COLORING JORDAN REGIONS AND CURVES*

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Abstract. A Jordan region is a subset of the plane that is homeomorphic to a closed disk. Consider a family $\mathcal{F}$ of Jordan regions whose interiors are pairwise disjoint, and such that any two Jordan regions intersect in at most one point. If any point of the plane is contained in at most k elements of $\mathcal{F}$ (with k sufficiently large), then we show that the elements of $\mathcal{F}$ can be colored with at most $k + 1$ colors so that intersecting Jordan regions are assigned distinct colors. This is best possible and answers a question raised by Reed and Shepherd in 1996. As a simple corollary, we also obtain a positive answer to a problem of Hliněný (1998) on the chromatic number of contact systems of strings. We also investigate the chromatic number of families of touching Jordan curves. This can be used to bound the ratio between the maximum number of vertex-disjoint directed cycles in a planar digraph, and its fractional counterpart.

Key words. graph coloring, geometric graphs, Jordan curves, Jordan regions

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1. Introduction. In this paper, a *Jordan region* is a subset of the plane that is homeomorphic to a closed disk. A family $\mathcal{F}$ of Jordan regions is *touching* if their interiors are pairwise disjoint. If any point of the plane is contained in at most k Jordan regions of $\mathcal{F}$, then we say that $\mathcal{F}$ is *k-touching*. If any two elements of $\mathcal{F}$ intersect in at most one point, then $\mathcal{F}$ is said to be *simple*. All the families of Jordan regions and curves we consider in this paper are assumed to have a finite number of intersection points. The first part of this paper is concerned with the *chromatic number* of simple *k*-touching families of Jordan regions, i.e., the minimum number of colors needed to color the Jordan regions, so that intersecting Jordan regions receive different colors. This can also be defined as the chromatic number of the *intersection graph* $G(\mathcal{F})$ of $\mathcal{F}$, which is the graph with vertex set $\mathcal{F}$ in which two vertices are adjacent if and only if the corresponding elements of $\mathcal{F}$ intersect. Recall that the *chromatic number* of a graph G, denoted by $\chi(G)$, is the least number of colors needed to color the vertices of G, so that adjacent vertices receive different colors. The chromatic number of a graph G is at least the *clique number* of G, denoted by $\omega(G)$, which is the maximum number of pairwise adjacent vertices in G, but the difference between the two parameters can be arbitrarily large (see [8] for a survey on the chromatic and clique numbers of geometric intersection graphs).

The following question was raised by Reed and Shepherd [10].

PROBLEM 1.1 (see [10]). *Is there a constant C such that for any simple touching family $\mathcal{F}$ of Jordan regions, $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + C$? Can we take $C = 1$?*

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Our main result is the following (we made no real effort to optimize the constant 490, which is certainly far from optimal; our main concern was to give a proof that is as simple as possible).

THEOREM 1.2. *For $k \geq 490$, any simple k-touching family of Jordan regions is $(k+1)$-colorable.*

Note that apart from the constant 490, Theorem 1.2 is best possible. Figure 1 depicts two examples of simple k-touching families of Jordan regions of chromatic number $k + 1$.
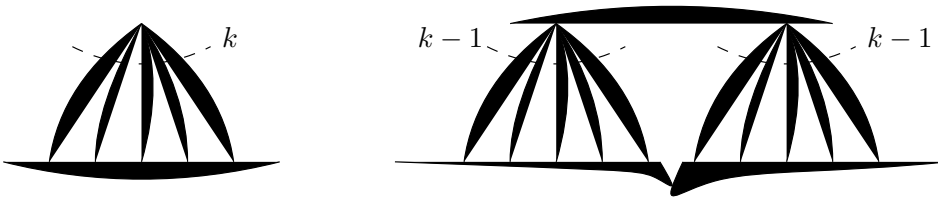

FIG. 1. *Two simple k-touching families of Jordan regions with chromatic number $k + 1$.*

It was proved in [2] that every simple k-touching family of Jordan regions is $3k$-colorable (their result is actually stated for k-touching families of strings, but it easily implies the result on Jordan regions). We obtain the next result as a simple consequence.

COROLLARY 1.3. *Any simple k-touching family of Jordan regions is $(k + 327)$-colorable.*

Proof. Let $\mathcal{F}$ be a simple k-touching family $\mathcal{F}$ of Jordan regions. If $k \leq 163$, then $\mathcal{F}$ can be colored with at most $3k \leq k+327$ colors by the result of [2] mentioned above. If $164 \leq k \leq 489$, then $\mathcal{F}$ is also 490-touching, and it follows from Theorem 1.2 that $\mathcal{F}$ can be colored with at most $491 \leq k + 327$ colors. Finally, if $k \geq 490$, Theorem 1.2 implies that $\mathcal{F}$ can be colored with at most $k + 1 \leq k + 327$ colors. □

Observe that for a given simple touching family $\mathcal{F}$ of Jordan regions, if we denote by k the least integer so that $\mathcal{F}$ is k-touching, then $\omega(G(\mathcal{F})) \geq k$, since k Jordan regions intersecting some point p of the plane are pairwise intersecting. Therefore, we obtain the following immediate corollary, which is a positive answer to the problem raised by Reed and Shepherd.

COROLLARY 1.4. *For any simple touching family $\mathcal{F}$ of Jordan regions, $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 327$ (and $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ if $\omega(G(\mathcal{F})) \geq 490$).*

Note that the bound $\chi(G(\mathcal{F})) \leq \omega(G(\mathcal{F})) + 1$ is also best possible (as shown by Figure 1, right).

It turns out that our main result also implies a positive answer to a question raised by Hliněný in 1998 [6]. A *string* is the image of some continuous injective function from $[0, 1]$ to $\mathbb{R}^2$, and the *interior* of a string is the string minus its two endpoints. A *contact system of strings* is a set of strings such that the interiors of any two strings have empty intersection. In other words, if c is a contact point in the interior of a string s, all the strings containing c distinct from s end at c. A contact system of strings is said to be *one-sided* if for any contact point c as above, all the strings ending at c leave from the same side of s (see Figure 2, left). Hliněný [6] raised the following problem.

PROBLEM 1.5 (see [6]). *Let $\mathcal{S}$ be a one-sided contact system of strings, such that any point of the plane is in at most k strings, and any two strings intersect in at most one point. Is it true that $G(\mathcal{S})$ has chromatic number at most $k + o(k)$ (or even $k + c$, for some constant c)?*
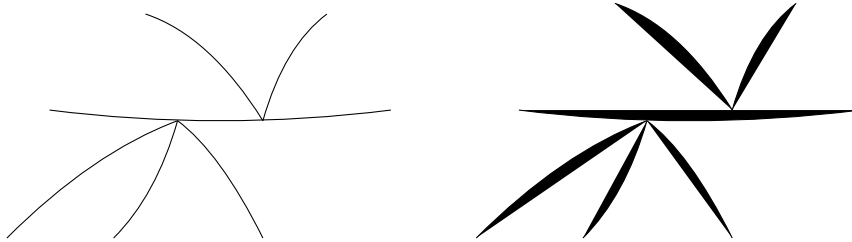

FIG. 2. *Turning a one-sided contact system of strings into a simple touching family of Jordan regions.*

The following simple corollary of Theorem 1.2 gives a positive answer to Problem 1.5.

