the domain of* F*), each* $\sim_j \subseteq W \times W$ *is an equivalence relation, and the* independence of agency *condition obtains:*

(IOA) for all $w_1, \ldots, w_n \in W$, *there is a* w' *such that* $w_j \sim_j w'$ *for all* $j \in N$.

For a given choice frame $F = \langle W, \langle \sim_j \rangle_{j \in N} \rangle$ *and non-empty* $G \subseteq N$, *we define* $\sim_G = \bigcap_{j \in G} \sim_j$.[7]

A choice model M *is a triple* $\langle W, \langle \sim_j \rangle_{j \in N}, V \rangle$ *where* $\langle W, \langle \sim_j \rangle_{j \in N} \rangle$ *is a choice frame and* $V : \mathfrak{P} \to \wp(W)$ *is a valuation function.*

We say that M *is* deterministic *iff* $\sim_N = \{(w, w) : w \in W\}$. M *is* finite *iff* W *is finite.*

For every non-empty $G \subseteq N$, the equivalence relation $\sim_G$ in a choice frame induces a partition of W: $\mathit{Choice}_G(M) =_{\mathsf{df}} \{\{w' \mid w' \sim_G w\} \mid w \in W\}$. The members of $\mathit{Choice}_G(M)$ are referred to as the *choices* of the group G in M. Note that $\mathit{Choice}_N(M) = \{X_1 \cap \ldots \cap X_n \mid X_1 \in \mathit{Choice}_1(M), \ldots, X_n \in \mathit{Choice}_n(M)\}$. It is obvious that a choice model M is deterministic iff every member of $\mathit{Choice}_N(M)$ is a singleton. We use the common term *cells* to refer to the members of a $\mathit{Choice}_N(M)$.

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[5] $[G]$ is also known as the *Chellas STIT*, after the seminal work in the logic of agency by Chellas [4].

[6] Given our semantics, $\Box\varphi$ is definable as $[i][j]\varphi$ for $i \neq j$. We will however treat $\Box$ as primitive for reasons of clarity.

[7] This property may be called the intersection property. In Sect. 6 we briefly mention how it relates to some completeness results.

Definition 2. *Where $M = \langle W, \langle \sim_j \rangle_{j \in N}, V \rangle$ is a choice model and $w \in W$,*

$$
\begin{array}{ll}
M, w \models p & \textit{iff } w \in V(p) \\
M, w \models \neg\phi & \textit{iff } M, w \not\models \phi \\
M, w \models \phi \wedge \psi & \textit{iff } M, w \models \phi \textit{ and } M, w \models \psi \\
M, w \models \Box\phi & \textit{iff for all } w' \in W \textit{ it holds that } M, w' \models \phi \\
M, w \models [G]\phi & \textit{iff for all } w' \in W \textit{ with } w' \sim_G w \textit{ it holds that } M, w' \models \phi.
\end{array}
$$

As usual, $\|\varphi\|^M = \{w \in W \mid M, w \models \varphi\}$.

That G is effective for a state of affairs φ in the choice model M can be expressed by means of the formula $\Diamond[G]\varphi$. This formula expresses that, for some world w in the model, $[G]\varphi$ is true. Since $Choice_G(M)$ is a partition of W, this is equivalent to saying that there is a choice $X \in Choice_G(M)$ such that $X \subseteq \|\varphi\|^M$.

2.3 Normal Game Models

In this section, we briefly spell out the semantics for group STIT using normal game forms, following [16]. Subsequently, we discuss the relation between normal game forms and deterministic choice models.

Definition 3. *A* normal game form *for the set of agents $N = \{1, \ldots, n\}$ is a tuple $G = \langle A_i \rangle_{i \in N}$, where each A_i is a non-empty set of actions $a, a', \ldots$ available to agent i. We call $\times_{i \in N} A_i$ the set of* action profiles *of the game, and denote its members by σ, σ', etc. Where $\sigma = \langle a_1, \ldots, a_n \rangle \in \times_{i \in N} A_i$ and $j \in N$, let $\pi^j(\sigma) = a_j$. Where $\emptyset \neq G \subseteq N$, let $\pi^G(\sigma)$ denote G's part in the action profile σ, i.e., $\pi^G(\sigma) = \langle \pi^j(\sigma) \rangle_{j \in G}$.*

A normal game model *is a tuple $S = \langle \langle A_i \rangle_{i \in N}, V \rangle$, where $\langle A_i \rangle_{i \in N}$ is a normal game form and $V : \mathfrak{P} \rightarrow \wp(\times_{i \in N} A_i)$ is a valuation function.*

