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Induced random β-transformation

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1. Introduction. Let $\beta \in (1,2)$ and $I_\beta := [0, 1/(\beta-1)]$. Given $x \in I_\beta$, we call a sequence $(b_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$ a *β-expansion* for x if

$$x = \sum_{n=1}^{\infty} \frac{b_n}{\beta^n}.$$

Noninteger representations of real numbers were pioneered in the papers of Rényi [13] and Parry [12]. Since then they have been studied by many authors and have connections with ergodic theory, fractal geometry, and number theory (see the survey articles [10] and [15]). Perhaps one of the most interesting objects to study within expansions in noninteger bases is the set of expansions, i.e.,

$$\varSigma_\beta(x) := \left\{ (b_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}} : \sum_{n=1}^{\infty} \frac{b_n}{\beta^n} = x \right\}.$$

A result of Sidorov [14] states that if $\beta \in (1,2)$ then Lebesgue almost every $x \in I_\beta$ satisfies $\operatorname{card} \varSigma_\beta(x) = 2^{\aleph_0}$. Moreover, for any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $\beta \in (1,2)$ and $x \in I_\beta$ such that $\operatorname{card} \varSigma_\beta(x) = k$ (see [6, 8, 9]). The situation described above is completely different from the case of integer base expansions where every number has a unique expansion except for a countable set of exceptions which have precisely two.

A useful observation when studying expansions in noninteger bases is that a β-expansion has a natural dynamical interpretation. Namely, let $T_0(x) = \beta x$, $T_1(x) = \beta x - 1$, and

$$\varGamma_\beta(x) := \left\{ (a_n)_{n=1}^\infty \in \{T_0, T_1\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in I_\beta \text{ for all } n \in \mathbb{N} \right\}.$$

It was shown in [1] that $\operatorname{card} \varSigma_\beta(x) = \operatorname{card} \varGamma_\beta(x)$ and the map sending

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(b_n) to (T_{b_n}) is a bijection between these two sets. Hence, performing T_0 corresponds to taking the digit 0, and T_1 corresponds to taking the digit 1.

An all encompassing method by which we can use the maps T_0 and T_1 to generate β-expansions is the *random β-transformation*. This map is defined as follows. Set $\Omega = \{0,1\}^{\mathbb{N}}$ and denote by σ the left shift on Ω. Consider the transformation $K_\beta : \Omega \times [0, 1/(\beta-1)] \to \Omega \times [0, 1/(\beta-1)]$ defined by

$$K_\beta(\omega, x) = \begin{cases} (\omega, T_0 x) & \text{if } 0 \leq x < 1/\beta, \\ (\sigma\omega, T_{\omega_1} x) & \text{if } 1/\beta \leq x \leq 1/(\beta(\beta-1)), \\ (\omega, T_1 x) & \text{if } 1/(\beta(\beta-1)) < x \leq 1/(\beta-1). \end{cases}$$

The random β-transformation K_β was introduced and studied in [5–4]. Given $x \in I_\beta$, the map K_β generates all possible β-expansions of x. Furthermore, it is a random mix of the classical *greedy map* and *lazy map* defined by

$$G_\beta(x) = \begin{cases} T_0(x) & \text{if } 0 \leq x < 1/\beta, \\ T_1(x) & \text{if } 1/\beta \leq x \leq 1/(\beta-1) \end{cases}$$

and

$$L_\beta(x) = \begin{cases} T_0(x) & \text{if } 0 \leq x < 1/(\beta(\beta-1)), \\ T_1(x), & \text{if } 1/(\beta(\beta-1)) \leq x \leq 1/(\beta-1), \end{cases}$$

respectively. Let $S := [1/\beta, 1/(\beta(\beta-1))]$; we refer to S as the *switch region*. This is the region where the greedy map G_β and the lazy map L_β differ, and where the coordinates of ω are used to decide which map to use. Understanding the dynamics of T_0 and T_1 on the switch region provides valuable insight into the possible $\Gamma_\beta(x)$, and thus the possible $\Sigma_\beta(x)$.

This paper is concerned with the dynamics of the first return map defined on the switch region. We consider the induced transformation U_β of K_β on the set $\Omega \times S$. More precisely, $U_\beta : \Omega \times S \to \Omega \times S$ is defined as follows:

$$U_\beta(\omega, x) := K_\beta^{r_1(\omega,x)}(\omega, x), \text{ where } r_1(\omega, x) = \inf\{m \geq 1 : K_\beta^m(\omega, x) \in \Omega \times S\}.$$

Similarly we set

$$U_{\beta,0}(x) := U_\beta((0)^\infty, x) \quad \text{and} \quad U_{\beta,1}(x) := U_\beta((1)^\infty, x).$$

Note that when we have fixed the sequence ω to equal $(0)^\infty$ or $(1)^\infty$, the maps $U_{\beta,0}$ and $U_{\beta,1}$ are well defined maps from S to S.

Remark 1.1. The map U_β is defined on $\Omega \times S$, and both $U_{\beta,0}$ and $U_{\beta,1}$ are defined on S. However, there exist ω and x for which $K_\beta(\omega, x)$ is never mapped back into $\Omega \times S$, thus for this choice of ω and x the map U_β is not well defined. Similarly, there exists x for which U_{β,ω_i} is not well defined. However, it is a consequence of the work of Sidorov [14] that the set of x for which $U_\beta^n(w, x)$ is well defined for all $n \in \mathbb{N}$ and $\omega \in \Omega$ is of full Lebesgue measure within S. Similarly, the set of x for which $U_{\beta,\omega_i}^n(x)$ is well defined for every $n \in \mathbb{N}$ is of full Lebesgue measure within S. Throughout this article we will abuse notation and let S denote both the switch region and the full measure subset of S for which U_β and U_{β,ω_i} are well defined. It should be clear which interpretation of S we mean from the context.

