Ergodicity versus non-ergodicity for Probabilistic Cellular Automata on rooted trees

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October 3, 2017

Abstract

In this article we study a class of shift-invariant and positive rate probabilistic cellular automata (PCA) on rooted d-regular trees $\mathbb{T}^d$. In a first result we extend the results of [10] on trees, namely we prove that to every stationary measure ν of the PCA we can associate a space-time Gibbs measure μ_ν on $\mathbb{Z} \times \mathbb{T}^d$. Under certain assumptions on the dynamics the converse is also true.

A second result concerns proving sufficient conditions for ergodicity and non-ergodicity of our PCA on d-ary trees for $d \in \{1, 2, 3\}$ and characterizing the invariant product Bernoulli measures.

1 Introduction

Cellular Automata (CAs) are discrete-time dynamical systems on a spatially extended discrete space. They are well known for being easy to implement and for exhibiting a rich and complex nonlinear behaviour as emphasized for instance in [32, 34, 14, 13], and they can give rise to multiple levels of organization [12]. Probabilistic Cellular Automata (PCA) are a straightforward generalization of CAs where now the updating rule is considered to be stochastic, see [27]. are employed as modeling tools in a wide range of applications, e.g. HIV infection [23], biological immune system [28], weather forecast [6], heart pacemaker tissue [21], and opinion forming [1]. Moreover, a natural context in which the PCA main ideas are of interest is that of evolutionary games [24, 25, 26].

Strong relations exist as well between PCA and the general equilibrium statistical mechanics framework [33, 22, 15, 5, 9, 10, 11, 29, 30]. A central question is the characterization of the equilibrium behavior of a general PCA dynamics. For instance, one primary interest is the study of its ergodic properties, e.g. the long-term behavior of the PCA and its dependence on the initial probability distribution. Regarding the ergodicity for PCA on infinite lattices, see for instance [29] for details and references. Moreover, conditions for ergodicity for general PCA can be found in the following papers: [15, 7, 16, 19, 20]. Furthermore, in case of a translation-invariant PCA on $\mathbb{Z}^d$ with positive rates, it has been shown in [10] that the law of the trajectories, starting from any stationary distribution, is the Boltzmann-Gibbs distribution for some space-time associated potential (in $\mathbb{Z}^{d+1}$). Moreover, it has also been proven that the converse is true: all the translation-invariant Gibbs states for such potential correspond to statistical space-time histories for the PCA. Therefore, phase transition for the space-time potential is closely related to the PCA ergodicity: non-uniqueness of translation invariant Gibbs states is equivalent to non-uniqueness of stationary measures for the PCA. The main ingredient for proving this result is the use of the local variational principle for the entropy density of the Gibbs measure. However, as it has been proved in [2], the variational principle for

Gibbs states fails for nearest neighbor finite state statistical mechanics systems on 3-ary trees. Hence, a first result to this paper is to extend the results presented by [10] for a class of PCA on infinite rooted trees. In particular, the PCA considered in this paper have non-degenerate shift-invariant local transition probabilities such that each local probabilistic rule depends only on the spins of the children of the node. This class of PCA has generally the Bernoulli product measure as invariant measure, and they are the natural generalization on trees of the models considered in [17].

A second type of results in this paper is to give conditions for ergodicity in case of d-ary trees, with $d \in \{1, 2, 3\}$. Our positive rate PCA satisfy indeed such conditions (i.e. (3.1) and (3.3)) that, when iterating the dynamics from the Bernoulli product measure, the resulting space-time diagram defines non trivial random fields with very weak dependences. This fact allows us to give a detailed analysis of the ergodicity problem and, for two relevant examples of PCA dynamics, we are able to find the critical parameters.

The paper is organized as follows. In Section 2 we extend the results of [10] in case of infinite rooted d-ary trees. We first define the PCA on a countably infinite set and in this general framework we show how stationary measures for a PCA can be naturally associated to Gibbs measures (Theorem 2.1). In order to state the converse result, we first restrict ourselves to the case of infinite rooted trees and to PCA with nondegenerate shift-invariant local transition probabilities that depends only on the spins of the children of the node. For this class of PCA, we state that all the time-invariant Gibbs states for the potential correspond to statistical space-time histories for the PCA (Theorem 2.2). In Section 3 we give results concerning conditions for the ergodicity of the PCA on d-ary trees. First we characterize Bernoulli product stationary measures via Lemma 3.1. In Theorem 3.1 we show that for $d = 1$ the PCA is always ergodic, and the same occurs for $d = 2$ with the additional assumption of spin-flip symmetry of the transition probabilities. In Theorem 3.2 the case of $d = 3$ is studied. We give two examples (Section 3.1.1, Section 3.1.2) where the critical parameters can be computed. Section 5 and the Appendices are devoted to the proofs of the main results.

