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Random N-continued fraction expansions

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Abstract

The N-continued fraction expansion is a generalization of the regular continued fraction expansion, where the digits 1 in the numerators are replaced by the natural number N. Each real number has uncountably many expansions of this form. In this article we focus on the case $N = 2$, and we consider a random algorithm that generates all such expansions. This is done by viewing the random system as a dynamical system, and then using tools from ergodic theory to analyse these expansions. In particular, we use a recent Theorem of Inoue (2012) to prove the existence of an invariant measure of product type whose marginal in the second coordinate is absolutely continuous with respect to Lebesgue measure. Also some dynamical properties of the system are shown and the asymptotic behaviour of such expansions is investigated. Furthermore, we show that the theory can be extended to the random 3-continued fraction expansion.

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1. Introduction

In 2008 Burger et al. introduced in [4] the N-continued fraction expansion. Given an $x \in \mathbb{R}$ and an $N \geq 1$ they showed that x can be represented in the following way:

$$x = d_0 + \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}}, \tag{1}$$

where the digits $d_i \in \mathbb{N}$. Anselm and Weintraub showed in [2] that every $x \in \mathbb{R}$ has in fact infinitely many such expansions. Dajani et al. obtained in [7] the N-continued fraction expansions from transformations of the form

$$S_{N,i}(x) = \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor + i & \text{if } x \in \left(0, \dfrac{N}{i+1}\right] \\ \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor & \text{if } x \in \left(\dfrac{N}{i+1}, N\right] \\ 0 & \text{if } x = 0 \end{cases}, \tag{2}$$

where $N \in \mathbb{N}$ and $i \in \{0, 1, \ldots, N-1\}$. Approaching the N-continued fraction expansions as a dynamical system Dajani et al. showed that the result obtained in [2] is immediate. They also gave invariant measures for several transformations generating N-expansions.

In this paper we will consider N-continued fraction expansions generated by a random dynamical system. A random dynamical system consists of a family of transformations on a state space and a probability distribution on the family of transformations. For each iterate a transformation of the family is chosen according to the probability distribution. In this paper we use the family of transformations $\{S_{N,i}, i \in \{0, 1, \ldots, N-1\}\}$ where $S_{N,i}$ are given by (2). The main question is whether we can find an invariant measure for this random system. The existence of invariant measures for random systems has been studied frequently over the past decades. We will use a recent theorem of Inoue, [9] to ensure the existence of an invariant measure.

Defining the random dynamical system as a skew product allows one to use results from ergodic theory in order to gain information about the asymptotic behaviour of the expansions. This is done in [10] for expansions like (1), where $N \in \{-1, 1\}$. In [8] more invariant measures for random β-expansions are obtained by constructing an isomorphism between the skew product for the random β-expansion and the digit sequences it induces. In this paper we will prove the existence of an invariant measure for the random transformation generating 2-continued fraction expansions, so expansions of the form (1) where $N = 2$. We will use the approach of [10] to show that an accelerated version of our system has an invariant measure of the form $m \times \mu$, where m is a Bernoulli measure and μ is equivalent with the Lebesgue measure. Using standard techniques, we lift the obtained invariant measure for the accelerated system to an invariant measure ρ for the original random system. We will write the random dynamical system as a skew product to obtain asymptotic properties of expansions like (1) using ergodic theoretic methods. Constructing an isomorphism between the skew product and the digit sequences obtained from the random dynamical system, allows us to show the existence of invariant measures which are singular with respect to the measure ρ mentioned above.

The paper is organized as follows. In Section 2 we define the random N-continued fraction transformation. In Section 3, we state the existence theorem of invariant measures for random transformations given by Inoue in [9], and show how we can apply this theorem to an induced transformation of the random 2-continued fraction transformation. In Section 4 we will define

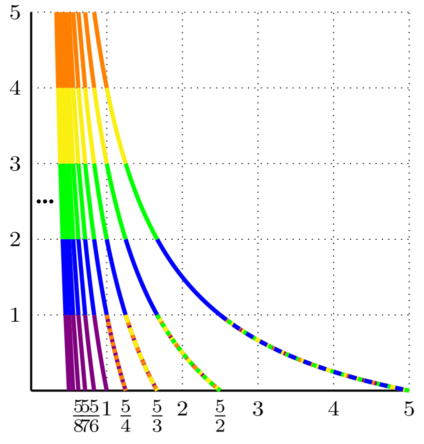

Fig. 1. The random 5-continued fraction transformation, violet, blue, green, yellow and orange illustrate the maps S_0, S_1, S_2, S_3, S_4 respectively. At each iterate one map is chosen according to some probability distribution.

the random 2-continued fraction transformation as a skew product. Subsequently, we introduce a skew product for the induced system and show that there exists an invariant product measure for this skew product which can be lifted to an invariant measure for random 2-continued fraction transformation. Section 5 shows the ergodicity of the random 2-continued fraction transformation, and in Section 6 some asymptotic properties are derived as well as the proof that the random system has finite entropy. In Section 7 we construct an isomorphism with a left shift to obtain more invariant measures. Finally in Section 8 we show how the theory developed in this paper can be generalized to the random 3-continued fraction transformation.

2. Random N-continued fraction transformation

Definition 2.1. Let $N \in \mathbb{N}$, we define for $0 \le i \le N-1$ transformations $S_i : [0, N] \to [0, N]$ by:

$$S_i(x) = \begin{cases} \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor + i & \text{if } x \in \left(0, \dfrac{N}{i+1}\right], \\ \dfrac{N}{x} - \left\lfloor \dfrac{N}{x} \right\rfloor & \text{if } x \in \left(\dfrac{N}{i+1}, N\right], \\ 0 & \text{if } x = 0. \end{cases} \qquad \diamond$$

We depicted the case $N = 5$ in Fig. 1. Note that in the case $N = 1$ we obtain the regular continued fraction transformation. The transformation $S_0(x)$ is called the greedy transformation and the transformation S_{N-1} the lazy transformation. To each transformation S_i we associate digits $d_{n,i}(x)$ which are defined by

$$d_{1,i}(x) = \begin{cases} k - i & \text{if } x \in \left(\dfrac{N}{k+1}, \dfrac{N}{k}\right], \; k \ge i+1 \\ k & \text{if } x \in \left(\dfrac{N}{k+1}, \dfrac{N}{k}\right], \; k \le i \\ \infty & \text{if } x = 0, \end{cases}$$

$$d_{n,i}(x) = d_{1,i}(S_i^{n-1}x).$$

Using these digits we can write

$$S_i(x) = \begin{cases} \frac{N}{x} - d_{1,i}(x) & \text{if } x \in (0, N] \\ 0 & \text{if } x = 0. \end{cases}$$

As in the case of regular continued fractions, we can use the transformations S_i to obtain an expansion for $x \in [0, N]$. For ease of notation, when i is fixed we simply write $d_n = d_{n,i}$. Then

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots + \cfrac{N}{d_n + S_i^n(x)}}}}.$$

In [2] it was noted that the N-continued fraction expansions can be related to the regular continued fraction expansions in the following way:

$$x = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cfrac{1}{d_4 + \ddots + \cfrac{1}{d_n}}}}} = \cfrac{N}{Nd_1 + \cfrac{N}{d_2 + \cfrac{N}{Nd_3 + \cfrac{N}{d_4 + \ddots \cfrac{N}{kd_n}}}}},$$

where $k = 1$ if n is even and $k = N$ if n is odd. Using this equality, it was shown in [2] that every x has infinitely many expansions of the form

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots + \cfrac{N}{d_n + \ddots}}}}. \tag{3}$$

More precisely if we truncate the expansion at level n the truncations will converge to x as $n \to \infty$. Dajani et al. showed in [7], that if we endow $[0, N]$ with the Borel-σ-algebra each transformation S_i has an invariant measure.

We extend these N-continued fraction transformations to a random transformation. Let $\{0, 1, 2, \dots, N-1\}$ be the parameter space, and let $(p_0, p_1, \dots, p_{N-1})$ be a probability vector on the parameter space, so with probability p_i we choose the transformation S_i.

Definition 2.2. Let $(W, \mathcal{B}, \nu)$ be a σ-finite parameter space and $(X, \mathcal{A}, \mu)$ a state space. Let X be the interval $[0, 1] \subset \mathbb{R}$, $\mathcal{A}$ the Borel-σ-algebra and μ the Lebesgue measure. For each $t \in W$ let $S_t : [0, 1] \to [0, 1]$ be a $\mathcal{B}$ measurable and non-singular map, i.e. $\mu(S^{-1}(A)) = 0$ if $\mu(A) = 0$. Let $p(t, x) : W \times [0, 1] \to [0, \infty)$ be a probability density function for each $x \in [0, 1]$ so $\int_W p(t, x) d\nu = 1$. Then we define the random transformation $S = \{S_t, p(t, x)\}$ as the Markov process with transition probabilities $\mathbb{P}(x, A) = \int_W 1_A(S_t(x))\nu(dt)$, where 1_A is the indicator function. $\diamond$

A piecewise monotone random transformation is defined as follows.

Definition 2.3. Let $S = \{S_t, p(t, x)\}$ be a random transformation on $(W, \mathcal{B}, \nu) \times ([0, 1], \mathcal{A}, \mu)$. Let Λ be a countable set of indices and for each $t \in W$, $\Lambda_t \subset \Lambda$. For each $t \in W$, $\{I_{i,t}\}_{i \in \Lambda_t}$ is a collection of subintervals in $[0, 1]$ such that $\mu([0, 1]\setminus\bigcup_{i\in\Lambda_t}) = 0$ and $I_j \cap I_i = \emptyset$ for all $j \neq i$, $i, j \in \Lambda_t$. For notational reasons we define $I_{i,t} = \emptyset$ if $i \in \Lambda\setminus\Lambda_t$ and define $\emptyset$ to be closed and let $\mathrm{int}(I_{i,t})$ denote the interior of $I_{i,t}$. We assume two conditions for the random map $\{S_t, p(t, x), \{I_{t,x}\}, t \in W\}$:

(1) The restriction of S_t to $\mathrm{int}(I_{i,t})$ is C^1 and monotone for each $i \in \Lambda$ and $t \in W$.
(2) Let $S_{t,i}$ be the restriction of S_t to $\mathrm{int}(I_{t,i})$ for each $t \in W$ and $i \in \Lambda$. Put

$$\phi_{t,i}(x) = \begin{cases} S_{t,i}^{-1}(x) & \text{if } x \in S_{t,i}(\mathrm{int}(I_{t,i})) \\ 0 & \text{if } x \in [0,1]\setminus S_{t,i}(\mathrm{int}(I_{t,i})) \end{cases}$$

for each $t \in W$ and $i \in \Lambda$. Note that $\phi_{t,i}(x) = 0$ if $i \in \Lambda\setminus\Lambda_t$. We assume that for each $x \in [0, 1]$ and $i \in \Lambda$, $w_{x,i}(t) := \phi_{t,i}(x)$ is a measurable function of t.

