Published for SISSA by Springer

Received: December 6, 2016
Revised: April 25, 2017
Accepted: June 26, 2017
Published: July 10, 2017

Supersymmetric Casimir energy and $\mathrm{SL}(3,\mathbb{Z})$ transformations

Frederic Brünner,[a,b] **Diego Regalado**[b,c] **and Vyacheslav P. Spiridonov**[d,e]

[a] *Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria*
[b] *Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, D-80805 Munich, Germany*
[c] *Institute for Theoretical Physics and Center for Extreme Matter and Emergent Phenomena, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*
[d] *Laboratory of Theoretical Physics, JINR, J. Curie str. 6, Dubna, Moscow region, 141980, Russia*
[e] *Laboratory of Mirror Symmetry, National Research University Higher School of Economics, 6 Usacheva str., Moscow, 119048, Russia*

E-mail: bruenner@hep.itp.tuwien.ac.at, regalado@mpp.mpg.de, spiridon@theor.jinr.ru

Abstract: We provide a recipe to extract the supersymmetric Casimir energy of theories defined on primary Hopf surfaces directly from the superconformal index. It involves an $\mathrm{SL}(3,\mathbb{Z})$ transformation acting on the complex structure moduli of the background geometry. In particular, the known relation between Casimir energy, index and partition function emerges naturally from this framework, allowing rewriting of the latter as a modified elliptic hypergeometric integral. We show this explicitly for $\mathcal{N}=1$ SQCD and $\mathcal{N}=4$ supersymmetric Yang-Mills theory for all classical gauge groups, and conjecture that it holds more generally. We also use our method to derive an expression for the Casimir energy of the nonlagrangian $\mathcal{N}=2$ SCFT with E_6 flavour symmetry. Furthermore, we predict an expression for Casimir energy of the $\mathcal{N}=1$ SP(2N) theory with $\mathrm{SU}(8)\times\mathrm{U}(1)$ flavour symmetry that is part of a multiple duality network, and for the doubled $\mathcal{N}=1$ theory with enhanced E_7 flavour symmetry.

Keywords: Supersymmetric Gauge Theory, Extended Supersymmetry

ArXiv ePrint: 1611.03831

https://doi.org/10.1007/JHEP07(2017)041

JHEP07(2017)041

Contents

1 Introduction

Supersymmetric gauge theories on curved spacetime manifolds have been studied intensely in recent years. The rise of localization as a tool to compute exact partition functions, especially in four-dimensional theories [1], has led to many interesting results and allowed the field to flourish (see [2] for a recent collection of reviews). Another quantity of interest is the superconformal index [3, 4], which can be defined for theories on Hopf surfaces, complex manifolds with $S^1 \times S^3$ topology. It is not affected by supersymmetry-preserving deformations and as such is invariant under the renormalization group flow. It receives contributions only from short representations and has served as a powerful check for many dualities. By applying the technique of localization [5, 6] (see also [7, 8]), it was found that the index $\mathcal{I}_{\rm SC}$ is related to the partition function $\mathcal{Z}_{\rm SUSY}$ by

$$\mathcal{Z}_{\rm SUSY} = e^{-\beta E_{\rm Casimir}}\, \mathcal{I}_{\rm SC}\,, \tag{1.1}$$

where β determines the size of the S^1 submanifold and $E_{\rm Casimir}$ is the supersymmetric Casimir energy. The latter determines the leading order behavior of the partition function in the $\beta \to \infty$ limit and is given in even dimensions by an equivariant integral of the anomaly polynomial [9]. It has also been investigated in the context of holography in [10].

Earlier, in an independent long term study inspired by quantum mechanical considerations, the third named author discovered the elliptic hypergeometric integrals [11, 12], the top level special functions of hypergeometric type. They generalize the plain hypergeometric functions and their q-analogs by adding one more (elliptic) deformation parameter p, also called a basic parameter. The original physical motivation was justified by application of these integrals in relativistic integrable many-body models [13, 14]. The relation to supersymmetric field theories was discovered in [15]. More precisely, superconformal indices appeared to be identical to certain elliptic hypergeometric integrals whose symmetry transformations rigorously confirm Seiberg dualities [16] in the sector of BPS states. From (1.1) it follows that the same integrals describe supersymmetric partition functions of four-dimensional field theories. As observed in [17], the superconformal indices of quiver theories represent partition functions of solvable two-dimensional lattice models. An explanation of this fact was given in [18] through the relation to two-dimensional topological field theories of [19]. A recent survey of this subject with references to relevant papers is given in [20]. All of this shows the universal relevance of elliptic hypergeometric functions for applications both in physics and mathematics.

One of the key checks of the validity of Seiberg dualities was a verification of 't Hooft anomaly matching conditions for dual theories [16]. As shown in [21], $\mathrm{SL}(3,\mathbb{Z})$ transformations are related to these matching conditions, since they reproduce all anomaly coefficients through the cocycle phase factor emerging in the corresponding transformation rule for the elliptic gamma function [22].

The partition function of a two-dimensional conformal field theory on a torus is invariant under the modular group $\mathrm{SL}(2,\mathbb{Z})$. The Casimir energy determines its leading order behavior as well, in complete analogy to the four-dimensional case. However, it is easy to see that the partition function of eq. (1.1) is not modular invariant. The same holds true for the superconformal index. E.g., in the so-called Schur limit degeneration of $\mathcal{N}=2$ index it turns out [23] that $\mathrm{SL}(2,\mathbb{Z})$ modular invariance is obtained only after modifying the index. Geometrically, the modular transformation acts on one fugacity parameter of the reduced index and corresponds to the usual action on a torus. The general superconformal index depends on a larger number of parameters. As such, it admits the action of the group $\mathrm{SL}(3,\mathbb{Z})$ transforming the complex structure moduli of the Hopf surface in a particular way, affecting β and the other "isometry parameters".

In this article, we study the transformation behavior of the superconformal index under $\mathrm{SL}(3,\mathbb{Z})$ in more detail. Our first observation is that different $\mathrm{SL}(3,\mathbb{Z})$ transformations act on complex structure moduli differently. While none leave them invariant, some map real ones into real, some real into complex parameters. In particular, there exists a transformation that partially relates the high and low "temperature" [1] asymptotics of the indices. The high temperature limit, when both basic parameters p and q tend to 1, maps indices to the hyperbolic integrals describing three-dimensional partition functions [24], while the low temperature limit, when p and q go to zero, degenerates the indices to Hilbert series counting the gauge invariant operators [25, 26]. The intermediate case, when only one of

[1]This is not really a temperature, since the boundary conditions for fermions on the S^1 are periodic.

the base parameters p or q goes to zero, reduces to the Macdonald theory — a relation which was noticed first in [27] and later developed in detail in [28].

Furthermore, we observe that the structure of the right hand side of eq. (1.1) arises as a consequence of the same $\mathrm{SL}(3,\mathbb{Z})$ transformation. This is because the elliptic gamma functions in the transformed superconformal index can be rewritten as a product of the modified elliptic gamma functions and exponentials consisting of the Bernoulli polynomials. The leading order contribution from those polynomials in the $\tilde{\beta} \to \infty$ limit, $\tilde{\beta}$ being the circle size of the transformed background, agrees precisely with the corresponding supersymmetric Casimir energy. As a direct consequence of eq. (1.1), the partition function can be written in terms of a modified elliptic hypergeometric integral. We demonstrate this explicitly for $\mathcal{N} = 1$ SQCD and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory for all classical gauge groups. We also apply this technique to the superconformal index of $\mathcal{N} = 2$ nonlagrangian theory with E_6 flavour symmetry that arises in the context of Argyres-Seiberg duality [29–31].

We use the $\mathrm{SL}(3,\mathbb{Z})$ transformation to predict the supersymmetric Casimir energy of the $\mathcal{N} = 1$ SP(2N) theory with $\mathrm{SU}(8) \times \mathrm{U}(1)$ flavour symmetry that was studied in [32] and that is part of a larger duality network. The other prediction we make concerns the Casimir energy of the $\mathcal{N} = 1$ theory with enhanced E_7 flavour symmetry constructed in [33], and we also discuss a related 6d/4d theory.

The article is organized as follows: in section 2, we define the $\mathrm{SL}(3,\mathbb{Z})$ transformations and explain how they act on the kernels of elliptic hypergeometric integrals, the basic building blocks of superconformal indices. We also recall important facts about Hopf surfaces. In section 3, we introduce the superconformal index, study its behavior under $\mathrm{SL}(3,\mathbb{Z})$ transformations and show how the Casimir energy can be extracted, leading to eq. (1.1). In section 4, we confirm this scheme explicitly for $\mathcal{N} = 1$ SQCD, $\mathcal{N} = 4$ SYM, $\mathcal{N} = 2$ E_6, and give our predictions regarding the $\mathcal{N} = 1$ SP(2N) theory with $\mathrm{SU}(8) \times \mathrm{U}(1)$ flavour symmetry, the theory with enhanced E_7 flavour symmetry and the 6d/4d model. We conclude and give a list of open questions in section 5.

