

# Shift-symmetries and gauge coupling functions in orientifolds and F-theory

Pierre Corvilain, Thomas W. Grimm and Diego Regalado

*Max Planck Institute for Physics,*

*Föhringer Ring 6, 80805 Munich, Germany*

*Institute for Theoretical Physics and Center for Extreme Matter and Emergent Phenomena,  
Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*

*E-mail:* [pcorvila@mpp.mpg.de](mailto:pcorvila@mpp.mpg.de), [grimm@mpp.mpg.de](mailto:grimm@mpp.mpg.de), [regalado@mpp.mpg.de](mailto:regalado@mpp.mpg.de)

**ABSTRACT:** We investigate the field dependence of the gauge coupling functions of four-dimensional Type IIB orientifold and F-theory compactifications with space-time filling seven-branes. In particular, we analyze the constraints imposed by holomorphicity and covariance under shift-symmetries of the bulk and brane axions. This requires introducing quantum corrections that necessarily contain Riemann theta functions on the complex torus spanned by the D7-brane Wilson line moduli. Our findings hint towards a new underlying geometric structure for gauge coupling functions in string compactifications. We generalize this discussion to a genuine F-theory compactification on an elliptically fibered Calabi-Yau fourfold. We perform the first general dimensional reduction of eleven-dimensional supergravity and dualization to the F-theory frame. The resulting effective action is compared with the circle reduction of a four-dimensional  $\mathcal{N} = 1$  supergravity theory. The F-theory geometry elegantly unifies bulk and brane degrees of freedom and allows us to infer non-trivial results about holomorphicity and shift-symmetries. For instance, we gain new insight into kinetic mixing of bulk and brane gauge fields.

**KEYWORDS:** D-branes, Discrete Symmetries, F-Theory, Superstring Vacua

**ARXIV EPRINT:** [1607.03897](https://arxiv.org/abs/1607.03897)

---

**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The D7-brane gauge coupling function and kinetic mixing</b>	<b>4</b>
2.1	Complex coordinates and the Kähler potential in Type IIB orientifolds	4
2.2	Continuous and discrete shift-symmetries	8
2.3	The $\mathcal{N} = 1$ gauge coupling function	10
2.4	One-loop corrections and theta-functions	12
2.4.1	A simple case with one Wilson line modulus	12
2.4.2	The general case with several Wilson line moduli	14
2.5	Comments on kinetic mixing and gaugings	15
<b>3</b>	<b>M-theory on Calabi-Yau fourfolds and the F-theory frame</b>	<b>17</b>
3.1	Dimensional reduction of M-theory on a smooth fourfold	17
3.2	The three-dimensional $\mathcal{N} = 2$ action and its symmetries	20
3.3	Dualization of fields to the F-theory frame	21
3.4	Symmetries of the dual Lagrangian	23
<b>4</b>	<b>Determining F-theory gauge coupling functions</b>	<b>25</b>
4.1	Transformation rules of the gauge coupling functions	25
4.2	Gauge coupling functions from dimensional reduction	26
4.2.1	On the weak string-coupling limit	27
4.2.2	Gauge coupling function for R-R vectors	27
4.2.3	Kinetic mixing between R-R and 7-brane vectors	28
4.2.4	Gauge coupling function for 7-brane vectors	29
4.3	Shift symmetries, quantum corrections, and theta functions	30
<b>5</b>	<b>Conclusions</b>	<b>32</b>
<b>A</b>	<b>Dualization of three-dimensional actions</b>	<b>35</b>
<b>B</b>	<b>Circle reduction of four-dimensional <math>\mathcal{N} = 1</math> supergravity</b>	<b>38</b>

---

**1 Introduction**

In four-dimensional effective actions with minimal  $\mathcal{N} = 1$  supersymmetry, the dynamics of the vector fields crucially depends on the gauge coupling functions determining their kinetic terms. Supersymmetry requires this function to be holomorphic in the complex scalars that arise as the bosonic parts of chiral multiplets [1]. This holomorphicity allows to infer certain non-renormalization theorems for this coupling function. In particular, one can show that it

only receives perturbative corrections to one-loop order, while non-perturbative corrections can be generally present. In effective theories arising from string theory, the gauge coupling function can depend on scalars admitting classical shift-symmetries. While this is key in the implementation of anomaly cancellation via the Green-Schwarz mechanism [2, 3], these symmetries can also constrain the functional form of the coupling independent of any gauging. In this work we exploit the interplay between holomorphicity and symmetries in the study of gauge coupling functions of brane and R-R gauge fields.

Deriving the gauge coupling function in a full-fledged string model can be challenging. In intersecting D-brane models this function has been investigated since their first construction [4, 5]. Of particular interest in this work will be intersecting Type IIB D-brane models with space-time filling D7-branes and O7-planes and their generalizations to F-theory models with seven-branes of general type. We furthermore focus on compactifications yielding a four-dimensional effective theory with  $\mathcal{N} = 1$  supersymmetry. At weak string coupling, i.e. when D7-branes and O7-planes are considered, the gauge coupling function can be studied by dimensionally reducing the D7-brane effective action as done in [6, 7].

Interestingly, it was already pointed out in [6] (and for the mirror-dual configurations in [8]) that the gauge coupling functions determined by direct classical reduction are not holomorphic in the complex coordinates determined for the rest of the effective action. First, this was observed for the D7-brane gauge coupling function in the presence of D7-brane Wilson line moduli. A solution to this problem was, however, suggested in [6], by arguing that the missing terms arise at one-string-loop order by using the orbifold results of [9, 10]. Second, including the mixing with R-R bulk  $U(1)$ 's, a further seeming conflict with holomorphicity in the independently derived complex coordinates is encountered. Given these gaps in our understanding of these basic couplings, one might wonder if there is a more systematic approach to determine and analyze these couplings. In this paper we suggest that by carefully studying the shift-symmetries of the axions in the theory, one can significantly constrain the gauge coupling function of both closed and open string gauge fields. This is done for the Type IIB weak string coupling setting in detail in section 2, while the generalization to F-theory can be found in section 3 and section 4. We should note however, that the F-theory analysis is not simply a generalization, but it is also useful in uncovering new interesting facts about the Type IIB case.

In general, the gauge coupling functions  $\hat{f}$  of D-branes depend on R-R form axions of the underlying supergravity theory. Since these forms admit shift-symmetries, they can be used to constrain the functional dependence of  $\hat{f}$  on the R-R-form axions. Using holomorphicity, one is then lead to constraints on the dependence of  $\hat{f}$  on the complex coordinates. Clearly, exploiting the symmetry properties for determining the gauge coupling function in string compactifications is not new and has, for example, already been discussed intensively in heterotic models (for early works on this subject, see e.g. [11, 12] and references therein). However, one fact that has not been exploited systematically is that higher-degree R-R forms can transform non-trivially under the shift-transformations of lower-degree R-R or D-brane gauge transformations. This is a direct consequence of having Chern-Simons terms in higher dimensions which, as we will discuss in detail, trans-

lates into having non-Abelian shift-symmetries among the axions in the lower-dimensional effective field theory.

Our strategy to constrain the corrections to the gauge coupling function is to combine our knowledge of the appropriate  $\mathcal{N} = 1$  complex coordinates with the expected symmetry properties of the gauge coupling function. More precisely, we first note that the gauge coupling function  $\hat{f}_{D7}$  is proportional to the Kähler coordinates  $T_\alpha$  in the absence of R-R and NS-NS two-form scalars  $G^a$  and D7-brane Wilson line scalars  $a_p$ . Including these fields, one finds corrections to  $T_\alpha$  depending on  $G^a, a_p$  as well as their complex conjugates  $\bar{G}^a, \bar{a}_p$ . We argue that once these moduli are included, the gauge coupling function cannot be simply given by  $T_\alpha$ , since that would break the discrete shift symmetries. However, just by using holomorphicity and such discrete symmetries, we can derive that the correction to  $\hat{f}_{D7}$  is a holomorphic section of a certain line bundle over the complex torus spanned by the axions. Finally, this fixes the form of the corrections, which consist of logarithms of Riemann theta functions depending on the Wilson lines.

The improved understanding of D7-brane gauge coupling functions finds an elegant description when moving to F-theory models studied via M-theory. In the F-theory description, the seven-brane dynamics is encoded by the geometry of an elliptically fibered Calabi-Yau fourfold  $Y_4$ . In particular, the complex structure moduli, seven-brane positions, and the axio-dilaton reside in a joint moduli space: the moduli space of complex structure deformations of  $Y_4$ . We also have that the two-form scalars  $G^a$ , Wilson lines  $a_p$  and R-R gauge fields are unified as arising from elements of the third cohomology of  $Y_4$ . In fact, they parameterize the complex torus  $H^{2,1}(Y_4)/H^3(Y_4, \mathbb{Z})$ . The gauge coupling function can then be determined via the duality to M-theory on the same fourfold by the following procedure: (1) compactify a general four-dimensional  $\mathcal{N} = 1$  theory on a circle, (2) integrate out all massive modes in the three-dimensional Coulomb branch, (3) compare the result with an M-theory compactification on a smooth Calabi-Yau fourfold. Using this procedure, the leading seven-brane gauge coupling function was found in [13] and some first results on corrections to this result have been obtained using this duality in [14]. As for the Type IIB case, we expect that in general the gauge coupling function depends on the scalars  $G^a$  and Wilson lines. As of now, however, the contribution from two-form scalars and Wilson lines has not been obtained via an M-theory reduction. Thus, in this work we will perform an M-theory reduction on a generic elliptically fibered Calabi-Yau fourfold keeping track of all fields including the two-form scalars, the Wilson lines and the R-R gauge fields, thereby generalising the results in [13]. We also explain in great detail the relevance of having an elliptically fibered space and the dualization procedure to bring the effective action to the correct F-theory duality frame to compare with a four-dimensional theory. Exploiting the shift-symmetries in the M-theory reduction and the F-theory frame we present a detailed discussion of the F-theory gauge coupling function. We extend the analysis of [15] and propose quantum corrections to ensure holomorphicity and shift-symmetry invariance.

This work is organized as follows. In section 2 we discuss the  $\mathcal{N} = 1$  effective action of a Type IIB orientifold compactification with a space-time filling D7-brane. We introduce the complex coordinates and Kähler potential capturing the dynamics of a rigid D7-brane with Wilson line moduli. We then study the symmetries of the moduli space and their

action on the gauge coupling function, which allows us to derive certain constraints for  $\hat{f}$ . In section 3, we perform the dimensional reduction of M-theory on a generic smooth Calabi-Yau fourfold and dualize to the correct F-theory duality frame. We carefully derive the shift-symmetries of the effective theory and the effect of the dualization on them. In section 4 we determine the gauge coupling function by matching the M-theory reduction with a circle reduction of a four-dimensional theory. Finally, we discuss the constraints that holomorphicity and gauge-invariance imposes on it. We leave a detailed discussion of the dualization of three-dimensional action to appendix A and of the circle reduction of a four-dimensional theory to appendix B.

## 2 The D7-brane gauge coupling function and kinetic mixing

In this section we consider the four-dimensional effective action that arises from Calabi-Yau orientifold compactifications of Type IIB with D7-branes and O7-planes. In particular, we aim to determine the characteristic functions determining the standard  $\mathcal{N} = 1$  supergravity with bosonic action [1]

$$\begin{aligned}
 S^{(4)} = & \int \frac{1}{2} \hat{R} \hat{\star} 1 - \hat{K}_{\mathbf{A}\mathbf{B}} d\hat{M}^{\mathbf{A}} \wedge \hat{\star} d\overline{\hat{M}}^{\mathbf{B}} - \frac{1}{4} \text{Re} \hat{f}_{\mathbf{I}\mathbf{J}}(\hat{M}) \hat{F}^{\mathbf{I}} \wedge \hat{\star} \hat{F}^{\mathbf{J}} \\
 & - \frac{1}{4} \text{Im} \hat{f}_{\mathbf{I}\mathbf{J}}(\hat{M}) \hat{F}^{\mathbf{I}} \wedge \hat{F}^{\mathbf{J}},
 \end{aligned}
 \tag{2.1}$$

where  $\hat{K}_{\mathbf{A}\mathbf{B}}$  are the second derivatives of a real Kähler potential  $\hat{K}(\hat{M}, \overline{\hat{M}})$  and  $\hat{f}_{\mathbf{I}\mathbf{J}}(\hat{M})$  is the holomorphic gauge coupling function. We will denote four-dimensional quantities with a hat. The functions  $\hat{K}$ ,  $\hat{f}_{\mathbf{I}\mathbf{J}}$  as well as the complex coordinates  $\hat{M}^{\mathbf{A}}$  are determined by reducing Type IIB supergravity coupled to the D7-brane and O7-plane world-volume actions following and extending [6, 7, 16]. We will also discuss the shift-symmetries and certain quantum corrections of the effective theory.

### 2.1 Complex coordinates and the Kähler potential in Type IIB orientifolds

The general form of the effective action for the bulk fields in such compactifications was determined in [16] by reducing Type IIB supergravity on a Calabi-Yau manifold  $Y_3$ , while also including the action of an holomorphic involution  $\sigma : Y_3 \rightarrow Y_3$ . The action of  $\sigma^*$  on the cohomology groups splits them into eigenspaces  $H^{p,q}(Y_3) = H_+^{p,q}(Y_3) \oplus H_-^{p,q}(Y_3)$ . The basis used to span these cohomologies is listed in table 1. This leads to the following expansion of the Kähler form  $J$  of  $Y_3$ , and the NS-NS and R-R form fields

$$\begin{aligned}
 J = v^\alpha \omega_\alpha, \quad B_2 = b^a \omega_a, \quad C_2 = c^a \omega_a, \\
 C_4 = C_2^\alpha \wedge \omega_\alpha + \rho_\alpha \tilde{\omega}^\alpha + A^\kappa \wedge \alpha_\kappa + \tilde{A}_\kappa \wedge \beta^\kappa,
 \end{aligned}
 \tag{2.2}$$

where  $c^a$ ,  $b^a$ , and  $\rho_\alpha$  are scalars,  $C_2^\alpha$  are two-forms, and  $(A^\kappa, \tilde{A}_\kappa)$  are vectors in the four-dimensional effective theory. It is crucial to stress that  $C_4$  has a self-dual field-strength, given by  $F_5 = dC_4 + \frac{1}{2} B_2 \wedge dC_2 - \frac{1}{2} C_2 \wedge dB_2$ .<sup>1</sup> This yields a duality between the two-

<sup>1</sup>Notice that we use a different convention than [6] for the field  $C_4$ . In particular, they are related by  $C_4^{\text{here}} = C_4^{\text{there}} - \frac{1}{2} B_2 \wedge C_2$ . In order to compare with the results obtained from the F-theory reduction, it is more convenient to use this convention, which makes  $C_4^{\text{here}}$  invariant under  $Sl(2, \mathbb{Z})$ .

Cohomology group	Basis elements	Fields	Index range
$H_+^{1,1}(Y_3)$	$\omega_\alpha$	$v^\alpha, C_2^\alpha$	$\alpha = 1, \dots, h_+^{1,1}(Y_3)$
$H_-^{1,1}(Y_3)$	$\omega_a$	$b^a, c^a$	$a = 1, \dots, h_-^{1,1}(Y_3)$
$H_+^{2,2}(Y_3)$	$\tilde{\omega}^\alpha$	$\rho_\alpha$	$\alpha = 1, \dots, h_+^{1,1}(Y_3)$
$H_+^3(Y_3)$	$(\alpha_\kappa, \beta^\kappa)$	$A^\kappa, \tilde{A}_\kappa$	$\kappa = 1, \dots, h_+^{2,1}(Y_3)$
$H_-^3(Y_3)$	$(\alpha_{\hat{k}}, \beta^{\hat{k}})$	$z^k$	$\hat{k} = 1, \dots, h_-^{2,1}(Y_3) + 1$

**Table 1.** Real basis for the cohomology groups. The dimensions are denoted by  $h_\pm^{p,q}(Y_3) = \dim H_\pm^{p,q}(Y_3)$ .  $(\alpha_\kappa, \beta^\kappa)$  and  $(\alpha_{\hat{k}}, \beta^{\hat{k}})$  are symplectic basis. Our index conventions include  $k = 1, \dots, h_-^{2,1}$ , while the hat on  $\hat{k}$  indicates the labeling of one further element. We also list the four-dimensional fields associated to these basis elements in the expansions (2.2).

forms  $C_2^\alpha$  and scalars  $\rho_\alpha$ , and identifies  $\tilde{A}_\kappa$  as the magnetic dual of  $A^\kappa$ . Therefore, we can eliminate the two-forms  $C_2^\alpha$  in favor of  $\rho_\alpha$  and the vector  $\tilde{A}_\kappa$  in favor of  $A^\kappa$ . It is, however, interesting to point out that the structures we discuss later on can be also analyzed in the dual frames as we will see in section 3. In addition to the zero modes of the forms (2.2), also the axio-dilaton  $\tau = C_0 + ie^{-\phi}$  reduces to a four-dimensional field. Finally, the deformations of the Calabi-Yau metric compatible with  $\sigma$  are the Kähler structure deformations  $v^\alpha$  and the complex structure deformations  $z^k$  parameterizing forms in  $H_-^{2,1}(Y_3, \mathbb{C})$ . Note that  $\tau$  and  $z^k$  are complex fields.

Before turning to the D7-branes let us note that a general  $\mathcal{N} = 1$  compactification can include background fluxes  $H_3$  and  $F_3$  [17, 18]. These transform negatively under  $\sigma^*$  and therefore admit an expansion

$$H_3 = m_H^{\hat{k}} \alpha_{\hat{k}} + e_{\hat{k}}^H \beta^{\hat{k}}, \quad F_3 = m_F^{\hat{k}} \alpha_{\hat{k}} + m_{\hat{k}}^F \beta^{\hat{k}}, \quad (2.3)$$

with the basis introduced in table 1. It is well-known that these fluxes induce a non-trivial superpotential in this Type IIB setting [19]. In the following we will not discuss background fluxes in much detail. While they can be included in the bulk sector without much effort, we will require however that they do not alter the couplings of the D7-brane.

The coupling to a single space-time filling D7-brane was studied in detail in [6, 7] by dimensionally reducing the D7-brane Born-Infeld and Chern-Simons actions. In order to review the results we will make some simplifying assumptions. In particular, we will analyze on the dynamics of a single D7-brane while being aware that a tadpole canceling configuration requires the inclusion of other D7-branes.<sup>2</sup> This will allow us to focus on the structures relevant to this work. Some interesting generalizations will appear in the study of the F-theory vacua of section 3. In particular, the F-theory analysis contains the proper inclusion of the seven-brane deformation (or position) moduli.

Let us consider a D7-brane wrapped on a divisor  $S$  in  $Y_3$  and denote its orientifold image by  $S' = \sigma(S)$ . It is useful to introduce  $S_+ = S \cup \sigma(S)$  and  $S_- = S \cup -\sigma(S)$ ,

<sup>2</sup>A more thorough discussion of the global constraints on such settings can be found, for example, in [20]. We refer the reader to these works especially for the discussion of the D5-brane tadpole constraint and the appropriate quantization conditions.

Cohomology group	Basis element	Fields	Index range
$H_-^{1,0}(S_+)$	$\gamma^p$	$a_p$	$p = 1, \dots, h_-^{1,0}(S_+)$
$H_-^{2,0}(S_+)$	$s^K$	$\zeta_K$	$K = 1, \dots, h_-^{2,0}(S_+)$
$H_-^1(S_+)$	$(\hat{\alpha}_p, \hat{\beta}^p)$	$(\tilde{c}^p, c_p)$	$p = 1, \dots, h_-^{1,0}(S_+)$

**Table 2.** Cohomology groups on the D7-brane divisor  $S_+$ . The dimensions are denoted by  $h_{\pm}^{p,q} = \dim H_{\pm}^{p,q}$ . While  $\gamma^p$  and  $s^K$  are complex basis elements, the forms  $(\hat{\alpha}_p, \hat{\beta}^p)$  constitute a real basis.

where the minus sign stands for orientation reversal. This allows to split the cohomologies  $H^{p,q}(S_+) = H_+^{p,q}(S_+) \oplus H_-^{p,q}(S_+)$  under  $\sigma$ . Then, the eight-dimensional gauge field  $A$  and embedding  $\zeta$  of the D7-brane image pair can be expanded as [6, 7]

$$A = A_{D7}P_- + a_p \gamma^p + \bar{a}_p \bar{\gamma}^p, \quad p = 1, \dots, h_-^{1,0}(S_+), \quad (2.4)$$

$$\zeta = \zeta_K s^K + \bar{\zeta}_K \bar{s}^K, \quad K = 1, \dots, h_-^{2,0}(S_+), \quad (2.5)$$

where  $P_-$  is a function equal to  $+1$  on  $S$  and  $-1$  on  $\sigma(S)$ . The fact that these have to be expanded into  $H_-^{1,0}(S_+)$  and  $H_-^{2,0}(S_+)$ , respectively, follows from the action of the orientifold on the open string states.

