

ON THE TREewidth OF RANDOM GEOMETRIC GRAPHS AND PERCOLATED GRIDS

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Abstract

In this paper we study the treewidth of the random geometric graph, obtained by dropping n points onto the square $[0, \sqrt{n}]^2$ and connecting pairs of points by an edge if their distance is at most $r = r(n)$. We prove a conjecture of Mitsche and Perarnau (2014) stating that, with probability going to 1 as $n \rightarrow \infty$, the treewidth of the random geometric graph is $\Theta(r\sqrt{n})$ when $\liminf r > r_c$, where r_c is the critical radius for the appearance of the giant component. The proof makes use of a comparison to standard bond percolation and with a little bit of extra work we are also able to show that, with probability tending to 1 as $k \rightarrow \infty$, the treewidth of the graph we obtain by retaining each edge of the $k \times k$ grid with probability p is $\Theta(k)$ if $p > \frac{1}{2}$ and $\Theta(\sqrt{\log k})$ if $p < \frac{1}{2}$.

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1. Introduction and main results

The *random geometric graph* $\mathcal{G}(n, r)$ is the random graph obtained by taking n points X_1, \dots, X_n independent and identically distributed (i.i.d.) uniformly at random from the square $[0, \sqrt{n}]^2$, and joining X_i and X_j by an edge if their Euclidean distance is at most r . Here $r = r(n)$ may and often does depend on n . To avoid having to deal with annoying trivial special cases we assume that $r \leq \sqrt{2n}$ throughout the paper. The study of random geometric graphs essentially goes back to Gilbert [7] who defined a very similar model in 1961. For this reason random geometric graphs are often also called the *Gilbert model* of random graphs. Random geometric graphs have been the subject of a considerable research effort in the last two decades. As a result, detailed information is now known on various aspects such as (k) -connectivity [22], [23], the largest component [24], the chromatic number and clique number [17], [20], the (non-)existence of Hamilton cycles [2], [21], monotone graph properties in general [8], and the simple random walk on the graph [4]. One of the most well-known phenomena in random geometric graphs is the ‘sudden emergence of a giant component’. By this we mean that there exists a critical value r_c such that if $\limsup r < r_c$ then, a.s., every component of $G(n, r)$ has $O(\log n)$ vertices, whereas if $\liminf r > r_c$ then, a.s., there exists a ‘giant’ component with $\Omega(n)$ vertices. Here and in the rest of the paper, we say that a sequence of events $(E_n)_n$ holds *asymptotically almost surely* (a.s.) if $\lim_{n \rightarrow \infty} \mathbb{P}(E_n) = 1$. The exact value of r_c is not known at this time, but simulations suggest that the exact value is approximately 1.2 (see [24]).

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For more details, and proofs, on the giant component phenomenon and background on random geometric graphs in general we refer the reader to [24].

In this paper we consider the *treewidth* of random geometric graphs. The treewidth of a graph was introduced by Halin [10] and independently, but later, by Robertson and Seymour [25]. It is a graph parameter that in a sense measures how similar a given graph is to a tree. (We postpone the precise—and technical—definition of treewidth until Section 2.) Treewidth plays an important role in modern algorithmic graph theory. Many NP-hard algorithmic decision problems have, for instance, been shown to be polynomially solvable when restricted to the class of instances with a bounded tree-width. In fact, a striking result of Courcelle [5] states that any algorithmic decision problem that can be expressed in monadic second-order logic can be solved in linear time for the class of graphs with bounded treewidth. An example of a decision problem that is NP-hard, in general, and can be expressed in monadic second-order is k -colorability (for any fixed k). As random geometric graphs have been used extensively as models for modeling communication networks, this motivated Mitsche and Perarnau [19] to consider the treewidth (tw) of random geometric graphs. They proved that if $r \in (0, r_c)$ is fixed then, a.a.s., $\text{tw}(G(n, r)) = \Theta(\log n / \log \log n)$, while if $r > C$, where C is a large constant, then, a.a.s., $\text{tw}(G(n, r)) = \Theta(r\sqrt{n})$. Mitsche and Perarnau [19] also conjectured that the second result should extend all the way to the critical value. Here we will prove their conjecture.

Theorem 1.1. *If $r = r(n)$ is such that $\liminf r > r_c$, where r_c is the critical value for the emergence of the giant component, then, a.a.s. as $n \rightarrow \infty$, $\text{tw}(G(n, r)) = \Theta(r\sqrt{n})$.*

Our proof of Theorem 1.1 makes use of a comparison to bond percolation on \mathbb{Z}^2 . Recall that this refers to the infinite random graph obtained by retaining each edge of the familiar integer lattice with probability p and discarding it with probability $1 - p$, independently of the choices for all other edges. We will denote by $\Gamma(k, p)$ the restriction of this process to the $k \times k$ integer grid. That is, $\Gamma(k, p)$ has vertex set $[k]^2$ and for every pair of points $u, v \in [k]^2$ with Euclidean distance equal to 1, we add an edge with probability p , independently of the choices for all other pairs. (Here and in the rest of the paper we use the notation $[k] := \{1, \dots, k\}$.) For the proof of Theorem 1.1 we only need to consider the treewidth of $\Gamma(k, p)$ when p is very close to 1, but with a little bit of extra work we are able to obtain the following result in addition to Theorem 1.1.

Theorem 1.2. *If $p \in (0, 1)$ is fixed then, a.a.s. as $k \rightarrow \infty$,*

$$\text{tw}(\Gamma(k, p)) = \begin{cases} \Theta(k) & \text{if } p > \frac{1}{2}, \\ \Theta(\sqrt{\log k}) & \text{if } p < \frac{1}{2}. \end{cases} \quad (1.1)$$

Note that k is the *square root* of the number of vertices of $\Gamma(k, p)$.