2 Mathematical preliminaries

In this section, we summarize mathematical statements required in the later parts of the paper.

2.1 Elliptic hypergeometric functions

Superconformal indices can be written as contour integrals of a product of elliptic gamma functions [15], forming the elliptic hypergeometric integrals [11, 12], a fact that has lead to a very fruitful interrelation between mathematics and physics. Mathematical properties of elliptic hypergeometric integrals confirm Seiberg dualities by showing that they have identical sets of BPS states [15, 25, 26]. Vice versa, known physical dualities lead to large number of new conjectural important mathematical identities requiring rigorous proofs [25, 26]. In the following, we will introduce these special functions and discuss their behavior under $\mathrm{SL}(3,\mathbb{Z})$ transformations.

JHEP07(2017)041

The elliptic gamma function is the unique (up to the multiplication by a constant) meromorphic solution to the finite difference equations

$$\begin{aligned} f(u+\omega_1) &= \theta(z;p)f(u), \\ f(u+\omega_3) &= \theta(z;q)f(u), \\ f(u+\omega_2) &= f(u) \end{aligned} \tag{2.1}$$

with $z = \exp(2\pi i u/\omega_2)$, $z \in \mathbb{C}^*$, and incommensurate $\omega_j \in \mathbb{C}$. Here

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty, \qquad (z;p)_\infty = \prod_{j=0}^{\infty}(1-zp^j),$$

is the Jacobi theta function. The bases p and q with $|p|, |q| < 1$ are related to the complex parameters ω_i by $p = \exp(2\pi i\omega_3/\omega_2)$ and $q = \exp(2\pi i\omega_1/\omega_2)$. An explicit form of the elliptic gamma function $f(u) = \Gamma(z;p,q)$ is given by the infinite product

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^iq^j}. \tag{2.2}$$

The elliptic hypergeometric integrals defining the transcendental elliptic hypergeometric functions are formed as contour integrals of particular products of $\Gamma(z;p,q)$ with special choice of the arguments z. The key characteristic property of these integrals [34] is that their integrand functions are defined as solutions of the first order finite-difference equations in the integration variables with the coefficients given by elliptic functions (i.e., meromorphic double periodic functions).

As shown in [34] there is another solution to the first line equation in eq. (2.1), such that the other equations are modified to

$$\begin{aligned} f(u+\omega_2) &= \theta(e^{2\pi i u/\omega_1};r)f(u), \\ f(u+\omega_3) &= e^{-\pi i B_{2,2}(u,\omega_1,\omega_2)}f(u), \end{aligned} \tag{2.3}$$

where $r = \exp(2\pi i\omega_3/\omega_1)$ is an additional base, and $B_{2,2}$ is a second order Bernoulli polynomial given by the expression

$$B_{2,2}(u,\omega_1,\omega_2) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \tag{2.4}$$

This solution was called the *modified elliptic gamma function* $f(u) = \mathcal{G}(u;\omega)$, with $\omega = (\omega_1,\omega_2,\omega_3)$. For $|p|, |q| < 1$ it is related to the standard elliptic gamma function by

$$\mathcal{G}(u;\omega) = \Gamma(re^{-2\pi i u/\omega_1};\tilde{q},r)\Gamma(e^{2\pi i u/\omega_2};p,q), \tag{2.5}$$

with $\tilde{q} = \exp(-2\pi i\omega_2/\omega_1)$. As follows from an identity derived in [22], the modified elliptic gamma function can be rewritten as

$$\mathcal{G}(u;\omega) = e^{-\frac{\pi i}{3}B_{3,3}(u,\omega)}\Gamma(e^{-2\pi i u/\omega_3};\tilde{r},\tilde{p}), \tag{2.6}$$

where $\tilde{r} = \exp(-2\pi i\omega_1/\omega_3)$, $\tilde{p} = \exp(-2\pi i\omega_2/\omega_3)$ and $B_{3,3}(u,\omega)$ is a third order Bernoulli polynomial given by

$$B_{3,3}(u,\omega) = \frac{1}{\omega_1\omega_2\omega_3}\left(u - \frac{1}{2}\sum_{k=1}^{3}\omega_k\right)\left(\left(u - \frac{1}{2}\sum_{k=1}^{3}\omega_k\right)^2 - \frac{1}{4}\sum_{k}^{3}\omega_k^2\right). \tag{2.7}$$

It is remarkable that $\mathcal{G}(u;\omega)$ remains a well defined meromorphic function of u even when the base q lies on the unit circle, $|q| = 1$, which is easily seen from the representation (2.6).

A key reason for introduction of the function $\mathcal{G}(u;\omega)$ as an additional gamma function is the fact that many useful identities for elliptic hypergeometric integrals are derived using only the first equation in the set (2.1). Therefore it is natural to expect that there should exist analogous identities formulated in terms of the function $\mathcal{G}(u;\omega)$, which, in contrast to the original relations, will be well defined for $|q| = 1$.

In section 4.3 we will need a double elliptic gamma function, given by

$$\Gamma(z;p,q,t) = \prod_{i,j,k=0}^{\infty}(1 - zp^iq^jt^k)(1 - z^{-1}p^{i+1}q^{j+1}t^{k+1}), \tag{2.8}$$

where $t = \exp(2\pi i\omega_4/\omega_2)$ with $|t| < 1$. It also possesses a modified version in a similar spirit as that of the original elliptic gamma function. This modified double elliptic gamma function is defined by [35]

$$\mathcal{G}(u;\omega_1,\ldots,\omega_4) = \frac{\Gamma(e^{2\pi iu/\omega_2};q,p,t)}{\Gamma(\tilde{q}e^{2\pi iu/\omega_1};\tilde{q},r,s)}, \tag{2.9}$$

with $s = \exp(2\pi i\omega_4/\omega_1)$. The vector ω now also includes ω_4. There exists an analog of eq. (2.6), namely the relation

$$\mathcal{G}(u;\omega_1,\ldots,\omega_4) = e^{-\frac{\pi i}{12}B_{4,4}}\frac{\Gamma(e^{-2\pi iu/\omega_3};\tilde{p},\tilde{r},\tilde{w})}{\Gamma(we^{-2\pi iu/\omega_4};\tilde{s},\tilde{t},w)}, \tag{2.10}$$

with $w = \exp(2\pi i\omega_3/\omega_4)$ and $\tilde{w} = \exp(-2\pi i\omega_4/\omega_3)$. The Bernoulli polynomial $B_{4,4} \equiv B_{4,4}(u,\omega)$ is given by

$$B_{4,4} = \frac{1}{\omega_1\omega_2\omega_3\omega_4}\left[\left(\left(u - \frac{1}{2}\sum_{k=1}^{4}\omega_k\right)^2 - \frac{1}{4}\sum_{k=1}^{4}\omega_k^2\right)^2 - \frac{1}{30}\sum_{k=1}^{4}\omega_k^4 - \frac{1}{12}\sum_{1\leq j<k\leq 4}\omega_j^2\omega_k^2\right]. \tag{2.11}$$

As it is again easy to see, the function $\mathcal{G}(u;\omega_1,\ldots,\omega_4)$ is well defined when the base q lies on the unit circle, $|q| = 1$.