Definition 4. *Where $S = \langle \langle A_i \rangle_{i \in N}, V \rangle$ is a normal game model and $\sigma \in \times_{i \in N} A_i$:*

$$
\begin{array}{ll}
S, \sigma \models p & \textit{iff } \sigma \in V(p) \\
S, \sigma \models \neg\phi & \textit{iff } S, \sigma \not\models \phi \\
S, \sigma \models \phi \wedge \psi & \textit{iff } S, \sigma \models \phi \textit{ and } S, \sigma \models \psi \\
S, \sigma \models \Box\phi & \textit{iff for all } \sigma' \in \times_{i \in N} A_i, \textit{ it holds that } S, \sigma' \models \phi \\
S, \sigma \models [G]\phi & \textit{iff for all } \sigma' \in W \textit{ with } \pi^G(\sigma) = \pi^G(\sigma') \textit{ it holds that } S, \sigma' \models \phi.
\end{array}
$$

As was the case for choice models, one can use the language $\mathfrak{L}$ to formalize statements concerning the effectivity of a given (group) agent G for a certain outcome, where outcomes are represented by propositions φ. The formula $\Diamond[G]\varphi$ expresses that there is an action profile σ such that, for all σ' with $\pi^G(\sigma) = \pi^G(\sigma')$, $S, \sigma' \models \varphi$. In other words, the group G has a combined choice $\pi^G(\sigma)$ such that, whatever the other agents do, φ is guaranteed.

2.4 A Correspondence Result

As mentioned in the introduction, there is a well-known correspondence between deterministic choice models on the one hand, and normal game models on the other – see e.g. [5,13,14].[8] To clarify the purpose of our new results, these correspondence results are explicated here.

Definition 5. *Let $S = \langle \langle A_i \rangle_{i\in N}, V\rangle$ be a normal game model. The corresponding choice model $M^S = \langle W, \langle \sim_i \rangle_{i\in N}, V\rangle$ is defined as follows:*

1. *$W = \times_{i\in N} A_i$*
2. *for all $i \in N$ and $\sigma, \sigma' \in W$, $\sigma \sim_i \sigma'$ iff $\pi^i(\sigma) = \pi^i(\sigma')$.*

Theorem 1. *Let $S = \langle \langle A_i \rangle_{i\in N}, V\rangle$ be a normal game model. Then (a) M^S is a deterministic choice model. Moreover, (b) where $\varphi \in \mathfrak{L}$ and $\sigma \in \times_{i\in N} A_i$: $S, \sigma \models \varphi$ iff $M^S, \sigma \models \varphi$.*

Proof. Suppose the antecedent holds. To obtain (a), note first that each relation $\sim_i$ is an equivalence relation. To see why the condition (IOA) holds for M^S, let $\sigma_1, \ldots, \sigma_n \in \times_{i\in N} A_i$. Let $\sigma = \langle \pi^1(\sigma_1), \ldots, \pi^n(\sigma_n)\rangle$. It can easily be verified that, for all $j \in N$, $\sigma \sim_j \sigma_j$. Finally, to see why M^S is deterministic, note that $\sigma \sim_N \sigma'$ iff for all $j \in N$, $\pi^j(\sigma) = \pi^j(\sigma')$ iff $\sigma = \sigma'$.

The proof of (b) is by a standard induction on the complexity of φ; it suffices to apply the truth conditions from Definitions 2 and 4. □

Definition 6. *Let $M = \langle W, \langle \sim_i \rangle_{i\in N}, V\rangle$ be a deterministic choice model. The corresponding normal game model $S^M = \langle \langle A_i \rangle_{i\in N}, V'\rangle$ is such that the following holds:*

1. *for all $i \in N$, $A_i = Choice_i(M)$*
2. *where $\sigma = \langle X_1, \ldots, X_n\rangle \in \times_{i\in N} A_i$ and $X_1 \cap \ldots \cap X_n = \{w\}$: $\sigma \in V'(p)$ iff $w \in V(p)$.*

Theorem 2. *Let $M = \langle W, \langle \sim_i \rangle_{i\in N}, V\rangle$ be a deterministic choice model and $\varphi \in \mathfrak{L}$. Then (a) $S^M = \langle \langle A_i \rangle_{i\in N}, V'\rangle$ is a normal game model. Moreover, where $\sigma = \langle X_1, \ldots, X_n\rangle \in \times_{i\in N} A_i$, $X_1 \cap \ldots \cap X_n = \{w\}$, and $\varphi \in \mathfrak{L}$: $S^M, \sigma \models \varphi$ iff $M, w \models \varphi$.*

Proof. Suppose the antecedent holds. To obtain (a), it suffices to check that V' is a valuation function. This follows immediately in view of the fact that for every $\sigma \in \times_{i\in N} Choice_i(M)$, there is a world w such that w is the only member of the intersection of all the choices that make up σ. To prove (b), we again apply a standard induction on the complexity of φ, together with the semantic clauses from Definitions 2 and 4. □

Theorems 1 and 2 give us at once:

Corollary 1. *Let $\varphi \in \mathfrak{L}$. Then φ is valid in all deterministic choice models if and only if φ is valid in all normal game models.*

[8] In [13,14], the authors actually establish a correspondence between strategic games and choice models enriched with preference relations $\preceq_i$ for the agents $i \in N$. Ignoring this extra dimension, one obtains exactly the correspondence that we spell out in the present section.