COROLLARY 1.6. *Let $\mathcal{S}$ be a one-sided contact system of strings, such that any point of the plane is in at most k strings, and any two strings intersect in at most one point. Then $G(\mathcal{S})$ has chromatic number at most $k + 127$ (and at most $k + 1$ if $k \geq 490$).*

Proof. Assume first that $k \leq 363$. It was proved in [2] that $G(\mathcal{S})$ has chromatic number at most $\lceil \frac{4}{3}k \rceil + 6$, so in this case at most $k + 127$, as desired. Assume now that $k \geq 364$. Let $\mathcal{F}$ be obtained from $\mathcal{S}$ by thickening each string s of $\mathcal{S}$, turning s into a (very thin) Jordan region (see Figure 2, from left to right). Since $\mathcal{S}$ is one-sided, each intersection point contains precisely the same elements in $\mathcal{S}$ and $\mathcal{F}$, and therefore $G(\mathcal{S})$ and $G(\mathcal{F})$ are equal, while $\mathcal{F}$ is a simple k-touching family of Jordan regions. If $364 \leq k \leq 489$, then $\mathcal{F}$ is also 490-touching and it follows from Theorem 1.2 that $G(\mathcal{S}) = G(\mathcal{F})$ has chromatic number at most $491 \leq k + 127$. Finally, if $k \geq 490$, then by Theorem 1.2, $G(\mathcal{S}) = G(\mathcal{F})$ has chromatic number at most $k + 1$, as desired. □

A *Jordan curve* is the boundary of some Jordan region of the plane. We say that a family of Jordan curves is *touching* if for any two Jordan curves a, b, the curves a and b do not cross (equivalently, either the interiors of the regions bounded by a and b are disjoint, or one is contained in the other). Moreover, if any point of the plane is on at most k Jordan curves, we say that the family is k-touching. Note that unlike above, the families of Jordan curves we consider here are not required to be simple (two Jordan curves may intersect in several points). Note that previous works on intersection of Jordan curves have usually considered the opposite case, where every two curves that intersect also cross (see, for instance, [7] and the references therein).

Let $\mathcal{F}$ be a k-touching family of Jordan curves. For any two intersecting Jordan curves $a, b \in \mathcal{F}$, let $\mathcal{D}(a, b)$ be the set of Jordan curves c distinct from a, b such that the (closed) region bounded by c contains exactly one of a, b. The cardinality of $\mathcal{D}(a, b)$ is called *the distance between a and b* and is denoted by $d(a, b)$. Note that since $\mathcal{F}$ is k-touching, any two intersecting Jordan curves are at distance at most $k - 2$ from each other.

Given a k-touching family $\mathcal{F}$, the *average distance in $\mathcal{F}$* is the average of $d(a, b)$, over all pairs of intersecting Jordan curves $a, b \in \mathcal{F}$. We conjecture the following.

CONJECTURE 1.7. *For any k-touching family $\mathcal{F}$ of Jordan curves, the average distance in $\mathcal{F}$ is at most $\frac{k}{2}$.*

It was proved by Fox and Pach [4] that each k-touching family of strings is $(6ek + 1)$-colorable, which directly implies that each k-touching family of Jordan curves is $(6ek + 1)$-colorable (note that $6e \approx 16.31$). We show how to improve this bound when the average distance is at most αk for some $\alpha \leq 1$.

THEOREM 1.8. *Let $\mathcal{F}$ be a k-touching family of Jordan curves, such that the average distance in $\mathcal{F}$ is at most αk for some constant $0 \leq \alpha \leq 1$. Then the chromatic number of $\mathcal{F}$ is at most $\frac{6e^\delta}{\delta + \delta^2(1-\alpha)}\, k$, where $\delta = \delta(\alpha) = \frac{1}{2-2\alpha}(1 - 2\alpha + \sqrt{4\alpha^2 - 8\alpha + 5})$ for $\alpha < 1$ and $\delta(1) = 1$.*

Note that $\delta(1) = 1 = \lim_{\alpha \to 1} \frac{1}{2-2\alpha}(1 - 2\alpha + \sqrt{4\alpha^2 - 8\alpha + 5})$. Theorem 1.8 has the following direct corollary.

COROLLARY 1.9. *Let $\mathcal{F}$ be a k-touching family of Jordan curves, such that the average distance in $\mathcal{F}$ is at most αk. Then $\mathcal{F}$ is βk-colorable, where*

$$
\beta = \begin{cases}
12.76 & \text{if} \quad \alpha \leq 3/4, \\
10.22 & \text{if} \quad \alpha \leq 1/2, \\
8.43 & \text{if} \quad \alpha \leq 1/4.
\end{cases}
$$

By Corollary 1.9, a direct consequence of Conjecture 1.7 would be that every k-touching family of Jordan curves is $10.22\,k$-colorable.

For any k-touching family of Jordan curves, the average distance is at most k. Theorem 1.8 implies that every family of Jordan curves is $6ek$-colorable, which is the bound of Fox and Pach [4] (without the $+1$). To understand the limitation of Theorem 1.8 it is interesting to consider the case $\alpha = o(1)$. Then δ tends to $\frac{1}{2}(1+\sqrt{5})$, and we obtain in this case that $\mathcal{F}$ is $7.14\,k$-colorable. A particular case is when $\alpha = 0$. This is equivalent to saying that any two intersecting Jordan curves are at distance 0 from each other, and therefore the family $\mathcal{F}$ of Jordan curves can be turned into a k-touching family of Jordan regions (here and everywhere else in this manuscript, it is crucial that the curves are pairwise noncrossing). Note that it was proved in [1] (see also [2]) that k-touching families of Jordan regions are $(\frac{3k}{2} + o(k))$-colorable.

We now prove the following weaker version of Conjecture 1.7.

THEOREM 1.10. *Let $\mathcal{F}$ be a family of k-touching Jordan curves. Then the average distance in $\mathcal{F}$ is at most $k/(1 + \frac{1}{16e})$.*

An immediate consequence of Theorems 1.8 and 1.10 is the following small improvement over the bound of Fox and Pach [4] in the case of Jordan curves.