For the description of the superconformal index, we also need the Euler function $\phi(q)$, which can be expressed in terms of the q-Pochhammer symbol as

$$\phi(q) = (q;q)_\infty.$$

In [21], it was discovered that 't Hooft anomaly matching conditions for Seiberg duality in $\mathcal{N} = 1$ theories are related to $SL(3,\mathbb{Z})$ transformations acting on the homogeneous coordinates ω by

$$\omega = (\omega_1,\omega_2,\omega_3) \longrightarrow \tilde{\omega} = (\omega_1,-\omega_3,\omega_2). \tag{2.12}$$

JHEP07(2017)041

Under this transformation, the bases p, q, r change as $p \to \tilde{p}$, $q \to \tilde{r}$ and $r \to \tilde{q}$. For the elliptic gamma function, we get

$$\begin{aligned}\Gamma(e^{2\pi i u/\omega_2}; p, q) &\longrightarrow \Gamma(e^{-2\pi i u/\omega_3}; \tilde{p}, \tilde{r}) \\ &= e^{\frac{\pi i}{3} B_{3,3}(u,w)} \mathcal{G}(u;\omega),\end{aligned} \tag{2.13}$$

as follows from eq. (2.6). The transformation law of the Euler function follows from the transformation properties of the Dedekind eta-function, and it is given by

$$\phi(e^{-2\pi i \frac{\omega_2}{\omega_1}}) = \left(-i\frac{\omega_1}{\omega_2}\right)^{1/2} e^{\frac{\pi i}{12}(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1})} \phi(e^{2\pi i \frac{\omega_1}{\omega_2}}), \tag{2.14}$$

where we assume that $\sqrt{-i} = e^{-\frac{\pi i}{4}}$. For the modified double elliptic gamma function, the modular transformations are more involved. The bases q, p, t in of the double elliptic gamma function in the numerator in (2.9) change to $\tilde{p}, \tilde{r}, \tilde{w}$ in (2.10) corresponding to the transformation $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$. However, for the function in the denominator one has the changes $\tilde{q}, r, s \to \tilde{t}, w, \tilde{s}$ corresponding to another $\mathrm{SL}(2,\mathbb{Z})$ subgroup action $(\omega_4, \omega_1) \to (-\omega_1, \omega_4)$.

2.2 Hopf surfaces

In this section, we discuss basic facts about Hopf surfaces as described in [5] and [36] (see also [37]). This is needed to understand the action of $\mathrm{SL}(3,\mathbb{Z})$ on the superconformal index geometrically. To avoid confusion, we mostly stick to the notation of [5].

One of the possible background geometries that preserve supersymmetry is $\mathrm{S}^1 \times \mathrm{S}^3$ [38]. In this case it is possible to have two complex Killing spinors of opposite R-charge, a requirement for $\mathcal{N} = 1$ supersymmetry. A class of complex manifolds of $\mathrm{S}^1 \times \mathrm{S}^3$ topology is given by *Hopf surfaces*, compact complex surfaces with universal covering $\mathbb{C}^2 - (0,0)$. There are different types of Hopf surfaces; we will restrict ourselves to *primary* Hopf surfaces, which possess a fundamental group isomorphic to $\mathbb{Z}$. All primary Hopf surfaces are diffeomorphic to $\mathrm{S}^1 \times \mathrm{S}^3$. More general Hopf surfaces arise as quotients by specific finite groups. A prerequisite for the existence of two complex Killing spinors of opposite R-charge is that the manifold admits two complex structures of opposite orientation. Such primary Hopf surfaces are given by quotients of $\mathbb{C}^2 - (0,0)$ of the form

$$(z_1^\pm, z_2^\pm) \sim (p_\pm z_1^\pm, q_\pm z_2^\pm), \tag{2.15}$$

where $p_\pm, q_\pm$ are complex structure moduli with $0 < |p_\pm| \leq |q_\pm| < 1$, and $z_1^\pm, z_2^\pm$ are complex coordinates. The signs $+$ and $-$ refer to the complex structures I_+ and I_-, respectively, and since we require them to be of opposite orientation with respect to each other, the manifold is said to possess ambihermitian structure.

To study Killing spinors on primary Hopf surfaces explicitly, a non-singular metric compatible with integrable complex structures was introduced in [5]. This metric admits a complex Killing vector K that satisfies $K_\mu K^\mu = 0$ and commutes with its own complex conjugate. As a consequence, there exist two Killing spinors. Furthermore, the existence of an additional real Killing vector is assumed, leading to a $\mathrm{U}(1)^3$ isometry group. The metric is given by

$$ds^2 = \Omega(\rho)^2 d\tau^2 + f^2(\rho)d\rho^2 + m_{IJ}(\rho)d\varphi_I d\varphi_J, \tag{2.16}$$

with $I, J = 1, 2$, where $\tau \sim \tau + \beta$ parametrizes the S^1 part, while $\rho, \varphi_1, \varphi_2$ for $0 \leq \rho \leq 1$, $0 \leq \varphi_1 \leq 2\pi$ and $0 \leq \varphi_2 \leq 2\pi$ are coordinates on the S^3. We describe the S^3 in terms of a torus fibration over the interval $0 \leq \rho \leq 1$, where $m_{IJ}(\rho)$ are the positive definite metric components of the torus, and $f(\rho), \Omega(\rho) > 0$. The complex Killing vector is given by

$$K = \frac{1}{2}\left(b_1 \frac{\partial}{\partial \varphi_1} + b_2 \frac{\partial}{\partial \varphi_2} - i\frac{\partial}{\partial \tau}\right), \tag{2.17}$$

with b_1 and b_2 being two real parameters. They are related to the complex structure moduli by

$$p_\pm = e^{\pm\beta|b_1|}, \qquad q_\pm = e^{\pm\beta|b_2|}. \tag{2.18}$$

While these parameters are real for a direct product metric like the one given above, for a non-direct product metric, they will in general be complex. The use of the letters p and q both in section 2 and section 3 is not a mere coincidence.

3 The superconformal index and SL$(3, \mathbb{Z})$

In this section, we discuss the SL$(3, \mathbb{Z})$ transformation properties of the superconformal index and present the main ideas of the article.

3.1 The index

The general expression for the superconformal index of an $\mathcal{N} = 1$ theory defined on a primary Hopf surface is given by [3, 4]

$$\mathcal{I} = \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\gamma H} p^{\frac{R}{2}+J_R+J_L} q^{\frac{R}{2}+J_R-J_L} \prod_i y_i^{F_i}; \tag{3.1}$$

where $\mathcal{F}$ is the fermion number, R is the R-charge, J_L and J_R are the Cartan generators of the rotation group $\mathrm{SU(2)_L} \times \mathrm{SU(2)_R}$, and F_j are maximal torus generators of the flavor group. The index only receives contributions from states with $H = E - 2J_L - \frac{3}{2}R = 0$, E being the energy, and is independent of the chemical potential γ. While the parameters y_i are fugacities for the flavor group, the basic parameters p and q are complex structure moduli of the Hopf surface and are given by

$$p = p_-, \qquad q = q_-. \tag{3.2}$$

For Lagrangian theories, its generic form is given by an integral over the gauge group:

$$\mathcal{I}(p, q, y) = \int_G d\mu(z) \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i(p^n, q^n, y^n, z^n)\right), \tag{3.3}$$

where y and z schematically denote all possible flavor or gauge fugacities. The single particle index $i(p, q, y, z)$ only depends on the characters of the representations of fields.

It was shown that for many theories, this integral can be rewritten in terms of elliptic hypergeometric functions that depend on the complex structure moduli p and q as described in section 2. The fugacities will in general satisfy a constraint relation that is required for the consistency of the integral, and is related to the absence of gauge anomalies. For Seiberg dual theories the indices (3.3) coincide despite of having quite different formal expressions. This was shown first for $\mathcal{N}=1$ SQCD in [15]. For brevity, we will not show how this comes about explicitly, and refer to [25, 26], where many examples can be found.

3.2 Mapping Hopf surfaces and asymptotics

As described in the previous sections, the superconformal index for a theory defined on a primary Hopf surface depends on its complex structure moduli, which arise as arguments of elliptic gamma functions. In order to see how the $\mathrm{SL}(3,\mathbb{Z})$ transformations described in section 2.1 act on them, we need to make the identifications $2\pi i\omega_3/\omega_2 = -\beta|b_1|$ and $2\pi i\omega_1/\omega_2 = -\beta|b_2|$. This is achieved by setting $\omega_1 = i|b_2|$, $\omega_3 = i|b_1|$ and $\beta = 2\pi/\omega_2$. Note that one has $\omega_3/\omega_1 \geq 0$, i.e. $|r| = 1$. Applying the transformation of eq. (2.12) to these quantities, we get

$$p = e^{-\beta|b_1|} \longrightarrow \tilde{p} = e^{-2\pi i\frac{\omega_2}{\omega_3}} = e^{-\tilde{\beta}\omega_2}, \tag{3.4}$$

and

$$q = e^{-\beta|b_2|} \longrightarrow \tilde{r} = e^{-2\pi i\frac{\omega_1}{\omega_3}} = e^{-\tilde{\beta}i|b_2|}, \tag{3.5}$$

with $\tilde{\beta} = 2\pi i/\omega_3 = 2\pi/|b_1|$. As we can see, the transformation does two things: it switches the parameters $|b_1|$ and ω_2, interchanging geometric information about the S^1 and the S^3. Furthermore, it turns the real parameter q into a complex one, corresponding to a Hopf surface with a non-direct product metric. The parameter p, however, is mapped from real to a real variable. Conversely, it is also possible to transform a complex parameter into a real one by simply replacing $|b_2|$ with $i|b_2|$ in the original expression.