3 Informal Sketch of the Proof

In the remainder we prove that, relative to the fragment $\mathfrak{L}^{-N}$, every (indeterministic) choice model M^i is a bounded morphic image of some deterministic choice model M^d (cf. Theorems 4 and 6 below).[9] In other words, for every world w in M^i there is a world w' in M^d such that, for every formula $\varphi \in \mathfrak{L}^{-N}$, φ is true at w in M^i iff φ is true at w' in M^d. More briefly, M^i and M^d are pointwise equivalent, relative to $\mathcal{L}^{-N}$. In view of the preceding, this result implies that the set of all formulas from $\mathfrak{L}^{-N}$ that are valid in all choice models coincides with the set of all formulas from $\mathfrak{L}^{-N}$ that are valid in all normal game models.[10] This way, we fill an important gap in the comparison of normal game models on the one hand, and choice models (and other traditional semantics of STIT logic) on the other.

Our proof generalizes a construction by Van Benthem and Pacuit [16] that only applies to the case where we have two agents and M^i is finite. The construction by Van Benthem and Pacuit is in turn based on known methods from modal product logics.[11] To guide the reader's intuitions and to explain our own contribution, Van Benthem and Pacuit's construction is explained in the current section.

Consider the model M^i depicted in Fig. 3 – as before, we abstract from the valuation function in our pictures of the models. Note that the cell that contains the highest number of possible worlds is the one containing three worlds: h, i, and j.

a	b,c	d
e,f	g	h,i,j

Fig. 3. An indeterministic choice model.

The proof by Van Benthem and Pacuit basically consists of two steps. The first step is to construct an $m \times m$ matrix $\mathsf{M}(X)$ for every cell X in M^i, where m is the highest number of worlds that occur in one cell of M^i. The points in this matrix are copies of the members of X, and the matrix is constructed in such a way that every $x \in X$ occurs at least once in each row and in each column of $\mathsf{M}(X)$. For instance, the cells in the second row of M^i give us the 3×3 matrices depicted in Fig. 4.

To obtain such $m \times m$ matrices for every cell X in the model, Van Benthem and Pacuit apply a simple arithmetical trick. We give a variant of theirs, that generalizes easily to the case of $n > 2$ agents.

[9] See e.g. [2] for an introduction to the notion of bounded morphisms in modal logic.

[10] For the grand coalition N, determinism obviously makes a difference. That is, within the class of all choice frames, determinism is characterized by the axiom $[N]\varphi \leftrightarrow \varphi$, which is not valid on indeterministic choice frames.

[11] See e.g. [7] for an introduction to modal product logics.

e	f	e
f	e	f
e	f	e

g	g	g
g	g	g
g	g	g

h	i	j
i	j	h
j	h	i

Fig. 4. Some 3×3 matrices.

For every cell X in the model, fix a surjective function $g_m^X : \{1, \ldots, m\} \to X$. Every point in $\mathsf{M}(X)$ is identified by its coordinates $\langle k, l \rangle$, where $k, l \in \{1, \ldots, m\}$. The world $x \in X$ that corresponds to the point $\langle k, l \rangle$ in $\mathsf{M}(X)$ is defined by $f : \{1, \ldots, m\}^2 \to X$ as follows:[12]

$$f(\langle k, l \rangle) = g_m^X(((k + l) \mathsf{mod}\ m) + 1)$$

This way, every row k of $M(X)$ is guaranteed to contain all members of X, and likewise for every column l of $M(X)$.

The second step in the construction from [16] consists in substituting the new, "small" matrices for the cells in the matrix that corresponds to the original model M^i. Applied to the above example, this gives us a single 6×9 matrix (Fig. 5).

a	a	a	b	c	b	d	d	d
a	a	a	c	b	c	d	d	d
a	a	a	b	c	b	d	d	d
e	f	e	g	g	g	h	i	j
f	e	f	g	g	g	i	j	h
e	f	e	g	g	g	j	h	i

Fig. 5. A deterministic choice model.

This new matrix corresponds to a deterministic choice model: rows are again the actions of one agent, columns are the actions of the other agent. The crucial point to note is that, as far as the effectivity of both agents is concerned, M^i and M^d are equivalent, that is, they validate exactly the same formulas in $\mathfrak{L}^{-N}$. Only the effectivity of the grand coalition N is affected.

To see how this new matrix can be accurately defined, we first need to explain how worlds in M^d are defined. For the two-agent case, the worlds in M^d are defined by (a) an index X that refers to a cell in M^i, and (b) the coordinates $k, l \in \{1, \ldots, m\}$ that specify a point in $\mathsf{M}(X)$. Two worlds $\langle X, k, l \rangle$ and $\langle X', k', l' \rangle$ are connected for agent 1, iff (a) the cells X and X' are included in a single choice $Y \in \mathit{Choice}_1(M^i)$, and (b) $k = k'$. Analogously, the new choices of the second agent are defined in terms of $\mathit{Choice}_2(M^i)$ and the indexes l, l' of the new worlds.