It is important to stress that the notion of  $\gamma^p$  being  $(0, 1)$  implies that the forms depend on the complex structure moduli  $z^k$  of the ambient Calabi-Yau space  $Y_3$ . To make this dependence more explicit, we can expand

$$\gamma^p = \frac{1}{2} \text{Re } f^{pq} (\hat{\alpha}_q - i \bar{f}_{qr} \hat{\beta}^r), \quad (2.6)$$

where  $(\hat{\alpha}_p, \hat{\beta}^p)$  is a real basis of  $H^1(S)$ . Here  $f_{pq}$  is a holomorphic function in the complex structure moduli  $z^k$ . For an appropriate basis, its real part  $\text{Re } f_{pq}$  is invertible and we denote the inverse by  $\text{Re } f^{pq}$ . This ansatz can be justified in the F-theory reduction as argued in [13, 15, 21] and was recently used in Type IIB orientifolds in [22]. While not a priori obvious a parametrization of the form (2.6) will allow us to bring the effective action into standard  $\mathcal{N} = 1$  form. This is most clearly seen in the F-theory treatment to which we will come back in section 3. Clearly, one can also expand  $A$  into the real basis  $(\hat{\alpha}_p, \hat{\beta}^p)$  such that

$$A = A_{D7}P_- + \tilde{c}^p \hat{\alpha}_p + c_p \hat{\beta}^p, \quad (2.7)$$

$$a_p = i c_p + f_{pq} \tilde{c}^q.$$

The basis  $(\hat{\alpha}_p, \hat{\beta}^p)$  is independent of the complex structure deformations and therefore all complex structure dependence in  $a_p$  is again captured by the function  $f_{pq}$ . We summarize our notation for the open string sector in table 2.

We are now in the position of stating our simplifying assumptions. First, we will assume that<sup>3</sup>

$$[\sigma(S)] = [S], \quad (2.8)$$

---

<sup>3</sup>Here and in the following we will denote by  $[D]$  the two-form class Poincaré dual to the divisor  $D$ .

i.e. that  $S$  and its orientifold image  $S'$  are in the same homology class. This implies that the  $U(1)$  gauge field of the D7-brane is not massive by a geometric Stückelberg mechanism [6, 23, 24]. And second, we will assume the vanishing of the intersections

$$\int_{S_+} i^* \alpha_{\hat{k}} \wedge \hat{\alpha}_p = \int_{S_+} i^* \alpha_{\hat{k}} \wedge \hat{\beta}^p = \int_{S_+} i^* \beta^{\hat{k}} \wedge \hat{\alpha}_p = \int_{S_+} i^* \beta^{\hat{k}} \wedge \hat{\beta}^p = 0, \quad (2.9)$$

where  $i$  denotes the embedding map of  $S_+$  into  $Y_3$ ,  $i : S_+ \hookrightarrow Y_3$ . This condition ensures that there is no superpotential that obstructs complex structure and Wilson line deformations.<sup>4</sup> The considered D7-branes can admit an arbitrary number  $h_-^{2,0}(S_+)$  of deformations  $\zeta_K$  and  $h_-^{1,0}(S_+)$  of Wilson line moduli  $a_p$ . To keep the presentation simple, we will freeze the fields  $\zeta_K$  as well as all matter fields arising at the intersections among D7-branes. This will allow us to focus the following discussion the couplings of the Wilson line moduli  $a_p$ . In the F-theory reduction, presented in section 3, a general dependence on the seven-brane deformations will be included and also charged matter states are (implicitly) accounted for.

Let us note that the condition (2.9) is only imposed for the orientifold negative forms  $(\alpha_{\hat{k}}, \beta^{\hat{k}})$  in  $Y_3$ . The positive forms  $(\alpha_\kappa, \beta^\kappa)$  can non-trivially intersect the negative one-forms on  $S_-$ . Thus, we introduce the intersection numbers

$$\begin{aligned} M_{\kappa p} &= \int_{S_-} i^* \alpha_\kappa \wedge \hat{\alpha}_p, & M_{\kappa}^p &= \int_{S_-} i^* \alpha_\kappa \wedge \hat{\beta}^p, \\ M_p^\kappa &= \int_{S_-} \hat{\alpha}_p \wedge i^* \beta^\kappa, & M^{p\kappa} &= \int_{S_-} \hat{\beta}^p \wedge i^* \beta^\kappa. \end{aligned} \quad (2.10)$$

As we discuss in subsection 2.5, these couplings control the kinetic mixing of the D7-brane  $U(1)$   $A_{D7}$  with the R-R gauge fields  $A^\kappa$  of the bulk theory.

We are now in the position to display the four-dimensional  $\mathcal{N} = 1$  complex coordinates. First, we have the complex fields

$$\text{Set 1:} \quad \tau = C_0 + i e^{-\phi}, \quad z^k, \quad (2.11)$$

which are already complex in our reduction ansatz. Their complex structure does not depend on other fields in the reduction. Note that the D7-brane deformations  $\zeta_K$  are part of Set 1, but have been frozen to keep the presentation simpler. Second, there are the complex fields

$$\text{Set 2:} \quad G^a = c^a - \tau b^a, \quad a_p, \quad (2.12)$$

which admit a complex structure that changes with the values of the fields in Set 1 given in (2.11). This is obvious from the definition of  $G^a$  and readily inferred for the  $a_p$ 's by noting that they are coefficients of complex structure dependent  $(0, 1)$ -forms in (2.4). Finally, there is a third set of fields:

$$\text{Set 3:} \quad T_\alpha = \frac{1}{2} \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma + i \rho_\alpha + \frac{i}{2(\tau - \bar{\tau})} \mathcal{K}_{\alpha ab} G^a (G - \bar{G})^b + \frac{1}{2} d_\alpha^{pq} a_p (a + \bar{a})_q, \quad (2.13)$$

---

<sup>4</sup>This was discussed in [6] from the perspective of relative cohomology and was derived in [25] from backreaction effects in supergravity.

which non-trivially depends on the fields in Set 1 and Set 2. The  $T_\alpha$  are often termed the complexified Kähler structure moduli. The introduced couplings are given by the  $Y_3$  intersection numbers

$$\mathcal{K}_{\alpha\beta\gamma} = \int_{Y_3} \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma, \quad \mathcal{K}_{\alpha ab} = \int_{Y_3} \omega_\alpha \wedge \omega_a \wedge \omega_b, \quad (2.14)$$

as well as the complex structure dependent function

$$d_\alpha^{pq} = i \int_{S_+} i^* \omega_\alpha \wedge \gamma^p \wedge \bar{\gamma}^q = -\frac{1}{2} \text{Re } f^{qr} Q_{\alpha r}{}^p, \quad Q_{\alpha r}{}^p = M_{\alpha r}{}^p + i f_{rs} M_\alpha{}^{sp}, \quad (2.15)$$

with

$$M_{\alpha p}{}^q = \int_{S_+} i^* \omega_\alpha \wedge \hat{\alpha}_p \wedge \hat{\beta}^q, \quad M_\alpha{}^{pq} = \int_{S_+} i^* \omega_\alpha \wedge \hat{\beta}^p \wedge \hat{\beta}^q. \quad (2.16)$$

For completeness, let us note that the Kähler potential takes the seemingly simple form

$$\hat{K} = -2 \log \mathcal{V} - \log(\tau - \bar{\tau}) - \log \left( \int_{Y_3} \Omega \wedge \bar{\Omega} \right). \quad (2.17)$$

This Kähler potential depends on the complex coordinates (2.11)–(2.12), i.e. we identify in (2.1) that

$$\hat{M}^{\mathbf{A}} = (\tau, z^k, G^a, a_p, T_\alpha). \quad (2.18)$$

All the field dependence of this  $\hat{K}$  on the fields of Set 2, i.e. the  $G^a$  and  $a_p$ , arises only through the definition of  $T_\alpha$ . In fact, we note that the volume  $\mathcal{V} = \frac{1}{6} \mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma$  in (2.17) depends on  $T_\alpha$  by solving (2.13) for  $v^\alpha$ , which then introduces a dependence on  $G^a, a_p$  mixed with  $\tau, z^k$ .

To conclude this subsection we discuss a special case for the above compactification separately in which several of the couplings simplify. More precisely, we briefly summarize the above result for  $h_-^{1,0}(S_+) = 1$  and  $h_-^{1,1}(Y_3) = 0$ , i.e. the case in which the rigid D7-branes only admits a single complex Wilson line modulus  $a$ . In this case the dynamics of  $a$  is encoded by the correction to  $T_\alpha$  given by

$$T_\alpha = \frac{1}{2} \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma + i \rho_\alpha - \frac{1}{4} (\text{Re } f)^{-1} M_\alpha a (a + \bar{a}), \quad (2.19)$$

where we have used that  $M_{\alpha p}{}^q$  in (2.16) reduces to a vector denoted by  $M_\alpha$  and that  $M_\alpha{}^{pq}$  vanishes due to antisymmetry for one modulus. The kinetic terms of  $a$  depend non-trivially on the complex structure moduli  $z^k$  through the holomorphic function  $f$ .

## 2.2 Continuous and discrete shift-symmetries

Having introduced the complex coordinates (2.11), (2.12), and (2.13) we are now in the position to discuss the symmetries. In order to do that we first recall that  $G^a$  and  $T_\alpha$  contain zero modes of R-R and NS-NS forms and therefore inherit discrete symmetries from large gauge transformations of  $C_2$ ,  $B_2$  and  $C_4$ . These are shifts by *integral* closed 2-forms, namely

$$\delta C_2 = \lambda^a \omega_a, \quad \delta B_2 = \tilde{\lambda}^a \omega_a, \quad (2.20)$$

where  $\lambda^a$  and  $\tilde{\lambda}^a$  are appropriately quantized constants.<sup>5</sup> Turning to  $C_4$ , an obvious large gauge transformation is  $\delta C_4 = \lambda_\alpha \tilde{\omega}^\alpha$ , for constant  $\lambda^\alpha$ . However, we note that the field-strength  $F_5 = dC_4 + \frac{1}{2}B_2 \wedge dC_2 - \frac{1}{2}C_2 \wedge dB_2$  actually contains terms depending on  $C_2$  and  $B_2$ . Therefore, the shifts (2.20) induce a shift of  $C_4$  as

$$\delta C_4 = \lambda_\alpha \tilde{\omega}^\alpha - \frac{1}{2} \tilde{\lambda}^a \omega_a \wedge C_2 + \frac{1}{2} \lambda^a \omega_a \wedge B_2. \quad (2.21)$$

A second set of symmetries arises from internal gauge transformations on the D7-brane world-volume. For constants  $\lambda^p, \tilde{\lambda}_p$  these are parameterized by

$$\delta A = \tilde{\lambda}^p \hat{\alpha}_p + \lambda_p \hat{\beta}^p. \quad (2.22)$$

Also in this case one finds that the four-form  $C_4$  has to shift. While we will not give the transformation of  $C_4$  directly, let us point out that it can be inferred by noting the NS-NS two-form  $B_2$  naturally combines with  $F = dA$  on the D7-brane world-volume as

$$\mathcal{F} = i^* B_2 - 2\pi\alpha' F, \quad (2.23)$$

where we have temporarily restored the  $\alpha'$  dependence. This implies that one can capture the gauge degrees of freedom of an Abelian D-brane with  $B_2$ , and the fact that the field  $C_4$  shifts under (2.22) is already contained in (2.21). A more detailed discussion how this is done in practice can be found in [26]. The transformations can be simply inferred when investigating the  $\mathcal{N} = 1$  coordinates as we will see next. Furthermore, since  $\mathcal{F}$  is gauge invariant, under a shift of the B-field (2.20), we have to shift the worldvolume flux on the brane accordingly

$$\delta F = \frac{1}{2\pi\alpha'} \tilde{\lambda}^a i^* \omega_a. \quad (2.24)$$

To examine the shifts of the  $\mathcal{N} = 1$  chiral coordinates, we first focus on the fields of Set 2 defined in (2.12). Performing the transformations (2.20) and (2.22) we find that

$$\delta G^a = \lambda^a - \tau \tilde{\lambda}^a, \quad \delta a_p = i\lambda_p + f_{pq} \tilde{\lambda}^q, \quad (2.25)$$

where we have used that  $a_p$  arises in the expansions (2.4) and (2.7). Both shifts are holomorphic in the moduli of Set 1 given in (2.11) and are shown to unify when using the F-theory description in terms of a Calabi-Yau fourfold (see section 3). The fields of Set 3 have the most involved transformation properties:

$$\delta T_\alpha = i\lambda_\alpha - \frac{i}{2} \mathcal{K}_{\alpha ab} \tilde{\lambda}^a (2G^b + \delta G^b) - \frac{1}{2} \tilde{\lambda}^p Q_{\alpha p}{}^q (a_q + \delta a_q) - \frac{1}{2} a_q (M_\alpha{}^{pq} \lambda_p + M_{\alpha p}{}^q \tilde{\lambda}^p), \quad (2.26)$$

which can be inferred by investigating the isometries of the Kähler manifold spanned by all complex fields with Kähler potential (2.17). Notice that this is valid for finite values of the transformation parameters and that the shift is holomorphic. It is also important to stress that (2.26) implies that the shift in  $\delta\rho_\alpha$  not only depends on  $\lambda_\alpha$  but also on  $\lambda^a$ ,

---

<sup>5</sup>As usual, the four-dimensional theory obtained from dimensional reduction is invariant under a continuous version of the symmetry, while quantum effects break it to the discrete subgroup.

$\tilde{\lambda}^a, \lambda_p, \tilde{\lambda}^p$ . As mentioned above, this is a consequence of the transformation rule for  $C_4$  given in (2.21), together with the shift induced by (2.22). This, in turn, implies that the isometry group generated by the transformation is actually a non-Abelian. To see this, we introduce the Killing vectors  $t_a, \tilde{t}_a, t^p, \tilde{t}_p$  and  $t^\alpha$  for the symmetries parameterized by  $\lambda^a, \tilde{\lambda}^a, \lambda_p, \tilde{\lambda}^p$ , and  $\lambda_\alpha$ . These are then found to respect the non-trivial commutators [15]

$$[t_a, \tilde{t}_b] = -\mathcal{K}_{\alpha ab} t^\alpha, \quad [t_p, \tilde{t}^q] = -M_{\alpha p}{}^q t^\alpha. \quad (2.27)$$

This algebra is a generalization of the well-known Heisenberg algebra. It is an interesting challenge to gauge this algebra while preserving supersymmetry [15, 27].

As mentioned earlier, in the absence of gaugings for the isometries (2.25) and (2.26), one expects that the continuous global shift-symmetries are actually broken to discrete symmetries at the quantum level. Since the discrete version of the symmetries comes from large gauge transformations in the higher-dimensional  $p$ -form fields, such shifts actually identify field configurations in the Set 2 to parameterize complex tori  $\mathbb{T}_{\text{closed}}^{2h_-^{1,1}}$  and  $\mathbb{T}_{\text{open}}^{2h_-^{1,0}}$ , e.g. one finds the identifications

$$\begin{aligned} c^a &\simeq c^a + 1, & b^a &\simeq b^a + 1, \\ c_p &\simeq c_p + 1, & \tilde{c}^p &\simeq \tilde{c}^p + 1, \end{aligned} \quad (2.28)$$

and  $G^a, a_p$  parameterizing<sup>6</sup>

$$\mathbb{T}_{\text{closed}}^{2h_-^{1,1}} = \frac{H_-^{1,1}(Y_3, \mathbb{C})}{H_-^2(Y_3, \mathbb{Z})}, \quad \mathbb{T}_{\text{open}}^{2h_-^{1,0}} = \frac{H^{1,0}(S, \mathbb{C})}{H^1(S, \mathbb{Z})}. \quad (2.29)$$

The complex structure on  $\mathbb{T}_{\text{closed}}^{2h_-^{1,1}}$  is simply given by  $\tau$ , while the complex structure on  $\mathbb{T}_{\text{open}}^{2h_-^{1,0}}$  is encoded in the holomorphic function  $f_{pq}$ . Finally, also  $\rho_\alpha$  is periodic  $\rho_\alpha \simeq \rho_\alpha + 1$ , but one has to additionally impose identifications under (2.28) using  $\delta\rho_\alpha$  obtained from (2.26). These identifications render the field space spanned by  $c^a, b^a, c_p, \tilde{c}^p$  and  $\rho_\alpha$  to be compact.

### 2.3 The $\mathcal{N} = 1$ gauge coupling function

We turn now to the  $\mathcal{N} = 1$  gauge coupling function for the Type IIB orientifold setting and study its symmetries. To keep the discussion simple, we first focus on the case in which the kinetic mixing is absent, i.e. the case in which the couplings (2.10) are zero

$$M_{\kappa p} = M_{\kappa}{}^p = M_p{}^\kappa = M^{p\kappa} = 0. \quad (2.30)$$

We will comment on the more general situation in subsection 2.5.

A first way to obtain the gauge coupling function is by performing a direct dimensional reduction. For the R-R gauge fields  $A^\kappa$  one then finds [16]

$$\hat{f}_{\kappa\lambda} = \mathcal{F}_{\kappa\lambda}|_{z^\kappa=0}, \quad (2.31)$$

---

<sup>6</sup>We are sloppy here by assuming that the  $\sigma^*$  split is compatible with restricting to integer homology and by neglecting cohomological torsion.

where  $\mathcal{F}_{\kappa\lambda} = \partial_{z^\kappa} \partial_{z^\lambda} \mathcal{F}$  is the second derivative of the holomorphic  $\mathcal{N} = 2$  pre-potential  $\mathcal{F}$  for  $Y_3$  of the underlying theory. The restriction in (2.31) is to the slice of complex structure deformations that are compatible with the orientifold condition  $\sigma^* \Omega = -\Omega$ , for the  $(3, 0)$ -form of  $Y_3$ . The function  $\hat{f}_{\kappa\lambda}$  is thus holomorphic in the complex structure deformations  $z^k$ .

Let us next include the D7-brane. In the absence of the moduli  $G^a$  and  $a_p$  in Set 2, one finds by a reduction of the Dirac-Born-Infeld and Chern-Simons action that

$$\hat{f}_{D7} = \delta_{D7}^\alpha \left( \frac{1}{2} \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma + i \rho_\alpha \right). \quad (2.32)$$

Here  $\delta_{D7}^\alpha$  is the restriction to the world-volume  $S_+$  and can be obtained by expanding the Poincaré-dual two-form  $[S_+]$  to  $S_+$  into the basis  $\omega_\alpha$ , i.e.  $[S_+] = \delta_{D7}^\alpha \omega_\alpha$ . The real part of  $\hat{f}_{D7}$  is determined by using the calibration conditions for supersymmetric cycles and thus obtained from the volume of  $S_+$  measured in the ten-dimensional Einstein-frame metric. In the string frame one has  $\text{Re} \hat{f}_{D7} \propto g_s^{-1}$ . Clearly, in the absence of fields of Set 2 the gauge-coupling is  $\hat{f}_{D7} = \delta_{D7}^\alpha T_\alpha$  and thus holomorphic in the  $\mathcal{N} = 1$  coordinates. Its imaginary part non-trivially shifts with  $\lambda_\alpha$  under (2.21), which are the standard constant shifts of the theta-angle.