In terms of the superconformal index, the transformation can be written schematically as

$$\mathcal{I}(e^{-\beta|b_1|}, e^{-\beta|b_2|}, y) \xrightarrow{\mathrm{SL}(3,\mathbb{Z})} \mathcal{I}(e^{-\tilde{\beta}\omega_2}, e^{-\tilde{\beta}i|b_2|}, \tilde{y}), \tag{3.6}$$

where $\tilde{y}$ stands for an arbitrary $\mathrm{SL}(3,\mathbb{Z})$ transformed set of fugacities that may include both flavor and R-symmetry. We shall give explicit expressions for $\tilde{y}$ for the examples considered below. The index on the right hand side is simply the index of the same theory placed on a Hopf surface with parameters exchanged as described above. Note, however, that whereas the original Hopf surface was defined using two contractions $z_1 \to pz_1$, $z_2 \to qz_2$ with $|p|, |q| < 1$, for the modular transformed surface the non-contractive regime $|q| = 1$ becomes admissible. This will result in the fact that modular transformed indices become well-defined meromorphic functions even for $|q| = 1$, i.e. effectively we cover a wider domain of values of the moduli.

Since the parameters the transformed index depends on are the same but are now placed in different combinations, it is natural to ask what happens to the transformed index in a particular limit of the original one. For example, one may consider the low "temperature" limit, i.e. $\beta \to \infty$, which corresponds to $\omega_2 \to 0$, or $p \to 0$ and $q \to 0$. This limit was considered in [25, 26], where it was indicated that for an appropriate choice of the flavour group fugacities the indices reduce to the Hilbert series that counts gauge invariant operators [39]. In the study of $\mathcal{N}=2$ indices in [28] it was called the Hall-Littlewood limit.

As mentioned above, this is the limit in which the Casimir energy is defined. In the transformed index, the only fugacity that depends on ω_2 is $p=\exp(-\tilde{\beta}\omega_2)$, which behaves as $p\to 1$. This limit is diverging and is not well understood on its own. However, the "high temperature" limit for indices, $\beta\to 0$, or equivalently $\omega_2\to\infty$, or $p\to 1$ and $q\to 1$, is well known. As noticed first in [24], in this case the indices reduce to partition functions of three-dimensional field theories up to a diverging exponential, which was shown in [40] to be related to anomaly coefficients in the combination $a-c$. The results of [21] were applied also in the context of formal holomorphic block factorisations of $4d$ superconformal indices [41] with the corresponding $\beta\to 0$ $3d$-reduction. A more detailed consideration of this limit for several different theories was given in [42]. Analytically, the limit $p,q\to 1$ reduces the elliptic hypergeometric integrals to the hyperbolic integrals.

We may also ask what happens for different $\mathrm{SL}(3,\mathbb{Z})$ transformations, for example the combinations $\omega_3\to-\omega_1$ and $\omega_1\to\omega_3$ or $\omega_1\to-\omega_2$ and $\omega_2\to\omega_1$. In the former case, we get the transformations $p\to\exp(\beta|b_2|)$ and $q\to\exp(-\beta|b_1|)$, while in the latter, we get $p\to\exp(\beta' i|b_1|)$ and $p\to\exp(-\beta'\omega_2)$, with $\beta'=2\pi/|b_2|$. It is straightforward to see how the transformed parameters behave in the $\beta\to 0$ and $\beta\to\infty$ limits. However, none of these additional transformations will lead to the supersymmetric Casimir energy in the way outlined below.

3.3 The recipe and a conjecture

The $\mathrm{SL}(3,\mathbb{Z})$ transformation properties of the kernels of superconformal indices were applied in [21] in the explanation of 't Hooft anomaly matching conditions for Seiberg duality of $\mathcal{N}=1$ SQCD. It was based on the equivalence of the transformed indices for both electric and magnetic theories. To this end, the transformed index has to be rewritten with the help of eq. (2.6) as the product of an exponential containing Bernoulli polynomials and a so-called modified elliptic hypergeometric integral:

$$\mathcal{I}(\tilde{p},\tilde{r},\tilde{y})=e^{\varphi}I^{\mathrm{mod}}(\tilde{p},\tilde{r},\tilde{y}). \tag{3.7}$$

The structure of the modified integral is essentially the same as that of the superconformal index, with elliptic gamma functions replaced by their modified counterparts and the prefactor in front of the integral slightly modified. To see this explicitly, consider section 4 for a number of concrete examples. A crucial point is that the dependence of the exponent on the integration variables vanishes, i.e. it is independent of fugacities of the gauge group. While it is a priori not guaranteed, from the physical point of view it is a consequence of the absence of gauge anomalies while from the mathematical point of view, it is the consequence of the balancing condition needed for the original definition of elliptic hypergeometric integrals themselves [34]. This is true for all dualities considered in [25, 26]. Matching of global anomalies is in general equivalent to the condition that the exponent φ matches for dual theories. In light of the recent discovery of the relationship between the anomaly polynomial and the Casimir energy through an equivariant integral, it is natural to conjecture that the Casimir energy is contained in φ. This is precisely what we find. The function φ consists of a term proportional to $\tilde{\beta}$, which agrees with the Casimir energy, and a residual function $\mathcal{R}(\tilde{\beta})$ that contains terms subleading in $\tilde{\beta}$. This leads us to propose the following recipe for calculating the Casimir energy:

$$\mathcal{I}(p,q,y) \xrightarrow{\mathrm{SL}(3,\mathbb{Z})} \mathcal{I}(\tilde{p},\tilde{r},\tilde{y}) = e^{\tilde{\beta}E_{\mathrm{Casimir}}+\mathcal{R}(\tilde{\beta})} I^{\mathrm{mod}}(\tilde{p},\tilde{r},\tilde{y}). \tag{3.8}$$

In words, there are three steps: i) transform the superconformal index according to eq. (2.12) and eq. (3.6), ii) rewrite it with the help of eq. (2.6) and iii) pull out the exponential factor and identify the leading order term as the Casimir energy. Even though we do not prove this recipe in full generality, we confirm it for several different theories in the next section.

Inspecting eq. (3.8), one can see that upon bringing the leading order exponential to the other side of the equation yields precisely the form of the right hand side of eq. (1.1). This leads us to conclude that the partition function of the corresponding $\mathrm{SL}(3,\mathbb{Z})$ transformed theory can be written as

$$\tilde{\mathcal{Z}}_{\mathrm{SUSY}} = e^{\mathcal{R}(\tilde{\beta})} I^{\mathrm{mod}}(\tilde{p},\tilde{r},\tilde{y}). \tag{3.9}$$

Since we could have also started the recipe with a transformed index, we get

$$\mathcal{Z}_{\mathrm{SUSY}} = e^{\mathcal{R}(\beta)} I^{\mathrm{mod}}. \tag{3.10}$$

We conjecture that the recipe can be applied to all theories for which the partition function can be written in the form of eq. (1.1). The partition function can then be rewritten in terms of a modified elliptic hypergeometric integral introduced in [43].

4 Examples

In this section, we give several examples for the application of the above recipe.

4.1 $\mathcal{N}=1$ SQCD

Supersymmetric QCD with $\mathcal{N}=1$ superalgebra is a gauge theory with gauge group $\mathrm{SU(N_c)}$, flavor symmetry $\mathrm{SU(N_f)_L} \times \mathrm{SU(N_f)_R} \times \mathrm{U(1)_B}$ and R-symmetry $\mathrm{U(1)_R}$. We have summarized the matter content in table 1. In its conformal window, i.e. for $3N_c/2 < N_f < 3N_c$, it is subject to Seiberg duality [16]. The superconformal index, which serves as a check for this duality, is given by

$$\mathcal{I}_{\mathrm{SQCD}}(p,q,y) = \kappa_{N_c} \int_{\mathbb{T}^{N_c-1}} \frac{\prod_{j=1}^{N_c}\prod_{l=1}^{N_f}\Gamma(s_l z_j, t_l^{-1}z_j^{-1};p,q)}{\prod_{1\le j<k\le N_c}\Gamma(z_j z_k^{-1}, z_j^{-1}z_k;p,q)} \prod_{i=1}^{N_c-1}\frac{dz_i}{2\pi i z_i}, \tag{4.1}$$

where we use the notation $\Gamma(a,b;p,q) := \Gamma(a;p,q)\Gamma(b;p,q)$ and $\kappa_{N_c} = \phi(p)^{N_c-1}\phi(q)^{N_c-1}/N_c!$. To see the transformation behavior, we rewrite the fugacities satisfying the balancing condition $\prod_{k=1}^{N_f} s_k t_k^{-1} = (pq)^{N_f-N_c}$ as $s_l = \exp(2\pi i\sigma_l/\omega_2)$,

	$\mathrm{SU(N_c)}$	$\mathrm{SU(N_f)}$	$\mathrm{SU(N_f)}$	$\mathrm{U(1)_B}$	$\mathrm{U(1)_R}$
$\mathbf{Q}^i$	f	f	1	1	$(N_f-N_c)/N_f$
$\tilde{\mathbf{Q}}_i$	$\bar{f}$	1	$\bar{f}$	-1	$(N_f-N_c)/N_f$
$\mathbf{V}$	adj	1	1	0	1

Table 1. The matter content of $\mathcal{N}=1$ SQCD. f denotes the fundamental, $\bar{f}$ the antifundamental and adj the adjoint representation.

