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Euclidean supergravity

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Abstract: Supergravity with eight supercharges in a four-dimensional Euclidean space is constructed at the full non-linear level by performing an off-shell time-like reduction of five-dimensional supergravity. The resulting four-dimensional theory is realized off-shell with the Weyl, vector and tensor supermultiplets and a corresponding multiplet calculus. Hypermultiplets are included as well, but they are themselves only realized with on-shell supersymmetry. We also briefly discuss the non-linear supermultiplet. The off-shell reduction leads to a full understanding of the Euclidean theory. A complete multiplet calculus is presented along the lines of the Minkowskian theory. Unlike in Minkowski space, chiral and anti-chiral multiplets are real and supersymmetric actions are generally unbounded from below. Precisely as in the Minkowski case, where one has different formulations of Poincaré supergravity upon introducing different compensating supermultiplets, one can also obtain different versions of Euclidean supergravity.

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Contents

1 Introduction

Euclidean versions of supersymmetric field theories are of interest in the study of a variety of physical problems. An early application concerned the many-instanton problem, which was first discussed by Zumino on the basis of a supersymmetric version of Euclidean Yang-Mills theory [1]. This theory is not equivalent to the Wick-rotated version of super-Yang-Mills theory. As is well known, Majorana spinors cannot exist in a Euclidean four-space, but there exists a version with eight supercharges that constitute a symplectic Majorana spinor, whose R-symmetry group equals $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$. The presence of the non-compact factor, which acts on spinors by chiral transformations, was already noted by Schwinger when discussing Euclidean quantum-electrodynamics [2]. Subsequent studies of the multi-instanton solutions in the context of supersymmetry were, for instance, reported in [3, 4].

Also superconformal transformations played a role in the analysis of [1]. The conformal group associated with $4D$ Euclidean space equals $\mathrm{SO}(5,1)$, whose Lie algebra can be extended to a superalgebra known as $\mathrm{SU}^*(4|2)$, according to the classification presented in [5].[1] Neither conformal nor non-conformal supergravities have been fully constructed in the context of Euclidean space-times. It is the purpose of this paper to fill this gap, motivated by many other applications where Euclidean supergravity is relevant. This will be done by carrying out an *off-shell* time-like reduction on five-dimensional (off-shell) Minkowski supergravity. In principle this yields a complete off-shell dictionary of the $5D$ Minkowski fields in terms of the $4D$ Euclidean fields. Because the reduction is off-shell, it can also be used in the context of higher-derivative $4D$ couplings, as was, for instance, done in [6, 7]. The same strategy for studying aspects of the $4D$ Euclidean supergravities by reduction from the $5D$ Minkowski theory was already used in [8, 9], but there the reduction was performed *on-shell* and the emphasis was more directed toward target-space aspects. Time-like reductions have also been carried out starting from higher-dimensional theories with more than eight supercharges. Starting from $11D$ supergravity and reducing on a Lorentzian-signature torus, the authors of [10] provided insights into the rigid dualities and local symmetries of Euclidean supergravities in arbitrary dimensions. Compactification of M-theory on a time-like circle exhibited generalized T-dualities relating type-II string theories and their BPS branes in various signatures [11], which was also understood in the context of different versions of the $\mathrm{OSp}(1|32)$ superalgbra [12]. These reductions are mainly based on theories with maximal supersymmmetry, and as a consequence the reductions cannot be realized off-shell and will only pertain to specific actions. In contrast, in this paper we derive the Euclidean $4D$ theory without the need to start from a specific higher-dimensional action. Therefore the $4D$ theories that we can construct do not necessarily have a higher-dimensional origin.

As we already mentioned, the above $4D$ Euclidean field theories are not equivalent to the theories that one obtains by analytic continuation to imaginary times of a $4D$ Minkowski theory. The latter play an important role in the more rigorous treatment of quantum field theories in the context of the Euclidean postulate, as well as in field theories at finite temperature and lattice gauge theories. The earliest discussion of such an analytically continued supersymmetric theory was presented in [13]. In a $4D$ Minkowski theory one has at least four supercharges which constitute a Majorana spinor, and under analytic continuation the number of supercharges will remain the same. In contradistinction, the $4D$ Euclidean theories discussed in this paper have at least twice as many supercharges, which constitute a symplectic Majorana spinor.

Euclidean supersymmetric field theories have already played a role in many applications and we briefly review some of them. First of all, a consistent definition of a Euclidean supersymmetric gauge theory was introduced in [14, 15]. Upon twisting the $\mathrm{SU}(2)$ R-symmetry group with one of the $\mathrm{SU}(2)$ factors belonging to the $\mathrm{SO}(4)$ tangent space

[1] See in particular table 16 of this work. We thank G. Inverso for pointing this out to us.

group, the Lagrangian remains invariant under a singlet supersymmetry for non-trivial Euclidean spaces. The singlet supersymmetry closes into a uniform gauge transformation and this property leads to a topological theory with a Q-exact energy-momentum tensor, so that correlation functions and the partition function become independent of the metric. Subsequent work on the relation between the supersymmetric field theories and their twisted version can be found in e.g. [16, 17].

More recently, Euclidean supersymmetric theories became of interest because of the possibility of applying supersymmetric localization, which potentially leads to a way of obtaining exact results in quantum field theories. In the seminal work of [18], $(9+1)$-dimensional super-Yang-Mills theory was dimensionally reduced to 4-dimensional gauge theory in flat Euclidean space with 16 supersymmetries. This approach is similar to the one followed in [8] and in the present paper. Under a subsequent conformal mapping $\mathbb{R}^4$ is changed to S^4. Because the path integral plays a central role in localization, the convergence of the latter was then ensured by deforming the integration contour of the scalar fields originating from the time component of the $10D$ gauge field to be over imaginary values.

A rather general analysis of $N=2$ conformal supergravity theories on four-manifolds was conducted in [19] with a view towards localization. In most cases one is interested in coupling the matter multiplets to a given supergravity background that will be frozen so that some rigid supersymmetry will remain. Only the matter fields are involved in the localization and the residual rigid supersymmetry is essential for setting up actual calculations. The strategy for defining the Euclidean theory was to perform a Wick rotation of the Minkowski theory to imaginary time, followed by a doubling of the field components and the supersymmetry spinor parameters. Subsequently one restricts the spinor parameters to symplectic Majorana spinors and adopts a set of appropriate reality conditions on the fields to ensure $N=2$ Euclidean supersymmetry. Reality and boundedness of the resulting actions for vector multiplets and hypermultiplets were not discussed. The structure of the Euclidean supergravity only plays an ancillary role, as one is primarily interested in a suitable fixed supergravity background. This approach for deriving the Euclidean supergravity from the Minkowskian one has become quite common in this context (in four dimensions see, for instance, [20]; for applications in other than four-dimensional Euclidean spaces, the reader may consult the review [21]). However, the approach based on time-like reduction, used in [8, 18] and in this paper, is perhaps more systematic, especially in the case where the non-linear aspects of the theory will be important. In most cases the various methods seem to lead to equivalent results.

The approach of [22] also makes use of a Wick rotation to imaginary times, followed by an adjustment of the various reality conditions on the fields of the Minkowski theory. A more explicit construction was presented in the follow-up paper [23]. This approach differs in that it restricts itself to *four* supersymmetries, which seemingly excludes to make reference to a consistent version of Euclidean supersymmetry, as that would require at least *eight* supersymmetries. It is quite conceivable that the resulting theory can nevertheless be related to the full Euclidean supergravity in the context of suitable truncations or by considering supergravity solutions with partially broken supersymmetry. Another topic addressed in [22] concerns the extra terms that one needs in the action when coupling to

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curved spaces (which were determined by making use of supercurrent techniques). Having the full supergravity available, these terms will alternatively follow from standard supergravity Lagrangians. Here we emphasize that the standard introduction of compensating fields to conformal supergravity proceeds in a way that is completely equivalent to what has been done in the Minkowskian theories.

In most cases localization has been applied to compact spaces, but there are also applications to anti-de Sitter spaces [24, 25]. In the first reference localization was applied to the evaluation of the quantum entropy function [26] to obtain the exact entropy of certain supersymmetric black holes, which can be compared to exact results that follow from string theory, such as given in [27, 28]. The second reference used localization to compute the partition function of anti-de Sitter space in four dimensions and related it to the partition function of the dual three-dimensional boundary conformal field theory [29]. The reader may also consult [30, 31] for additional contributions. As demonstrated by these applications, localization does not necessarily involve only matter fields in a fixed supergravity background. This fact provides a specific motivation for the construction of the full non-linear Euclidean $4D$ supergravity that we intend to present in this paper.

The construction of $4D$ Euclidean supergravity will be based on an *off-shell* time-like reduction of $5D$ $N=1$ Minkowski supergravity to four Euclidean dimensions. This reduction is based on a corresponding reduction of the off-shell supersymmetry algebra and can therefore be performed systematically on separate supermultiplets. To accomplish this one maps a supermultiplet in a higher dimension on a corresponding, not necessarily irreducible, supermultiplet in lower dimension. When considering the supersymmetry algebra in the context of a lower-dimensional space-time, the dimension of the automorphism group of the algebra (the R-symmetry group) usually increases, and this has to be taken into account when casting the resulting supermultiplet in its standard form. The fact that an irreducible multiplet in higher dimension can possibly become reducible in lower dimensions tends to complicate the reduction procedure. The same technique has been followed in [6] to establish the precise relation between the $5D$ and $4D$ Minkowskian supergravities with higher-order derivatives.

The construction is facilitated by the fact that the spinor dimension is the same in five and in four dimensions. In fact, both the $5D$ Minkowski and $4D$ Euclidean supergravities are based on symplectic Majorana spinors, which share a common SU(2) factor in the R-symmetry group. We will exhibit in detail how the additional SO(1,1) factor will emerge in four dimensions. Here we recall that in conformal supergravity, R-symmetry is realized as a local symmetry.

The whole reduction scheme is subtle, especially in view of the fact that the $5D$ Weyl multiplet decomposes into a $4D$ Weyl multiplet and an additional vector multiplet. In spite of this, both in five and in four dimensions, the matter multiplets are defined in a generic superconformal background expressed in the Weyl multiplet fields. To fully establish this fact requires to also consider the transformation rules beyond the linearized approximation. Once we obtain the $4D$ Euclidean off-shell transformation rules, we write them in close analogy with the Minkowskian fields in similar notation and normalization factors. After this it is in principle straightforward to obtain the effective actions and other results, either by substituting the Euclidean fields or by comparison to the Minkowski formulae. However, there remain some subtle differences that require special care, such as the fact that chiral and anti-chiral multiplets are real in the Euclidean setting.

This paper is organized as follows. Section 2 presents a brief summary of $5D$ supergravity with Minkowski signature, followed by a brief explanation of some key features of the time-like reduction. The actual details of the reduction have been relegated to appendices, so that section 3 will start with the definition of the $4D$ Euclidean fields and their transformation rules. Apart from various definitions this section presents a full discussion of the Euclidean Weyl supermultiplet. The subsequent sections also make heavy use of the Euclidean transformation rules that follow from the reduction. Chiral and anti-chiral multiplets are discussed in section 4 and section 5 gives a summary of the vector and tensor supermultiplets and the hypermultiplet. Vector multiplet systems are discussed in section 6, including the Euclidean electric-magnetic duality group. Finally our conclusions and summary are presented in section 7.

The details of the off-shell reduction are relegated to two appendices. Appendix A deals with the Weyl multiplet and the disentanglement of a $4D$ Kaluza-Klein vector multiplet. Appendix B deals with the various matter multiplets. For completeness we also include the transformations of the so-called 'non-linear multiplet' in five dimensions, which was originally discovered in four dimensions, to demonstrate that the various formulations of $4D$ Minkowskian supergravity each have their corresponding counterparts in $4D$ Euclidean supergravity.

2 Euclidean supergravity from off-shell time-like dimensional reduction

A convenient and systematic way to obtain the full Euclidean supergravity in four dimensions is to start from five-dimensional Minkowskian supergravity with eight supersymmetries and perform a time-like reduction. To obtain a complete off-shell formulation of Euclidean supergravity, the dimensional reduction has to be applied without referring to specific Lagrangians or to field equations. The primary purpose is to fully uncover the structure of the four-dimensional Euclidean theory whose off-shell features have not been pursued systematically so far.

Let us therefore start by briefly summarizing the salient features of the $5D$ Weyl multiplet. Subsequently we will indicate the starting points for the time-like dimensional reduction scheme. The more detailed derivation of the off-shell reduction is relegated to appendix A. The resulting formulation of $4D$ off-shell Euclidean supergravity is presented in the next section 3.

The independent fields of the $5D$ Weyl multiplet consist of the fünfbein $e_M{}^A$, the gravitino fields $\psi_M{}^i$, the dilatational gauge field b_M, the R-symmetry gauge fields $V_{Mi}{}^j$ (which is an anti-hermitian, traceless matrix in the SU(2) indices i, j), a tensor T_{AB}, a scalar D and a spinor χ^i. All spinor fields are symplectic Majorana spinors. Our conventions are as in [6]. The three gauge fields $\omega_M{}^{AB}$, $f_M{}^A$ and $\phi_M{}^i$, associated with local Lorentz transformations, conformal boosts and S-supersymmetry, respectively, are composite fields as will be discussed later. The infinitesimal Q, S and K transformations of the independent

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	Weyl multiplet										parameters	
field	$e_M{}^A$	$\psi_M{}^i$	b_M	$V_M{}_i{}^j$	T_{AB}	χ^i	D	$\omega_M{}^{AB}$	$f_M{}^A$	$\phi_M{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Table 1. Weyl weights w of the $5D$ Weyl multiplet component fields and the supersymmetry transformation parameters.

fields, parametrized by spinors ϵ^i and η^i and a vector $\Lambda_{\rm K}{}^A$, respectively, are as follows,[2]

$$\begin{aligned}
\delta e_M{}^A &= \bar\epsilon_i\gamma^A\psi_M{}^i\,,\\
\delta\psi_M{}^i &= 2\,\mathcal{D}_M\epsilon^i + \frac{1}{2}\mathrm{i}\,T_{AB}(3\,\gamma^{AB}\gamma_M - \gamma_M\gamma^{AB})\epsilon^i - \mathrm{i}\gamma_M\eta^i\,,\\
\delta V_{Mi}{}^j &= 6\mathrm{i}\,\bar\epsilon_i\phi_M{}^j - 16\,\bar\epsilon_i\gamma_M\chi^j - 3\mathrm{i}\,\bar\eta_i\psi_M{}^j + \delta^i{}_j\Big[-3\mathrm{i}\,\bar\epsilon_k\phi_M{}^k + 8\,\bar\epsilon_k\gamma_M\chi^k + \frac{3}{2}\mathrm{i}\,\bar\eta_k\psi_M{}^k\Big]\,,\\
\delta b_M &= \mathrm{i}\bar\epsilon_i\phi_M{}^i - 4\bar\epsilon_i\gamma_M\chi^i + \frac{1}{2}\mathrm{i}\bar\eta_i\psi_M{}^i + 2\,\Lambda_{\rm K}{}^A\,e_{MA}\,,\\
\delta T_{AB} &= \frac{4}{3}\mathrm{i}\,\bar\epsilon_i\gamma_{AB}\chi^i - \frac{1}{4}\mathrm{i}\,\bar\epsilon_i R_{AB}{}^i(Q)\,,\\
\delta\chi^i &= \frac{1}{2}\epsilon^i D + \frac{1}{64}R_{MNj}{}^i(V)\,\gamma^{MN}\epsilon^j + \frac{3}{64}\mathrm{i}(3\,\gamma^{AB}\not{D} + \not{D}\gamma^{AB})T_{AB}\,\epsilon^i\\
&\quad - \frac{3}{16}T_{AB}T_{CD}\gamma^{ABCD}\epsilon^i + \frac{3}{16}T_{AB}\gamma^{AB}\eta^i\,,\\
\delta D &= 2\,\bar\epsilon_i\not{D}\chi^i - 2\mathrm{i}\,\bar\epsilon_i T_{AB}\,\gamma^{AB}\chi^i - \mathrm{i}\bar\eta_i\chi^i\,.
\end{aligned}\tag{2.1}$$

Under local scale transformations the fields and transformation parameters transform as indicated in table 1. The derivatives $\mathcal{D}_M$ are covariant with respect to all the bosonic gauge symmetries with the exception of the conformal boosts. In particular we note

$$\mathcal{D}_M\epsilon^i = \Big(\partial_M - \frac{1}{4}\omega_M{}^{CD}\,\gamma_{CD} + \frac{1}{2}\,b_M\Big)\epsilon^i + \frac{1}{2}\,V_{Mj}{}^i\,\epsilon^j\,,\tag{2.2}$$

where the gauge fields transform under their respective gauge transformations according to $\delta\omega_M{}^{AB} = \mathcal{D}_M\epsilon^{AB}$, $\delta b_M = \mathcal{D}_M\Lambda_D$ and $\delta V_{Mi}{}^j = -2\,\mathcal{D}_M\Lambda_i{}^j$, with $(\Lambda_i{}^j)^* \equiv \Lambda^i{}_j = -\Lambda_j{}^i$. The derivatives D_M are covariant with respect to all the superconformal symmetries.

The above supersymmetry variations and also the conventional constraints involve a number of supercovariant curvature tensors, denoted by $R(P)_{MN}{}^A$, $R(M)_{MN}{}^{AB}$, $R(D)_{MN}$, $R(K)_{AB}{}^A$ $R(V)_{MNi}{}^j$, $R(Q)_{MN}{}^i$ and $R(S)_{MN}{}^i$ whose explicit form can be found in [6]. The so-called conventional constraints,

$$R(P)_{MN}{}^A = 0\,,\qquad \gamma^M R(Q)_{MN}{}^i = 0\,,\qquad e_A{}^M\,R(M)_{MN}{}^{AB} = 0\,,\tag{2.3}$$

determine the gauge fields $\omega_M{}^{AB}$, $f_M{}^A$ and $\phi_M{}^i$. These constraints lead to additional conditions on the curvatures when combined with the Bianchi identities. In this way one

[2]In five dimensions we consistently use world indices $M, N, \ldots$ and tangent space indices $A, B, \ldots$. For fields that do not carry such indices the distinction between $5D$ and $4D$ fields may not always be manifest, but it will be indicated in the text whenever necessary.

can derive $R(M)_{[ABC]D} = 0 = R(D)_{AB}$ and the pair-exchange property $R(M)_{ABCD} = R(M)_{CDAB}$ from the first and the third constraint. The second constraint, which implies also that $\gamma_{[MN}R(Q)_{PQ]}{}^i = 0$, determines the curvature $R(S)_{MN}{}^i$.

The reduction to four space-time dimensions is effected by first carrying out the standard Kaluza-Klein decompositions on the various fields that will ensure that the resulting $4D$ fields will transform consistently under four-dimensional diffeomorphisms. The $5D$ space-time coordinates x^M are decomposed into four coordinates x^μ and a fifth coordinate $x^{\hat{5}}$. The dependence on this fifth coordinate will be suppressed in the reduction. Likewise the tangent-space indices A decompose into the four indices $a = 1, 2, 3, 4$ and a fifth index $A = 5$. In Pauli-Källén notation one of the coordinates is imaginary so that the $5D$ space-time signature will be a permutation of $(-++++)$. In [6] the fifth coordinate $x^{\hat{5}}$ was real, so that the reduced theory was based on a four-dimensional Minkowskian space-time. In this paper we consider the time-like reduction where the fifth coordinate is purely imaginary. Upon the reduction, where the dependence on the fifth coordinate is suppressed, the resulting theory will then be based on a four-dimensional Euclidean space. An important observation is that the results of [6] were obtained with Pauli-Källén convention, which enables a direct conversion into the Euclidean theory by an appropriate change of the reality conditions on the fields. One simply has to include factors $\pm\mathrm{i}$ whenever dealing with the fifth world or tangent-space component. For instance, the fifth coordinate of x^M takes the form $x^{\hat{5}} = \mathrm{i}x^0$, so that the fifth component of a contravariant vector field $V^{\hat{5}}$ will be imaginary and can be written as $\mathrm{i}V^0$, where V^0 is real. For a covariant vector the fifth component $W_{\hat{5}}$ will instead be equal to $-\mathrm{i}W_0$, where W_0 is real. A corresponding rule applies to tangent-space vectors.

After this general introduction we will exhibit the consequences of the above strategy. As is standard, the vielbein field and the dilatational gauge field are first written in special form, by means of an appropriate local Lorentz transformation and a conformal boost in the time direction, respectively. In obvious notation,

$$e_M{}^A = \begin{pmatrix} e_\mu{}^a & \mathrm{i}B_\mu\phi^{-1} \\ 0 & \phi^{-1} \end{pmatrix}, \qquad e_A{}^M = \begin{pmatrix} e_a{}^\mu & -\mathrm{i}e_a{}^\nu B_\nu \\ 0 & \phi \end{pmatrix}, \qquad b_M = \begin{pmatrix} b_\mu \\ 0 \end{pmatrix}. \tag{2.4}$$

Note that the vielbein field is not real because we will keep using the tangent-space indices $A = 1, \ldots, 5$. As compared to the space-like reduction the field ϕ has remained unchanged while the Kaluza-Klein gauge field B_μ requires a factor i so that it remains real. All the fields on the right-hand side of (2.4) are now real and possible sign factors depend on whether we have suppressed an upper coordinate $A = 5$ and/or a lower coordinate $M = \hat{5}$. The fields now carry only four-dimensional world and tangent-space indices, $\mu, \nu, \ldots$ and $a, b, \ldots$, taking four values while the components referring to the fifth direction will be suppressed. Observe that the scaling weights for $e_M{}^A$ and $e_\mu{}^a$ are equal to $w = -1$, while for ϕ we have $w = 1$. The fields b_M, b_μ and B_μ carry weight $w = 0$.