2 From PCA to Gibbs measures and back

2.1 PCA on countably infinite sets

Let the single spin space be a nonempty finite set S and let V denote a countably infinite set (for example, the d-dimensional cubic lattice $\mathbb{Z}^d$ or, more generally, the vertex set of a countably infinite graph). In the following we introduce a special class of discrete-time Markov chains on the state space $\Omega_0 = S^{\mathsf{V}}$ whose main feature is the fact that all spins are simultaneously and independently updated (*parallel updating*), the so-called probabilistic cellular automata.

We define our probabilistic cellular automaton as follows.

Definition 2.1. *A PCA is a discrete-time Markov chain on Ω_0 with the following properties. At each site i in V*

 (a) *corresponding to each configuration $x \in \Omega_0$ we associate a probability measure $p_i(\cdot|x)$ on S, and*

 (b) *assume that for every spin s, the map*
$$x \mapsto p_i(s|x)$$
  is a local function. So, there is a finite subset $U(i)$ of V such that the equality $p_i(s|x) = p_i(s|y)$ holds for every s whenever x and y satisfy $x_j = y_j$ for each j in $U(i)$.

In this setting, we associate to each point x in Ω_0 the product measure
$$P(dy|x) = \bigotimes_{i \in \mathsf{V}} p_i(dy_i|x), \tag{2.1}$$
and introduce the probabilistic cellular automaton dynamics on our state space Ω_0 by considering the Markov kernel P given by the expression
$$P(x, B) = P(B|x) \tag{2.2}$$
where B is a Borel set of Ω_0.

Now, we recall the definition of a stationary measure for the dynamics P.

Definition 2.2. *A probability measure ν on Ω_0 is called stationary for the dynamics P defined above if and only if*
$$\int P(x, B)\nu(dx) = \nu(B)$$
holds for every Borel set B of Ω_0.

2.2 From PCA to Gibbs measures...

In this section we will show how stationary measures for a PCA can be naturally associated to Gibbs measures for a corresponding equilibrium statistical mechanical model. Let us consider the set of sites given by the countably infinite set $\mathbb{Z} \times \mathsf{V}$, the collection $\mathscr{S}$ consisting of all nonempty finite subsets of $\mathbb{Z} \times \mathsf{V}$. We also consider the configuration space $\Omega = S^{\mathbb{Z} \times \mathsf{V}}$ together with its product σ-algebra $\mathscr{F}$. Given an arbitrary space-time spin configuration ω in Ω, for each site x in $\mathbb{Z} \times \mathsf{V}$, say $\mathsf{x} = (n, i)$, let $\omega_{n,i}$ denote the value ω_{x} of the spin at this site, just for simplicity. Furthermore, for each integer n and each configuration ω, we define the configuration at time n as the element ω_n of Ω_0 given by $\omega_n = (\omega_{n,i})_{i \in \mathsf{V}}$.

Now, let us consider again the setting from the previous section. We will assume that our PCA dynamics is non-degenerate, that is, the local transition probabilities have positive rates: $p_i(s|x) > 0$ holds for all $i \in \mathsf{V}$, $s \in S$ and $x \in \Omega_0$. Furthermore, we also suppose that for each site i, the set

$$\{j \in \mathsf{V} : i \in U(j)\} \tag{2.3}$$

is finite, which means that at each step in the dynamics of the PCA, each spin can have influence only on the future state of a finite number of spins. Given a stationary measure ν for P, it is possible to construct a probability measure μ_ν on $(\Omega, \mathscr{F})$ uniquely determined by the identity

$$\mu_\nu(\omega_t \in B_0, \omega_{t+1} \in B_1, \ldots, \omega_{t+n} \in B_n) = \int_{B_0} \nu(dx_0) \int_{B_1} P(x_0, dx_1) \cdots \int_{B_n} P(x_{n-1}, dx_n), \tag{2.4}$$

where t is an integer, n a positive integer, and $B_0, B_1, \ldots, B_n$ are Borel sets of Ω_0. In the following, given a site x in $\mathbb{Z} \times \mathsf{V}$, say $\mathsf{x} = (n, i)$, we will use $U(\mathsf{x})$ to denote the set

$$U(\mathsf{x}) = \{(n-1, j) : j \in U(i)\}.$$

Observe that our assumption (2.3) is equivalent to say that for each point x, the set

$$\{\mathsf{y} \in \mathbb{Z} \times \mathsf{V} : \mathsf{x} \in U(\mathsf{y})\}$$

is finite. This remark is very useful in the proof of the following theorem, whose prove is given in Appendix A.