The inclusion on the  $G^a$  moduli is also straightforward, since the corrections in  $G^a$  are at the same order of  $g_s$  as the volume part. Indeed, dimensionally reducing the D7 action one finds that, with vanishing worldvolume flux, the gauge coupling function is [6]<sup>7</sup>

$$\hat{f}_{D7} = \delta_{D7}^\alpha \left[ \frac{1}{2} \mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta + \frac{1}{2} e^{-\phi} \mathcal{K}_{\alpha ab} b^a b^b + i \left( \rho_\alpha - \frac{1}{2} \mathcal{K}_{\alpha ab} c^a b^b + C_0 \frac{1}{2} \mathcal{K}_{\alpha ab} b^a b^b \right) \right], \quad (2.33)$$

which is holomorphic in the  $T_\alpha$  coordinates (2.13) in the absence of Wilson line moduli. We note that, naively, the gauge coupling function is now transforming non-trivially under the symmetries (2.26) since, in addition to the constant shifts with  $\lambda_\alpha$ , one also finds shifts with  $\lambda^a$  holomorphic in  $G^a$  and  $\tau$ . However, (2.33) is only valid when the gauge flux on the D7-brane is zero, i.e.  $F = 0$ , which as noted above, is not a gauge invariant condition since it shifts according to eq. (2.24). Thus, the gauge invariant version of (2.33) is actually

$$\hat{f}_{D7} = \delta_{D7}^\alpha \left( T_\alpha + i \mathcal{K}_{\alpha ab} f^a G^b + \frac{i}{2} \tau \mathcal{K}_{\alpha ab} f^a f^b \right), \quad (2.34)$$

where we defined the worldvolume fluxes  $f^a$  as

$$F = \frac{1}{2\pi\alpha'} f^a i^* \omega_a. \quad (2.35)$$

Since these transform according to (2.24), we find that the gauge coupling function is both holomorphic and invariant under the whole set of shift symmetries (modulo a constant imaginary shift), as it should.

Finally, when including the Wilson line moduli for the D7-brane, we immediately face a problem. At first, one might think that the gauge coupling function is given in this

---

<sup>7</sup>The slightly odd factor of 1/2 in the term proportional to  $\mathcal{K}_{\alpha ab} c^a b^b$  arises due to fact that our  $C_4$  is shifted such that it is  $Sl(2, \mathbb{Z})$  invariant in Type IIB.

case by (2.34), where  $T_\alpha$  contains a quadratic term in the Wilson lines (2.13). However, the dimensional reduction of the D7-brane action does not give such a term and we find again (2.32). As argued in [6], a contribution quadratic in the Wilson lines is generated at one loop in  $g_s$  and is therefore natural that it is not captured by the Dirac-Born-Infeld action, which is only valid at tree level in open string amplitudes.<sup>8</sup> Such corrections were computed in [9, 10] in toroidal models, which show that indeed, a quadratic term arises at one loop level. It is therefore natural to split  $\hat{f}_{D7}$  as

$$\hat{f}_{D7} = \hat{f}_{D7}^{\text{red}} + \hat{f}_{D7}^{1\text{-loop}}, \tag{2.36}$$

where  $\hat{f}_{D7}^{\text{red}}$  is obtained by direct dimensional reduction of the D7-brane action. Comparing (2.36) with (2.34) one is lead to make the ansatz

$$\hat{f}_{D7}^{1\text{-loop}} = \frac{1}{2} \delta_{D7}^\alpha d_\alpha{}^{pq} a_p (a_q + \bar{a}_q) + \log \Theta, \tag{2.37}$$

where  $\Theta$  is a holomorphic function. Note that our analysis of the shift symmetries implies that the quadratic term in (2.37) cannot be the full result, since under shifts of the Wilson line moduli, the field  $T_\alpha$  shifts by a non-constant term, which would make the gauge coupling function non-gauge invariant. We therefore introduced the non-vanishing holomorphic function  $\Theta$  in the moduli  $a_p$  and  $z^k$ . In the next section we discuss the properties of this completion in more detail.

## 2.4 One-loop corrections and theta-functions

Let us have a closer look at the inclusion of the Wilson line moduli in the discussion of the D7-brane gauge coupling function  $\hat{f}_{D7}$ . As stressed above the quadratic term in the  $a_p$  arise at order  $g_s^0$ , i.e. is only visible at the open string one loop level. In toroidal models [9, 10] it was furthermore shown that the fully corrected gauge coupling function contains a Riemann theta function depending on the D-brane moduli. In toroidal models, these theta functions arise due to the underlying toroidal compactification space. While we are not dealing with such a simple geometry, we have stressed in (2.29) that the Wilson lines in this more general orientifold compactification also parameterize a higher-dimensional complex torus. In the following we will use this fact together with the transformation property (2.26) to infer the general form of  $\hat{f}_{D7}$  as a function of  $a_p$ . More precisely, we suggest that  $\Psi = e^{\hat{f}_{D7}^{1\text{-loop}}}$  introduced in (2.37) can be viewed as a holomorphic section of a certain line bundle on the torus  $\mathbb{T}_{\text{open}}^{2h_{-}^{1,0}}$  introduced in (2.29). Our construction is inspired by the discussion of the M5-brane action first given in [32]. It has been extended and applied relevantly for our orientifold setting, for example, in refs. [33, 34]. A similar strategy has been also suggested in the construction of the non-perturbative  $\mathcal{N} = 1$  superpotential [35–38].

### 2.4.1 A simple case with one Wilson line modulus

Before discussing the general case let us exemplify our reasoning for a single Wilson line  $a$ , i.e. for the situation discussed around (2.19). The complex field  $a$  parameterizes a complex

---

<sup>8</sup>This was also noticed in the mirror dual configurations [8], which were also studied in [28]. Corrections in Type IIA orbifolds have been studied, for example, in [29–31].

two-torus  $\mathbb{T}_{\text{open}}^2$  with complex structure given by the function  $f$ . As above we can write  $a = ic + f\tilde{c}$  with  $c \cong c + 1$ ,  $\tilde{c} \cong \tilde{c} + 1$ . We then introduce the following connection on this torus

$$\mathfrak{A} = \frac{iM}{4\text{Re}f} (a d\bar{a} - \bar{a} da). \tag{2.38}$$

with  $M \in 2\pi\mathbb{Z}$ . The field strength  $\mathfrak{F} = \frac{iM}{2\text{Re}f} da \wedge d\bar{a}$  is a (1,1)-form, so  $\mathfrak{A}$  is a connection on a holomorphic line bundle  $\mathcal{L}$ . Holomorphic sections of  $\mathcal{L}$  are defined as sections that satisfy

$$\bar{\psi}_{\mathfrak{A}} \Psi = (\bar{\psi} - i\mathfrak{A}_{\bar{a}}) \Psi = 0, \tag{2.39}$$

where the differential is with respect to  $\bar{a}$ . Note that  $\Psi$  is defined on a torus and thus has to respect appropriate boundary conditions. Compatibility of (2.39) with the torus shifts  $a \cong a + ni + mf$ , with  $n, m \in \mathbb{Z}$ , implies that  $\Psi$  has to transform as

$$\Psi(a + ni + mf) = \exp\left(-\frac{iM}{2\text{Re}f} \text{Im}[(in + fm)\bar{a}]\right) \Psi(a), \tag{2.40}$$

where we kept  $f$  constant, therefore ignoring the dependence on complex structure. One can now simply solve the differential equation (2.39) together with the boundary conditions (2.40). There are  $|M|/2\pi$  linearly independent solutions given by (see e.g. [39])<sup>9</sup>

$$\Psi_j = e^{-\frac{M}{4\text{Re}f} a(a+\bar{a})} \vartheta \left[ \begin{matrix} \frac{2\pi j}{M} \\ 0 \end{matrix} \right] \left( \frac{iMf}{2\pi}, iMa \right), \quad j = 0, 1, \dots, |M|/2\pi - 1, \tag{2.41}$$

with

$$\vartheta \left[ \begin{matrix} \mu \\ \nu \end{matrix} \right] (\tau, a) = \sum_{l \in \mathbb{Z}} e^{i\pi\tau(\mu+l)^2} e^{2\pi i(\mu+l)(a+\nu)} \tag{2.42}$$

the Jacobi theta function. Notice that the theta functions above can be seen as holomorphic sections of the bundle defined by (2.38) in *holomorphic gauge*, i.e. with  $\mathfrak{A}^{0,1} = 0$  but  $\mathfrak{A}^{1,0} \neq 0$ , defined by the following complex gauge transformation<sup>10</sup>

$$\mathfrak{A}_h = \mathfrak{A} - d\left(\frac{iM}{2\text{Re}f} a \text{Re} a\right) = -\frac{iM}{\text{Re}f} \text{Re} a da. \tag{2.43}$$

One thus recovers the standard transformation behavior of the theta functions under the torus shifts.

In order to relate the  $\Psi_j$  given in (2.41) to the gauge coupling function we next consider taking the logarithm of an arbitrary solution  $\Psi = \sum_{j=0}^{|M|-1} C_j \Psi_j$ ,

$$\log \Psi = -\frac{M}{4\text{Re}f} a(a+\bar{a}) + \log \Theta, \quad \Theta = \sum_{j=0}^{|M|-1} C_j \vartheta \left[ \begin{matrix} \frac{2\pi j}{M} \\ 0 \end{matrix} \right] \left( \frac{iMf}{2\pi}, iMa \right). \tag{2.44}$$

<sup>9</sup>One can show that there are  $|M|$  independent solutions without having to solve the equation. This follows from an index theorem, see e.g. [40], which in this case is  $\int_{T^2} \mathfrak{F} = -M$ .

<sup>10</sup>Usually we consider only gauge transformations  $A \rightarrow A + d\chi$  with  $\chi$  a real function. However, the eq. (2.39) is invariant under the complexified gauge group so we may take  $\chi$  complex.

This equation is already quite illuminating. The first piece is precisely the correction to the  $T_\alpha$  coordinate proportional to the moduli  $a$ , as in eq. (2.19). The second term,  $\log \Theta$ , is holomorphic in  $a$  and transforms precisely in the right way to render  $\delta_{D7}^\alpha T_\alpha + \log \Theta$  invariant under shifts in  $a$ . Therefore, identifying

$$\hat{f}_{D7} = \delta_{D7}^\alpha T_\alpha + \log \Theta, \quad (2.45)$$

with  $M = \delta_{D7}^\alpha M_\alpha$  and appropriate  $C_j$ , yields a suitable completion of the gauge coupling function of a D7-brane. As promised, we have identified  $\Psi = e^{\hat{f}_{D7}^{1-\text{loop}}}$  as a holomorphic section of a line bundle on a two-torus, when viewing the one-loop part of the  $T_\alpha$  coordinates as functions of  $a, \bar{a}$ .

Note that we have only focused on the  $a$ -dependence of  $\hat{f}_{D7}$  in the above discussion. We know, however, that supersymmetry implies that  $\hat{f}_{D7}$  also has to be holomorphic in the complex structure moduli  $z^k$ . Indeed, we find that our construction appropriately yields such a holomorphic dependence through the theta functions  $\vartheta$  in (2.44) due to the holomorphic function  $f(z^k)$ . In general, however, the coefficients  $C_j$  can also depend holomorphically on the moduli  $z^k$ . This dependence is not constrained by our considerations of shift-symmetries. It can be constrained by including further symmetries, such as monodromy symmetries in the complex structure moduli space, but considerations of this type are beyond the scope of this work.

#### 2.4.2 The general case with several Wilson line moduli

Let us now repeat the same arguments for the more general situation with several Wilson line moduli  $a_p$ . The first step consists of constructing the line bundle  $\mathcal{L}$  on  $\mathbb{T}_{\text{open}}^{2h-1,0}$ , by defining an appropriate connection. We do this by analyzing the general transformations (2.26) of  $T_\alpha$  under the torus shifts. Then we can follow the same strategy as above to constrain the expected one-loop correction.

We would like to consider a holomorphic function  $\Theta(z^k, a_p)$  such that under the shift (2.25) of the  $a_p$  satisfies

$$\Theta(z^k, a_p + \delta a_p) = \exp(-\delta_{D7}^\alpha \delta T_\alpha) \Theta(z^k, a_p), \quad (2.46)$$

with  $\delta T_\alpha$  given in (2.26). The existence of such a  $\Theta$  implies that

$$\hat{f}_{D7} = \delta_{D7}^\alpha T_\alpha + \log \Theta, \quad (2.47)$$

remains invariant. As above, when viewing  $T_\alpha$  as functions of  $a_p$  we can identify  $\Psi = e^{\hat{f}_{D7}^{1-\text{loop}}}$  as a holomorphic section of the line bundle  $\mathcal{L}$  satisfying (2.39) for some connection  $\mathfrak{A}$ .

It is easier to determine the connection  $\mathfrak{A}$  in holomorphic gauge which reads

$$\mathfrak{A}_h = \frac{i}{4} \delta_{D7}^\alpha (2M_{\alpha p}{}^q \text{Re } f^{pr} \text{Re } a_r + M_\alpha{}^{pq} a_p) da_q. \quad (2.48)$$

Indeed, one checks that (freezing complex structure) the connection transforms as

$$\mathfrak{A}_h(a_p + \delta a_p) = \mathfrak{A}_h(a_p) + d\chi, \quad \chi = -i\delta_{D7}^\alpha \delta T_\alpha. \quad (2.49)$$

The field strength of  $\mathfrak{A}_h$  is

$$\mathfrak{F} = -\frac{i}{4} \delta_{D7}^\alpha M_{\alpha p}{}^r \operatorname{Re} f^{pq} da_r \wedge d\bar{a}_q, \quad (2.50)$$

where we imposed that

$$\delta_{D7}^\alpha \left( M_\alpha{}^{pq} + \operatorname{Re} f^{r[p} M_{\alpha r}{}^{q]} \right) = 0 \quad (2.51)$$

such that  $F^{2,0} = F^{0,2} = 0$ . Notice that the field strength does not depend on  $M_\alpha{}^{pq}$  which, in particular, means that the number of solutions of (2.39) is independent of  $M_\alpha{}^{pq}$ .<sup>11</sup> Note that the constraint (2.51) can actually always be satisfied for a single D7-brane when choosing a basis  $(\hat{\alpha}_p, \hat{\beta}^p)$  in (2.6) that is symplectic with respect to the inner product  $\langle \alpha, \beta \rangle = \int_{S_+} \delta_{D7}^\alpha \omega_\alpha \wedge \alpha \wedge \beta$ .<sup>12</sup>

We can thus infer the form of the solution  $\Theta$  is a sum over the Riemann theta functions

$$\vartheta \begin{bmatrix} \mu^p \\ \nu_p \end{bmatrix} (f_{pq}, a_p) = \sum_{l^p \in \Gamma} e^{i\pi f_{pq}(\mu^p + l^p)(\mu^q + l^q)} e^{2\pi i(\mu^p + l^p)(a_p + \nu_p)}, \quad (2.52)$$

where  $\Gamma$  is a  $h_-^{1,0}$ -dimensional integer lattice. As in the simpler case considered before, the coefficients in this sum can be complex structure dependent and are not constrained by the torus shift-symmetries. In fact, in this section we worked in a fixed complex structure of the Calabi-Yau threefold. For a proper treatment of the dependence on complex structure moduli, we should consider a line bundle over  $\mathbb{T}_{\text{open}}^{2h_-^{1,0}}$ , which is itself fibered over the space of complex structures.

This concludes our discussion on the interplay between holomorphicity of the gauge coupling function and its behavior under the shift-symmetries of the axions when there is no kinetic mixing among the open and closed string gauge bosons.

## 2.5 Comments on kinetic mixing and gaugings

Up to now we have assumed that the kinetic mixing between the open and closed string  $U(1)$ 's vanishes, cf. (2.30). In this section we comment briefly on how the presence of mixing changes the situation (see [41, 42] for a discussion on kinetic mixing in D-brane models from a different perspective).

As shown in [6], the mixing is controlled by the couplings defined in (2.10). In our notation, the result that one obtains from reducing the D7-brane action is

$$\hat{f}_{\kappa D7} = \operatorname{Re} f^{pq} \operatorname{Re} f_{\kappa\lambda} (M_q{}^\lambda - i\bar{f}_{qr} M^{r\lambda}) a_p. \quad (2.53)$$

Since both  $f_{\kappa\lambda}$  and  $f_{pq}$  depend holomorphically on complex structure, we find that  $\hat{f}_{\kappa D7}$  has a complicated dependence on the complex structure moduli, which does not seem holomorphic. However, the M-theory computation done in the next section shows that

<sup>11</sup>This is related to the fact that in (2.27) the couplings  $M_{\alpha p}{}^q$  (and not  $M_\alpha{}^{pq}$ ) determine the structure constants of the isometries of the scalar manifold.

<sup>12</sup>Note that this inner product can be degenerate on the full set  $(\hat{\alpha}_p, \hat{\beta}^p)$ .

there is an identity which proves that this quantity is actually holomorphic. Indeed, one can show that

$$\text{Re } f^{pq} \text{Re } f_{\kappa\lambda} (M_q^\lambda - i \bar{f}_{qr} M^{r\lambda}) = M_\kappa^p + i f_{\kappa\lambda} M^{\lambda p}, \quad (2.54)$$

and so the mixing becomes

$$\hat{f}_{\kappa D7} = \left( M_\kappa^p + i f_{\kappa\lambda} M^{\lambda p} \right) a_p, \quad (2.55)$$

which is now manifestly holomorphic. Notice that from the type IIB perspective this is a highly non-trivial identity among (2,1)-forms in the internal space and (0,1)-forms on the worldvolume of the brane. However, in the F-theory description, both of them lift to three-forms in the Calabi-Yau fourfold, where the identity (2.54) becomes obvious (see the discussion around (3.18)).

Now we can analyze how the kinetic mixing behaves under the shift symmetries of the axions  $a_p$ . Clearly,  $\hat{f}_{\kappa D7}$  is not invariant, which might be a reason to think that this cannot be correct, or at least not the full result. However, the presence of mixing has an interesting consequence for the symmetries, which implies that the gauge coupling function must transform non-trivially under shifts in the Wilson lines. Again, since this is most easily seen from the M/F-theory description done in the next section, we will just quote the result here. Under a shift (2.25), we have that<sup>13</sup>

$$\begin{aligned} \delta A^\kappa &= d\Lambda^\kappa - (M_p^\kappa \tilde{\lambda}^p + M^{p\kappa} \lambda_p) A_{D7}, \\ \delta A_{D7} &= d\Lambda_{D7}, \end{aligned} \quad (2.56)$$

where we included the corresponding gauge transformations of the vectors,  $\Lambda^\kappa$  and  $\Lambda_{D7}$ . Thus, a shift in the axions induces a *constant* change of basis in the space of U(1)'s, which mixes the open and closed gauge bosons. For integer values of  $\tilde{\lambda}^p$  and  $\lambda_p$ , we find that the change of basis for the vectors is also integral, as expected from charge quantization. This, in turn, implies that the gauge coupling function has to depend on the Wilson lines and should not be invariant under the symmetries, unlike in the case where the mixing vanishes. We leave a more detailed discussion to subsection 3.4.

Let us close this section with some remarks about the interplay between the transformation (2.56) and the gauging of the isometries (2.27) of the scalar manifold from a purely field-theoretical perspective. As we stressed earlier, the isometries of the scalar manifold are non-Abelian, while the gauge symmetry of the vectors is Abelian. This suggests that one cannot gauge such isometries without introducing extra vectors or structure. However, this is not the case, precisely because the vectors transform as in (2.56). Indeed, suppose that we gauge the isometries

$$X_A = \Theta_A^p \tilde{t}_p + \Theta_{Ap} t^p + \Theta_{A\alpha} t^\alpha, \quad (2.57)$$

where  $A$  runs over  $\kappa$  and the D7-brane gauge boson, and  $\Theta$  is the embedding tensor. This means that, under a gauge transformation, we have to perform a shift in the corresponding

---

<sup>13</sup>Notice that the couplings that appear in (2.56) are not exactly those in (2.55). However, using the identity (2.54) we see that the transformation of the vectors is trivial if and only if the kinetic mixing is zero. This was already observed in [42].

axions, namely

$$\tilde{\lambda}^p = \Theta_A{}^p \Lambda^A, \quad \lambda_p = \Theta_{Ap} \Lambda^A, \quad \lambda_\alpha = \Theta_{A\alpha} \Lambda^A. \quad (2.58)$$

Thus, the parameters  $\tilde{\lambda}^p$  and  $\lambda_p$  are generically no longer constant and the transformation (2.56) is not simply a constant change of basis. Instead, using (2.58) it becomes

$$\begin{aligned} \delta A^\kappa &= d\Lambda^\kappa - (M_p{}^\kappa \Theta_A{}^p + M^{p\kappa} \Theta_{Ap}) A_{D7} \Lambda^A, \\ \delta A_{D7} &= d\Lambda_{D7}, \end{aligned} \quad (2.59)$$

which can be readily recognized as the gauge transformation of a *non-Abelian* gauge group. Thus, we see that the transformation (2.56) allows to gauge certain non-Abelian isometry starting with an Abelian gauge group. Finally, since the resulting gauge group is non-compact and non-semisimple, the gauge coupling function cannot be constant [27], which fits nicely with what we find from the reduction.<sup>14</sup> See [15, 43, 44] for more details on the gauging of such isometries.