For $i \geq 1$ let $r_i(\omega, x) := r_1(U_\beta^{i-1}(\omega, x))$ be the *ith return time* to the switch region $\Omega \times S$. Note that for any β and ω, the set $\{r_1(\omega, x)\}_{x \in S}$ equals $\mathcal{R}_\beta := \{m, m+1, \ldots\}$ where m is some natural number that only depends upon β. We emphasise that $\mathcal{R}_\beta$ has no dependence on ω. One of the goals of this paper is to understand the sequences $(r_i(\omega, x))_{i=1}^\infty$ and to answer the following question: given $\omega \in \Omega$ and a sequence of integers $(j_i)_{i=1}^\infty \in \mathcal{R}_\beta^{\mathbb{N}}$, when is it possible to find $x \in S$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$? The following theorem provides an answer to this question. To state it, we have to introduce two classes of algebraic integers. Let α_k denote the unique solution in $(1, 2)$ of the equation

$$x^{k+1} - 2x^k + x - 1 = 0,$$

and let γ_k denote the kth *multinacci number*, the unique root of

$$x^{k+1} - x^k - x^{k-1} - \cdots - x - 1 = 0$$

contained in $(1, 2)$.

THEOREM 1.1. *Let $\beta \in (\alpha_k, \gamma_k]$ for some $k \geq 2$. Then for any $\omega \in \Omega$ and $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ there exists $x \in S$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$. Moreover, if $\beta \notin (\alpha_k, \gamma_k]$ for all $k \geq 2$, then there exist $\omega \in \Omega$ and $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ such that no $x \in S$ satisfies $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$.*

As we will see, the algebraic properties of α_k and γ_k correspond naturally to conditions on the orbit of 1 and its reflection $1/(\beta - 1) - 1$. These points completely determine the dynamics of the greedy map G_β and the lazy map L_β respectively, and hence it is not surprising that these points play a crucial role in our situation as well. For values of β lying outside of the intervals $(\alpha_k, \gamma_k]$ it is natural to ask whether the following weaker condition is satisfied: given $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$, do there exist $\omega \in \Omega$ and $x \in S$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$? Let η_k denote the unique root of the equation

$$2x^{k+1} - 4x^k + 1 = 0$$

contained in $(1, 2)$.

THEOREM 1.2. *Let $\beta \in (\alpha_k, \eta_k]$ for some $k \geq 1$. Then for any sequence $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ there exist $\omega \in \Omega$ and $x \in S$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$.*

If β satisfies the hypothesis of Theorem 1.2 then the orbits of 1 and $1/(\beta - 1) - 1$ satisfy a *cross over property.* This property is sufficient to prove Theorem 1.2. Note that $\alpha_k \leq \gamma_k \leq \eta_k$ for each $k \geq 1$. Table 1 lists some initial values of α_k, γ_k and η_k.

Table 1. Values of α_k, γ_k and η_k

k	α_k	γ_k	η_k
1	$\frac{1+\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$1+2^{-1/2}$
2	$1.7549\ldots$	$1.8393\ldots$	$1.8546\ldots$
3	$1.8668\ldots$	$1.9276\ldots$	$1.9305\ldots$
4	$1.9332\ldots$	$1.9660\ldots$	$1.9666\ldots$
5	$1.9672\ldots$	$1.9836\ldots$	$1.9837\ldots$

The second half of this paper is concerned with the maps $U_{\beta,0}$ and $U_{\beta,1}$. To state our results, it is necessary to make a definition. Given a closed interval $[a,b]$, we call a map $T:[a,b]\to[a,b]$ a *generalized Lüroth series transformation* (abbreviated to GLST) if there exists a countable set $\{I_n\}_{n=1}^{\infty}$ of bounded subintervals ($I_n=(l_n,r_n),[l_n,r_n],(l_n,r_n]$, or $[l_n,r_n)$) for which the following criteria are satisfied:

(1) $I_n\cap I_m=\emptyset$ for $n\neq m$.

(2) $\displaystyle\sum_{n=1}^{\infty}(r_n-l_n)=b-a.$

(3) $\displaystyle T(x)=a+\frac{(x-l_n)(b-a)}{r_n-l_n}$ for $x\in I_n$.

Property (3) is equivalent to the map T restricted to I_n being the unique surjective linear orientation preserving map from I_n into S.

The traditional *Lüroth expansion* of a number $x\in(0,1]$ is a sequence $(a_n)_{n=1}^{\infty}$ of natural numbers where $a_n\geq 2$ for each n and

$$x=\frac{1}{a_1}+\frac{1}{a_1(a_1-1)a_2}+\cdots+\frac{1}{a_1(a_1-1)a_2(a_2-1)\cdots a_n}+\cdots.$$

This expansion can be seen to be generated by the map $T:[0,1]\to[0,1]$ where

$$T(x)=\begin{cases} n(n+1)x-n & \text{if } x\in(1/(n+1),1/n],\\ 0 & \text{if } x=0.\end{cases}$$

GLSTs were introduced in [2]. Our definition is slightly different from the one appearing in that paper but all of the main results translate over into our context. Namely if $T:[a,b]\to[a,b]$ is a GLST then the normalised Lebesgue measure on $[a,b]$ is a T-invariant ergodic measure. Our main result for the maps $U_{\beta,0}$ and $U_{\beta,1}$ is the following theorem.