$t_l=\exp(2\pi i\tau_l/\omega_2)$ and $z_l=\exp(2\pi i u_l/\omega_2)$. Then, their transforms $\tilde{s}_l$, $\tilde{t}_l$ and $\tilde{z}_l$ are related simply by $\omega_2\to-\omega_3$.

Using eq. (2.6), the transformed index $\mathcal{I}_{\mathrm{SQCD}}(\tilde{y},\tilde{p},\tilde{r})$ can now be rewritten in the form indicated in eq. (3.8), with the Casimir energy E_{Casimir} given by

$$\begin{aligned}E_{\mathrm{Casimir}}=\lim_{\omega_3\to0}\frac{\omega_3}{6}\Bigg(&\sum_{i=1}^{N_f}\sum_{j=1}^{N_c}\big(B_{3,3}(\sigma_i+u_j,\omega)+B_{3,3}(-\tau_i-u_j,\omega)\big)\\&-\sum_{1\le i<j\le N_c}\big(B_{3,3}(u_i-u_j,\omega)+B_{3,3}(u_j-u_i,\omega)\big)\Bigg)\\&+\frac{N-1}{24}(\omega_1+\omega_2),\end{aligned}\tag{4.2}$$

in complete agreement with the result of [9]. Notice that the dependence of eq. (4.2) on the integration variables u_i drops once we impose the balancing condition, which in the additive notation should be chosen precisely as $\sum_{i=1}^{N_f}(\sigma_i-\tau_i)=(N_f-N_c)\sum_{k=1}^{3}\omega_k$ (the transition from the multiplicative to additive notation is slightly ambiguous due to the existence of a natural period for the exponential function). Such a property serves as a criterion for the absence of gauge anomalies [21].

The residual function $\mathcal{R}(\tilde{\beta})$ reads

$$\mathcal{R}(\tilde{\beta})=-\frac{i\pi N_c}{3}\sum_{i=1}^{N_f}\big(C(\sigma_i)+C(-\tau_i)\big)+\frac{i\pi(N_c^2-1)}{3}C(0)\,,\tag{4.3}$$

where we have defined [2]

$$C(x)=B_{3,3}(x)-\frac{1}{\omega_3}\lim_{\omega_3\to0}\omega_3B_{3,3}(x).\tag{4.4}$$

Finally, the modified elliptic hypergeometric integral is given by

$$\mathcal{I}_{\mathrm{SQCD}}^{\mathrm{mod}}=\kappa_{N_c}^{\mathrm{mod}}\int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}}\frac{\prod_{j=1}^{N_c}\prod_{l=1}^{N_f}\mathcal{G}(\sigma_i+u_j,-\tau_i-u_j;\omega)}{\prod_{1\le j<k\le N_c}\mathcal{G}(u_i-u_j,u_j-u,i;\omega)}\prod_{k}^{N_c-1}\frac{du_k}{\omega_3},\tag{4.5}$$

with

$$\kappa_{N_c}^{\mathrm{mod}}=\frac{(2\kappa_2^{\mathrm{mod}})^{N_c-1}}{N_c!},\qquad\kappa_2^{\mathrm{mod}}:=-\frac{\omega_3}{\omega_2}\frac{\phi(p)\phi(q)\phi(r)}{2\phi(\tilde{q})}.\tag{4.6}$$

[2] Notice that $C(x)$ is invariant under $x\to\alpha x$, $\omega_i\to\alpha\omega_i$, so it only depends on two ratios of three quasiperiods ω_i.

	SP(2N)	SU(8)	U(1)	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	$\frac{1}{4}$
X	T_A	1	1	0
V	adj	1	0	$\frac{1}{2}$

Table 2. The matter content of the $\mathcal{N}=1$ theory with SP(2N) gauge symmetry and SU(8) $\times$ U(1) flavour symmetry. T_A denotes the antisymmetric tensor representation.

Evidently, this integral is well defined for $|q| = 1$ [43]. The prefactor (4.6) arises from eq. (4.1) in such a way that the Casimir energy comes out correctly in the absence of flavor symmetries. While this choice seems to be arbitrary at this point, the same pattern also appears in all the other cases we have checked.

4.2 $\mathcal{N}=1$ SP(2N) theory with SU(8) $\times$ U(1) flavour symmetry

In [32], Vartanov and one of the present authors studied a network of dualities for a certain set of $\mathcal{N}=1$ theories. The starting point is an "electric" model described by SQCD-like theory with symplectic SP(2N) gauge symmetry and SU(8) $\times$ U(1) flavour symmetry. Its field content is summarized in table 2. The corresponding superconformal index is given by

$$\mathcal{I}^{SP(2N)} = \psi_N \int \prod_{1\leq j<k\leq N} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^{N} \frac{\prod_{k=1}^{8} \Gamma(t^{\frac{1-N}{4}} (pq)^{\frac{1}{4}} y_k z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{2\pi i z_j}, \tag{4.7}$$

where

$$\psi_N = \frac{\phi(p)^N \phi(q)^N}{2^N N!} \Gamma(t; p, q)^{N-1}, \tag{4.8}$$

with the fugacities $y_j := \exp(2\pi i \alpha_j/\omega_2)$ and $t := \exp(2\pi i \tau/\omega_2)$ satisfying the balancing condition

$$t^{2N-2} \prod_{j=1}^{8} t_j = (pq)^2. \tag{4.9}$$

Note that a $\Gamma(az^{\pm 1}; p, q) := \Gamma(az, az^{-1}; p, q)$.

The resulting expression for the Casimir energy is

$$E_{\text{Casimir}} = \frac{N}{24}(\omega_1+\omega_2) + \lim_{\omega_3\to 0} \frac{\omega_3}{6} \Bigg(\sum_{1\leq j<k\leq N} (B_{3,3}(\tau \pm u_j \pm u_k) - B_{3,3}(\pm u_j \pm u_k)) \\ + \sum_{j=1}^{N} \Bigg(\sum_{k=1}^{8} B_{3,3}\left(\frac{1-N}{4}\tau + \frac{\omega_1+\omega_3}{4} + \alpha_k \pm u_j\right) - B_{3,3}(\pm 2u_j)\Bigg) + (N-1)B_{3,3}(\tau)\Bigg).$$

Terms containing factors of u_j again vanish completely due to the balancing condition, corresponding to the absence of gauge anomalies. The residual function is given by terms in higher powers of ω_3. The modified elliptic hypergeometric integral for this case can be read off from the expression given in the end of the next section.

4.3 $\mathcal{N}=1$ E_7 flavor enhanced theories

In [33], Dimofte and Gaiotto constructed an $\mathcal{N}=1$ theory that possesses a point in its moduli space where flavour symmetry is enhanced to E_7. This can be done by deforming a product of two copies of an SU(2) gauge theory $\mathcal{T}$ with SU(8) flavour symmetry. The superconformal index of one copy is given by

$$\mathcal{I}_{\mathcal{T}}(y) = \frac{\phi(p)\phi(q)}{2} \int \frac{\prod_{i=1}^{8} \Gamma((pq)^{1/4} y_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i}, \tag{4.10}$$

where the fugacities $y_j := \exp(2\pi i \alpha_j/\omega_2)$ satisfying $\prod_{i=1}^{8} y_i = 1$ correspond to the SU(8) flavour symmetry. This index possesses a nontrivial symmetry transformation described by the Weyl group $W(E_7)$ [34]. The model of [33] with enhanced E_7 flavor symmetry has now the index given by the product

$$\mathcal{I}_{\mathrm{E}_7} = \mathcal{I}_{\mathcal{T}}(y) \times \mathcal{I}_{\mathcal{T}}(y^{-1}). \tag{4.11}$$