Theorem 2.1. *The space-time measure μ_ν obtained from a stationary measure ν for our PCA is a Gibbs measure for the interaction $\Phi = (\Phi_A)_{A \in \mathscr{S}}$, where each $\Phi_A : \Omega \to \mathbb{R}$ is given by*

$$\Phi_A(\omega) = \begin{cases} -\log p_i(\omega_{\mathsf{x}}|\omega_{n-1}) & \text{if } A = \{\mathsf{x}\} \cup U(\mathsf{x}) \text{ for some } \mathsf{x} = (n, i), \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

2.3 PCA on infinite rooted trees

We specify now the class of PCA that will be considered in this paper. We introduce indeed probabilistic cellular automata on d-ary trees $\mathsf{V} = \mathbb{T}^d$ with root $\mathbf{o}$ and degree $\deg(x) = d+1$ for all vertices $x \neq \mathbf{o}$ and $\deg(\mathbf{o}) = d$. Without loss of generality, the d-ary tree $\mathbb{T}^d$ can be regarded as the set

$$\bigcup_{n \geq 0} \{0, \ldots, d-1\}^n$$

consisting of all finite sequence of integers from 0 to $d-1$. Given finite sequences i in $\{0, \ldots, d-1\}^n$ and j in $\{0, \ldots, d-1\}^m$, say $i = (i_k)_{k=0}^{n-1}$ and $j = (j_k)_{k=0}^{m-1}$, we naturally define their sum $i+j$ as the concatenation of these sequences, i.e., the sum is the element of $\{0, \ldots, d-1\}^{m+n}$ given by

$$(i+j)_k = \begin{cases} i_k & \text{if } k \in \{0, \ldots, n-1\}, \\ j_{k-n} & \text{if } k \in \{n, \ldots, m+n-1\}. \end{cases}$$

Once defined the translation on $\mathbb{T}^d$, then we are allowed to associate to each site i in $\mathbb{T}^d$ the shift map $\Theta_i : S^{\mathbb{T}^d} \to S^{\mathbb{T}^d}$ defined by

$$\Theta_i x = (x_{i+j})_{j \in \mathbb{T}^d} \tag{2.6}$$

at each point $x = (x_j)_{j \in \mathbb{T}_d}$. Furthermore, for each $k \in \{0, \ldots, d-1\}$, we denote by e_k the sequence $e_k = (k)$ consisting only of the number k, therefore, the e_k's are the neighbors of the root $\mathbf{o}$ of $\mathbb{T}^d$.

From now on, we consider the single spin space $S = \{-1, +1\}$, so, our state space Ω_0 is described as $\Omega_0 = \{-1, +1\}^{\mathbb{T}^d}$. Moreover, following [31], we give the definitions of attractive dynamics and of repulsive dynamics.

Definition 2.3. *We call the dynamics P attractive if for every positive integer n, for all configurations x, y such that $x \leq y$ and each nondecreasing local function f, we have*

$$P^n(x, f) \leq P^n(y, f). \tag{2.7}$$

Definition 2.4. *We call the dynamics P repulsive if for every positive integer n, for all configurations x, y such that $x \leq y$ and each nondecreasing local function f, we have*

$$P^n(x, f) \geq P^n(y, f). \tag{2.8}$$

By [16, 31] it follows that the dynamics is *attractive* if and only if for all configurations x, y such that $x \leq y$ we have $p_{\mathbf{o}}(+1|x) \leq p_{\mathbf{o}}(+1|y)$, furthermore, it is *repulsive* if and only if for all configurations x, y such that $x \leq y$ we have $p_{\mathbf{o}}(+1|x) \geq p_{\mathbf{o}}(+1|y)$.

The PCA considered in this paper has nondegenerate shift-invariant local transition probabilities such that each probabilistic rule $p_i(\cdot|x)$ depends only on the spins of the children of i. More precisely, we will state the following assumptions on the transition kernel.

Assumptions:

(A1) each $p_{\mathbf{o}}(\cdot|x)$ be a probability measure such that $p_{\mathbf{o}}(s|x) > 0$ holds for all $s \in \{-1, +1\}$,

(A2) the map $x \mapsto p_{\mathbf{o}}(s|x)$ depends only on the values of x on $U(\mathbf{o}) = \{e_0, \ldots, e_{d-1}\}$, and

(A3) for each i in $\mathbb{T}^d \backslash \{\mathbf{o}\}$, the local transition probability $p_i(\cdot|x)$ satisfies

$$p_i(s|x) = p_{\mathbf{o}}(s|\Theta_i x). \tag{2.9}$$

Note that Assumption (A1) is the so-called nondegeneracy property, while Assumption (A3) is the invariance of the PCA dynamics under tree shifts. We remark as well that, it follows from $(A2)$ and $(A3)$ that the map $x \mapsto p_i(s|x)$ depends only on the values assumed by the spins of x on $U(i) = i + \{e_0, \ldots, e_{d-1}\}$. One of the crucial features of this dynamics P is that under Assumptions (A2) and (A3) the relation

$$P^n(x, \{y_F = \xi\}) = \prod_{i \in F} P^n(\Theta_i x, \{y_{\mathbf{o}} = \xi_i\}) \tag{2.10}$$

holds for every configuration x, finite volume configuration $(\xi_i)_{i \in F}$ for some $F \subseteq \mathbb{T}^d$, and positive integer n.