### 3 M-theory on Calabi-Yau fourfolds and the F-theory frame

In this section, we perform the dimensional reduction of M-theory on a smooth Calabi-Yau fourfold  $Y_4$  without fluxes. Then, by restricting to the case in which  $Y_4$  is elliptically fibered, we perform the necessary dualization to compare the resulting three-dimensional theory to the circle reduction of an arbitrary four-dimensional  $\mathcal{N} = 1$  supergravity theory. Let us note that this approach has been already successfully applied in previous works, see e.g. [13, 15, 23, 45]. However, it is crucial to stress that the reduction and comparison that we present is the most general analysis carried out so far.<sup>15</sup> In particular, we will cover the cases that capture kinetic mixing between R-R bulk and 7-brane gauge fields.

#### 3.1 Dimensional reduction of M-theory on a smooth fourfold

We begin our analysis by performing the dimensional reduction of eleven-dimensional supergravity on  $Y_4$ . Such reductions were performed already in [46–48], and we will deviate from these works only by considering a more explicit ansatz for the (2, 1)-forms on  $Y_4$ .

The starting point is the bosonic part of eleven-dimensional supergravity given by

$$S^{(11)} = \frac{1}{2} \int \left( \hat{R} \hat{\star} 1 - \frac{1}{2} \hat{G} \wedge \hat{\star} \hat{G} - \frac{1}{6} \hat{C} \wedge \hat{G} \wedge \hat{G} \right), \quad (3.1)$$

where  $\hat{R}$  is the eleven-dimensional Ricci scalar and  $\hat{G} = d\hat{C}$  is the four-form field strength for the three-form  $\hat{C}$ . We will consider backgrounds of the form

$$\begin{aligned} \langle d\hat{s}^2 \rangle &= \eta_{\mu\nu} dx^\mu dx^\nu + 2g_{m\bar{n}} dy^m dy^{\bar{n}}, \\ \langle d\hat{C} \rangle &= 0, \end{aligned} \quad (3.2)$$

<sup>14</sup>Actually, as shown in [15], the gauge coupling function does not depend on the gaugings.

<sup>15</sup>Also in comparison to [21] we will drop simplifying assumptions.

Cohomology group	Basis element	Fields	Index range
$H^{1,1}(Y_4)$	$\omega_\Sigma$	$(L^\Sigma, A^\Sigma)$	$\Sigma = 1, \dots, h^{1,1}(Y_4)$
$H^{3,1}(Y_4)$	$\chi_{\mathcal{K}}$	$z^{\mathcal{K}}$	$\mathcal{K} = 1, \dots, h^{3,1}(Y_4)$
$H^{2,1}(Y_4)$	$\Psi^{\mathcal{A}}$	$N_{\mathcal{A}}$	$\mathcal{A} = 1, \dots, h^{2,1}(Y_4)$
$H^3(Y_4)$	$(\alpha_{\mathcal{A}}, \beta^{\mathcal{A}})$	$(\tilde{c}^{\mathcal{A}}, c_{\mathcal{A}})$	$\mathcal{A} = 1, \dots, h^{2,1}(Y_4)$

**Table 3.** Relevant cohomology groups in the reduction on  $Y_4$ . The dimensions are denoted by  $h^{p,q}(Y_4) = \dim H^{p,q}(Y_4)$ . While  $\Psi^{\mathcal{A}}$  and  $\chi_{\mathcal{K}}$  are complex basis elements, the forms  $\omega_\Sigma$  and  $(\alpha_{\mathcal{A}}, \beta^{\mathcal{A}})$  constitute real basis elements.

where  $g_{m\bar{n}}$  is a Calabi-Yau metric on the fourfold  $Y_4$ . This choice of background ensures that the resulting effective theory is a three-dimensional  $\mathcal{N} = 2$  supergravity.

The effective theory of interest include all massless fluctuations around the background solution (3.2). The massless modes arising from fluctuations of the metric can be encoded in terms of the Kähler form  $J$  expanded as

$$J = v^\Sigma \omega_\Sigma, \quad \Sigma = 1, \dots, h^{1,1}(Y_4), \quad (3.3)$$

where  $\omega_\Sigma$  form a basis of harmonic two-forms. The fields  $v^\Sigma$  are three-dimensional real scalar fields that parametrize the Kähler structure deformations of  $Y_4$ . We also have  $h^{3,1}(Y_4)$  complex fields  $z^{\mathcal{K}}$ ,  $\mathcal{K} = 1, \dots, h^{3,1}(Y_4)$ , that encode the complex structure deformations of  $Y_4$ .

The massless modes that come from fluctuations of the M-theory three-form  $\hat{C}$  are given by

$$\hat{C} = A^\Sigma \wedge \omega_\Sigma + N_{\mathcal{A}} \Psi^{\mathcal{A}} + \bar{N}_{\mathcal{A}} \bar{\Psi}^{\mathcal{A}}, \quad \mathcal{A} = 1, \dots, h^{2,1}(Y_4), \quad (3.4)$$

where we introduced  $\Psi^{\mathcal{A}}$ , a basis of harmonic (1,2)-forms. We note that  $A^\Sigma$  are three-dimensional vector fields and  $N^{\mathcal{A}}$  are three-dimensional complex scalars. Following [13], we choose the following parametrization of the (1,2)-forms

$$\Psi^{\mathcal{A}} = \frac{1}{2} \text{Re} f^{\mathcal{A}\mathcal{B}} (\alpha_{\mathcal{B}} - i \bar{f}_{\mathcal{B}\mathcal{C}} \beta^{\mathcal{C}}), \quad (3.5)$$

where  $(\alpha_{\mathcal{A}}, \beta^{\mathcal{B}})$  are a basis of integral harmonic real three-forms and  $f_{\mathcal{A}\mathcal{B}}$  is holomorphic in complex structure. We also defined  $\text{Re} f^{\mathcal{A}\mathcal{B}}$ , which is the inverse of  $\text{Re} f_{\mathcal{A}\mathcal{B}}$ . Thus,  $\hat{G}$  is given by

$$\hat{G} = dA^\Sigma \wedge \omega_\Sigma + DN_{\mathcal{A}} \wedge \Psi^{\mathcal{A}} + D\bar{N}_{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{A}}, \quad (3.6)$$

with

$$DN_{\mathcal{A}} = dN_{\mathcal{A}} - \text{Re} N_{\mathcal{B}} \text{Re} f^{\mathcal{B}\mathcal{C}} \partial_{\mathcal{K}} f_{\mathcal{C}\mathcal{A}} dz^{\mathcal{K}}, \quad D\bar{N}_{\mathcal{A}} = \overline{DN_{\mathcal{A}}}. \quad (3.7)$$

Note that we could have also chosen to use the real basis  $(\alpha_{\mathcal{A}}, \beta^{\mathcal{B}})$  in the expansion of  $\hat{C}$ . This would introduce real scalars  $(\tilde{c}^{\mathcal{A}}, c_{\mathcal{A}})$ , which are related to the complex scalars  $N_{\mathcal{A}}$  via  $N_{\mathcal{A}} = ic_{\mathcal{A}} + f_{\mathcal{A}\mathcal{B}} \tilde{c}^{\mathcal{A}}$ , but we will work directly with  $N_{\mathcal{A}}$ . The basis form and corresponding fields are summarized in table 3.

Substituting the ansatz (3.3) and (3.4) into the action (3.1) and performing a Weyl rescaling, which brings the effective action into the Einstein frame, we find that the three-dimensional effective theory is given by

$$\begin{aligned}
 S^{(3)} = & \int \frac{1}{2} R \star 1 - G_{\mathcal{K}\bar{\mathcal{L}}} dz^{\mathcal{K}} \wedge \star d\bar{z}^{\bar{\mathcal{L}}} - G_{\Sigma\Lambda} dL^{\Sigma} \wedge \star dL^{\Lambda} - G_{\Sigma\Lambda} dA^{\Sigma} \wedge \star dA^{\Lambda} \\
 & - \frac{1}{2} L^{\Sigma} d_{\Sigma}{}^{AB} DN_{\mathcal{A}} \wedge \star D\bar{N}_{\mathcal{B}} - \frac{1}{4i} d_{\Sigma}{}^{AB} F^{\Sigma} \wedge (N_{\mathcal{A}} D\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}} DN_{\mathcal{A}}).
 \end{aligned} \tag{3.8}$$

Let us introduce the different objects that appear in this expression. We introduced the rescaled Kähler moduli

$$L^{\Sigma} = \frac{v^{\Sigma}}{\hat{\mathcal{V}}}, \quad \mathcal{V} = \frac{1}{4!} \mathcal{K}_{\Sigma\Lambda\Gamma\Delta} L^{\Sigma} L^{\Lambda} L^{\Gamma} L^{\Delta}, \quad \hat{\mathcal{V}} = \frac{1}{4!} \int_{Y_4} J^4, \tag{3.9}$$

with

$$\mathcal{K}_{\Sigma\Lambda\Gamma\Delta} = \int_{Y_4} \omega_{\Sigma} \wedge \omega_{\Lambda} \wedge \omega_{\Gamma} \wedge \omega_{\Delta}, \tag{3.10}$$

the intersection number of two-forms. The kinetic term for the complex structure moduli  $z^{\mathcal{K}}$  depends on a Kähler metric given by

$$G_{\mathcal{K}\bar{\mathcal{L}}} = -\frac{\int_{Y_4} \chi_{\mathcal{K}} \wedge \chi_{\bar{\mathcal{L}}}}{\int_{Y_4} \Omega \wedge \bar{\Omega}} = -\partial_{z^{\mathcal{K}}} \partial_{\bar{z}^{\bar{\mathcal{L}}}} \log \left( \int_{Y_4} \Omega \wedge \bar{\Omega} \right), \tag{3.11}$$

where  $\chi_{\mathcal{L}}$  are a basis of harmonic (3,1)-forms, with  $\mathcal{L} = 1, \dots, h^{1,3}(Y_4)$ . Regarding the kinetic terms for the vector multiplets ( $L^{\Sigma}$ ,  $A^{\Sigma}$ ), we have that

$$G_{\Sigma\Lambda} = \frac{\hat{\mathcal{V}}}{4} \int_{Y_4} \omega_{\Sigma} \wedge \star \omega_{\Lambda} = -\frac{1}{8\mathcal{V}} \left( \mathcal{K}_{\Sigma\Lambda} - \frac{1}{18\mathcal{V}} \mathcal{K}_{\Sigma} \mathcal{K}_{\Lambda} \right) = -\frac{1}{4} \partial_{L^{\Sigma}} \partial_{L^{\Lambda}} \log \mathcal{V}, \tag{3.12}$$

where we defined

$$\mathcal{K}_{\Sigma} = \mathcal{K}_{\Sigma\Lambda\Gamma\Delta} L^{\Lambda} L^{\Gamma} L^{\Delta}, \quad \mathcal{K}_{\Sigma\Lambda} = \mathcal{K}_{\Sigma\Lambda\Gamma\Delta} L^{\Gamma} L^{\Delta}, \tag{3.13}$$

and used that

$$\star \omega_{\Sigma} = -\frac{1}{2} J \wedge J \wedge \omega_{\Sigma} + \frac{\hat{\mathcal{V}}^2}{36} \mathcal{K}_{\Sigma} J \wedge J \wedge J. \tag{3.14}$$

Finally, we introduced the couplings

$$\int_{Y_4} \Psi^{\mathcal{A}} \wedge \star \bar{\Psi}^{\mathcal{B}} = L^{\Sigma} d_{\Sigma}{}^{\mathcal{A}\mathcal{B}}, \quad d_{\Sigma}{}^{\mathcal{A}\mathcal{B}} = i \int_{Y_4} \omega_{\Sigma} \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}}, \tag{3.15}$$

where we used that  $\star \Psi^{\mathcal{A}} = -iJ \wedge \Psi^{\mathcal{A}}$ . These can be written as

$$d_{\Sigma}{}^{\mathcal{A}\mathcal{B}} = -\frac{1}{2} \text{Re} f^{\mathcal{B}\mathcal{C}} Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}}, \quad Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}} = M_{\Sigma\mathcal{C}}{}^{\mathcal{A}} + i f_{\mathcal{C}\mathcal{B}} M_{\Sigma}{}^{\mathcal{B}\mathcal{A}}, \tag{3.16}$$

when using the intersection numbers

$$M_{\Sigma\mathcal{A}}{}^{\mathcal{B}} = \int_{Y_4} \omega_{\Sigma} \wedge \alpha_{\mathcal{A}} \wedge \beta^{\mathcal{B}}, \quad M_{\Sigma}{}^{\mathcal{A}\mathcal{B}} = \int_{Y_4} \omega_{\Sigma} \wedge \beta^{\mathcal{A}} \wedge \beta^{\mathcal{B}}, \tag{3.17}$$

which are independent of the Kähler and complex structure moduli. Notice that there are two important properties of the  $\Psi^A$  that we have used numerous throughout the derivation:

$$d_\Sigma^{AB} = \overline{d_\Sigma^{BA}}, \quad \int_{Y_4} \omega_\Sigma \wedge \Psi^A \wedge \Psi^B = 0, \quad (3.18)$$

The first relation implies that  $\text{Re } f^{AB} \text{Re } f_{CD} \overline{Q_{\Sigma B}^D} = Q_{\Sigma C}^A$  and is the origin of the identity (2.54). The second identity allows to remove the intersection numbers involving  $\alpha_A \wedge \alpha_B$ , such that the result only depends on  $M_{\Sigma A}^B$  and  $M_\Sigma^{AB}$  defined in (3.17).

### 3.2 The three-dimensional $\mathcal{N} = 2$ action and its symmetries

Before manipulating the three-dimensional effective theory (3.8) further, it is important to stress that it can be written in an  $\mathcal{N} = 2$  form with three-dimensional Yang-Mills terms [48]. This implies that all couplings are determined by a real function  $K$ , which we will call the kinetic potential. Explicitly the bosonic part of the  $\mathcal{N} = 2$  action takes the form

$$S_{\mathcal{N}=2}^{(3)} = \int \frac{1}{2} R \star 1 - \tilde{K}^{\hat{A}\hat{B}} d\phi_{\hat{A}} \wedge \star d\bar{\phi}_{\hat{B}} + \frac{1}{4} \tilde{K}_{\Sigma\Lambda} (dL^\Sigma \wedge \star dL^\Lambda + F^\Sigma \wedge \star F^\Lambda) + F^\Sigma \wedge \text{Im}(\tilde{K}_\Sigma^{\hat{A}} d\phi_{\hat{A}}), \quad (3.19)$$

where  $\phi_{\hat{A}}$  denotes the different complex scalar multiplets,  $z^{\mathcal{K}}$  and  $N_{\mathcal{A}}$ , and  $(L^\Sigma, A^\Sigma)$  corresponds to vector multiplets. Comparing (3.8) with (3.19) one infers that the kinetic potential is given by

$$\tilde{K}(N_{\mathcal{A}}, z^{\mathcal{K}} | L^\Sigma) = -\log \left( \int_{Y_4} \Omega \wedge \bar{\Omega} \right) + \log \mathcal{V} + L^\Sigma \text{Re } N_{\mathcal{A}} \text{Re}[d_\Sigma^{BA} N_B]. \quad (3.20)$$

It is worth pointing out that (3.20) is valid without any further assumptions about the real three-forms  $(\alpha_A, \beta^B)$  appearing in (3.5).<sup>16</sup>

Let us briefly discuss the symmetries of the effective action. First of all, it has an Abelian gauge symmetry given by

$$\delta A^\Sigma = d\Lambda^\Sigma, \quad (3.21)$$

where  $\Lambda^\Sigma$  is an arbitrary function. Furthermore, as advanced earlier, it has a global Abelian symmetry acting on the scalars  $N_{\mathcal{A}}$  as

$$\delta N_{\mathcal{A}} = i\lambda_{\mathcal{A}} + f_{AB} \tilde{\lambda}^B, \quad (3.22)$$

with  $\lambda_{\mathcal{A}}$  and  $\tilde{\lambda}^A$  real constants. These symmetries descend from large gauge transformations of the  $\hat{C}$ -field, namely  $\delta \hat{C} = \tilde{\lambda}^A \alpha_A + \lambda_{\mathcal{A}} \beta^{\mathcal{A}}$  with  $\tilde{\lambda}^A, \lambda_{\mathcal{A}} \in \mathbb{Z}$ . As usual, the classical supergravity analysis is invariant under a continuous version of the symmetry, while quantum effects break it to the discrete group. Using this discrete version one identifies the scalars  $N_{\mathcal{A}}$  to parameterize a complex torus

$$\mathbb{T}_M^{2h^{2,1}(Y_4)} = \frac{H^{2,1}(Y_4)}{H^3(Y_4, \mathbb{Z})}, \quad (3.23)$$

<sup>16</sup>In [21] it was assumed that a basis can be chosen such that  $\beta^A \wedge \beta^B = 0$  in cohomology. While this simplifies the computations significantly and is compatible with the weak coupling limit, it needs not necessarily be imposed in general.

with a complex structure encoded by the function  $f_{AB}$ . Since  $f_{AB}$  and  $N_A$  vary with  $z^{\mathcal{K}}$ , this torus is non-trivially fibered over the complex structure moduli space. This is reminiscent of the complex tori discussed in (2.29), since one of the  $z^{\mathcal{K}}$  of the Calabi-Yau fourfold will translate to the  $\tau$  in the orientifold limit. However, the three-dimensional action (3.19) with (3.20) is not yet in the correct duality frame in order to make the connection with the four-dimensional F-theory setting manifest.

We will turn to the dualization and the match with a four-dimensional theory in the next subsection. Before doing this, let us point out another interesting feature of the above formulation. It is not difficult to check that the kinetic potential (3.20) is not invariant under (3.22), but rather transforms as

$$\delta\tilde{K} = -\frac{1}{2}L^\Sigma \operatorname{Re} \left[ Q_{\Sigma\mathcal{A}}{}^{\mathcal{B}}(N_{\mathcal{B}} + \delta N_{\mathcal{B}})\tilde{\lambda}^{\mathcal{A}} + (M_{\Sigma\mathcal{A}}{}^{\mathcal{B}}\tilde{\lambda}^{\mathcal{A}} + M_{\Sigma}{}^{\mathcal{A}\mathcal{B}}\lambda_{\mathcal{A}})N_{\mathcal{B}} \right]. \quad (3.24)$$

However, we check that this transformation yields a boundary term in the action and can therefore be neglected. The reason for this fact is that, in general, the kinetic potential in (3.19) is unique up to

$$\delta\tilde{K} = \operatorname{Re} g(\phi) + L^\Sigma \operatorname{Re} h_\Sigma(\phi), \quad (3.25)$$

where  $g(\phi), h_\Sigma(\phi)$  are holomorphic functions of  $\phi_{\hat{A}}$ . Indeed, using that  $f_{AB}$  is holomorphic in  $z^{\mathcal{K}}$  this is precisely what happens in (3.24). While being in three dimensions, we have thus found a natural set of holomorphic functions in our setting. As we will see later, these play a key role in the up-lift to four dimensions and indeed reappear in the holomorphic gauge coupling function.

### 3.3 Dualization of fields to the F-theory frame

The previous reduction is valid for any smooth Calabi-Yau fourfold. In order to have an F-theory background, we have to restrict to the cases in which  $Y_4$  is elliptically fibered, which imposes certain conditions on the geometric data. In turn, these translate into restrictions on the three-dimensional effective action that ensure that it comes from the compactification of a four-dimensional theory on a circle. This is expected from the M-theory to F-theory duality and the main tool to infer information about F-theory effective actions. However, performing the  $Y_4$  reduction as in subsection 3.1 the resulting three-dimensional theory is generally not in the correct duality frame to lift it to a four-dimensional theory, so a Hodge star duality is usually required. Before going into the details of the dualization, let us illustrate this with an example.