If $\{S_t, p(t, x), \{I_{t,x}\}, t \in W\}$ satisfies the conditions (1) and (2) then we call the system $W = \{S_t, p(t, x), \{I_{t,x}\}, t \in W\}$ a piecewise monotonic random transformation. ◇

3. Invariant measure in the case $N = 2$

We will use the recent theorem of Inoue [9] to show that there exists an invariant measure for the case $N = 2$. An invariant measure for a random transformation is defined as follows.

Definition 3.1. Let $S = \{S_t, p(t, x), t \in W\}$ be a random system of transformations on $(X, \mathcal{A}, \mu) \times (W, \mathcal{B}, \nu)$ and define

$$\mathbf{P}_*\mu(A) := \int_X \int_W p(t, x)\mathbf{1}_A(S_t(x))d\nu(t)d\mu(x)$$

If $\mathbf{P}_*\mu = \mu$ then we call μ an invariant measure for the random system S. ◇

Let $P_S : L^1 \to L^1$ be the density of $\mathbf{P}_*(\mu)$ with respect to μ. Then P_S is the random Perron Frobenius operator and an invariant density for the transformation S corresponds to a fixed point of P_S. For random piecewise monotonic transformations we can define the Perron Frobenius operator explicitly by:

$$P_S f(x) = \int_W \sum_{i\in\Lambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i(x)})|\phi'_{t,i}(x)|\mathbf{1}_{S_t(\mathrm{int} I_{i,t})}(x)d\nu. \tag{4}$$

In [9] the existence of invariant measure is shown for piecewise monotonic random transformations satisfying certain conditions (see Theorem 3.2), by showing the existence of a fixed point of the random Perron Frobenius operator.

Let $\bigvee_{[a,b]} f$ denote the variation of $f : [a, b] \to \mathbb{R}$ on $[a, b]$, i.e.

$$\bigvee_{[a,b]} = \sup_{a=x_0,x_1,\ldots,x_n=b} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| .$$

The theorem for the existence of invariant measure as stated by Inoue in [9] is the following.

Theorem 3.2. *Let $S = \{S_t, p(t,x), \{I_{i,t}\}_{i\in\Lambda} : t \in W\}$ be a random transformation as defined in Definition 2.3. For $t \in W$ and $x \in [0,1]$, put*

$$g(t,x) = \begin{cases} \dfrac{p(t,x)}{|S_t'(x)|}, & \text{if } x \in \bigcup_i int(I_{t,i}) \\ 0, & \text{if } x \in [0,1] \backslash \bigcup_i int(I_{t,i}). \end{cases} \tag{5}$$

Assume the following conditions hold:

1. $\sup_{x\in[0,1]} \int_W g(t,x) d\nu(t) < \alpha < 1$, *i.e. on average the functions S_i are expanding.*
2. *There exists a constant M such that $\bigvee_{[0,1]} g(t,\cdot) < M$ for almost all $t \in W$, that is, there exists a ν-measurable set $W_0 \subset W$ such that $\int_{W_0} p(t,x)\nu(dt) = 1$ and $\bigvee_{[0,1]} g(t,\cdot) < M$ for all $t \in W_0$.*

Then S has an invariant probability measure μ_p which is absolutely continuous with respect to the Lebesgue measure. Moreover μ_p admits a probability density function h_p which is of bounded variation and satisfies for all $A \in \mathcal{A}$:

$$\mu_p(A) = \int_X \int_W p(t,x) \mathbf{1}_A(S_t(x)) h_p(x) d\nu d\lambda. \tag{6}$$

Let us see whether Theorem 3.2 provides the existence of an invariant measure for the random continued fraction transformation as defined in Section 2. We immediately encounter two problems. In the first place the random N-continued fraction transformation goes from $[0,N] \to [0,N]$ instead of $[0,1] \to [0,1]$. In the second place the N-continued fraction transformation does not satisfy condition 1 of Theorem 3.2. For example for points $x \in (\sqrt{N}, N]$ we have

$$\int g(t,x)\nu(dt) = \int p(t,x)\frac{x^2}{N}\nu(dt) > 1.$$

Therefore we need to adjust the random N-continued fraction transformation.

3.1. An invariant measure for the accelerated 2-random continued fraction transformation

We go back to the case $N = 2$. The random 2-continued fraction transformation is given by the pair of maps $S_0, S_1 : [0,2] \to [0,2]$ defined by

$$S_0 = \begin{cases} \dfrac{2}{x} - \left\lfloor \dfrac{2}{x} \right\rfloor & \text{if } x \in (0,2] \\ 0 & \text{if } x = 0 \end{cases}$$
$$S_1 = \begin{cases} \dfrac{2}{x} - \left(\left\lfloor \dfrac{2}{x} \right\rfloor - 1 \right) & \text{if } x \in (0,1] \\ \dfrac{2}{x} - \left\lfloor \dfrac{2}{x} \right\rfloor & \text{if } x \in (1,2] \\ 0 & \text{if } x = 0. \end{cases} \tag{7}$$

We shall refer to S_0 as the lower transformation and to S_1 as the upper transformation. Let $p \in (0,1)$ and let transformation S_0 occur with probability p and S_1 with probability $1-p$, we then have the random transformation $S = \{S_i, p_i, i \in \{0,1\}\}$ where $p_0 = p = 1 - p_1$. We start with reducing this transformation to a transformation from $[0,1] \to [0,1]$. Note that

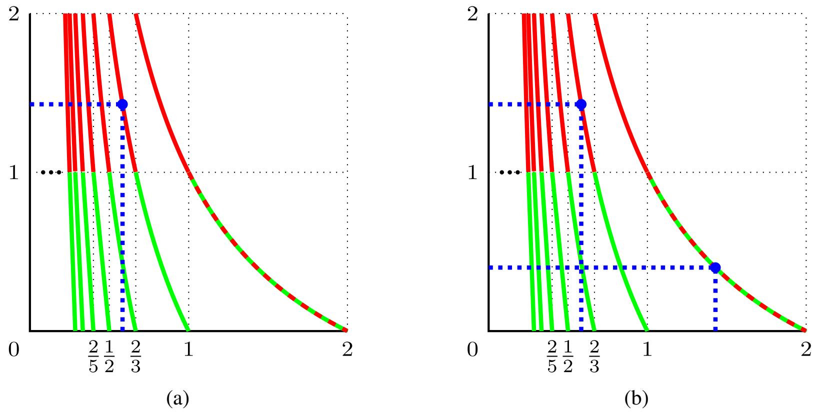

Fig. 2. (a) x after one iteration by S, (b) x after two times applying S.

for $x \in [0, 1]$ we have $S_0(x) \in [0, 1]$, hence S_0 restricted to $[0, 1]$ is already a transformation from $[0, 1]$ to $[0, 1]$. On the other hand if $x \in [0, 1]$, then $S_1(x) \in (1, 2]$. However, on $(1, 2]$ the transformation S_0 and S_1 coincide, and for $x \in (1, 2]$, $S_0(x) = S_1(x) \in [0, 1]$. Therefore, if we start with the transformation S_1, then we can apply the transformation S_1 once more and we will always end up in $[0, 1]$. So modifying the random 2-continued fraction transformation to a transformation from $[0, 1]$ to $[0, 1]$ comes down to accelerating the transformation when we choose the upper transformation, see Fig. 2. Therefore we define the accelerated random 2-continued fraction transformation by the pair $T_0, T_1 : [0, 1] \to [0, 1]$ given by

$$\begin{aligned} T_0(x) =& S_0(x) = \begin{cases} \dfrac{2}{x} - \left\lfloor \dfrac{2}{x} \right\rfloor & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases} \\ T_1(x) =& S_1 \circ S_1(x) = S_0 \circ S_1(x) = \begin{cases} \dfrac{2}{\frac{2}{x} - (\lfloor \frac{2}{x} \rfloor - 1)} - 1 & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases}. \end{aligned} \tag{8}$$

Let $p \in (0, 1)$, then we use transformation T_0 with probability p, so $p_0 = p$ and we use T_1 with probability $1 - p$, so $p_1 = 1 - p$. Endow $[0, 1]$ with the Borel-σ-algebra. The existence of an invariant measure for the random transformation $T = \{T_i, p_i, i \in \{0, 1\}\}$ which is absolutely continuous with the Lebesgue measure follows now by Theorem 3.2.

Proposition 3.3. *The random transformation $T = \{T_0, T_1; p, 1 - p\}$ satisfies the conditions of Theorem 3.2 and therefore has an invariant measure μ_p which is absolutely continuous with respect to the Lebesgue measure.*

Proof. We set $\{I_{0,k}\} = \{I_{1,k}\} = \left\{\left(\frac{2}{k+1}, \frac{2}{k}\right], k \in \mathbb{N}\right\}$. The derivatives of T_0, T_1 are given by $T_0'(x) = \frac{-2}{x^2}$ for $x \in (0, 1]$, and $T_1'(x) = \frac{4}{(2-(k-1)x)^2}$ for $x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]$. Therefore, the restriction of T_0 to $(\frac{2}{k+1}, \frac{2}{k}]$ is a continuous monotone decreasing function and the restriction of T_1 to $(\frac{2}{k+1}, \frac{2}{k}]$ is a continuous monotone increasing function, so condition (1) is satisfied. For condition (2) note:

$$g(0,x) = \frac{p}{2}x^2,$$
$$g(1,x) = \frac{1-p}{4}(2-x(k-1))^2 \qquad \text{for } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right].$$

Suppose that $x = 0$, then $g(0,0)+g(1,0) = 0 < 1$. If $x \in (0,1]$ we have $g(0,x) < \frac{p}{2}$. Therefore it is enough to show that $g(1,x) \le (1-\frac{p}{2})$ for $x \in (0,1]$. For $x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]$ we find:

$$\frac{1-p}{4}\left(2-\frac{2}{k}(k-1)\right)^2 < g(1,x) < \frac{1-p}{4}\left(2-\frac{2}{k+1}(k-1)\right)^2.$$

Since $k \ge 2$ it follows that:

$$\sup_{x\in[0,1]} (g(0,x)+g(1,x)) < 1$$

and condition (1) is satisfied.