Theorem 1.3. *There exists a set $M\subseteq(1,2)$ of Hausdorff dimension* 1 *and Lebesgue measure zero such that:*

(1) *If $\beta \in M$ then both $U_{\beta,0}$ and $U_{\beta,1}$ are GLSTs.*
(2) *If $\beta \notin M$ then neither $U_{\beta,0}$ nor $U_{\beta,1}$ is a GLST.*

What is more, we can describe the set M explicitly.

Before we move on to the proofs of Theorems 1.1–1.3, we provide a worked example. Namely, we consider the case where $\beta = (1+\sqrt{5})/2$, which exhibits some of the important features of our later proofs.

EXAMPLE 1.1. If $\beta = (1+\sqrt{5})/2$ then $S = [1/\beta, 1]$. Let $C_j = \{\omega \in \Omega : \omega_1 = j\}$, $j = 0,1$. Then for any $\omega \in C_0$, we have $r_1(\omega, 1) = \infty$ and $r_1(\omega, 1/\beta) = 1$, while for any $\omega \in C_1$, $r_1(\omega, 1) = 1$ and $r_1(\omega, 1/\beta) = \infty$. If $x \in (1/\beta, 1)$, then $r_1(\omega, x) \geq 2$ for all $\omega \in \Omega$.

For $i \geq 2$, let

$$B_i^0 := \{x \in S : U_{\beta,0}(x) = (T_1^{i-1} \circ T_0)(x)\}, \tag{1.1}$$

$$B_i^1 := \{x \in S : U_{\beta,1}(x) = (T_0^{i-1} \circ T_1)(x)\}. \tag{1.2}$$

A simple calculation shows that for $i \geq 2$,

$$B_i^0 = \Big(\sum_{n=2}^{i+1} \frac{1}{\beta^n}, \sum_{n=2}^{i+2} \frac{1}{\beta^n}\Big] = (T_1^{i-1} \circ T_0)^{-1}(1/\beta, 1], \tag{1.3}$$

$$B_i^1 = \left[\frac{1}{\beta} + \frac{1}{\beta^{i+1}}, \frac{1}{\beta} + \frac{1}{\beta^i}\right) = (T_0^{i-1} \circ T_1)^{-1}[1/\beta, 1). \tag{1.4}$$

The collection $\{B_i^0 : i \geq 2\}$ is a partition of $(1/\beta, 1)$, and $\{B_i^1 : i \geq 2\}$ is a partition of $(1/\beta, 1)$. Equation (1.3) demonstrates that $U_{\beta,0}$ restricted to B_i^0 is a full branch, thus $U_{\beta,0}$ is a GLST. Similarly equation (1.4) implies $U_{\beta,1}$ is a GLST. We include a diagram of the graph of $U_{\beta,0}$ in Figure 1.

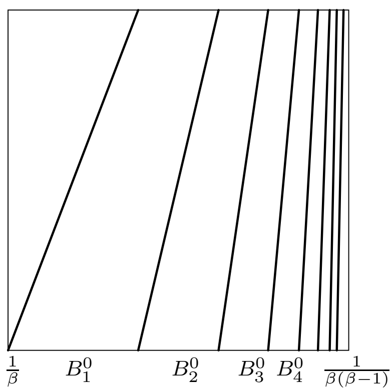

Fig. 1. The graph of $U_{\beta,0}$ when $\beta = (1+\sqrt{5})/2$

By the aforementioned results of [2], a GLST is ergodic with respect to the normalised Lebesgue measure μ. Hence we can compute the average return time. For $\beta = (1+\sqrt{5})/2$, Lebesgue almost every $x \in S$ satisfies

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} r_j((0)^\infty, x) &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=2}^{\infty} i\chi_{B_i^0}((U_{\beta,0})^j(x)) \\ &= \int \sum_{i=2}^{\infty} i\chi_{B_i^0}\, d\mu = 2\beta^2 - \beta = 3.6178\ldots,\end{aligned}$$

where $\chi_{B_i^0}$ denotes the characteristic function of B_i^0. Note that the above also holds with $r_j((0)^\infty, x)$ replaced with $r_j((1)^\infty, x)$.

By Theorem 1.1, there exist $\omega \in \Omega$ and $(j_i)_{i=1}^\infty \in R^{\mathbb{N}}_{(1+\sqrt{5})/2}$ for which no x satisfies $r_i(\omega, x) = j_i$ for $i = 1, 2, \ldots$. This is essentially a consequence of the fact mentioned above that if $\omega \in C_0$ then $r_1(\omega, 1) = \infty$ and $r_1(\omega, 1/\beta) = 1$, while for any $\omega \in C_1$, we have $r_1(\omega, 1) = 1$ and $r_1(\omega, 1/\beta) = \infty$. This statement implies that we cannot have a return time 1 followed by any other natural number.

2. Sequences of return times

2.1. Proof of Theorem 1.1. The proofs of Theorems 1.1 and 1.2 both make use of a nested interval construction. We begin by examining the condition $\beta \in (\alpha_k, \gamma_k]$. It is easy to show that

$$\begin{aligned}\beta \in (\alpha_k, 2) \;&\Leftrightarrow\; (T_1^{k-1}\circ T_0)\Big(\frac{1}{\beta}\Big) > \frac{1}{\beta(\beta-1)} \\ &\Leftrightarrow\; (T_0^{k-1}\circ T_1)\Big(\frac{1}{\beta(\beta-1)}\Big) < \frac{1}{\beta}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\beta \in (1, \gamma_k] \;&\Leftrightarrow\; (T_1^{k}\circ T_0)\Big(\frac{1}{\beta}\Big) \le \frac{1}{\beta} \\ &\Leftrightarrow\; (T_0^{k}\circ T_1)\Big(\frac{1}{\beta(\beta-1)}\Big) \ge \frac{1}{\beta(\beta-1)}.\end{aligned} \tag{2.2}$$