COROLLARY 1.11. *Any k-touching family of Jordan curves is $15.95k$-colorable.*

An interesting connection between the chromatic number of k-touching families of Jordan curves and the packing number of directed cycles in directed planar graphs was observed by Reed and Shepherd in [10]. In a planar digraph G, let $\nu(G)$ be the maximum number of vertex-disjoint directed cycles. This quantity has a natural linear relaxation, where we seek the maximum $\nu^*(G)$ for which there are weights in $[0, 1]$ on each directed cycle of G, summing up to $\nu^*(G)$, such that for each vertex v of G, the sum of the weights of the directed cycles containing v is at most 1. It was observed by Reed and Shepherd [10] that for any G there are integers n and k such that $\nu^*(G) = \frac{n}{k}$ and G contains a collection of n pairwise noncrossing directed cycles (counted with multiplicities) such that each vertex is in at most k of the directed cycles. If we replace

each directed cycle of the collection by its image in the plane, we obtain a k-touching family of Jordan curves. Assume that this family is βk-colorable for some constant β. Then the family contains an independent set (a set of pairwise nonintersecting Jordan curves) of size at least $n/(\beta k)$. This independent set corresponds to a packing of directed cycles in G. As a consequence, $\nu(G) \geq n/(\beta k) = \nu^*(G)/\beta$, and then $\nu^*(G) \leq \beta \nu(G)$. The following is therefore a direct consequence of Corollaries 1.9 and 1.11.

THEOREM 1.12. *For any planar directed graph G, $\nu^*(G) \leq 15.95 \cdot \nu(G)$. Moreover, if Conjecture 1.7 holds, then $\nu^*(G) \leq 10.22 \cdot \nu(G)$.*

This improves a result of Reed and Shepherd [10], who proved that for any planar directed graph G, $\nu^*(G) \leq 28 \cdot \nu(G)$. The same result with a constant factor of 16.31 essentially followed from the result of Fox and Pach [4] (and the discussion above). Using classical results of Goemans and Williamson [5], Theorem 1.12 also gives improved bounds on the ratio between the maximum packing of directed cycles in planar digraphs and the dual version of the problem, namely the minimum number of vertices that need to be removed from a planar digraph in order to obtain an acyclic digraph.

Organization of the paper. The proofs of Theorems 1.2, 1.8, and 1.10 are given in sections 2, 3, and 4, respectively. Section 5 concludes the paper with some remarks and open problems.

2. Proof of Theorem 1.2. In the proof below we will use the following parameters instead of their numerical values (for the sake of readability): $\varepsilon = \frac{1}{4}$, $b = \frac{18}{\varepsilon} = 72$, and $k \geq 7b - 14 = 490$.

The proof proceeds by contradiction. Assume that there exists a counterexample $\mathcal{F}$, and take it with a minimum number of Jordan regions.

We will construct a bipartite planar graph G from $\mathcal{F}$ as follows: for any Jordan region d of $\mathcal{F}$ we add a vertex in the interior of d (such a vertex will be called a *disk vertex*), and for any contact point p (i.e., any point on at least two Jordan regions), we add a new vertex at p (such a vertex will be called a *contact vertex*). Now, for every Jordan region d and contact point p on d, we add an edge between the disk vertex corresponding to d and the contact vertex corresponding to p.

We now start with some remarks on the structure of G.

CLAIM 2.1. *G is a connected bipartite planar graph.*

Proof. The fact that G is planar and bipartite easily follows from the construction. If G is disconnected, then $G(\mathcal{F})$ itself is disconnected, and some connected component contradicts the minimality of $\mathcal{F}$. □

CLAIM 2.2. *All the faces of G have degree (number of edges in a boundary walk counted with multiplicity) at least 6.*

Proof. Note that by construction, the graph G is simple (i.e., there are no parallel edges). Assume for the sake of contradiction that G has a face f of degree 4. Then either f bounds three vertices (and $\mathcal{F}$ consists of two Jordan curves intersecting in a single point, in which case the theorem trivially holds), or the face f corresponds to two Jordan regions of $\mathcal{F}$ sharing two distinct points, which contradicts the fact that $\mathcal{F}$ is simple. Since G is bipartite, it follows that each face has degree at least 6. □

Two disk vertices having a common neighbor are said to be *loose neighbors* in G (this corresponds to intersecting Jordan regions in $\mathcal{F}$).

CLAIM 2.3. *Every disk vertex has at least $k+1$ loose neighbors in G.*

Proof. Assume that some disk vertex has at most k loose neighbors in G. Then the corresponding Jordan region d of $\mathcal{F}$ intersects at most k other Jordan regions in $\mathcal{F}$. By minimality of $\mathcal{F}$, the family $\mathcal{F} \setminus \{d\}$ is $(k+1)$-colorable, and any $(k+1)$-coloring easily extends to d, since d intersects at most k other Jordan regions. We obtain a $(k+1)$-coloring of $\mathcal{F}$, which is a contradiction. □

CLAIM 2.4. *G has minimum degree at least 2, and each contact vertex has degree at most k.*

Proof. The fact that each contact vertex has degree at least 2 and at most k directly follows from the definition of a k-touching family. If G contains a disk vertex v of degree at most 1, then since contact vertices have degree at most k, v has at most $k-1$ loose neighbors in G, which contradicts Claim 2.3. □

CLAIM 2.5. *For any edge uv, at least one of u, v has degree at least 3.*

Proof. Assume that a disk vertex u of degree 2 is adjacent to a contact vertex of degree 2. Then u has at most $1 + k - 1 = k$ loose neighbors, which contradicts Claim 2.3. □

A d-vertex (resp., $\leq d$-vertex, $\geq d$-vertex) is a vertex of degree d (resp., at most d, at least d). A $\geq b$-vertex is also said to be a *big vertex*. A vertex that is not big is said to be *small*.

CLAIM 2.6. *Each disk vertex of degree at most 7 has at least one big neighbor.*

Proof. Assume that some disk vertex v of degree at most 7 has no big neighbor. It follows that all the neighbors of v have degree at most $b-1$, and so v has at most $7(b-2) \leq k$ loose neighbors, which contradicts Claim 2.3. □

We now assign to each vertex v of G a charge $\omega(v) = 2d(v) - 6$, and to each face f of G a charge $\omega(f) = d(f) - 6$ (here the function d refers to the degree of a vertex or a face). By Euler's formula, the total charge assigned to the vertices and edges of G is precisely -12. We now proceed by locally moving the charges (while preserving the total charge) until all vertices and faces have nonnegative charge. In this case we obtain that $-12 \geq 0$, which is a contradiction. The charges are locally redistributed according to the following rules (for rule (R2), we need the following definition: a *bad vertex* is a disk 3-vertex v adjacent to two contact 2-vertices u, w, such that the three faces incident to v have degree 6 and the neighbors of u and w have degree 3).

(R1) For each big contact vertex v and each sequence of three consecutive neighbors u_1, u_2, u_3 of v in clockwise order around v, we do the following. If u_2 has a unique big neighbor (namely, v), then v gives $2 - \varepsilon$ to u_2. Otherwise v gives 1 to u_2, and $(1-\varepsilon)/2$ to each of u_1 and u_3.

(R2) Each big contact vertex gives ε to each bad neighbor.

(R3) Each small contact vertex of degree at least 4 gives $\frac{1}{2}$ to each neighbor.