[12] Where $i, j \in \mathbb{N}$, $i\ \mathsf{mod}\ j$ is shorthand for "i modulo j", i.e., the remainder after division of i by j.

Even though it does the job, the matrix depicted in Fig. 5 is somewhat large for our purposes: e.g. the third row is superfluous, since it is identical to the first row. Still, the advantage of this construction is that it can easily be generalized to models with n agents. Just as for 2 agents, every cell X in a game for n agents can be replaced with a new, n-dimensional matrix. This is exactly what happens in the proof to which we now turn.

4 Product Construction, Finite Case

In this section and the next one, we prove our main results. In the current section, we will consider the case where M^i is finite and construct an $\mathfrak{L}^{-N}$-equivalent, deterministic *and finite* model M^d from it. In the next section we consider the case where M^i is infinite; using a slightly more complex construction, we construct an $\mathfrak{L}^{-N}$-equivalent deterministic model M^d from M^i. For reasons that will be explained in Sect. 5, the second type of construction will always render an infinite model M^d, even when M^i is finite. Hence, the proof in the present section is not just a special case of the one from the next section.

Recall that we hold the number n of agents fixed in this paper; the construction of M^d will depend in part on this number (see Definition 8). We first lift the equivalence relations $\sim^i_j$ to relations $\approx^i_j$ between the cells in M^i:

Definition 7. *Where $X, Y \in Choice_N(M^i)$: $X \approx^i_j Y$ iff for every $x \in X$ and every $y \in Y$ it holds that $x \sim^i_j y$.*

Each of the following can be easily verified:

Proposition 1. *$X \approx^i_j Y$ iff there are $x \in X$ and $y \in Y$ such that $x \sim^i_j y$.*

Proposition 2. *$\approx^i_j$ is an equivalence relation.*

Definition 8. *Let $M^i = \langle W^i, \langle \sim^i_j \rangle_{j \in N}, V^i \rangle$ be a finite indeterministic choice model. Let m be the number of worlds in the cell in $Choice_N(M^i)$ with the highest cardinality. For every $X \in Choice_N(M^i)$, fix a surjective function $g^X_m : \{1, \ldots, m\} \to X$. Where $X \in Choice_N(M^i)$ and $k_1, \ldots, k_n \in \{1, \ldots, m\}$, let*

$$f(\langle X, k_1, \ldots, k_n \rangle) = g^X_m(((k_1 + \ldots + k_n) \mathsf{mod}\ m) + 1)$$

The model $M^d = \langle W^d, \langle \sim^d_j \rangle_{j \in N}, V^d \rangle$ is defined as follows:

$$\begin{aligned} W^d &= \{\langle X, k_1, \ldots, k_n \rangle : X \in Choice_N(M^i) \text{ and } \{k_1, \ldots, k_n\} \subseteq \{1, \ldots, m\}\} \\ \sim^d_j &= \{(\langle X, k_1, \ldots, k_n \rangle, \langle Y, l_1, \ldots, l_n \rangle) \in (W^d)^2 : X \approx^i_j Y \text{ and } k_j = l_j\} \\ V^d(p) &= \{\langle X, k_1, \ldots, k_n \rangle \in W^d : f(\langle X, k_1, \ldots, k_n \rangle) \in V^i(p)\}. \end{aligned}$$

We first make two basic observations about the set W^d as given by Definition 8. The proofs are safely left to the reader.

Proposition 3. *$f : W^d \to W^i$ is onto.*

If X is a set, let $\mathsf{card}(X)$ denote the number of elements in X.[13]

Proposition 4. $\mathsf{card}(W^d) = m^n \times \mathsf{card}(Choice_N(M^i))$. *Hence, M^d is finite.*

Theorem 3. *M^d is a deterministic choice model.*

Proof. We need to prove a number of things:

1. "$W^d \neq \emptyset$." Immediate, by Proposition 3 and since $W^i \neq \emptyset$.
2. "Every $\sim^d_j$ is an equivalence relation." Immediate in view of Proposition 2 and because of the definition of $\sim^d_j$.
3. "M^d satisfies Independence of Agents." Let $\langle X_j, k^j_1, \ldots, k^j_n\rangle \in W^d$ for all $j \in N$. We have to show that there is a $\langle Y, l_1, \ldots, l_n\rangle \in W^d$ such that for all $j \in N$ it holds that $\langle Y, l_1, \ldots, l_n\rangle \sim^d_j \langle X_j, k^j_1, \ldots, k^j_n\rangle$. First, set $l_j = k^j_j$ for all $j \in N$. Second, fix an arbitrary $x_j \in X_j$ for all $j \in N$. Because of Independence of Agents for M^i, there is a $y \in W^i$ such that $x_j \sim^i_j y$ for all $j \in N$. Let Y be the cell $Y \in Choice_N(M^i)$ that contains y. Then, because of Proposition 1, it must be that $Y \approx^i_j X_j$ for all $j \in N$. By the definition of W^d, it must be that $\langle Y, l_1, \ldots, l_n\rangle \in W^d$. By the definition of $\sim^d_j$, it must be that $\langle Y, l_1, \ldots, l_n\rangle \sim^d_j \langle X_j, k^j_1, \ldots, k^j_n\rangle$ for all $j \in N$. Hence, M^d satisfies Independence of Agents.
4. "$\sim^d_N$ is the identity relation over W^d." Suppose that $\langle X, k_1, \ldots, k_n\rangle \sim^d_N \langle Y, l_1, \ldots, l_n\rangle$. Hence, $\langle X, k_1, \ldots, k_n\rangle \sim^d_j \langle Y, l_1, \ldots, l_n\rangle$ for all $j \in N$. Then $X, Y \in Choice_N(M^i)$ and for all $j \in N$ both $X \approx^i_j Y$ and $k_j = l_j$. Hence it must be that (a) $\langle k_1, \ldots, k_n\rangle = \langle l_1, \ldots, l_n\rangle$. Because $X \approx^i_j Y$ for all $j \in N$, it must be that for all $j \in N$ and for all $x \in X$ and all $y \in Y$ it holds that $x \sim^i_j y$. Hence, for all $x \in X$ and all $y \in Y$ it holds that $x \sim^i_N y$. Because $X, Y \in Choice_N(M^i)$, it must be that (b) $X = Y$. From (a) and (b) we conclude that $\langle X, k_1, \ldots, k_n\rangle = \langle Y, l_1, \ldots, l_n\rangle$.

By (i)-(iv), M^d is a deterministic choice model. □

Theorem 4. *f is a bounded morphism from M^d to M^i in $\mathfrak{L}^{-N}$, i.e.:*

1. *f is onto.*
2. *For all $w, w' \in W^d$ and all non-empty $G \subset N$: if $w \sim^d_G w'$, then $f(w) \sim^i_G f(w')$*
3. *For all $u \in W^d$, all non-empty $G \subset N$, and all $y \in W^i$: if $f(u) \sim^i_G y$, then there is a $v \in W^d$ such that $f(v) = y$ and $u \sim^d_G v$.*
4. *For all $w \in W^d$ and $p \in \mathfrak{P}$: $w \in V^d(p)$ iff $f(w) \in V^i(p)$.*

[13] Note that we apply the card function both to finite and infinite (even uncountable) sets.

Proof. Ad 1. This is Proposition 3.
Ad 2. Let $w = \langle X, k_1, \ldots, k_n\rangle$ and $w' = \langle Y, l_1, \ldots, l_n\rangle$ be arbitrary members of W^d, and suppose that $w \sim^d_G w'$. Let $f(w) = x$ and $f(w') = y$. It follows that $x \in X$, $y \in Y$, and for all $j \in G$, $X \approx^i_j Y$. By Definition 7, for all $j \in G$, $x \sim^i_j y$. Hence, $x \sim^i_G y$.
Ad 3. Suppose the antecedent holds for $u = \langle X, k_1, \ldots, k_n\rangle$. Note that $f(u) \in X$. Let $Y \in \mathit{Choice}_N(M^i)$ be such that $y \in Y$. By Proposition 1 and the supposition, (†) for all $j \in G$, $X \approx^i_j Y$. Fix a $t \in N - G$. For all $j \in N - \{t\}$, let $l_j = k_j$. Let $l_t \in \{1, \ldots, m\}$ be such that $g^X_m(((l_1 + \ldots + l_n) \mathsf{mod}\ m) + 1) = y$. It is a matter of basic arithmetic to check that there is indeed such an l_t. Let now $v = \langle Y, l_1, \ldots, l_n\rangle$. It follows that $f(v) = y$. By (†) and in view of the construction, $u \sim^d_j v$ for all $j \in G$. Hence, $u \sim^d_G v$.
Ad 4. Immediate in view of the definition of V^d. □

It is well-known that, whenever there is a bounded morphism between two models M and M', then these models are pointwise equivalent – see e.g. [2, Proposition 2.14]. Hence, Theorem 4 gives us:

Corollary 2. *For all $\varphi \in \mathfrak{L}^{-N}$ and all $w \in W^d$: $M^d, w \models \varphi$ iff $M^i, f(w) \models \varphi$.*

5 Product Construction, Infinite Case

The proof in Sect. 4 makes essential use of the upper bound m on the cardinality of each $X \in \mathit{Choice}_N(M^i)$. As a result, we can apply well-known arithmetic techniques to construct the n-dimensional matrices that form the core of the construction of M^d. For the infinite case, a slightly different construction is needed.