For the fermions there is yet no need to introduce new notation, because the spinors have an equal number of components in five and four space-time dimensions. We will thus employ symplectic Majorana spinors ψ^i with $i = 1, 2$ subject to the reality constraint,[3]

$$C^{-1}\,\bar{\chi}_i{}^{\mathrm{T}} = \varepsilon_{ij}\,\chi^j\,. \tag{2.5}$$

The Dirac conjugate is defined by $\bar{\psi} = \psi^\dagger\gamma^5$, where $\gamma^5 = \mathrm{i}\gamma^0$. Observe that we adhere to the convention according to which raising or lowering of SU(2) indices is effected by complex conjugation. For fermionic bilinears, with spinor fields ψ^i and φ^i and a spinor matrix Γ constructed from products of gamma matrices, we note the following result,

$$(\bar{\varphi}_j\,\gamma_5\,\Gamma^\dagger\,\gamma_5\,\psi^i)^\dagger = \bar{\psi}_i\Gamma\varphi^j = -\delta_i{}^j\,\bar{\varphi}_k\,C^{-1}\,\Gamma^{\mathrm{T}}\,C\,\psi^k + \bar{\varphi}_i\,C^{-1}\,\Gamma^{\mathrm{T}}\,C\,\psi^j\,. \tag{2.6}$$

Hence the bilinears $O_i{}^j$ equal to $\mathrm{i}\,\bar{\psi}_i\,\varphi^j$, $\bar{\psi}_i\gamma_A\varphi^j$ and $\mathrm{i}\,\bar{\psi}_i\gamma_{AB}\varphi^j$ are pseudo-hermitean: $O^i{}_j = \varepsilon^{ik}\varepsilon_{jl}\,O_k{}^l$, provided $A, B, \ldots = 1, \ldots, 4$. In the context of the spinors special care is required in converting the Minkoswki ones into the Euclidean signature, because (Fierz) reordering of the spinors depends sensitively on whether the spinor is a Majorana or an anti-Majorana field. Observe that the gravitino field $\psi_{\hat{5}}$, with its world index in the fifth direction, will be an anti-Majorana field. This will be properly accounted for in the Kaluza-Klein ansätze, which will include the proper factors of the imaginary unit, as can be seen in appendix A.

3 Supergravity with eight supercharges in four Euclidean dimensions

Here we present the superconformal transformation rules in $4D$ Euclidean supergravity that have been derived in appendix A by means of an off-shell reduction of the $5D$ theory. The $5D$ R-symmetry is then extended to chiral $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$. Unlike in $4D$ Minkowski supergravity the spinors of the Euclidean theory will be symplectic Majorana spinors. For Euclidean spinors it is natural to define the Dirac conjugate by hermitian conjugation, so that $\bar{\chi}_i \equiv \chi_i^\dagger$. With this definition the symplectic Majorana spinors satisfy the condition

$$C^{-1}\,\bar{\chi}_i{}^{\mathrm{T}} = \varepsilon_{ij}\,\chi^j\,, \tag{3.1}$$

The charge conjugation matrix C thus differs from the $5D$ one. While it is still anti-symmetric and unitary, just as in five dimensions, the (hermitian) gamma matrices γ_a now satisfy the relation

$$C\,\gamma_a\,C^{-1} = -\gamma_a{}^{\mathrm{T}}\,, \qquad (a = 1, 2, 3, 4). \tag{3.2}$$

Note here that $\gamma_5 \equiv \gamma_1\,\gamma_2\,\gamma_3\,\gamma_4$ satisfies

$$C\,\gamma_5\,C^{-1} = \gamma_5{}^{\mathrm{T}}\,. \tag{3.3}$$

In appendix A.1 it is explained how these definitions emerge in the reduction from five dimensions (see, in particular, (A.37)). As before, raising or lowering of SU(2) indices is effected by complex conjugation. For fermionic bilinears $\bar{\psi}_i\Gamma\varphi^j$ we note the following result,

$$(\bar{\varphi}_j\,\Gamma^\dagger\,\psi^i)^\dagger = \bar{\psi}_i\Gamma\varphi^j = -\varepsilon_{ik}\,\varepsilon^{jl}\,\bar{\varphi}_l\,C^{-1}\,\Gamma^{\mathrm{T}}\,C\,\psi^k\,, \tag{3.4}$$

[3]The charge conjugation matrix C has the properties $C\gamma_A C^{-1} = \gamma_A{}^{\mathrm{T}}$, with $C^{\mathrm{T}} = -C$ and $C^\dagger = C^{-1}$. The $5D$ hermitian gamma matrices in Pauli-Källén notation satisfy $\gamma_{ABCDE} = \mathbf{1}\,\varepsilon_{ABCDE}$.

which is rather similar but not identical to the $5D$ result (2.6). We note that (3.4) applies also to chiral spinors. For convenience we also specify the Fierz rearrangement formulae for chiral spinors,

$$
\begin{aligned}
\varphi^j{}_\pm\,\bar\psi_{i\pm} &= -\frac{1}{4}(1\pm\gamma^5)\left(\bar\psi_{i\pm}\,\varphi^j{}_\pm\right)+\frac{1}{8}\gamma^{ab}\left(\bar\psi_{i\pm}\gamma_{ab}\varphi^j{}_\pm\right),\\
\varphi^j{}_\mp\,\bar\psi_{i\pm} &= -\frac{1}{4}\gamma^a(1\pm\gamma^5)\left(\bar\psi_{i\pm}\gamma_a\varphi^j{}_\mp\right).
\end{aligned}
\tag{3.5}
$$

Let us now consider the transformations of the fields of the Euclidean $4D$ Weyl multiplet. The superconformal algebra comprises the generators of the spatial diffeomorphisms, local tangent space rotations, dilatation, special conformal, chiral SU(2) and SO(1,1), supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations ($e_\mu{}^a$), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu{}^i{}_j$ and A_μ) and Q-supersymmetry ($\psi_\mu{}^i$) are independent fields. The remaining gauge fields associated with the Lorentz ($\omega_\mu{}^{ab}$), special conformal ($f_\mu{}^a$) and S-supersymmetry transformations ($\phi_\mu{}^i$) are composite fields. Obviously, world and tangent-space indices μ and a, respectively, take the values $1,2,3,4$. The corresponding supercovariant curvatures and covariant fields are contained in a tensor chiral multiplet, which comprises $24\oplus 24$ off-shell degrees of freedom. This relation with the chiral supermultiplet is decribed in section 4. In addition to the independent superconformal gauge fields, it contains three other fields: a symplectic Majorana spinor doublet χ^i, a scalar D, and a Lorentz antisymmetric tensor T_{ab}, which decomposes into a self-dual and an anti-self-dual field. The Weyl and chiral weights have been collected in table 2.

As derived in detail in appendix A the Euclidean fields, which comprise $24\oplus 24$ bosonic and fermionic degrees of freedom, transform as follows under the Q-, S- and K- transformations of the superconformal algebra,

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon_i\,\gamma^5\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i+\frac{1}{16}\mathrm{i}\,(T_{ab}{}^+ + T_{ab}{}^-)\gamma^{ab}\gamma_\mu\epsilon^i-\mathrm{i}\gamma_\mu\eta^i\,,\\
\delta b_\mu &= \frac{1}{2}\mathrm{i}\,\bar\epsilon_i\,\gamma^5\phi_\mu{}^i-\frac{3}{4}\bar\epsilon_i\,\gamma^5\gamma_\mu\chi^i+\frac{1}{2}\mathrm{i}\,\bar\eta_i\,\gamma^5\psi_\mu{}^i+\Lambda_{\mathrm{K}}{}^a\,e_{\mu a}\,,\\
\delta A_\mu &= -\frac{1}{2}\mathrm{i}\,\bar\epsilon_i\,\phi_\mu{}^i-\frac{3}{4}\,\bar\epsilon_i\,\gamma_\mu\chi^i-\frac{1}{2}\mathrm{i}\,\bar\eta_i\,\psi_\mu{}^i\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\mathrm{i}\,\bar\epsilon_j\,\gamma^5\phi_\mu{}^i-3\,\bar\epsilon_j\,\gamma^5\gamma_\mu\chi^i-2\mathrm{i}\,\bar\eta_j\,\gamma^5\psi_\mu{}^i\\
&\quad-\frac{1}{2}\delta^i{}_j\big(2\mathrm{i}\,\bar\epsilon_k\,\gamma^5\phi_\mu{}^k-3\,\bar\epsilon_k\,\gamma^5\gamma_\mu\chi^k-2\mathrm{i}\,\bar\eta_k\,\gamma^5\psi_\mu{}^k\big)\,,\\
\delta T_{ab}{}^\pm &= -8\mathrm{i}\,\bar\epsilon_i\,\gamma^5 R(Q)_{ab}{}^{i\,\pm}\,,\\
\delta\chi^i &= \frac{1}{24}\mathrm{i}\,\gamma^{ab}\,\not{D}(T^+_{ab}+T^-_{ab})\epsilon^i+\frac{1}{6}R(\mathcal{V})_{ab}{}^i{}_j\,\gamma^{ab}\epsilon^j-\frac{1}{3}R(A)_{ab}\,\gamma^{ab}\gamma^5\epsilon^i\\
&\quad+D\,\epsilon^i+\frac{1}{24}(T^+_{ab}+T^-_{ab})\gamma^{ab}\eta^i\,,\\
\delta D &= \bar\epsilon_i\,\gamma^5\not{D}\chi^i\,,
\end{aligned}
\tag{3.6}
$$

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	Weyl multiplet											parameters	
field	$e_\mu{}^a$	ψ_μ^i	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	$T_{ab}^{\pm}$	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	ϕ_μ^i	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$\mp\frac{1}{2}$	0	0	0	± 1	$\mp\frac{1}{2}$	0	0	0	$\pm\frac{1}{2}$	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$
$\tilde\gamma_5$		$\pm$		$\pm$		$\pm$	$\pm$		$\pm$		$\pm$	$\pm$	$\pm$

Table 2. Weyl weights w and chiral SO(1,1) weights c, chirality/duality $\tilde\gamma_5$ of the spinors and the antisymmetric tensor field for the $4D$ Euclidean Weyl multiplet.

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Note that we have refrained from decomposing the spinors into chiral components, to keep the equations as compact as possible. Only when referring specifically to (anti)chiral spinors will we add the subscript $\pm$. Note that ϵ^i denotes the symplectic Majorana parameter of Q-supersymmetry, η^i the symplectic Majorana parameter of S-supersymmetry, and $\Lambda_{\rm K}{}^a$ is the transformation parameter for special conformal boosts. The full superconformal covariant derivative is denoted by D_μ, while $\mathcal{D}_\mu$ denotes a derivative covariant with respect to Lorentz, dilatation, chiral SO(1,1), and SU(2) transformations. In particular,

$$\begin{aligned}\mathcal{D}_\mu\epsilon^i &= \Big(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} + \frac{1}{2}\,b_\mu + \frac{1}{2}\,A_\mu\gamma^5\Big)\epsilon^i + \frac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,,\\ \mathcal{D}_\mu\eta^i &= \Big(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} - \frac{1}{2}\,b_\mu - \frac{1}{2}\,A_\mu\gamma^5\Big)\eta^i + \frac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\eta^j\,.\end{aligned} \tag{3.7}$$

There are three additional gauge fields, namely $\omega_\mu{}^{ab}$, ϕ_μ^i and $f_\mu{}^a$, associated with SO(4) tangent space rotations, S-supersymmetry and conformal boosts, respectively, with corresponding parameters ϵ^{ab}, η^i and $\Lambda_{\rm K}{}^a$. These fields are composite and depend on the other fields. This is expressed in terms of so-called conventional constraints which contain various superconformal curvature tensors that will be presented momentarily. We first present the explicit solutions for $\omega_\mu{}^{ab}$, ϕ_μ^i and $f_\mu{}^a$,

$$\begin{aligned}\omega_\mu{}^{ab} &= -\,2\,e^{\nu[a}\,(\partial_{[\mu} + b_{[\mu})e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\rho}\,e_{\mu c}\,(\partial_\rho + b_\rho)e_\nu{}^c - \frac{1}{4}\big(2\,\bar\psi_{\mu\,i}\,\gamma^5\gamma^{[a}\psi^{b]i} + \bar\psi^a{}_i\,\gamma^5\gamma_\mu\psi^{b\,i}\big)\,,\\ \phi_\mu{}^i &= -\,\frac{1}{2}\mathrm{i}\left(\gamma^{\rho\sigma}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i + \frac{1}{32}\mathrm{i}\,(T^+_{ab} + T^-_{ab})\gamma^{ab}\gamma_\rho\psi_\sigma{}^i + \frac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,,\\ f_\mu{}^a &= \frac{1}{2}\,R(\omega,e)_\mu{}^a - \frac{1}{4}\left(D + \frac{1}{3}R(\omega,e)\right)e_\mu{}^a - \frac{1}{2}\,\widetilde{R}(A)_\mu{}^a - \frac{1}{32}\,T^-_{\mu b}\,T^{+\,ba} + \cdots\,,\end{aligned} \tag{3.8}$$

where the ellipses in the last equation denote fermionic contributions. Here and elsewhere we use the notation where $\widetilde{R}_{ab}$ equals the dual tensor $\frac{1}{2}\varepsilon_{abcd}\,R^{cd}$. Furthermore, $R(\omega,e)_\mu{}^a = R(\omega)_{\mu\nu}{}^{ab}e_b{}^\nu$ is the non-symmetric Ricci tensor, and $R(\omega,e)$ the corresponding Ricci scalar. The uncontracted curvature $R(\omega)_{\mu\nu}{}^{ab}$ is defined by

$$R(\omega)_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^{ac}\,\omega_{\nu c}{}^b + \omega_\nu{}^{ac}\,\omega_{\mu c}{}^b\,. \tag{3.9}$$

The Q- and S-supersymmetry variations of the composite gauge fields, as well as their transformations under conformal boosts, take the following form,

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} ={}& -\frac{1}{2}\mathrm{i}\,\bar\epsilon_i\gamma^5\gamma^{ab}\phi_\mu{}^i + \frac{1}{4}\mathrm{i}\,\bar\epsilon_i\gamma^5\psi_\mu{}^i(T^{+ab}+T^{-ab}) + \frac{3}{4}\,\bar\epsilon_i\gamma^5\gamma_\mu\gamma^{ab}\chi^i \\
&+ \bar\epsilon_i\gamma^5\gamma_\mu R(Q)^{ab\,i} + \frac{1}{2}\mathrm{i}\,\bar\eta_i\gamma^5\gamma^{ab}\psi_\mu{}^i + 2\,\Lambda_K{}^{[a}e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i ={}& 2\,\mathcal{D}_\mu\eta^i + 2\mathrm{i}\,f_\mu{}^a\gamma_a\epsilon^i + \frac{1}{16}\not{\mathcal{D}}(T_{ab}{}^+ + T_{ab}{}^-)\gamma^{ab}\gamma_\mu\,\epsilon^i \\
&- \frac{1}{4}\mathrm{i}\gamma^{ab}\gamma_\mu R(\mathcal{V})_{ab}{}^i{}_j\,\epsilon^j - \frac{1}{2}\mathrm{i}\gamma^{ab}\gamma_\mu\gamma^5 R(A)_{ab}\,\epsilon^i \\
&- \frac{3}{2}\mathrm{i}[(\bar\chi_j\gamma^5\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^5\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] - \mathrm{i}\Lambda_{\mathrm{K}}{}^a\,\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a ={}& \frac{1}{4}\mathrm{i}\,\bar\epsilon_i\,\gamma^5\psi_\mu{}^i D_b(T^{+ba}+T^{-ba}) - \frac{3}{4}e_\mu{}^a\,\bar\epsilon_i\gamma^5\not{D}\chi^i - \frac{3}{4}\bar\epsilon_i\,\gamma^5\gamma^a\psi_\mu{}^i\,D \\
&+ \bar\epsilon_i\,\gamma^5\gamma_\mu D_b R(Q)^{ba\,i} + \frac{1}{2}\,\bar\eta_i\,\gamma^5\gamma^a\phi_\mu{}^i + \mathcal{D}_\mu\Lambda_{\mathrm{K}}{}^a\,.
\end{aligned}
\tag{3.10}
$$

A systematic way to derive these variations is by noting that the composite gauge fields follow from imposing three conventional constraints (whose solutions are given by (3.8)),

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 0\,, \\
\gamma^\mu R(Q)_{\mu\nu}{}^i + \frac{3}{2}\gamma_\nu\chi^i &= 0\,, \\
e^\nu{}_b\,R(M)_{\mu\nu}{}^{ab} - \widetilde{R}(A)_\mu{}^a + \frac{1}{16}T_{\mu b}{}^-\,T^{ab+} - \frac{3}{2}\,D\,e_\mu{}^a &= 0\,.
\end{aligned}
\tag{3.11}
$$

These constraints are invariant under all the symmetries associated with the SU*(4|2) superalgebra with the exception of Q-supersymmetry. Therefore they will only induce changes in the Q-supersymmetry transformations and in the supersymmetry commutators that involve the Q-supersymmetry generators. This phenomenon is well known from the 4D Minkowskian conformal supergravities.

For completeness we present the definitions of all the supercovariant curvature tensors,

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}\,e_{\nu]}{}^a - \frac{1}{2}\,\bar\psi_{i[\mu}\gamma^5\gamma^a\psi_{\nu]}{}^i\,, \\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \mathrm{i}\,\gamma_{[\mu}\phi_{\nu]}{}^i + \frac{1}{16}\mathrm{i}\,(T^+_{ab}+T^-_{ab})\,\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,, \\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2\,f_{[\mu}{}^a e_{\nu]a} - \frac{1}{2}\mathrm{i}\,\bar\psi_{i[\mu}\gamma^5\phi_{\nu]}{}^i + \frac{3}{4}\,\bar\psi_{i[\mu}\gamma^5\gamma_{\nu]}\chi^i\,, \\
R(A)_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} + \frac{1}{2}\mathrm{i}\,\bar\psi_{i[\mu}\phi_{\nu]}{}^i + \frac{3}{4}\,\bar\psi_{i[\mu}\gamma_{\nu]}\chi^i\,,
\end{aligned}
$$

$$
\begin{aligned}
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j \\
&\quad - 2\mathrm{i}\,\bar\psi_{j[\mu}\gamma^5\phi_{\nu]}{}^i + 3\,\bar\psi_{j[\mu}\gamma^5\gamma_{\nu]}\chi^i + \frac{1}{2}\delta^i{}_j\big(2\mathrm{i}\,\bar\psi_{k[\mu}\gamma^5\phi_{\nu]}{}^k - 3\,\bar\psi_{k[\mu}\gamma^5\gamma_{\nu]}\chi^k\big)\,, \\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \frac{1}{2}\mathrm{i}\,\bar\psi_{i[\mu}\gamma^5\phi_{\nu]}{}^i \\
&\quad - \frac{1}{8}\mathrm{i}\,\bar\psi_{\mu\,i}\gamma^5\psi_\nu{}^i\,(T^{ab+} + T^{ab-}) - \frac{3}{4}\bar\psi_{i[\mu}\gamma^5\gamma_{\nu]}\gamma^{ab}\chi^i - \bar\psi_{i[\mu}\gamma^5\gamma_{\nu]}R(Q)^{ab\,i}\,, \\
R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i + 2\mathrm{i}\,f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i + \frac{1}{16}\mathrm{i}\,\not{D}(T^+_{ab} + T^-_{ab})\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i \\
&\quad + \frac{3}{2}\mathrm{i}\,\gamma_a\psi_{[\mu}{}^i\bar\psi_{\nu]j}\gamma^5\gamma^a\chi^j - \frac{1}{4}\mathrm{i}\,R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j - \frac{1}{2}\mathrm{i}\,R(A)_{ab}\gamma^5\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,, \\
R(K)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \frac{1}{4}\,\bar\phi_{i[\mu}\gamma^5\gamma^a\phi_{\nu]}{}^i \\
&\quad - \frac{1}{8}\Big[\mathrm{i}\,\bar\psi_{i[\mu}\gamma^5 D_b(T^{+ba} + T^{-ba})\psi_{\nu]}{}^i + 6\,e_{[\mu}{}^a\bar\psi_{\nu]i}\gamma^5\not{D}\chi^i \\
&\quad - 3\,D\,\bar\psi_{i[\mu}\gamma^5\gamma^a\psi_{\nu]}{}^i + 8\,\bar\psi_{i[\mu}\gamma^5\gamma_{\nu]}D_bR(Q)^{ba\,i}\Big]\,.
\end{aligned} \tag{3.12}
$$

It is also convenient to introduce two modified curvatures by including suitable covariant terms,

$$
\begin{aligned}
\mathcal{R}(M)_{ab}{}^{cd} &= R(M)_{ab}{}^{cd} + \frac{1}{32}\big(T^-_{ab}\,T^{+cd} + T^+_{ab}\,T^{-cd}\big)\,, \\
\mathcal{R}(S)_{ab}{}^i{}_\pm &= R(S)_{ab}{}^i{}_\pm - \frac{3}{8}\,T^\pm_{ab}\,\chi^i_\pm\,,
\end{aligned} \tag{3.13}
$$

where we note that $\gamma^{ab}\big(\mathcal{R}(S) - R(S)\big)_{ab}{}^i = 0$. By making use of the conventional constraints and the Bianchi identities of the superconformal algebra, one can show that the modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ satisfies the following relations:

$$
\begin{aligned}
\mathcal{R}(M)_{\mu\nu}{}^{ab}\,e^\nu{}_b &= \widetilde{R}(A)_\mu{}^a + \frac{3}{2}\,D\,e_\mu{}^a\,, \\
\frac{1}{4}\,\varepsilon_{ab}{}^{ef}\,\varepsilon^{cd}{}_{gh}\,\mathcal{R}(M)_{ef}{}^{gh} &= \mathcal{R}(M)_{ab}{}^{cd}\,, \\
\varepsilon_{cdea}\,\mathcal{R}(M)^{cd\,e}{}_b = \varepsilon_{becd}\,\mathcal{R}(M)_a{}^{e\,cd} &= 2\,\widetilde{R}(D)_{ab} = 2\,R(A)_{ab}\,.
\end{aligned} \tag{3.14}
$$

Note that this modified curvature does not satisfy the pair exchange property,

$$
\mathcal{R}(M)_{ab}{}^{cd} - \mathcal{R}(M)^{cd}{}_{ab} = 4\,\delta^{[c}_{[a}\,\widetilde{R}(A)_{b]}{}^{d]}\,. \tag{3.15}
$$