2. Notation and preliminaries

In this section we give some definitions and results which we will need in the sequel. We start with the precise definition of treewidth. For a graph $G = (V, E)$ on n vertices, we call (T, \mathcal{W}) a *tree decomposition* of G , where \mathcal{W} is a set of vertex subsets $W_1, \dots, W_s \subset V$, called *bags*, and T is a forest with vertices in \mathcal{W} , such that

- $\bigcup_{i=1}^s W_i = V$;
- for any $e = uv \in E$ there exists a set $W_i \in \mathcal{W}$ such that $u, v \in W_i$;
- for any $v \in V$, the subgraph induced by the $W_i \ni v$ is connected as a subgraph of T .

The *width* of a tree-decomposition is $w(T, \mathcal{W}) = \max_{1 \leq i \leq s} |W_i| - 1$, and the *treewidth* of a graph G can be defined as

$$\text{tw}(G) := \min_{(T, \mathcal{W})} w(T, \mathcal{W}),$$

where the minimum is taken over all tree decompositions (T, \mathcal{W}) of G . From the definition of treewidth, one can see that the treewidth of a tree is one, while the treewidth of a k -clique is $k - 1$. We also observe that if H is a subgraph of G , then $\text{tw}(H) \leq \text{tw}(G)$, and if G is a graph with connected components G_1, \dots, G_m , then $\text{tw}(G) = \max_{1 \leq i \leq m} \text{tw}(G_i)$.

Given an edge xy of graph G , the graph G/xy is obtained from G by *contracting* the edge xy . That is, to obtain G/xy , we identify the vertices x and y and remove all resulting loops and duplicate edges. A graph H is a *minor* of G if it is a subgraph of the graph obtained from G by a sequence of edge-contractions. Again, one can see from the definitions that if H is a *minor* of G , $\text{tw}(H) \leq \text{tw}(G)$.

Alon *et al.* [1] proved the following powerful result, bounding the treewidth of graphs without a given minor.

Theorem 2.1. (See [1].) *If G does not have H as a minor, then $\text{tw}(G) \leq |V(H)|^{3/2} \sqrt{|V(G)|}$.*

In this paper we will make use of the following immediate corollary.

Corollary 2.1. *There exists a constant $C > 0$ such that every planar graph G satisfies $\text{tw}(G) \leq C \sqrt{|V(G)|}$.*

Throughout the paper we will denote by $\Gamma(k) (:= \Gamma(k, 1))$ the $k \times k$ grid. The next observation appears as Exercise 16 in [6, Chapter 12].

Lemma 2.1. *We have $\text{tw}(\Gamma(k)) = k$.*

For one of our lower bounds on the treewidth, we will need the following lemma which links the treewidth of a graph and the existence of a partition of its vertex set with special properties. A vertex partition $V = \{A, S, B\}$ is a *balanced k -partition* if $|S| = k + 1$, there are no edges in G between a vertex in A and a vertex in B , and $\frac{1}{3}(n - k - 1) \leq |A|, |B| \leq \frac{2}{3}(n - k - 1)$. In this case, S is called a *balanced separator*. The following result connecting balanced partitions and treewidth is due to Kloks [14], which provides a necessary condition for a graph to have a treewidth of certain size.

Lemma 2.2. (See [14].) *Let G be a graph and suppose that $\text{tw}(G) \leq k \leq |V(G)| - 1$. Then G has a balanced k -partition.*

We say that $A \subseteq \{0, 1\}^n$ is an *up-set* if whenever we take a point of A and we change one of its coordinates into a one, then the resulting point is still in A . We will use the following lemma later on.

Lemma 2.3. (Harris' lemma, [11].) *Let $A, B \subseteq \{0, 1\}^n$ be up-sets and let $X = (X_1, \dots, X_n)$ be a vector of independent Bernoulli random variables. Then $\mathbb{P}(X \in A \cap B) \geq \mathbb{P}(X \in A) \mathbb{P}(X \in B)$.*

By a slight abuse of notation, throughout this paper we will denote the graph with vertex set \mathbb{Z}^2 and an edge $vw \in E(\mathbb{Z}^2)$ if and only if $\|v - w\| = 1$ also by \mathbb{Z}^2 . Recall that bond percolation on \mathbb{Z}^2 refers to the random process where we keep each edge of \mathbb{Z}^2 with probability p and discard it with probability $1 - p$, independently of all other edges. The edges that are kept are referred to as *open* and the discarded edges as *closed*. If $R := \{a, \dots, b\} \times \{c, \dots, d\}$ is an axis-parallel rectangle, then we say that R has a *horizontal crossing* if there is an open path

that stays inside R and connects the left side $\{a\} \times \{c, \dots, d\}$ to the right side $\{b\} \times \{c, \dots, d\}$. A vertical crossing is defined similarly. We denote by $H(R)$ the event that there is a horizontal crossing of R , and $V(R)$ the event that there is a vertical crossing of R . In the sequel, we will use the following well-known result on bond percolation on \mathbb{Z}^2 with $p > \frac{1}{2}$. A proof can, for instance, be found in [3, Lemma 8, p. 64].

Lemma 2.4. *If $p > \frac{1}{2}$ then $\lim_{k \rightarrow \infty} \mathbb{P}(H([3k] \times [k])) = 1$.*

In words, when $p > \frac{1}{2}$ then the probability of crossing a $3k \times k$ rectangle in the long direction can be made arbitrarily close to 1 by making k large.