(R4) Each contact 3-vertex adjacent to some ≥ 3-vertex gives ε to each neighbor of degree 2.

(R5) Each disk vertex of degree at least 4 gives $1 + \varepsilon$ to each neighbor of degree at most 3.

(R6) For each disk vertex v of degree 3 and each neighbor u of v with $d(u) \leq 3$, we do the following. If either u has degree 3, or u has degree 2 and the neighbor of u distinct from v has degree at least 4, then v gives $1 - \varepsilon$ to u. Otherwise, v gives 1 to u.

(R7) Each face f of degree at least 8 gives $\frac{1}{2}$ to each disk vertex incident with f.

We now analyze the new charge of each vertex and face after all these rules have been applied.

By Claim 2.2, all faces have degree at least 6. Since faces of degree 6 start with a charge of 0, and do not give any charge, their new charge is still 0. Let f be a face of degree $d \geq 8$. Then f starts with a charge of $d - 6$ and gives at most $\frac{d}{2} \cdot \frac{1}{2}$ by rule (R7). The new charge is then at least $d - 6 - \frac{d}{2} \cdot \frac{1}{2} = \frac{3d}{4} - 6 \geq 0$, as desired.

We now consider disk vertices. Note that these vertices receive charge by rules (R1)–(R4) and (R7) and give charge by rules (R5)–(R6). Consider first a disk vertex v of degree $d \geq 8$. Then v starts with a charge of $2d - 6$ and gives at most $d(1 + \varepsilon)$ (by rule (R5)), so the new charge of v is at least $2d - 6 - d(1 + \varepsilon) = d(1 - \varepsilon) - 6 \geq 0$ (since $\varepsilon = \frac{1}{4}$).

Assume now that v is a disk vertex of degree $4 \leq d \leq 7$. Then by Claim 2.6, v has at least one big neighbor. The vertex v starts with a charge of $2d - 6$, receives at least $2 - \varepsilon$ by rule (R1), and gives at most $(d-1)(1+\varepsilon)$ by rule (R5). The new charge of v is then at least $2d - 6 + 2 - \varepsilon - (d-1)(1+\varepsilon) = d(1-\varepsilon) - 3 \geq 0$ (since $\varepsilon = \frac{1}{4}$).

We now consider a disk vertex v of degree 3. Again, it follows from Claim 2.6 that v has at least one big neighbor. The vertex v starts with a charge of 0, and since v has at least one big neighbor, v receives at least $2 - \varepsilon$ from its big neighbors by rule (R1). Let w be a big neighbor of v, and assume first that at least one of the two neighbors of v distinct from w (call them u_1, u_2) is not a 2-vertex adjacent to two 3-vertices. Then by rule (R6), v gives at most $2 - \varepsilon$ to u_1, u_2 (recall that by Claim 2.5, no two vertices of degree 2 are adjacent in G). In this case the new charge of v is at least $2 - \varepsilon - (2 - \varepsilon) \geq 0$, as desired. Assume now that u_1, u_2 both have degree 2 and their neighbors all have degree 3. In this case v gives 1 to each of u_1, u_2 and the new charge of v is at least $2 - \varepsilon - 2 \geq -\varepsilon$. If v is incident to a face of degree at least 8, v receives at least $\frac{1}{2}$ from such a face, and its new charge is at least $-\varepsilon + \frac{1}{2} \geq 0$, as desired. So we can assume that all the faces incident to v are faces of degree 6. In other words, v is a bad vertex. Then w gives an additional charge of ε to v by rule (R2), and the new charge of v in this last case is at least $-\varepsilon + \varepsilon \geq 0$, as desired.

Assume now that v is a disk vertex of degree 2. Then the vertex v starts with a charge of -2. By Claim 2.6, v has a big neighbor; call it w. By Claim 2.5, the neighbor of v distinct from w (call it u) has degree at least 3. If u is big, then v receives a charge of $1 + 1 = 2$ by rule (R1), and its new charge is thus at least $-2 + 2 = 0$, so we can assume that u is small (in particular, v receives $2 - \varepsilon$ from w by rule (R1)). If u has degree at least 4, then u gives a charge of $\frac{1}{2}$ to v by rule (R3), and the new charge of v is then at least $-2 + 2 - \varepsilon + \frac{1}{2} \geq 0$. If v lies on a face of degree at least 8, then v receives $\frac{1}{2}$ from this face by rule (R7), and its new charge is then at least $-2 + 2 - \varepsilon + \frac{1}{2} \geq 0$. So we can assume that u has degree 3 and all the faces containing v have degree 6. If u is adjacent to some ≥ 3-vertex, then u gives ε to v by rule (R4), and in this case the new charge of v is at least $-2 + 2 - \varepsilon + \varepsilon \geq 0$. So we can further assume that all the neighbors of u are 2-vertices. Call u_1, u_2 the neighbors of u distinct from v, and for $i = 1, 2$ let v_i be the neighbor of u_i distinct from u. Since u has degree 3, it follows from Claim 2.6 that v_1 and v_2 are big. Let v^+ (resp., v^-) be the neighbor of w immediately succeeding (resp., preceding) v in clockwise order around w. The faces containing v have degree 6, and since G is bipartite with minimum degree at least 2 (by Claims 2.1 and 2.4), each of these two faces is bounded by six vertices. As a consequence, we can assume that v^+ is adjacent to v_1 and v^- is adjacent to v_2 (see Figure 3). It follows that each of v^+, v^- has at least two big

neighbors. Therefore, by rule (R1), v received from w (in addition to the $2 - \varepsilon$ that were taken into account earlier) $2 \cdot (1 - \varepsilon)/2 = 1 - \varepsilon$. So the new charge of v is at least $-2 + 2 - \varepsilon + 1 - \varepsilon = 1 - 2\varepsilon \geq 0$, as desired.
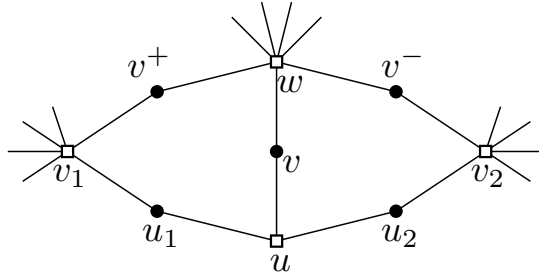

FIG. 3. *Contact vertices are depicted with white squares, and disk vertices are depicted with black dots.*

We now study the new charge of contact vertices. Note that contact vertices give charge by rules (R1)–(R4) and receive charge by rules (R5)–(R7). Consider a contact vertex v of degree 2. Then v starts with a charge of -2. By Claim 2.5, the two neighbors of v (call them u and w) have degree at least 3. If they both have degree at least 4, then they both give $1 + \varepsilon$ to v by rule (R5), and the new charge of v is at least $-2 + 2(1 + \varepsilon) \geq \varepsilon$. If one of u, w has degree at least 4 and the other has degree 3, then v receives $1 + \varepsilon$ by rule (R5) and $1 - \varepsilon$ by rule (R6). In this case the new charge of v is at least $-2 + 1 + \varepsilon + 1 - \varepsilon = 0$. Finally, if u and w both have degree 3, then they both give 1 to v by rule (R6), and the new charge of v is at least $-2 + 1 + 1 = 0$, as desired.