Thus $\beta \in (\alpha_k, \gamma_k]$ is equivalent to the orbit of $1/\beta$ either jumping over the switch region, or satisfying $U_{\beta,0}(1/\beta) = (T_1^k \circ T_0)(1/\beta) = 1/\beta$. Similarly, $\beta \in (\alpha_k, \gamma_k]$ is equivalent to the orbit of $1/(\beta(\beta-1))$ either jumping over the switch region, or satisfying

$$U_{\beta,1}\Big(\frac{1}{\beta(\beta-1)}\Big) = (T_0^k\circ T_1)\Big(\frac{1}{\beta(\beta-1)}\Big) = \frac{1}{\beta(\beta-1)}.$$

The following properties are important consequences of the above. First of all, it is straightforward to see that for $\beta \in (\alpha_k, \gamma_k]$ we have $R_\beta = \{k+1,$

$k+2,\dots\}$. Secondly, for $i \ge k+1$ we have

$$B_i^0 = (T_1^{i-1} \circ T_0)^{-1}(S), \tag{2.3}$$

$$B_i^1 = (T_0^{i-1} \circ T_1)^{-1}(S), \tag{2.4}$$

where B_i^0 and B_i^1 are as in Example 1.1, but in this case they do not form a partition of S.

We now prove Theorem 1.1; we separate the proof into the following propositions.

Proposition 2.1. *Let $\beta \in (\alpha_k, \gamma_k]$ for some $k \ge 2$. Then for any $\omega \in \Omega$ and $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ there exists $x \in S$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \dots$.*

Proof. Let $\beta \in (\alpha_k, \gamma_k]$ and fix $(\omega_i) \in \Omega$ and $(j_i) \in \{k+1, k+2, \dots, \}^{\mathbb{N}}$. We set $\mathcal{I}_1 = B_{j_1}^{\omega_1}$ and

$$\begin{aligned}\mathcal{I}_i := B_{j_1}^{\omega_1} &\cap (T_{\overline{\omega_1}}^{j_1-1} \circ T_{\omega_1})^{-1}(B_{j_2}^{\omega_2}) \\ &\cap \dots \cap \big((T_{\overline{\omega_{i-1}}}^{j_{i-1}-1} \circ T_{\omega_{i-1}}) \circ \dots \circ (T_{\overline{\omega_1}}^{j_1-1} \circ T_{\omega_1})\big)^{-1}(B_{j_i}^{\omega_i})\end{aligned} \tag{2.5}$$

for $i \ge 2$. In the above and throughout we let $\overline{\omega_i} = 1 - \omega_i$. Any element of $\mathcal{I}_i$ satisfies $r_l(\omega, x) = j_l$ for $1 \le l \le i$. By (2.3) and (2.4) we have $(T_{\overline{\omega_1}}^{j_1-1} \circ T_{\omega_1})(\mathcal{I}_1) = S$, and by an induction argument it can be shown that

$$\big((T_{\overline{\omega_i}}^{j_i-1} \circ T_{\omega_i}) \circ \dots \circ (T_{\overline{\omega_1}}^{j_1-1} \circ T_{\omega_1})\big)(\mathcal{I}_i) = S \tag{2.6}$$

for all $i \in \mathbb{N}$. This guarantees that $\mathcal{I}_i$ is nonempty and well defined for each $i \in \mathbb{N}$. Moreover, $\mathcal{I}_{i+1} \subseteq \mathcal{I}_i$ by (2.5). Thus $(\mathcal{I}_i)$ is a decreasing sequence of compact intervals and

$$E = \bigcap_{i=1}^{\infty} \mathcal{I}_i$$

is nonempty. Finally, any $x \in E$ satisfies $r_i(\omega, x) = j_i$ for all $i \in \mathbb{N}$. ∎

Proposition 2.2. *Let $\beta \in (1, (1+\sqrt{5})/2]$. Then there exist $\omega \in \Omega$ and $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ such that no $x \in S$ satisfies $r_i(\omega, x) = j_i$ for $i = 1, 2, \dots$.*

Proof. Any $\beta \in (1, (1+\sqrt{5})/2]$ satisfies $R_\beta = \{1, 2, \dots\}$. We now fix the sequence $\omega = (0)^\infty$ and $(j_i) = (1)^\infty$. There exists no $x \in S$ satisfying $r_i((0)^\infty, x) = 1$ for all $i \ge 1$, as this would imply there exists $x \in S$ satisfying $T_0^i(x) \in S$ for all $i \ge 1$. This is not possible as repeated iteration of T_0 eventually maps any element of S outside of S. ∎

Proposition 2.3. *Let $\beta \in ((1+\sqrt{5})/2, 2) \setminus \bigcup_{k=2}^{\infty} (\alpha_k, \gamma_k]$. Then there exist $\omega \in \Omega$ and $(j_i) \in \mathcal{R}_\beta^{\mathbb{N}}$ such that no $x \in S$ satisfies $r_i(\omega, x) = j_i$ for $i = 1, 2, \dots$.*