The fermionic conventional constraint implies that the chiral and anti-chiral projections of the $R(Q)_{ab}{}^i$ curvature are anti-self-dual and self-dual, respectively:

$$
\gamma^5 R(Q)_{ab}{}^i = -\widetilde{R}(Q)_{ab}{}^i\,. \tag{3.16}
$$

In addition, combining the constraint with the Bianchi identities yields

$$
\gamma^a\,\widetilde{\mathcal{R}}(S)_{ab}{}^i{}_\pm = 2\mathrm{i}\,D^a R(Q)_{ab}{}^i{}_\mp\,. \tag{3.17}
$$

Finally we present the commutation relations of the superconformal algebra. The first one is the commutator between two Q-supersymmetry variations,

$$\big[\delta_{\rm Q}(\epsilon_1),\, \delta_{\rm Q}(\epsilon_2)\big] = \xi^\mu D_\mu + \delta_{\rm M}(\varepsilon_{ab}) + \delta_{\rm S}(\eta^i) + \delta_{\rm K}(\Lambda_{\rm K}{}^a) + \delta_{\rm gauge}\,, \tag{3.18}$$

where the parameters of the various transformations appearing on the r.h.s. are given by

$$\begin{aligned}
\xi^\mu &= 2\,\bar\epsilon_{2i}\,\gamma^5\gamma^\mu\epsilon_1{}^i\,,\\
\varepsilon_{ab} &= \frac{1}{2}\mathrm{i}\,\bar\epsilon_{2i-}\,\epsilon_1{}^i_-\,T^-_{ab} - \frac{1}{2}\mathrm{i}\,\bar\epsilon_{2i+}\,\epsilon_1{}^i_+\,T^+_{ab}\,,\\
\eta^i &= -\,6\mathrm{i}\,\bar\epsilon_{[1j-}\,\epsilon_{2]}{}^i_-\,\chi^j{}_+ + 6\mathrm{i}\,\bar\epsilon_{[1j+}\,\epsilon_{2]}{}^i_+\,\chi^j{}_-\,,\\
\Lambda_{\rm K}{}^a &= -\,\frac{1}{2}\mathrm{i}\,\bar\epsilon_{2i-}\,\epsilon_1{}^i_-\,D_b T^{-ba} + \frac{1}{2}\mathrm{i}\,\bar\epsilon_{2i+}\,\epsilon_1{}^i_+\,D_b T^{+ba} + \frac{3}{2}\,\bar\epsilon_{2i}\gamma^a\gamma^5\epsilon_1{}^i\,D\,.
\end{aligned} \tag{3.19}$$

We also present the commutator of a Q-supersymmetry variation with either an S-supersymmetry variation or a conformal boost variation,

$$\begin{aligned}
\big[\delta_{\rm Q}(\epsilon),\, \delta_{\rm S}(\eta)\big] &= \delta_{\rm M}(\mathrm{i}\,\bar\epsilon_i\,\gamma^5\gamma^{ab}\eta^i) + \delta_{\rm D}(-\mathrm{i}\,\bar\epsilon_i\,\gamma^5\eta^i) + \delta_{\rm SO(1,1)}(\mathrm{i}\,\bar\epsilon_i\,\eta^i)\\
&\quad + \delta_{\rm SU(2)}(2\mathrm{i}\,\bar\epsilon_j\,\gamma^5\eta^i - \delta_j^i\,\mathrm{i}\,\bar\epsilon_k\,\gamma^5\eta^k)\,,\\
\big[\delta_{\rm Q}(\epsilon),\, \delta_{\rm K}(\Lambda_K)\big] &= \delta_{\rm S}(-\mathrm{i}\Lambda_{\rm K}{}^a\gamma_a\epsilon^i)\,.
\end{aligned} \tag{3.20}$$

In the presence of matter multiplets that contain gauge fields, such as the vector and the tensor multiplets, the gauge transformations of the latter will also appear in the commutator of two Q-supersymmetry transformations. We will indicate these contributions when discussing the vector and tensor multiplets in section 5.

4 Chiral and anti-chiral multiplets

Chiral multiplets are defined by the condition that the scalar field of lowest Weyl weight w transforms into a chiral spinor field. The scalar also carries a chiral weight c with respect to SO(1, 1) transformations, which must be equal to $c = -w$. Likewise one may also define an anti-chiral multiplet where the scalar field transforms into an anti-chiral spinor. In that case one must have $c = w$. To explain the reason for this restriction on the chiral weights one parametrizes the possible Q- and S-supersymmetry transformations on the first two fields, denoted by $A_\pm$ and $\Psi^i{}_\pm$, where the plus (minus) index refers to the chiral (anti-chiral) multiplet (and thus determines the chirality of the spinor),

$$\begin{aligned}
\delta A_\pm &= \pm\,\mathrm{i}\bar\epsilon_{i\pm}\Psi^i{}_\pm\,,\\
\delta\Psi^i{}_\pm &= -\,2\mathrm{i}\,\not{D}A_\pm\epsilon^i_\mp + \varepsilon_{jk}\,B^{ij}{}_\pm\,\epsilon^k_\pm - \frac{1}{2}F_{ab}{}^\mp\,\gamma^{ab}\epsilon^i_\pm + 2w\,A_\pm\,\eta^i_\pm\,,
\end{aligned} \tag{4.1}$$

where we note the supercovariant derivative of $A_\pm$,

$$D_\mu A_\pm = (\partial_\mu - w\,b_\mu - c_\pm\,A_\mu)A_\pm \mp \frac{1}{2}\mathrm{i}\,\bar\psi_{\mu\,i\pm}\,\Psi^i{}_\pm\,. \tag{4.2}$$

Imposing the commutation relation (3.18) on $A_\pm$ shows that $\delta\Psi^i{}_\pm$ is correct, provided that $B^{ij}{}_\pm$ is symmetric in (ij). Furthermore the second commutation relation in (3.20) shows

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	Scalar (anti-)chiral multiplet					
field	$A_\pm$	$\Psi^i{}_\pm$	$B^{ij}{}_\pm$	$F_{ab}{}^\mp$	$\Lambda^i{}_\pm$	$C_\pm$
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$\mp w$	$\mp\left(w-\frac{1}{2}\right)$	$\mp(w-1)$	$\mp(w-1)$	$\mp\left(w-\frac{3}{2}\right)$	$\mp(w-2)$
$\tilde\gamma_5$		$\pm$		$\mp$	$\pm$	

Table 3. Weyl and chiral weights (w and c) and fermion chirality/two-form duality ($\tilde\gamma_5$) of scalar (anti-)chiral multiplet component fields. Note that the chiral (anti-chiral) multiplet contains fermion components of positive (negative) chirality and Lorentz two-forms that are anti-self-dual (self-dual). All bosonic fields are (pseudo-)real.

that the S-supersymmetry variation of $\Psi^i{}_\pm$ is correct, while the first one shows that the algebra can only close provided that the chiral charges $c_\pm$ are equal to $c_\pm = \mp w$, thus confirming the claim above.

It is clear that the (anti-)chiral multiplet contains $8\oplus 8$ bosonic and fermionc degrees of freedom. Their Weyl and chiral weights are straightforward to determine and are shown in table 3. Moreover the (anti-)chiral multiplets are real, unlike in Minkowski space where they are complex and where the complex conjugate of a chiral multiplet will constitute an anti-chiral multiplet. Hence in the context of Euclidean supersymmetry the chiral and anti-chiral fields should be treated as independent.

The continuation of the above analysis straightforwardly leads to all the Q- and S-supersymmetry transformations of the fields belonging to a chiral or anti-chiral super-multiplet,

$$\begin{aligned}
\delta A_\pm =&\ \pm\,\mathrm{i}\bar\epsilon_{i\pm}\Psi^i{}_\pm\,,\\
\delta\Psi^i{}_\pm =&\ -2\mathrm{i}\,\not{D}A_\pm\,\epsilon^i{}_\mp + \varepsilon_{jk}\,B^{ij}{}_\pm\,\epsilon^k{}_\pm - \frac{1}{2}F_{ab}{}^\mp\,\gamma^{ab}\epsilon^i{}_\pm + 2w\,A_\pm\,\eta^i{}_\pm\,,\\
\delta B^{ij}{}_\pm =&\ \mp 2\,\bar\epsilon_{k\mp}\,\not{D}\Psi^{(i}{}_\pm\,\varepsilon^{j)k} \mp 2\mathrm{i}\,\bar\epsilon_{k\pm}\Lambda^{(i}{}_\pm\,\varepsilon^{j)k} \mp 2\mathrm{i}(1-w)\,\bar\eta_{k\pm}\Psi^{(i}{}_\pm\,\varepsilon^{j)k}\,,\\
\delta F_{ab}{}^\mp =&\ \mp\frac{1}{2}\bar\epsilon_{i\mp}\,\not{D}\gamma_{ab}\Psi^i{}_\pm \pm \frac{1}{2}\mathrm{i}\,\bar\epsilon_{i\pm}\gamma_{ab}\Lambda^i{}_\pm \pm \frac{1}{2}\mathrm{i}(1+w)\,\bar\eta_{i\pm}\gamma_{ab}\,\Psi^i{}_\pm\,,\\
\delta\Lambda^i{}_\pm =&\ \frac{1}{2}\mathrm{i}\gamma^{ab}\not{D}F^\mp_{ab}\,\epsilon^i{}_\mp + \mathrm{i}\,\not{D}B^{ij}{}_\pm\,\varepsilon_{jk}\,\epsilon^k{}_\mp + C_\pm\,\epsilon^i{}_\pm\\
&+\frac{1}{8}\mathrm{i}\left(\not{D}A_\pm\,T^\pm_{ab} + w\,A_\pm\,\not{D}T^\pm_{ab}\right)\gamma^{ab}\,\epsilon^i{}_\mp \pm \frac{3}{2}\mathrm{i}\,\gamma_a\,\epsilon^i{}_\mp\,\bar\chi_{j\mp}\,\gamma^a\,\Psi^j{}_\pm\\
&-(1+w)\,B^{ij}{}_\pm\,\varepsilon_{jk}\,\eta^k{}_\pm + \frac{1}{2}(1-w)\,\gamma^{ab}\,F^\mp_{ab}\,\eta^i{}_\pm\,,\\
\delta C_\pm =&\ \mp 2\,\bar\epsilon_{i\mp}\,\not{D}\Lambda^i{}_\pm \pm 6\mathrm{i}\,\bar\epsilon_{i\mp}\,\chi^k{}_\mp\,\varepsilon_{kj}B^{ij}{}_\pm\\
&\pm\frac{1}{8}\big((w-1)\,\bar\epsilon_{i\mp}\,\gamma^{ab}\,\not{D}T^\pm_{ab}\,\Psi^i_\pm + \bar\epsilon_{i\mp}\,\gamma^{ab}T^\pm_{ab}\,\not{D}\Psi^i{}_\pm\big) \pm 2\mathrm{i}w\,\bar\eta_{i\pm}\,\Lambda^i{}_\pm\,.
\end{aligned}\tag{4.3}$$

At this point we should point out that there are three special values for w. One is $w = 1$ where the fields $B^{ij}{}_\pm$ and $F_{ab}{}^\mp$ are both neutral under the SO(1,1) R-symmetry. In this situation it is possible to define a reduced multiplet by first combining a chiral and an anti-chiral multiplet with $w = 1$, which together comprise $16 \oplus 16$ bosonic and fermionic fields. Subsequently one imposes a constraint which again reduces the number of components to $8 \oplus 8$. This constraint is only consistent with supersymmetry provided $w = 1$ and implies, for instance, that $F_{ab}{}^{\mp}$ are subject to a Bianchi identity, which reads $D_a(F^{ab+} - F^{ab-}) = 0$ modulo terms that depend non-linearly on the fields . We will return to this in the next section when we discuss the vector multiplet.

The second one is $w = 2$. In that case the field $C_\pm$ has Weyl weight equal to 4 and zero SO(1, 1) weight. Both the chiral and the anti-chiral multiplet with the proper weight will define a superconformally invariant action. For this one makes use of the following density formula,

$$
\begin{aligned}
\mathcal{L}_\pm = e\Big[& C_\pm \mp \bar\psi_{\mu i\mp}\,\gamma^\mu\Lambda^i{}_\pm \pm \frac{1}{16}\bar\psi_{\mu i\mp}\,T_{ab}{}^\pm\,\gamma^{ab}\gamma^\mu\Psi^i{}_\pm + \frac{1}{16}A_\pm\,(T_{ab}{}^\pm)^2 \\
& \pm \frac{1}{2}\mathrm{i}\,\bar\psi_{\mu i\mp}\,\gamma^{\mu\nu}\,\psi_\nu{}^j{}_\mp\,\varepsilon_{jk}\,B^{ki}{}_\pm \mp \mathrm{i}\,\bar\psi_{\mu i\mp}\,\psi_\nu{}^i{}_\mp\,(F^{\mu\nu\pm} - \frac{1}{2}A_\pm\,T^{\mu\nu\,\pm})\Big] \\
& \mp \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\,\bar\psi_{\mu i\mp}\,\psi_\nu{}^i{}_\mp\,\big(\mathrm{i}\,\bar\psi_{\rho j\mp}\,\gamma_\sigma\,\Psi^j{}_\pm + \bar\psi_{\rho j\mp}\,\psi_\sigma{}^j{}_\mp\,A_\pm\big)\,.
\end{aligned} \tag{4.4}
$$

Another special case is an (anti-)chiral multiplet with $w = 0$. In that case the highest component $C_\pm$ is invariant under S-supersymmetry. Therefore C_+ itself can be regarded as the lowest component of a new anti-chiral multiplet; likewise C_- can be regarded as the lowest component of a new chiral multiplet. Both these new multiplets carry Weyl weight $w = 2$. This multiplet is known as the *kinetic chiral multiplet*. When applying the density formula to this multiplet, the answer will be equal to a total derivative. On the other hand, applying the density formula to the product of a $w = 0$ (anti-)chiral multiplet times a kinetic (anti-)chiral multiplet, will lead to a higher derivative action. We refer to [32, 33] for a more detailed discussion of the kinetic multiplet and variants thereof in the context of the 4D Minkowski theory. We stress that the generic features of these multiplets are the same as in the Euclidean theory. In the remainder of this section we consider a number of additional noteworthy features of chiral and anti-chiral supermultiplets.

4.1 Product rule for Euclidean chiral multiplets

The product of two (anti-)chiral multiplets of weights w_1 and w_2, with components denoted by $(A_\pm,\,\Psi^i{}_\pm,\,B^{ij}{}_\pm,\,F_{ab}{}^\mp,\,\Lambda^i{}_\pm,\,C_\pm)$ and $(a_\pm,\,\psi^i{}_\pm,\,b^{ij}{}_\pm,\,f_{ab}{}^\mp,\,\lambda^i{}_\pm,\,c_\pm)$, yields an (anti-)chiral multiplet of weight $w = w_1 + w_2$ according to the following product rule:

$$
\begin{aligned}
&(A_\pm, \Psi^i{}_\pm,\, B^{ij}{}_\pm,\, F_{ab}{}^\mp,\, \Lambda^i{}_\pm,\, C_\pm) \otimes (a_\pm,\, \psi^i{}_\pm,\, b^{ij}{}_\pm,\, f_{ab}{}^\mp,\, \lambda^i{}_\pm,\, c_\pm) = \\
&\quad\Big(A_\pm\,a_\pm,\; A_\pm\,\psi^i{}_\pm + a_\pm\,\Psi^i{}_\pm,\; A_\pm\,b^{ij}{}_\pm + a_\pm\,B^{ij}{}_\pm \mp \mathrm{i}\,\bar\psi_{k\pm}\,\varepsilon^{k(i}\Psi^{j)}{}_\pm, \\
&\qquad A_\pm\,f_{ab}{}^\mp + a_\pm\,F_{ab}{}^\mp \pm \frac{1}{4}\mathrm{i}\,\bar\psi_{i\pm}\,\gamma_{ab}\Psi^i{}_\pm, \\
&\qquad A_\pm\,\lambda^i{}_\pm + a_\pm\,\Lambda^i{}_\pm - \frac{1}{2}\,\varepsilon_{jk}(B^{ij}{}_\pm\,\psi^k{}_\pm + b^{ij}{}_\pm\,\Psi^k{}_\pm) - \frac{1}{4}(F_{ab}{}^\mp\gamma^{ab}\psi^i{}_\pm + f_{ab}{}^\mp\gamma^{ab}\Psi^i{}_\pm), \\
&\qquad A_\pm\,c_\pm + a_\pm\,C_\pm - \frac{1}{2}\,\varepsilon_{ik}\varepsilon_{jl}\,B^{ij}{}_\pm\,b^{kl}{}_\pm - f_{ab}{}^\mp\,F^{ab\mp} \pm \mathrm{i}\,\bar\psi_{i\pm}\Lambda^i{}_\pm \pm \mathrm{i}\,\bar\Psi_{i\pm}\lambda^i{}_\pm \Big)\,.
\end{aligned} \tag{4.5}
$$

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Note that the product of a chiral multiplet with an anti-chiral multiplet does not yield a chiral or anti-chiral multiplet, because $A_\pm a_\mp$ does not transform into an (anti-)chiral fermion under supersymmetry, but into $a_\mp \Psi^i{}_\pm - A_\pm \psi^i{}_\mp$. Therefore the result is a general scalar supermultiplet which comprises $128 \oplus 128$ bosonic and fermionic components.

A homogeneous function $\mathcal{G}^\pm(\Phi_\pm)$ of several (anti-)chiral superfields $\Phi^\Lambda{}_\pm$ defines an (anti-)chiral superfield, whose components take the following form,

$$
\begin{aligned}
A_\pm|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm(A_\pm)\,, \\
\Psi^i{}_\pm|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm{}_\Lambda\, \Psi^i{}_\pm{}^\Lambda\,, \\
B^{ij}{}_\pm|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm{}_\Lambda\, B^{ij}{}_\pm{}^\Lambda \mp \frac{1}{2}\mathrm{i}\, \mathcal{G}^\pm{}_{\Lambda\Sigma}\, \bar\Psi_{k\pm}{}^\Lambda\, \varepsilon^{k(i}\, \Psi^{j)}{}_\pm{}^\Sigma\,, \\
F_{ab}{}^\mp|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm{}_\Lambda\, F_{ab}{}^{\mp\,\Lambda} \pm \frac{1}{8}\mathrm{i}\, \mathcal{G}^\pm{}_{\Lambda\Sigma}\, \bar\Psi_{i\pm}{}^\Lambda \gamma_{ab} \Psi^i{}_\pm{}^\Sigma\,, \\
\Lambda^i{}_\pm|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm{}_\Lambda\, \Lambda^i{}_\pm{}^\Lambda - \frac{1}{2}\, \mathcal{G}^\pm{}_{\Lambda\Sigma} \left[\varepsilon_{jk}\, B^{ij}{}_\pm{}^\Lambda\, \Psi^k{}_\pm{}^\Sigma + \frac{1}{2}\, F_{ab}{}^{\mp\,\Lambda}\, \gamma^{ab} \Psi^i{}_\pm{}^\Sigma\right] \\
&\quad \mp \frac{1}{48}\mathrm{i}\, \mathcal{G}^\pm{}_{\Lambda\Sigma\Gamma}\, \gamma^{ab} \Psi^i{}_\pm{}^\Lambda\; \bar\Psi_{j\,\pm}{}^\Sigma \gamma_{ab} \Psi^j{}_\pm{}^\Gamma\,, \\
C_\pm|_{\mathcal{G}^\pm} &= \mathcal{G}^\pm{}_\Lambda\, C_\pm{}^\Lambda - \frac{1}{4}\, \mathcal{G}^\pm{}_{\Lambda\Sigma} \left[\varepsilon_{ik}\varepsilon_{jl}\, B^{ij}{}_\pm{}^\Lambda\, B^{kl}{}_\pm{}^\Sigma + 2\, F_{ab}{}^{\mp\,\Lambda} F^{ab\mp\,\Sigma} \pm 4\mathrm{i}\, \bar\Lambda_i^{\pm\Lambda}\, \Psi^i{}_\pm{}^\Sigma\right] \\
&\quad \mp \frac{1}{4}\mathrm{i}\, \mathcal{G}^\pm{}_{\Lambda\Sigma\Gamma} \left[\varepsilon_{jk}\, B^{ij}{}_\pm{}^\Lambda\, \bar\Psi_{i\pm}{}^\Sigma\, \Psi^k{}_\pm{}^\Gamma + \frac{1}{2}\, \bar\Psi^\Lambda_{i\pm} F_{ab}{}^{\mp\,\Sigma}\, \gamma^{ab}\, \Psi^i{}_\pm{}^\Gamma\right] \\
&\quad \pm \frac{1}{192} \mathcal{G}^\pm{}_{\Lambda\Sigma\Gamma\Xi}\, \bar\Psi_{i\pm}{}^\Lambda \gamma_{ab} \Psi^i{}_\pm{}^\Sigma\; \bar\Psi_{j\pm}{}^\Gamma\, \gamma^{ab}\, \Psi^j{}_\pm{}^\Xi\,.
\end{aligned}
\tag{4.6}
$$

Here $\mathcal{G}^\pm{}_{\Lambda\Sigma\cdots}$ denote multiple derivatives of the function $\mathcal{G}^\pm$ with respect to the $\Phi^\Lambda{}_\pm, \Phi^\Sigma{}_\pm, \ldots$

4.2 The Weyl multiplet as a reduced chiral multiplet

The Weyl multiplet is related to a sum of a chiral anti-self-dual tensor multiplet and an anti-chiral self-dual multiplet. Because a self-dual and an anti-self-dual tensor carry three components, both these multiplets contain $24 \oplus 24$ degrees of freedom. The Weyl multiplet is then obtained by imposing a constraint on this reducible field representation that leads to a different supermultiplet of $24 \oplus 24$ degrees of freedom. From the transformation rules of the Weyl multiplet it is easy to identify this structure. Namely, the lowest components of the chiral and anti-chiral multiplets are the fields $T_{ab}{}^\pm$ and the fermionic field strengths $R(Q)_{ab}{}^{i\pm}$. As shown in (3.16), these fermionic field strengths are (anti-)chiral and their chirality and duality phases are opposite. The fact that the chiral and anti-chiral components get more and more entangled for the higher components of these multiplets, is an indication of the constraint that holds between the chiral and the anti-chiral multiplet.