Consider a massless chiral multiplet  $\hat{\Phi}$  and a massless vector multiplet  $\hat{A}$  of a four-dimensional  $\mathcal{N} = 1$  supersymmetric theory that cannot be dualized into each other. When we dimensionally reduce on a circle, we find that the chiral multiplet gives an  $\mathcal{N} = 2$  chiral multiplet  $\Phi$  in three dimensions and the vector  $\hat{A}$  yields an  $\mathcal{N} = 2$  vector multiplet, that consists of a three-dimensional vector field  $A$  together with a real scalar  $a$ . Since the vector field  $A$  is massless, it can be dualized to a real scalar  $\tilde{a}$  which, together with  $a$ , corresponds to a chiral multiplet  $\Phi_A$ . Conversely, we can also dualize the chiral multiplet  $\Phi$  into a vector multiplet if it appears in the three-dimensional action with a real continuous shift symmetry. In general, after performing such a dualization, we can no longer lift the theory

back to four dimensions. Thus, if we start with an  $\mathcal{N} = 2$  three-dimensional theory (with massless scalars and vectors) and wish to lift it to four dimensions, we first have to make sure we are in the correct duality frame.

In our case, the structure of the elliptic fibration, together with the expectations from Type IIB compactifications, is enough to find the correct frame. Following [13, 45], we split the three- and two-forms as

$$\Psi^{\mathcal{A}} = (\Psi^A, \Psi^\kappa), \quad \omega_\Sigma = (\omega_{\hat{i}}, \omega_\alpha), \quad (3.26)$$

where  $\Psi^\kappa$  correspond to three-forms on the base of the fibration and  $\Psi^A$  have components on the fiber. Similarly, the two-forms  $\omega_\alpha$ , which are dual to vertical divisors, come from the base whereas  $\omega_{\hat{i}}$  do not. In particular, the latter can be further split as

$$\omega_{\hat{i}} = (\omega_0, \omega_i), \quad (3.27)$$

where  $\omega_0$  is dual to the base and  $\omega_i$  include the exceptional divisors and the extra sections. We can give a rough characterization of these forms by counting how many ‘legs’ their components have in the elliptic fiber. In fact,  $\omega_\alpha, \Psi^\kappa$  have no legs in the elliptic fiber.  $\Psi^A, \omega_i$  have generically components with one and zero legs in the elliptic fiber, while  $\omega_0$  has generically components with two, one and zero legs in the elliptic fiber. In order to have a non-vanishing coupling depending on an  $Y_4$ -integral over the above forms, one has to have a wedge-product of forms that admits at least some components with two legs along the elliptic fiber. One thus immediately finds the vanishing conditions

$$\mathcal{K}_{\alpha\beta\gamma\delta} = 0, \quad \mathcal{K}_{i\alpha\beta\gamma} = 0, \quad M_{\alpha\kappa}{}^{\mathcal{A}} = M_{\alpha\mathcal{A}}{}^\kappa = M_\alpha{}^{\kappa\mathcal{A}} = M_i{}^{\kappa\lambda} = 0, \quad (3.28)$$

The intersections  $M_0{}^{\kappa\lambda}$  and  $M_{0\kappa}{}^\lambda$  are in general non-vanishing. However, we can always chose a special three-form basis  $(\alpha_\kappa, \beta^\kappa)$  such that

$$M_0{}^{\kappa\lambda} = 0, \quad M_{0\kappa}{}^\lambda = \delta_\kappa^\lambda, \quad (3.29)$$

The split of the forms induces a split of the different fields as follows

$$N_{\mathcal{A}} = (N_A, N_\kappa), \quad L^\Sigma = (L^{\hat{i}}, L^\alpha), \quad A^\Sigma = (A^{\hat{i}}, A^\alpha). \quad (3.30)$$

On the one hand, the complex fields  $N_\kappa$  lift to a four-dimensional vectors  $A^\kappa$  (R-R vectors) and so have to be dualized. On the other hand, the scalars  $N_A$  correspond to both the  $G^a$  moduli and 7-brane Wilson lines, so they remain as scalars. Regarding the three-dimensional vector multiplets, the  $(A^\alpha, L^\alpha)$  lift to the four-dimensional complex scalars  $T_\alpha$ , so  $A^\alpha$  should be dualized into a scalar. Finally, the vectors  $(A^{\hat{i}}, L^{\hat{i}})$  include the 7-brane vectors as well as the Kaluza-Klein vector coming from the reduction of the metric, so they are not dualized.

We are now ready to perform the dualization that brings the action (3.19) into the appropriate frame to lift to four dimensions. As usual, this can be done in a manifestly supersymmetric way by performing a Legendre transform of the kinetic potential  $\tilde{K}$  (see appendix A for a detailed discussion). In order to dualize the scalars  $N_\kappa$  into vectors, we

need to make sure that the kinetic potential does not depend on  $\text{Im } N_\kappa$ . At first, this is not the case for  $\tilde{K}$  given in (3.20). However, we may remove such a dependence by performing a transformation of the form (3.25), which yields

$$\begin{aligned} \tilde{K} = & -\log\left(\int_{Y_4} \Omega \wedge \bar{\Omega}\right) + \log \mathcal{V} + L^\Sigma \text{Re } d_\Sigma{}^{AB} \text{Re } N_A \text{Re } N_B \\ & + L^\Sigma (2 \text{Im } d_\Sigma{}^{\kappa A} \text{Re } N_\kappa \text{Im } N_A + \text{Im } d_\Sigma{}^{AB} \text{Re } N_A \text{Im } N_B). \end{aligned} \quad (3.31)$$

We denote the dual kinetic potential by  $K(N_A, T_\alpha, z^\mathcal{K}|L^\hat{\ell}, n^\kappa)$  and is given by

$$K(N_A, T_\alpha, z^\mathcal{K}|L^\hat{\ell}, n^\kappa) = \tilde{K}(N_A, N_\kappa, z^\mathcal{K}|L^\hat{\ell}, L^\alpha) - L^\alpha \text{Re } T_\alpha - \text{Re } N_\kappa n^\kappa, \quad (3.32)$$

where the new variables are defined as

$$\text{Re } T_\alpha \equiv \frac{\partial \tilde{K}}{\partial L^\alpha}, \quad n^\kappa \equiv \frac{\partial \tilde{K}}{\partial \text{Re } N_\kappa}, \quad (3.33)$$

The dualized action can then be derived by inserting (3.32) and (3.33) into the general action (3.19). Notice that  $\text{Re } N_\kappa$  and  $L^\alpha$  in (3.32) should be understood as functions of  $L^\hat{\ell}, N_A, \text{Re } T_\alpha$ , and  $n^\kappa$ . This requires inverting the maps (3.33), which can be done explicitly for  $\text{Re } N_\kappa$ . We find the identify

$$\text{Re } N_\kappa = \text{Re } d_{\kappa\lambda} \left( \frac{1}{2} n^\lambda - L^\hat{\ell} \text{Re} \left[ d_i{}^{A\lambda} N_A \right] \right), \quad (3.34)$$

where  $\text{Re } d_{\lambda\kappa}$  is defined as the inverse of  $L^\hat{\ell} \text{Re } d_i{}^{\lambda\kappa}$ . For the complex scalars  $T_\alpha$  we only find an implicit expression given by

$$\text{Re } T_\alpha = \frac{\partial}{\partial L^\alpha} \log \mathcal{V} + \text{Re} [d_\alpha{}^{AB} N_A] \text{Re } N_B, \quad (3.35)$$

This implicit form of the coordinates and kinetic potential is familiar already from the orientifold setting (2.13) and (2.17). However, it should be stressed that the M-theory result is more involved, since it contains the scalars  $L^\hat{\ell}, n^\kappa$  such that  $K$  is not a Kähler potential.

Determining the dual Lagrangian is technically involved but straightforward. In order to do that, we have to compute the derivatives of  $K(N_A, T_\alpha, z^\mathcal{K}|L^\hat{\ell}, n^\kappa)$  and express them in terms of derivatives of the original kinetic potential  $\tilde{K}(N_A, N_\kappa, z^\mathcal{K}|L^\hat{\ell}, L^\alpha)$ . The details of this computation are summarized in appendix A.

### 3.4 Symmetries of the dual Lagrangian

Before we continue analyzing the three-dimensional Lagrangian, let us first discuss the symmetries of the dual Lagrangian. For the original Lagrangian, we found a set of Abelian symmetries given by (3.21) and (3.22), so one might think that the symmetries of the dual Lagrangian are also Abelian. However, this is not the case [15], which can be traced back to the existence of a Chern-Simons term in the eleven-dimensional supergravity action. In the democratic formulation we find that, due to the Chern-Simons term, the large gauge

transformations of the three-form and the dual six-form potentials are not independent, but rather given by

$$\delta\hat{C}_3 = \omega_3, \quad \delta\hat{C}_6 = \omega_6 - \frac{1}{2}\omega_3 \wedge \hat{C}_3, \quad (3.36)$$

with  $\omega_3$  and  $\omega_6$  integral closed forms. Upon dimensional reduction of the democratic action, one can check that the symmetries may be Abelian or non-Abelian, depending on how one eliminates the redundant degrees of freedom. A detailed field theory analysis in arbitrary dimension of this fact can be found in [44].

Explicitly we can investigate the symmetries of the dual Lagrangian by translating the ones (3.21) and (3.22) from the original one into this new frame. In addition, one directly checks the new symmetries of the vectors  $A^\kappa$  by using (3.19) with (3.32) and shows perfect match with the symmetries of  $n^\kappa$  as expected by supersymmetry. The set of gauge and global symmetries is then found to be

$$\begin{aligned} \delta N_A &= i\lambda_A + f_{AA}\tilde{\lambda}^A, \\ \delta T_\alpha &= i\lambda_\alpha - \frac{1}{2}\tilde{\lambda}^A Q_{\alpha A}{}^B (N_B + \delta N_B) - \frac{1}{2}(M_\alpha{}^{AB}\lambda_A + M_{\alpha A}{}^B\tilde{\lambda}^A)N_B, \\ \delta n^\kappa &= -L^{\hat{i}}(M_{\hat{i}A}{}^\kappa\tilde{\lambda}^A + M_{\hat{i}}{}^{A\kappa}\lambda_A), \\ \delta A^\kappa &= d\Lambda^\kappa - A^{\hat{i}}(M_{\hat{i}A}{}^\kappa\tilde{\lambda}^A + M_{\hat{i}}{}^{A\kappa}\lambda_A), \\ \delta A^{\hat{i}} &= d\Lambda^{\hat{i}}, \end{aligned} \quad (3.37)$$

where  $\lambda_\alpha, \lambda_A, \tilde{\lambda}^A$  are arbitrary real constants and  $\Lambda^\kappa, \Lambda^{\hat{i}}$  are arbitrary real functions. Notice that the right hand side of  $\delta N_A, \delta T_\alpha$  is holomorphic and that the transformation is valid for finite values of  $\lambda_\alpha, \lambda_A$ , and  $\tilde{\lambda}^A$ .

The symmetry group is now non-Abelian and, in particular, it is a generalization of the Heisenberg group. Notice also that, unlike for the original Lagrangian, the symmetries of the scalars and vectors are mixed. This can be seen from the transformation rule for  $A^\kappa$ , that depends on  $\tilde{\lambda}^A$  and  $\lambda_A$ , inducing a *constant* change of basis in the space of U(1)'s (see also [42]). This necessarily implies that the gauge coupling function must depend on the scalars and transform under the symmetries appropriately in order to make the whole Lagrangian invariant. Furthermore, if we were to gauge the global (non-Abelian) symmetry by promoting  $\tilde{\lambda}^A$  and  $\lambda_A$  to be arbitrary functions, we find that the transformation of the vectors is no longer constant and precisely matches that of a non-Abelian vector field [15].

More explicitly, in order that the three-dimensional kinetic terms are invariant under (3.37), i.e.

$$\delta(K_{\mathcal{I}\mathcal{J}}F^{\mathcal{I}} \wedge \star F^{\mathcal{J}}) = 0, \quad (3.38)$$

the three-dimensional gauge kinetic terms should transform, for finite  $\tilde{\lambda}^A = (\tilde{\lambda}^A, \tilde{\lambda}^\kappa)$  and  $\lambda_A$ , as

$$K_{\mathcal{I}\mathcal{J}} \longrightarrow m_{\mathcal{I}}^{\mathcal{K}} m_{\mathcal{J}}^{\mathcal{L}} K_{\mathcal{K}\mathcal{L}}, \quad (3.39)$$

with

$$m_{\mathcal{I}}^{\mathcal{J}} = \begin{pmatrix} \delta_{\lambda}^{\kappa} & M_{\hat{j}A}{}^{\kappa}\tilde{\lambda}^A + M_{\hat{j}}{}^{A\kappa}\lambda_A \\ 0 & \delta_{\hat{j}}^{\hat{i}} \end{pmatrix}. \quad (3.40)$$

Here the indices  $\mathcal{I}, \mathcal{J}, \dots$  run over all the three-dimensional vectors, namely  $A^\kappa$  and  $A^{\hat{i}}$ .

As we will see in the next section, the couplings  $M_{iA}{}^\kappa$  and  $M_j{}^{A\kappa}$  are related to kinetic mixing of 7-brane and bulk gauge fields, while  $M_{0I}{}^\kappa$  and  $M_0{}^{A\kappa}$  have no immediate four-dimensional meaning. We would like to stress at this point that, in three dimensions, the coefficient of the kinetic terms of the vectors  $K_{\mathcal{I}\mathcal{J}}$  is invariant if and only if the kinetic mixing is zero and  $M_{0A}{}^\kappa$  and  $M_0{}^{A\kappa}$  vanish. This carries over to a property of the four-dimensional gauge coupling function, as we show in the following.

#### 4 Determining F-theory gauge coupling functions

Having determined the three-dimensional action in the correct duality frame, we can compare it with the circle reduction of an arbitrary four-dimensional action. As shown in appendix B, the circle reduction of a four-dimensional  $\mathcal{N} = 1$  supergravity action (2.1) yields a three-dimensional  $\mathcal{N} = 2$  supergravity given by (3.19), with kinetic potential

$$K(M|R, \xi) = \hat{K}(M, \bar{M}) + \log R - \frac{1}{2R} \text{Re} \hat{f}_{\mathbf{I}\mathbf{J}}(M) \xi^{\mathbf{I}} \xi^{\mathbf{J}}. \quad (4.1)$$

Here we set  $R = r^{-2}$ , with  $r$  being the radius of the circle, and introduced the scalars  $\xi^{\mathbf{I}}$  that come from reducing the four-dimensional vector fields. The index  $\mathbf{I}$  runs over the four-dimensional vector fields and is split as  $\{\kappa, i\}$ . From now on, we denote four-dimensional quantities by a hat.

##### 4.1 Transformation rules of the gauge coupling functions

Before we proceed to compare the result obtained from the (dualized) M-theory reduction with a generic four-dimensional theory on a circle, let us discuss the transformation properties of the four-dimensional gauge coupling functions. In the last section we saw that, in general, the kinetic terms of the three-dimensional vectors  $K_{\mathcal{I}\mathcal{J}}$  transform under the shift-symmetries of the scalars. Clearly, the four dimensional gauge coupling function shares a similar property. Indeed, consider the ansatz for a four-dimensional vector on a circle, namely

$$\hat{A}^{\mathbf{I}} = A^{\mathbf{I}} - \frac{\xi^{\mathbf{I}}}{R} (dy + A^0), \quad (4.2)$$

where  $dy$  is the non-trivial one-form on the circle. We also introduced the Kaluza-Klein vector  $A^0$  coming from the reduction of the metric on a circle, namely

$$d\hat{s}^2 = ds^2 + \frac{1}{R} (dy + A^0)^2. \quad (4.3)$$

Using (3.37) together with (4.2), we find that the transformation of the four-dimensional vector on  $\mathbb{R}^{1,2} \times S^1$  is

$$\delta \hat{A}^\kappa = d\Lambda^\kappa - A^i (M_{iA}{}^\kappa \tilde{\lambda}^A + M_i{}^{A\kappa} \lambda_A) + (M_{0I}{}^\kappa \tilde{\lambda}^I + M_0{}^{A\kappa} \lambda_A) dy, \quad (4.4a)$$

$$\delta \hat{A}^i = d\Lambda^i, \quad (4.4b)$$

where we used that  $L^0 = R$  and  $\frac{L^i}{R} \rightarrow 0$ . Since  $dy$  is the non-trivial one-form on  $S^1$ , we recognize the last term in (4.4a) as a large gauge transformation. These transformations

along the circle are often key in investigating the properties of the F-theory effective action as recently demonstrated in [49, 50] for 7-brane gauge fields. Here we find a non-trivial completion of these transformations to include R-R bulk gauge fields. In the decompactification limit, large gauge transformations are meaningless since there are no non-trivial one-forms in  $\mathbb{R}^{1,3}$ . Thus, we find that the transformation of the vectors in  $\mathbb{R}^{1,3}$  is

$$\begin{aligned}\delta\hat{A}^\kappa &= d\hat{\Lambda}^\kappa - A^i(M_{iA}{}^\kappa\tilde{\lambda}^A + M_i{}^{A\kappa}\lambda_A), \\ \delta\hat{A}^i &= d\hat{\Lambda}^i,\end{aligned}\tag{4.5}$$

where now  $\hat{\Lambda}^\kappa$  and  $\hat{\Lambda}^i$  are arbitrary functions in four-dimensions. This shows that, under shifts of the four-dimensional scalars  $\hat{N}_A$ , the vectors  $\hat{A}^\kappa$  transform non-trivially only when  $M_{iA}{}^\kappa$  or  $M_i{}^{A\kappa}$  are different from zero. This is the M-theory derivation of the result given in eq. (2.56).

Now from (4.5) and (2.1), we can readily determine the transformation rules for the four-dimensional gauge coupling function  $\hat{f}_{\mathbf{IJ}}$ , namely

$$\hat{f}_{\mathbf{IJ}} \longrightarrow \hat{m}_{\mathbf{I}}^{\mathbf{K}} \hat{m}_{\mathbf{J}}^{\mathbf{L}} \hat{f}_{\mathbf{KL}} + iC_{\mathbf{IJ}},\tag{4.6}$$

with

$$\hat{m}_{\mathbf{I}}^{\mathbf{J}} = \begin{pmatrix} \delta_\lambda^\kappa & M_{jA}{}^\kappa\tilde{\lambda}^A + M_j{}^{A\kappa}\lambda_A \\ 0 & \delta_j^i \end{pmatrix},\tag{4.7}$$

We included the possibility of having a constant shift  $C_{\mathbf{IJ}}$  in  $\text{Im}\hat{f}_{\mathbf{IJ}}$ . Splitting the indices, this corresponds to

$$\begin{aligned}\delta\hat{f}_{\kappa\lambda} &= iC_{\kappa\lambda} \\ \delta\hat{f}_{i\kappa} &= p_i^\kappa\hat{f}_{\kappa\lambda} + iC_{\kappa\lambda} \\ \delta\hat{f}_{ij} &= p_i^\kappa\hat{f}_{j\kappa} + p_j^\kappa\hat{f}_{i\kappa} + p_i^\kappa p_j^\lambda\hat{f}_{\kappa\lambda} + iC_{ij},\end{aligned}$$

with

$$p_i^\kappa = M_{iA}{}^\kappa\tilde{\lambda}^A + M_i{}^{A\kappa}\lambda_A.\tag{4.8}$$

Finally, notice that when  $M_{iA}{}^\kappa = M_i{}^{A\kappa} = 0$ , we find that the gauge coupling function must be invariant, up to possibly constant shifts of its imaginary part. In the following we will see that this corresponds to the case in which the kinetic mixing between the four-dimensional vectors  $\hat{A}^\kappa$  and  $\hat{A}^i$  vanishes.

## 4.2 Gauge coupling functions from dimensional reduction

In the following, we compare the action derived from the kinetic potential (3.32) with the one derived from (4.1), paying special attention to the gauge coupling function. In order to do so, we will need the derivatives of the dual kinetic potential  $K(N_A, T_\alpha, z^\mathcal{K}|L^i, n^\kappa)$ , which are given in appendix A.