The Casimir energy can now be calculated in a straightforward manner, with the result

$$E_{\mathrm{Casimir}} = \frac{\omega_1+\omega_2}{\omega_1\omega_2} \left[\frac{1}{4}(\omega_1^2+\omega_2^2) + \frac{\omega_1\omega_2}{24} - \frac{1}{2}\sum_{i=1}^{8} \alpha_i^2 \right], \tag{4.12}$$

and the residual function

$$\mathcal{R}(\tilde{\beta}) = \frac{\pi i}{3} \left[\frac{4\omega_3^2}{\omega_1\omega_2} + \frac{1}{4\omega_1\omega_2} \left(13\omega_3(\omega_1+\omega_2) + 7\omega_1^2 - 9\omega_1\omega_2 + 7\omega_2^2 - 24\sum_{i=1}^{8}\alpha_i^2 \right) \right]. \tag{4.13}$$

The corresponding modified elliptic hypergeometric integral is given by

$$\mathcal{I}_{\mathrm{E}_7}^{\mathrm{mod}} = \left(\kappa_2^{\mathrm{mod}}\right)^2 \int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}} \frac{\prod_{j=1}^{8} \mathcal{G}(\frac{1}{4}(\omega_1+\omega_3)+\alpha_j \pm u, \frac{1}{4}(\omega_1+\omega_3)-\alpha_j \pm u'; \omega)}{\mathcal{G}(\pm 2u, \pm 2u'; \omega)} \frac{du}{\omega_3}\frac{du'}{\omega_3}. \tag{4.14}$$

In [33] another model with extended E_7 flavor symmetry was suggested on the basis of a $5d$ chiral hypermultiplet interacting with a $4d$ theory living on codimension 1 space. It is not clear how to compute the Casimir energy for this system using our approach, since the corresponding half-indices do not obey clear modular transformation properties. However, using a similar idea, in [35] a $6d/4d$ theory was suggested where a $6d$ chiral hypermultiplet was interacting with the $4d$ model described in the previous section which lives in a "corner" of the $6d$ space. The corresponding superconformal index has the form

$$I_{6d/4d} = \frac{\mathcal{I}^{SP(2N)}}{\prod_{1\le j<k\le 8} \Gamma(t^{\frac{N+1}{2}}(pq)^{\frac{1}{2}} y_j y_k; p, q, t)}, \tag{4.15}$$

where $\mathcal{I}^{SP(2N)}$ is given in (4.7).

The partition function of $6D$ chiral hypermultiplet has not yet been computed using the localization method. However, we may speculate that the result is given by the modified superconformal index proposed in [35], in analogy with the relation between $4d$ indices and partition functions considered in [5]–[9] and the present work. It is given by

$$I_{6d/4d}^{\text{mod}} = \psi_{6d/4d}^{\text{mod}} \int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}} \prod_{1\leq i<j\leq N} \frac{\mathcal{G}(\tau \pm u_i \pm u_j;\omega)}{\mathcal{G}(\pm u_i \pm u_j;\omega)} \prod_{j=1}^{N} \frac{\prod_{k=1}^{8} \mathcal{G}(\alpha_k \pm u_j;\omega)}{\mathcal{G}(\pm 2u_j;\omega)} \frac{du_j}{\omega_3}, \tag{4.16}$$

where $\mathcal{G}(u;\omega) := \mathcal{G}(u;\omega_1,\ldots,\omega_3)$ and the prefactor has the form for $\tau := \omega_4$:

$$\psi_{6d/4d}^{\text{mod}} = \frac{\left(\kappa_2^{\text{mod}}\right)^N}{N!} \frac{G(\omega_4;\omega_1,\ldots,\omega_3)^{N-1}}{\prod_{1\leq j<k\leq 8} \mathcal{G}(\frac{N}{4}\omega_4 + \frac{1}{4}\sum_{i=1}^{4}\omega_i + \alpha_j + \alpha_k;\omega_1,\ldots,\omega_4)} \tag{4.17}$$

with a product of modified elliptic gamma functions in the denominator. Applying the logical line of the previous considerations, we can expect that the supersymmetric Casimir energy of the described $6d/4d$ system will be given by

$$E_{\text{Casimir}}^{6d/4d} = E_{\text{Casimir}}^{\text{SP(2N)}} - \lim_{\omega_3\to 0} \frac{\omega_3}{24} \sum_{1\leq j<k\leq 8} B_{4,4}\left(\frac{N}{4}\omega_4 + \frac{1}{4}\sum_{i=1}^{4}\omega_i + \alpha_j + \alpha_k\right), \tag{4.18}$$

which receives now also contributions from the Bernoulli polynomial of fourth order, as it is given in section 2.1.

4.4 $\mathcal{N} = 2$ strongly coupled E_6 SCFT

Argyres-Seiberg duality [29] maps an $\mathcal{N} = 2$ supersymmetric SU(3) gauge theory with $N_f = 6$ flavour symmetry with a weakly coupled description to a theory that involves a strongly coupled sector without a Lagrangian description. This sector is a superconformal theory with an E_6 flavour symmetry [30]. While the superconformal index of such a theory cannot be written down by conventional means, the problem was circumvented in [31] in an elegant manner by employing mathematical techniques developed in [44]. The resulting index can be written as

$$\begin{aligned}\mathcal{I}_{E_6} &= \frac{1}{2}\frac{\phi(p)\phi(q)}{\Gamma\left(\hat{r}(\hat{s}^4\hat{t}^{-2})^{\pm 1}\right)} \int_{\mathbb{T}} \frac{ds}{s} \frac{\Gamma\left((\hat{s}\hat{t})^{-\frac{1}{2}}(\hat{s}^2\hat{t}^{-1})^{\pm 1}s^{\pm 1}\right)}{\Gamma\left((\hat{s}\hat{t})^{-1}, s^{\pm 2}\right)} \mathcal{I}(s,r,\mathbf{y},\mathbf{z}) \\ &+ \frac{1}{2}\frac{\Gamma\left(\hat{t}^2\hat{s}^{-4}\right)}{\Gamma\left(\hat{r}\hat{t}^2\hat{s}^{-4}\right)}\left[\mathcal{I}\left(s=(\hat{s}\hat{t}^{-1})^{\frac{3}{2}}, r,\mathbf{y},\mathbf{z}\right) + \mathcal{I}\left(s=(\hat{s}^{-1}\hat{t})^{\frac{3}{2}}, r,\mathbf{y},\mathbf{z}\right)\right] \\ &+ \frac{1}{2}\frac{\Gamma\left(\hat{t}^{-2}\hat{s}^{4}\right)}{\Gamma\left(\hat{r}\hat{t}^{-2}\hat{s}^{4}\right)}\left[\mathcal{I}\left(s=\hat{s}^{-\frac{5}{2}}\hat{t}^{\frac{1}{2}}, r,\mathbf{y},\mathbf{z}\right) + \mathcal{I}\left(s=\hat{s}^{\frac{5}{2}}\hat{t}^{-\frac{1}{2}}, r,\mathbf{y},\mathbf{z}\right)\right],\end{aligned} \tag{4.19}$$

where $\Gamma(z) := \Gamma(z;p,q)$ and $\mathcal{I}(s,r,\mathbf{y},\mathbf{z})$ is the index of a weakly coupled SU(3) theory with six hypermultiplets, and is given by

$$\begin{aligned}\mathcal{I}(s,r,\mathbf{y},\mathbf{z}) = \frac{\phi(p)^2\phi(q)^2}{6}\Gamma(\hat{r})^2 \int \prod_{i,j=1}^{3} \Gamma\left((\hat{s}\hat{t})^{\frac{1}{2}}\left(s^{\frac{1}{3}}\frac{\hat{z}_i}{x_j}\right)^{\pm 1}, (\hat{s}\hat{t})^{\frac{1}{2}}\left(s^{-\frac{1}{3}}\hat{y}_i x_j\right)^{\pm 1}\right) \\ \times \prod_{i\neq j} \frac{\Gamma(\hat{r}x_i x_j^{-1})}{\Gamma(x_i x_j^{-1})} \prod_{i=1}^{2} \frac{dx_i}{2\pi i x_i}.\end{aligned} \tag{4.20}$$

The fugacities $\hat{r} := \exp(2\pi i\rho/\omega_2)$, $\hat{s} := \exp(2\pi i\sigma/\omega_2)$ and $\hat{t} := \exp(2\pi i\tau/\omega_2)$ satisfy the balancing condition $\hat{r}\hat{s}\hat{t} = pq$ and are related to the parameters t, v and w in [31] by $\hat{r} = t^2 v$, $\hat{s} = t^{\frac{8}{3}} v^{-\frac{2}{3}} w^{-\frac{1}{3}}$. Furthermore, we have $\hat{y}_j := \exp(2\pi i\alpha_j/\omega_2) = \tilde{r}^{-1} y_j$ and $\hat{z}_j := \exp(2\pi i\beta_j/\omega_2) = \tilde{r}^{-1} z_j$, where $\prod_{j=1}^3 x_j = \prod_{j=1}^3 y_j = \prod_{j=1}^3 z_j = 1$ and $\tilde{r} := \exp(2\pi i r/\omega_2)$.