2.4   ...and back

According to Theorem 2.1, every stationary measure for the PCA defined above can be associated to a Gibbs measure for the corresponding statistical mechanical model Φ defined by (2.5). Next, we show that for our class PCA on trees, under suitable conditions, the converse of this problem is also valid.

Theorem 2.2. *Under the Assumption (A1)-(A3), let μ be a Gibbs measure for the interaction Φ defined by (2.5), such that it is invariant under time translations, i.e., μ is a Gibbs measure that satisfies*

$$\mu(\omega_m \in B) = \mu(\omega_{m-1} \in B)$$

for each integer m and each Borel subset B of Ω_0. Then, there is a stationary measure ν for our PCA such that $\mu = \mu_\nu$.

Therefore, thanks to Theorem 2.2 the study of the ergodicity of the PCA can be closely related to the study the uniqueness of the Gibbs measure associated to it.

Remark. *In Appedix A, we give a more general proof for Theorem 2.2. It actually holds for any PCA on $\Omega_0 = S^{\mathsf{V}}$, where S is a nonempty finite set and V is a (locally finite) infinite rooted tree, satisfying $(A1)$ and*

(A2') *Let $d : \mathsf{V} \times \mathsf{V} \to \mathbb{R}$ be the distance function that assigns to each pair (i, j) of vertices the length of the unique path connecting them. Corresponding to each point i that belongs to V the set $U(i)$ is a finite set such that*

$$U(i) \subseteq \{j \in \mathsf{V} : d(\mathbf{o}, i) < d(\mathbf{o}, j)\}. \tag{2.11}$$

3 Conditions for ergodicity for PCA on trees

In this section we will present some results regarding sufficient conditions for the ergodicity for our class of PCA described previously. We first state a lemma regarding the characterization for stationary product measures, whose proof is given in Appendix B.

Lemma 3.1. *A Bernoulli product measure $\nu = \mathsf{Bern}(\mathsf{p})^{\otimes \mathbb{T}^d}$, $\mathsf{p} \in [0,1]$, is an stationary measure for P if and only if*

$$\int p_{\mathbf{o}}(+1|x)\nu(dx) = \mathsf{p} \tag{3.1}$$

i.e. if and only if

$$\sum_{l=0}^{d}(-1)^l\left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\|I|=l}}\left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_k=-1\text{ for all }k\in I}}(-1)^{\#\{m:\xi_m=-1\}}p_{\mathbf{o}}(+1|\xi)\right)\right]\mathsf{p}^{d-l} = \mathsf{p}. \tag{3.2}$$

Moreover, the probability to find the spin $+1$ at the root of $\mathbb{T}^d$ after $n+1$ steps of this dynamics starting from the configuration x can be written as

$$P^{n+1}(x,\{y_{\mathbf{o}}=+1\}) \tag{3.3}$$

$$=\sum_{l=0}^{d}(-1)^l\left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\|I|=l}}\left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_k=-1\text{ for all }k\in I}}(-1)^{\#\{m:\xi_m=-1\}}p_{\mathbf{o}}(+1|\xi)\right)\prod_{k\in\{0,\dots,d-1\}\setminus I}P^n(\Theta_{e_k}x,\{y_{\mathbf{o}}=+1\})\right].$$

3.1 Results end Examples

From now on, we will abbreviate $+1$ by $+$ resp. -1 by $-$. In the first theorem we prove ergodicity results for the line and the binary trees, while in the second theorem we prove ergodicity and non-ergodicity results for the 3-ary trees.

Theorem 3.1. *Let us consider the PCA with transition probabilities satisfying (A1)-(A3). Then, we have the following results.*

(a) *If $d=1$, then the PCA is ergodic. The unique attractive stationary Bernoulli product measure has parameter*

$$\mathsf{p} = \frac{p_{\mathbf{o}}(+|-)}{p_{\mathbf{o}}(-|+)+p_{\mathbf{o}}(+|-)}.$$

(b) *Let $d=2$ and the transition probabilities being symmetric under spin-flip, i.e., $p_{\mathbf{o}}(s|x)=p_{\mathbf{o}}(-s|-x)$. Then the PCA is ergodic, where its unique stationary measure is $\mathsf{Bern}\left(\frac{1}{2}\right)^{\otimes\mathbb{T}^2}$.*