Formally speaking, we can describe bond percolation on \mathbb{Z}^2 as a random vector X taking values in $\{0, 1\}^{E(\mathbb{Z}^2)}$. Here $X_e = 1$ if e is open, and $X_e = 0$ otherwise. In the standard setup, the coordinates X_e are i.i.d. Bernoulli random variables. One can also consider more general bond percolation models in which the coordinates are not independent. We say that such a bond percolation model Y is *1-independent* if, for every pair of sets $S, T \subseteq E(\mathbb{Z}^2)$ with the property that no edge in S shares an endpoint with any edge in T , the random vectors $(Y_e)_{e \in S}$ and $(Y_e)_{e \in T}$ are independent. Recall that a coupling of two random objects X, Y is a joint probability space on which both are defined (and have the correct marginal distributions). The following result is a reformulation of a special case of a result by Liggett *et al.* [15].

Theorem 2.2. (See [15].) *There exists a function $\pi : [0, 1] \rightarrow [0, 1]$ such that, $\lim_{p \uparrow 1} \pi(p) = 1$, and the following holds. Suppose that Y follows a 1-independent bond percolation model on \mathbb{Z}^2 and set $p := \inf_{e \in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$. Then there exists a coupling of Y with standard (i.e. independent) bond percolation X with $\mathbb{P}(X_e = 1) = \pi(p)$, such that, almost surely, $X_e \leq Y_e$ for all $e \in E(\mathbb{Z}^2)$.*

In words, this last theorem says that every 1-independent bond percolation model contains the edges of an independent bond percolation model, and the edge probability $\pi(p)$ of this independent bond percolation approaches 1 as $p := \inf_{e \in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$ approaches 1.

When working with random geometric graphs, it is often useful to consider a *Poissonized* version of the random geometric graph. We define $G_{\text{Po}}(n, r)$ analogously to $G(n, r)$ except that now we take the points of a Poisson process of intensity 1 on $[0, \sqrt{n}]^2$ and then build our graph on that as before. Equivalently, we can say that we throw $N_n \stackrel{\text{D}}{=} \text{Po}(n)$ i.i.d. uniform points onto $[0, \sqrt{n}]^2$ and then build the graph on those as before where, ‘ $\stackrel{\text{D}}{=}$ ’ denotes equality in distribution. Working with the Poissonized version is often useful in proofs because of the convenient independence properties of the Poisson process. Recall that if $N_n \stackrel{\text{D}}{=} \text{Po}(n)$ then $\mathbb{P}(N_n > (1 + \varepsilon)n) = o(1)$, as can for instance be seen by Chebyshev’s inequality. Using a straightforward coupling and rescaling, this gives the following observation.

Corollary 2.2. *There is a coupling such that for every $r = r(n)$, a.a.s., $G_{\text{Po}}((1 - \varepsilon)n, r\sqrt{1 - \varepsilon})$ is a subgraph of $G(n, r)$.*

It, of course, also makes sense to simply consider the random geometric graph built on a Poisson process \mathcal{P} of intensity 1 on all of the plane \mathbb{R}^2 . This is the well-known *continuum percolation* model defined originally by Gilbert [7]. We remark that Gilbert and several other sources in the literature fix $r = 1$ and allow the intensity of the Poisson process to vary, but it is easily seen that this is equivalent to the setting where we vary r and the intensity of the Poisson process is fixed to be 1. Note that $G_{\text{Po}}(n, r)$ is just the restriction of continuum percolation to the square $[0, \sqrt{n}]^2$. We also remark that the critical r_c for the ‘emergence of a giant component’ in $G(n, r)$ is the same as the critical value for the existence of an infinite

component in continuum percolation (see [24, Chapters 9 and 10]). Similarly to the case of bond percolation on \mathbb{Z}^2 , we can define crossing events for continuum percolation. Our definition follows Meester and Roy [18]. For $R = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ an axis-parallel rectangle, we say that $H(R)$ holds (i.e. there is a horizontal crossing of R) if it is possible to draw a continuous curve between the right and left side that stays inside R and is completely covered by the balls of radius $r/2$ centered on the points of \mathcal{P} . Note that this in particular implies that there is a path between a vertex that is within $r/2$ of the left side of R , and a vertex within $r/2$ of the right side of R such that all other vertices of the path are either inside R or within distance $r/2$ of R . We have the following analogue of Lemma 2.4.

Lemma 2.5. (See [18, Corollary 4.1].) *If $r > r_c$ then $\lim_{a \rightarrow \infty} \mathbb{P}(H([0, 3a] \times [0, a])) = 1$.*

We say that an event A defined with respect to the Poisson process \mathcal{P} is *increasing* if whenever A holds for some set of points $X = \{x_1, x_2, \dots\} \subseteq \mathbb{R}^2$ (i.e. some realization of \mathcal{P}), then A also holds for any set X' that contains X . We have the following analogue of Lemma 2.3 above.

Lemma 2.6. (See [18, Theorem 2.2].) *If A, B are increasing events (with respect to \mathcal{P}) then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$.*

3. The treewidth of the percolated grid $\Gamma(k, p)$

3.1. When p is large

Instead of proving the $p > \frac{1}{2}$ part of Theorem 1.2 directly, we first prove the following weaker version.