Consider a contact vertex v of degree 3. Then v starts with a charge of 0 and only gives charge if rule (R4) applies. In this case, v gives a charge of ε to at most two of its neighbors. However, if rule (R4) applies, then, by definition, v has a neighbor of degree at least 3. Then v receives at least $1 - \varepsilon$ from such a neighbor by rules (R5)–(R6). In this case, the new charge of v is at least $0 - 2\varepsilon + 1 - \varepsilon \geq 0$ (since $\varepsilon = \frac{1}{4}$).

Assume now that v is a contact vertex of degree $d \geq 4$. Then v starts with a charge of $2d - 6$. If v is small, then v gives at most $d \cdot \frac{1}{2}$ by rule (R3), and the new charge of v is then at least $2d - 6 - d \cdot \frac{1}{2} = \frac{3d}{2} - 6 \geq 0$. Assume now that v is big. In this case, applications of rule (R1) cost v no more than $d(2 - \varepsilon)$ charge. We claim the following.

CLAIM 2.7. *For every big contact vertex v of degree d, applications of rule (R2) cost v no more than $\frac{2d}{3} \cdot \varepsilon$ charge.*

Proof. We will show that v never gives a charge of ε to three consecutive neighbors of v, which implies the claim. Assume for the sake of contradiction that v gives a charge of ε to three consecutive neighbors u, w, x of v (in clockwise order around v). Assume that the neighbors of u are v, u_1, u_2 (in clockwise order around u), and the neighbors of w are v, w_1, w_2 (in clockwise order around w). Recall that by the definition of a bad vertex, each of u_1, u_2, w_1, w_2 has degree 2, and all the faces incident to u or w have degree 6. Let u_1', u_2' be the neighbors of u_1, u_2 distinct from u, and let w_1', w_2' be the neighbors of w_1, w_2 distinct from w. By the definition of a bad vertex, each of u_1', u_2', w_1', w_2' has degree 3, and since all the faces incident to u or w have degree 6, $u_2' = w_1'$ and the vertices u_1', u_2', w_2' have a common neighbor, which we call
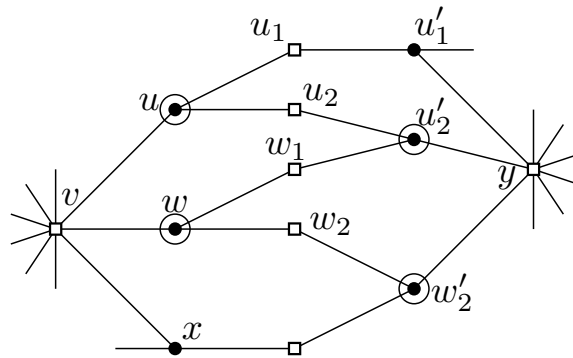
Fig. 4. *Illustration of the proof of Claim* 2.7. *The four removed vertices are circled.*

y. Again, by the definition of a bad vertex, the neighbor of w_2' distinct from y has degree 2 and is adjacent to x (see Figure 4). Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ by removing the disks corresponding to u, w, u_2', w_2'. By minimality of $\mathcal{F}$, $\mathcal{F}'$ has a $(k+1)$-coloring c, which we seek to extend to u, w, u_2', w_2' (by a slight abuse of notation we identify a disk vertex of G with the corresponding disk of $\mathcal{F}$). Note that u and w_2' have at most $k-1$ colored neighbors, while w and u_2' have at most $k-2$ colored neighbors. Since $k+1$ colors are available, it follows that each of u, w_2' has a list of at least two available colors, while each of w, u_2' has a list of at least three available colors. We must choose a color in each of the four lists such that vertices in each pair of vertices among u, w, u_2', w_2', except the pair uw_2', are assigned different colors. This is equivalent to the following problem: take H to be the complete graph on four vertices minus an edge, assign to each vertex z of H an arbitrary list of at least $d_H(z)$ colors, and then choose a color in each list such that adjacent vertices are assigned different colors. It follows from a classical result of Erdős, Rubin, and Taylor [3] that this is possible for any 2-connected graph distinct from a complete graph and an odd cycle (and in particular, this holds for H). Therefore, the $(k+1)$-coloring c of $\mathcal{F}'$ can be extended to u, w, u_2', w_2' to obtain a $(k+1)$-coloring of $\mathcal{F}$, which is a contradiction. This proves Claim 2.7. ☐

Hence, if v is a big contact vertex of degree d, then the new charge of v is at least $2d - 6 - d(2 - \varepsilon) - \frac{2d}{3} \cdot \varepsilon = \frac{d}{3} \cdot \varepsilon - 6$. Since v is big, $d \geq b$, and so the new charge of v is at least $b\varepsilon/3 - 6 = 0$ (since $b = 18/\varepsilon$). It follows that the new charge of all vertices and faces is nonnegative, and then the total charge (which equals -12) is nonnegative, which is a contradiction. This concludes the proof of Theorem 1.2.

3. Proof of Theorem 1.8. We start with a simple lemma showing that in order to bound the chromatic number of k-touching families of Jordan curves, it is enough to asymptotically bound the number of edges in their intersection graphs.

LEMMA 3.1. *Assume that there is a constant $a > 0$ and a function $f = o(1)$ such that for any integers k, n and any k-touching family $\mathcal{F}$ of n Jordan curves, the graph $G(\mathcal{F})$ has at most $ak(1 + f(k))n$ edges. Then for any integer k, any k-touching family of Jordan curves is $2ak$-colorable.*

Proof. Let $\mathcal{F}$ be a k-touching family of n Jordan curves, and let m denote the number of edges of $G(\mathcal{F})$. For some integer ℓ, replace each element $c \in \mathcal{F}$ by ℓ concentric copies of c, without creating any new intersection point (i.e., any portion of Jordan curve between two intersection points is replaced by ℓ parallel portions of

Jordan curves). Let $\ell\mathcal{F}$ denote the resulting family. Note that $\ell\mathcal{F}$ is ℓk-touching and contains ℓn elements, and $G(\ell\mathcal{F})$ contains $\binom{\ell}{2}n + \ell^2 m$ edges. Hence, we have $\binom{\ell}{2}n + \ell^2 m < a \cdot \ell k(1 + f(\ell k)) \cdot \ell n$. Therefore, $m < (ak(1 + f(\ell k)) - \frac{1}{2} + \frac{1}{2\ell})n$, and $G(\mathcal{F})$ contains a vertex of degree at most $2ak(1 + f(\ell k)) - 1 + \frac{1}{\ell}$. This holds for any ℓ, and since the degree of a vertex is an integer and $f = o(1)$, $G(\mathcal{F})$ indeed contains a vertex of degree at most $2ak - 1$. We proved that k-touching families of Jordan curves are $(2ak - 1)$-degenerate and therefore $2ak$-colorable.   □

We will also need the following two lemmas.