Theorem 3.2. *Let $d=3$ and the transition probabilities be symmetric under spin-flip symmetry. Denote by $\alpha := p_{\mathbf{o}}(+|+++)$ and $\gamma := p_{\mathbf{o}}(+|-++)+p_{\mathbf{o}}(+|+-+)+p_{\mathbf{o}}(+|++-)$. Then the PCA is*

(a) *ergodic, if α and γ satisfy*

   (i) *$1+\alpha-\gamma=0$, or*

   (ii) *the PCA is attractive and $1+\alpha-\gamma\neq0$ and $3\alpha+\gamma\leq5$, or*

   (iii) *the PCA is repulsive and $1+\alpha-\gamma\neq0$ and $3\alpha+\gamma\geq1$.*

  In this case the unique stationary measure is given by $\mathsf{Bern}\left(\frac{1}{2}\right)^{\otimes\mathbb{T}^3}$.

(b) *non-ergodic, if α and γ satisfy*

   (i) *$1+\alpha-\gamma\neq0$ and $3\alpha+\gamma>5$. In this case, we have several stationary Bernoulli product measures with parameter*

$$\mathsf{p}\in\left\{\frac{1}{2},\frac{1+\sqrt{1+\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2},\frac{\sqrt{1-\frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2}\right\},$$

  or

(ii) the PCA is repulsive and $1 + \alpha - \gamma \neq 0$ and $3\alpha + \gamma < 1$.

Remark. *In the last case (Theorem 3.1 (b) − (ii)), we can actually prove that the PCA oscillates between two Bernoulli product measures with distinct paremeters* p.

Before we pass to the proofs of the theorems we will discuss some examples.

3.1.1 Example 1

For $d = 3$ and $\beta > 0$, consider the PCA with transition probabilities given by

$$p_i(s|x) = \frac{1}{2}\left(1 + s\tanh\left(\beta\sum_{k=0}^{2} J_k x_{i+e_k}\right)\right) \tag{3.4}$$

where J_0, J_1 and $J_2 \in \mathbb{R}$. Hence, for suitable values of the constants, there exists a critical $\beta_c \in (0, \infty)$ such that the PCA is ergodic for $\beta \leq \beta_c$ and non-ergodic otherwise. In fact the following result holds.

Proposition 3.1. *Suppose that one of the following conditions on the coupling constants J_0, J_1, J_2 is fulfilled.*

(C1) $J_0, J_1, J_2 > 0$ and $J_0 \leq J_1 + J_2$, $J_1 \leq J_0 + J_2$, and $J_2 \leq J_0 + J_1$.

(C2) $J_0, J_1, J_2 < 0$ and $J_0 \geq J_1 + J_2$, $J_1 \geq J_0 + J_2$, and $J_2 \geq J_0 + J_1$.

Let α, γ be defined as in Theorem 3.2, and let function $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f(\beta) = 3\alpha + \gamma.$$

Then, there exists β_c depending on the constants constants J_0, J_1, J_2 such that for

(a) $\beta \leq \beta_c$ the PCA with transition probabilities given by (4.1) is ergodic, and

(b) $\beta > \beta_c$ the PCA is non-ergodic.

Remark 1. *Note that, thanks to the spin-flip symmetry of the kernel (4.1), we can apply Theorem 3.2. Moreover, we remark that the lattice equivalent of the kernel (4.1) has been extensively studied in [4].*

Remark 2. *If condition (C1) holds, then $\beta_c = f^{-1}(5)$. Otherwise, if (C2) holds, then $\beta_c = f^{-1}(1)$. In particular, if $J_0 = J_1 = J_2 = J \in \mathbb{R}\backslash\{0\}$, it follows that $\beta_c = \frac{1}{2|J|}\log(1 + 2^{2/3})$. In [16] a similar ferromagnetic PCA has been studied on $\mathbb{Z}^2$ where $\beta_c = \frac{1}{2J}\log(1 + \sqrt{2})$.*

3.1.2 Example 2

Let us consider the PCA on the 3-ary tree defined as follows. Suppose that at each step every spin assume the value corresponding to the majority among their children. After that each spin make an error with a probability $\epsilon \in (0, 1)$ independently of each other, that is, if the spin at the site i assumed the value $+1$ (resp. -1), then it will change to -1 (resp. $+1$) with probability ϵ and keep the value $+1$ (resp. -1) with probability $1 - \epsilon$. This type of PCA on trees has been first studied in [31], where non-ergodicity has been proven only for sufficiently small ϵ. In our example we have

$$p_{\mathbf{o}}(+|+++) = p_{\mathbf{o}}(+|++-) = p_{\mathbf{o}}(+|+-+) = p_{\mathbf{o}}(+|-++) = 1 - \epsilon.$$

Proposition 3.2. *There exist two critical values $\epsilon_c^{(1)} = \frac{1}{6}$ and $\epsilon_c^{(2)} = \frac{5}{6}$ such that*

(a) the PCA is ergodic if $\epsilon_c^{(1)} \leq \epsilon \leq \epsilon_c^{(2)}$, and

(b) non-ergodic for $\epsilon \notin [\epsilon_c^{(1)}, \epsilon_c^{(2)}]$.