Because the Weyl multiplet is the only tensor (anti-)chiral multiplet that we encounter, we will not exhibit the nature of its chiral multiplet constraint in any detail. But the relation with the chiral multiplet implies that one can construct a *scalar* chiral and anti-chiral multiplet of weight $w = 2$ by considering the square of the Weyl multiplet. These multiplets can then easily be coupled to other (anti-)chiral scalar multiplets. Therefore

	vector multiplet				tensor multiplet				hypermultiplet	
field	$X_\pm$	W_μ	$\Omega^i{}_\pm$	Y^{ij}	L^{ij}	E_a	φ^i	$G_\pm$	$A_i{}^\alpha$	$\zeta^\alpha{}_\pm$
w	1	0	$\frac{3}{2}$	2	2	3	$\frac{5}{2}$	3	1	$\frac{3}{2}$
c	∓ 1	0	$\mp\frac{1}{2}$	0	0	0	$\pm\frac{1}{2}$	± 1	0	$\pm\frac{1}{2}$
γ_5			$\pm$				$\pm$			$\pm$

Table 4. Weyl weights w, chiral $\mathrm{SO}(1,1)$ weights c, chirality γ_5 of the spinors for the $4D$ Euclidean vector multiplet, the tensor multiplet and the hypermultiplet.

we present this square of the Weyl multiplet, and list its explicit (anti-)chiral multiplet components,

$$\begin{aligned}
A_\pm|_{\mathrm{W}^2} &= \big(T_{ab}{}^{\mp}\big)^2\,,\\
\Psi^i{}_\pm|_{\mathrm{W}^2} &= -\,16\,T^{\mp\,ab}\,R(Q)_{ab}{}^{i}{}_{\pm}\,,\\
B^{ij}{}_\pm|_{\mathrm{W}^2} &= 16\,T^{\mp\,ab}\varepsilon^{k(i}\,R(\mathcal{V})_{ab}{}^{j)}{}_k \mp 64\mathrm{i}\,\varepsilon^{k(i}\,\bar R(Q)_{ab\,k\,\pm}\,R(Q)^{ab\,j)}{}_\pm\,,\\
F_{ab}{}^{\mp}|_{\mathrm{W}^2} &= -\,16\,\mathcal{R}(M)^{cd}{}_{ab}\,T_{cd}{}^{\mp} \pm 16\mathrm{i}\,\bar R(Q)_{cd\,i\,\pm}\,\gamma_{ab}\,R(Q)^{cd\,i}{}_\pm\,,\\
\Lambda^i{}_\pm|_{\mathrm{W}^2} &= -\,32\,\gamma^{ab}R(Q)_{cd}{}^{i}{}_\pm\mathcal{R}(M)^{cd}{}_{ab} + 16\,\big(\mathcal{R}(S)_{ab}{}^{i}{}_\pm - 3\mathrm{i}\,\gamma_{[a}D_{b]}\,\chi^i_\mp\big)T^{\mp ab}\\
&\quad -\,64\,R(\mathcal{V})_{ab}{}^{i}{}_j\,R(Q)^{ab\,j}{}_\pm\,,\\
C_\pm|_{\mathrm{W}^2} &= -\,64\,\mathcal{R}(M)^{cd\mp}{}_{ab}\,\mathcal{R}(M)_{cd}{}^{\mp\,ab} - 32\,R(\mathcal{V})^{\mp\,ab\,i}{}_j\,R(\mathcal{V})_{ab}{}^{\mp j}{}_i\\
&\quad +16\,T^{\mp\,ab}D_aD^cT_{cb}{}^{\pm} \mp 128\mathrm{i}\,\bar{\mathcal{R}}(S)^{ab}{}_{i\pm}\,R(Q)_{ab}{}^{i}{}_\pm \mp 384\,\bar R(Q)^{ab}{}_{i\pm}\gamma_aD_b\chi^i{}_\mp\,.
\end{aligned}\qquad (4.7)$$

In view of its Weyl weight one can construct an invariant action that is linearly proportional to the various components indicated above, by making use of the density formula (4.4). This is the action of conformal $N=2$ supergravity. It is worth noting that the corresponding Lagrangians $\mathcal{L}_+$ and $\mathcal{L}_-$ will differ by a total derivative.

5 The vector and tensor multiplets and the hypermultiplet

In this section we briefly discuss the three relevant matter multiplets in $N=2$ supersymmetry. The vector and the tensor multiplets are defined off-shell and each comprise $8\oplus 8$ off-shell degrees of freedom. The hypermultiplet is only defined in an on-shell version. The field content of these multiplets is summarized in table 4 together with their Weyl and chiral weights. The derivation follows again by dimensional reduction of the $5D$ Minkowski theory, which is presented in appendix B. Below we directly proceed to the results for the $4D$ Euclidean supermultiplets.

5.1 The vector supermultiplet

The $4D$ Euclidean vector supermultiplet involves two real scalar fields X_+ and X_-, one symplectic Majorana spinor Ω^i, the gauge field W_μ and an auxiliary field Y^{ij} obeying the pseudo-reality condition $(Y^{ij})^* \equiv Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}\,Y^{kl}$. Here we present the non-abelian version, where g will denote the gauge coupling constant. Therefore we write all the vector multiplet components as matrices that take their values in the Lie algebra of the gauge group. The Q- and S-supersymmetry transformations of the $4D$ Euclidean supermultiplet components are as follows,

$$
\begin{aligned}
\delta X_\pm &= \pm\,\mathrm{i}\,\bar\epsilon_{i\pm}\,\Omega^i{}_\pm\,,\\
\delta W_\mu &= \bar\epsilon_{i+}\big(\gamma_\mu\Omega^i{}_- - 2\mathrm{i}\,X_-\psi_\mu{}^i{}_+\big) - \bar\epsilon_{i-}\big(\gamma_\mu\Omega^i{}_+ - 2\mathrm{i}\,X_+\psi_\mu{}^i{}_-\big)\\
\delta\Omega^i{}_\pm &= -\,2\mathrm{i}\,\not{D}X_\pm\,\epsilon^i{}_\mp - \frac{1}{2}\Big[\hat F(W)_{ab}{}^\mp - \frac14 X_\mp\,T_{ab}{}^\mp\Big]\gamma^{ab}\epsilon^i{}_\pm - \varepsilon_{kj}\,Y^{ik}\epsilon^j{}_\pm\\
&\quad + 2g[X_\pm,\,X_\mp]\epsilon^i{}_\pm + 2\,X_\pm\,\eta^i{}_\pm\,,\\
\delta Y^{ij} &= 2\,\varepsilon^{k(i}\,\bar\epsilon_{k+}\not{D}\Omega^{j)}{}_- - 2\,\varepsilon^{k(i}\,\bar\epsilon_{k-}\not{D}\Omega^{j)}{}_+\\
&\quad - 4\mathrm{i}g\,\varepsilon^{k(i}\,\bar\epsilon_{k+}[X_-,\,\Omega^{j)}_+] + 4\mathrm{i}g\,\varepsilon^{k(i}\,\bar\epsilon_{k-}[X_+,\,\Omega^{j)}_-]\,.
\end{aligned}
\tag{5.1}
$$

The supercovariant field strength $\hat F(W)_{\mu\nu}$ equals the non-abelian version of the expression (B.2),

$$
\begin{aligned}
\hat F(W)_{\mu\nu} &= 2\,\partial_{[\mu}W_{\nu]} - g[W_\mu,\,W_\nu] + \bar\psi_{i[\mu}\gamma_{\nu]}\Omega^i{}_+ - \bar\psi_{i[\mu}\gamma_{\nu]}\Omega^i{}_-\\
&\quad + \mathrm{i}\,X_-\bar\psi_{\mu\,i}\,\psi_\nu{}^i{}_+ - \mathrm{i}\,X_+\bar\psi_{\mu\,i}\,\psi_\nu{}^i{}_-\,.
\end{aligned}
\tag{5.2}
$$

For convenience, we also note the following result for the superconformally invariant field strength,

$$
\delta\Big[\hat F(W)_{\mu\nu} - \frac14 X_+\,T_{\mu\nu}{}^+ - \frac14 X_-\,T_{\mu\nu}{}^-\Big] = -2\,\bar\epsilon_i\gamma^5\gamma_{[\mu}D_{\nu]}\Omega^i + \mathrm{i}\,\bar\eta_i\gamma^5\gamma_{\mu\nu}\Omega^i\,, \tag{5.3}
$$

where on the right-hand side we refrained from writing the spinors in terms of chiral components to make the expression shorter.

The transformation rules (5.1) close according to the superconformal algebra indicated in the equations (3.18), (3.19) and (3.20), where the gauge transformation $\delta_{\rm gauge}$ in (3.18) is associated with the vector multiplet, whose field-dependent parameter θ equals

$$
\theta = 4\mathrm{i}\,\bar\epsilon_{2i-}\epsilon_1{}^i{}_-X_+ - 4\mathrm{i}\,\bar\epsilon_{2i+}\epsilon_1{}^i{}_+X_-\,, \tag{5.4}
$$

where $X_\pm$ are the two real scalar fields of the vector multiplet. For every independent vector multiplet there will be a corresponding gauge transformation.

5.2 The tensor supermultiplet

The $4D$ Euclidean tensor supermultiplet comprises a triplet of scalar fields $L_{ij} = \varepsilon_{ik}\varepsilon_{jl}\,L^{kl}$, a symplectic Majorana spinor φ^i, an anti-symmetric tensor gauge field $E_{\mu\nu}$ and two (auxiliary) scalars $G_\pm$. Their Q- and S-supersymmetry transformations are

$$
\begin{aligned}
\delta L^{ij} &= 2\mathrm{i}\,\bar\epsilon_{k+}\,\varphi^{(i}{}_+\,\varepsilon^{j)k} - 2\mathrm{i}\,\bar\epsilon_{k-}\,\varphi^{(i}{}_-\,\varepsilon^{j)k}\,,\\
\delta E_{\mu\nu} &= \mathrm{i}\,\bar\epsilon_{i+}\,\gamma_{\mu\nu}\,\varphi^i_+ + \mathrm{i}\,\bar\epsilon_{i-}\,\gamma_{\mu\nu}\,\varphi^i_- + 2\,\varepsilon_{jk}\,L^{ij}\big(\bar\epsilon_{i+}\,\gamma_{[\mu}\,\psi_{\nu]-}{}^k + \bar\epsilon_{i-}\,\gamma_{[\mu}\,\psi_{\nu]+}{}^k\big)\,,\\
\delta\varphi^i{}_\pm &= -\,\mathrm{i}\,\varepsilon_{jk}\not{D}L^{ij}\,\epsilon^k{}_\mp - \mathrm{i}\,\hat{\not{E}}\,\epsilon^i{}_\mp + G_\pm\,\epsilon^i{}_\pm + 2\,L^{ij}\varepsilon_{jk}\,\eta^k{}_\pm\,,\\
\delta G_\pm &= \mp\,2\,\bar\epsilon_{i\mp}\not{D}\varphi^i{}_\pm \pm 6\mathrm{i}\,\varepsilon_{jk}L^{ij}\,\bar\epsilon_{i\mp}\chi^k{}_\mp \mp \frac18\mathrm{i}\,T^\pm_{ab}\,\bar\epsilon_{i\mp}\gamma^{ab}\varphi^i{}_\mp \pm 2\mathrm{i}\,\bar\eta_{i\pm}\varphi^i{}_\pm\,.
\end{aligned}
\tag{5.5}
$$

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where $\hat{E}^\mu$ denotes the supercovariant field strength associated with $E_{\mu\nu}$,

$$\hat{E}^\mu = \frac{1}{2}\, e^{-1} \varepsilon^{\mu\nu\rho\sigma} \Big[\partial_\nu E_{\rho\sigma} - \frac{1}{2} \mathrm{i}\, \bar{\psi}_{\nu i} \gamma_{\rho\sigma} \varphi^i - \frac{1}{2}\, \varepsilon_{jk} L^{ij}\, \bar{\psi}_{\nu i} \gamma_\rho \psi_\sigma{}^k \Big] \,. \tag{5.6}$$

This vector defines the fully supersymmetric field strength of the tensor field, which is subject to a Bianchi identity that involves $D_\mu \hat{E}^\mu$.

The above transformation rules close according to the superconformal algebra, and just as in the case of the vector multiplet, there is a new contribution to the gauge transformation δ_{gauge} in (3.18) related to a tensor gauge transformation $\delta E_{\mu\nu} = 2\, \partial_{[\mu} \theta_{\nu]}$ with parameter,

$$\theta_\mu = 2\, \bar{\epsilon}_{2i} \gamma_\mu \epsilon_1{}^k\, \varepsilon_{kj}\, L^{ij} \,, \tag{5.7}$$

where L^{ij} is the scalar triplet field of the tensor multiplet. For every independent tensor multiplet there will be such a corresponding tensor gauge transformation.

There is an intriguing relationship between the abelian vector multiplet and the tensor multiplet. Namely, the vector multiplet field Y^{ij} transforms identically as the scalar L^{ij} of the tensor field: they are both SU(2) triplets, their Weyl and chiral weights are identical and they are invariant under S-supersymmetry. Furthermore, under Q-supersymmetry they transform into a spinor that transforms as an SU(2) doublet $\varphi^i{}_\pm$ and $\mathrm{i}\, \not{D} \Omega^i{}_\mp$, respectively. This shows that Y^{ij} transforms as the lowest component of a tensor multiplet. This is analogous to the situation that we encountered for a chiral supermultiplet of weight $w = 0$, whose highest component transforms as a $w = 2$ anti-chiral supermultiplet.

A related feature is that there exists a supersymmetric density for the product of a vector with a tensor multiplet,

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{VT}} = {} & X_+\, G_+ + X_-\, G_- + \frac{1}{2} Y^{ij} L_{ij} + \frac{1}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}\, E_{\rho\sigma} \\ & + \big(\mathrm{i}\, \bar{\Omega}_{i+} + X_+\, \bar{\psi}_{\mu i -} \gamma^\mu \big)\, \varphi^i{}_+ - \big(\mathrm{i}\, \bar{\Omega}_{i-} + X_-\, \bar{\psi}_{\mu i +} \gamma^\mu \big)\, \varphi^i{}_- \\ & + \frac{1}{2} \big(\bar{\Omega}_{i+} \gamma^\mu \psi_\mu{}^k{}_- - \bar{\Omega}_{i-} \gamma^\mu \psi_\mu{}^k{}_+ \\ & + \mathrm{i}\, X_+\, \bar{\psi}_{\mu i -} \gamma^{\mu\nu} \psi_\nu{}^k{}_- - \mathrm{i}\, X_-\, \bar{\psi}_{\mu i +} \gamma^{\mu\nu} \psi_\nu{}^k{}_+ \big)\, \varepsilon^{ij}\, L_{jk} \,. \end{aligned} \tag{5.8}$$

Observe that the term proportional to the vector field strength times the tensor gauge field is invariant under tensor gauge transformations up to a total derivative. This expression takes a similar form as in the Minkowski theory [34].

We should point out that it is also possible to define a vector multiplet in terms of the components of a tensor multiplet. However, in that case one has to multiply by non-trivial functions of the tensor multiplet scalars in order to bridge the gap in the conformal dimensions. The resulting Lagrangians will then contain higher-derivative couplings [35].

5.3 Hypermultiplets

The supersymmetry transformations for the $4D$ Euclidean hypermultiplets read as follows,

$$\begin{aligned} \delta A_i{}^\alpha &= 2\mathrm{i}\, \bar{\epsilon}_{i+}\, \zeta^\alpha{}_+ - 2\mathrm{i}\, \bar{\epsilon}_{i-}\, \zeta^\alpha{}_- \,, \\ \delta \zeta^\alpha_\pm &= -\,\mathrm{i}\, \not{D} A_i^\alpha\, \epsilon^i_\mp - 2g\, X_\mp{}^\alpha{}_\beta\, A_i^\beta \epsilon^i_\pm + A_i{}^\alpha \eta^i_\pm \,. \end{aligned} \tag{5.9}$$

Here, the $A_i{}^\alpha$ are the local sections of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle. We note the existence of a covariantly constant anti-symmetric tensor $\Omega_{\alpha\beta}$ (and its complex conjugate $\Omega^{\alpha\beta}$ satisfying the reality condition $\Omega_{\alpha\gamma}\,\Omega^{\beta\gamma} = \delta_\alpha{}^\beta$), which in principle depends on the scalars. The symplectic Majorana condition for the spinors reads as $C^{-1}\bar\zeta_\alpha{}^{\mathrm{T}} = \Omega_{\alpha\beta}\,\zeta^\beta$. Covariant derivatives contain the $\mathrm{Sp}(r)$ connection $\Gamma_A{}^\alpha{}_\beta$, associated with field-dependent $\mathrm{Sp}(r)$ rotations of the fermions, while the $\mathrm{Sp}(1)$ connections are provided by the $\mathrm{SU}(2)$ R-symmetry gauge fields $\mathcal{V}_\mu{}^i{}_j$. The sections $A_i{}^\alpha$ are pseudo-real, i.e. they are subject to the constraint, $A_i{}^\alpha \varepsilon^{ij}\Omega_{\alpha\beta} = A^j{}_\beta \equiv (A_j{}^\beta)^*$. The target-space is a hyper-Kähler cone whose metric is encoded in the so-called hyper-Kähler potential. In (B.13) we wrote the transformation rules for the case that the hyper-Kähler cone is flat. The extension to non-trivial hyper-Kähler cone geometries is straightforward and has been described in [36, 37]. We have also included an optional coupling to a number of abelian or non-abelian vector multiplets with a generic gauge coupling g. The corresponding vector multiplets are then written as matrices which take their values in the Lie algebra associated with the gauge group. For instance, we have the vector multiplet scalars $X_\pm{}^\alpha{}_\beta \equiv X_\pm{}^I\,(t_I)^\alpha{}_\beta$, where the t_I denote the gauge group generators. Of course, this requires that the tensor $\Omega_{\alpha\beta}$, as well as related quantities, will be invariant under the gauge group. In addition the covariant derivatives must be covariantized also with respect to the gauge group. Note that the latter must be a subgroup of $\mathrm{Sp}(r)$.

It is possible to construct a tensor multiplet from the product of two hypermultiplets [34]. This product is defined in terms of a numerical tensor $\eta_{\alpha\beta}$ satisfying

$$\eta_{\alpha\beta} = \Omega_{\gamma\alpha}\,\eta^{\gamma\delta}\,\Omega_{\delta\beta}\,, \tag{5.10}$$

with $\eta^{\alpha\beta} \equiv (\eta_{\alpha\beta})^*$. The tensor multiplet components are then given by the following combinations of hypermultiplet components $(A_i{}^\alpha,\, \zeta^\alpha)$ and $(A_i^{\prime\alpha},\, \zeta^{\prime\alpha})$:

$$\begin{aligned}
L^{ij} &= -\,\varepsilon^{k(i}\varepsilon^{j)l}\,A_k{}^\alpha\,A_l^{\prime\beta}\,\eta_{\alpha\beta}\,,\\
\varphi^i &= \varepsilon^{ij}\big(A_j{}^\alpha\,\zeta^{\prime\beta} + A_j^{\prime\beta}\,\zeta^\alpha\big)\eta_{\alpha\beta}\,,\\
\hat E_\mu &= \frac{1}{2}\,\varepsilon^{ij}\big(A_i{}^\alpha\,D_\mu A_j^{\prime\beta} - D_\mu A_i{}^\alpha\,A_j^{\prime\beta}\big)\eta_{\alpha\beta} + \Omega^{\alpha\gamma}\eta_{\alpha\beta}\,\bar\zeta_{\gamma-}\gamma_\mu\,\zeta^{\prime\beta}_{+} - \Omega^{\alpha\gamma}\eta_{\alpha\beta}\,\bar\zeta_{\gamma+}\gamma_\mu\,\zeta^{\prime\beta}_{-}\,,\\
G_\pm &= \mp\,2\mathrm{i}\,\Omega^{\alpha\gamma}\eta_{\alpha\beta}\,\bar\zeta_{\gamma\pm}\zeta^{\prime\beta}_{\pm}
\end{aligned} \tag{5.11}$$

The hypermultiplet is only defined on shell, and this will reflect itself as a constraint on the field $\hat E_\mu$, which can actually be identified with the Bianchi identity for the tensor field that we mentioned earlier. Note that in the above equations we have suppressed the optional gauging for the underlying hypermultiplets.

Finally we present the supersymmetric Lagrangian density for hypermultiplets that

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transform under a certain local gauge group,

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm H} = {}& \frac{1}{2}\,\varepsilon^{ij}\,\Omega_{\alpha\beta}\,A_i{}^\alpha\Big(D^a D_a + \frac{3}{2}\,D\Big)A_j{}^\beta \\
&- \Big(\bar\zeta_\alpha + \frac{1}{2}{\rm i}\,\bar\psi_{\mu i}\gamma^\mu A^i{}_\alpha\Big)\,\gamma^5\Big[\not{D}\zeta^\alpha - {\rm i}\,\Big(\frac{3}{2}\,\chi^k A_k{}^\alpha - \frac{1}{4}\,T_{ab}\gamma^{ab}\zeta^\alpha\Big) + {\rm i}\,g\,\Omega^{k\,\alpha}{}_\beta A_k{}^\beta\Big] \\
&+ 2{\rm i}\,g\,\bar\zeta_{\alpha -}\,X_-{}^\alpha{}_\beta\,\zeta_-^\beta - 2{\rm i}\,g\,\bar\zeta_{\alpha +}\,X_+{}^\alpha{}_\beta\,\zeta_+^\beta \\
&+ \Omega_{\alpha\beta}\Big(2\,g^2\,\varepsilon^{ij}\,A_i{}^\alpha X_-{}^\beta{}_\gamma X_+{}^\gamma{}_\delta\,A_j{}^\delta - \frac{1}{2}\,g\,A_i{}^\alpha\,Y^{ij\,\beta}{}_\gamma\,A_j{}^\gamma\Big) \\
&+ \frac{1}{2}{\rm i}\,g\,A^i{}_\alpha\,\bar\Omega_i{}^\alpha{}_\beta\,\gamma^5\,\zeta^\beta\,,
\end{aligned}
\tag{5.12}
$$

where the covariant derivatives contain the gauge fields associated with the optional gauge invariance.