Applying our $\mathrm{SL}(3,\mathbb{Z})$ method as in the examples above yields an expression for the Casimir energy,

$$E_{\text{Casimir}} = \frac{-\rho}{\omega_1\omega_2}\Bigg[\frac{3}{2}\sum_{i=1}^{3}\left(\alpha_i^2+\beta_i^2\right) + \frac{71}{6}\rho^2 - \frac{91}{4}(\omega_1\rho+\omega_2\rho-\omega_1\omega_2) + \frac{131}{12}(\omega_1^2+\omega_2^2) - 36(\omega_1\tau+\omega_2\tau-\rho\tau) + 27\tau^2\Bigg], \tag{4.21}$$

which is in precise agreement with the results of [9]. This can be seen after the identifications $\rho = -\sigma$, $\tau = \frac{1}{3}(-e + 2(\omega_1+\omega_2+\omega_3+\sigma)$, $\alpha_j = y_j - r$ and $\beta_j = z_j - r$, where we have used the notation of [9]. It is again straightforward to write down the residual function and the modified elliptic hypergeometric integral, but we refrain from doing so explicitly for the sake of brevity.

4.5 $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

As a final example, we consider $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with SP(2N) gauge group. One has to start with a more general expression than (3.1), see [3]. The explicit expression [19, 27] is given by

$$\mathcal{I}_{\text{SP(2N)}} = \chi_N \int \prod_{1\le i<j\le N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 1} z_j^{\pm 1}; p, q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_{i=1}^{N} \frac{\prod_{k=1}^3 \Gamma(s_k z_i^{\pm 2}; p, q)}{\Gamma(z_i^{\pm 2})} \frac{dz_i}{2\pi i z_i}, \tag{4.22}$$

with $s_k = \exp(2\pi i\alpha_k/\omega_2)$ and

$$\chi_N = \frac{\phi(p)^N \phi(q)^N}{2^N N!} \prod_{k=1}^{3} \Gamma(s_k; p, q)^N. \tag{4.23}$$

The balancing condition reads $s_1 s_2 s_3 = pq$. Applying the same transformation as above leads, in complete analogy, to

$$E_{\text{Casimir}} = (2N^2 + N)\frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}, \tag{4.24}$$

while the residual function takes the simple form

$$\mathcal{R}(\tilde{\beta}) = \frac{3\pi i N}{2}. \tag{4.25}$$

The modified elliptic hypergeometric integral is given by

$$I_{\mathcal{N}=4}^{\text{mod}} = \chi_N^{\text{mod}} \int_{-\frac{\omega_3}{2}}^{\frac{\omega_3}{2}} \prod_{1\le i<j\le N} \frac{\prod_{k=1}^3 \mathcal{G}(\alpha_k \pm u_i \pm u_j; \omega)}{\mathcal{G}(\pm u_i \pm u_j; \omega)} \prod_{i=1}^{N} \frac{\prod_{k=1}^3 \mathcal{G}(\alpha_k \pm 2u_i; \omega)}{\mathcal{G}(\pm 2u_i; \omega)} \frac{du_i}{\omega_3}, \tag{4.26}$$

with

$$\chi_N^{\rm mod} = \frac{1}{2^N N!} \left(-\frac{\omega_3}{\omega_2}\right)^N \left(\frac{\phi(p)\phi(q)\phi(r)}{\phi(\tilde{q})}\right)^N \prod_{k=1}^{3} \mathcal{G}(\alpha_k;\omega)^N. \tag{4.27}$$

and $\sum_{k=1}^{3} \alpha_k = \sum_{i=1}^{3} \omega_i$. We have also performed the calculation for all the other classical groups, with the result being

$$E_{\rm Casimir} = d_G \frac{\alpha_1\alpha_2\alpha_3}{2\omega_1\omega_2}, \tag{4.28}$$

where d_G is the dimension of the group. This is again in full agreement with [9]. The residual function is the same in all cases.

5 Discussion

In this article, we have studied the connection between $\mathrm{SL}(3,\mathbb{Z})$ transformations, the supersymmetric partition function and the superconformal index of four-dimensional theories. We have proposed a new recipe to extract the supersymmetric Casimir energy from the index alone, and confirmed it for several theories with $\mathcal{N} = 1, 2, 4$ supersymmetry. We have also predicted the Casimir energy for two very interesting $\mathcal{N} = 1$ theories. Moreover, we have shown that the structure of eq. (1.1) emerges from $\mathrm{SL}(3,\mathbb{Z})$ transformations and, given that the localization result holds, found a way to write the partition function in terms of a modified elliptic hypergeometric integral. It is tempting to state that the modified elliptic hypergeometric integrals actually coincide with partition functions, as the exponent in the computations of the latter for example in the case of a chiral superfield in [7] is similar to the $\mathrm{SL}(3,\mathbb{Z})$ transformation factor. However, due to the complicated nature of the regularization procedure such a statement would require rigorous mathematical justification (see [5–8] for detailed considerations of this problem).

Finally, we want to comment on the geometric and physical interpretation of the $\mathrm{SL}(3,\mathbb{Z})$ transformation and the emergence of the Casimir energy. Consider the action of the transformations $\omega_2 \to -\omega_3$ and $\omega_3 \to \omega_2$ on the identification of eq. (2.15). We see that the resulting identification still gives a primary Hopf surface, but with different defining parameters. As the transformation is not continuously connected to the identity, it is a large diffeomorphism of the manifold. The failure of the superconformal index/partition function to be invariant under this diffeomorphism points towards the presence of a gravitational anomaly (see [7] for a similar phenomenon).

There are many open questions and avenues to pursue in the future, some of which we want to mention in the following:

- It would be interesting to find a physical interpretation of the residual phase polynomial $\mathcal{R}(\tilde{\beta})$. One can say that the original cocycle phase function emerging from the $\mathrm{SL}(3,\mathbb{Z})$ transformation represents some kind of a modified Casimir energy whose low temperature $\beta \to \infty$ leading term yields the true Casimir energy. According to the considerations of [41], one can interpret the expression (2.5) as a combination of two superconformal indices defined on two different manifolds arising from a Heegard-decomposition of the original manifold. Then it remains to clarify the meaning of the result of such a gluing given by the expression (2.6).

- In section 3.2, we have shown that a particular $\mathrm{SL}(3,\mathbb{Z})$ transformation takes the $\beta \to \infty$ limit of the original index into the $p \to 1$ limit of the transformed index. It would be interesting to investigate analytically such a diverging limit. The high temperature limit $\beta \to \infty$, with $p, q \to 1$ taken simultaneously, the degeneration leads to three-dimensional theories. It remains to investigate in detail all regimes that can be reached by the $\mathrm{SL}(3,\mathbb{Z})$ transformations.

- In [23], the $\mathcal{N} = 2$ Schur index was modified in such a way that it is invariant under modular transformations. It is in principle conceivable that a similar modification exists for the superconformal index and $\mathrm{SL}(3,\mathbb{Z})$. We have not discussed such a modification in the present article.

- There are other ways of computing the supersymmetric Casimir energy, e.g. as an equivariant integral over the anomaly polynomial [9] or as a limit of an index-character counting twisted holomorphic modes [36]. Even though the relationship between $\mathrm{SL}(3,\mathbb{Z})$ and anomaly coefficients was found already in [21], it would be important to clarify what is the precise relation to these approaches.

- In [45], an intriguing connection between partition functions, topological strings and $\mathrm{SL}(3,\mathbb{Z})$ transformations was uncovered. It would be of interest to know if these insights are in any way related to our results.

- Finally, it would be interesting to have, even in the absence of a proof, more explicit checks of the recipe. For example, it would be desirable to see that it works for other theories with $\mathcal{N} = 2$ supersymmetry [19] or with more complicated superconformal indices, like the linear quivers of [46].

Acknowledgments

The authors would like to thank H.-C. Kim and J. Sparks for helpful discussions. V.S. is indebted to A. Kapustin for a discussion of the structure of partition functions computed via localization. D.R. thanks the University of Wisconsin-Madison, and F.B. the Mathematical Institute of the University of Oxford for hospitality during the completion of this work. F.B. was supported by the Austrian Science Fund FWF, project no. P26366, and the FWF doctoral program Particles & Interactions, project no. W1252. D.R. was supported by a grant of the Max Planck Society. Results of section 3 have been worked out within the Russian Science Foundation project no. 14-11-00598. V.S. is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF government grant, ag. no. 14.641.31.0001.