4 Discussion

In this work we proved the correspondence between stationary measures for PCA on infinite rooted trees and time-invariant Gibbs measures for a corresponding statistical mechanical model. The main implication of this fact is once we establish conditions for uniqueness of Gibbs measures for such system, we guarantee the uniqueness of stationary distributions for the associated PCA. On the other hand, the existence of multiple stationary measures implies on the phase transition in the statistical mechanical model. In this way we provide a partial relationship between ergodicity and phase transition extending, the results from [10].

Restricting to the study of PCA on d-ary trees, we also provide the ergodicity properties of such PCA, fully characterizing the behavior for $d = 1$, $d = 2$ assuming spin-flip symmetry, and $d = 3$ together with the property of attractiveness (resp. repulsiveness) also under the assumption of spin-flip symmetry. The main idea of such study can also be employed for the study of such PCA for any d, since it relies on equations (3.2) and (3.3). For further generalizations, it is necessary to investigate the properties of fixed points the polynom in the left-hand side of equation (3.2).

It is also worth investigating generalizations of PCA from Examples 1 and 2. Note that Theorem 2.1 together with Dobrushin's uniqueness theorem implies that the PCA on $\mathbb{T}^d$ whose transition probabilities are given by

$$p_i(s|x) = \frac{1}{2}\left(1 + s \tanh\left(\beta \sum_{k=0}^{d-1} J_k x_{i+e_k}\right)\right) \tag{4.1}$$

there is a unique stationary measure given by $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^d}$ for β large enough, suggesting the ergodicity at high temperatures.

5 Proofs of Ergodicity results

5.1 Proof of Theorem 3.1

5.1.1 Case (a)

Proof. Note that the PCA on $\mathbb{T}^1$ is equivalent to a PCA model on $\mathbb{Z}_+$. In order to simplify the computations, let us use a and b to denote $p_\mathbf{o}(+1|+1)$ and $p_\mathbf{o}(+1|-1)$, respectively. Since the local transition probabilities have positive rates, then, we have $|a - b| < 1$. It follows that for each point x in Ω_0, we have

$$\begin{aligned} P^{n+1}(x, \{y_\mathbf{o} = +1\}) &= \int P(z, \{y_\mathbf{o} = +1\}) P^n(x, dz) \\ &= a \cdot P^n(x, \{y_{e_0} = +1\}) + b \cdot P^n(x, \{y_{e_0} = -1\}) \\ &= (a-b) \cdot P^n(x, \{y_{e_0} = +1\}) + b \\ &= (a-b) \cdot P^n(\Theta_{e_0} x, \{y_\mathbf{o} = +1\}) + b \end{aligned}$$

for each positive integer n. Note that the relation above can also be obtained by means of equation (3.3). Thus, the quantity above can be expressed as

$$P^n(x, \{y_\mathbf{o} = +1\}) = (a-b)^{n-1} \cdot p_\mathbf{o}(+1|\Theta_{\underbrace{e_0 + \cdots + e_0}_{n-1 \text{ times}}} x) + b \cdot \sum_{k=0}^{n-2} (a-b)^k.$$

It follows that for any initial configuration x, $P^n(x, \{y_\mathbf{o} = +1\})$ converges to $\mathsf{p} = \frac{b}{1-(a-b)}$ as n approaches infinity. Therefore, using equation (2.10), we conclude that this PCA is ergodic, where its unique attractive stationary measure is $\mathsf{Bern}(\mathsf{p})^{\otimes \mathbb{T}^1}$. $\qquad\square$

5.1.2 Case (b)

Proof. Let $a, b \in (0, 1)$ defined by $a = p_\mathbf{o}(+|-, -) = 1 - p_\mathbf{o}(+|+, +)$ and $b = p_\mathbf{o}(+|-, +) = 1 - p_\mathbf{o}(+|+, -)$, respectively. Let us show that $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^2}$, in fact, is the unique attractive stationary measure. According to equation (3.3), we have

$$P^{n+1}(x, \{y_\mathbf{o} = +1\}) = (1 - b - a) P^n(\Theta_{e_0} x, \{y_\mathbf{o} = +1\}) + (b - a) P^n(\Theta_{e_1} x, \{y_\mathbf{o} = +1\}) + a.$$