Proof. For $\beta \in ((1+\sqrt{5})/2, 2)$ we have $r_1(\omega, 1/\beta) \geq 2$ for any $\omega \in C_0$. Moreover, by our assumption that $\beta \notin (\alpha_k, \gamma_k]$ for any $k \geq 2$, we must have

$$(T_1^k \circ T_0)\left(\frac{1}{\beta}\right) \in \left(\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$$

for some $k \geq 1$. For such a β we have $R_\beta := \{k+1, k+2, \ldots\}$. Let $\omega = (0)^\infty$ and $(j_i) = (k+1)^\infty$. We now show that there exists no $x \in S$ satisfying $r_i((0)^\infty, x) = k+1$ for all i. Since $k+1$ is the earliest return time, there exists a single interval $\mathcal{I}$ for which $\mathcal{I} := \{x \in S : r_1((0)^\infty, x) = k+1\}$; moreover for any $x \in \mathcal{I}$ we have $U_{\beta,0}(x) = (T_1^k \circ T_0)(x)$. Thus, any x satisfying $r_i((0)^\infty, x) = k+1$ for all $i \in \mathbb{N}$ must satisfy

$$(T_1^k \circ T_0)^i(x) \in S \quad \text{for all } i \in \mathbb{N}. \tag{2.7}$$

We now explain why this is not possible.

The map $T_1^k \circ T_0$ scales distances by a factor of β^{k+1} and satisfies $(T_1^k \circ T_0)(x) > x$ for x to the right of the fixed point of $T_1^k \circ T_0$. We previously observed that $(T_1^k \circ T_0)(1/\beta) \in (1/\beta, 1/(\beta(\beta-1))]$, thus the fixed point of $T_1^k \circ T_0$ is to the left of S. Therefore under repeated iteration of $T_1^k \circ T_0$ every $x \in S$ is eventually mapped outside of S. This implies that (2.7) cannot hold, and we have proved our result. ∎

Combining Propositions 2.1–2.3 yields Theorem 1.1.

2.2. Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1 in that we make use of a nested interval construction. However, in the proof we do not explicitly construct the desired ω, but only show its existence; as such, the proof takes on an added degree of abstraction.

Let us start by examining the consequences of $\beta \in (\alpha_k, \eta_k]$ for some $k \geq 1$. For $k \geq 2$ we ignore the intervals $(\alpha_k, \gamma_k]$ as they are covered by Theorem 1.1. For β in the remaining parameter space we have

$$\begin{aligned} (T_1^k \circ T_0)\left(\frac{1}{\beta}\right) &\in \left(\frac{1}{\beta}, \frac{1}{2(\beta-1)}\right], \\ (T_0^k \circ T_1)\left(\frac{1}{\beta(\beta-1)}\right) &\in \left[\frac{1}{2(\beta-1)}, \frac{1}{\beta(\beta-1)}\right). \end{aligned} \tag{2.8}$$

We emphasise that for any $\beta \in (1,2)$ the point $1/(2(\beta-1))$ is the midpoint of the interval S and is thus always in the interior of S. Equation (2.8) is equivalent to $U_{\beta,0}(1/\beta)$ being in the left hand side of S, and $U_{\beta,1}(1/(\beta(\beta-1)))$ being in the right hand side. As such, the two orbits cross over when they return to S.

The cross over property described by (2.8) implies

$$(T_1^k \circ T_0)(B_{k+1}^0) \cup (T_0^k \circ T_1)(B_{k+1}^1) = S. \tag{2.9}$$

Moreover, for any $i \geq k+1$ we have

$$(T_1^i \circ T_0)(B_{i+1}^0) = S. \tag{2.10}$$

With the identities (2.9) and (2.10) we may now prove Theorem 1.2.

Proof of Theorem 1.2. Let $\beta \in (\gamma_k, \eta_k]$ and fix a sequence of return times $(j_i) \in R_\beta^{\mathbb{N}} = \{k+1, k+2, \dots\}^{\mathbb{N}}$. We will construct a set J such that for any $x \in J$ there exists a sequence ω satisfying $r_i(\omega, x) = j_i$ for all $i \in \mathbb{N}$. We construct J by building a sequence of levels $J_1, J_2, \dots$. Each J_i will be a finite collection $\{\mathcal{I}_l^i\}_{l=1}^{2^i}$ of compact intervals. Moreover,

$$\bigcup_{l=1}^{2^{i+1}} \mathcal{I}_l^{i+1} \subseteq \bigcup_{l=1}^{2^i} \mathcal{I}_l^i \tag{2.11}$$

for each $i = 1, 2, \dots$. Thus

$$J = \bigcap_{i=1}^{\infty} \bigcup_{l=1}^{2^i} \mathcal{I}_l^i$$

is nonempty, and as we will see, for each $x \in J$ there exists an $\omega \in \Omega$ such that $r_i(\omega, x) = j_i$ for $i = 1, 2, \dots$. We emphasise that in our construction not every $\mathcal{I}_j^i$ will necessarily be nonempty.

For each level J_i it is useful to define a collection of maps, $M_i = \{f_l^i\}_{l=1}^{2^i}$. Each f_l^i will be a map from $\mathcal{I}_l^i$ into S. These maps will also satisfy

$$\bigcup_{l=1}^{2^i} f_l^i(\mathcal{I}_l^i) = S. \tag{2.12}$$

We start by letting

$$J_1 = \{B_{j_1}^0, B_{j_1}^1\} \quad \text{and} \quad M_1 = \{T_1^{j_1-1} \circ T_0, T_0^{j_1-1} \circ T_1\}.$$

By (2.9) and (2.10) we have

$$(T_1^{j_1-1} \circ T_0)(B_{j_1}^0) \cup (T_0^{j_1-1} \circ T_1)(B_{j_1}^1) = S.$$

So (2.12) holds when $i = 1$. Assume we have constructed J_i and M_i for $1 \leq i \leq N$, and (2.11) holds for $1 \leq i \leq N-1$, and (2.12) holds for $1 \leq i \leq N$. We now construct J_{N+1} and M_{N+1}. To each $f_l^N \in M_N$ we associate the compact intervals $(f_l^N)^{-1}(B_{j_{N+1}}^0)$ and $(f_l^N)^{-1}(B_{j_{N+1}}^1)$; the set of these new intervals is our J_{N+1}. By (2.12) the collection $\{(f_l^N)^{-1}(B_{j_{N+1}}^0), (f_l^N)^{-1}(B_{j_{N+1}}^1)\}$ of intervals is nonempty. Each f_l^N is a map from $\mathcal{I}_l^N$ into S, thus $(f_l^N)^{-1}(B_{j_{N+1}}^1) \subseteq \mathcal{I}_l^N$, and (2.11) holds for $i = N$.