Proposition 3.1. *There exist constants $c > 0$ and $p < 1$ such that $\text{tw}(\Gamma(k, p)) \geq ck$ a.a.s.*

If $A \subseteq \mathbb{Z}^2$ is finite and connected (as a subgraph of \mathbb{Z}^2) then there is a well defined ‘surrounding cycle’ $\text{surr}(A)$ in the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ (that separates A from ∞ , and every other cycle in $(\mathbb{Z}^2)^*$ that separates A from ∞ contains $\text{surr}(A)$ in its interior). For $A \subseteq [k]^2$ connected, we define $\text{outer}(A)$ to be the set of edges of $\Gamma(k)$ that cross $\text{surr}(A)$. (See Figure 1 for a depiction.)

We will make use of the following straightforward observation. We include a proof for completeness.

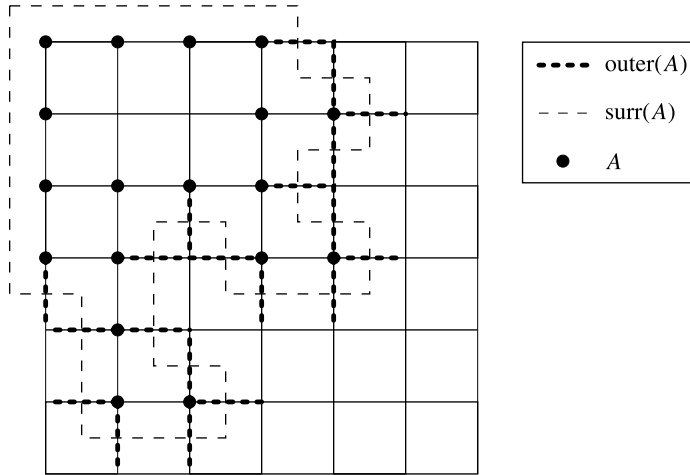
Lemma 3.1. *Suppose that $A \subseteq [k]^2$ is connected (as a subgraph of $\Gamma(k)$) and does not contain a horizontal crossing of $[k]^2$. Then $|\text{outer}(A)| \geq \max(\sqrt{|A|}, |\text{surr}(A)|/4)$.*

Proof. First suppose that A contains a vertical crossing. Since A does not contain a horizontal crossing, $\text{surr}(A)$ must contain a (dual) path that separates the left edge of $[k]^2$ from its right edge. This implies that $\text{outer}(A)$ contains at least k edges. Hence, $|\text{outer}(A)| \geq \sqrt{|A|}$. Note that, since A can intersect at most one of the vertical sides of $[k]^2$, we have that the number of edges of $\text{surr}(A)$ that do not intersect edges of $\Gamma(k)$ is at most $|\text{surr}(A)| - |\text{outer}(A)| \leq 3k$. This shows $|\text{outer}(A)| \geq |\text{surr}(A)|/4$.

Let us then assume that A contains neither a horizontal nor a vertical crossing. Let $a := |\pi_x(A)|$, $b := |\pi_y(A)|$, where π_x , respectively π_y , denote the projection onto the x -axis, respectively the y -axis. Clearly, we have $|A| \leq ab$ and $|\text{outer}(A)| \geq a + b$. Thus,

$$\sqrt{|A|} \leq \max(a, b) \leq a + b \leq |\text{outer}(A)|.$$

Also note that $|\text{surr}(A)| - |\text{outer}(A)| \leq a + b$. So certainly $|\text{outer}(A)| \geq |\text{surr}(A)|/4$. □

FIGURE 1: Depiction of $\text{surr}(A)$ and $\text{outer}(A)$ for a set $A \subseteq [7]^2$.

We say a set $A \subseteq [k]^2$ is *dirty* (with respect to $\Gamma(k, p)$) if

- A is connected (as a subgraph of $\Gamma(k)$),
- A intersects at most three sides of $[k]^2$, and
- at most half of the edges of $\text{outer}(A)$ are open in $\Gamma(k, p)$.

We say that a vertex $v \in [k]^2$ is dirty if it is contained in some dirty set.

Lemma 3.2. *There exists a $p_0 < 1$ such that whenever $p \geq p_0$ then, a.a.s., there are at most $k^2/10^{10}$ dirty vertices.*

Proof. Let Y denote the number of vertices contained in some dirty set A with $|\text{surr}(A)| \geq k^{0.01}$ and let Z denote the number of vertices contained in some dirty set A with $|\text{surr}(A)| < k^{0.01}$. It is easy to see that the number of cycles in $(\mathbb{Z}^2)^*$ that have length ℓ and that surround a given vertex $v \in \mathbb{Z}^2$ is at most $\ell 4^\ell$. This allows us to bound the expectation of Y as

$$\begin{aligned} \mathbb{E}(Y) &\leq k^2 \sum_{\ell \geq k^{0.01}} 4^\ell \binom{\ell}{\ell/8} (1-p)^{\ell/8} \\ &\leq k^2 \sum_{\ell \geq k^{0.01}} 8^\ell (1-p)^{\ell/8} \\ &= \frac{k^2 (8(1-p)^{1/8})^{k^{0.01}}}{1 - 8(1-p)^{1/8}} \\ &= o(1), \end{aligned}$$

where we have used the fact that $|\text{outer}(A)| \geq |\text{surr}(A)|/8$ in the first line, that $\binom{\ell}{\ell/8} \leq 2^\ell$ in the second line, and where the last equality holds provided p_0 was chosen sufficiently close to 1 (and $p \geq p_0$). In particular, for p_0 sufficiently close to 1 and $p > p_0$, we have $Y = 0$ a.a.s.