LEMMA 3.2. *For any integers ℓ, k, d such that $d+2 \le \ell \le k$, and for any $p \in [0,1)$,*

$$(1-p)^{\ell-2} + p(1-p)^{\ell-3}(\ell - d - 2) \ge (1-p)^{k-2} + p(1-p)^{k-3}(k - d - 2).$$

Proof. For fixed $d \in \mathbb{R}$ and $p \in [0,1)$ we write $f(\ell) := (1-p)^{\ell-2} + p(1-p)^{\ell-3}(\ell - d - 2)$. Note that $f(d + 2) = (1-p)^d = f(d + 3)$. Furthermore, for all reals $\ell \ge d+3$, $\frac{d}{d\ell}f(\ell) = (1-p)^{\ell-3} \cdot (\log(1-p) \cdot (1 + p \cdot (\ell - d - 3)) + p) \le (1-p)^{\ell-3}(-p \cdot (1 + p \cdot (\ell - d - 3)) + p) \le 0$. So $f(\ell) \ge f(k)$ for all integers $d+2 \le \ell \le k$. □

LEMMA 3.3. *For any reals $1 \le \delta < 2$ and $k \ge 2$, we have $(1 - \frac{\delta}{k})^{k-3} \ge (1 - \frac{\delta}{k})^{k-2} \ge e^{-\delta}$.*

Proof. We clearly have $(1 - \frac{\delta}{k})^{k-3} \ge (1 - \frac{\delta}{k})^{k-2}$. To see that the second part of the inequality holds, observe first that for any real $0 \le x \le 2$, we have $e^{-x} \le 1 - \frac{x}{3}$, and thus the desired inequality holds for $k = 2, 3$.

Assume now that $k \ge 4$. Note that for any real $x \ge 0$, we have $e^{-x} \le 1 - x + \frac{x^2}{2}$. Thus,

$$\exp(-\tfrac{\delta}{k-2}) \le 1 - \tfrac{\delta}{k-2} + \tfrac{\delta^2}{2(k-2)^2} = 1 - \tfrac{\delta}{k} + \tfrac{\delta}{2k(k-2)^2}((\delta - 4)k + 8) \le 1 - \tfrac{\delta}{k},$$

with the rightmost inequality holding since $(4 - \delta)k \ge 2k \ge 8$. It follows that $\exp(-\delta) \le (1 - \frac{\delta}{k})^{k-2}$, as desired. □

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. Let $\mathcal{F}$ be a k-touching family of n Jordan curves, with average distance at most αk, and let $\delta = \delta(\alpha)$ be as provided by Theorem 1.8. Note that since $0 \le \alpha \le 1$, we have $1 \le \delta \le \frac{1}{2}(1 + \sqrt{5}) < 2$. We denote by E the edge-set of $G(\mathcal{F})$ and by m the cardinality of E. We will prove that $m < \frac{3e^\delta k}{\delta + \delta^2(1 - \alpha - \frac{2}{k})}$. Using Lemma 3.1, this implies that the chromatic number of any k-touching family of Jordan curves with average distance at most αk is at most $\frac{6e^\delta k}{\delta + \delta^2(1-\alpha)}$. Note that the chosen value $\delta(\alpha)$ of δ minimizes the value of $\frac{6e^\delta}{\delta + \delta^2(1-\alpha)}$. In the remainder of the proof, we will only use the fact that $1 \le \delta < 2$.

As observed in [2], we can assume without loss of generality that each Jordan curve is a polygon (this is a simple consequence of the fact that any simple plane graph can be drawn with straight-line edges).

We recall that for two intersecting Jordan curves $a, b \in \mathcal{F}$, $\mathcal{D}(a, b)$ is the set of Jordan curves c distinct from a, b such that the (closed) region bounded by c contains exactly one of a, b, and the cardinality of $\mathcal{D}(a, b)$ (which is called *the distance between a and b*) is denoted by $d(a, b)$. For each edge $ab \in E$, we choose an arbitrary point $x(a, b)$ in the intersection of the Jordan curves corresponding to a and b. Observe that since the curves are pairwise noncrossing, $x(a, b)$ is contained in all the curves of $\mathcal{D}(a, b)$. We now select each Jordan curve of $\mathcal{F}$ uniformly at random, with probability

$p = \frac{\delta}{k}$. Let $\mathcal{F}'$ be the obtained family. The expectation of the number of Jordan curves in $\mathcal{F}'$ is pn. For any pair of intersecting Jordan curves a, b, we denote by P_{ab} the probability that the set S of Jordan curves of $\mathcal{F}'$ containing $x(a, b)$ satisfies

(1) S has size at most 3,

(2) $a, b \in S$, and

(3) if $|S| = 3$, then the Jordan curve of S distinct from a and b is not an element of $\mathcal{D}(a, b)$.

Observe that

$$P_{ab} = p^2(1-p)^{\ell-2} + p^3(1-p)^{\ell-3}(\ell - d(a, b) - 2),$$

where $\ell \in \{d(a, b) + 2, \ldots, k\}$ denotes the number of Jordan curves containing $x(a, b)$ in $\mathcal{F}$.

We say that an edge $ab \in E$ is *good* if a, b satisfy (1), (2), and (3) above. It follows from Lemmas 3.2 and 3.3 that the expectation of the number of good edges is

$$\sum_{ab \in E} P_{ab} \geq \sum_{ab \in E} \left(p^2(1-p)^{k-2} + p^3(1-p)^{k-3}(k - d(a, b) - 2) \right)$$

$$\geq p^2 e^{-\delta} m + p^3 e^{-\delta} \sum_{ab \in E} (k - d(a, b) - 2)$$

$$= p^2 e^{-\delta} m \left(1 + p \left(k - 2 - \tfrac{1}{m} \sum_{ab \in E} d(a, b) \right) \right).$$

$$\geq p^2 e^{-\delta} m \left(1 + \delta(1 - \alpha - \tfrac{2}{k}) \right),$$

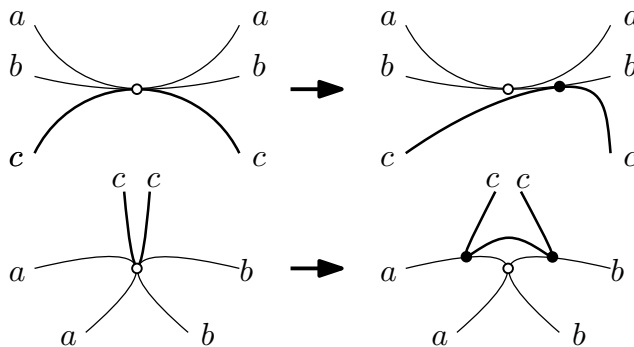since $\sum_{ab \in E} d(a, b) \leq \alpha k m$.