5.4 The vector multiplet as a reduced chiral multiplet

The vector multiplet is related to the reducible combination of a chiral and an anti-chiral scalar multiplet, just as the Weyl multiplet was related to a chiral and an anti-chiral tensor multiplet. On this reducible multiplet one imposes a supersymmetric constraint. This can only be done provided the (anti-)chiral multiplets carry weight $w = 1$. Since a scalar (anti-) chiral multiplet carries $8 \oplus 8$ degrees of freedom, one ends up again with $8 \oplus 8$ degrees of freedom.

Denoting the components of the (anti-)chiral multiplets by $(A_\pm,\, \Psi^i{}_\pm,\, B^{ij}{}_\pm,\, F^\mp_{ab},\, \Lambda^i{}_\pm, C_\pm)|_V$, the supersymmetric constraint identifies the fields $B^{ij}{}_+$ and $B^{ij}{}_-$, and it leads to a Bianchi identity that involves $F_{ab}{}^+$ and $F_{ab}{}^-$, modified by extra terms induced by supergravity; the higher components $\Lambda^i{}_\pm$ and $C_\pm|_V$ are expressed in terms of the lower components.

Conversely, this construction enables one to define a chiral and an anti-chiral scalar multiplet in terms of the vector multiplet components. This leads to the following result,

$$
\begin{aligned}
A_\pm|_{\rm V} ={}& X_\pm\,, \\
\Psi^i{}_\pm|_{\rm V} ={}& \Omega^i{}_\pm\,, \\
B^{ij}{}_\pm|_{\rm V} ={}& -\,Y^{ij}\,, \\
F_{ab}{}^\mp|_{\rm V} ={}& \hat F_{ab}{}^\mp - \frac{1}{4}\,X_\mp\,T_{ab}{}^\mp\,, \\
\Lambda^i{}_\pm|_{\rm V} ={}& {\rm i}\,\not{D}\Omega^i{}_\mp\,, \\
C_\pm|_{\rm V} ={}& 2\,\Box_c X_\mp + \frac{1}{4}\,\Big(\hat F_{ab}{}^\pm - \frac{1}{4}\,X_\pm T_{ab}{}^\pm\Big)\,T^{ab\,\pm} \mp 3{\rm i}\,\bar\chi_{i\,\mp}\,\Omega^i{}_\mp\,,
\end{aligned}
\tag{5.13}
$$

where $\Box_c \equiv D^a D_a$ is the superconformal d'Alembertian obtained by the product of two fully superconformal derivatives. With these identifications, the transformation rules (5.1) induce the proper transformations for the (anti-)chiral multiplets with $w = 1$. We will make use of this result in the next section.

6 Locally supersymmetric actions of vector multiplets

Following the standard procedure that was originally used for locally supersymmetric vector multiplet actions in Minkowski space [38, 39], on chooses two functions $\mathcal{F}^\pm(X_\pm)$ depending on several fields $X_\pm{}^I$ which are the lowest components of a set of vector multiplets labeled by an index I. According to (5.13) the chiral and the anti-chiral multiplets both involve all the vector multiplet fields, albeit in mutually different ways. The functions $\mathcal{F}^\pm(X_\pm)$ may also depend on some independent (anti-)chiral multiplets which can serve as a background. For simplicity we suppress such background fields for now and restrict ourselves to vector multiplets only. A possible dependence of the functions on chiral background fields, crucial for the construction of a large class of higher-derivative actions, will be presented later in this section.

In order to have local supersymmetry, the functions $\mathcal{F}^\pm$ must be homogeneous functions of degree two, so that

$$\mathcal{F}^\pm(\lambda\, X_\pm{}^I) = \lambda^2\, \mathcal{F}^\pm(X_\pm{}^I)\,. \tag{6.1}$$

Using (4.6) and (5.13) we determine the various components of the (anti-)chiral multiplets defined by $\mathcal{F}^\pm(X_\pm{}^I)$ which have Weyl weight $w=2$. Subsequently one uses the invariant density formula (4.4) to obtain the Lagrangian,

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-\,, \tag{6.2}$$

which leads to the following bosonic terms,

$$\begin{aligned} e^{-1}\mathcal{L}_\pm =& -\mathcal{F}^\pm{}_I\, \Box_c\, X_\mp{}^I - \frac{1}{8}\mathcal{F}^\pm{}_I \left(F_{ab}{}^{\pm I} - \frac{1}{4}\, X_\pm{}^I T_{ab}{}^\pm\right) T^{ab\,\pm} \\ &+ \frac{1}{4}\,\mathcal{F}^\pm{}_{IJ} \left(F_{ab}{}^{\mp I} - \frac{1}{4}\, X_\mp{}^I\, T_{ab}{}^\mp\right) \left(F^{ab\,\mp J} - \frac{1}{4}\, X_\mp{}^J\, T^{ab\,\mp}\right) \\ &+ \frac{1}{8}\,\mathcal{F}^\pm{}_{IJ}\, Y^{ij\,I}\, Y_{ij}{}^J - \frac{1}{32}\,\mathcal{F}^\pm\,(T_{ab}{}^\pm)^2\,, \end{aligned} \tag{6.3}$$

where $\mathcal{F}^\pm{}_{IJ\cdots}$ denotes derivatives of $\mathcal{F}^\pm$ with respect to $X_\pm{}^I, X_\pm{}^J,\ldots$. Here the bosonic terms of the superconformal d'Alembertian $\Box_c$ acting on $X_\mp{}^I$ are equal to

$$\Box_c X_\mp{}^I = (\partial_\mu \mp A_\mu)^2 X_\mp{}^I + \left(\frac{1}{6}\, R - D\right) X_\mp{}^I, \tag{6.4}$$

where we have used the bosonic part of $f_\mu{}^a$ defined in (3.8). Note that the dependence on the dilatational gauge field b_μ has disappeared and R denotes the standard Ricci scalar. Upon dropping a total derivative the kinetic term for the scalars can be written as

$$e^{-1}\mathcal{L}_{\text{kinetic}} = N_{IJ}\left[(\partial_\mu + A_\mu) X_+{}^I\, (\partial^\mu - A^\mu) X_-{}^J - \left(\frac{1}{6}\, R - D\right) X_+{}^I\, X_-{}^J\right], \tag{6.5}$$

where

$$N_{IJ} = \mathcal{F}^+{}_{IJ} + \mathcal{F}^-{}_{IJ}\,, \tag{6.6}$$

which is a symmetric tensor that can be written in a suggestive way as the second derivative of $K(X_+, X_-)$, defined by

$$N_{IJ} = \frac{\partial^2 K(X_+, X_-)}{\partial X_+{}^I\, \partial X_-{}^J}\,, \tag{6.7}$$

where

$$K(X_+, X_-) = X_-{}^I\, \mathcal{F}^+{}_I(X_+) + X_+{}^I\, \mathcal{F}^-{}_I(X_-)\,, \tag{6.8}$$

These equations look rather similar to the ones that one encounters in Minkowski space, except that $\mathcal{F}^+$ and $\mathcal{F}^-$ are in principle two unrelated homogeneous functions of a number of separate real variables. The quantity (6.8) has already appeared in [8], where it was shown that it defines a prepotential for the metric (6.6) on a (affine) special para-Kähler manifold.

Apart from the fact that we have two independent real functions $\mathcal{F}^\pm(X_\pm)$ in the Euclidean theory, the structure of these Lagrangians is very similar to the structure of the Minkowski theory (see, e.g. [38, 39]). Therefore we will not try to clarify various other terms in these Lagrangians. We will also not consider the case of non-abelian vector multiplet Lagrangians, which can be derived along the same lines.

An interesting feature of these Lagrangians is that both the Euclidean and the Minkowkian $4D$ gauge theories are subject to electric-magnetic duality. Since there are subtle differences between Euclidean and Minkowskian duality (for instance a selfdual tensor is real in Euclidean space, while in Minowski space it is complex), it is of interest to investigate electric-magnetic duality in the Euclidean context. Hence let us start from a Lagrangian $\mathcal{L}(F)$ depending on n abelian self-dual and anti-self-dual field strengths $F_{\mu\nu}{}^{\pm I}$ (but not on their derivatives) and possibly on other fields. The field equations are defined in terms of the tensors

$$G^{\mu\nu\pm}{}_I = \pm\frac{1}{e}\,\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}{}^{\pm I}}\,. \tag{6.9}$$

For instance, the tensors $G^{\mu\nu\pm}{}_I$ for the model at the beginning of this section read as follows,

$$\pm\, G^{\mu\nu\pm}{}_I = \mathcal{F}^{\mp}{}_{IJ}\, F^{\mu\nu\,\pm J} - \frac{1}{4} N_{IJ}\, X_\pm{}^J\, T^{\mu\nu\,\pm}\,. \tag{6.10}$$

The corresponding Bianchi identities and equations of motion then take a similar form,

$$\partial^\mu\big(F_{\mu\nu}{}^{+I} - F_{\mu\nu}{}^{-I}\big) = 0 = \partial^\mu\big(G_{\mu\nu}{}^+{}_I - G_{\mu\nu}{}^-{}_I\big)\,. \tag{6.11}$$

Obviously these equations are invariant under the electric-magnetic duality transformations,

$$\begin{pmatrix} F_{\mu\nu}{}^{\pm I} \\ G_{\mu\nu}{}^\pm{}_J \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde F_{\mu\nu}{}^{\pm I} \\ \tilde G_{\mu\nu}{}^\pm{}_J \end{pmatrix} = \begin{pmatrix} U^I{}_K & Z^{IL} \\ W_{JK} & V_J{}^L \end{pmatrix} \begin{pmatrix} F_{\mu\nu}{}^{\pm K} \\ G_{\mu\nu}{}^\pm{}_L \end{pmatrix}\,, \tag{6.12}$$

where $\tilde F_{\mu\nu}{}^{I\pm}$, and $\tilde G_{\mu\nu J}{}^\pm$ denote the transformed field strengths (and not the Hodge dual), and the $n\times n$ submatrices $U^I{}_K$, Z^{IL}, W_{JK} and $V_J{}^L$ are real. The relevant question is whether the rotated tensors $\tilde G_{\mu\nu}{}^\pm{}_J$ can again follow from a new Lagrangian $\tilde{\mathcal{L}}(\tilde F)$ in analogy with (6.9). In that case there may be a different Lagrangian that leads to an equivalent

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set of Bianchi identities and equations of motion. As it turns out, this poses the following restriction on the matrices in (6.12),

$$\begin{aligned} &U^{\mathrm{T}} V - W^{\mathrm{T}} Z = V U^{\mathrm{T}} - W Z^{\mathrm{T}} = \mathbb{1}\,, \\ &U^{\mathrm{T}} W = W^{\mathrm{T}} U\,, \qquad\qquad Z^{\mathrm{T}} V = V^{\mathrm{T}} Z\,. \end{aligned} \tag{6.13}$$

which are equivalent to

$$\begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} U & Z \\ W & V \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

Hence the electric-magnetic dualities form a group of equivalence transformations that connect different Lagrangians which describe the same physics. As is clear from this group is equal to $\mathrm{Sp}(2n;\mathbb{R})$ which is the same group that is relevant for the Minkowski theory [40].

The above results follow from the observation that the two Lagrangians $\tilde{\mathcal{L}}$ and $\mathcal{L}$ are related by

$$\tilde{\mathcal{L}}(\tilde{F}) - \frac{1}{4}\tilde{F}_{\mu\nu}{}^{+I}\,\tilde{G}^{\mu\nu}{}_I{}^{+} - \frac{1}{4}\tilde{F}_{\mu\nu}{}^{-I}\,\tilde{G}^{\mu\nu}{}_I{}^{-} = \mathcal{L}(F) - \frac{1}{4}F_{\mu\nu}{}^{I+}\,G^{\mu\nu}{}_I{}^{+} - \frac{1}{4}F_{\mu\nu}{}^{I-}\,G^{\mu\nu}{}_I{}^{-}\,, \tag{6.14}$$

up to terms that are independent of $F_{\mu\nu}{}^{\pm I}$. However, by carrying out the electric-magnetic duality transformation one only redefines the field strengths and not the other fields. As a consequence supersymmetry will no longer be manifest. To avoid this unwelcome feature, one can also make a similar transformation on the scalar fields. This transformation takes the following form,

$$\begin{pmatrix} X_\pm{}^I \\ \mp\mathcal{F}^\pm{}_J \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}_\pm{}^I \\ \mp\tilde{\mathcal{F}}^\pm{}_J \end{pmatrix} = \begin{pmatrix} U^I{}_K & Z^{IL} \\ W_{JK} & V_J{}^L \end{pmatrix} \begin{pmatrix} X_\pm{}^K \\ \mp\mathcal{F}^\pm{}_L \end{pmatrix}, \tag{6.15}$$

where $\tilde{X}_\pm{}^I$ and $\tilde{\mathcal{F}}^\pm{}_J$ denote the new scalar fields and the first derivatives of a new function $\tilde{\mathcal{F}}^\pm(\tilde{X}_\pm)$. This function will only exist provided (6.13) is satisfied. This is all in line with what is known from the Minkowski theory [38, 41, 42].

We close this section by discussing the possibility of including an additional (anti-)chiral superfield of Weyl weight w into the functions $\mathcal{F}^\pm$. This new field will not correspond to a vector multiplet and can be considered as a background. In that case the homogeneity requirement for $\mathcal{F}^\pm$ must involve the background field,

$$\mathcal{F}^\pm(\lambda X_\pm{}^I, \lambda^w \widehat{A}_\pm) = \lambda^2\,\mathcal{F}^\pm(X_\pm{}^I, \widehat{A}_\pm)\,. \tag{6.16}$$

We can then construct the corresponding invariant action following the same procedure outlined at the beginning of this section. The resulting bosonic action is (6.2), supplemented by terms that depend on the derivatives of the prepotentials with respect to the (anti-)chiral superfield,

$$\mathcal{L}_A = \mathcal{L}_{A+} + \mathcal{L}_{A-}\,, \tag{6.17}$$

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with

$$\mathcal{L}_{A\pm} = -\frac{1}{2}\,\mathcal{F}^{\pm}{}_{\widehat{A}}\,\widehat{C}_{\pm} - \frac{1}{4}\,\mathcal{F}^{\pm}{}_{\widehat{A}I}\,\widehat{B}_{\pm}{}^{ij}\,Y_{ij}^{\ I} + \frac{1}{2}\,\mathcal{F}^{\pm}{}_{\widehat{A}I}\,\widehat{F}^{ab\,\mp}\Big(F_{ab}{}^{\mp I} - \frac{1}{4}\,X_{\mp}{}^{I}T_{ab}{}^{\mp}\Big)$$
$$+\frac{1}{8}\,\mathcal{F}^{\pm}{}_{\widehat{A}\widehat{A}}\,\widehat{B}_{\pm}{}^{ij}\,\widehat{B}_{\pm}{}^{kl}\,\varepsilon_{ik}\,\varepsilon_{jl} + \frac{1}{4}\,\mathcal{F}^{\pm}{}_{\widehat{A}\widehat{A}}\,\widehat{F}_{ab}{}^{\mp}\,\widehat{F}^{ab\,\mp}\,. \tag{6.18}$$

The possibility of including such a dependence on an (anti-)chiral superfield allows for the inclusion of higher-derivative terms in the action of Euclidean $4D$, $N = 2$ superconformal gravity coupled to vector multiplets, by choosing the square of the Weyl multiplet as the (anti-)chiral background. In this case, the first term in (6.18) contains the square of the modified Lorentz curvature according to (4.7).

7 Summary and conclusions

In this paper we have derived off-shell supergravity in four Euclidean dimensions with eight supersymmetries, together with an extended superconformal multiplet calculus that can be used to construct a large variety of invariant actions. Its general structure closely resembles that of Minkowski $N = 2$ supergravity. The two theories have an equal number of supersymmetries and therefore they have a corrreponding variety of supermultiplets. Nevertheless there are a number of crucial differences. The Euclidean theory has symplectic Majorana spinors and an $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$ R-symmetry group, whereas the Minkowski theory has Majorana fermions and an $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry group. These differences are then also reflected in different reality conditions on the various fields. For instance the chiral and anti-chiral supermultiplets are real and they are not related by complex conjugation. The latter fact implies that the action of the theory is built on two, not necessarily related, real functions for the chiral and the anti-chiral multiplet sector.

The non-linear transformations of Euclidean supergravity were initially derived by time-like reduction from $5D$ Minkowski supergravity, the results of which were then subsequently extended to a complete superconformal multiplet calculus. As a by-product we have obtained a dictionary between the five- and four-dimensional fields, but in this paper the five-dimensional origin only plays a secondary role.

Just as for $N = 2$ Minkowski supergravity the multiplet calculus is built on superconformal multiplets. This implies that the resulting actions are initially superconformally invariant. However, by including the actions for an appropriate set of compensating multiplets the combined action will become gauge equivalent to an off-shell Poincaré supergravity. As a result there exist different formulations of Poincaré supergravity corresponding to inequivalent compensating multiplet configurations. In [43] three choices were pointed out. One always needs one compensating vector multiplet and a second one which is either a hypermultiplet, a tensor multiplet or a so-called non-linear multiplet. Each of these multiplets contains $8\oplus 8$ bosonic and fermionic degrees of freedom. Since the superconformal multiplet contains $24\oplus 24$ bosonic and fermionic degrees of freedom, off-shell Poincaré supergravity will be based on $40\oplus 40$ degrees of freedom. Note that this list is not necessarily exhaustive.

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As it turns out the same situation exists for Euclidean supergravity. There are also three off-shell field representations with the first compensator equal to a vector multiplet and the second one a hypermultiplet, a tensor multiplet or a non-linear multiplet. The latter exists also in $5D$ Minkowski theory [44] and upon a time-like reduction can thus be realized in $4D$ Euclidean supergravity as well. We have refrained from giving its explicit transformation rules. Instead, in appendix B.4, we have presented the $5D$ transformation rules of this multiplet in the conventions of section 2.

The theory presented in this paper can be used to define Euclidean supersymmetric theories on curved spaces upon freezing out the supergravity fields, which is of interest to localization problems in field theory. A related area of interest concerns the quantum entropy of black hole solutions in supergravity obtained via localization of the quantum entropy function, where the supergravity fields are not frozen completely. This question can now be considered without having to rely on analytic continuation of the Minkowski theory. As a first step in this program, one should derive the BPS black hole solutions of the Euclidean theory that we have just presented. We intend to return to this issue in a forthcoming paper.

We should point out that the analytically continued theories that appeared in [19, 24] are related to the Euclidean supergravity theory of this paper modulo some straightforward field redefinitions. Evidently, our results are also in line with the theory of Euclidean vector multiplets presented in [8], since both make use of a time-like reduction of a $5D$ Minkowski theory.

A Off-shell dimensional reduction; the Weyl multiplet

Starting from the off-shell superconformal transformations for $5D$ supermultiplets presented in section 2, we perform a reduction of the time coordinate and obtain the $4D$ Euclidean superconformal transformations for the corresponding supermultiplets. The Weyl multiplet contains the gauge fields associated with the superconformal transformations as well as additional supercovariant fields, which act as a background for all other supermultiplets. Therefore this multiplet must be considered first. Here a subtle complication is that the Weyl multiplet becomes reducible upon the reduction. In $D = 5$ it comprises $32 \oplus 32$ bosonic and fermionic degrees of freedom, which, in the reduction to $D = 4$ dimensions, decompose into the Weyl multiplet comprising $24 \oplus 24$ degrees of freedom and a vector multiplet comprising $8 \oplus 8$ degrees of freedom.

In section 2 we also described the Kaluza-Klein decomposition of the metric and the dilatational gauge field that ensure that the $4D$ fields transform covariantly under the $4D$ diffeomorphisms. Since these decompositions involve gauge choices on the vielbein and the dilatational gauge field, compensating Lorentz and special conformal transformations must be included when deriving the $4D$ Q-supersymmetry transformations to ensure that these gauge conditions are preserved. Here the parameter of the compensating Lorentz transformation is most relevant. It is equal to

$$\varepsilon^{a5} = -\varepsilon^{5a} = \mathrm{i}\phi\,\bar{\epsilon}_i\gamma^a\psi^i \iff \varepsilon^{a0} = -\varepsilon^{0a} = \phi\,\bar{\epsilon}_i\gamma^a\psi^i\,, \tag{A.1}$$

where we assumed the standard Kaluza-Klein decomposition on the gravitino fields,

$$\psi_M{}^i = \begin{pmatrix} \psi_\mu{}^i + B_\mu \psi^i \\ -\mathrm{i}\psi^i \end{pmatrix} , \tag{A.2}$$

which ensures that $\psi_\mu{}^i$ on the right-hand side transforms as a $4D$ vector. Owing to the factor i in this decomposition both ψ_μ^i and ψ^i are symplectic Majorana spinors. Upon including this extra term, one can write down the Q- and S-supersymmetry transformations on the $4D$ fields defined above. As a result of this, the $4D$ and $5D$ supersymmetry transformation will be different. For instance, the supersymmetry transformations of the $4D$ fields $e_\mu{}^a$, ϕ and B_μ read,

$$\begin{aligned} \delta e_\mu{}^a &= \bar\epsilon_i \gamma^a \psi_\mu{}^i , \\ \delta\phi &= \mathrm{i}\phi^2\, \bar\epsilon_i \gamma^5 \psi^i , \\ \delta B_\mu &= -\, \phi^2\, \bar\epsilon_i \gamma_\mu \psi^i - \mathrm{i}\phi\, \bar\epsilon_i \gamma^5 \psi_\mu{}^i , \end{aligned} \tag{A.3}$$

where the first term in δB_μ originates from the compensating transformation (A.1). Consequently the supercovariant field strength of B_μ contains a term that is not contained in the supercovariant five-dimensional curvature $R(P)_{MN}{}^A$. Therefore the $5D$ spin-connection components are not supercovariant with respect to $4D$ supersymmetry, as is reflected in the second formula below,

$$\begin{aligned} \omega_M{}^{ab} &= \begin{pmatrix} \omega_\mu{}^{ab} \\ 0 \end{pmatrix} - \frac{1}{2}\phi^{-2}\hat F(B)^{ab} \begin{pmatrix} B_\mu \\ -\mathrm{i} \end{pmatrix} , \\ \omega_M{}^{a5} &= -\,\frac{1}{2}\mathrm{i} \begin{pmatrix} \phi^{-1}\hat F(B)_\mu{}^a - \phi\,\bar\psi_{\mu i}\gamma^a\psi^i \\ 0 \end{pmatrix} - \mathrm{i}\phi^{-2} D^a\phi \begin{pmatrix} B_\mu \\ -\mathrm{i} \end{pmatrix} . \end{aligned} \tag{A.4}$$

Here we introduced a supercovariant field strength and derivative (with respect to $4D$ supersymmetry),

$$\begin{aligned} \hat F(B)_{\mu\nu} &= 2\,\partial_{[\mu} B_{\nu]} + \phi^2\, \bar\psi_{[\mu i}\gamma_{\nu]}\psi^i + \frac{1}{2}\mathrm{i}\phi\,\bar\psi_{\mu i}\gamma_5\psi_\nu{}^i , \\ D_\mu\phi &= (\partial_\mu - b_\mu)\phi - \frac{1}{2}\mathrm{i}\phi^2\,\bar\psi_{\mu i}\gamma_5\psi^i . \end{aligned} \tag{A.5}$$

We should mention that the dilatational gauge field (as well as the composite gauge fields, such as $\omega_\mu{}^{ab}$, that depend on it) will not necessarily acquire the form that is familiar from $4D$. This may require to include an additional compensating conformal boost transformation.