Finally, we show that the functions $g(i,x) : [0,1] \to \mathbb{R},\ i \in \{0,1\}$ are of bounded variation. Note $\bigvee_{[0,1]} g(0,x) = \frac{p}{2}$. Since $g(1,x)$ is a monotone continuous function on $(\frac{2}{k+1}, \frac{2}{k}]$ for each $k \ge 2,\ k \in \mathbb{N}$ we find,

$$\bigvee_{[0,1]} g(1,x) = \frac{1-p}{4}\sum_{k\ge 2}\frac{16}{(k+1)^2} - \frac{4}{k^2} < \infty.$$

Therefore, all conditions of Theorem 3.2 are satisfied. We conclude that there exists a probability measure μ_p on $[0,1]$ which is absolutely continuous with respect to the Lebesgue measure λ, and has a density function h_p that is of bounded variation. Moreover, μ_p has the property that

$$\mu_p(A) = p\mu_p(T_0^{-1}A) + (1-p)\mu_p(T_1^{-1}A)$$

for each Borel measurable set $A \subset [0,1]$. □

In the next Section 4 we will show how we can use the invariant measure of the induced map to obtain an invariant measure for the original transformation.

4. Dynamical properties of the random 2-continued fraction transformation

We define the 2-continued fraction transformation as a skew product, in this way we de-randomize the transformation, and therefore obtain the opportunity to apply theorems and techniques from ergodic theory.

Let $S_0, S_1 : [0,2] \to [0,2]$ be as in Eq. (7), $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the left shift and set $\Omega = \{0,1\}^{\mathbb{N}}$. We define the transformation $R : \Omega \times [0,2] \to \Omega \times [0,2]$ by

$$R(\omega,x) = \begin{cases} (\omega, S_1(x)) & x \in (1,2] \\ (\sigma(\omega), S_{\omega_1}(x)) & x \in (0,1] \\ (\sigma(\omega), 0) & x = 0. \end{cases}$$

Note that for $x \in (1,2]$, $S_1(x) = S_0(x)$. Define the digits of R by

$$b_1(\omega,x) = \begin{cases} 1 & x \in (1,2] \\ k & x \in \left(\frac{2}{k+1}, \frac{2}{k}\right], \quad \omega_1 = 0 \\ k-1 & x \in \left(\frac{2}{k+1}, \frac{2}{k}\right], \quad \omega_1 = 1 \\ \infty & x = 0. \end{cases} \tag{9}$$

Set $b_n(\omega, x) = b_1(R^{n-1}(\omega, x))$. Let π_2 denote the projection on the second coordinate, then we can write $\pi_2(R(\omega, x)) = \frac{2}{x} - b_1(\omega, x)$. Denoting $b_i = b_i(\omega, x)$, we can write

$$x = \frac{2}{b_1 + \pi_2(R(\omega, x))} = \cdots = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{2}{\ddots + \cfrac{2}{b_n + \pi_2(R^n(\omega, x))}}}}.$$

Let $\frac{p_n}{q_n} = \frac{p_n}{q_n}(\omega, x)$ denote the partial fractions obtained by applying the transformation R n-times, so

$$\frac{p_n}{q_n} = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{2}{\ddots + \cfrac{2}{b_n}}}}. \tag{10}$$

Using the Moebius transformation in a similar way as for the regular continued fraction expansions (see [6], chapter 1), we obtain the following recurrence relations:

$$\begin{aligned} &p_{-1} = 1 \quad p_0 = 0 \quad p_n = 2p_{n-2} + b_n p_{n-1}, \\ &q_{-1} = 0 \quad q_0 = 1 \quad q_n = 2q_{n-2} + b_n q_{n-1}. \end{aligned} \tag{11}$$

Like for the regular continued fraction we can express x with the help of these recurrence relations,

$$x = \frac{p_n + p_{n-1}(\pi_2(R^n(\omega, x)))}{q_n + q_{n-1}(\pi_2(R^n(\omega, x)))}.$$

We obtain

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{2^{n+1}}{q_n^2}. \tag{12}$$

From the recurrence relations the following proposition holds by induction:

Proposition 4.1. *Define q_n and p_n as above, then $q_n \ge 2^{n-1}$ and $p_n \ge 2^{n-1}$ $\forall n \in \mathbb{N}$.*

Hence using Eq. (12) the following proposition holds.

Proposition 4.2. *If $x \in [0, 1]$ then $\lim_{n\to\infty} \left| x - \frac{p_n}{q_n} \right| = 0$. Therefore we can expand x like*

$$x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{2}{\ddots + \cfrac{2}{b_n + \ddots}}}}.$$

To obtain an invariant measure for R we link this transformation to the accelerated transformation.

4.1. The accelerated random 2-continued fraction transformation

Recall the accelerated random 2-continued fraction transformation is the random transformation $\{T_0, T_1, p_0, p_1\}$ where $p_0 = p = 1 - p_1$, $T_0, T_1 : [0, 1] \to [0, 1]$ are as given in (8). We define the skew product $K : \Omega \times [0, 1] \to \Omega \times [0, 1]$ for the accelerated random 2-continued fraction transformation by

$$K(\omega, x) = \begin{cases} \left(\sigma(\omega), T_{\omega_1} x\right) & \text{if } x \in (0, 1] \\ (\sigma(\omega), 0) & \text{if } x = 0. \end{cases}$$

Notice that

$$K(\omega, x) = \begin{cases} R(\omega, x) & \text{if } \omega_1 = 0, x \in [0, 1] \\ R^2(\omega, x) & \text{if } \omega_1 = 1, x \in [0, 1]. \end{cases}$$

Let $\tau : \Omega \times [0, 1] \to \mathbb{N}$ denote the first return time defined by

$$\tau(\omega, x) = \inf\left\{n \geq 1 : R^n(\omega, x) \in \Omega \times [0, 1]\right\} = \begin{cases} 1 & \text{if } \omega_1 = 0 \\ 2 & \text{if } \omega_1 = 1. \end{cases}$$

Then

$$K(\omega, x) = R^{\tau(\omega,x)}(\omega, x),$$

and we see that K is indeed the induced transformation of R. Notice that for $x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]$, $k \in \mathbb{N},\ k \geq 2$ we can write $T_0(x) = \frac{2}{x} - k$, and $T_1(x) = \frac{2}{\frac{2}{x} - (k-1)} - 1$. Therefore, given (ω, x) we have

$$x = \begin{cases} \dfrac{2}{k + T_0(x)} & \text{if } \omega_1 = 0 \text{ and } x \in \left(\dfrac{2}{k+1}, \dfrac{2}{k}\right] \\ \dfrac{2}{(k-1) + \frac{2}{1+T_1(x)}} & \text{if } \omega_1 = 1 \text{ and } x \in \left(\dfrac{2}{k+1}, \dfrac{2}{k}\right]. \end{cases} \tag{13}$$

We define digits a_i by

$$a_1(\omega, x) = \begin{cases} k & \text{if } x \in \left(\dfrac{2}{k+1}, \dfrac{2}{k}\right] \text{ and } \omega_1 = 0 \\ (k-1, 1) & \text{if } x \in \left(\dfrac{2}{k+1}, \dfrac{2}{k}\right] \text{ and } \omega_1 = 1 \\ \infty & x = 0 \end{cases}$$
$$a_i = a_1\left(K^{i-1}(\omega, x)\right).$$

From Eq. (13) we see that K induces the same expansions as R does. When $\omega_1 = 0$, R and K generate the same expression, while when $\omega_1 = 1$ the expression generated by K is the same expression as the one obtained when using R twice. To relate the expansions obtained by K and R we introduce the variable $\tilde{n}$,

$$\tilde{n} : \mathbb{N} \times \Omega \times [0, 1] \to \mathbb{N}$$
$$\tilde{n}(n, \omega, x) = \sum_{i=1}^{n} \mathbf{1}_{\{\omega_i = 0\}}(\omega, x) + 2 \cdot \mathbf{1}_{\{\omega_i = 1\}}(\omega, x) = \sum_{i=0}^{n-1} \tau(K^i(\omega, x)). \tag{14}$$

Applying K n times gives the same initial block in the expansion as applying R $\tilde{n}$ times. Therefore the partial fractions obtained by applying K n times equals $\frac{p_{\tilde{n}}}{q_{\tilde{n}}}$, and hence form a subsequence of the partial fractions $\frac{p_n}{q_n}$ for R.

A nice property of the transformation K is that with a given block of digits $a_1, \ldots, a_n$, where $a_i \in \{k, (k-1, 1) : k \geq 2\}$ associated with a point x, corresponds a unique block $\omega_1 \cdots \omega_n \in \{0, 1\}^n$ such that for any $\omega \in \Omega$ starting with this block, one has $d_1(\omega, x) = a_1, \ldots, d_n(\omega, x) = a_n$. To be more precise, set

$$\omega_i = \begin{cases} 0 & \text{if } a_i = k \text{ for some } k \geq 2 \\ 1 & \text{if } a_i = (k-1, 1) \text{ for some } k \geq 2, \end{cases}$$

and denote the cylinder set corresponding to the sequence $\omega_1 \cdots \omega_n$, by $[\omega]_n$. Define

$$\Delta(a_1, \ldots, a_n) = \{x \in [0, 1] : \forall \omega \in [\omega]_n, d_1(\omega, x) = a_1, \ldots, d_n(\omega, x) = a_n\}.$$

Proposition 4.3. *The set $\Delta(a_1, \ldots, a_n)$ is an interval of length $\frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}} + q_{\tilde{n}-1})}$, where $\tilde{n} = \sum_{i=1}^{n} 1_{\{\omega_i = 0\}} + 2 \cdot 1_{\{\omega_i = 1\}} = \sum_{i=0}^{n-1} \tau(K^i(\omega, x))$ for any $\omega \in [\omega]_n$.*

Proof. The proof is similar to the proof for regular continued fractions, see [6]. □

4.2. Invariant measures for R and K

Endow $\Omega \times [0, 1]$ with the σ-algebra $\sigma(\mathcal{C} \times \mathcal{B}[0, 1])$, where $\mathcal{C}$ is the σ-algebra generated by the cylinders on $\{0, 1\}^{\mathbb{N}}$ and $\mathcal{B}[0, 1]$ the Borel σ-algebra restricted to $[0, 1]$. Let m_p be the product measure on $\mathcal{C}$ and μ_p the invariant measure obtained in Section 3.1 for the accelerated random 2-continued fraction transformation. We have the following proposition.