To each $(f_l^N)^{-1}(B_{j_{N+1}}^0)$ we associate the map $(T_1^{j_{N+1}-1} \circ T_0) \circ f_l^N$, and to each $(f_l^N)^{-1}(B_{j_{N+1}}^1)$ the map $(T_0^{j_{N+1}-1} \circ T_1) \circ f_l^N$. This collection of maps is our new M_{N+1}.

Moreover,

$$\begin{aligned}
&\Big(\bigcup_{l=1}^{2^N}((T_1^{j_{N+1}-1}\circ T_0)\circ f_l^N)\circ (f_l^N)^{-1}(B^0_{j_{N+1}})\Big)\\
&\qquad\qquad\qquad\qquad \cup\Big(\bigcup_{l=1}^{2^N}((T_0^{j_{N+1}-1}\circ T_1)\circ f_l^N)\circ (f_l^N)^{-1}(B^1_{j_{N+1}})\Big)\\
&= (T_1^{j_{N+1}-1}\circ T_0)\Big(\bigcup_{l=1}^{2^N} f_l^N (f_l^N)^{-1}(B^0_{j_{N+1}})\Big)\\
&\quad \cup (T_0^{j_{N+1}-1}\circ T_1)\Big(\bigcup_{l=1}^{2^N} f_l^N (f_l^N)^{-1}(B^1_{j_{N+1}})\Big)\\
&= (T_1^{j_{N+1}-1}\circ T_0)(B^0_{j_{N+1}})\cup (T_0^{j_{N+1}-1}\circ T_1)(B^1_{j_{N+1}}) \quad \text{(by (2.12) for } i=N)\\
&= S \quad \text{(by (2.9) and (2.10)).}
\end{aligned}$$

Therefore (2.12) holds for $i = N+1$. Hence we can repeat the above steps indefinitely, and J_i and M_i are well defined for all $i \in \mathbb{N}$ and satisfy (2.11) and (2.12). This implies that the set J is well defined and nonempty.

It is not immediately obvious why an $x \in J$ admits an $\omega \in \Omega$ such that $r_i(\omega, x) = j_i$ for all $i \geq 1$. We now explain why. If $x \in J$, then by our construction for each $n \in \mathbb{N}$ there exists $(\omega_i^n)_{i=1}^n \in \{0,1\}^n$ such that

$$(T_{\omega_i^n}^{j_i}\circ T_{\omega_i^n})\circ\cdots\circ(T_{\omega_1^n}^{j_1}\circ T_{\omega_1^n})(x)\in S \tag{2.13}$$

for all $1 \leq i \leq n$. We identify the finite sequence (ω_i^n) with the infinite sequence $\upsilon_n = (\omega_1^n, \ldots, \omega_n^n, (0)^\infty)$. We equip Ω with the usual metric $d(\cdot,\cdot)$ where $d((\epsilon_i),(\delta_i)) = 2^{-\mathsf{n}((\epsilon_i),(\delta_i))}$ where $\mathsf{n}(x,y) = \inf\{i : \epsilon_i \neq \delta_i\}$. With respect to this metric, Ω is a compact metric space, thus there exist $\upsilon \in \Omega$ and a subsequence of (υ_n) such that $\upsilon_{n_k} \to \upsilon$. Then

$$(T_{\upsilon_i}^{j_i}\circ T_{\upsilon_i})\circ\cdots\circ(T_{\upsilon_1}^{j_1}\circ T_{\upsilon_1})(x)\in S \tag{2.14}$$

for all $i \in \mathbb{N}$, since υ is the limit of a sequence satisfying (2.13). Clearly (2.14) implies that $r_i(\upsilon, x) = j_i$ for all $i \in \mathbb{N}$. ■

Remark 2.1. We end this section by pointing out that there are nontrivial examples of $\beta \in (1,2)$ for which there exists $(j_i) \in R_\beta^{\mathbb{N}}$ and no $x \in S$ and $\omega \in \Omega$ for which $r_i(\omega, x) = j_i$ for all $i \in \mathbb{N}$. For example take $\beta = 1.754$. We chose this value because it is slightly less than α_2. Thus $T_1\circ T_0(1/\beta) \in S$, but it is only slightly less than the right end point of the switch region. Clearly $R_\beta := \{2, 3, \ldots\}$. However, any point that can have a return time 2 gets mapped close to the endpoints of S under the corresponding map. Being close to the endpoints of the switch suggests either a large return time or a small return time. This is the case for $\beta = 1.754$, and a simple calculation shows that it is not possible for $r_1(\omega, x) = 2$ and $r_2(\omega, x) = 3$.