Next, we consider Z . For $v \in [k]^2$, we denote by E_v the event that v is contained in a dirty A with $|\text{surr}(A)| < k^{0.01}$. We have

$$\begin{aligned}\mathbb{P}(E_v) &\leq \sum_{\ell \leq k^{0.01}} \ell 4^\ell \binom{\ell}{\ell/8} (1-p)^{\ell/8} \\ &\leq \sum_{\ell \leq k^{0.01}} \ell (8(1-p)^{1/8})^\ell \\ &\leq \frac{8(1-p)^{1/8}}{(1-8(1-p)^{1/8})^2} \\ &\leq 10^{-11},\end{aligned}$$

where the last inequality holds provided p_0 is chosen sufficiently close to 1 (and $p \geq p_0$).

On the other hand, we clearly have

$$\mathbb{P}(E_v) \geq (1-p)^4.$$

Hence, we have $k^2(1-p)^4 \leq \mathbb{E}Z = \sum_v \mathbb{P}(E_v) \leq k^2/10^{11}$. In particular, $\mathbb{E}Z = \Theta(k^2)$.

Next we consider the second moment of Z . Observe that if $|u-v|_\infty \geq 3k^{0.01}$, then E_u and E_v are independent. (Here $|(x, y)|_\infty = \max(|x|, |y|)$ denotes the familiar L_∞ -norm.) This allows us to write

$$\begin{aligned}\mathbb{E}Z^2 &= \sum_{u,v} \mathbb{P}(E_u \cap E_v) \\ &= \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty < 3k^{0.01}} \mathbb{P}(E_u | E_v) + \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty \geq 3k^{0.01}} \mathbb{P}(E_u | E_v) \\ &\leq \sum_v \mathbb{P}(E_v) 36k^{0.02} + \sum_{u,v} \mathbb{P}(E_v) \mathbb{P}(E_u) \\ &= \mathbb{E}(Z) \cdot o(k^2) + (\mathbb{E}Z)^2 \\ &= (1+o(1))(\mathbb{E}Z)^2.\end{aligned}$$

This shows that $\text{var}(Z) = o((\mathbb{E}Z)^2)$. An application of Chebyshev's inequality shows that

$$\mathbb{P}\left(Z > \frac{k^2}{10^{10}}\right) \leq \mathbb{P}\left(|Z - \mathbb{E}Z| \geq \frac{9}{10} \mathbb{E}Z\right) \leq \left(\frac{10}{9}\right)^2 \frac{\text{var}(Z)}{(\mathbb{E}Z)^2} = o(1).$$

In conclusion, we have seen that, when p_0 is sufficiently close to 1 and $p_0 < p \leq 1$, then $Y = 0$ a.s. and $Z \leq k^2/10^{10}$ a.s., which obviously implies the lemma. \square

Proof of Proposition 3.1. Let us pick $1 > p > p_0$ with p_0 as provided by Lemma 3.2. Then, a.s., $\Gamma(k, p)$ has no more than $k^2/10^{10}$ dirty vertices. In the remainder of the proof we therefore assume we are given a subgraph $G \subseteq \Gamma(k)$ for which there are at most $k^2/10^{10}$ dirty vertices, but which is otherwise arbitrary. We will show that any such G satisfies $\text{tw}(G) \geq k/1000$.

Aiming for a contradiction, we assume that there exists some balanced partition $\{A, S, B\}$ of $V(G) = [k]^2$ with $|S| < k/1000$.

We first observe that we can assume, without loss of generality, that A does not contain a horizontal crossing. For, if it does then B cannot contain a vertical crossing (otherwise A, B

would not be disjoint). Hence, by applying symmetry (switching the roles of A , B and rotating by 90 degrees) we can indeed assume A does not contain a horizontal crossing. Observe that

$$|A| \geq k^2 - |B| - |S| \geq k^2 - \frac{2}{3}k^2 - \frac{1}{1000}k \geq \frac{1}{10}k^2.$$

Let A_1, \dots, A_m denote the connected components of A (connected when considered as subgraphs of G). We set

$$\mathcal{I} := \{i : A_i \text{ is not dirty}\}, \quad A' := \bigcup_{i \in \mathcal{I}} A_i.$$

Since the total number of dirty vertices is less than $k^2/10^{10}$, we have

$$|A'| \geq |A| - \frac{k^2}{10^{10}} \geq \frac{k^2}{100}.$$

Note that every edge (in G) between a vertex of A' and a vertex of $[k^2] \setminus A'$ must in fact connect a vertex of A' to a vertex of S . Hence, it follows that

$$4|S| \geq \frac{1}{2} \sum_{i \in \mathcal{I}} |\text{outer}(A_i)| \geq \frac{1}{2} \sum_{i \in \mathcal{I}} \sqrt{|A_i|} \geq \frac{1}{2} \sqrt{|A'|} \geq \frac{1}{20}k.$$

(Here we have used Lemma 3.1 for the third expression and the concavity of the square root function for the fourth expression.) So it follows that $k/1000 \geq |S| \geq k/80$, a contradiction.

This shows that there is no balanced partition with $|S| < k/1000$, which implies that $\text{tw}(G) \geq k/1000$ by Klops' lemma (Lemma 2.2). \square

3.2. When $p > \frac{1}{2}$

We are now ready to prove the first part of Theorem 1.2 with the help of Proposition 3.1.

Proof of Theorem 1.2. (The $p > \frac{1}{2}$ case.) Our proof is an application of a standard technique for comparing supercritical percolation to percolation with p close to 1, by means of Lemma 2.4 and Theorem 2.2. See, for instance, [3, pp. 74–75].

Let p_0 be as provided by Proposition 3.1, and let π be as provided by Theorem 2.2. We now pick p_1 such that $\pi(p_1) > p_0$. By Lemma 2.4, we can find an $a \in \mathbb{N}$ such that $\mathbb{P}(H([3a] \times [a])) > \sqrt[3]{p_1}$.