Proposition 4.4. *The product measure $m_p \times \mu_p$ is an invariant measure for the map K.*

Proof. Let $A \in \sigma(\mathcal{C} \times \mathcal{B}([0, 1]))$, such that $A = B \times [a, b]$, where

$$B = \{\omega \in \Omega : \omega_1 = i_1, \ldots, \omega_n = i_n,\ i_1, \ldots, i_n \in \{0, 1\}\}$$

is a cylinder set. Hence A is in the set of generators of $\mathcal{C} \times \mathcal{B}[0, 1]$. By Proposition 3.3,

$$\begin{aligned} m_p \times \mu_p(K^{-1}A) &= p m_p(B) \mu_p \left(T_0^{-1}([a, b])\right) + (1-p) m_p(B) \mu_p \left(T_1^{-1}([a, b])\right) \\ &= m_p \times \mu_p(A). \end{aligned}$$

We conclude $m_p \times \mu_p$ is indeed an invariant measure for the map K. □

Since K is an induced transformation of R we can use standard techniques, see e.g. [12] to obtain a finite invariant measure ρ for R defined by,

$$\begin{aligned} \rho(E) &= \frac{1}{\int_{\Omega \times [0,1]} \tau \, dm_p \times \mu_p} \sum_{n=0}^{\infty} \\ &\quad m_p \times \mu_p(\{(\omega, x) \in \Omega \times [0, 1]; \tau(\omega, x) > n\} \cap R^{-n}(E)) \\ &= \frac{1}{2-p} [m_p \times \mu_p(\Omega \times [0, 1] \cap E) + m_p \times \mu_p([1] \times [0, 1] \cap R^{-1}(E))]. \end{aligned}$$

In the following, we use a similar technique from [10] to show that μ_p is in fact equivalent with the Lebesgue measure.

Proposition 4.5. *Let $I \subset [0, 1]$ be a non-trivial interval. Then $\forall \omega \in \Omega$, there is an $n \geq 1, n \in \mathbb{N}$ such that $(0, 1) \subset \left(T_{\omega_n} \circ \cdots \circ T_{\omega_1}\right) I \subset [0, 1]$.*

Proof. Let $J \subset [0, 1]$ be a non-trivial open interval, so $J = (c, d)$, $c, d \in (0, 1)$, $c < d$. First assume $\exists k \in \mathbb{N}$ such that $\frac{2}{k} \in J$. Notice that $T_0(\frac{2}{k}) = 0$, so $T_0(J) = (b, 1) \cup [0, c)$ for some $b, c \in (0, 1)$. Therefore $\exists k \in \mathbb{N}$ such that $(\frac{2}{k+1}, \frac{2}{k}] \subset [0, c)$ and hence $(0, 1) \subset T_{\omega_2}\left(T_{\omega_1}(J)\right) \subset [0, 1]$. If $\omega_1 = 1$, then $T_1(J) = (b, 1] \cup (0, c)$ for some $b, c \in (0, 1)$, since $T_1(\frac{2}{k}) = 1$. By the same reasoning as before $(0, 1) \subset T_{\omega_2}(T_{\omega_1}(J)) \subset [0, 1]$.

Suppose for all $k \in \mathbb{N}$ we have $\frac{2}{k} \notin J$. Set $J_1 = T_{\omega_1}(c, d) = (c_1, d_1)$ and $J_i = T_{\omega_i}(c_{i-1}, d_{i-1})$ then it follows by induction that $\lambda(J_i) \geq (\frac{1}{1-(d-c)})^i (d - c)$ as long as $\frac{2}{k} \notin J_j$ for all $1 \leq j \leq i$ and $k \in \mathbb{N}$. Hence the size of the interval grows exponentially and therefore for all J there exists an $n \in \mathbb{N}$ such that $\frac{2}{k} \in J_n$ and $(0, 1) \subset T_{\omega_{(n+2)}} \circ T_{\omega_{(n+1)}} J_n$. □

Recall that if f is a function of bounded variation on I, then it can be redefined on a countable set to become a lower semi-continuous function. Moreover if f is lower semicontinuous on $I = [a, b] \subset \mathbb{R}$, then it is bounded from below and assumes its minimum value, see [3]. Using this two statements we can prove the following proposition.

Proposition 4.6. *Let h_p be the probability density function from Theorem 3.2. Then $h_p > 0$ for all $y \in (0, 1)$.*

Proof. Since h_p is a density function of bounded variation we can find an interval I such that $h_p > \alpha$ on I. Using the fact that $h_p = P_T h_p$, see Section 3, linearity of the Perron Frobenius operator and $P_T^n = P_{T^n}$, see [9] we can write:

$$\begin{aligned} h_p(y) &= P_{T^n} h_p(y) \\ &> \alpha P_{T^n} \mathbf{1}_I(y) \\ &= \alpha \sum_{(\omega_1,\ldots,\omega_n)\in\Omega} \ \sum_{x\in\left(T_{\omega_1}\circ\cdots\circ T_{\omega_n}\right)^{-1}\{y\}} 1_I(x) \left| \frac{p_{\omega_1}\cdots p_{\omega_n}}{\left(T_{\omega_1}\circ\cdots\circ T_{\omega_n}\right)'(x)} \right|. \end{aligned}$$

By Proposition 4.5 we know there exists a n such that $T_{\omega_n} \circ \cdots \circ T_{\omega_n}(I) = (0, 1)$ and therefore there exists an $x \in I$ such that $T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x) = y$. We conclude $h_p(y) > 0$ for all $y \in (0, 1)$. □

Using that h_p is of bounded variation the following standard proof shows that h_p is equivalent with the Lebesgue measure.

Proposition 4.7. *The density function h_p is bounded from above and away from* 0.

Proof. Since $[0, 1]$ is a closed and bounded subset in $\mathbb{R}$ and h_p is of bounded variation, h_p is bounded from above. We can redefine h_p on a countable set to get a lower semi-continuous function. A lower semi-continuous functions attains its minimum on [0,1]. By Proposition 4.6 we see that $h_p > 0$ on $(0, 1)$. Therefore we are left to show that $h_p(1) > 0$ and $h_p(0) > 0$. Let $\epsilon > 0$ and consider $T_0^{-1}(1 - \epsilon, 1)$. Note that for $k \geq 2$, $k \in \mathbb{N}$

$$(\frac{2}{1+k}, \frac{2}{1-\epsilon+k}) \subset T_0^{-1}(1-\epsilon, 1)$$

and

$$\lambda\left((\frac{2}{1+k},\frac{2}{1-\epsilon+k})\right)=\frac{2\epsilon}{(1+k)(1-\epsilon+k)}.$$

Hence

$$\frac{k^2}{2}\lambda\left((\frac{2}{1+k},\frac{2}{1-\epsilon+k})\right)<\lambda((1-\epsilon,1))<\frac{(k+1)^2}{2}\lambda\left((\frac{2}{1+k},\frac{2}{1-\epsilon+k})\right).$$

Therefore,

$$\begin{aligned}\lim_{x\uparrow 1}h_p(x)&=\lim_{\epsilon\to 0}\frac{1}{\lambda((1-\epsilon,1))}\int_{1-\epsilon}^{1}h_p(x)dx\\&=\lim_{\epsilon\to 0}\frac{p\mu_p(T_0^{-1}(1-\epsilon,1))+(1-p)\mu_p(T_1^{-1}(1-\epsilon,1))}{\lambda((1-\epsilon,1))}\\&\geq\lim_{\epsilon\to 0}\frac{p\mu_p((\frac{2}{1+k},\frac{2}{1-\epsilon+k}))}{\frac{(k+1)^2}{2}\lambda\left((\frac{2}{1+k},\frac{2}{1-\epsilon+k})\right)}\\&=\frac{2ph_p(\frac{2}{k+1})}{(k+1)^2}>0.\end{aligned}$$

The case $h_p(0)>0$ follows in the same way, choosing $(0,\epsilon)$ as starting interval and taking $\lim_{x\downarrow 0}$. □

5. Ergodicity of *R* and *K*

To prove that the map K is ergodic we will use the following proposition of Aimino, see [1].

Proposition 5.1 (*See [1] Proposition 3.1*)**.** *There exist constants $C\geq 0$ and $\gamma<1$ such that for all functions f of bounded variation and all $g\in L^\infty(\lambda)$,*

$$\lim_{n\to\infty}\left|\int_{[0,1]}P_{T^n}f\cdot g\,d\mu_p-\int_{[0,1]}f\,d\mu_p\int_{[0,1]}g\,d\mu_p\right|\leq C\gamma^n\|f\|_{BV}\|g\|_\infty.$$

Here $\|f\|_{BV}=\|f\|_{L^1}+\bigvee_{[0,1]}f$ and $\|g\|_\infty=\sup_{x\in[0,1]}|g(x)|$.

Proposition 5.2. *The map K is mixing with respect to the measure $m_p\times\mu_p$.*

Proof. We define the cylinders as follows:

$$\begin{aligned}[\bar{\omega}]_n\times\Delta_n a&=[\bar{\omega_1},\ldots,\bar{\omega_n}]\times\Delta(a_1,\ldots,a_n)_{\bar{\omega_1},\ldots,\bar{\omega_n}}\\&=\{(\omega,x):\omega_1=\bar{\omega_1},\ldots,\omega_n=\bar{\omega_n},d_1(\omega,x)=a_1,\ldots,d_n(\omega,x)=a_n\}\\&=\{(\omega,x):\omega_1=\bar{\omega_1},\ldots,\omega_n=\bar{\omega_n},x\in\bigcap_{i=1}^{n}(T_{\omega_{i-1}}\circ\cdots\circ T_{\omega_1})^{-1}\\&\qquad\times(\frac{2}{k_i+1},\frac{2}{k_i}]\}.\end{aligned}\tag{15}$$

Here the k_i's in the last line are the k_i associated with a_i, i.e. if $a_i=k_i$ or $a_i=(k_i-1,1)$ we use in both cases the interval $\left(\frac{2}{k_i+1},\frac{2}{k_i}\right]$. These cylinders form a generating set for the σ-algebra $\sigma(\mathcal{C}\times\mathcal{B}[0,1])$. Therefore, to prove that K is mixing it is enough to show that for all cylinders

$$\lim_{l\to\infty}(m_p \times \mu_p)(K^{-l}([\omega]_n \times \Delta_n(a)) \cap [v]_m \times \Delta_m(b))$$
$$= m_p \times \mu_p([\omega]_n \times \Delta_n(a))m_p \times \mu_p([v]_m \times \Delta_m(b)).$$

This follows from Proposition 5.1 using the properties of the random and non-random Perron Frobenius operator which can be found in [3], chapter 4 and [9]. □

Proposition 5.3. *The measure ρ is ergodic with respect to the transformation R.*

Proof. Note $R^{-1}(\Omega \times [0,1]) \cup \Omega \times [0,1] = \Omega \times [0,2]$ so we have $\rho\left(\bigcup_{k\geq 0} R^{-k}(\Omega \times [0,1])\right) = 1$ and by standard result of ergodic theory see [3] chapter 3, we conclude that R is ergodic. □

6. Asymptotic properties

The fact that the measure ρ is ergodic gives us the possibility to prove some asymptotic properties of the transformation R. Let q_n denote the denominator of the partial fractions as defined by (10). Then the following proposition holds:

Proposition 6.1. $\lim_{n\to\infty}\frac{1}{n}\log q_n(\omega, x)$ *exists ρ a.e.*

Proof. First we show by induction that if $\frac{p_n}{q_n} = \frac{p_n(\omega,x)}{q_n(\omega,x)}$ denote the partial fractions, then

$$p_n(\omega, x) = 2q_{n-1}(R(\omega, x))$$

for all $n \in \mathbb{N}$. By the recursion relations, see Eq. (11), $p_1(\omega, x) = 2$ and $q_0(R(\omega, x)) = 1$. Suppose the result holds true for all $n \leq N$ then

$$\begin{aligned} p_{N+1}(\omega, x) &= 2p_{N-1}(\omega, x) + b_{N+1}(\omega, x)p_N(\omega, x) \\ &= 4q_{N-2}(R(\omega, x)) + b_N(R(\omega, x)) \cdot 2 \cdot q_{N-1}(R(\omega, x)) \\ &= 2q_N(R(\omega, x)). \end{aligned}$$

Using this we can write

$$\frac{1}{q_n(\omega, x)} = \frac{p_n(\omega, x)}{q_n(\omega, x)}\frac{p_{n-1}(R(\omega, x))}{q_{n-1}(R(\omega, x))}\cdots\frac{p_1(R^{n-1}(\omega, x))}{q_1(R^{n-1}(\omega, x))}\cdot\left(\frac{1}{2}\right)^{n-1}.$$

Taking the logarithm and using the ergodicity of R with respect to the measure ρ, we continue the proof in the same way as the proof of Paul Levy's theorem for regular continued fractions. For example see [6] Chapter 3. □

Let C be the collection of cylinders defined in (15) and let I_C be the information function with respect to these cylinders, $I_{\mathcal{C}} : X \to \mathbb{R}$, $I_{\mathcal{C}} = \sum_{A\in\mathcal{C}} \mathbf{1}_A(\omega, x)\log(m_p \times \mu_p(A))$, with respect to the cylinder $\mathcal{C}$. We write

$$I_{\mathcal{C}} = \log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n)(\omega, x))),$$

where $[\omega]_n \times \Delta_n(a_n)(\omega, x)$ denotes the cylinder set to which (ω, x) belongs. Then the following proposition holds.

Proposition 6.2. $\lim_{n\to\infty}\frac{1}{n}\log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n)(\omega, x)))$ *exists and is finite $m_p \times \mu_p$ a.e.*

Proof.

$$\lim_{n\to\infty}\frac{1}{n}\log(m_p\times\mu_p([\omega]_n\times\Delta_n(a_n)(\omega,x)))=\lim_{n\to\infty}\frac{1}{n}\log(m_p([\omega]_n(\omega)))+\lim_{n\to\infty}\frac{1}{n}\log(\mu_p(\Delta a_n)(x)).$$

Since the left shift is ergodic with respect to m_p we can use the Birkhoff ergodic theorem to calculate the first limit.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log(m_p([\omega]_n(\omega)))&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log(p^{\mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega))})+\log((1-p)^{\mathbf{1}_{\{\omega_1=1\}}(\sigma^i(\omega))})\\&=\int_\Omega \mathbf{1}_{\{\omega_1=0\}}(\omega)\log(p)+\mathbf{1}_{\{\omega_1=1\}}(\omega)\log((1-p))dm_p\\&=p\log p+(1-p)\log(1-p).\end{aligned}$$

For the second limit we use $c\lambda(\Delta a_n)<\mu_p(\Delta a_n)<C\lambda(\Delta a_n)$, and therefore by Proposition 4.3

$$\lim_{n\to\infty}\frac{1}{n}\log(\mu_p(\Delta a_n)(x))=\lim_{n\to\infty}\frac{1}{n}\log(\lambda(\Delta a_n))=\lim_{n\to\infty}\frac{1}{n}\log(\frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}}+q_{\tilde{n}-1})}).$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{2^{\tilde{n}}}{2q_{\tilde{n}}^2}\right)<\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}}+q_{\tilde{n}-1})}\right)<\lim_{n\to\infty}\frac{1}{n}\log\left(\frac{2^{\tilde{n}}}{q_{\tilde{n}}^2}\right)$$

Note that by the Birkhoff ergodic theorem we get

$$\lim_{n\to\infty}\frac{1}{n}\log(2^{\tilde{n}})=\lim_{n\to\infty}\frac{\tilde{n}}{n}\log(2)=\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}\tau(K^i(\omega,x))}{n}\log(2)=(2-p)\log 2.$$

We conclude that

$$\lim_{n\to\infty}\frac{1}{n}\log(\mu_p(\Delta a_n)(x))=(2-p)\log 2-\lim_{n\to\infty}\frac{1}{n+1}2\log(q_{\tilde{n}})<\infty,$$

where in the last step we use Proposition 6.1, and the fact that $\lim_{n\to\infty}\frac{\tilde{n}}{n}=2-p$. Therefore, $\lim_{n\to\infty}\frac{1}{n}\log(m_p\times\mu_p([\omega_n]\times\Delta_n a_n))$ exists and is finite. □

In order to show that the map R has finite entropy, we will first show with the Shannon–McMillan–Breiman Theorem, see e.g. [12], that the map K has finite entropy.

Theorem 6.3. *For any $0\le p\le 1$ the transformation K has finite metric entropy under the measure $m_p\times\mu_p$.*

Proof. Let $\alpha=\{[\omega_i]_1\times\Delta_1(a_i),\omega_i\in\{0,1\},a_i\in\{k_i,(k_i-1,1):k_i\in\mathbb{N}\}\}$ be the collection of cylinders of length 1. First we show that cylinders of the form $[\omega]_1\times\Delta_1(a)$ form a generating partition. Denoting by k_n the first coordinate of a_i we can write

$$\begin{aligned}&[\omega_{i_0}]_1\times\Delta_1(a_{i_0})\cap K^{-1}([\omega_{i_1}]_1\times\Delta_1(a_{i_1}))\cap\cdots\cap K^{-(n-1)}([\omega_{i_n}]_1\times\Delta_1(a_{i_n}))\\&\quad=\{(\omega,x)\in\Omega\times[0,1]:\\&\qquad\omega_1=\omega_{i_0},x\in(\frac{2}{k_{i_0}},\frac{2}{k_{i_0}+1}],\omega_2=\omega_{i_1},T_{\omega_0}(x)\in(\frac{2}{k_{i_1}+1},\frac{2}{k_{i_1}}],\end{aligned}$$

$$\cdots \omega_n = \omega_{i_{n-1}}, T_{\omega_{i_{n-2}}} \circ \cdots \circ T_{\omega_{i_0}}(x) \in (\frac{2}{k_{i_n}+1}, \frac{2}{k_{i_n}}]\}$$
$$= [\omega]_n \times \Delta_n(a_n).$$

Since the cylinders of the form $[\omega]_n \times \Delta_n(a_n)$ generate $\sigma(\mathcal{C} \times [0, 1])$, α is a generating partition. To apply the Shannon–McMillan–Breiman theorem we have to check that $H(\alpha) < \infty$:

$$\begin{aligned} H(\alpha) &= - \sum_{[\omega]_1 \times \Delta_1(a), \omega \in \{0,1\}, a \in \mathbb{N}} m_p \times \mu_p([\omega]_1 \times \Delta_1(a)) \log(m_p \times \mu([\omega]_1 \times \Delta_1(a))) \\ &= - \sum_{[\omega]_1 \times \Delta_1(a), \omega \in \{0,1\}, a \in \mathbb{N}} m_p([\omega]_1)\mu_p(\Delta_1(a)) \log(m_p([\omega]_1)\mu_p(\Delta_1(a))) \\ &= -p \log p - (1-p)\log(1-p) - \sum_{a \in \mathbb{N}} \mu_p(\Delta a) \log(\mu_p(\Delta a)) \\ &< \infty. \end{aligned}$$

Were we used in the inequality that by equivalence of the Lebesgue measure we have

$$c\lambda(\Delta a) \log(c\lambda(\Delta a)) < \mu_p(\Delta a) \log \mu_p(\Delta a) < C\lambda(\Delta a) \log(C\lambda(\Delta a))$$

and

$$\sum_{k=1}^{n} C\lambda(\Delta a) \log(C\lambda(\Delta a)) = 2C \sum_{k=1}^{n} \frac{1}{k(k+1)} (\log(2C) - \log(k) - \log(k+1)) < \infty.$$

We conclude that α has finite entropy and therefore by the Shannon–McMillan–Breiman theorem and Proposition 6.2. We conclude that $h(K(\omega, x)) = \beta < \infty$, for some $\beta \in \mathbb{R}$. $\square$

From the above result it follows directly by Abramov's formula, see e.g. [12], for the entropy of an induced function that the entropy of R also exists and is finite. Finally we obtain some result for the digits of R as defined in Eq. (9).

Proposition 6.4. *Let b_i be the digits induced by R as defined in (9), then for $\rho - a.e.(\omega, x) \in \Omega \times [0, 1]$, we have*

$$1 < \lim_{n \to \infty} (b_1(x, \omega), \ldots, b_n(x, \omega))^{\frac{1}{n}} < \infty$$

and

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} b_i(x, \omega)}{n} = \infty.$$

The proof of the above proposition is the same as for the regular continued fractions and a full detailed proof is given in [11].

7. More invariant measures

The results obtained for the regular 2-continued fractions are only existence results and moreover they hold only ρ almost everywhere. So we do not know anything about the behaviour of the random 2-continued fraction transformation on the ρ null-sets. Therefore it is interesting to look for other invariant measures for the dynamical system $(\Omega \times [0, 1], \sigma(\mathcal{C} \times \mathcal{B}[0, 1]), R)$. We do this by constructing a commuting bijection between our space $\Omega \times [0, 2]$ and $\mathbb{N}^{\mathbb{N}}$, the space where the digits sequences induced by R live. By definition ∞ is a possible digit, however this only occurs when the orbit of a point (ω, x) is mapped under R to a point whose second coordinate is a zero. Although in this case the expansion is finite, but formally the associated digit sequence ends with a tail of ∞'s. These points will not be in the support of the measures that we will construct. To this end, let

$$M \subset \Omega \times [0,2],\ M = \{(\omega, x) : \pi_2(R^n(\omega, x)) \neq 0\ \forall n \in \mathbb{N}\}.$$

Define $\psi : M \to \mathbb{N}^{\mathbb{N}}$ by $\psi(\omega, x) = (b_1(\omega, x), b_2(\omega, x), \ldots)$. We will show that ψ is indeed an isomorphism, and that we have the following commuting diagram

$$\begin{array}{ccc} M & \xrightarrow{R} & M \\ \downarrow \psi & & \downarrow \psi \\ \mathbb{N}^{\mathbb{N}} & \xrightarrow{\sigma} & \mathbb{N}^{\mathbb{N}} \end{array}$$

where σ denotes the left shift. The strategy used in this section uses the same techniques as used in [8].