FIG. 5. *The point $x(a, b)$ is depicted by a white dot, and the newly created points are depicted by black dots.*

Let $\mathcal{F}''$ be obtained from $\mathcal{F}'$ by slightly modifying the Jordan curves around each intersection point x as follows. If $x = x(a, b)$ for some good edge ab, then we do the following. Note that by (1) and (2), $a, b \in \mathcal{F}'$, and $x = x(a, b)$ is contained in at most one Jordan curve of $\mathcal{F}'$ distinct from a and b. Assume that such a Jordan curve exists, and call it c. By (3), a and b are at distance 0 from each other in $\mathcal{F}'$. We then slightly modify c in a small disk centered in $x(a, b)$ so that for any $d \in \{a, b\}$, if c and d are at distance 0 from each other in $\mathcal{F}'$, then they remain at distance 0 from each other in $\mathcal{F}''$. Moreover, the point $x(a, b)$ and the newly created points are 2-touching

in $\mathcal{F}''$ (see Figure 5). If ac (resp., bc) is also a good edge with $x(a, c) = x$ (resp., $x(b, c) = x$), then note that the conclusion above also holds with a, b replaced by a, c (resp., b, c). Now, for any other intersection point y of Jordan curves of $\mathcal{F}'$ that is not equal to $x(a, b)$ for some good edge ab, we make the Jordan curves disjoint at y. It follows from the definition of a good edge that the family $\mathcal{F}''$ obtained from $\mathcal{F}'$ after these modifications is 2-touching, and for any good edge ab, a and b are at distance 0 from each other in $\mathcal{F}''$. Note that $G(\mathcal{F}'')$ is planar, since $\mathcal{F}''$ is 2-touching, and its expected number of edges is $\sum_{ab \in E} P_{ab}$. Since the number of edges of a planar graph is less than three times its number of vertices, we obtain

$$3pn > \sum_{ab \in E} P_{ab} \geq p^2 e^{-\delta} m \left(1 + \delta(1 - \alpha - \tfrac{2}{k}) \right).$$

As a consequence,

$$m < \frac{3e^\delta\, k}{\delta + \delta^2 \left(1 - \alpha - \tfrac{2}{k} \right)}\, n,$$

as desired. This concludes the proof of Theorem 1.8.

4. Proof of Theorem 1.10. The following is an easy variation of the main result of Fox and Pach [4]. Consider three Jordan curves a, b, c such that a is outside the region bounded by c, b is inside the region bounded by c, and a intersects b. Then we say that the pair a, b is *c-crossing*.

LEMMA 4.1. *Let c be a Jordan curve, and let $\mathcal{F}$ be a family of n Jordan curves such that $\mathcal{F} \cup \{c\}$ is k-touching and all the elements of $\mathcal{F}$ intersect c. Then the number of c-crossing pairs in $\mathcal{F}$ is at most $2ekn$.*

Proof. Let m be the number of c-crossing pairs in $\mathcal{F}$. For each c-crossing pair a, b in $\mathcal{F}$, we consider an arbitrary point $x(a, b)$ in $a \cap b$. We now select each Jordan curve of $\mathcal{F}$ uniformly at random with probability $p = \frac{1}{k}$. Let $\mathcal{F}'$ be the resulting family. A c-crossing pair a, b in $\mathcal{F}$ is *good* if $\mathcal{F}'$ contains a and b, but does not contain any other Jordan curve of $\mathcal{F}$ containing $x(a, b)$. Note that the probability that a given c-crossing pair a, b is good is at least $p^2 (1 - p)^{k-3}$, and therefore the expectation of the number of good c-crossing pairs is at least $p^2 (1 - p)^{k-3} m$. For any intersection point y of Jordan curves of $\mathcal{F}'$ that is not equal to $x(a, b)$ for some good c-crossing pair a, b, we make the Jordan curves disjoint at y (this is possible since the Jordan curves are pairwise noncrossing). Let $\mathcal{F}''$ be the obtained family. Observe that $\mathcal{F}''$ is 2-touching and each intersection point contains one Jordan curve lying outside the region bounded by c and one Jordan curve lying inside the region bounded by c. The graph $G(\mathcal{F}'')$ is therefore planar and bipartite. The expectation of the number of vertices of $G(\mathcal{F}'')$ is pn, and the expectation of the number of edges of $G(\mathcal{F}'')$ is at least $p^2(1-p)^{k-3} m$. Since any planar bipartite graph on N vertices contains at most $2N$ edges, it follows that $p^2(1-p)^{k-3} m < 2pn$. Since $(1 - \frac{1}{k})^{k-3} > e^{-1}$, we obtain that $m < 2ekn$, as desired. $\qquad\square$

Some planar quadrangulations can be represented as 2-touching families of Jordan curves intersecting a given Jordan curve c (so that each edge of the quadrangulation corresponds to a c-crossing pair of Jordan curves). Therefore, the bound $2N$ cannot be decreased (by more than an additive constant) in the proof of Lemma 4.1. Furthermore, the possibly near-extremal example in Figure 6 shows that the bound $2ekn$ in Lemma 4.1 cannot be improved to less than $(2k - 4)n$.
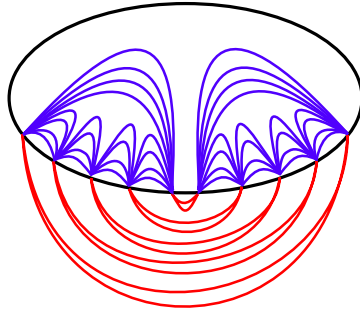
We are now ready to prove Theorem 1.10.

FIG. 6. *A family $\mathcal{F}$ of n Jordan curves that all intersect a fixed Jordan curve c, such that $\mathcal{F} \cup \{c\}$ is k-touching. Each Jordan curve in the interior of c touches each Jordan curve in the exterior of c. In the interior, there are only two sets of $k-2$ concentric Jordan curves. The remaining $n-2k+4$ Jordan curves are in the exterior of c. As n goes to infinity, the number of c-crossing pairs divided by n converges to $2k-4$.*

Proof of Theorem 1.10. Let E denote the edge-set of $G(\mathcal{F})$, and let $m = |E|$. Let $\alpha = \frac{1}{km} \sum_{ab \in E} d(a,b)$. Note that the average distance in $\mathcal{F}$ is αk.