3. Proof of Theorem 1.3. Let us begin our proof of Theorem 1.3 by defining the set M that appears in its statement. Let

$$\begin{aligned} M := \big\{ &\beta \in (1,2) : \operatorname{card} \Sigma_\beta(1) = 1\} \\ &\cup \{\beta \in (1,2) : U_{\beta,0}(1/\beta) \in \{1/\beta, 1/(\beta(\beta-1))\}\big\}. \end{aligned}$$

The first set in this union is the set of univoque bases; the study of this set is classical within expansions in noninteger bases—we refer the reader to [7, 8, 9, 11]. Erdős and Joó [9] showed that the set of univoque bases has Hausdorff dimension 1 and Lebesgue measure zero.

The second set in the above union is a countable set of algebraic numbers, thus M has Hausdorff dimension 1 and Lebesgue measure zero. It is worth noting that if $\beta \in (1,2)$ and $U_{\beta,0}(1/\beta) \in \{1/\beta, 1/(\beta(\beta-1))\}$ then $\operatorname{card} \Sigma_\beta(1) = \aleph_0$. The important observation to make from the definition of M is that

$$\beta \in M \;\Leftrightarrow\; \frac{1}{\beta} \text{ and } \frac{1}{\beta(\beta-1)} \text{ are never mapped into the interior of } S.$$

This property will be sufficient to prove that both $U_{\beta,0}$ and $U_{\beta,1}$ are GLSTs. Our proof of Theorem 1.3 is split into the following propositions.

Proposition 3.1. *If $\beta \notin M$ then neither $U_{\beta,0}$ nor $U_{\beta,1}$ is a GLST.*

Proof. If $\beta \notin M$ then $U_{\beta,0}(1/\beta) \in S^0$. In this case at the left endpoint of S the graph of $U_{\beta,0}$ has an incomplete branch. Thus $U_{\beta,0}$ is not a GLST, as all of the branches are full for this class of transformations. The proof that $U_{\beta,1}$ is not a GLST is similar and appeals to the fact that $U_{\beta,1}(1/(\beta(\beta-1))) \in S^0$. ∎

Proposition 3.2. *If $\beta \in M$ then $U_{\beta,0}$ and $U_{\beta,1}$ are GLSTs.*

We will only show that if $\beta \in M$ then $U_{\beta,0}$ is a GLST, the proof for $U_{\beta,1}$ being analogous. Moreover, as we previously demonstrated in Example 1.1 that the maps $U_{\beta,0}$ and $U_{\beta,1}$ are GLSTs for $\beta = (1+\sqrt{5})/2$, we restrict our attention to the interval $((1+\sqrt{5})/2, 2)$, where the rest of the set M exists.

Before proceeding with our proof that $U_{\beta,0}$ is a GLST, we make several observations. Let $\beta \in ((1+\sqrt{5})/2, 2)$ and $x \in S$ be such that $U_{\beta,0}$ is well defined. Then

$$U_{\beta,0}(x) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(x) \tag{3.1}$$

for some $\omega_i \in \{0,1\}$ that alternate digits with $\omega_1 = 1$. Equation (3.1) holds because the map T_0 maps every element of S outside of S. The quantity $i-1$ is the number of times x jumps over S before eventually being mapped inside. Note that if i is even then $\omega_i = 0$, and if i is odd then $\omega_i = 1$.

Let

$$C_n := T_0^{-n}(S) \quad \text{and} \quad D_n := T_1^{-n}(S)$$

where $n \in \mathbb{N}$. Equation (3.1) demonstrates that if $U_{\beta,0}(x)$ is well defined then x must eventually map into a C_n or a D_n. Note that for $\beta \in ((1+\sqrt{5})/2, 2)$ the C_n are all disjoint and contained in $(0, 1/\beta)$, and similarly the D_n are all disjoint and contained in $(1/(\beta(\beta-1)), 1/(\beta-1))$

It is instructive here to make a final notational remark before entering into the proof of Proposition 3.2. As we will see, the proof relies heavily on understanding the trajectories of certain intervals under certain maps and where they lie relative to C_n, D_n and S. Often we will be in a situation where a relation ($I \cap J = \emptyset$, $I \subseteq J$) is true only if we ignore the endpoints of these intervals. For ease of exposition, instead of repeatedly emphasising the fact that this relation holds modulo the endpoints, we will simply state that the equation holds. This is technically not correct, but our proof is then far more succinct.

Proof of Proposition 3.2. To prove $U_{\beta,0}$ is a GLST it suffices to show that for any $x \in S$ such that $U_{\beta,0}(x)$ is well defined, we have

$$\{y \in S : U_{\beta,0}(y) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(y)\} = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S). \tag{3.2}$$

Here we have assumed $U_{\beta,0}(x) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(x)$. We now explain why (3.2) implies $U_{\beta,0}$ is a GLST. The intervals on the left hand side of (3.2) are all disjoint, thus part (1) of the definition of a GLST holds. By Sidorov's result we know that for Lebesgue almost every $x \in S$ the map $U_{\beta,0}(x)$ is well defined, thus the lengths of the intervals on the left hand side of (3.2) sum up to the length of S, so we get part (2) of the definition of a GLST. Lastly, the right hand side of (3.2) demonstrates that $U_{\beta,0}$ restricted to this interval is surjective onto S; since there is a unique surjective linear orientation preserving map from this interval onto S, we also deduce part (3) of the definition of a GLST.