Proposition 7.1. *Let $x \in M$ and let $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ with $b_i \in \mathbb{N}$. Then there exists an $\omega \in \Omega$ such that $b_i = d_i(\omega, x)$.*

To prove the above proposition we will first prove a helpful lemma.

Lemma 7.2. *For all $k \in \mathbb{N}$ let I_k denote the interval $(\frac{2}{k+1}, \frac{2}{k}]$. Then,*

(1) *If $x \in I_1$ we have $b_1 = 1$.*
(2) *If $x \in I_k$ for $k \geq 2$ we have $b_1 \in \{k-1, k\}$.*

Proof. Since $b_1 \in \mathbb{N}$ we have that if $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ then, $0 < x < \frac{2}{b_1} \leq 2$, so $x \in (0, 2)$.

For case (1), suppose $x \in I_1 = (1, 2]$ and $b_1 > 1$, then

$$x \leq \cfrac{2}{2 + \cfrac{2}{b_2 + \ddots}} < \frac{2}{2} = 1,$$

which is a contradiction. We conclude that $b_1 = 1$. For case (2), we assume that $x \in I_k$ and $b_1 < k - 1$. Then,

$$x \geq \cfrac{2}{k - 2 + \cfrac{2}{b_2 + \ddots}} > \frac{2}{k-2+2} = \frac{2}{k}.$$

So, $x \notin I_k$ and hence $b_1 \geq k - 1$. Now suppose $b_1 > k$, then $x \leq \cfrac{2}{k+1+\cfrac{2}{b_2+\ddots}} < \frac{2}{k+1}$, so $x \notin I_k$.

We conclude $b_1 \leq k$ and therefore $b_1 \in \{k-1, k\}$. $\square$

We now prove Proposition 7.1.

Proof. Denote by $x_n = \cfrac{2}{b_n + \cfrac{2}{b_{n+1} + \ddots}}$. Let $l_n(x)$ be a variable which counts the number of times $x_i \in [0, 1]$ for $1 \le i \le n$. We will show by induction that for each $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ we can find $\omega \in \Omega$ such that $d_i(\omega, x) = b_i$ for all $i \in \mathbb{N}$. For the base step, note the following

(1) If $x \in I_1 = (1, 2]$ then by Lemma 7.2 we have $b_1 = d_1(\omega, x) = 1$ for all $\omega \in \Omega$. Since $x \in (1, 2]$, $l_1(x) = 0$ and $[\omega]_{l_1} = \Omega$.
(2) If $x \in I_k$, for $k \ge 2$ then we have by Lemma 7.2 that $b_1 \in \{k - 1, k\}$.

- If $b_1 = k$, then we have $d_1(\omega, x) = k$ for all $\omega \in [0]$, so we set $\omega_1 = 0$, $l_1(x) = 1$, since $x \in [0, 1]$ and $[\omega]_{l_1} = [0]$.
- If $b_1 = k - 1$, then for all $\omega \in [1]$ we have $d_1(\omega, x) = k - 1$ and hence we set $\omega_1 = 1$. Again $l_1(x) = 1$, since $x \in [0, 1]$, and $[\omega]_{l_1} = [1]$.

Therefore, we have found a cylinder $[\omega]_{l_1}$ such that $\forall \omega \in [\omega]_{l_1}$, $d_1(\omega, x) = b_1$, where the cylinder $[\omega]_0 = \Omega$. Suppose we have found a cylinder $[\omega]_{l_n}$ such that $\forall \omega \in [\omega]_{l_n}$ we have

$$(d_1(\omega, x), d_2(\omega, x), \ldots, d_n(\omega, x)) = (b_1, b_2, \ldots, b_n).$$

Consider x_{n+1} and note that b_{n+1} is b_1 for x_{n+1}. If $x_{n+1} \in [0, 1]$ we find by the above procedure a cylinder $[\omega]_{l_{n+1}}$, such that $[\omega]_{l_{n+1}} \subset [\omega]_{l_n}$ and

$$(d_1(\omega, x), d_2(\omega, x), \ldots, d_n(\omega, x), d_{n+1}(\omega, x)) = (b_1, b_2, \ldots, b_n, b_{n+1}).$$

If $x_{n+1} \in (1, 2]$, then $l_{n+1} = l_n$ so $[\omega]_{l_n} = [\omega]_{l_{n+1}}$ and we do not refine the cylinder. Notice each time $x_n \in (1, 2]$ we know that $x_{n+1} \in [0, 1]$, so $l_n \ge \frac{n}{2}$. Therefore if $n \to \infty$, then $l_n \to \infty$ and $[\omega]_{l_{n+1}} \subset [\omega]_{l_n}$. Therefore $\bigcap_n [\omega]_{l_n} = \{\omega\}$, for some $\omega \in \Omega$. This concludes the lemma. $\square$

Remark 7.3. The proof of Proposition 7.1 shows that for any continued fraction expansion $(b_1, b_2, \ldots)$ of x there exists a unique $\omega \in \Omega$, such that

$$(d_1(\omega, x), d_2(\omega, x), \ldots) = (b_1, b_2, \ldots). \qquad \diamond$$

Now we are able to show that $\psi : (M, \sigma(\Omega \times [0, 2]) \cap M, \mu, R) \to (\mathbb{N}^{\mathbb{N}}, \mathcal{C}, \nu, \sigma)$ is indeed an isomorphism. Recall that $\sigma(\mathcal{C} \times \mathcal{B}[0, 1])$ is the product σ-algebra generated by cylinder sets of the form $[\omega]_n \times \Delta a_n$, which were defined in terms of the digits induced by the transformation K. The same kind of cylinder sets we define for the transformation R. We start with the partition:

$$\mathcal{P} = \{\Omega \times I_1, [0] \times I_k, [1] \times I_k,\ k \in \mathbb{N}\},$$

which we call the time-0-partition. Let

$$\mathcal{P}_n = \mathcal{P} \vee R^{-1}\mathcal{P} \vee \cdots \vee R^{-(n-1)}\mathcal{P}$$

be the time-n-partition, an element of $C \in \mathcal{P}_n$ is then of the form

$$C = A_1 \vee R^{-1} A_2 \vee \cdots \vee R^{-n-1} A_n$$

for $A_i \in \mathcal{P}$. For each $(\omega, x) \in C$, the value $l_n(\omega, x) = \sum_{i=0}^{n-1} \mathbf{1}_{(\Omega\times[0,1])}(R^i(\omega, x))$ is the same, as well as $(\omega_1, \omega_2, \ldots, \omega_{l_n})$ and the first n digits in the expansion. The elements of $\mathcal{P}$ are the cylinders of length 1 and the elements of $\mathcal{P}_n$ are the cylinders of length n.

In contrast to the cylinders for the transformation K, we do not work here with the digits b_i. This is because for each digit induced by the function K we know precisely which ω is used. For the function R we do not know this. For example a digit $b_1 = k,\ k \geq 2$ could be induced by R if $(\omega, x) \in [0] \times (\frac{2}{k+1}, \frac{2}{k}]$ or if $(\omega, x) \in [1] \times (\frac{2}{k+2}, \frac{2}{k+1}]$. Clearly the cylinder sets for R generate $\sigma(\mathcal{C} \times \mathcal{B}[0, 2])$.

Now, let ν be any shift invariant measure on the product σ-algebra of $\mathbb{N}^{\mathbb{N}}$, and define $\mu = \nu \circ \psi$. We have the following result.

Proposition 7.4. *Let M be the subset defined in the begin of this section. The function $\psi : (M, \sigma(\Omega \times [0, 2]) \cap M, \mu, R) \to (\mathbb{N}^{\mathbb{N}}, \mathcal{C}, \nu, \sigma)$ defined by $\psi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \ldots)$ is an isomorphism.*

Proof. First we show that ψ is one-to-one and onto. By Proposition 7.1 we have that ψ is onto. We show ψ is injective by constructing its inverse. Given a sequence $(b_1, b_2, \ldots)$ we can write $r_n = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots + \cfrac{2}{b_n}}}$. Using the Moebius transformation we see that $r_n = A_1 \cdot A_2 \cdots A_n(0)$, where $A_i = \begin{bmatrix} 0 & 2 \\ 1 & b_i \end{bmatrix}$ and therefore $r_n = \frac{p_n}{q_n}$. Hence

$$r_n = \sum_{i=1}^{n} \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \sum_{i=1}^{n} \frac{-(-2)^n}{q_n q_{n-1}}$$

and $\lim_{n\to\infty} r_n = \sum_{i=1}^{\infty} \frac{-(-2)^n}{q_n q_{n-1}}$ exists by the alternating series test and Proposition 4.1. We conclude that for each series $(b_1, b_2, \ldots)$ there exists a unique x such that $x = \lim_{n\to\infty} r_n$. On the other hand, we have already proved in Proposition 7.1 that there exists a unique ω, such that $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$. We set $\psi'(b_1, b_2, \ldots) = (\omega, x)$. To show that we have constructed an inverse of ψ we have to show that $\psi \circ \psi' = \psi' \circ \psi = \mathrm{id}$ where id denotes the identity function. Now,

$$\psi \circ \psi'(b_1, b_2, \ldots) = \psi(\omega, \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}) = (b_1, b_2, \ldots),$$

where the last equation follows just by construction of ω and the fact that $b_i = d_i(\omega, x)$. On the other hand,

$$\psi' \circ \psi(\omega, x) = \psi'(d_1(\omega, x), d_2(\omega, x), \ldots) = (\omega, \cfrac{2}{d_1 + \cfrac{2}{d_2 + \ddots}}).$$

Since ω is the unique element in Ω generating the digit sequence $(d_1(\omega, x), d_2(\omega, x) \cdots)$, we have $\psi' \circ \psi(\omega, x) = (\omega, x)$.

We prove that ψ is a measurable bijection. First we proof that $\psi : M \to \mathbb{N}^{\mathbb{N}}$ is measurable. We distinguish 2 cases.

- If $b_1 \neq 1$ then
$$\psi^{-1}([b_1]) = \{(\omega, x) \in M : d_1(\omega, x) = b_1\}$$
$$= ((\frac{2}{b_1+1}, \frac{2}{b_1}] \times [0] \cup (\frac{2}{b_1+2}, \frac{2}{b_1+1}] \times [1]) \cap M,$$
so $\psi^{-1}([b_1]) \in \sigma(\mathcal{C} \times \mathcal{B})$.
- If $b_1 = 1$ then
$$\psi^{-1}([1]) = \{(\omega, x) \in M : d_1(\omega, x) = 1\}$$
$$= \left(\left(\frac{2}{3}, 1\right] \times [1] \cup (1, 2] \times \Omega\right) \cap M,$$
so $\psi^{-1}([1]) \in \sigma(\mathcal{C} \times \mathcal{B})$.