For R a $3a \times a$ rectangle, we define the event $E(R) := H(R) \cap V(R_L) \cap V(R_R)$, where R_L denotes the leftmost $a \times a$ subrectangle, and R_R denotes the rightmost $a \times a$ rectangle (see Figure 13 of [3, p. 74] for a depiction). If R is a $a \times 3a$ rectangle then we define $E(R) := V(R) \cap H(R_B) \cap H(R_T)$ with R_B , respectively R_T , the bottom, respectively top, $a \times a$ subrectangle of A . Note that, by choice of a and Harris' lemma, we have $\mathbb{P}(E(R)) > p_1$ for every $3a \times a$ or $a \times 3a$ rectangle R .

We now define a (dependent) bond percolation model Y on \mathbb{Z}^2 as follows. We declare the horizontal edge between (i, j) and $(i + 1, j)$ open in Y if $E(\{2ai + 1, \dots, 2ai + 3a\} \times \{2aj + 1, \dots, 2aj + a\})$ holds; similarly, the edge between (i, j) and $(i, j + 1)$ is open in Y if $E(\{2ai + 1, \dots, 2ai + a\} \times \{2aj + 1, \dots, 2aj + 3a\})$ holds. To clarify the construction, let us mention that one could think of the square $\{2ai + 1, \dots, 2ai + a\} \times \{2aj + 1, \dots, 2aj + a\}$ as representing the point (i, j) and in the $3a \times a$ rectangle R that represents the edge between (i, j) and $(i + 1, j)$, we have that R_L represents (i, j) and R_R represents $(i + 1, j)$. It is not difficult to see that Y is in fact 1-independent. Hence, by Theorem 2.2, $Y \geq X$, where X is standard (independent) percolation on \mathbb{Z}^2 with edge-probability $> p_0$.

We can view $\Gamma(k, p)$ as the restriction of the (independent, edge-probability p) percolation process to the $k \times k$ grid $[k]^2$. We let Γ_X , respectively Γ_Y , denote the subgraph that X , respectively Y , defines on $[\ell]^2$, where $\ell := \lfloor k/2a \rfloor - 1$. As the reader can easily check, we have chosen ℓ so that each of the rectangles corresponding to the edges of Γ_Y is contained in $[k]^2$. Observe that by construction (and Proposition 3.1) we have a.s.,

$$\text{tw}(\Gamma_Y) \geq \text{tw}(\Gamma_X) = \Omega(\ell) = \Omega(k).$$

Next, we remark that Γ_Y is in fact a minor of $\Gamma(k, p)$ (under the natural coupling associated with the construction of Y). To see this, we can proceed as follows. If $E(R)$ holds with R a $3a \times a$ rectangle that corresponds to some edge of Γ_Y , then we perform a sequence of contractions that will identify all vertices of R_L that participate in (horizontal or vertical) crossings of R_L into a single vertex x , we produce a vertex y via contractions in R_R similarly, and then we contract the remaining edges of a long, horizontal crossing of R into a single edge that connects x and y . If we carry this out for each rectangle corresponding to an edge of Γ_Y and discard any unneeded vertices (making sure to keep exactly one vertex in each $a \times a$ square that corresponds to a vertex $(i, j) \in [\ell]^2$ that was not incident to any edge of Y), then we obtain a graph isomorphic to Γ_Y .

Since Γ_Y is a minor of $\Gamma(k, p)$, we have $\text{tw}(\Gamma(k, p)) \geq \text{tw}(\Gamma_Y) = \Omega(k)$, a.s., as required, completing the proof. \square

3.3. When $p < \frac{1}{2}$

In this section we prove the upper and lower bound of the treewidth $\Gamma(k, p)$ for $p < \frac{1}{2}$. We need the following result from percolation theory, that is originally due to Kesten [12], [13].

Theorem 3.1. (See [12], [13].) *Consider bond percolation on \mathbb{Z}^2 and let C_0 denote the number of vertices in the cluster (component) of the origin. For each $p < \frac{1}{2}$ there exists $\lambda(p) > 0$ such that*

$$\mathbb{P}(|C_0| \geq n) \leq e^{-n\lambda(p)} \quad \text{for all } n \geq 0.$$

This has the following easy consequence.

Corollary 3.1. *If $0 < p < \frac{1}{2}$ then, a.s., all components of $\Gamma(k, p)$ have $O(\log k)$ vertices.*

Proof. Let us fix $0 < p < \frac{1}{2}$ and let $\lambda(p)$ be as provided by Theorem 3.1. Let $K := 100/\lambda(p)$. Observe that, for every $v \in [k]^2$ and $\ell \in \mathbb{N}$, the probability that it is in a component of order $\geq \ell$ in $\Gamma(k, p)$ is no more than the probability that $|C_0|$ exceeds ℓ . Thus, we can conclude that

$$\begin{aligned} \mathbb{P}(\Gamma(k, p) \text{ has a component of size } \geq K \log k) &\leq k^2 \exp[-100 \log k] \\ &= \exp[-98 \log k] \\ &= o(1). \end{aligned} \quad \square$$

Since the treewidth of a graph is equal to the maximum of the treewidth of its components, and all components of $\Gamma(k, p)$ are planar, the required upper bound for $\text{tw}(\Gamma(k, p))$ in the case when $p < \frac{1}{2}$ follows immediately using Corollary 2.1.