Subsequently one writes corresponding Kaluza-Klein decompositions for some of the other fields of the Weyl multiplet, which do not require special gauge choices,

$$V_{Mi}{}^j = \begin{pmatrix} V_{\mu i}{}^j + B_\mu V_i{}^j \\ -\mathrm{i}V_i{}^j \end{pmatrix} , \quad \phi_M{}^i = \begin{pmatrix} \phi_\mu{}^i + B_\mu\phi^i \\ -\mathrm{i}\phi^i \end{pmatrix} \quad T_{AB} = \begin{pmatrix} T_{ab} \\ T_{a5} \equiv \frac{1}{6}\mathrm{i}A_a \end{pmatrix} . \tag{A.6}$$

Hence we are now ready to consider the Q- and S-supersymmetry transformations of the spinor fields originating from the $5D$ gravitino fields. Up to possible higher-order spinor terms, one derives the following results from (2.1),

$$
\begin{aligned}
\delta(\phi^2\,\psi^i) =& -\frac{1}{2}\Big[-\hat F(B)_{ab} + \gamma_5\phi(3\,T_{ab} + \frac{1}{4}\phi^{-1}\hat F(B)_{ab}\gamma_5)\Big]\gamma^{ab}\epsilon^i \\
&+ \mathrm{i}\big[\not{D}\phi\,\gamma^5 - \not{A}\phi\big]\epsilon^i - \phi^2 V^i{}_j\,\epsilon^j \\
&+ \gamma_5\phi\Big[\eta^i - \frac{1}{3}\mathrm{i}\not{A}\gamma_5\epsilon^i - \frac{1}{8}\gamma_5\phi^{-1}(\hat F(B)_{ab} - 4\phi T_{ab}\gamma_5)\gamma^{ab}\epsilon^i\Big]\,, \\
\delta\psi_\mu{}^i =& \,2\left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}b_\mu + \frac{1}{2}e_\mu{}^a\,A_a\gamma_5\right)\epsilon^i + V_{\mu j}{}^i\epsilon^j \\
&+ \frac{1}{2}\mathrm{i}\Big[3\,T_{ab} + \frac{1}{4}\phi^{-1}\hat F(B)_{ab}\gamma_5\Big]\gamma_{ab}\gamma_\mu\epsilon^i \\
&- \mathrm{i}\gamma_\mu\Big[\eta^i - \frac{1}{3}\mathrm{i}\not{A}\gamma_5\epsilon^i - \frac{1}{8}\gamma_5\phi^{-1}(\hat F(B)_{ab} - 4\phi T_{ab}\gamma_5)\gamma^{ab}\epsilon^i\Big]\,.
\end{aligned}
\tag{A.7}
$$

Clearly, the fields $e_\mu{}^a$ and $\psi_\mu{}^i$ must belong to the Weyl multiplet, whereas ϕ, B_μ and $\phi^2\psi^i$ correspond to the Kaluza-Klein vector multiplet, as the transformations shown in (A.3) and (A.7) have many features in common with the expected $4D$ transformations of these supermultiplets. Note that we have multiplied ψ^i with a factor ϕ^2 to give it the expected Weyl weight $w = \frac{3}{2}$. At this stage we have only identified one of the two $w = 1$ scalars that must reside in a $4D$ vector multiplet. The field A_a seems to play the role of an R-symmetry connection because it appears to covariantize the derivatives on ϕ and ϵ^i in (A.7). Furhtermore, a particular linear combination of the $5D$ tensor components T_{ab} and the (dual) supercovariant field strength $\hat F(B)_{ab}$ appears in the transformations (A.7) in precisely the same form as the $4D$ auxiliary tensor T_{ab}, so that the latter is not just proportional to the original $5D$ tensor field. The same combination will also appear in other transformation rules, as we shall see in, for instance, appendix B. Finally, S-supersymmetry transformations are accompanied by extra contributions characterized by a field-dependent parameter proportional to ϵ^i.

However, the result (A.7) is not yet complete as we have suppressed variation contributions quadratic in the spinor fields. First of all we did not include the non-covariant term in (A.4) and we ignored the compensating Lorentz transformation (A.1). Secondly we ignored the variation of the field B_μ in the decomposition of the $4D$ gravitino (A.2), and thirdly the multiplication of ψ^i with ϕ^2 will also generate a variation quadratic in ψ^i. Since these terms will play an important role we summarize them below,

$$
\begin{aligned}
\delta(\phi^2\psi^i)\big|_{\text{non-linear}} =& \,\frac{1}{2}\mathrm{i}\phi^3\,\bar\epsilon_j\gamma^a\psi^j\,\gamma_a\gamma_5\psi^i + 2\mathrm{i}\,\phi^3\,\bar\epsilon_j\gamma^5\psi^j\,\psi^i\,, \\
\delta\psi_\mu{}^i\big|_{\text{non-linear}} =& -\frac{1}{2}\mathrm{i}\phi\,\bar\psi_{\mu j}\gamma^a\psi^j\,\gamma_a\gamma^5\epsilon^i + \frac{1}{2}\mathrm{i}\phi\,\bar\epsilon_j\gamma^a\psi^j\,\gamma_a\gamma_5\psi_\mu{}^i \\
&+ \big(\phi^2\,\bar\epsilon_j\gamma_\mu\psi^j + \mathrm{i}\phi\,\bar\epsilon_j\gamma^5\psi_\mu{}^j\big)\psi^i\,.
\end{aligned}
\tag{A.8}
$$

The systematic pattern already noticed in [6] for the space-like reduction is that the $5D$ supersymmetry transformations can uniformally be decomposed in terms of the $4D$ supersymmetry transformations, and S-supersymmetry and SU(2) R-symmetry transformations with field-dependent parameters. Since the derivation is identical to what was carried out in [6], we just present the universal formula for $5D$ Q-supersymmetry transformations of fields Φ that transform covariantly in the $4D$ setting,

$$\delta_{\rm Q}(\epsilon)\big|_{5D}^{\rm reduced}\Phi = \delta_{\rm Q}(\epsilon)\big|_{4D}\Phi + \delta_{\rm S}(\tilde\eta)\big|_{4D}\Phi + \delta_{\rm SU(2)}(\tilde\Lambda)\big|_{4D}\Phi + \delta'(\tilde\Lambda^0)\Phi\,. \tag{A.9}$$

Here the first term on the right-hand side defines the $4D$ supersymmetry transformation, while $\tilde\eta$ and $\tilde\Lambda$ denote the (universal) field-dependent parameters of accompanying S-supersymmetry and SU(2) R-symmetry transformations. The last variation denoted by $\delta'(\tilde\Lambda^0)$ is a linear transformation on the fields Φ that signals the emergence of an extra component in the $4D$ Euclidean R-symmetry group. Note that $\tilde\eta$, $\tilde\Lambda$ and $\tilde\Lambda^0$ are all linearly proportional to the supersymmetry parameter ϵ^i. The explicit form of these field-dependent parameters is as follows,

$$\begin{aligned}
\tilde\eta^i =& -\frac{1}{3}{\rm i}\not{A}\gamma_5\epsilon^i - \frac{1}{8}\gamma_5\phi^{-1}\big(\hat F(B)_{ab} - 4\phi T_{ab}\gamma_5\big)\gamma^{ab}\epsilon^i \\
& -\frac{1}{4}{\rm i}\,\phi^2\Big(\bar\psi_j\gamma^5\psi^i\gamma_5 - \bar\psi_j\psi^i + \bar\psi_j\gamma^a\psi^i\gamma_a + \frac{1}{2}\bar\psi_k\gamma^5\gamma^a\psi^k\gamma_5\gamma_a\delta_j{}^i\Big)\epsilon^j\,,\\
\tilde\Lambda_j{}^i =& -{\rm i}\,\phi\Big(\bar\epsilon_j\gamma^5\psi^i - \frac{1}{2}\delta_j{}^i\bar\epsilon_k\gamma^5\psi^k\Big)\,,\\
\tilde\Lambda^0 =&\,{\rm i}\,\phi\,\bar\epsilon_k\psi^k\,.
\end{aligned} \tag{A.10}$$

After verifying that the decomposition is universally realized these extra symmetries with field-dependent coefficients can be dropped provided they define local symmetries of the $4D$ theory.

Evaluating the terms of higher order in the fermions is subtle; here we can only partly rely on the results of [6] because the phases of the spinor bilinears cannot always be converted directly from $5D$ (as noted below equation (2.6)). It leads to the following redefinitions of the various bosonic fields,

$$\begin{aligned}
\hat A_\mu =&\, A_a\,e_\mu{}^a - \frac{1}{2}{\rm i}\,\phi\,\bar\psi_j\psi_\mu{}^j - \frac{1}{4}\phi^2\,\bar\psi_j\gamma^5\gamma_\mu\psi^j\,,\\
\hat T_{ab} =&\, 24\,T_{ab} - \phi^{-1}\,\varepsilon_{abcd}\,\hat F(B)^{cd} + {\rm i}\phi^2\,\bar\psi_i\gamma_{ab}\psi^i\,,\\
\hat V_j{}^i =&\, \phi^2\,V_j{}^i + \frac{3}{2}{\rm i}\phi^3\,\bar\psi_j\,\gamma^5\psi^i\,,\\
\hat V_{\mu j}{}^i =&\, V_{\mu j}{}^i + {\rm i}\phi\Big(\bar\psi_{\mu j}\gamma^5\psi^i - \frac{1}{2}\delta_j{}^i\,\bar\psi_{\mu k}\gamma^5\psi^k\Big) + \frac{1}{2}\phi^2\,\bar\psi_j\gamma_\mu\psi^i\,.
\end{aligned} \tag{A.11}$$

Note that in the last two equations possible contributions proportional to $\bar\psi_k\gamma^5\psi^k$ and $\bar\psi_k\gamma_\mu\psi^k$ do not appear as they vanish owing to the symplectic Majorana condition.

The modifications given in (A.11) lead to important changes in the supersymmetry transformations. For instance, the S-supersymmetry transformations are given by

$$\begin{aligned}\delta\hat{A}_\mu &= \frac{1}{2}\mathrm{i}\bar{\psi}_{\mu j}\gamma^5\eta^j\,,\\ \delta\hat{T}_{ab} &= 0\,,\\ \delta\hat{V}_j{}^i &= 0\,,\\ \delta\hat{V}_{\mu j}{}^i &= -2\mathrm{i}\Big(\bar{\psi}_{\mu j}\,\eta^i - \frac{1}{2}\delta_j{}^i\,\bar{\psi}_{\mu k}\,\eta^k\Big)\,. \end{aligned}\tag{A.12}$$

In particular, note that the factor in the variation of $\hat{V}_{\mu i}{}^j$ has now changed as compared to the corresponding $5D$ S-variation given in (2.1). Furthermore, $\hat{A}_\mu$ is not supercovariant because its Q-supersymmetry variation contains a term proportional to the derivative of the supersymmetry parameter. This suggest that $\hat{A}_\mu$ will be related to a gauge field associated with an extra $4D$ R-symmetry, which will indeed be consistent with the fact that $\hat{A}_\mu$ transforms into the gravitino fields under S-supersymmetry.

Let us now present the supersymmetry transformations for the redefined fields, suppressing the field-dependent S-supersymmetry and SU(2) transformations indicated in (A.9). For the vierbein and gravitini, we find

$$\begin{aligned}\delta e_\mu{}^a &= \bar{\epsilon}_i\gamma^a\psi_\mu{}^i\,,\\ \delta\psi_\mu{}^i &= 2\left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}b_\mu + \frac{1}{2}\hat{A}_\mu\gamma_5\right)\epsilon^i + \hat{V}_{\mu j}{}^i\,\epsilon^j + \frac{1}{16}\mathrm{i}\,\hat{T}_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \mathrm{i}\gamma_\mu\eta^i\,. \end{aligned}\tag{A.13}$$

For the scalar ϕ, the spinor $\hat{\psi}^i \equiv \phi^2\psi^i$ and the Kaluza-Klein photon field B_μ we have the following Q- and S-supersymmetry transformations,

$$\begin{aligned}\delta\phi &= \mathrm{i}\,\bar{\epsilon}_i\gamma^5\hat{\psi}^i\,,\\ \delta B_\mu &= -\bar{\epsilon}_i\gamma_\mu\hat{\psi}^i - \mathrm{i}\phi\,\bar{\epsilon}_i\gamma^5\psi_\mu{}^i\,,\\ \delta\hat{\psi}^i &= \frac{1}{2}\left[\hat{F}(B)_{ab} - \frac{1}{8}\phi\,\hat{T}_{ab}\gamma_5\right]\gamma^{ab}\epsilon^i\\ &\quad - \mathrm{i}\gamma^5\gamma^\mu\left[\mathcal{D}_\mu\phi - \frac{1}{2}\mathrm{i}(\bar{\psi}_{\mu j}\gamma^5\hat{\psi}^j - \bar{\psi}_{\mu j}\hat{\psi}^j\gamma^5) - \hat{A}_\mu\phi\,\gamma^5\right]\epsilon^i + \hat{V}_j{}^i\,\epsilon^j + \phi\gamma^5\eta^i\,, \end{aligned}\tag{A.14}$$

where the derivative $\mathcal{D}_\mu$ is covariant with respect to $4D$ local Lorentz, dilatation and SU(2) transformations.

At this point we briefly make a few observations. First of all, we have suppressed the chiral transformations proportional to the field-dependent parameter $\tilde{\Lambda}^0$,

$$\delta\psi_\mu{}^i = -\frac{1}{2}\tilde{\Lambda}^0\,\gamma^5\psi_\mu{}^i\,,\quad \delta\hat{\psi}^i = -\frac{1}{2}\tilde{\Lambda}^0\,\gamma^5\hat{\psi}^i\,.\tag{A.15}$$

Clearly we were not allowed to drop these terms in view of the fact they do not correspond to local transformation of the $4D$ theory at this stage. Furthermore, the variations of $\hat{\psi}^i$ proportional to $\bar{\psi}_{\mu j}\hat{\psi}^j$ are not part of a supercovariant derivative of the field ϕ. And finally the field $\hat{A}_\mu$ is not a gauge field associated with the chiral transformations (although it appears in a suggestive way). However, it is not a proper matter field either as it does not transform supercovariantly. As it turns out these issues have a common origin.

Before resolving these questions it is better to first proceed and take a closer look at a composite fermionic gauge field $\hat{\phi}_\mu{}^i$ that serves as a $4D$ connection for S-supersymmetry. It is the solution of the equation (in the ensuing analysis we will not exhibit terms quadratic in the spinor fields)

$$\gamma^\mu\Big[(\mathcal{D}_{[\mu}+\frac{1}{2}\hat{A}_{[\mu}\gamma^5)\psi_{\nu]}{}^i-\frac{1}{2}\mathrm{i}\,\gamma_{[\mu}\,\hat{\phi}_{\nu]}{}^i+\frac{1}{32}\mathrm{i}\,\hat{T}_{ab}\gamma^{ab}\,\gamma_{[\mu}\,\psi_{\nu]}{}^i\Big]=0\,, \tag{A.16}$$

and transforms under S- and Q-supersymmetry as

$$\begin{aligned}\delta\hat{\phi}_\mu{}^i =&\, 2\left(\mathcal{D}_\mu-\frac{1}{2}\hat{A}_\mu\gamma_5\right)\eta^i+\frac{1}{48}\mathrm{i}\gamma_\mu\hat{T}_{ab}\gamma^{ab}\eta^i\\ &+2\mathrm{i}\,\hat{f}_\mu{}^a\gamma_a\epsilon^i+\frac{1}{16}\left(\gamma^\nu\gamma^{ab}\gamma_\mu-\frac{1}{3}\gamma_\mu\gamma^{ab}\gamma^\nu\right)D_\nu\hat{T}_{ab}\epsilon^i\\ &-\frac{1}{4}\mathrm{i}\left(\gamma^{ab}\gamma_\mu-\frac{1}{3}\gamma_\mu\gamma^{ab}\right)R(\hat{V})_{abj}{}^i\epsilon^j-\frac{1}{2}\mathrm{i}\left(\gamma^{ab}\gamma_\mu+\frac{1}{3}\gamma_\mu\gamma^{ab}\right)R(\hat{A})_{ab}\gamma^5\epsilon^i\,,\end{aligned} \tag{A.17}$$

where $\hat{f}_\mu{}^a$ reads

$$\hat{f}_\mu{}^a=\frac{1}{2}R(\omega,e)_\mu{}^a-\frac{1}{12}R(\omega,e)\,e_\mu{}^a-\frac{1}{2}\widetilde{R}(\hat{A})_\mu{}^a-\frac{1}{128}(\hat{T}-\widetilde{\hat{T}})_{\mu b}\,(\hat{T}+\widetilde{\hat{T}})^{ba}\,, \tag{A.18}$$

where $R(\omega,e)_\mu{}^a = R(\omega)_{\mu\nu}{}^{ab}\,e_b{}^\nu$ is the generalized (non-symmetric) Ricci tensor. Its anti-symmetric part is equal to $R(\omega,e)_{[\mu\nu]} = R(b)_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. This follows from the identity $R(\omega)_{[ab,c]}{}^d = -R(b)_{[ab}\,\delta_{c]}{}^d$, which reflects the fact that the spin connection $\omega_\mu{}^{ab}$ depends on the dilatational gauge field b_μ. As a result the generalized Riemann tensor $R(\omega)_{\mu\nu}{}^{ab}$ is not symmetric under pair-exchange,

$$R(\omega)_{ab,cd}-R(\omega)_{cd,ab}=-2\,\eta_{[a[c}\,R(\omega,e)_{d]b]}+2\,\eta_{[c[a}\,R(\omega,e)_{b]d]}\,. \tag{A.19}$$

Finally, $\widetilde{R}(\hat{A})_{\mu\nu}$ denotes the dual of $R(\hat{A})_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu$.

The $5D$ S-supersymmetry gauge field $\phi_M{}^i$ follows from the fermionic conventional constraint given in (2.3) and can be decomposed as follows under the $4D$ reduction,

$$\begin{aligned}&\phi_\mu{}^i|_{5D}-\frac{1}{6}\mathrm{i}\,\hat{A\!\!\!/}\gamma_5\psi_\mu{}^i+\frac{1}{96}\hat{T}_{ab}\gamma^{ab}\psi_\mu{}^i-\frac{1}{12}\phi^{-1}\hat{F}_{ab}\gamma^{ab}\gamma_5\psi_\mu{}^i\\ &=\frac{1}{2}\hat{\phi}_\mu{}^i+\frac{1}{3}\phi^{-1}\gamma_5D_\mu\hat{\psi}^i+\frac{1}{12}\phi^{-1}\gamma_\mu\gamma_5\not{D}\hat{\psi}^i-\frac{2}{3}\phi^{-1}\hat{A}_\mu\hat{\psi}^i+\frac{1}{6}\phi^{-1}\gamma_\mu\hat{A\!\!\!/}\hat{\psi}^i\\ &\quad+\frac{1}{3}\mathrm{i}\phi^{-2}\gamma^\nu\hat{F}_{\mu\nu}\hat{\psi}^i-\frac{1}{24}\mathrm{i}\phi^{-2}\gamma_\mu\hat{F}_{ab}\gamma^{ab}\hat{\psi}^i-\frac{1}{96}\mathrm{i}\phi^{-1}\hat{T}_{ab}\gamma^{ab}\gamma_\mu\gamma_5\hat{\psi}^i\\ &\quad-\frac{2}{3}\phi^{-2}\gamma_5(\mathcal{D}_\mu\phi-\hat{A}_\mu\phi\gamma_5)\hat{\psi}^i-\frac{1}{6}\phi^{-2}\gamma_\mu\gamma_5\left(\not{\mathcal{D}}\phi-\hat{A\!\!\!/}\phi\gamma_5\right)\hat{\psi}^i\,.\end{aligned} \tag{A.20}$$

The right-hand side of this equation contains only supercovariant $4D$ expressions, with the exception of the field $\hat{\phi}_\mu{}^i$ which is a gauge field. For instance $D_\mu\hat{\psi}^i$ is the $4D$ fully supercovariant derivative given by (at linear order in the spinor fields)

$$D_\mu\hat\psi^i = \Big(\mathcal{D}_\mu + \frac{1}{2}\hat A_\mu\gamma^5\Big)\hat\psi^i - \frac{1}{2}\phi\,\gamma^5\hat\phi_\mu{}^i - \frac{1}{4}\Big[\hat F(B)_{ab} - \frac{1}{8}\phi\,\hat T_{ab}\gamma_5\Big]\gamma^{ab}\psi_\mu{}^i \\ + \frac{1}{2}\mathrm{i}\gamma^5\gamma^\nu\big[\mathcal{D}_\nu\phi - \hat A_\nu\phi\,\gamma^5\big]\psi_\mu{}^i - \frac{1}{2}\hat V_j{}^i\,\psi_\mu{}^j\,, \tag{A.21}$$

which also contains the S-supersymmetry gauge field $\hat\phi_\mu{}^i$. The terms on the left-hand side of (A.20) that depend explicitly on $\psi_\mu{}^i$ seem to affect the covariance under Q-supersymmetry. However, they are to be expected because, according to (A.9), the $5D$ Q-supersymmetry differs from the $4D$ one by a field dependent S-supersymmetry transformation parametrized by $\tilde\eta^i$ given in (A.10).