By induction, we can show that

$$P^n(x, \{y_\mathbf{o} = +1\}) = \sum_{i \in \{0,1\}^{n-1}} (1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}} P(\Theta_i x, \{y_\mathbf{o} = +1\})$$

$$+a \sum_{l=0}^{n-2} \sum_{i \in \{0,1\}^l} (1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}}.$$

Using the fact that for any real numbers p and q, the relation

$$\sum_{i \in \{0,1\}^l} p^{\#\{k:i_k=0\}} q^{\#\{k:i_k=1\}} = (p+q)^l$$

holds for every nonnegative integer l, it follows that

$$P^n(x, \{y_\mathbf{o} = +1\}) = \sum_{i \in \{0,1\}^{n-1}} (1-b-a)^{\#\{k:i_k=0\}}(b-a)^{\#\{k:i_k=1\}} P(\Theta_i x, \{y_\mathbf{o} = +1\}) + a \sum_{l=0}^{n-2} (1-2a)^l. \quad (5.1)$$

Since the absolute value of the first term of equation (5.1) is bounded by

$$\sum_{i \in \{0,1\}^{n-1}} |1-b-a|^{\#\{k:i_k=0\}} |b-a|^{\#\{k:i_k=1\}} = (|1-b-a|+|b-a|)^{n-1},$$

then

$$\lim_{n \to \infty} P^n(x, \{y_\mathbf{o} = +1\}) = a \sum_{l=0}^{\infty} (1-2a)^l = \frac{1}{2}. \quad (5.2)$$

Therefore, by means of equation (2.10), we conclude that $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^2}$ as the unique attractive stationary measure of the PCA. □

5.2 Proof of Theorem 3.2

5.2.1 Case (a)-(i) and (b)-(i)

Proof. Recall we abbreviated $\alpha = p_\mathbf{o}(+|+++)$ and $\gamma = p_\mathbf{o}(+|-++) + p_\mathbf{o}(+|+-+) + p_\mathbf{o}(+|++-)$. From Lemma 3.1 we know that a stationary product measure has to satisfy

$$\int p_\mathbf{o}(+1|x)\nu(dx) = \mathsf{p} \quad (5.3)$$

which was equivalent to solving equation (3.2), i.e.

$$2(1+\alpha-\gamma)\mathsf{p}^3 - 3(1+\alpha-\gamma)\mathsf{p}^2 + (3\alpha-\gamma-1)\mathsf{p} + (1-\alpha) = 0. \quad (5.4)$$

Since $\mathsf{p} = \frac{1}{2}$ is a solution for the equation above, then, it can be written as

$$2\left(\mathsf{p} - \frac{1}{2}\right)\left[(1+\alpha-\gamma)\mathsf{p}^2 - (1+\alpha-\gamma)\mathsf{p} - (1-\alpha)\right] = 0. \quad (5.5)$$

Suppose that $1+\alpha-\gamma = 0$. Let $0 < \alpha < 1$, then, analogously as in the previous case, we have

$$P^n(x, \{y_\mathbf{o} = +1\})$$

$$= \sum_{i \in \{0,1,2\}^{n-1}} (\alpha - p_\mathbf{o}(+|++-))^{\#\{k:i_k=0\}}(\alpha - p_\mathbf{o}(+|+-+))^{\#\{k:i_k=1\}}(\alpha - p_\mathbf{o}(+|-++))^{\#\{k:i_k=2\}} P(\Theta_i x, \{y_\mathbf{o} = +1\})$$

$$+(1-\alpha)\sum_{l=0}^{n-2} \sum_{i \in \{0,1,2\}^l} (\alpha - p_\mathbf{o}(+|++-))^{\#\{k:i_k=0\}}(\alpha - p_\mathbf{o}(+|+-+))^{\#\{k:i_k=1\}}(\alpha - p_\mathbf{o}(+|-++))^{\#\{k:i_k=2\}}.$$

The equation above implies that $P^n(x, \{y_\mathbf{o} = +1\}) \to \frac{1}{2}$, therefore, the PCA is ergodic.

Now, if $1+\alpha-\gamma \neq 0$, we have two other complex solutions

$$\mathsf{p}_+ = \frac{1 + \sqrt{1 + \frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2} \quad (5.6)$$

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and

$$\mathsf{p}_- = \frac{1 - \sqrt{1 + \frac{4(1-\alpha)}{1+\alpha-\gamma}}}{2}. \tag{5.7}$$

Therefore, both p_+ and p_+ are inside the interval $(0, 1)$ and are different from $\frac{1}{2}$ if and only if $3\alpha + \gamma > 5$. Note that it is excluded that $\mathsf{p}_- \notin (0, 1)$ and $\mathsf{p}_+ \in (0, 1)$ resp. the other way around. $\qquad\square$