Corollary 3.2. *If $0 < p < \frac{1}{2}$ then, a.s., $\text{tw}(\Gamma(k, p)) = O(\sqrt{\log k})$.*

The following lemma now completes the proof of Theorem 1.2.

Lemma 3.3. *Fix $0 < p < \frac{1}{2}$ then, a.s., $\text{tw}(\Gamma(k, p)) = \Omega(\sqrt{\log k})$.*

Proof. We fix a $\varepsilon = \varepsilon(p)$ (small, to be determined later), and we set $\ell := \lceil \sqrt{\varepsilon \log k} \rceil$. We now fix $N := \lfloor k/(\ell + 1) \rfloor^2 = \Omega(k^2/\log k)$ (vertex-)disjoint $\ell \times \ell$ -subgrids G_1, \dots, G_N in $[k]^2$. We will say that the subgrid G_i is *intact* if all of its edges are present in $\Gamma(k, p)$. By independence of the events that the G_i -s are intact, we have

$$\begin{aligned} \mathbb{P}(\text{at least one } G_i \text{ is intact}) &= 1 - (1 - p^{2\ell(\ell-1)})^N \\ &\geq 1 - \exp[-Np^{2\ell(\ell-1)}] \\ &\geq 1 - \exp[-Np^{\ell^2}] \\ &\geq 1 - \exp[-Np^{\varepsilon \log k}]. \end{aligned}$$

Next, note that

$$Np^{\varepsilon \log k} = \Omega\left(\frac{k^2}{\log k} \exp[\varepsilon \log p \log k]\right) = \Omega(\exp[2 \log k - \log \log k + \varepsilon \log p \log k]).$$

Hence, provided we choose $\varepsilon < -2/\log p$, we have $Np^{\varepsilon \log k} \rightarrow \infty$ and, hence, also

$$\mathbb{P}(\text{at least one } G_i \text{ is intact}) = 1 - o(1).$$

Hence, by Lemma 2.1, and since $\text{tw}(H) \leq \text{tw}(G)$ if $H \subseteq G$, it follows that $\text{tw}(\Gamma(k, p)) \geq \ell = \Omega(\sqrt{\log k})$ a.a.s. \square

Corollary 3.2 and Lemma 3.3 together give the $p < \frac{1}{2}$ part of Theorem 1.2.

4. Proof of Theorem 1.1

Since Mitsche and Perarnau [19] have already shown the result holds when $r = r(n)$ is larger than some fixed constant C , we only need to consider the case when $r_c < \liminf r \leq \limsup r \leq C$. Note that in this case $\Theta(r\sqrt{n})$ simplifies to $\Theta(\sqrt{n})$. Moreover, by monotonicity, we see that for any such sequence r , a.a.s., $\text{tw}(G(n, r)) \leq \text{tw}(G(n, C)) = O(\sqrt{n})$ by Mitsche and Perarnau's result. Hence, we only need to prove an a.a.s. lower bound for the treewidth of order $\Omega(\sqrt{n})$. Using Corollary 2.2 and monotonicity, Theorem 1.1 follows if we can establish the following lemma.

Lemma 4.1. *For every fixed $r > r_c$, we have $\text{tw}(G_{p_0}(n, r)) = \Omega(\sqrt{n})$ a.a.s.*

Proof. The proof is almost exactly the same as the proof of the $p > \frac{1}{2}$ case of Theorem 1.2 above. Again, we let p_0 be as provided by Proposition 3.1, we let π be as provided by Theorem 2.2, and we pick p_1 such that $\pi(p_1) > p_0$. Using Lemma 2.5, we find an a such that $\mathbb{P}(H([0, 3a] \times [0, a])) > \sqrt[3]{p_1}$. For R a $3a \times a$ or $a \times 3a$ rectangle we define $E(R)$ as in the proof of the $p > \frac{1}{2}$ case of Theorem 1.2 above. By choice of a and Lemma 2.6 we have $\mathbb{P}(E(R)) > p_1$ for any such rectangle.

We again define a 1-independent bond percolation model Y on \mathbb{Z}^2 , by declaring the horizontal edge between (i, j) and $(i + 1, j)$ open in Y if $E([2ai, 2ai + 3a] \times [2aj, 2aj + a])$ holds; and the edge between (i, j) and $(i, j + 1)$ is open in Y if $E([2ai, 2ai + a] \times [2aj, 2aj + 3a])$ holds. (Note that 1-independence holds provided we chose a sufficiently large.) Again from Theorem 2.2 it follows that $Y \geq X$, where X is standard (independent) percolation on \mathbb{Z}^2 with edge probability $> p_0$.

We set $k := \lfloor \sqrt{n}/2a \rfloor - 1$, and we let Γ_X , respectively Γ_Y , be the restriction of X , respectively Y , to $[k]^2$. Arguing analogously to the way we did in the proof of the $p > \frac{1}{2}$

case of Theorem 1.2, we see that Γ_Y is in fact a minor of $G_{P_0}(n, r)$ (under the natural coupling we get from the construction of Y). Hence, using Proposition 3.1, we have, a.s.,

$$\text{tw}(G_{P_0}(n, r)) \geq \text{tw}(\Gamma_Y) \geq \text{tw}(\Gamma_X) = \Omega(k).$$

Since $k = \Theta(\sqrt{n})$, this concludes the proof. \square

5. Discussion and further work

Together with the work of Mitsche and Perarnau [19], our Theorem 1.1 provides an almost complete picture of the behavior of the treewidth of random geometric graphs, up to the order of the leading constants.