The correctness of this result can be verified by considering the Q- and S-supersymmetry variations of the $4D$ SU(2) gauge fields $\hat V_\mu{}_i{}^j$. After taking into account the Kaluza-Klein decomposition, one has to correct for the field-dependent S-supersymmetry transformation indicated in (A.9), which precisely cancels against the terms in (A.20) that depend explicitly on ψ_μ^i. Furthermore one has to take into account the redefinitions in (A.11) and the field-dependent SU(2) transformation in (A.9). Their combined effect will only lead to terms such as

$$\mathrm{i}(\delta\bar\psi_{\mu i} - 2\,\mathcal{D}_\mu\epsilon_i)\,\gamma^5\phi^{-1}\hat\psi^j\,,\quad -2\mathrm{i}\,\bar\epsilon_i\Big[\gamma^5 D_\mu(\phi^{-1}\hat\psi^j) + \frac{1}{2}\hat\phi_\mu{}^j\Big]\,,\quad \phi^{-1}\bar{\hat\psi}_i\,\delta(\phi^{-1}\hat\psi^j)\,, \tag{A.22}$$

where the derivative D_μ is supercovariant. Combining this with the result of the Kaluza-Klein decomposition and with (A.20), one obtains

$$\delta\hat V_\mu{}_i{}^j = 2\mathrm{i}\,\bar\epsilon_i\hat\phi_\mu{}^j - 2\,\bar\epsilon_i\gamma_\mu\hat\chi^j - 2\mathrm{i}\,\bar\eta_i\psi_\mu{}^j - \frac{1}{2}\delta_i{}^j\big(2\mathrm{i}\,\bar\epsilon_k\hat\phi_\mu{}^k - 2\,\bar\epsilon_k\gamma_\mu\hat\chi^k - 2\mathrm{i}\,\bar\eta_k\psi_\mu{}^k\big)\,, \tag{A.23}$$

where $\hat\chi^i$ is a supercovariant spinor field equal to

$$\hat\chi^i = 8\,\chi^i\big|_{5D} - \frac{1}{4}\mathrm{i}\phi^{-1}\gamma^5\not{D}\hat\psi^i - \frac{1}{2}\phi^{-2}\hat V_k{}^i\hat\psi^k \\ + \frac{1}{8}\phi^{-2}\Big[\hat F_{ab} - \frac{1}{4}\phi\hat T_{ab}\gamma^5\Big]\gamma^{ab}\hat\psi^i - \frac{1}{2}\mathrm{i}\phi^{-1}\not{\hat A}\hat\psi^i\,. \tag{A.24}$$

Let us subsequently turn to the Q- and S-supersymmetry transformations of the field $\hat\chi^i$, which contains the remaining independent fermion field $\chi^i|_{5\mathrm{D}}$ of the $5D$ Weyl multiplet according to the equation above. When writing its variation in terms of the $4D$ quantities, we naturally obtain terms that depend exclusively on the $4D$ Weyl multiplet components and others that will involve both the Weyl multiplet and the Kaluza-Klein vector multiplet. The latter terms should then cancel by the variations of the additional terms in (A.24), because $\hat\chi^i$ must vary exclusively into the components of the $4D$ Weyl multiplet. Here one should again compensate for the composite S-supersymmetry variation parametrized in terms of $\tilde\eta^i$. This leads to the following expression,

$$\delta\hat\chi^i = 8\,\delta\chi^i\big|_{5D} - \frac{3}{2}T_{AB}\,\gamma^{AB}\tilde\eta^i - \frac{1}{4}\mathrm{i}\,\delta\big[\phi^{-1}\gamma^5\not{D}\hat\psi^i\big] \\ - \frac{1}{8}\,\delta\Big[4\,\phi^{-2}\hat V_j{}^i\,\hat\psi^j - \phi^{-2}\Big[\hat F(B)^{ab} - \frac{1}{4}\phi\,\hat T^{ab}\gamma^5\Big]\gamma_{ab}\,\hat\psi^i + 4\mathrm{i}\,\phi^{-1}\not{\hat A}\,\hat\psi^i\Big]\,, \tag{A.25}$$

where we use the definition (A.21) for the supercovariant derivative of $\hat\psi^i$ based on the S-supersymmetry gauge field $\hat\phi_\mu{}^i$. Eventually we will make another, more suitable, choice for this composite gauge field, but for the moment we adopt this definition.

Restricting ourselves to terms linearly proportional to fermion fields, the variation of $\hat\chi^i$ takes the following form,

$$\delta\hat\chi^i = \frac{1}{24}\hat T_{ab}\gamma^{ab}\eta^i + \frac{1}{6}R(V)_{ab j}{}^i\,\gamma^{ab}\epsilon^j + \frac{1}{24}\mathrm{i}\,\gamma^{ab}\,\not{D}\hat T_{ab}\epsilon^i - \frac{1}{3}R(A)_{ab}\,\gamma^{ab}\gamma^5\epsilon^i + \hat D\,\epsilon^i\,, \quad \text{(A.26)}$$

where $\hat D$ is defined as (up to terms quadratic in spinor fields)

$$\begin{aligned}\hat D = {}& 4\,D\big|_{5D} - \frac{1}{4}\phi^{-2}\,\hat V_j{}^k\,\hat V_k{}^j + \frac{1}{4}\phi^{-1}\Big[(\mathcal{D}_a)^2 + \frac{1}{6}R(\omega,e)\Big]\phi - \frac{1}{12}(\hat A_a)^2\\ &- \frac{1}{12}\phi^{-2}\hat F^{ab}\hat F_{ab} + \frac{1}{192}\phi^{-1}\epsilon_{abcd}\,\hat T^{ab}\hat F^{cd} + \frac{1}{384}\hat T^{ab}\,\hat T_{ab}\,, \end{aligned}\quad \text{(A.27)}$$

where we have made use of (A.18). Note that all bosonic terms in (A.26) have been included.

We conclude this part of the analysis by giving the Q- and S-supersymmetry transformations for the remaining fields, where we give also some further details about terms quadratic in the spinor fields,

$$\begin{aligned}\delta b_\mu ={}& \frac{1}{2}\mathrm{i}\,\bar\epsilon_i\,\hat\phi_\mu{}^i - \frac{1}{2}\epsilon_i\gamma_\mu\hat\chi^i + \frac{1}{2}\mathrm{i}\,\bar\eta_i\,\psi_\mu{}^i\,,\\ \delta\hat T_{ab} ={}& -8\mathrm{i}\,\bar\epsilon_i\,R(Q)_{ab}{}^i + 4\mathrm{i}\,\bar\epsilon_i\gamma_{ab}\hat\chi^i + \frac{1}{2}\varepsilon_{abcd}\,\hat T^{cd}\,\tilde\Lambda^0\,,\\ \delta\hat A_\mu ={}& \bar\epsilon_i\gamma_\mu\gamma^5\hat\chi^i - \frac{1}{2}\mathrm{i}\,\bar\epsilon_i\gamma^5\hat\phi_\mu{}^i - \frac{1}{2}\mathrm{i}\,\bar\eta_i\gamma^5\psi_\mu{}^i + \partial_\mu\tilde\Lambda^0\,,\\ \delta\hat V_j{}^i ={}& 2\,\bar\epsilon_j(\not{D}\hat\psi^i - \mathrm{i}\gamma^5\phi\,\hat\chi^i) - \delta_j{}^i\,\bar\epsilon_k(\not{D}\hat\psi^k - \mathrm{i}\gamma^5\phi\,\hat\chi^k)\,,\\ \delta\hat D ={}& \bar\epsilon_i\,\not{D}\hat\chi^i + \cdots\,. \end{aligned}\quad \text{(A.28)}$$

In the derivation of the first result for δb_μ we note that we have encountered the same phenomenon when deriving the transformation rules for $\hat V_\mu{}_i{}^j$ in (A.23). Namely the S-supersymmetry transformation with field-dependent parameter $\tilde\eta^i$ in (A.9) cancels against the terms in (A.20) that depend explicitly on ψ_μ^i. After that we use the definition of $\hat\chi^i$ in (A.24), and the remaining terms are absorbed into the $4D$ conformal boost transformation. Since b_μ is the only field that transforms under conformal boosts, this will only affect the explicit form of the supersymmetry algebra. The transformation rules of $\hat T_{ab}$, $\hat A_\mu$ and $\hat V_j{}^i$ do not involve further subtleties, except that $\hat A_\mu$ does not seem to transform supercovariantly. The transformation rule of $\hat D$, however, cannot be reliably calculated at this stage, because we have not yet determined the contributions quadratic in the spinor fields in its definition (A.27). In view of the fact that the original $5D$ theory as well as its reduced $4D$ version is consistent, there is no doubt that the present calculation can be completed to all orders.

We have thus shown in considerable detail how the $5D$ Weyl multiplet reduces to the $4D$ Euclidean Weyl multiplet and a Kaluza-Klein vector supermultiplet. However, the latter multiplet involves only seven bosonic and eight fermionic degrees of freedom, so

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that one bosonic field seems to be missing in the Kaluza-Klein vector multiplet. A similar counting for the Weyl multiplet reveals that the Weyl multiplet has twenty-five bosonic and twenty-four fermionic degrees of freedom (in off-shell counting one always corrects for the number of gauge invariances, so that, for instance, each gravitino contributes only eight fermionic degrees of freedom).

The reason for the mismatch is well known; under dimensional reduction one obtains the lower-dimensional theory in a partially gauge-fixed form. The R-symmetry is extended to $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$, where the non-compact $\mathrm{SO}(1,1)$ factor acts by a chiral transformations on the fermions (it will also act on some of the bosonic fields). At this point the $\mathrm{SO}(1,1)$ group is, however, not realized as a local invariance. Although the vector field $\hat{A}_\mu$ seems to play the role of an $\mathrm{SO}(1,1)$ gauge field, it is not transforming under a corresponding gauge symmetry and represents four bosonic dergrees of freedom. This is the underlying reason why the Kaluza-Klein supermultiplet is not yet realized as an irreducible multiplet.

Full irreducibility can be obtained by introducing a compensating scalar field φ and writing

$$\hat{A}_\mu = A_\mu - \partial_\mu\varphi\,, \tag{A.29}$$

where A_μ and φ transform under *local* $\mathrm{SO}(1,1)$ gauge transformations as

$$A_\mu \to A_\mu + \partial_\mu\Lambda^0\,, \qquad \varphi\to\varphi+\Lambda^0\,, \tag{A.30}$$

so that $\hat{A}_\mu$ remains invariant. Under supersymmetry we assume that φ changes according to

$$\delta\varphi = -\tilde{\Lambda}^0 = -\mathrm{i}\phi^{-1}\bar{\epsilon}_i\,\hat{\psi}^i\,. \tag{A.31}$$

Subsequently one uniformly redefines all fields and parameters with a suitable φ-dependent $\mathrm{SO}(1,1)$ transformation, which will remove all explicit terms in the transformation rules proportional to $\tilde{\Lambda}^0$ and furthermore resolve various questions raised previously (see e.g. the discussion following (A.15)). Upon imposing the gauge condition $\varphi=0$, all the $\tilde{\Lambda}^0$-terms will re-emerge in the form of compensating gauge transformations.

We now summarize all the φ-dependent field redefinitions. The R-covariant spinors, transforming under local $\mathrm{SU}(2)\times\mathrm{SO}(1,1)$ R-symmetry transformations, are as follows,

$$\begin{aligned}
\epsilon^i|^{\mathrm{Rcov}} &= \exp\Big[-\tfrac12\varphi\,\gamma^5\Big]\,\epsilon^i\,, & \psi_\mu{}^i|^{\mathrm{Rcov}} &= \exp\Big[-\tfrac12\varphi\,\gamma^5\Big]\,\psi_\mu{}^i\,,\\
\eta^i|^{\mathrm{Rcov}} &= \exp\Big[\tfrac12\varphi\,\gamma^5\Big]\,\eta^i\,, & \phi_\mu{}^i|^{\mathrm{Rcov}} &= \exp\Big[\tfrac12\varphi\,\gamma^5\Big]\,\hat{\phi}_\mu{}^i\,,\\
\chi^i|^{\mathrm{Rcov}} &= \exp\Big[-\tfrac12\varphi\,\gamma^5\Big]\,\hat{\chi}^i\,, & \psi^i|^{\mathrm{Rcov}} &= \exp\Big[-\tfrac12\varphi\,\gamma^5\Big]\,\hat{\psi}^i\,.
\end{aligned} \tag{A.32}$$

Also some of the bosons will have to be redefined so that they transform covariantly under $\mathrm{SO}(1,1)$. First of all the tensor fields $\hat{T}_{ab}$, when decomposed into the self-dual and anti-self-dual (real) components, take the form

$$T_{ab}{}^{\pm\mathrm{Rcov}} = \exp[\pm\varphi]\,\hat{T}_{ab}{}^{\pm}\,. \tag{A.33}$$

Furthermore the scalars ϕ and φ are combined into the fields $\phi\exp[\mp\varphi]$ which transform into R-covariant spinors according to

$$\delta\big(\phi\,e^{\mp\varphi}\big) = \pm\mathrm{i}\big[\bar{\epsilon}_i(1\pm\gamma^5)\psi^i\big]^{\mathrm{Rcov}}\,, \tag{A.34}$$

where the fermions on the right-hand side are R-covariant. After these last redefinitions the Weyl multiplet has become irreducible. It now includes the $\mathrm{SO}(1,1)$ gauge field A_μ, so that it comprises $24\oplus 24$ off-shell bosonic and fermionic degrees of freedom. The compensator φ belongs to the Kaluza-Klein vector multiplet, which is defined in a background consisting of the Weyl multiplet and comprises $8\oplus 8$ degrees of freedom.

In the next subsection we describe how to bring the theory in a form that is closely related to the $4D$ Minkowski theory, first by modifying the expressions for the dependent fields of the Weyl multiplet, $\phi_\mu{}^i$, and $f_\mu{}^a$, which implies that we modify the conventional constraints. In a second subsection we will then briefly disucuss the vector, tensor and hyper multiplets.

A.1 S-invariant conventional constraints

At this stage we will make some field redefinitions to bring the results in closer contact with the Minkowski version of $N=2$ supergravity. First of all we will redefine the S-supersymmetry gauge field according to

$$\phi_\mu{}^i = \phi_\mu{}^i|_{\rm old} - \frac{1}{2}\mathrm{i}\gamma_\mu\chi^i\,. \tag{A.35}$$

This will correspond to a different conventional constraint (the previous one was given by (A.16)) that is S-supersymmetric. At the same time we make use of the R-covariant fields defined previously. As a result the transformation rules will acquire a simpler form.

Because of the redefinition (A.35) the explicit expressions for the the dependent gauge fields ϕ_μ^i and f_μ^a become

$$\begin{aligned}\phi_\mu{}^i =&-\mathrm{i}\Big(\frac{1}{2}\gamma^{\rho\sigma}\gamma_\mu-\frac{1}{6}\gamma_\mu\gamma^{\rho\sigma}\Big)\Big(\mathcal{D}_\rho\psi_\sigma{}^i+\frac{1}{32}\mathrm{i}\,T_{ab}\gamma^{ab}\gamma_\rho\psi_\sigma{}^i+\frac{1}{4}\gamma_{\rho\sigma}\chi^i\Big)\,,\\ f_\mu{}^a =&\frac{1}{2}R(\omega,e)_\mu{}^a-\frac{1}{4}\Big(D+\frac{1}{3}R(\omega,e)\Big)\,e_\mu{}^a-\frac{1}{2}\tilde{R}(A)_\mu{}^a-\frac{1}{32}T_{\mu b}^-\,T^{+ba}\,,\end{aligned} \tag{A.36}$$

where, in the last expression, we have only shown the bosonic terms. Note that we have been using R-covariant spinors and tensor fields in the above results. Furthermore we have suppressed the carets on the various fields and therefore use the proper $4D$ covariant fields wherever possible. The derivative D_μ is fully supercovariant whereas the derivative $\mathcal{D}_\mu$ is covariant with respect to local Lorentz, dilatations and the full R-symmetry $\mathrm{SO}(1,1)\times\mathrm{SU}(2)$. All the field strengths are also supercovariant. The presence of the fully supersymmetric covariant quantities has not been verified in every possible detail, but these covariantizations are implied by the supersymmetry algebra.

In view of the fact that we are now dealing with a Euclidean theory, we prefer to change the definition of the Dirac conjugate fields accordingly, so that $\bar\chi_i\equiv\chi_i^\dagger$. This requires to replace all the spinors $\bar\chi_i$ in previous equations by $\bar\chi_i\,\gamma^5$. The symplectic Majorana condition on the spinors then preserves its form, provided the charge conjugation matrix is redefined according to

$$C|^{\rm new} = C\,\gamma^5\,. \tag{A.37}$$

The correponding symplectic Majorana condition for the spinors is given in (3.1). As a result of the redefinition (A.37) the hermitian gamma matrices γ_a now satisfy

$$C\,\gamma_a\,C^{-1} = -\gamma_a{}^{\mathrm{T}}\,, \qquad (a=1,2,3,4). \tag{A.38}$$

For spinor bilinears $\bar\psi_i\Gamma\varphi^j$, a useful formula is given in (3.4).

With these new conventions we summarize the supersymmetry transformations in terms of the R-covariant fields,

$$\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon_i\,\gamma^5\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i + \frac{1}{16}\mathrm{i}\,(T_{ab}{}^+ + T_{ab}{}^-)\gamma^{ab}\gamma_\mu\epsilon^i - \mathrm{i}\gamma_\mu\eta^i\,,\\
\delta b_\mu &= \frac12\mathrm{i}\,\bar\epsilon_i\,\gamma^5\phi_\mu{}^i - \frac34\bar\epsilon_i\,\gamma^5\gamma_\mu\chi^i + \frac12\mathrm{i}\,\bar\eta_i\,\gamma^5\psi_\mu{}^i + \Lambda_{\mathrm{K}}{}^a\,e_{\mu a}\,,\\
\delta A_\mu &= -\frac12\mathrm{i}\,\bar\epsilon_i\,\phi_\mu{}^i - \frac34\,\bar\epsilon_i\,\gamma_\mu\chi^i - \frac12\mathrm{i}\,\bar\eta_i\,\psi_\mu{}^i\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\mathrm{i}\,\bar\epsilon_j\,\gamma^5\phi_\mu{}^i - 3\,\bar\epsilon_j\,\gamma^5\gamma_\mu\chi^i - 2\mathrm{i}\,\bar\eta_j\,\gamma^5\psi_\mu{}^i\\
&\quad -\frac12\delta^i{}_j\big(2\mathrm{i}\,\bar\epsilon_k\,\gamma^5\phi_\mu{}^k - 3\,\bar\epsilon_k\,\gamma^5\gamma_\mu\chi^k - 2\mathrm{i}\,\bar\eta_k\,\gamma^5\psi_\mu{}^k\big)\,,\\
\delta T_{ab}{}^\pm &= -8\mathrm{i}\,\bar\epsilon_i\,\gamma^5 R(Q)_{ab}{}^{i\,\pm}\,,\\
\delta\chi^i &= \frac{1}{24}\mathrm{i}\,\gamma^{ab}\,\not{D}(T^+_{ab} + T^-_{ab})\gamma^{ab}\epsilon^i + \frac16 R(\mathcal{V})_{ab}{}^i{}_j\,\gamma^{ab}\epsilon^j - \frac13 R(A)_{ab}\,\gamma^{ab}\gamma^5\epsilon^i\\
&\quad + D\,\epsilon^i + \frac{1}{24}(T^+_{ab} + T^-_{ab})\gamma^{ab}\eta^i\,,\\
\delta D &= \bar\epsilon_i\,\gamma^5\not{D}\chi^i\,,
\end{aligned} \tag{A.39}$$

where we changed notation for the SU(2) R-symmetry gauge field to remain in line with the $4D$ Minkowski theory and used the (anti-)self-dual components of the field strength $R(Q)_{ab}{}^i$,

$$\begin{aligned}
\mathcal{V}_\mu{}^i{}_j &= V_{\mu\,j}{}^i\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \mathrm{i}\,\gamma_{[\mu}\,\phi_{\nu]}{}^i + \frac{1}{16}\mathrm{i}\,(T_{ab}{}^+ + T_{ab}{}^-)\gamma^{ab}\,\gamma_{[\mu}\,\psi_{\nu]}{}^i\,.
\end{aligned} \tag{A.40}$$

We also present the transformation rule for the two dependent gauge fields, $\phi_\mu{}^i$ and $f_\mu{}^a$, up to terms quadratic in fermions:

$$\begin{aligned}
\delta\phi_\mu{}^i &= 2\,\mathcal{D}_\mu\eta^i + 2\mathrm{i}\,f_\mu{}^a\gamma_a\epsilon^i + \frac{1}{16}\not{D}(T_{ab}{}^+ + T_{ab}{}^-)\gamma^{ab}\gamma_\mu\,\epsilon^i\\
&\quad - \frac14\mathrm{i}\gamma^{ab}\gamma_\mu R(\mathcal{V})_{ab}{}^i{}_j\,\epsilon^j - \frac12\mathrm{i}\gamma^{ab}\gamma_\mu\gamma^5 R(A)_{ab}\,\epsilon^i - \mathrm{i}\,\Lambda_{\mathrm{K}}{}^a\,\gamma_a\psi_\mu{}^i\,,
\end{aligned} \tag{A.41}$$

$$\begin{aligned}
\delta f_\mu{}^a &= \frac14\mathrm{i}\,\bar\epsilon_i\,\gamma^5\psi_\mu{}^i D_b(T^{+ba} + T^{-ba}) - \frac34 e_\mu{}^a\,\bar\epsilon_i\gamma^5\not{D}\chi^i - \frac34\bar\epsilon_i\,\gamma^5\gamma^a\psi_\mu{}^i\,D\\
&\quad + \bar\epsilon_i\,\gamma^5\gamma_\mu D_b R(Q)^{ba\,i} + \frac12\,\bar\eta_i\,\gamma^5\gamma^a\phi_\mu{}^i + \mathcal{D}_\mu\Lambda_{\mathrm{K}}{}^a\,.
\end{aligned}$$

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For completeness we also list the transformation rules for the Kaluza-Klein multiplet,

$$
\begin{aligned}
\delta\big(\phi\, e^{\mp\varphi}\big) &= \mathrm{i}\bar\epsilon_i(1\pm\gamma^5)\hat\psi^i\,,\\
\delta B_\mu &= -\,\bar\epsilon_i\,\gamma^5\gamma_\mu\hat\psi^i - \mathrm{i}\,\bar\epsilon_i\,\phi\,\mathrm{e}^{\varphi\gamma^5}\psi_\mu{}^i\,,\\
\delta\hat\psi^i &= \frac{1}{2}\Big[\hat F(B)_{ab} - \frac{1}{8}\phi\,\hat T_{ab}\gamma_5\Big]\gamma^{ab}\epsilon^i - \mathrm{i}\gamma^5 \not{D}\big(\phi\,\mathrm{e}^{\varphi\gamma^5}\big)\epsilon^i + \hat V_j{}^i\,\epsilon^j + \phi\,\mathrm{e}^{-\varphi\gamma^5}\gamma^5\eta^i\,,\\
\delta\hat V_j{}^i &= 2\,\bar\epsilon_j\,\gamma^5\not{D}\hat\psi^i - \delta_j{}^i\,\bar\epsilon_k\,\gamma^5\not{D}\hat\psi^k\,,
\end{aligned}
\tag{A.42}
$$

where the supercovariant field strength associated with the vector field B_μ

$$
\hat F(B)_{\mu\nu} = 2\,\partial_{[\mu}B_{\nu]} + \bar\psi_{[\mu i}\,\gamma^5\gamma_{\nu]}\hat\psi^i + \frac{1}{2}\mathrm{i}\bar\psi_{\mu i}\,\phi\,\mathrm{e}^{\varphi\gamma^5}\psi_\nu{}^i\,. \tag{A.43}
$$

Observe that for the Kaluza-Klein vector multiplet we still use the fields $\hat\psi^i$, $\hat V_j{}^i$ and $\hat T_{ab}$ but the spinors are R-covariant fields.