The idea behind Definition 9 below can be explained as follows. In every world $u \in W^d$, each of the agents gets to choose exactly one world from W^i, and one natural number $k \in \mathbb{N}$. The output for this world, given by f, depends on the one hand on the index X, on the other hand, on which agent $t \in N$ chose the highest number $k_t \in \mathbb{N}$, and the world w_t that this agent t chose.

We first introduce some more notation. Where $\overline{x} = \langle x_1, \ldots, x_k\rangle$ is a k-tuple and $1 \leq j \leq k$, let $\pi^j(\overline{x}) = x_j$. Where $X \subset \mathbb{N}$ is a finite set of natural numbers, let $\mathsf{max}_<(X)$ denote the largest element in X.

Definition 9. *Let $M^i = \langle W^i, \langle \sim^i_j\rangle_{j\in N}, V^i\rangle$ be an infinite indeterministic choice model. For every $X \in \mathit{Choice}_N(M^i)$, fix a surjective function $g^X : W^i \to X$.*[14] *Where $X \in \mathit{Choice}_N(M^i)$, $w_1, \ldots, w_n \in W^i$, and $k_1, \ldots, k_n \in \mathbb{N}$, let*

$$f(\langle X, \langle w_1, k_1\rangle, \ldots, \langle w_n, k_n\rangle\rangle) = g^X(w_l)$$

where $l \in \{1, \ldots, n\}$ is the smallest natural number such that $k_l = \mathsf{max}_<\{k_1, \ldots, k_n\}$. The model $M^d = \langle W^d, \langle \sim^d_j\rangle_{j\in N}, V^d\rangle$ is defined as follows:

[14] Since $X \subseteq W^i$, it can be easily verified that there is at least one such function g^X.

$$\begin{aligned} W^d &= \{\langle X, \langle w_1, k_1\rangle, \ldots, \langle w_n, k_n\rangle\rangle : X \in \mathit{Choice}_N(M^i), w_1, \ldots, w_n \in W^i, \\ &\qquad \mathit{and}\ k_1, \ldots, k_n \in \mathbb{N}\} \\ \sim_j^d &= \{(\langle X, \overline{\epsilon}\rangle, \langle Y, \overline{\epsilon'}\rangle) \in (W^d)^2 : X \approx_j^i Y \ \mathit{and}\ \pi^j(\overline{\epsilon}) = \pi^j(\overline{\epsilon'})\} \\ V^d(p) &= \{w \in W^d : f(w) \in V^i(p)\}. \end{aligned}$$

Note that we use $\overline{\epsilon}, \overline{\epsilon'}, \ldots$ as metavariables for tuples of the form $\langle\langle w_1, k_1\rangle, \ldots, \langle w_n, k_n\rangle\rangle$ that are part of a larger tuple $w \in W^d$.

Theorem 5. *M^d is a deterministic choice model.*

Proof. We need to prove a number of things:

1. "$W^d \neq \emptyset$." Since $W^i \neq \emptyset$, also $\mathit{Choice}_N(M^i) \neq \emptyset$. By Definition 9, $W^d \neq \emptyset$.
2. "Every $\sim_j^d$ is an equivalence relation." Immediate in view of Proposition 2 and by Definition 9.
3. "M^d satisfies Independence of Agents." Consider arbitrary $w_1, \ldots, w_n \in W^d$, where each $w_j = \langle X_j, \langle w_1^j, k_1^j\rangle, \ldots, \langle w_n^j, k_n^j\rangle\rangle$. Fix an arbitrary $x_j \in X_j$ for all $j \in N$. Because of Independence of Agents for M^i, there is a $y \in W^i$ such that $x_j \sim_j^i y$ for all $j \in N$. Let Y be the cell $Y \in \mathit{Choice}_N(M^i)$ that contains y. Then, because of Proposition 1, it must be that (a) $Y \approx_j^i X_j$ for all $j \in N$. Let $w' = \langle Y, \overline{\epsilon}\rangle = \langle Y, \langle w_1^1, k_1^1\rangle, \ldots, \langle w_n^n, k_n^n\rangle\rangle$. By (a) and the definition of $\sim_j^d$, for all $j \in N$, $w_j \sim_j^d w'$.
4. "$\sim_N^d$ is the identity relation over W^d." Suppose that $\langle X, \overline{\epsilon}\rangle \sim_N^d \langle Y, \overline{\epsilon'}\rangle$. Then, for all $j \in N$, $\pi^j(\overline{\epsilon}) = \pi^j(\overline{\epsilon'})$ and hence (a) $\overline{\epsilon} = \overline{\epsilon'}$. Because $X \approx_j^i Y$ for all $j \in N$, it must be that for all $j \in N$ and for all $x \in X$ and all $y \in Y$ it holds that $x \sim_j^i y$. Hence, for all $x \in X$ and all $y \in Y$ it holds that $x \sim_N^i y$. Because $X, Y \in \mathit{Choice}_N(M^i)$, it must be that (b) $X = Y$. From (a) and (b) we conclude that $\langle X, \overline{\epsilon}\rangle = \langle Y, \overline{\epsilon'}\rangle$.