We begin with the simplest case that $U_{\beta,0}(x) = (T_1^{n_1} \circ T_0)(x)$, i.e. $T_0(x) \in D_{n_1}$. Importantly, since $\beta \in M$, we know that $1 \notin D_{n_1}^0$. Thus $T_0(S) \cap D_{n_1} = [1, 1/(\beta-1)] \cap D_{n_1} = D_{n_1}$. Therefore $T_0^{-1}(D_{n_1}) \subseteq S$ and any y in this interval satisfies $U_{\beta,0}(y) = (T_1^{n_1} \circ T_0)(y)$. This implies that

$$\{y \in S : U_{\beta,0}(x) = (T_1^{n_1} \circ T_0)(y)\} = (T_1^{n_1} \circ T_0)^{-1}(S). \tag{3.3}$$

It remains to show that (3.2) holds in the general case. Obviously

$$\{y \in S : U_{\beta,0}(y) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(y)\} \subseteq (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S). \tag{3.4}$$

To show the opposite inclusion, we examine the formula for $U_{\beta,0}$ more closely. We assume $U_{\beta,0}(x) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(x)$ for some $i \geq 2$. Since $i \geq 2$, $T_0(x)$ lies in a connected component of $[1, 1/(\beta-1)) \setminus \bigcup_{n=1}^{\infty} D_n$, say $\mathcal{I}_1$. We let

$$E := \left\{T_0^{-n}\left(\frac{1}{\beta}\right), T_0^{-n}\left(\frac{1}{\beta(\beta-1)}\right), T_1^{-n}\left(\frac{1}{\beta}\right), T_1^{-n}\left(\frac{1}{\beta(\beta-1)}\right),\right.$$
$$\left. G_\beta^n(1), G_\beta^n\left(\frac{1}{\beta-1}-1\right) : n \geq 0\right\}.$$

Here G_β is the greedy map defined earlier. Since $\beta \in M$, no element of E is in the interior of a C_n, a D_n, or S.

Importantly, $\mathcal{I}_1 = (a_1, b_1)$ where $a_1, b_1 \in E$. In this case, either

$$(a_1,b_1)=\left(1,T_1^{-n_1}\left(\frac{1}{\beta}\right)\right) \quad \text{or} \quad (a_1,b_1)=\left(T_1^{-(n_1-1)}\left(\frac{1}{\beta(\beta-1)}\right),T_1^{-n_1}\left(\frac{1}{\beta}\right)\right).$$

Therefore

$$T_1^k(\mathcal{I}_1) \cap S = \emptyset \quad \text{for } 1 \leq k \leq n_1 - 1 \quad \text{and} \quad T_1^{n_1}(\mathcal{I}_1) \subseteq \left(\frac{2-\beta}{\beta-1}, \frac{1}{\beta}\right).$$

The endpoints of $T_1^{n_1}(\mathcal{I}_1)$ are elements of E, and so not in the interior of any C_n. The point $(T_1^{n_1} \circ T_0)(x)$ is either in C_n for some n, or in $T_1^{n_1}(\mathcal{I}_1) \setminus \bigcup_{n=1}^\infty C_n$. In the latter case, let $\mathcal{I}_2$ be the connected component it is in. Let $\mathcal{I}_2 = (a_2, b_2)$. Then again $a_2, b_2 \in E$. Thus

$$T_0^k(\mathcal{I}_2) \cap S = \emptyset \text{ for } 1 \leq k \leq n_2 - 1 \quad \text{and} \quad T_0^{n_2}(\mathcal{I}_2) \subseteq \left(\frac{1}{\beta(\beta-1)}, 1\right). \tag{3.5}$$

The endpoints of $T_0^{n_2}(\mathcal{I}_2)$ are again in E, and therefore not in the interior of any D_n. The point x is either mapped into a D_n, or contained in a connected component of $T_0^{n_2}(\mathcal{I}_2) \setminus \bigcup_{n=1}^\infty D_n$. In the latter case, we repeat the previous steps. Eventually, x is mapped into either C_{n_i} or D_{n_i}, and our algorithm terminates. Without loss of generality we assume x is eventually mapped into D_{n_i}. The above algorithm yields a finite sequence $(\mathcal{I}_j)_{j=1}^{i-1}$ of intervals which satisfy the following properties:

(1) $\mathcal{I}_1 \subseteq T_0(S)$.
(2) For $1 \leq j \leq i-1$ we have $T_{\omega_j}^k(\mathcal{I}_{n_j}) \cap S = \emptyset$ for $1 \leq k \leq n_j$.
(3) For $1 \leq j \leq i-2$ we have $\mathcal{I}_{j+1} \subseteq T_{\omega_j}^{n_j}(\mathcal{I}_j)$.
(4) $D_{n_i} \subseteq T_{\omega_{i-1}}^{n_{i-1}}(\mathcal{I}_{i-1})$.

Above, $\omega_j = 0$ if j is even and $\omega_j = 1$ if j is odd. These properties have the following consequences:

(5) $(T_{\omega_i}^{n_i})^{-1}(S) \subseteq \mathcal{I}_{i-1}$.
(6) For $1 \leq j \leq i-1$ we have $(T_{\omega_i}^{n_i} \circ \cdots \circ T_{\omega_j}^k)^{-1}(S) \cap S = \emptyset$ for $1 \leq k \leq n_j$.
(7) For $1 \leq j \leq i-1$ we have $(T_{\omega_i}^{n_i} \circ \cdots \circ T_{\omega_j}^{n_j})^{-1}(S) \subseteq \mathcal{I}_{n_j}$.
(8) $(T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S) \subseteq S$.

Properties (5)–(7) imply that every $y \in (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S)$ satisfies $U_{\beta,0}(y) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(y)$. Thus, by (8),

$$(T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)^{-1}(S) \subseteq \{y \in S : U_{\beta,0}(y) = (T_{\omega_i}^{n_i} \circ \cdots \circ T_1^{n_1} \circ T_0)(y)\},$$

which when combined with (3.4) yields (3.2). ∎

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