### 4.2.1 On the weak string-coupling limit

In addition to presenting the F-theory result we will also study the restriction to the weak string-coupling limit discussed in section 2. In order to do that it is useful to point out the matching of the moduli. First, note that the complex structure moduli  $z^{\mathcal{K}}$  of  $Y_4$  correspond to the complex structure moduli of the double cover  $Y_3$  of  $B_3$ , the axio-dilaton  $\tau$ , and the D7-brane deformations  $\zeta_K$ :

$$z^{\mathcal{K}} \xrightarrow{\text{weak coupl.}} z^k, \tau, \zeta_K, \tag{4.9}$$

which are the fields in the Set 1 given in (2.11).<sup>17</sup> Second, the F-theory moduli  $N_A$  are naturally split as

$$N_A \xrightarrow{\text{weak coupl.}} (a_p, G^a), \tag{4.10}$$

where  $a_p$  are the D7-brane Wilson line moduli and  $G^a$  are the R-R and NS-NS two-form moduli constituting the Set 2 given in (2.12). Third, recalling the result (2.31) and the definitions (2.7), (2.12) we note that one identifies<sup>18</sup>

$$\begin{pmatrix} f_{\kappa\lambda} & f_{\kappa B} \\ f_{A\lambda} & f_{AB} \end{pmatrix} \xrightarrow{\text{weak coupl.}} \begin{pmatrix} -\mathcal{F}_{\kappa\lambda}|_{z^k=0} & 0 \\ 0 & (f_{pq}, -i\tau\delta^{ab}) \end{pmatrix}, \tag{4.11}$$

where we stress that  $\mathcal{F}_{\kappa\lambda}$  and  $f_{pq}$  are only determined as functions of the complex structure moduli of  $Y_3$ . The F-theory result is significantly more general, since it encodes the full dependence on all complex structure moduli  $z^{\mathcal{K}}$  of  $Y_4$ . Applying the split (4.9) it can be used to derive corrections to the orientifold result.

### 4.2.2 Gauge coupling function for R-R vectors

Let us start with the derivation of the four-dimensional gauge coupling function for the R-R vectors, namely  $\hat{f}_{\kappa\lambda}$ . From the results in appendix B, we immediately see that the real part of the gauge coupling function is encoded in  $K_{\kappa\lambda}$ , which is the kinetic term for the three-dimensional vectors  $A^\kappa$ . According to eq. (A.13), it is given by

$$K_{\kappa\lambda} = \frac{1}{R} \text{Re } f_{\kappa\lambda}, \tag{4.12}$$

where we assumed that

$$f_{\kappa A} = 0, \quad L^0 = R. \tag{4.13}$$

These assumptions appear to be essential. They greatly simplify the results and, in particular, they make (4.12) into the real part of a holomorphic function, which matches the expectations from the Type IIB perspective. Thus, we will assert that (4.13) holds for the rest of the paper. It would be interesting to show that the vanishing condition  $f_{\kappa A} = 0$  can be proved for elliptic fibrations.

<sup>17</sup>Note that we have not included  $\zeta_K$  in the orientifold analysis. In F-theory a general  $z^{\mathcal{K}}$ -dependence automatically includes these moduli.

<sup>18</sup>The identification of  $f_{\kappa\lambda}$  with (2.31) will become apparent in the next paragraphs.

The computation of the imaginary part of the four-dimensional gauge coupling function is a bit more involved. However, by carefully tracking the circle reduction, we see that it is encoded in the three-dimensional action in the couplings

$$F^\kappa \wedge \text{Im}(K_\kappa^{\hat{A}} d\phi_{\hat{A}}) \quad (4.14)$$

in (3.19), where  $\hat{A}$  runs over all the chiral fields in three-dimensions. According to the results in appendix A, we find that

$$F^\kappa \wedge \text{Im}(K_\kappa^{\hat{A}} d\phi_{\hat{A}}) = \frac{n^\lambda}{2R} F^\kappa \wedge d\text{Im} f_{\kappa\lambda} + \frac{L^i}{2R} F^\kappa \wedge d\text{Im}(Q_{i\kappa}{}^a N_a). \quad (4.15)$$

In particular, the imaginary part of  $\hat{f}_{\kappa\lambda}$  is encoded in the coefficient that multiplies  $n^\lambda/R$  above. Thus, from (4.12) and (4.15), we conclude that the four-dimensional gauge coupling function for the R-R gauge bosons is given by

$$\hat{f}_{\kappa\lambda} = -f_{\kappa\lambda}, \quad (4.16)$$

which is holomorphic in the complex structure moduli of the Calabi-Yau fourfold, and therefore holomorphic with respect to the four-dimensional chiral fields. The result (4.16) is in accord with the expectations from the Type IIB orientifolds, cf. (2.31). However, it is important to note that the F-theory result (4.16) is significantly more general, since the function  $f_{\kappa\lambda}$  can depend on all complex structure moduli of  $Y_4$ .

### 4.2.3 Kinetic mixing between R-R and 7-brane vectors

Now we move on to considering the kinetic mixing  $\hat{f}_{\kappa i}$  between the open and closed string gauge bosons. From the circle reduction, we see that  $\text{Re} \hat{f}_{\kappa i}$  is encoded in  $K_{\kappa i}$ , the three-dimensional kinetic mixing between  $A^\kappa$  and  $A^i$ . We find that the M-theory reduction yields

$$K_{\kappa i} = \frac{1}{R} \text{Re} [Q_{i\kappa}{}^A N_A], \quad (4.17)$$

where  $Q_{i\kappa}{}^A$  is the holomorphic function defined in (3.16). Notice that (4.17) is again the real part of a holomorphic function of the complex moduli. This also shows that the mixing is proportional to the couplings  $M_{i\kappa}{}^A$  and  $M_i{}^{A\kappa}$ , which are related to the ones that appear in (4.5) by the identity (3.18). This proves the statement in the last section that the transformation for the vector is trivial if and only if the mixing vanishes.

Just like in the previous case, we can compute the imaginary part of the mixing  $\text{Im} \hat{f}_{\kappa i}$  by analyzing (4.15). In this case, it is given by the term proportional to  $L^i/R$ . Thus, we find that

$$\hat{f}_{\kappa i} = -Q_{i\kappa}{}^A N_A = -(M_{i\kappa}{}^A + i f_{\kappa\lambda} M_i{}^{\lambda A}) N_A, \quad (4.18)$$

which is holomorphic in both the complex structure moduli  $z^\mathcal{K}$  and the moduli  $N_A$ .

The identification (4.18) agrees with the result given in section 2.5, when asserting that  $M_i{}^{\lambda A}$  is only non-vanishing for the directions of the Wilson line moduli  $a_p$ . However, let us stress again that in order to match it with the results obtained in [6] from dimensional

reduction of the D7-brane action we had to use heavily the identities (3.18), which were not known in the Type IIB context (see the discussion around eq. (2.55)).

Let us briefly mention that we can compute the mixing between the Kaluza-Klein vector and the R-R vectors, which is

$$K_{\kappa 0} = -\frac{1}{R^2} \left( n^\lambda \text{Re } f_{\kappa\lambda} + L^i \text{Re } [Q_{i\kappa}^A N_A] \right). \quad (4.19)$$

Of course, this has no meaning in four dimensions. However, it is reassuring to check that it is what one would expect from a theory that comes from a circle reduction, given (4.12) and (4.17).

#### 4.2.4 Gauge coupling function for 7-brane vectors

Finally, let us discuss the gauge coupling function  $\hat{f}_{ij}$  for the seven-brane gauge fields that, as we saw in section 2.3, is the most involved coupling. In particular, we do not expect to obtain a holomorphic gauge coupling function  $\hat{f}_{ij}$  directly from dimensional reduction. In the following we simply give the result that we obtain from dimensional reduction and in the next subsection we then discuss how one can use holomorphicity and the discrete shift-symmetries of the axions to constrain the exact result.

Following the same strategy as before, we see that  $\text{Re } \hat{f}_{ij}$  is given by  $K_{ij}$ , the three-dimensional kinetic terms for the 7-brane gauge bosons. There is, however, a further complication when discussing this coupling that has to be addressed. As shown in appendix A, in terms of the original kinetic potential  $\tilde{K}$ , it reads

$$K_{ij} = \tilde{K}_{ij} - \tilde{K}_{i\alpha} \tilde{K}_{j\beta} (\tilde{K}_{\alpha\beta})^{-1} - \tilde{K}_i^\kappa \tilde{K}_j^\lambda (\tilde{K}^{\kappa\lambda})^{-1}, \quad (4.20)$$

with  $\tilde{K}$  given by (3.31). Thus, we immediately see that  $K_{ij}$  depends on all the possible intersection numbers (3.10), but we do not expect all of them to contribute to the gauge coupling function in four dimensions. In particular, the couplings  $\mathcal{K}_{ijkl}$  and  $\mathcal{K}_{ijk\alpha}$  induce a dependence of  $K_{ij}$  on the scalars  $L^i$ , which have no four-dimensional scalar analog. This suggests that, just like in [51–53], the classical M-theory reduction contains terms that correspond to one-loop effects from the circle reduction of the four-dimensional theory. However, notice that unlike in [51–53], we are performing a dimensional reduction without fluxes, so the four-dimensional theory is non-chiral in our case. Thus, the smooth Calabi-Yau fourfold encodes information about non-chiral states. We leave a more detailed study of these corrections and their interpretation for future work.

In order to match the classical circle reduction, we will compute the coupling (4.20) assuming that the only non-vanishing intersection numbers are

$$\mathcal{K}_{0\alpha\beta\gamma} \equiv \mathcal{K}_{\alpha\beta\gamma}, \quad \mathcal{K}_{\alpha\beta ij} \equiv -C_{ij}^\gamma \mathcal{K}_{\alpha\beta\gamma}, \quad (4.21)$$

where  $\mathcal{K}_{\alpha\beta\gamma}$  are the intersection numbers of the two-forms on the base of the elliptic fibration. We also expressed the intersection numbers  $\mathcal{K}_{\alpha\beta ij}$  in terms of those of the base. The precise interpretation of the divisors labeled with indices  $i, j$  depends on the model under consideration. The first possibility is that  $i, j$  are labeling exceptional divisors over

a single non-Abelian 7-brane wrapping a divisor  $S$  in the base  $B_3$ . In this case one can expand the Poincaré-dual two-form as  $[S] = \delta_7^\alpha \omega_\alpha|_{B_3}$  and split  $C_{ij}^\alpha = \delta_7^\alpha C_{ij}$ , where  $C_{ij}$  is the Cartan matrix of the non-Abelian gauge algebra.<sup>19</sup> A second possibility is that the indices  $i, j$  label multiple  $U(1)$  gauge factors stemming from several 7-branes on different divisors in  $B_3$ . In this case it is convenient to keep  $C_{ij}^\alpha$  in this general form, since this allows us to include kinetic mixing among the 7-brane  $U(1)$ 's. In either case, we compute to linear order in  $C_{ij}^\alpha$  that

$$K_{ij} = -\frac{3}{R\mathcal{K}^{\mathbf{b}}} C_{ij}^\alpha \mathcal{K}_\alpha^{\mathbf{b}} + \frac{1}{R} \text{Re}[d_i^{A^\kappa} N_A] \text{Re}[d_j^{B^\lambda} N_B] \text{Re} f_{\kappa\lambda}, \quad (4.22)$$

where we defined

$$\mathcal{K}^{\mathbf{b}} = \mathcal{K}_{\alpha\beta\gamma} L^\alpha L^\beta L^\gamma, \quad \mathcal{K}_\alpha^{\mathbf{b}} = \mathcal{K}_{\alpha\beta\gamma} L^\beta L^\gamma. \quad (4.23)$$

In this expression, on the one hand, the first term in (4.22) is proportional to the volumes of the divisors in  $B_3$  specified by  $C_{ij}^\alpha$ . From the Type IIB perspective this corresponds to the fact that the gauge coupling scales with the volumes of the cycles wrapped by the 7-branes. The second term, on the other hand, is proportional to the couplings  $M_{iA^\kappa}$  and  $M_{jB^\lambda}$  and, in particular, vanishes when there is no mixing between  $A^\kappa$  and  $A^i$ . Notice that, as expected from the Type IIB discussion, (4.22) is *not* the real part of a holomorphic function of the chiral fields, even in the absence of mixing. Indeed, from (3.33) we have that

$$C_{ij}^\alpha \text{Re} T_\alpha = \frac{3}{\mathcal{K}^{\mathbf{b}}} C_{ij}^\alpha \mathcal{K}_\alpha^{\mathbf{b}} + C_{ij}^\alpha \text{Re}[d_\alpha^{BA} N_B] \text{Re} N_A, \quad (4.24)$$

which contains a term proportional to the square of  $N_A$  that is missing in (4.22). This is precisely the same problem we encountered in the Type IIB setting of section 2.3, where the contribution proportional to the square of the Wilson lines does not arise from dimensional reduction.

Finally, let us mention that the second term in (4.22) is holomorphic in  $N_A$  if and only if we have that

$$Q_{i\kappa}{}^A Q_{jB}{}^\kappa = 0. \quad (4.25)$$

However, this is not sufficient to guarantee holomorphicity in complex structure moduli  $z^{\mathcal{K}}$  of  $Y_4$ . In the following we discuss in detail the corrections that are needed, in four dimensions, to have a holomorphic gauge coupling function. However, we focus on the case without kinetic mixing of 7-brane and R-R gauge fields and leave the general case to future work.

### 4.3 Shift symmetries, quantum corrections, and theta functions

In the previous subsection we have shown that a direct dimensional reduction of eleven-dimensional supergravity on a smooth Calabi-Yau fourfold yields vector kinetic terms and a complex moduli space that appear to be incompatible with a reduced four-dimensional holomorphic gauge coupling function. In the absence of kinetic mixing the missing terms in the completion to a holomorphic result are of the form  $\text{Re}(d_\alpha^{BA} N_B) \text{Re} N_A$ . From our

---

<sup>19</sup>In order to have this simple identification one has to restrict to ADE gauge algebras.

detailed discussion of the orientifold setting in section 2, however, we should be alerted that this apparent conflict was already encountered for D7-brane Wilson line moduli. In fact, we recalled in subsections 2.3 and 2.4 that the corrections to the gauge coupling functions quadratic in the Wilson line moduli are only generated at one string-loop order and therefore are not found by a dimensional reduction of the tree-level D7-brane effective action. We observe that in F-theory effective actions derived via eleven-dimensional supergravity a similar feature occurs for all moduli  $N_A$ , i.e. both the Wilson line moduli and the R-R and NS-NS two-form moduli in the split (4.10). This implies that to ensure holomorphicity of the gauge coupling function in  $T_\alpha$  one needs to include in the M-theory reduction a quantum correction of the form

$$f_{ij}^{\text{quant}} = C_{ij}^\alpha d_\alpha^{BA} N_B \text{Re } N_A + \dots, \tag{4.26}$$

In the M-theory setting it is much harder to identify the origin of such a correction. One expects that it arises due to certain M2-brane states, by following the F-theory to M-theory duality, but it remains an open question how to make this more precise. As we will see in the following we can nevertheless infer non-trivial constraints on  $f_{ij}^{\text{quant}}$  by using symmetries and the expected holomorphicity properties of the effective theory. For simplicity we will only discuss the case without kinetic mixing in the rest of this work.

In order to proceed we begin by collecting a few observations supporting the fact that important corrections have to be missing in the reduction of the supergravity action. On the one hand, it is clear from the outset that the three-dimensional reduction result is invariant under all the shift-symmetries (3.37) even when choosing continuous parameters  $\lambda_A, \lambda_\alpha, \tilde{\lambda}^A$ . Since these symmetries are inherited from the eleven-dimensional action and unbroken throughout the classical reduction, there is simply no way how they could be broken. On the other hand, we have argued in subsection 2.4 that in the presence of the fields  $N_A$  the *continuous* symmetries  $\lambda_A, \tilde{\lambda}^A$  acting non-trivially on the holomorphic gauge coupling function in four dimensions are always broken. The *discrete* symmetries are, however, manifest when including 7-brane fluxes or quantum corrections resulting in a theta function on the complex torus spanned by  $N_A$ . We expect that this is equally the case for a full-fledged F-theory compactification, such that indeed corrections must be missing in the above dimensional reduction.

Note that in the M-theory background (3.2) we did not include any background fluxes  $\langle d\hat{C} \rangle$  on  $Y_4$ . This implies that the F-theory setting will not contain background fluxes either and, in particular, we did not consider 7-branes with world-volume fluxes. This implies that the manifestation of the discrete symmetries for the  $G^a$  moduli obtained in (2.23) by completing  $i^*B_2 - 2\pi\alpha'F$ , requires an extension of our M-theory analysis. In fact, it was argued in [54] that such orientifold fluxes are precisely the ones that correspond to so-called hypercharge fluxes in F-theory GUTs [55, 56]. They neither induce a D-term nor an F-term potential for the considered moduli, but nevertheless can, for example, break a non-Abelian gauge group. In our context they are crucial to make the discrete symmetries manifest. It is of enormous importance to understand the manifestation of these fluxes in the M-theory reduction in greater detail.

The second possibility encountered in subsection 2.4 was a manifestation of the discrete shift-symmetries by completing the first term in (4.26) with a theta function. In fact, note that the  $N_A$  span a complex torus  $\mathbb{T}_{\mathbb{F}}^{2n}$  of real dimension  $2n = 2(h^{2,1}(Y_4) - h^{2,1}(B_3))$ . Its complex structure is determined by the holomorphic function  $f_{AB}$  and by using our assumption  $f_{\lambda A} = 0$ , as given in (4.13), and the restriction to a setting without kinetic mixing this torus arises trivially in the split of  $\mathbb{T}_{\mathbb{M}}^{2h^{2,1}(Y_4)}$  defined in (3.23). We then define a line bundle  $\mathcal{L}$  on this torus analog to the one in subsection 2.4. Freezing the complex structure moduli of  $Y_4$  one defines the connection in holomorphic gauge

$$\mathfrak{A}_{ij}^h = \frac{i}{4} C_{ij}^\alpha (2M_{\alpha A}{}^B \operatorname{Re} f^{AC} \operatorname{Re} N_C + M_\alpha{}^{AB} N_A) dN_B, \quad (4.27)$$

such that  $\mathfrak{F}_{ij} = d\mathfrak{A}_{ij}^h$  is a (1,1)-form. Note that this expression is still in the three-dimensional Coulomb branch as indicated by the indices  $i, j$ . While the lift with a non-Abelian gauge group is more involved, one realizes that for a single U(1) gauge group factor one finds the generalization of (2.48). In the following we will restrict to this Abelian case and drop the indices  $i, j$ . Arguing as in subsection 2.4 one can use this connection in the non-holomorphic gauge and look for holomorphic sections

$$\Psi = \exp\left(C^\alpha d_\alpha{}^{BA} N_B \operatorname{Re} N_A\right) \Theta(N_A, z^{\mathcal{K}}), \quad (4.28)$$

Here, as in subsection 2.4,  $\Theta$  is a sum of Riemann theta functions with  $z^{\mathcal{K}}$ -dependent coefficients, in general.

Unfortunately, we do not know an M-theory argument how  $\Theta$  can be fully determined. In addition to the ambiguities in the complex structure dependent coefficients, one also faces the fact that fluxes should be properly included into (4.28). One might speculate that the some of the constants  $(\nu_a, \mu^a)$  determining the shifts in the theta functions (2.52) might admit an interpretation as fluxes. However, we also expect that the non-holomorphic pre-factor and hence the line bundle and connection become modified. It would be very interesting to investigate the proper inclusion of fluxes in future work.

## 5 Conclusions

In this paper we have studied the gauge coupling functions arising in  $\mathcal{N} = 1$  Type IIB orientifolds with D7-branes and F-theory. First, we have analyzed the result that one obtains from dimensional reduction of Type IIB supergravity coupled to the D7-brane action without kinetic mixing between the open and closed string gauge fields following [6]. We have seen that this does not yield a gauge coupling function which is holomorphic in the chiral coordinates and, therefore, has to be modified. As already mentioned in [6], one expects that corrections coming from open-string one-loop effects generate precisely the missing terms to establish a holomorphic result. However, an explicit computation of such corrections is very challenging and has only been done in a related setting in toroidal orbifolds [9]. We have shown that by carefully analyzing the shift-symmetries of the closed string and open string axions in the effective field theory, one can severely constrain the specific structure of such corrections even in generic Calabi-Yau vacua.