By (i)-(iv), M^d is a deterministic choice model. □

Theorem 6. *f is a bounded morphism from M^d to M^i in $\mathfrak{L}^{-N}$, i.e.:*

1. *f is onto.*
2. *For all $w, w' \in W^d$ and all non-empty $G \subset N$: if $w \sim_G^d w'$, then $f(w) \sim_G^i f(w')$*
3. *Where $u \in W^d$, $G \subset N$, and $f(u) \sim_G^i y$ for a $y \in W^i$: there is a $v \in W^d$ such that $f(v) = y$ and $u \sim_G^d v$.*
4. *For all $w \in W^d$ and $p \in \mathfrak{P}$: $w \in V^d(p)$ iff $f(w) \in V^i(p)$.*

Proof. Ad 1. Let $x \in W^i$ be arbitrary. Let $X \in \mathit{Choice}_N(M^i)$ be such that $x \in X$. Let $y \in W^i$ be such that $g^X(y) = x$. Finally, let $u = \langle X, \langle y, 1\rangle, \ldots, \langle y, 1\rangle\rangle$ be a sequence of length $n+1$. Note that $u \in W^d$. Moreover, $f(u) = g^X(y) = x$.
Ad 2. Analogous to the proof of Theorem 4.3.
Ad 3. Suppose the antecedent holds for $u = \langle X, \langle w_1, k_1\rangle, \ldots, \langle w_n, k_n\rangle\rangle$. Note that $f(u) \in X$. Let $Y \in \mathit{Choice}_N(M^i)$ be such that $y \in Y$. Note that, by the supposition, (†) for all $j \in G$, $X \approx_j^i Y$. Fix a $t \in N - G$. For all $j \in N - \{t\}$, let $w'_j = w_j$ and $l_j = k_j$. Let l_t be an arbitrary natural number

such that $l_t > l_j$ for all $j \in N - \{t\}$. Fix $w'_t \in Y$ such that $g^Y(w'_t) = y$. Let $v = \langle Y, \langle w'_1, l_1\rangle, \ldots, \langle w'_n, l_n\rangle\rangle$. It follows that $f(v) = g^Y(w'_t) = y$. By (†) and in view of the construction, $u \sim^d_j v$ for all $j \in G$, and hence $u \sim^d_G v$.
Ad 4. Immediate in view of the definition of V^d. □

It is useful, at this point, to check how we can cut down the size of W^d given certain restrictions on M^i. One can e.g. easily observe that, if there is a $Y \in \mathit{Choice}_N(M^i)$ such that, for all $Z \in \mathit{Choice}_N(M^i)$, $\mathsf{card}(Y) \geq \mathsf{card}(Z)$, then we can replace W^i with Y in the definition of g^X and W^d.

A natural follow-up question is: what if W^i is finite? Can we construct one proof that works for both finite and infinite models M^i, and that guarantees that the constructed model M^d is finite whenever M^i is finite? Note that the change that we proposed in the previous paragraph will not do to obtain such a proof. That is, all the natural numbers can still be used for the indices $k_1, \ldots, k_n$, whence W^d is bound to be infinite under the present construction. Moreover, the complication with double indices $\langle k_j, w_j\rangle$ seems necessary in order to ensure that, whatever all the other agents do, any given agent $j \in N$ can still "enforce" every world $x \in X$ for a given cell $X \in \mathit{Choice}_N(M^i)$, by choosing a yet higher index k_j and the world $y \in Y$ with $g^X(y) = x$.

6 Concluding Remarks

In this paper, we have shown that one can retrieve determinism without adding "nature" as an agent. We generalized an earlier result by Van Benthem and Pacuit [16], and showed that every indeterministic n-agent choice model is pointwise equivalent to a deterministic n-agent choice model, as long as we ignore the grand coalition. As a corollary, any (possibly infinite) choice model for n agents can be translated into an $\mathfrak{L}^{-N}$-equivalent normal game form for n agents, where the latter is finite if the former is finite. Our result thus contributes to connecting STIT logic and game theory more generally.

A number of questions should be answered in future work. Let us start with the most technical ones. First, can we rephrase the proof for the infinite case in such a way that it also covers the finite case, ensuring that M^d is finite whenever M^i is? Second, what about STIT logic with infinitely (countably many) agents? Here, the results appear to be mixed. If we only allow for finite groups in the language, we can easily generalize the construction from Sect. 5. However, if we allow for infinite groups $G \subset N$, this construction no longer does the job.