By induction we show that the function is measurable. So suppose that the result holds for cylinders of length n. Then we obtain:

$$\begin{aligned}
&\psi^{-1}[b_1, b_2, \dots, b_n, b_{n+1}] \\
&= \{(\omega, x) \in M : d_1(\omega, x) = b_1, d_2(\omega, x) = b_2, \dots, d_n(\omega, x) = b_n, d_{n+1}(\omega, x) = b_{n+1}\} \\
&= \{(x, \omega) \in M : d_1(\omega, x) = b_1, d_2(\omega, x) = b_2, \dots, d_n(\omega, x) = b_n\} \\
&\quad \cap \{(x, \omega) \in M : d_1(R^n(\omega, x)) = b_1\} \\
&= \psi^{-1}[b_1, b_2, \dots, b_n] \cap R^{-n}(\psi^{-1}[b_1]) \cap M.
\end{aligned}$$

The last line is measurable since R is a measurable function.

Now we show that $\psi^{-1} = \psi'$ is a measurable function. To show the measurability of ψ' it is enough to check that $\mathrm{Im}(\psi([\omega]_{l_n} \times (\frac{3}{k+1} \frac{3}{k}]))$ is in the σ-algebra generated by the cylinder sets on $\mathbb{N}$. Define $A_{y,i} = \bigcup_{n\geq 2}[y, \underbrace{1, \dots, 1}_{2i+1\ times}, n]$ and $B_{y,i} = \bigcup_{n\geq 2}[y, \underbrace{1, \dots, 1}_{2i\ times}, n]$ and $C_{y,1} = (y, \underbrace{1, 1, 1, \dots}_{\text{infinitely many 1's}})$. Then

$$\psi'^{-1}([\omega]_{l_1} \times \Delta k_1) = \psi([\omega]_{l_1} \times (\frac{2}{k_1+1}, \frac{2}{k_1}])$$
$$= \begin{cases} \bigcup_{i\in\mathbb{N}_0} B_{k_1,i} \cup C_{k_1,1} & \text{if } k_1 \geq 2,\ [\omega]_{l_1} = [0] \\ \bigcup_{i\in\mathbb{N}} A_{k_1-1,i} \cup C_{k_1,1} & \text{if } k_1 \geq 2,\ [\omega]_{l_1} = [1] \\ \bigcup_{i\in\mathbb{N}} B_{1,i} \cup C_{k_1,1} & \text{if } k_1 = 1. \end{cases}$$

Hence $\psi'^{-1}([\omega]_{l_1} \times \Delta k_1) \in \mathcal{C}$. Now suppose the result holds for sets of length n so $\psi'^{-1}([\omega]_{l_n} \times \Delta k_n) \in \mathcal{C}$, we show that $\psi'^{-1}([\omega]_{l_{n+1}} \times \Delta k_{n+1}) \in \mathcal{C}$. First define:

$$A_{n,y,i} = \bigcup_{(b_1,\dots,b_n)\in\mathbb{N}^n} \bigcup_{m\geq 2}[b_1, \dots, b_n, y, \underbrace{1, \dots, 1}_{2i+1\ times}, m]$$
$$B_{n,y,i} = \bigcup_{(b_1,\dots,b_n)\in\mathbb{N}^n} \bigcup_{m\geq 2}[b_1, \dots, b_n, y, \underbrace{1, \dots, 1}_{2i\ times}, m]$$
$$C_{n,y,1} = \bigcup_{(b_1,\dots,b_n)} (b_1, \dots, b_n, y, \underbrace{1, 1, 1, \dots}_{\text{infinitely many 1's}}).$$

So we can write,

$$\begin{aligned}
&\psi'^{-1}([\omega]_{l_{n+1}} \times \Delta k_{n+1}) \\
&= \psi'^{-1}([\omega]_{l_n} \times \Delta k_n) \cap \psi'^{-1}(\{(\omega, x) : R^n(x, \omega) \in [\omega_{l_{n+1}}] \times (\frac{2}{k_n}, \frac{2}{k_{n+1}}]\}) \\
&= \mathrm{Im}\psi([\omega]_{l_n} \times \Delta k_n) \cap \mathrm{Im}\psi(\{(\omega, x) : R^n(x, \omega) \in [\omega_{l_{n+1}}] \times (\frac{2}{k_n}, \frac{2}{k_{n+1}}]\}) \\
&= \begin{cases} F \cap (\bigcup_{i \in \mathbb{N}_0} B_{n,k_n,i} \cup C_{n,y,1}) & \text{if } k_n \geq 2,\ [\omega_{l_n}] = [0] \\ F \cap (\bigcup_{i \in \mathbb{N}} A_{n,k_n-1,i} \cup C_{n,y,1}) & \text{if } k_n \geq 2,\ [\omega_{l_n}] = [1] \\ F \cap (\bigcup_{i \in \mathbb{N}} B_{n,1,i} \cup C_{n,y,1}) & \text{if } k_n = 1, \end{cases}
\end{aligned}$$

where $F \in \mathcal{C}$ is the element $\psi'^{-1}([\omega]_{l_n} \times \Delta k_n)$. We conclude that ψ' is indeed a measurable function. The fact that ψ preserves measure is immediately since we defined $\mu = \nu \circ \psi$. Finally, we have to show that $\sigma \circ \psi$ is the same operation as $\psi \circ R$. Since σ is the left shift we have

$$\sigma \circ \psi(\omega, x) = \sigma(d_1(\omega, x), d_2(\omega, x), d_3(\omega, x), \ldots) = (d_2(\omega, x), d_3(\omega, x), \ldots)$$

On the other hand

$$\psi \circ R(\omega, x) = (d_1(R(\omega, x)), d_2(R(\omega, x)), \ldots) = (d_2(\omega, x), d_3(\omega, x), \ldots)$$

so indeed $\sigma \circ \psi = \psi \circ R$. We conclude that ψ is indeed an isomorphism. □

Now we can define measures on $\Omega \times [0, 2]$ with support M as lifts of shift invariant probability measures on $\mathbb{N}^{\mathbb{N}}$. Examples of natural shift invariant measures on $\mathbb{N}^{\mathbb{N}}$ are obtained as follows. Let $(p_1, p_2, \ldots)$ be a probability vector on $\mathbb{N}$, and ν the product measure on $\mathbb{N}^{\mathbb{N}}$ with these weights, i.e. if $b_1, b_2, \ldots, b_n \in \mathbb{N}$, then

$$\nu_p(\{(y_1, y_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : y_1 = b_1, \ldots, y_n = b_n\}) = p_{b_1} p_{b_2} \cdots p_{b_n}.$$

Examples of probability-measures we can choose on $\mathbb{N}$ to construct the product measure ν are:

(i) $p_i = \frac{1}{i(i+1)}$,
(ii) $p_i = \frac{e^{-\lambda}\lambda^i}{i!}$, Poisson distribution,
(iii) $p_i = r^{i-1}(1-r)$, Geometric Distribution.

Since each product measure on the cylinder sets in $\mathbb{N}^{\mathbb{N}}$ is ergodic with respect to the left shift, the above probability distributions induce an ergodic measure ν. Therefore the measure $\mu = \nu \circ \psi$ on $\Omega \times [0, 2]$ is also ergodic. However since all the measures ν are different it follows that they are singular with respect to each other. Also the measure ρ, defined in Section 3 is singular with respect to these measures, since the mean of the digits with respect to ρ is infinite and the mean of the digits with respect to the measures ν constructed by the above probability distributions are all finite. In general we see that a finite arithmetic mean, seems to be a generic behaviour on ρ-null sets. Considering the entropy, it is shown in [5] that for measures ν which give finite mean digit sequences, the geometric measure with this mean is the measure of maximal entropy. Whether the product measure $m_p \times \mu_p$ has maximum entropy is not known.

8. Random 3-continued fraction transformation

The techniques and results for $N = 2$ can be extended to arbitrary integer N. The calculations however are very tedious. We illustrate this with the case $N = 3$. The 3 random continued fraction are defined as follows, see Section 2:

$$S_0, S_1, S_2 : (0, 3] \to (0, 3]$$
$$S_0 : x \to \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor$$
$$S_1 : x \to \begin{cases} \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor + 1 & \text{if } x \in \left(0, 1\frac{1}{2}\right] \\ \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor & \text{if } x \in \left(1\frac{1}{2}, 3\right] \end{cases}$$
$$S_2 : x \to \begin{cases} \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor + 2 & \text{if } x \in (0, 1] \\ \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor & \text{if } x \in (1, 3]. \end{cases}$$

We will refer to S_0 as the lower map, to S_1 as the middle map and S_2 as the upper map. Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ and define the function R as follows:

$$R : \Omega \times [0, 3] \to \Omega \times [0, 3],$$
$$R(\omega, x) = \begin{cases} (\sigma\omega, S_{\omega_1}(x)) & \text{if } x \in \left(0, 1\frac{1}{2}\right] \\ (\omega, S_{\omega_1}(x)) & \text{if } x \in \left(1\frac{1}{2}, 3\right] \\ (\sigma\omega, 0) & \text{if } x = 0. \end{cases}$$

Fig. 3 illustrates the transformation R. Notice R does not shift ω if $x \in (1\frac{1}{2}, 3]$. In the area $(1\frac{1}{2}, 3]$ the three maps S_0, S_1 and S_2 coincide and therefore we do not have to choose which map we use. As before the digits of R are given by:

$$b_1(\omega, x) = \begin{cases} 1 & x \in \left(1\frac{1}{2}, 3\right] \\ k & x \in \left(\frac{3}{k+1}, \frac{3}{k}\right] & \omega_1 = 0,\ k \in \mathbb{N}, k \geq 2 \\ k-1 & x \in \left(\frac{3}{k+1}, \frac{3}{k}\right] & \omega_1 = 1,\ k \in \mathbb{N},\ k \geq 2 \\ k-2 & x \in \left(\frac{3}{k+1}, \frac{3}{k}\right] & \omega_1 = 2,\ k \in \mathbb{N},\ k \geq 3 \\ 2 & x \in \left(1, 1\frac{1}{2}\right] & \omega_1 = 2 \end{cases}$$

and we define $b_n(\omega, x) = b_1(R^{n-1}(\omega, x))$. Using Moebius transformations we obtain the following recurrence relations:

$$p_{-1} = 1 \qquad p_0 = 0 \qquad p_n = 3p_{n-2} + b_n p_{n-1},$$
$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_n = 3q_{n-2} + b_n q_{n-1}.$$

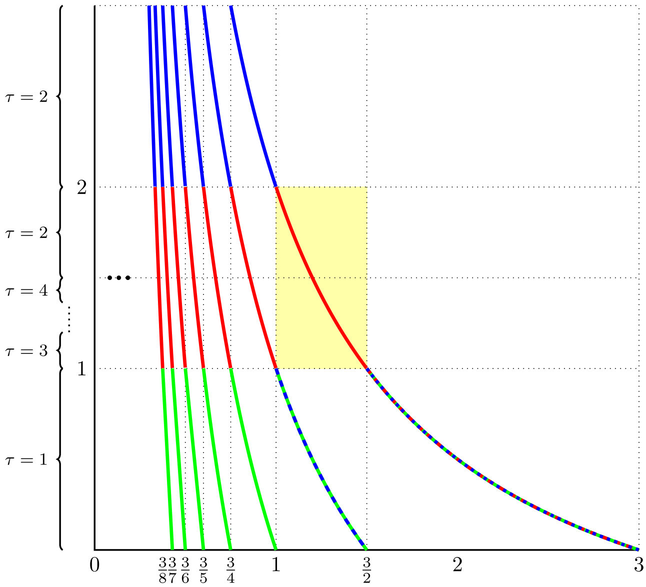

Fig. 3. Map T_0 in green, T_1 in red and T_2 in blue.