In Type IIB orientifolds we have discussed two mechanisms to ensure that the gauge coupling function  $\hat{f}_{D7}$  transforms appropriately under discrete shift-symmetries. On the one hand, we reviewed the inclusion of D7-brane world-volume flux to make the symmetries of the R-R and NS-NS two-form moduli  $G^a$  manifest. On the other hand, we have stressed that gauge coupling function in general also depends on the complex Wilson line moduli  $a_p$ , which also admit discrete shift-symmetries. In fact, they span a complex torus with complex structure determined by a function  $f_{pq}$ , which is itself holomorphic in the complex structure moduli. Then, by simply imposing holomorphicity and invariance under such symmetries, we obtain that the required one-loop corrections are encoded by a holomorphic section  $\Psi = \exp(\hat{f}_{D7}^{1\text{-loop}})$  of a certain line bundle defined on the torus spanned by the Wilson lines. Constructing the connection on this line bundle, such sections are then found to be comprised of a term quadratic in the Wilson lines, required for holomorphicity of the complete  $\hat{f}_{D7}$ , and a sum of Riemann theta functions with, in general, complex structure dependent coefficients. This form of the gauge coupling function is in agreement with the results in [9], even though in our setting the torus is in general not related to the compactification space.

It is important to stress that we did not unravel the precise physical interpretation of having to deal with holomorphic sections  $\Psi$  of the constructed line bundle. We were lead to this construction by holomorphicity and symmetries of the gauge coupling function, but we were not able to completely fix the choice of  $\Psi$  appearing in the gauge coupling function. Our construction, however, is reminiscent of the consideration first given in [32]. In this work the partition function of an M5-brane is constructed and a similar ambiguity of choosing the correct section had to be addressed. One might hope that the extensions of [32] to Type IIB supergravity with D-branes [33, 34] might shed new light on the significance of the choice of  $\Psi$  in our setting. It is also intriguing to point out that the complex structure dependence of  $\Psi$  might be fully constrained when identifying it as a wave-function of a quantum system along the lines of [57]. It would be interesting to check whether these ideas can be made more explicit for our setting.

Extending our analysis of the Type IIB orientifold setting we have also included the effects of kinetic mixing between D7-brane gauge fields and R-R gauge fields. In particular, we derived that when the mixing is non-zero, the gauge coupling function should not be invariant under the shift-symmetries, since these induce a constant change of basis in the space of gauge bosons, mixing open and closed string  $U(1)$ 's. Our systematic approach allowed us to clarify certain puzzles that appeared in [15]. In particular, we argued that it is indeed possible to gauge specific non-Abelian isometries by Abelian vectors even though the gauge coupling function is independent of such gaugings. The underlying structure is omnipresent in string theory models and stems from the fact that higher-degree R-R form potentials admit non-trivial symmetry transformations under lower-degree forms from the brane or bulk theory. It would be interesting to see whether the ideas to exploit the stringy symmetries for the axions and gauge fields can be generalized further.

In the second main part of the work, we have studied the gauge coupling function for genuine F-theory backgrounds via dimensional reduction of M-theory on a Calabi-Yau fourfold. One of the main advantages of this approach is that many of the moduli

that appear to be completely different from the Type IIB perspective, turn out to have a common origin in the Calabi-Yau fourfold. In addition to being also applicable away from weak string coupling, the F-theory settings also allow us (1) to fully include the dependence on the 7-brane position moduli, (2) derive interesting and useful relations between different moduli that are obscure in the IIB picture, and (3) provide geometric arguments for the properties of the various couplings in the bulk and 7-brane sector.

In order to investigate the gauge coupling function we have crucially extended the results in [13]. We performed the M-theory reduction in full generality and explained in detail the role of the elliptic fibration when performing the dualization to the F-theory frame. In doing so, we have paid special attention to the shift-symmetries of the axions coming from the M-theory three-form expanded into three-forms of the Calabi-Yau fourfold. We have shown explicitly that a direct reduction of eleven-dimensional supergravity at first only yields shift-symmetries that are Abelian. Due to the dualization of three-dimensional fields into the F-theory frame they become non-Abelian as already discussed in [15, 44]. As we have seen, this is a direct consequence of having a non-trivial Chern-Simons coupling in the eleven-dimensional supergravity action. Furthermore, it provides the M-theory origin of the more involved shift-symmetries in Type IIB compactifications.

We then determined the four-dimensional gauge coupling functions of the F-theory setting, by comparing the three-dimensional M-theory effective action with the circle reduction of a four-dimensional theory. As in the Type IIB orientifolds, the resulting gauge coupling function is at first not holomorphic. In fact, the reduction of eleven-dimensional supergravity does not capture any of the quadratic corrections in the  $T_\alpha$  coordinates determined from the scalar kinetic terms. This is compatible with the fact that the dimensional reduction does not break the continuous shift-symmetries and indicates that important quantum corrections are missed. However, by mimicking the arguments we made for the Wilson line moduli in Type IIB orientifolds, we derived that an appropriate correction to the F-theory gauge coupling function is again captured by holomorphic sections of a certain line bundle. Such sections include a quadratic correction required for holomorphicity in the  $T_\alpha$  coordinates, but also generally allow for as logarithm of a sum of Riemann theta functions with complex structure dependent coefficients. This line bundle and these theta functions are now defined on a complex torus spanned by the axions coming from the M-theory three-form that are not dualized into vector multiplets in the F-theory frame. This torus is thus a subspace of the complex torus  $H^{2,1}(Y_4)/H^3(Y_4, \mathbb{Z})$ , which also captures the degrees of freedom of the R-R bulk vector fields. A detailed study of this geometric object and its variation over the complex structure moduli space is therefore of key phenomenological interest. In this work we have already conjectured certain constraints on geometric data of elliptically fibered Calabi-Yau fourfolds. In particular, by demanding supersymmetry of the four-dimensional effective action, we have proposed that the function  $f_{\mathcal{AB}}$ , which is holomorphic in the complex structure moduli of  $Y_4$  and defined in (3.5), should satisfy some non-trivial relations. Our analysis has been done for a generic compactification space, without referring to a specific example. Thus, it would be interesting to analyze in detail different examples to check whether such relations are indeed satisfied.

Another interesting approach to derive the couplings relevant for the gauge coupling function in F-theory was presented in a series of papers [58–60]. It was shown in these

papers that the coefficient functions of the couplings of type  $F^4$ , where  $F$  is an eight-dimensional gauge field, satisfy certain Picard-Fuchs-type differential equations. It would be interesting to explore the relation of these findings to the results of this paper.

### Acknowledgments

We are grateful to Ralph Blumenhagen, Sebastian Greiner, Andreas Kapfer, Fernando Marchesano, Kepa Sousa, Stephan Stieberger, and Irene Valenzuela for useful discussions. This work was supported by a grant of the Max Planck Society.

## A Dualization of three-dimensional actions

In this appendix, we perform the dualization of  $\mathcal{N} = 2$  three-dimensional actions, where massless vectors and scalars are dual to each other. The dualization can be done explicitly by adding Lagrange multipliers to the action and integrating out the fields we want to dualize. In fact, this can be done in superspace, where the  $\mathcal{N} = 2$  supersymmetry is manifest, and which corresponds to a Legendre transform of the Kähler potential.

To illustrate this, let us consider the three-dimensional  $\mathcal{N} = 2$  action for a massless vector multiplet

$$S_1(V) = \int d^3x d^2\theta d^2\bar{\theta} \tilde{K}(G(V)), \tag{A.1}$$

where  $G$  is the linear multiplet ( $D^2G = \bar{D}^2G = 0$ ) that contains the field strength, namely  $G = \frac{i}{2} \bar{D}^\alpha D_\alpha V$ . To perform the duality transformation, we consider the parent action given by

$$S_P(G, \Phi) = \int d^3x d^2\theta d^2\bar{\theta} \left( \tilde{K}(G) - G \operatorname{Re} \Phi \right), \tag{A.2}$$

where  $G$  is now an unconstrained real superfield and  $\Phi$  is a chiral superfield. By varying  $\Phi$ , we find that  $G$  is a linear superfield and substituting this in the action, we obtain (A.1). On the other hand, varying the action with respect to  $G$  gives

$$\operatorname{Re} \Phi = \frac{\partial \tilde{K}(G)}{\partial G}, \tag{A.3}$$

which leads to the dual action

$$S_2(\Phi) = \int d^3x d^2\theta d^2\bar{\theta} K(\operatorname{Re} \Phi), \tag{A.4}$$

where  $K$  is the Legendre transform of  $\tilde{K}$ .

Therefore, we may dualize the action used in the main text,

$$S_{\mathcal{N}=2}^{(3)} = \int -\tilde{K}^{\hat{A}\hat{B}} d\phi_{\hat{A}} \wedge \star d\bar{\phi}_{\hat{B}} + \frac{1}{4} \tilde{K}_{\Sigma\Lambda} (dL^\Sigma \wedge \star dL^\Lambda + F^\Sigma \wedge \star F^\Lambda) + F^\Sigma \wedge \operatorname{Im}(\tilde{K}_{\Sigma}^{\hat{A}} d\phi_{\hat{A}}), \tag{A.5}$$

by performing a Legendre transform of the kinetic potential. Since we want to dualize some of the scalars into vectors, and vice versa, we split the fields as follows<sup>20</sup>

$$\phi_{\hat{A}} = (\phi_a, N_\kappa), \quad L^\Sigma = (L^\ell, L^\alpha), \quad A^\Sigma = (A^\ell, A^\alpha), \tag{A.6}$$

---

<sup>20</sup>Notice that, for the dualization in section 3, we have that  $\phi_a = (z^\kappa, N_A)$ .

and dualize the fields with Greek indices. We also assume that the kinetic potential does not depend on  $\text{Im } N_\kappa$ .<sup>21</sup> The appropriate Legendre transform is given by

$$K(\varphi_a, T_\alpha | l^i, n^\kappa) = \tilde{K}(\phi_a, N_\kappa | L^i, L^\alpha) - L^\alpha \text{Re } T_\alpha - \text{Re } N_\kappa n^\kappa, \quad (\text{A.7})$$

where the new variables are defined as

$$\text{Re } T_\alpha \equiv \tilde{K}_{L^\alpha} \equiv \tilde{K}_\alpha, \quad n^\kappa \equiv \tilde{K}_{\text{Re } N_\kappa} \equiv \tilde{K}^\kappa. \quad (\text{A.8})$$

The dual action takes exactly the same form as (A.5), but with field content changed,

$$\phi_{\hat{A}} = (\varphi_a, T_\alpha), \quad L^\Sigma = (l^i, n^\kappa), \quad A^\Sigma = (A^i, A^\kappa), \quad (\text{A.9})$$

and  $K$  replaced by its Legendre transform given by (A.7). Although the fields  $\phi_a$  and  $L^i$  were not dualized, we nevertheless changed their names to  $\varphi_a$  and  $l^i$  for clarity.

It is possible to express all the derivatives of  $K$  in terms of those of  $\tilde{K}$ , if one knows the derivatives of the old variables with respect to the new. The dualization gives us the opposite, i.e. the derivatives of the new variables with respect to the old, which we collect in a matrix

$$M_j^i = \frac{\partial x_{\text{new}}^i}{\partial x_{\text{old}}^j} = \begin{pmatrix} l^i & \varphi_a & \text{Re } T_\alpha & n^\kappa \\ \delta_j^i & 0 & \tilde{K}_{\alpha j} & \tilde{K}_j^\kappa \\ 0 & \delta_a^b & \tilde{K}_\alpha^b & \tilde{K}^{\kappa b} \\ 0 & 0 & \tilde{K}_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \tilde{K}^{\kappa\lambda} \end{pmatrix} \begin{matrix} L^j \\ \phi_b \\ L^\beta \\ \text{Re } N_\lambda \end{matrix} \equiv \begin{pmatrix} \mathbf{1} & B \\ 0 & D \end{pmatrix}, \quad (\text{A.10})$$

where we assumed that  $\tilde{K}_\alpha^\kappa = 0$ .<sup>22</sup> The derivatives of the old variables in terms of the new ones is given by the inverse of this matrix, namely

$$M^{-1} = \begin{pmatrix} \mathbf{1} & -BD^{-1} \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} L^i & \phi_a & L^\alpha & \text{Re } N_\kappa \\ \delta_j^i & 0 & -\tilde{K}_{\beta j}(\tilde{K}_{\alpha\beta})^{-1} & -\tilde{K}_j^\lambda(\tilde{K}^{\kappa\lambda})^{-1} \\ 0 & \delta_a^b & -\tilde{K}_\beta^b(\tilde{K}_{\alpha\beta})^{-1} & -\tilde{K}^{\lambda b}(\tilde{K}^{\kappa\lambda})^{-1} \\ 0 & 0 & (\tilde{K}_{\alpha\beta})^{-1} & 0 \\ 0 & 0 & 0 & (\tilde{K}^{\kappa\lambda})^{-1} \end{pmatrix} \begin{matrix} l^j \\ \varphi_b \\ \text{Re } T_\beta \\ n^\lambda \end{matrix}. \quad (\text{A.11})$$

Using this we find the derivatives of the new kinetic potential in terms of derivatives of the original one, which are given by

$$\begin{aligned} K_{\kappa\lambda} &= -(\tilde{K}^{\kappa\lambda})^{-1} & K_{\kappa i} &= \tilde{K}_i^\lambda(\tilde{K}^{\kappa\lambda})^{-1} \\ K^{\alpha\beta} &= -\frac{1}{4}(\tilde{K}_{\alpha\beta})^{-1} & K_\kappa^a &= \tilde{K}^{\lambda a}(\tilde{K}^{\kappa\lambda})^{-1} \\ K_{ij} &= \tilde{K}_{ij} - \tilde{K}_i^\kappa \tilde{K}_j^\lambda (\tilde{K}^{\kappa\lambda})^{-1} - \tilde{K}_{i\alpha} \tilde{K}_{j\beta} (\tilde{K}_{\alpha\beta})^{-1} & K^{\alpha a} &= \frac{1}{2} \tilde{K}_\beta^a (\tilde{K}_{\alpha\beta})^{-1} \\ K_i^a &= \tilde{K}_i^a - \tilde{K}_i^\kappa \tilde{K}^{\alpha\lambda} (\tilde{K}^{\kappa\lambda})^{-1} - \tilde{K}_{i\alpha} \tilde{K}_\beta^a (\tilde{K}_{\alpha\beta})^{-1} & K_i^\alpha &= \frac{1}{2} \tilde{K}_{\beta i} (\tilde{K}_{\alpha\beta})^{-1} \\ K^{ab} &= \tilde{K}^{ab} - \tilde{K}^{a\kappa} \tilde{K}^{\lambda b} (\tilde{K}^{\kappa\lambda})^{-1} - \tilde{K}_\alpha^a \tilde{K}_\beta^b (\tilde{K}_{\alpha\beta})^{-1} & K_\kappa^\alpha &= 0. \end{aligned} \quad (\text{A.12})$$

<sup>21</sup>This is actually not strictly necessary, however, it is true for the Kähler potential that we dualize in the main text.

<sup>22</sup>This is true for the Kähler potential (3.31), since  $d_\alpha \kappa^A = 0$ . It is straightforward to drop this assumption.

For the case analyzed in the main text, namely for  $\tilde{K}$  given in (3.31), we find the following derivatives of  $K$

$$K_{\kappa\lambda} = \frac{1}{R} \operatorname{Re} f_{\kappa\lambda} \quad (\text{A.13})$$

$$K_{\kappa i} = \frac{1}{R} \operatorname{Re}[Q_{i\kappa}{}^A N_A] \quad (\text{A.14})$$

$$K_{\kappa\mathcal{K}} = \frac{1}{2R} \partial_{\mathcal{K}} f_{\kappa\lambda} (n^\lambda + iL^i M_i{}^{\lambda A} N_A) = \frac{1}{2R} \left( \partial_{\mathcal{K}} f_{\kappa\lambda} n^\lambda + \partial_{\mathcal{K}} Q_{i\kappa}{}^A L^i N_A \right) \quad (\text{A.15})$$

$$K_{\kappa}^A = \frac{1}{2R} Q_{i\kappa}{}^A L^i \quad (\text{A.16})$$

$$K_i^A = \frac{1}{2R} Q_{i\kappa}{}^A n^\kappa - \frac{1}{R} \left( Q_{i\kappa}{}^A \operatorname{Re}[d_j{}^{B\kappa} N_B] + Q_{j\kappa}{}^A \operatorname{Re}[d_i{}^{B\kappa} N_B] \right) L^j \\ + \frac{1}{2R} \left( \operatorname{Re} d_\alpha{}^{AB} N_B + d_\alpha{}^{AB} \bar{N}_B \right) C_{ij}^\alpha L^j \quad (\text{A.17})$$

$$K_{i\mathcal{K}} = \frac{1}{R} \partial_{\mathcal{K}} f_{\kappa\lambda} \left[ -2 \left( \operatorname{Re}[d_j{}^{A\kappa} N_A] d_i{}^{\lambda B} \operatorname{Re} N_B + i \operatorname{Re}[d_i{}^{A\kappa} N_A] \operatorname{Im} d_j{}^{\lambda B} N_B \right) L^j \right. \\ \left. + n^\kappa i \operatorname{Im} d_i{}^{\lambda A} N_A \right] - \frac{1}{2R} \partial_{\mathcal{K}} f_{BC} \left( \operatorname{Re} f^{AB} d_\alpha{}^{CD} \operatorname{Re} N_A \operatorname{Re} N_D \right) C_{ij}^\alpha L^j \quad (\text{A.18})$$

$$K_{ij} = \frac{1}{R} \operatorname{Re} \left[ -C_{ij}^\alpha \left( T_\alpha - \operatorname{Re}[d_\alpha{}^{AB} N_B] \operatorname{Re} N_A \right) + 4 \operatorname{Re}[d_i{}^{A\kappa} N_A] \operatorname{Re}[d_j{}^{B\lambda} N_B] \operatorname{Re} f_{\kappa\lambda} \right] \quad (\text{A.19})$$

$$K_i^\alpha = -\frac{1}{2R} C_{ij}^\alpha L^j \quad (\text{A.20})$$

$$K^{\alpha\bar{\beta}} = \frac{1}{16} (G_{\alpha\beta})^{-1} + \frac{1}{16R} (G_{\alpha\gamma})^{-1} (G_{\beta\delta})^{-1} H_{\gamma\delta\epsilon} C_{ij}^\epsilon L^i L^j, \quad (\text{A.21})$$

where we defined

$$G_{\alpha\beta} \equiv -\frac{3}{2} \left( \frac{\mathcal{K}_{\alpha\beta}^b}{\mathcal{K}^b} - \frac{3}{2} \frac{\mathcal{K}_\alpha^b \mathcal{K}_\beta^b}{(\mathcal{K}^b)^2} \right) \quad (\text{A.22})$$

$$H_{\alpha\beta\gamma} \equiv -\frac{3}{4} \left( \frac{\mathcal{K}_{\alpha\beta\gamma}^b}{\mathcal{K}^b} - 3 \frac{\mathcal{K}_{\alpha\beta}^b \mathcal{K}_\gamma^b + \mathcal{K}_{\beta\gamma}^b \mathcal{K}_\alpha^b + \mathcal{K}_{\gamma\alpha}^b \mathcal{K}_\beta^b}{(\mathcal{K}^b)^2} + 9 \frac{\mathcal{K}_\alpha^b \mathcal{K}_\beta^b \mathcal{K}_\gamma^b}{(\mathcal{K}^b)^3} \right), \quad (\text{A.23})$$

and worked at leading order in  $C_{ij}^\alpha$ . It is also useful to consider the following combinations

$$F^\kappa \wedge \operatorname{Im}(K_\kappa^{\hat{A}} d\phi_{\hat{A}}) = \frac{n^\lambda}{2R} F^\kappa \wedge d\operatorname{Im} f_{\kappa\lambda} + \frac{L^i}{2R} F^\kappa \wedge d\operatorname{Im}(Q_{i\kappa}{}^A N_A) \\ F^i \wedge \operatorname{Im}(K_i^{\hat{A}} d\phi_{\hat{A}}) = \frac{n^\kappa}{2R} F^i \wedge d\operatorname{Im}(Q_{i\kappa}{}^A N_A) \\ + \frac{L^j}{R} F^i \wedge d\operatorname{Im} \left[ -C_{ij}^\alpha \left( \frac{1}{2} T_\alpha - \operatorname{Re}[d_\alpha{}^{BA} N_B] \operatorname{Re} N_A \right) \right. \\ \left. + 4 \operatorname{Re}[d_i{}^{A\kappa} N_A] \operatorname{Re}[d_j{}^{B\lambda} N_B] \operatorname{Re} f_{\kappa\lambda} \right].$$

## B Circle reduction of four-dimensional $\mathcal{N} = 1$ supergravity

In this appendix, we perform the circle reduction of the following four-dimensional  $\mathcal{N} = 1$  ungauged supergravity action,

$$S^{(4)} = \int_{\mathcal{M}_4} \frac{1}{2} \hat{R} \hat{\star} 1 - \hat{K}_{\mathbf{A}\mathbf{B}} d\hat{M}^{\mathbf{A}} \wedge \hat{\star} d\hat{M}^{\mathbf{B}} - \frac{1}{4} \text{Re } f_{\mathbf{I}\mathbf{J}}(\hat{M}) \hat{F}^{\mathbf{I}} \wedge \hat{\star} \hat{F}^{\mathbf{J}} - \frac{1}{4} \text{Im } f_{\mathbf{I}\mathbf{J}}(\hat{M}) \hat{F}^{\mathbf{I}} \wedge \hat{F}^{\mathbf{J}}, \quad (\text{B.1})$$

where  $\hat{K}_{\mathbf{A}\mathbf{B}}$  and  $\text{Re } f_{\mathbf{I}\mathbf{J}}$  are positive definite (we use the mostly minus metric convention), and hatted objects live in four dimensions. When  $\mathcal{M}_4 = \mathcal{M}_3 \times S^1$ , we can decompose the metric as

$$d\hat{s}^2 = ds^2 + r^2 (dy + A^0)^2, \quad y \sim y + 2\pi. \quad (\text{B.2})$$

With such a decomposition of the metric, one finds the following reduction of the Einstein-Hilbert term

$$\int_{\mathcal{M}_4} \frac{1}{2} R_4 \hat{\star} 1 = \int_{\mathcal{M}_3} \frac{1}{2} R_3 \star 1 - \frac{1}{4R^2} dR \wedge \star dR - \frac{1}{4R^2} F^0 \wedge \star F^0, \quad (\text{B.3})$$

where in addition, we performed a Weyl rescaling  $g_{\mu\nu}^{\text{new}} = r^2 g_{\mu\nu}^{\text{old}}$  to bring the action to the Einstein frame, and introduced the new variable  $R \equiv r^{-2}$ .