B Off-shell dimensional reduction; matter multiplets

We briefly consider the dimensional reduction for the vector and tensor multiplets and for the hypermultiplet. We first present their $5D$ transformation rules and then apply the reduction to the transformations of the corresponding $4D$ Euclidean multiplets. Using the standard Kaluza-Klein ansatz outlined earlier, one obtains the Q- and S-supersymmetry transformation rules upon including the compensating Lorentz transformation (A.1), and, at the end, suppressing the composite S-supersymmetry, SU(2) R-symmetry and SO(1, 1) transformations given in (A.9). The dictionary that expresses the $4D$ fields in terms of the $5D$ ones will then be presented to indicate how the $4D$ results were derived. We close this appendix by giving the $5D$ Minkowski non-linear multiplet transformation rules.

B.1 The vector supermultiplet

The $5D$ vector supermultiplet consists of a real scalar σ, a gauge field W_M, a triplet of (auxiliary) fields Y^{ij}, and a fermion field Ω^i. Under Q- and S-supersymmetry these fields transform as follows,

$$
\begin{aligned}
\delta\sigma &= \mathrm{i}\,\bar\epsilon_i\,\Omega^i\,,\\
\delta\Omega^i &= -\,\frac{1}{2}(\hat F_{AB} - 4\,\sigma T_{AB})\gamma^{AB}\epsilon^i - \mathrm{i}\not{D}\sigma\,\epsilon^i - 2\varepsilon_{jk}\,Y^{ij}\epsilon^k + \sigma\,\eta^i\,,\\
\delta W_M &= \bar\epsilon_i\gamma_M\Omega^i - \mathrm{i}\sigma\,\bar\epsilon_i\psi_M{}^i\,,\\
\delta Y^{ij} &= \varepsilon^{k(i}\,\bar\epsilon_k\,\not{D}\Omega^{j)} + 2\mathrm{i}\,\varepsilon^{k(i}\,\bar\epsilon_k\Big(-\frac{1}{4}T_{AB}\gamma^{AB}\Omega^{j)} + 4\sigma\chi^{j)}\Big) - \frac{1}{2}\mathrm{i}\varepsilon^{k(i}\,\bar\eta_k\,\Omega^{j)}\,.
\end{aligned}
\tag{B.1}
$$

where $(Y^{ij})^* \equiv Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}$, and the supercovariant field strength $\hat F_{MN}(W)$ is defined as,

$$
\hat F_{MN}(W) = 2\,\partial_{[M}W_{N]} - \bar\Omega_i\gamma_{[M}\psi_{N]}{}^i + \frac{1}{2}\mathrm{i}\sigma\,\bar\psi_{[Mi}\psi_{N]}{}^i\,. \tag{B.2}
$$

	vector multiplet				tensor multiplet				hypermultiplet	
field	σ	W_M	Ω_i	Y_{ij}	L^{ij}	E_A	φ^i	N	$A_i{}^\alpha$	ζ^α
w	1	0	$\frac{3}{2}$	2	3	4	$\frac{7}{2}$	4	$\frac{3}{2}$	2

Table 5. Weyl weights w of the vector multiplet, the tensor multiplet and the hypermultiplet component fields in five space-time dimensions.

The fields behave under local scale transformations according to the weights shown in table 5.

The dimensional reduction proceeds in the same way as before, except that we now have the advantage that we have already identified some of the $4D$ fields belonging to the $4D$ Weyl multiplet. We decompose the $5D$ gauge field W_M into a four-dimensional gauge field W_μ and a scalar $W = W_5$. The result for the $4D$ Euclidean supermultiplet thus involves two real fields X_+ and X_-, one spinor Ω^i, the gauge field W_μ and an auxiliary field Y^{ij}. The overall normalization of the multiplet is defined by the requirement that the $4D$ gauge field W_μ is precisely the one that follows from the $5D$ one upon adopting the Kaluza-Klein ansatz. The supersymmetry transformations of the $4D$ multiplet components take the following form,

$$\begin{aligned}
\delta X_\pm &= \pm\,\mathrm{i}\,\bar\epsilon_{i\pm}\Omega^i{}_\pm\,,\\
\delta W_\mu &= \bar\epsilon_{i+}\big(\gamma_\mu\Omega^i{}_- - 2\mathrm{i}\,X_-\psi_{\mu+}{}^i\big) - \bar\epsilon_{i-}\big(\gamma_\mu\Omega^i{}_+ - 2\mathrm{i}\,X_+\psi_{\mu-}{}^i\big)\\
\delta\Omega^i{}_\pm &= -\,2\mathrm{i}\,\not{D}X_\pm\,\epsilon^i{}_\mp - \frac{1}{2}\Big[\hat F(W)_{ab}^\mp - \frac{1}{4}X_\mp\,T_{ab}^\mp\Big]\gamma^{ab}\epsilon^i{}_\pm - \varepsilon_{kj}\,Y^{ik}\epsilon^j{}_\pm + 2\,X^\pm\eta^i{}_\pm\,,\\
\delta Y^{ij} &= 2\,\varepsilon^{k(i}\bar\epsilon_k\gamma^5\not{D}\Omega^{j)}\,.
\end{aligned} \tag{B.3}$$

The supercovariant field strength $\hat F(W)_{\mu\nu}$ is defined as

$$\hat F(W)_{\mu\nu} = 2\,\partial_{[\mu}W_{\nu]} + \bar\psi_{i[\mu}\gamma_{\nu]}\Omega^i{}_+ - \bar\psi_{i[\mu}\gamma_{\nu]}\Omega^i{}_- + \mathrm{i}\,X_-\bar\psi_{\mu\,i}\psi_{\nu+}{}^i - \mathrm{i}\,X_+\bar\psi_{\mu\,i}\psi_{\nu-}{}^i\,. \tag{B.4}$$

This result is based on the following identification of the $4D$ fields expressed in terms of the $5D$ ones,

$$\begin{aligned}
X_\pm &= \frac{1}{2}\mathrm{e}^{\mp\varphi}(\sigma\pm\phi W)\,,\\
\Omega^i &= \exp\Big[-\frac{1}{2}\varphi\,\gamma^5\Big]\,(\Omega^i + W\,\hat\psi^i)\,,\\
Y^{ij} &= 2\,Y^{ij} - W\,\hat V_k{}^i\,\varepsilon^{jk} + \mathrm{i}\,\phi^{-1}\,\Big(\bar\Omega_k\gamma^5 - \frac{1}{2}\,\sigma\phi^{-1}\,\bar{\hat\psi}_k\Big)\,\hat\psi^{(i}\,\varepsilon^{j)k}\,.
\end{aligned} \tag{B.5}$$

For the Kaluza-Klein supermultiplet the corresponding identification is as follows (cf. (A.42)),

$$\begin{aligned} X_\pm &= \mp\frac{1}{2}\mathrm{e}^{\mp\varphi}\phi\,,\\ W_\mu &= B_\mu\,,\\ \Omega^i &= -\,\hat\psi^i\,,\\ Y^{ij} &= \varepsilon^{ik}\,\hat V_k{}^j\,. \end{aligned} \tag{B.6}$$

Note that we have been dealing with the abelian vector multiplet. The non-abelian extension of the vector multiplet will be given in subsection 5.1.

B.2 The tensor supermultiplet

The linear multiplet consists of a triplet of scalars L^{ij}, a rank-three antisymmetric tensor gauge field E_{MNP}, a fermion doublet φ^i, and a real (auxiliary) scalar N. The supercovariant field strength associated with the tensor field is denoted by $\hat E_A$. The superconformal transformation rules for these fields are as follows,

$$\begin{aligned} \delta L^{ij} &= -\,2\mathrm{i}\,\varepsilon^{k(i}\,\bar\epsilon_k\varphi^{j)}\,,\\ \delta\varphi^i &= -\,\mathrm{i}\,\varepsilon_{jk}\not{D}L^{ij}\epsilon^k + (N-\mathrm{i}\hat{\not{E}})\epsilon^i + 3\,\varepsilon_{jk}L^{ij}\eta^k\,,\\ \delta E_{MNP} &= \bar\epsilon_i\gamma_{MNP}\varphi^i - 3\mathrm{i}\,\bar\epsilon_i\gamma_{[MN}\psi_{P]}{}^k\,\varepsilon_{jk}L^{ij}\,,\\ \delta\hat E_A &= -\,\mathrm{i}\,\bar\epsilon_i\gamma_{AB}D^B\varphi^i + \frac{1}{4}\bar\epsilon_i(3\gamma_A\gamma^{BC}+\gamma^{BC}\gamma_A)\varphi^i\,T_{BC} - 2\,\bar\eta_i\gamma_A\varphi^i\,,\\ \delta N &= \bar\epsilon_i\not{D}\varphi^i + \frac{3}{2}\mathrm{i}\bar\epsilon_i\gamma^{AB}\varphi^i\,T_{AB} - 8\mathrm{i}\,\varepsilon_{jk}\,\bar\epsilon_i\chi^k L^{ij} + \frac{3}{2}\mathrm{i}\,\bar\eta_i\varphi^i\,, \end{aligned} \tag{B.7}$$

where the supercovariant field strength $\hat E^A$ is is given by,

$$\hat E^M = \frac{1}{6}\mathrm{i}\,e^{-1}\varepsilon^{MNPQR}\Big[\partial_N E_{PQR} - \frac{1}{2}\bar\psi_{Ni}\gamma_{PQR}\varphi^i + \frac{3}{4}\mathrm{i}\,\bar\psi_{Ni}\gamma_{PQ}\psi_R{}^k\,\varepsilon_{jk}L^{ij}\Big]\,. \tag{B.8}$$

Its corresponding Bianchi identity reads $D_A\hat E^A = 0$. The behaviour under local scale transformations is indicated by the weights shown in table 5.

To have the standard $4D$ Weyl weight, we redefine the field L^{ij} with a multiplicative factor ϕ^{-1}. The supersymmetry transformation of $\phi^{-1}L^{ij}$ then yields the corresponding spinor field

$$\varphi^i|_{4D} = \phi^{-1}\varphi^i - L^{ik}\,\gamma^5\psi^l\,\varepsilon_{kl}\,, \tag{B.9}$$

where we took into account the composite SU(2) transformation specified in (A.9). Subsequently one introduces the phase factors, just as before, and collect the variations of the new spinor.

In this way one obtains the supersymmetry transformations for the $4D$ Euclidean supermultiplet,

$$\begin{aligned}
\delta L^{ij} &= 2\mathrm{i}\,\bar\epsilon_k\,\gamma^5\varphi^{(i}\,\varepsilon^{j)k}\,,\\
\delta E_{\mu\nu} &= \mathrm{i}\,\bar\epsilon_i\,\gamma_{\mu\nu}\,\varphi^i + 2\,\varepsilon_{jk}L^{ij}\,\bar\epsilon_i\gamma_{[\mu}\psi_{\nu]}{}^k\,,\\
\delta\varphi^i_\pm &= -\,\mathrm{i}\,\varepsilon_{jk}\not{D}L^{ij}\,\epsilon^k_\mp - \mathrm{i}\,\hat{\not{E}}\epsilon^i_\mp + G_\pm\epsilon^i_\pm + 2\,\varepsilon_{jk}L^{ij}\,\eta^k_\pm\,,\\
\delta G_\pm &= \mp\,2\,\bar\epsilon_{i\mp}\not{D}\varphi^i_\pm \pm 6\mathrm{i}\,\varepsilon_{jk}L^{ij}\,\bar\epsilon_{i\mp}\chi^k_\mp \mp \frac{1}{8}\mathrm{i}\,T^\pm_{ab}\,\bar\epsilon_{i\mp}\gamma^{ab}\varphi^i_\mp \pm 2\mathrm{i}\,\bar\eta_{i\pm}\varphi^i_\pm\,,
\end{aligned} \tag{B.10}$$

where $\hat{E}^\mu$ denotes the supercovariant field strength associated with the tensor field $E_{\mu\nu}$. Its definition is given in (5.6); $G_\pm$ are two real scalars. This result is based on the following identification of the $4D$ fields in terms of the $5D$ fields,

$$\begin{aligned}
L^{ij} &= \phi^{-1}L^{ij}\,,\\
E_{\mu\nu} &= \mathrm{i}\,E_{\mu\nu\hat5}\,,\\
\varphi^i &= \exp\left[\frac{1}{2}\varphi\gamma^5\right]\left(\phi^{-1}\varphi^i - L^{ik}\,\gamma^5\psi^l\,\varepsilon_{kl}\right),\\
G_\pm &= e^{\pm\varphi}\left(\phi^{-1}N \mp \mathrm{i}\,\hat{E}_{\hat5} - \frac{1}{2}\mathrm{i}\,\bar\psi_i\gamma^5\varphi^i \mp \mathrm{i}\,\bar\psi_i\varphi^i \pm \frac{1}{2}\,\phi^{-2}\varepsilon_{jk}L^{ij}\,\hat{V}_i{}^k\right).
\end{aligned} \tag{B.11}$$

B.3 Hypermultiplets

Hypermultiplets are associated with target spaces of dimension $4r$ that are hyper-Kähler cones [36]. The $5D$ supersymmetry transformations are most conveniently written in terms of the sections $A_i{}^\alpha(\phi)$, where $\alpha = 1, 2, \ldots, 2r$,

$$\begin{aligned}
\delta A_i{}^\alpha &= 2\mathrm{i}\,\bar\epsilon_i\zeta^\alpha\,,\\
\delta\zeta^\alpha &= -\,\mathrm{i}\not{D}A_i{}^\alpha\epsilon^i + \frac{3}{2}A_i{}^\alpha\eta^i\,.
\end{aligned} \tag{B.12}$$

The $A_i{}^\alpha$ are the local sections of an $\mathrm{Sp}(r)\times\mathrm{Sp}(1)$ bundle. We also note the existence of a covariantly constant skew-symmetric tensor $\Omega_{\alpha\beta}$ (and its complex conjugate $\Omega^{\alpha\beta}$ satisfying $\Omega_{\alpha\gamma}\,\Omega^{\beta\gamma} = \delta_\alpha{}^\beta$), and the symplectic Majorana condition for the spinors reads as $C^{-1}\bar\zeta_\alpha{}^{\mathrm{T}} = \Omega_{\alpha\beta}\,\zeta^\beta$. Covariant derivatives contain the $\mathrm{Sp}(r)$ connection $\Gamma_A{}^\alpha{}_\beta$, associated with rotations of the fermions. The sections $A_i{}^\alpha$ are pseudo-real, i.e. they are subject to the constraint, $A_i{}^\alpha\varepsilon^{ij}\Omega_{\alpha\beta} = A^j{}_\beta \equiv (A_j{}^\beta)^*$. The information on the target-space metric is contained in the so-called hyper-Kähler potential. For our purpose the geometry of the hyper-Kähler cone is not relevant. Hence we assume that the cone is flat, so that the target-space connections and curvatures will vanish. The extension to non-trivial hyper-Kähler cone geometries is straightforward.

The supersymmetry transformations for the $4D$ Euclidean hypermultipets read as follows,

$$\begin{aligned}
\delta A_i{}^\alpha &= 2\mathrm{i}\,\bar\epsilon_i\,\gamma^5\zeta^\alpha\,,\\
\delta\zeta^\alpha &= -\,\mathrm{i}\not{D}A_i{}^\alpha + A_i{}^\alpha\eta^i\,.
\end{aligned} \tag{B.13}$$

	non-linear multiplet			
field	${\Phi^i}_\alpha$	λ^i	M	V_A
w	0	$\frac{1}{2}$	1	1

Table 6. Weyl weights w of the non-linear multiplet component fields in five space-time dimensions.

This result is based on the following identification of the $4D$ fields in terms of the $5D$ fields,

$$\begin{aligned} {A_i}^\alpha &= \phi^{-1/2} {A_i}^\alpha \,, \\ \zeta^\alpha &= \exp\left[\frac{1}{2}\varphi\gamma^5\right] \left(\phi^{-1/2}\zeta^\alpha - \frac{1}{2}\,\phi^{-3/2} {A_i}^\alpha\,\gamma^5 \hat{\psi}^i\right) , \end{aligned} \tag{B.14}$$

In the deriviation of the transformation rules (B.13) we again made use of the composite SU(2) transformation specified in (A.9). Hypermultiplets charged under non-abelian vector multiplets are presented in subsection 5.3.

B.4 The non-linear multiplet

Here we present the so-called non-linear multiplet in $5D$ Minkowski space as presented in [44]. Upon the timelike dimensional reduction, this multiplet will yield the non-linear multiplet in $4D$ Minkowski space. Here we present the multiplet in the conventions of section 2.

The $5D$ multiplet is based on an SU(2) matrix of scalars ${\Phi^i}_\alpha$ with $\alpha = 1, 2$, a symplectic Majorana spinor λ^i, a scalar M and a vector field V_A, in a superconformal background. The weights of the fields under dilatations are shown in table 6. The Q- and S supersymmetry transformations and the conformal boost transformations are as follows,

$$\begin{aligned} \delta {\Phi^i}_\alpha &= \mathrm{i}\,(2\,\bar\epsilon_j\,\lambda^i - \delta^i_j\,\bar\epsilon_k\,\lambda^k)\,{\Phi^j}_\alpha \,, \\ \delta\lambda^i &= \mathrm{i}\,{\Phi^i}_\alpha\,\not{D}{\Phi^\alpha}_j\,\epsilon^j + \frac{1}{2}\,M\,\epsilon^i - \frac{1}{2}\mathrm{i}\,\not{V}\epsilon^i + 2\mathrm{i}\,(\bar\epsilon_j\lambda^i)\,\lambda^j - \frac{3}{2}\,\eta^i \,, \\ \delta M &= 2\,\bar\epsilon_i\,\not{D}\lambda^i - \bar\epsilon_i\,\not{V}\,\lambda^i + \mathrm{i}\,\bar\epsilon_i\,\lambda^i\,M + 2\,{\Phi^i}_\alpha\,D_A{\Phi^\alpha}_j\,\bar\epsilon_i\,\gamma^A\,\lambda^j + 4\mathrm{i}\,\bar\epsilon_i\,\chi^i + \frac{5}{2}\mathrm{i}\,\bar\epsilon_i\,\gamma^{AB}\,\lambda^i\,T_{AB} \\ \delta V_A &= -\,2\mathrm{i}\,\bar\epsilon_i\,\gamma_{AB}\,D^B\lambda^i + \mathrm{i}\,\bar\epsilon_i\,\gamma_A\not{V}\,\lambda^i + \bar\epsilon_i\,\gamma_A\,\lambda^i\,M - 2\mathrm{i}\,{\Phi^i}_\alpha\,D_B{\Phi^\alpha}_j\,\bar\epsilon_i\,\gamma_A\gamma^B\,\lambda^j \\ &\quad - 4\,\bar\epsilon_i\,\gamma_A\,\chi^i + 6\,\bar\epsilon_i\,\gamma^B\,\lambda^i\,T_{AB} + 2\,\bar\epsilon_i\,\gamma_A\gamma^{BC}\,\lambda^i\,T_{BC} + \bar\lambda_i\,\gamma_A\,\eta^i + 6\,\Lambda_{\mathrm{K}\,A} \,. \end{aligned} \tag{B.15}$$

There is also a constraint on the divergence of the vector field,

$$D_A V^A - \frac{1}{2}\,V_A V^A - 2\,D - \frac{1}{2}\,M^2 + D_A{\Phi^i}_\alpha\,D^A{\Phi^\alpha}_i + \text{fermions} = 0 \,, \tag{B.16}$$

which ensures that the multiplet comprises $8 \oplus 8$ bosonic and fermionic degrees of freedom and the transformation rules will close off-shell.

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