A different issue concerns the axiomatization of the logic we presented. Drawing on earlier results from [8,16], it can be shown that the $\mathfrak{L}^{-N}$-fragment of the logic of deterministic choice models is isomorphic to the modal product logic $\mathbf{S5^n}$. The latter logic is not decidable and cannot be finitely axiomatized for $n > 2$, cf. [8]. A non-standard axiomatization of $\mathbf{S5^n}$ has been presented in [17]. It remains to be seen how this axiomatization can be extended to the full language $\mathfrak{L}$ which includes $[N]$.

There are various ways for retrieving (finite) axiomatizability, decidability and acceptable complexity in the context of group STIT. First, one may restrict the formal language. For instance, it was proven in [11, Sect. 3] that when nesting of STIT operators is not allowed (i) the satisfiability problem becomes decidable in non-deterministic polynomial time (Corollary 1, p. 821), and (ii) the restricted logic becomes finitely axiomatizable (Corollary 2, p. 821).

Second, one may use different models to interpret the STIT language. Most importantly, one may weaken the *intersection property*, which says that $\sim_G = \bigcap_{j\in G} \sim_j$, to the requirement of monotonic effectivity: if $F \subseteq G$, then $\sim_G \subseteq \sim_F$. It has been shown in [3] that complete logics are readily available for these models, typically using Sahlqvist schemes [2].

A third route that was suggested in [16] is to give up the Independence of Agency (IOA) condition. If one does not impose (IOA) on the models, one obtains the non-deterministic counterpart of what Van Benthem and Pacuit call *general game models* [15,16]. Let us call such models *general choice models*. Note that in a general choice model, the choices of one agent may depend on the choices of other agents. The logic of general choice models coincides with the logic of distributed knowledge for arbitrary groups, which is known to be finitely axiomatizable and decidable [6, Chap. 3]. Now, as a matter of fact, our proofs in the current paper do not rely on (IOA), except where we show that the newly constructed model M^d also satisfies (IOA). Hence, our results reduce the problem of axiomatization of the logic of general game models to the axiomatization of the logic of general choice models.

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References

1. Belnap, N., Perloff, M., Xu, M., Bartha, P.: Facing the Future: Agents and Choice in Our Indeterminist World. Oxford University Press, Oxford (2001)
2. Blackburn, P., De Rijke, M., Venema, Y.: Modal Logic. Cambridge Tracts in Theoretical Computer Science (2001)
3. Broersen, J., Herzig, A., Troquard, N.: A normal simulation of coalition logic and an epistemic extension. In Samet, D. (ed.) Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge, pp. 92–101. ACM (2007)
4. Chellas, B.F.: Time and modality in the logic of agency. Stud. Log. **51**(3–4), 485–517 (1992)

5. Ciuni, R., Horty, J.: Stit logics, games, knowledge, and freedom. In: Baltag, A., Smets, S. (eds.) Johan van Benthem on Logic and Information Dynamics. OCL, vol. 5, pp. 631–656. Springer, Cham (2014). doi:10.1007/978-3-319-06025-5_23
6. Fagin, R., Halpern, J.Y., Moses, Y., Vardi, M.Y.: Reasoning About Knowledge. MIT Press, Cambridge (2003)
7. Gabbay, D., Kurucz, A., Wolter, F., Zakharyaschev, M.: Many-dimensional Modal Logics: Theory and Applications. Studies in Logic and the Foundations of Mathematics, vol. 148. North Holland Publishing Company (2003)
8. Herzig, A., Schwarzentruber, F.: Properties of logics of individual and group agency. In: Areces, C., Gobldblatt, R. (eds.) Advances in Modal Logic. College Publications (2008)
9. Horty, J.F.: Agency and Deontic Logic. Oxford University Press, New York (2001)
10. Kooi, B., Tamminga, A.: Moral conflicts between groups of agents. J. Philos. Log. **37**(1), 1–21 (2008)
11. Lorini, E., Schwarzentruber, F.: A logic for reasoning about counterfactual emotions. Artif. Intell. **175**(3), 814–847 (2011)
12. Osborne, M., Rubinstein, A.: A Course in Game Theory, 7th edn. MIT Press, Cambridge (2001)
13. Tamminga, A.: Deontic logic for strategic games. Erkenntnis **78**(1), 183–200 (2013)
14. Turrini, P.: Agreements as norms. In: Ågotnes, T., Broersen, J., Elgesem, D. (eds.) DEON 2012. LNCS, vol. 7393, pp. 31–45. Springer, Heidelberg (2012). doi:10.1007/978-3-642-31570-1_3
15. van Benthem, J.: Logic in Games. MIT Press, Cambridge (2014)
16. Benthem, J., Pacuit, E.: Connecting logics of choice and change. In: Müller, T. (ed.) Nuel Belnap on Indeterminism and Free Action. OCL, vol. 2, pp. 291–314. Springer, Cham (2014). doi:10.1007/978-3-319-01754-9_14
17. Venema, Y.: Rectangular games. J. Symbolic Log. **63**(4), 1549–1564 (1998)