Furthermore, the reduction ansatz for the vectors is,

$$\hat{A}^{\mathbf{I}} = A^{\mathbf{I}} - \zeta^{\mathbf{I}} (dy + A^0), \quad (\text{B.4})$$

where  $\zeta^{\mathbf{I}}$  are three-dimensional scalars. The reduction of the terms containing vectors is

$$\int_{\mathcal{M}_4} \text{Re } f_{\mathbf{I}\mathbf{J}} \hat{F}^{\mathbf{I}} \wedge \star \hat{F}^{\mathbf{J}} = \int_{\mathcal{M}_3} \frac{1}{R} \text{Re } f_{\mathbf{I}\mathbf{J}} \left[ \left( F^{\mathbf{I}} - \frac{\xi^{\mathbf{I}}}{R} F^0 \right) \wedge \star \left( F^{\mathbf{J}} - \frac{\xi^{\mathbf{J}}}{R} F^0 \right) - \left( d\xi^{\mathbf{I}} - \frac{\xi^{\mathbf{I}}}{R} dR \right) \wedge \star \left( d\xi^{\mathbf{J}} - \frac{\xi^{\mathbf{J}}}{R} dR \right) \right], \quad (\text{B.5})$$

$$\int_{\mathcal{M}_4} \text{Im } f_{\mathbf{I}\mathbf{J}} \hat{F}^{\mathbf{I}} \wedge \hat{F}^{\mathbf{J}} = 2 \int_{\mathcal{M}_3} d\text{Im } f_{\mathbf{I}\mathbf{J}} \wedge \left( F^{\mathbf{I}} - \frac{1}{2} \frac{\xi^{\mathbf{I}}}{R} F^0 \right) \frac{\xi^{\mathbf{J}}}{R}, \quad (\text{B.6})$$

where we introduced  $\xi^{\mathbf{I}} \equiv R \zeta^{\mathbf{I}}$ , which are the proper three-dimensional scalar fields (they form a vector multiplet together with the reduced vector  $A^{\mathbf{I}}$ ; similarly  $R$  and  $A^0$  form a vector multiplet).

Putting all this together we obtain the following three-dimensional action

$$S^{(3)} = \int_{\mathcal{M}_3} \frac{1}{2} R_3 \star 1 - \frac{1}{4R^2} (dR \wedge \star dR + F^0 \wedge \star F^0) - \hat{K}_{\mathbf{A}\mathbf{B}} dM^{\mathbf{A}} \wedge \star d\bar{M}^{\mathbf{B}} - \frac{1}{4R} \text{Re } f_{\mathbf{I}\mathbf{J}} \left( d\xi^{\mathbf{I}} - \frac{\xi^{\mathbf{I}}}{R} dR \right) \wedge \star \left( d\xi^{\mathbf{J}} - \frac{\xi^{\mathbf{J}}}{R} dR \right) - \frac{1}{4R} \text{Re } f_{\mathbf{I}\mathbf{J}} \left( F^{\mathbf{I}} - \frac{\xi^{\mathbf{I}}}{R} F^0 \right) \wedge \star \left( F^{\mathbf{J}} - \frac{\xi^{\mathbf{J}}}{R} F^0 \right) - \frac{1}{2} d\text{Im } f_{\mathbf{I}\mathbf{J}} \wedge \left( F^{\mathbf{I}} - \frac{1}{2} \frac{\xi^{\mathbf{I}}}{R} F^0 \right) \frac{\xi^{\mathbf{J}}}{R}. \quad (\text{B.7})$$

One can check that this action can be put into the standard  $\mathcal{N} = 2$  supergravity form,

$$\begin{aligned}
 S^{(3)} = & \int_{\mathcal{M}_3} \frac{1}{2} R_3 \star 1 - K_{\mathbf{A}\mathbf{B}} dM^{\mathbf{A}} \wedge \star d\bar{M}^{\mathbf{B}} \\
 & + \frac{1}{4} K_{\mathcal{I}\mathcal{J}} (d\xi^{\mathcal{I}} \wedge \star d\xi^{\mathcal{J}} + F^{\mathcal{I}} \wedge \star F^{\mathcal{J}}) + F^{\mathcal{I}} \wedge \text{Im}(K_{\mathcal{I}\mathbf{A}} dM^{\mathbf{A}}),
 \end{aligned}
 \tag{B.8}$$

with kinetic potential

$$K = \hat{K}(M, \bar{M}) + \log R - \frac{1}{2R} \text{Re} f_{\mathbf{I}\mathbf{J}} \xi^{\mathbf{I}} \xi^{\mathbf{J}},$$

where the indices  $(0, \mathbf{I})$  have been gathered into a single index  $\mathcal{I}$ ,

$$\xi^{\mathcal{I}} = (R, \xi^{\mathbf{I}}), \quad A^{\mathcal{I}} = (A^0, A^{\mathbf{I}}).$$

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992).
- [2] M.B. Green and J.H. Schwarz, *Anomaly cancellation in supersymmetric  $D = 10$  gauge theory and superstring theory*, *Phys. Lett.* **B 149** (1984) 117 [[INSPIRE](#)].
- [3] A. Sagnotti, *A note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett.* **B 294** (1992) 196 [[hep-th/9210127](#)] [[INSPIRE](#)].
- [4] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, *Phys. Rept.* **445** (2007) 1 [[hep-th/0610327](#)] [[INSPIRE](#)].
- [5] L.E. Ibáñez and A.M. Uranga, *String theory and particle physics: an introduction to string phenomenology*, Cambridge University Press, Cambridge U.K. (2012).
- [6] H. Jockers and J. Louis, *The effective action of D7-branes in  $N = 1$  Calabi-Yau orientifolds*, *Nucl. Phys.* **B 705** (2005) 167 [[hep-th/0409098](#)] [[INSPIRE](#)].
- [7] H. Jockers and J. Louis, *D-terms and F-terms from D7-brane fluxes*, *Nucl. Phys.* **B 718** (2005) 203 [[hep-th/0502059](#)] [[INSPIRE](#)].
- [8] T.W. Grimm and D. Vieira Lopes, *The  $N = 1$  effective actions of D-branes in Type IIA and IIB orientifolds*, *Nucl. Phys.* **B 855** (2012) 639 [[arXiv:1104.2328](#)] [[INSPIRE](#)].
- [9] M. Berg, M. Haack and B. Körs, *Loop corrections to volume moduli and inflation in string theory*, *Phys. Rev.* **D 71** (2005) 026005 [[hep-th/0404087](#)] [[INSPIRE](#)].
- [10] M. Berg, M. Haack and B. Körs, *On the moduli dependence of nonperturbative superpotentials in brane inflation*, [hep-th/0409282](#) [[INSPIRE](#)].
- [11] V. Kaplunovsky and J. Louis, *Field dependent gauge couplings in locally supersymmetric effective quantum field theories*, *Nucl. Phys.* **B 422** (1994) 57 [[hep-th/9402005](#)] [[INSPIRE](#)].
- [12] V. Kaplunovsky and J. Louis, *On gauge couplings in string theory*, *Nucl. Phys.* **B 444** (1995) 191 [[hep-th/9502077](#)] [[INSPIRE](#)].

- [13] T.W. Grimm, *The  $N = 1$  effective action of F-theory compactifications*, *Nucl. Phys. B* **845** (2011) 48 [[arXiv:1008.4133](#)] [[INSPIRE](#)].
- [14] T.W. Grimm, D. Klevers and M. Poretschkin, *Fluxes and warping for gauge couplings in F-theory*, *JHEP* **01** (2013) 023 [[arXiv:1202.0285](#)] [[INSPIRE](#)].
- [15] T.W. Grimm, T.G. Pugh and D. Regalado, *Non-abelian discrete gauge symmetries in F-theory*, *JHEP* **02** (2016) 066 [[arXiv:1504.06272](#)] [[INSPIRE](#)].
- [16] T.W. Grimm and J. Louis, *The effective action of  $N = 1$  Calabi-Yau orientifolds*, *Nucl. Phys. B* **699** (2004) 387 [[hep-th/0403067](#)] [[INSPIRE](#)].
- [17] M. Graña, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [[hep-th/0509003](#)] [[INSPIRE](#)].
- [18] M.R. Douglas and S. Kachru, *Flux compactification*, *Rev. Mod. Phys.* **79** (2007) 733 [[hep-th/0610102](#)] [[INSPIRE](#)].
- [19] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four folds*, *Nucl. Phys. B* **584** (2000) 69 [*Erratum ibid.* **B 608** (2001) 477] [[hep-th/9906070](#)] [[INSPIRE](#)].
- [20] R. Blumenhagen, V. Braun, T.W. Grimm and T. Weigand, *GUTs in type IIB orientifold compactifications*, *Nucl. Phys. B* **815** (2009) 1 [[arXiv:0811.2936](#)] [[INSPIRE](#)].
- [21] S. Greiner and T.W. Grimm, *On mirror symmetry for Calabi-Yau fourfolds with three-form cohomology*, *JHEP* **09** (2016) 073 [[arXiv:1512.04859](#)] [[INSPIRE](#)].
- [22] F. Carta, F. Marchesano, W. Staessens and G. Zoccarato, *Open string multi-branched and Kähler potentials*, *JHEP* **09** (2016) 062 [[arXiv:1606.00508](#)] [[INSPIRE](#)].
- [23] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *Massive abelian gauge symmetries and fluxes in F-theory*, *JHEP* **12** (2011) 004 [[arXiv:1107.3842](#)] [[INSPIRE](#)].
- [24] A.P. Braun, A. Collinucci and R. Valandro, *The fate of  $U(1)$ 's at strong coupling in F-theory*, *JHEP* **07** (2014) 028 [[arXiv:1402.4054](#)] [[INSPIRE](#)].
- [25] F. Marchesano, D. Regalado and G. Zoccarato, *On D-brane moduli stabilisation*, *JHEP* **11** (2014) 097 [[arXiv:1410.0209](#)] [[INSPIRE](#)].
- [26] M.R. Douglas, D.S. Park and C. Schnell, *The Cremmer-Scherk mechanism in F-theory compactifications on  $K3$  manifolds*, *JHEP* **05** (2014) 135 [[arXiv:1403.1595](#)] [[INSPIRE](#)].
- [27] C.M. Hull, A. Karlhede, U. Lindström and M. Roček, *Nonlinear  $\sigma$  models and their gauging in and out of superspace*, *Nucl. Phys. B* **266** (1986) 1 [[INSPIRE](#)].
- [28] M. Kerstan and T. Weigand, *The effective action of D6-branes in  $N = 1$  type IIA orientifolds*, *JHEP* **06** (2011) 105 [[arXiv:1104.2329](#)] [[INSPIRE](#)].
- [29] D. Lüst and S. Stieberger, *Gauge threshold corrections in intersecting brane world models*, *Fortsch. Phys.* **55** (2007) 427 [[hep-th/0302221](#)] [[INSPIRE](#)].
- [30] F. Gmeiner and G. Honecker, *Complete gauge threshold corrections for intersecting fractional D6-branes: The  $Z_6$  and  $Z_6'$  standard models*, *Nucl. Phys. B* **829** (2010) 225 [[arXiv:0910.0843](#)] [[INSPIRE](#)].
- [31] G. Honecker, *Kähler metrics and gauge kinetic functions for intersecting D6-branes on toroidal orbifolds — The complete perturbative story*, *Fortsch. Phys.* **60** (2012) 243 [[arXiv:1109.3192](#)] [[INSPIRE](#)].
- [32] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [[hep-th/9610234](#)] [[INSPIRE](#)].

- [33] D. Belov and G.W. Moore, *Holographic action for the self-dual field*, [hep-th/0605038](#) [[INSPIRE](#)].
- [34] D.M. Belov and G.W. Moore, *Type II actions from 11-dimensional Chern-Simons Theories*, [hep-th/0611020](#) [[INSPIRE](#)].
- [35] O.J. Ganor, *A note on zeros of superpotentials in F-theory*, *Nucl. Phys. B* **499** (1997) 55 [[hep-th/9612077](#)] [[INSPIRE](#)].
- [36] T.W. Grimm, *Non-perturbative corrections and modularity in  $N = 1$  type IIB compactifications*, *JHEP* **10** (2007) 004 [[arXiv:0705.3253](#)] [[INSPIRE](#)].
- [37] T.W. Grimm, M. Kerstan, E. Palti and T. Weigand, *On fluxed instantons and moduli stabilisation in IIB orientifolds and F-theory*, *Phys. Rev. D* **84** (2011) 066001 [[arXiv:1105.3193](#)] [[INSPIRE](#)].
- [38] M. Kerstan and T. Weigand, *Fluxed M5-instantons in F-theory*, *Nucl. Phys. B* **864** (2012) 597 [[arXiv:1205.4720](#)] [[INSPIRE](#)].
- [39] D. Cremades, L.E. Ibáñez and F. Marchesano, *Computing Yukawa couplings from magnetized extra dimensions*, *JHEP* **05** (2004) 079 [[hep-th/0404229](#)] [[INSPIRE](#)].
- [40] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge U.K. (2012).
- [41] S.A. Abel, M.D. Goodsell, J. Jaeckel, V.V. Khoze and A. Ringwald, *Kinetic mixing of the photon with hidden U(1)s in string phenomenology*, *JHEP* **07** (2008) 124 [[arXiv:0803.1449](#)] [[INSPIRE](#)].
- [42] F. Marchesano, D. Regalado and G. Zoccarato, *U(1) mixing and D-brane linear equivalence*, *JHEP* **08** (2014) 157 [[arXiv:1406.2729](#)] [[INSPIRE](#)].
- [43] M. Berasaluce-Gonzalez, P.G. Camara, F. Marchesano, D. Regalado and A.M. Uranga, *Non-Abelian discrete gauge symmetries in 4d string models*, *JHEP* **09** (2012) 059 [[arXiv:1206.2383](#)] [[INSPIRE](#)].
- [44] T.W. Grimm and D. Regalado, *On dualities for non-Abelian gauge theories with continuous center*, [arXiv:1509.04281](#) [[INSPIRE](#)].
- [45] F. Denef, *Les Houches lectures on constructing string vacua*, [arXiv:0803.1194](#) [[INSPIRE](#)].
- [46] M. Haack and J. Louis, *Duality in heterotic vacua with four supercharges*, *Nucl. Phys. B* **575** (2000) 107 [[hep-th/9912181](#)] [[INSPIRE](#)].
- [47] M. Haack and J. Louis, *M theory compactified on Calabi-Yau fourfolds with background flux*, *Phys. Lett. B* **507** (2001) 296 [[hep-th/0103068](#)] [[INSPIRE](#)].
- [48] M. Berg, M. Haack and H. Samtleben, *Calabi-Yau fourfolds with flux and supersymmetry breaking*, *JHEP* **04** (2003) 046 [[hep-th/0212255](#)] [[INSPIRE](#)].
- [49] T.W. Grimm, A. Kapfer and D. Klevers, *The Arithmetic of elliptic fibrations in gauge theories on a circle*, *JHEP* **06** (2016) 112 [[arXiv:1510.04281](#)] [[INSPIRE](#)].
- [50] T.W. Grimm and A. Kapfer, *Anomaly cancelation in field theory and F-theory on a circle*, *JHEP* **05** (2016) 102 [[arXiv:1502.05398](#)] [[INSPIRE](#)].
- [51] T.W. Grimm and H. Hayashi, *F-theory fluxes, chirality and Chern-Simons theories*, *JHEP* **03** (2012) 027 [[arXiv:1111.1232](#)] [[INSPIRE](#)].
- [52] M. Cvetič, T.W. Grimm and D. Klevers, *Anomaly cancellation and abelian gauge symmetries in F-theory*, *JHEP* **02** (2013) 101 [[arXiv:1210.6034](#)] [[INSPIRE](#)].

- [53] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, *Chiral four-dimensional F-theory compactifications with SU(5) and multiple U(1)-factors*, *JHEP* **04** (2014) 010 [[arXiv:1306.3987](#)] [[INSPIRE](#)].
- [54] C. Mayrhofer, E. Palti and T. Weigand, *Hypercharge flux in IIB and F-theory: anomalies and gauge coupling unification*, *JHEP* **09** (2013) 082 [[arXiv:1303.3589](#)] [[INSPIRE](#)].
- [55] C. Beasley, J.J. Heckman and C. Vafa, *GUTs and exceptional branes in F-theory — II: experimental predictions*, *JHEP* **01** (2009) 059 [[arXiv:0806.0102](#)] [[INSPIRE](#)].
- [56] R. Donagi and M. Wijnholt, *Breaking GUT groups in F-theory*, *Adv. Theor. Math. Phys.* **15** (2011) 1523 [[arXiv:0808.2223](#)] [[INSPIRE](#)].
- [57] E. Witten, *Quantum background independence in string theory*, [hep-th/9306122](#) [[INSPIRE](#)].
- [58] W. Lerche and S. Stieberger, *Prepotential, mirror map and F-theory on K3*, *Adv. Theor. Math. Phys.* **2** (1998) 1105 [*Erratum ibid.* **3** (1999) 1199] [[hep-th/9804176](#)] [[INSPIRE](#)].
- [59] W. Lerche, S. Stieberger and N.P. Warner, *Quartic gauge couplings from K3 geometry*, *Adv. Theor. Math. Phys.* **3** (1999) 1575 [[hep-th/9811228](#)] [[INSPIRE](#)].
- [60] W. Lerche, S. Stieberger and N.P. Warner, *Prepotentials from symmetric products*, *Adv. Theor. Math. Phys.* **3** (1999) 1613 [[hep-th/9901162](#)] [[INSPIRE](#)].