References

[1] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[2] V. Pestun et al., *Localization techniques in quantum field theories*, J. Phys. **A** (2017) in press [arXiv:1608.02952] [INSPIRE].

[3] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, Commun. Math. Phys. **275** (2007) 209 [hep-th/0510251] [INSPIRE].

[4] C. Römelsberger, *Counting chiral primaries in $N = 1$, $D = 4$ superconformal field theories*, Nucl. Phys. **B 747** (2006) 329 [hep-th/0510060] [INSPIRE].

[5] B. Assel, D. Cassani and D. Martelli, *Localization on Hopf surfaces*, JHEP **08** (2014) 123 [arXiv:1405.5144] [INSPIRE].

[6] B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen and D. Martelli, *The Casimir energy in curved space and its supersymmetric counterpart*, JHEP **07** (2015) 043 [arXiv:1503.05537] [INSPIRE].

[7] C. Closset and I. Shamir, *The $N = 1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization*, JHEP **03** (2014) 040 [arXiv:1311.2430] [INSPIRE].

[8] A. Arabi Ardehali, J.T. Liu and P. Szepietowski, *High-temperature expansion of supersymmetric partition functions*, JHEP **07** (2015) 113 [arXiv:1502.07737] [INSPIRE].

[9] N. Bobev, M. Bullimore and H.-C. Kim, *Supersymmetric Casimir energy and the anomaly polynomial*, JHEP **09** (2015) 142 [arXiv:1507.08553] [INSPIRE].

[10] P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *The holographic supersymmetric Casimir energy*, Phys. Rev. **D 95** (2017) 021902 [arXiv:1606.02724] [INSPIRE].

[11] V.P. Spiridonov, *On the elliptic beta function*, Russ. Math. Surv. **56** (2001) 185.

[12] V.P. Spiridonov, *Essays on the theory of elliptic hypergeometric functions*, Russ. Math. Surv. **63** (2008) 405 [arXiv:0805.3135].

[13] V.P. Spiridonov, *Classical elliptic hypergeometric functions and their applications*, *Rokko Lect. Math.* **18** (2005) 253 [math/0511579].

[14] V.P. Spiridonov, *Elliptic hypergeometric functions and Calogero-Sutherland-type models*, Theor. Math. Phys. **150** (2007) 266 [*Teor. Mat. Fiz.* **150** (2007) 311].

[15] F.A. Dolan and H. Osborn, *Applications of the superconformal index for protected operators and q-hypergeometric identities to $N = 1$ dual theories*, Nucl. Phys. **B 818** (2009) 137 [arXiv:0801.4947] [INSPIRE].

[16] N. Seiberg, *Electric-magnetic duality in supersymmetric non-Abelian gauge theories*, Nucl. Phys. **B 435** (1995) 129 [hep-th/9411149] [INSPIRE].

[17] V.P. Spiridonov, *Elliptic beta integrals and solvable models of statistical mechanics*, *Contemp. Math.* **563** (2012) 181 [arXiv:1011.3798] [INSPIRE].

[18] J. Yagi, *Quiver gauge theories and integrable lattice models*, JHEP **10** (2015) 065 [arXiv:1504.04055] [INSPIRE].

[19] A. Gadde, E. Pomoni, L. Rastelli and S.S. Razamat, *S-duality and 2d topological QFT*, JHEP **03** (2010) 032 [arXiv:0910.2225] [INSPIRE].

JHEP07(2017)041

[20] L. Rastelli and S.S. Razamat, *The supersymmetric index in four dimensions*, arXiv:1608.02965 [INSPIRE].

[21] V.P. Spiridonov and G.S. Vartanov, *Elliptic hypergeometric integrals and 't Hooft anomaly matching conditions*, *JHEP* **06** (2012) 016 [arXiv:1203.5677] [INSPIRE].

[22] G. Felder and A. Varchenko, *The elliptic gamma function and* $\mathrm{SL}(3, Z) \times Z^3$, *Adv. Math.* **156** (2000) 44 [math/9907061].

[23] S.S. Razamat, *On a modular property of* $N = 2$ *superconformal theories in four dimensions*, *JHEP* **10** (2012) 191 [arXiv:1208.5056] [INSPIRE].

[24] F.A.H. Dolan, V.P. Spiridonov and G.S. Vartanov, *From* 4d *superconformal indices to* 3d *partition functions*, *Phys. Lett.* **B 704** (2011) 234 [arXiv:1104.1787] [INSPIRE].

[25] V.P. Spiridonov and G.S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities*, *Commun. Math. Phys.* **304** (2011) 797 [arXiv:0910.5944] [INSPIRE].

[26] V.P. Spiridonov and G.S. Vartanov, *Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots and vortices*, *Commun. Math. Phys.* **325** (2014) 421 [arXiv:1107.5788] [INSPIRE].

[27] V.P. Spiridonov and G.S. Vartanov, *Superconformal indices of* $N = 4$ *SYM field theories*, *Lett. Math. Phys.* **100** (2012) 97 [arXiv:1005.4196] [INSPIRE].

[28] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *Gauge theories and Macdonald polynomials*, *Commun. Math. Phys.* **319** (2013) 147 [arXiv:1110.3740] [INSPIRE].

[29] P.C. Argyres and N. Seiberg, *S-duality in* $N = 2$ *supersymmetric gauge theories*, *JHEP* **12** (2007) 088 [arXiv:0711.0054] [INSPIRE].

[30] J.A. Minahan and D. Nemeschansky, *An* $N = 2$ *superconformal fixed point with* E_6 *global symmetry*, *Nucl. Phys.* **B 482** (1996) 142 [hep-th/9608047] [INSPIRE].

[31] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *The superconformal index of the* E_6 *SCFT*, *JHEP* **08** (2010) 107 [arXiv:1003.4244] [INSPIRE].

[32] V.P. Spiridonov and G.S. Vartanov, *Superconformal indices for* $N = 1$ *theories with multiple duals*, *Nucl. Phys.* **B 824** (2010) 192 [arXiv:0811.1909] [INSPIRE].

[33] T. Dimofte and D. Gaiotto, *An* E_7 *surprise*, *JHEP* **10** (2012) 129 [arXiv:1209.1404] [INSPIRE].

[34] V.P. Spiridonov, *Theta hypergeometric integrals*, *St. Petersburg Math. J.* **15** (2004) 929 [*Alg. Analiz* **15** (2003) 161] [math/0303205].

[35] V.P. Spiridonov, *Modified elliptic gamma functions and* 6d *superconformal indices*, *Lett. Math. Phys.* **104** (2014) 397 [arXiv:1211.2703] [INSPIRE].

[36] D. Martelli and J. Sparks, *The character of the supersymmetric Casimir energy*, *JHEP* **08** (2016) 117 [arXiv:1512.02521] [INSPIRE].

[37] C. Closset, T.T. Dumitrescu, G. Festuccia and Z. Komargodski, *The geometry of supersymmetric partition functions*, *JHEP* **01** (2014) 124 [arXiv:1309.5876] [INSPIRE].

[38] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[39] A. Hanany and N. Mekareeya, *Counting gauge invariant operators in SQCD with classical gauge groups*, *JHEP* **10** (2008) 012 [arXiv:0805.3728] [INSPIRE].

JHEP07(2017)041

[40] L. Di Pietro and Z. Komargodski, *Cardy formulae for SUSY theories in $d = 4$ and $d = 6$*, *JHEP* **12** (2014) 031 [arXiv:1407.6061] [INSPIRE].

[41] F. Nieri and S. Pasquetti, *Factorisation and holomorphic blocks in* 4*d*, *JHEP* **11** (2015) 155 [arXiv:1507.00261] [INSPIRE].

[42] A. Arabi Ardehali, *High-temperature asymptotics of supersymmetric partition functions*, *JHEP* **07** (2016) 025 [arXiv:1512.03376] [INSPIRE].

[43] J.F. van Diejen and V.P. Spiridonov, *Unit circle elliptic beta integrals*, *Ramanujan J.* **10** (2005) 187 [math/0309279].

[44] V.P. Spiridonov and S.O. Warnaar, *Inversions of integral operators and elliptic beta integrals on root systems*, *Adv. Math.* **207** (2006) 91 [math/0411044].

[45] G. Lockhart and C. Vafa, *Superconformal partition functions and non-perturbative topological strings*, arXiv:1210.5909 [INSPIRE].

[46] F. Brünner and V.P. Spiridonov, *A duality web of linear quivers*, *Phys. Lett.* **B 761** (2016) 261 [arXiv:1605.06991] [INSPIRE].