Corollary 5.1. *Asymptotically almost surely,*

$$\text{tw}(G(n, r)) = \begin{cases} \Theta\left(\frac{\log n}{\log \log n}\right) & \text{if } 0 < \liminf r \leq \limsup r < r_c, \\ \Theta(r\sqrt{n}) & \text{if } \liminf r > r_c. \end{cases}$$

Interestingly, by a result of McDiarmid [16], the clique number of random geometric graphs is a.s. equal to $(1 + o(1)) \log n / \log \log n$ when r is constant. This gives rise to the following natural questions.

Question 5.1. Suppose that $0 < \liminf r \leq \limsup r < r_c$.

- Is $\text{tw}(G(n, r)) = (1 + o(1)) \log n / \log \log n$ a.s.?
- Is $\text{tw}(G(n, r)) = \omega(G(n, r))$ a.s.?

Of course we would also be very interested to learn the precise leading constants for the supercritical case. With our methods and those of Mitsche and Perarnau [19], the following natural conjecture still seems out of reach.

Conjecture 5.1. *Suppose that $r > r_c$ is fixed. Then there exists a $c = c(r)$ such that $\text{tw}(G(n, r)) = (c + o(1))\sqrt{n}$ a.s.*

Another tantalizing question is what happens precisely at the critical point. Based on widely believed conjectures on the ‘critical exponents’ for two-dimensional percolation (see [9, Chapters 9 and 10]), we offer the following conjectures.

Conjecture 5.2. *Asymptotically almost surely, $\text{tw}(G(n, r_c)) = n^{91/192+o(1)}$.*

Conjecture 5.3. *Asymptotically almost surely, $\text{tw}(\Gamma(k, \frac{1}{2})) = k^{91/96+o(1)}$.*

We have made two separate conjectures and added some slack in the exponent so that there is a bit more hope that at least one of the conjectures will be solved.

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References

- [1] ALON, N., SEYMOUR, P. AND THOMAS, R. (1990). A separator theorem for nonplanar graphs. *J. Amer. Math. Soc.* **3**, 801–808.
- [2] BALOGH, J. *et al.* (2011). Hamilton cycles in random geometric graphs. *Ann. Appl. Prob.* **21**, 1053–1072.
- [3] BOLLOBÁS, B. AND RIORDAN, O. (2006). *Percolation*. Cambridge University Press.
- [4] COOPER, C. AND FRIEZE, A. (2011). The cover time of random geometric graphs. *Random Structures Algorithms* **38**, 324–349.
- [5] COURCELLE, B. (1990). The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Inf. Comput.* **85**, 12–75.
- [6] DIESTEL, R. (2010). *Graph Theory* (Grad. Texts Math. **173**), 4th edn. Springer, Heidelberg.
- [7] GILBERT, E. N. (1961). Random plane networks. *J. Soc. Indust. Appl. Math.* **9**, 533–543.
- [8] GOEL, A., RAI, S. AND KRISHNAMACHARI, B. (2005). Monotone properties of random geometric graphs have sharp thresholds. *Ann. Appl. Prob.* **15**, 2535–2552.
- [9] GRIMMETT, G. (1999). *Percolation* (Fundamental Principles Math. Sci. **321**), 2nd edn. Springer, Berlin.
- [10] HALIN, R. (1976). *S*-functions for graphs. *J. Geom.* **8**, 171–186.
- [11] HARRIS, T. E. (1960). A lower bound for the critical probability in a certain percolation process. *Proc. Camb. Phil. Soc.* **56**, 13–20.
- [12] KESTEN, H. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Commun. Math. Phys.* **74**, 41–59.
- [13] KESTEN, H. (1981). Analyticity properties and power law estimates of functions in percolation theory. *J. Statist. Phys.* **25**, 717–756.
- [14] KLOKS, T. (1994). *Treewidth: Computations and Approximations* (Lecture Notes Comput. Sci. **842**). Springer, Berlin.
- [15] LIGGETT, T. M., SCHONMANN, R. H. AND STACEY, A. M. (1997). Domination by product measures. *Ann. Prob.* **25**, 71–95.
- [16] MCDIARMID, C. J. H. (2003). Random channel assignment in the plane. *Random Structures Algorithms* **22**, 187–212.
- [17] MCDIARMID, C. AND MÜLLER, T. (2011). On the chromatic number of random geometric graphs. *Combinatorica* **31**, 423–488.
- [18] MEESTER, R. AND ROY, R. (1996). *Continuum Percolation* (Camb. Tracts Math. **119**). Cambridge University Press.
- [19] MITSCHKE, D. AND PERARNAU, G. (2012). On the treewidth and related parameters of random geometric graphs. In *29th Internat. Symp. on Theoretical Aspects of Computer Science (LIPIcs. Leibniz Internat. Proc. Inform.* **14**), pp. 408–419.
- [20] MÜLLER, T. (2008). Two-point concentration in random geometric graphs. *Combinatorica* **28**, 529–545.
- [21] MÜLLER, T., PÉREZ-GIMÉNEZ, X. AND WORMALD, N. (2011). Disjoint Hamilton cycles in the random geometric graph. *J. Graph Theory* **68**, 299–322.
- [22] PENROSE, M. D. (1997). The longest edge of the random minimal spanning tree. *Ann. Appl. Prob.* **7**, 340–361.
- [23] PENROSE, M. D. (1999). On *k*-connectivity for a geometric random graph. *Random Structures Algorithms* **15**, 145–164.
- [24] PENROSE, M. D. (2003). *Random Geometric Graphs* (Oxford Stud. Prob. **5**). Oxford University Press.
- [25] ROBERTSON, N. AND SEYMOUR, P. (1986). Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms* **7**, 309–322.