Like we did in the case $N = 2$, we will define the transformation $K : \Omega \times [0, 1] \to \Omega \times [0, 1]$ as the induced function $R_{\Omega\times[0,1]}(\omega, x)$. In order to do this we introduce the return time

$$\begin{aligned} \tau &: \Omega \times [0, 1] \to \mathbb{N}, \\ &(\omega, x) \to \inf\{n \in \mathbb{N} : R^n(\omega, x) \in \Omega \times [0, 1]\}. \end{aligned}$$

If the transformation enters the area $\Omega \times [1, \frac{3}{2}]$ it can stay there for an infinitely long time when $(\omega, x) = ((1, 1, 1, \ldots), x)$. Therefore τ will take all values in $\mathbb{N}$. Note there are two ways for the transformation to leave the area $\Omega \times [1, \frac{3}{2}]$, namely the occurrence of an $\omega_1 \neq 1$ or the transformation S_1^i enters the area $[\frac{3}{2}, 2]$ after some time i. In this way we obtain an explicit expression for the return time τ.

$$\tau(\omega, x) = \begin{cases} 1 & \text{if } \omega_1 = 0,\ x \in [0, 1] \\ 2 & \text{if } \omega_1 = 2,\ x \in [0, 1] \\ n+1 & \text{if } (\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n, \omega_{n+1}) \in \{(\underbrace{1, 1, \ldots, 1}_{n\ times}, 0), (\underbrace{1, 1, \ldots, 1}_{n\ times}, 2)\} \\ & \text{and } x \in \cap_{i=1}^{n} S_1^{-i}\left(1, 1\frac{1}{2}\right) \cap [0, 1] \\ n+1 & \text{if } (\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n) = \underbrace{(1, 1, \ldots, 1)}_{n\ times} \\ & \text{and } x \in \cap_{i=1}^{n-1} S_1^{-i}\left(1, 1\frac{1}{2}\right) \cap S_1^{-n}\left(1\frac{1}{2}, 2\right] \cap [0, 1]. \end{cases}$$

We write

$$\cap_{i=1}^{n} S_1^{-i}(1, 1\frac{1}{2}) = S_1^{-1}(\cap_{i=0}^{n-1} S_1^{-i}(1, 1\frac{1}{2})) = S_1^{-1} I_n,$$

where $I_n = (\cap_{i=0}^{n-1} S_1^{-i}(1, 1\frac{1}{2}))$. Notice that when (ω, x) is in $[\underbrace{(1, 1, \ldots, 1)}_{n\ times}] \times I_n$, we have that $b_i = 1$ for all $i \le n$. Therefore by the recursion relations we obtain

$$I_n = \left(\frac{p_{n-1} + 1\frac{1}{2} p_{n-2}}{q_{n-1} + 1\frac{1}{2} q_{n-2}}, \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right)$$

if n even and

$$I_n = \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}}, \frac{p_{n-1} + 1\frac{1}{2} p_{n-2}}{q_{n-1} + 1\frac{1}{2} q_{n-2}} \right)$$

if n is odd. Now if

$$(\omega, x) \in [\underbrace{1, \ldots, 1}_{n+1\ times}] \times S_1^{-1}(I_n \backslash I_{n+1}),$$

then $S_1^{n+1}(\omega, x) \in (1\frac{1}{2}, 2)$ and hence $\tau(\omega, x) = n + 2$. Let J_n denote the interval $J_n = I_n \backslash I_{n+1}$. Then $S_1^{-1} J_n = S_1^{-1}(I_n \backslash I_{n+1})$ and

$$J_n = \left(\frac{p_{n-1} + 1\frac{1}{2} p_{n-2}}{q_{n-1} + 1\frac{1}{2} q_{n-2}}, \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right) \Big\backslash \left(\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n + 1\frac{1}{2} p_{n-1}}{q_n + 1\frac{1}{2} q_{n-1}} \right)$$

when n is even and

$$J_n = \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}}, \frac{p_{n-1} + 1\frac{1}{2} p_{n-2}}{q_{n-1} + 1\frac{1}{2} q_{n-2}} \right) \Big\backslash \left(\frac{p_n + 1\frac{1}{2} p_{n-1}}{q_n + 1\frac{1}{2} q_{n-1}}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right)$$

when n is odd. Now we are ready to define the induced transformation K. Let

$$\begin{aligned}
T_i &: [0, 1] \to [0, 1] \\
T_0(x) &= S_0(x) = \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor \\
T_1(x) &= S_0 \circ S_2(x) = \frac{3}{\frac{3}{x} - \lfloor \frac{3}{x} \rfloor + 2} - 1
\end{aligned}$$

$$T_{(2,i)}(x) = \begin{cases} S_0 \circ S_1^{i+1}(x) & \text{if } x \in S_1^{-1} J_i \text{ for } i \in \mathbb{N} \cup \{0\} \\ 2x & \text{otherwise} \end{cases}$$

$$T_{(3,i)}(x) = \begin{cases} S_0 \circ S_1^{i}(x) & \text{if } x \in S_1^{-1} I_i \text{ for } i \in \mathbb{N} \\ 2x & \text{otherwise} \end{cases}$$

Note that $S_0 \circ S_2(x) = S_1 \circ S_2(x) = S_2 \circ S_2(x)$, and $S_0 \circ S_1^{i+1}(x) = S_1 \circ S_1^{i+1}(x) = S_2 \circ S_1^{i+1}(x)$ if $x \in S_1^{-1} J_i$ for $i \in \mathbb{N} \cup \{0\}$ and $S_0 \circ S_1^{i}(x) = S_2 \circ S_1^{i}(x)$ if $x \in S_1^{-1} I_i$ for $i \in \mathbb{N}$. We let $2x$ occur in both $T_{(2,i)}$ and $T_{(3,i)}$ with probability 0. So this transformation $2x$ will in fact never occur, but it just helpful to satisfy the conditions of Inoue's theorem. Note that we can write for the last two transformations:

$$T_{(2,i)}(x) = \begin{cases} \dfrac{3}{S_1^{i+1}(x)} - 1 & \text{if } x \in S_1^{-1} J_i \\ 2x & \text{otherwise} \end{cases}$$

$$T_{(3,i)}(x) = \begin{cases} \dfrac{3}{S_1^i(x)} - 2 & \text{if } x \in S_1^{-1} I_i \\ 2x & \text{otherwise.} \end{cases}$$

Given a probability vector (p_0, p_1, p_2) on $\{0, 1, 2\}$ we define a probability vector for the transformations $T_0, T_1, T_{(2,i)}, T_{(3,i)}, i \in \mathbb{N}$. We set

$$\begin{aligned} \mathbb{P}(T_0) &= p_0 \\ \mathbb{P}(T_1) &= p_2 \\ \mathbb{P}(T_{(2,i)})(x) &= p_1^{i+1} \mathbf{1}_{S_1^{-1} J_i}(x) \\ \mathbb{P}(T_{(3,i)})(x) &= p_1^i (1 - p_1) \mathbf{1}_{S_1^{-1} I_i}(x) \end{aligned}$$

One can verify that this is indeed a probability vector. For example if $x \in J_i$, so $x \in I_k$ for $1 \le k \le i$, then it follows that

$$p_0 + p_2 + p_1^{i+1} + (1 - p_1) \sum_{j=1}^{i} p_1^j = p_0 + p_2 + p_1^{i+1} + p_1 - p_1^{i+1} = 1.$$

Hence we see that for each $x \in [0, 1]$ we get a well defined probability vector. Now we can give an explicit expression for the transformation K,

$$K(\omega, x) = \begin{cases} (\sigma(\omega), T_0(x)) & \text{if } \omega_1 = 0 \\ (\sigma(\omega), T_1(x)) & \text{if } \omega_1 = 2 \\ (\sigma^{i+1}(\omega), T_{(2,i)}(x)) & \text{if } \omega_j = 1 \ \forall 1 \le j \le i+1 \text{ and } x \in S_1^{-1}(J_i) \\ (\sigma^{i+1}(\omega), T_{(3,i)}(x)) & \text{if } \omega_j = 1 \ \forall 1 \le j \le i,\ \omega_{i+1} \in \{0, 2\} \text{ and } x \in S_1^{-1} I_n. \end{cases}$$

We can define digits for K in a similar way we did in chapter 2, namely:

$$d_i(\omega, x) = \begin{cases} k & \text{if } \omega_1 = 0 \text{ and } x \in \left(\dfrac{3}{k+1}, \dfrac{3}{k}\right] \\ (k-2, 1) & \text{if } \omega_1 = 2 \text{ and } x \in \left(\dfrac{3}{k+1}, \dfrac{3}{k}\right] \\ (k-1, \underbrace{1, \dots, 1}_{n\ times}, 1) & \text{if } \omega \in [\underbrace{1, \dots, 1}_{n\ times}] \text{ and } x \in \left(\dfrac{3}{k+1}, \dfrac{3}{k}\right] \cap S_1^{-1} J_n \\ (k-1, \underbrace{1, \dots, 1}_{n-1\ times}, 2) & \text{if } \omega \in [\underbrace{1, \dots, 1}_{n\ times}, 0] \cup [\underbrace{1, \dots, 1}_{n\ times}, 2] \\ & \text{and } x \in \left(\dfrac{3}{k+1}, \dfrac{3}{k}\right] \cap S_1^{-1} I_n. \end{cases}$$

With this transformation we can continue like we have done before with the 2-random continued fraction transformation. However the computations will be rather tedious. For the details we refer to the thesis [11].

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