Fix some $0 < \epsilon < 1$, and set $\delta = \frac{1}{2}(1 - \epsilon)$. For any edge $ab \in E$ with $d(a,b) > 0$, we do the following. Note that there is a unique ordering $c_1, \ldots, c_d$ of the elements of $\mathcal{D}(a,b)$, such that for any $1 \leq i \leq d$, the distance between a and c_i is $i-1$. Then the edge ab gives a charge of 1 to each of the elements $c_{\lceil \delta d \rceil}, c_{\lceil \delta d \rceil + 1}, \ldots, c_{\lfloor (1-\delta)d \rfloor + 1}$. Let T be the total charge given during this process. Note that

$$T = \sum_{ab \in E} \left(\lfloor (1-\delta)d(a,b) \rfloor + 1 - \lceil \delta d(a,b) \rceil + 1 \right)$$
$$\geq \sum_{ab \in E} (1 - 2\delta)d(a,b) = \epsilon \sum_{ab \in E} d(a,b).$$

We now analyze how much charge was received by an arbitrary Jordan curve c. Let $N(c)$ denote the neighborhood of c, and let $N^+(c)$ (resp., $N^-(c)$) denote the set of neighbors of c lying outside (resp., inside) the region bounded by c. Observe that if c received a charge of 1 from some edge ab, then without loss of generality we have $a \in N^+(c)$, $b \in N^-(c)$, and both a and b are at distance at most $\max(\lfloor (1 - \delta)d(a,b) \rfloor, d(a,b) - \lceil \delta d(a,b) \rceil) \leq (1-\delta)d(a,b) \leq (1-\delta)k$ from c. Let $N_{1-\delta}(c)$ denote the set of neighbors of c that are at distance at most $(1-\delta)k$ from c. Then the charge received by c is at most the number of c-crossing pairs a, b in the subfamily of $\mathcal{F}$ induced by $N_{1-\delta}(c)$, which is at most $2ek|N_{1-\delta}(c)|$ by Lemma 4.1.

For any γ, let m_γ denote the number of edges $ab \in E$ such that a and b are at distance at most γk. It follows from the analysis above that $T \leq 4ekm_{1-\delta}$. Therefore, $\sum_{ab \in E} d(a,b) \leq \frac{4e}{\epsilon} km_{1-\delta}$. Since $\sum_{ab \in E} d(a,b) = \alpha km$, we have $m_{(1+\epsilon)/2} = m_{1-\delta} \geq \frac{\epsilon \alpha}{4e} m$.

We now study the contribution of an arbitrary edge ab to the sum $\sum_{ab \in E} d(a,b) = \alpha km$. Let t be some integer. If $d(a,b) \leq \frac{t+1}{2t}k$, then ab contributes at most $\frac{t+1}{2t}k$ to αkm, and therefore at most $\frac{t+1}{2t}$ to αm. Note that there are $m_{(t+1)/2t}$ such edges ab. For each $2 \leq i \leq t-1$, each edge ab such that $\frac{t+i-1}{2t}k < d(a,b) \leq \frac{t+i}{2t}k$ contributes at most $\frac{t+i}{2t}$ to αm, and there are $m_{(t+i)/2t} - m_{(t+i-1)/2t}$ such edges. Finally, each edge ab with $d(a,b) > \frac{2t-1}{2t}k$ contributes at most 1 to αm, and there are $m - m_{(2t-1)/2t}$

such edges. As a consequence,

$$\alpha m \le \tfrac{t+1}{2t} m_{(t+1)/2t} + \sum_{i=2}^{t-1} \left(\tfrac{t+i}{2t}(m_{(t+i)/2t} - m_{(t+i-1)/2t}) \right) + m - m_{(2t-1)/2t}$$

$$= \sum_{i=1}^{t-1} \left(m_{(t+i)/2t}\left(\tfrac{t+i}{2t} - \tfrac{t+i+1}{2t}\right) \right) + m$$

$$= m - \tfrac{1}{2t} \sum_{i=1}^{t-1} m_{(t+i)/2t}$$

$$\le m - \tfrac{1}{2t} \sum_{i=1}^{t-1} \tfrac{i}{t} \tfrac{\alpha}{4e}\, m,$$

since $m_{(1+\epsilon)/2} \ge \tfrac{\epsilon\alpha}{4e} m$ for every $0 < \epsilon < 1$. As a consequence, we obtain that $\alpha \le 1 - \tfrac{t-1}{t} \tfrac{\alpha}{16e}$. Since this holds for any integer t, we have $\alpha \le 1 - \tfrac{\alpha}{16e}$ and therefore $\alpha \le 1/(1 + \tfrac{1}{16e})$, as desired.

5. Remarks and open questions. Most of the proof of Theorem 1.2 proceeds by finding a Jordan region intersecting at most k other Jordan regions (see Claim 2.3). On a single occasion, we use a different reduction (via a list-coloring argument). A natural question is whether this could be avoided. Is it true that in any simple k-touching family of Jordan regions, if k is large enough, then there is a Jordan region which intersects at most k other Jordan regions? This turns out to be wrong, as depicted in Figure 7. However, a proof along the lines of that of Theorem 1.2 (but significantly simpler) shows that if k is large enough, then there is a Jordan region which intersects at most $k+1$ other Jordan regions. It was pointed out to us by Patrice Ossona de Mendez (after the original version of this manuscript was submitted) that he also obtained this result in 1999 (see [9]). His result and its proof are stated with a completely different terminology, but the ideas are essentially the same. In particular, his result also implies (relatives of) our Corollaries 1.3 and 1.4.
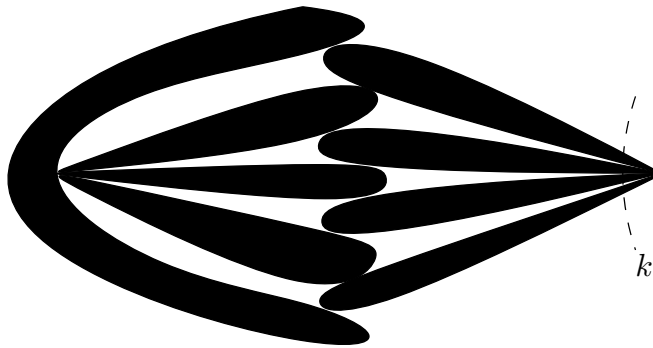

FIG. 7. *Every Jordan region intersects precisely $k + 1$ other Jordan regions.*

This can be used to obtain a result on the chromatic number of *simple* families of k-touching Jordan curves (families of k-touching Jordan curves such that any two Jordan curves intersect in at most one point). Using the result mentioned above (that if k is sufficiently large and the interiors are pairwise disjoint, then there is a Jordan region that intersects at most $k + 1$ other Jordan regions), it is not difficult to show

that the chromatic number of any simple family of k-touching Jordan curves is at most $2k$ plus a constant. We believe that the answer should be much smaller.

PROBLEM 5.1. *Is it true that for some constant c, any simple family of k-touching Jordan curves can be colored with at most $k + c$ colors?*

It was conjectured in [2] that if $\mathcal{S}$ is a family of pairwise noncrossing strings such that (i) any two strings intersect in at most one point and (ii) any point of the plane is on at most k strings, then $\mathcal{S}$ is $(k + c)$-colorable for some constant c. Note that, if true, this conjecture would give a positive answer to Problem 5.1.

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