5.2.2 Case (a)-(ii)

Proof. Let us consider an attractive PCA. Again, by using Lemma 3.1, we can find a map $F : [0, 1] \to \mathbb{R}$

$$F(\mathsf{p}) = 2(1 + \alpha - \gamma)\mathsf{p}^3 - 3(1 + \alpha - \gamma)\mathsf{p}^2 + (3\alpha - \gamma)\mathsf{p} + (1 - \alpha). \tag{5.8}$$

such that its fixed points are the parameters of the product Bernoulli measures. We will show that F has a unique attractive fixed point at $\mathsf{p} = \frac{1}{2}$. Note that

$$F'(\mathsf{p}) = 6(1 + \alpha - \gamma)\mathsf{p}^2 - 6(1 + \alpha - \gamma)\mathsf{p} + (3\alpha - \gamma), \tag{5.9}$$

$$F''(\mathsf{p}) = 12(1 + \alpha - \gamma)\mathsf{p} - 6(1 + \alpha - \gamma), \tag{5.10}$$

and

$$F''\left(\frac{1}{2}\right) = -\frac{1}{2}(-3 + 3\alpha + \gamma). \tag{5.11}$$

Let us prove that F is an increasing function that satisfies

$$\begin{cases} F(\mathsf{p}) > \mathsf{p} & \text{for all } \mathsf{p} < \frac{1}{2}, \\ F(\frac{1}{2}) = \frac{1}{2} & \text{and} \\ F(\mathsf{p}) < \mathsf{p} & \text{for all } \mathsf{p} > \frac{1}{2}. \end{cases} \tag{5.12}$$

Suppose that $1 + \alpha - \gamma < 0$. Since the PCA is attractive, it follows that $3\alpha \geq \gamma$ and the minimum value of F given by $F'(0) = F'(1) = 3\alpha - \gamma$ is nonnegative. Therefore, F is increasing. Moreover, the property (5.12) follows from the identity

$$F(\mathsf{p}) - \mathsf{p} = 2\left(\mathsf{p} - \frac{1}{2}\right)\left[(1 + \alpha - \gamma)\mathsf{p}^2 - (1 + \alpha - \gamma)\mathsf{p} - (1 - \alpha)\right] \tag{5.13}$$

where $(1 + \alpha - \gamma)\mathsf{p}^2 - (1 + \alpha - \gamma)\mathsf{p} - (1 - \alpha) < 0$ for all $\mathsf{p} \neq \frac{1}{2}$. Now, let us consider the case where $1 + \alpha - \gamma > 0$. The attractiveness of the PCA implies that $\gamma \geq 3(1 - \alpha)$, so, the minimum value of F' is $F'(\frac{1}{2}) = -3 + 3\alpha + \gamma \geq 0$. Again, we prove that F is increasing. Furthermore, we have (5.12) by means of the equation

$$F(\mathsf{p}) - \mathsf{p} = 2\left(\mathsf{p} - \frac{1}{2}\right)(1 + \alpha - \gamma)(\mathsf{p} - \mathsf{p}_-)(\mathsf{p} - \mathsf{p}_+) \tag{5.14}$$

where $\mathsf{p}_- < 0$ and $\mathsf{p}_+ > 1$ are given by equation (5.7) and (5.6), respectively. Since F is increasing, $F(0) = 1 - \alpha < \frac{1}{2}$ and $F(1) = \alpha > \frac{1}{2}$, then $F(x)$ belongs to $\left[1 - \alpha, \frac{1}{2}\right) \subseteq \left[0, \frac{1}{2}\right)$ for all x in $\left[0, \frac{1}{2}\right)$ and $F(x)$ belongs to $\left(\frac{1}{2}, \alpha\right] \subseteq \left(\frac{1}{2}, 1\right]$ for all x in $\left(\frac{1}{2}, 1\right]$. Using the continuity of F, we easily conclude that $\lim_{n\to\infty} F^n(\mathsf{p}) = \frac{1}{2}$ for every point p that belongs to the interval $[0, 1]$, therefore, $\mathsf{p} = \frac{1}{2}$ is the unique attractive fixed point for F.

It follows from the conclusion above that both $P^n(x_-, \{y_\mathbf{o} = +1\})$ and $P^n(x_+, \{y_\mathbf{o} = +1\})$ converge to $\frac{1}{2}$ as n approaches infinity, where x_- and x_+ are respectively the configurations with all spins -1 and $+1$ on $\mathbb{T}^3$. Therefore, since the inequality $x_- \leq x \leq x_+$ holds for every configuration x, it follows from Definition 2.3 that:

$$P^n(x_-, \{w_\mathbf{o} = +1\}) \leq P^n(x, \{w_\mathbf{o} = +1\}) \leq P^n(x_+, \{w_\mathbf{o} = +1\}), \tag{5.15}$$

therefore,

$$\lim_{n\to\infty} P^n(x, \{w_\mathbf{o} = +1\}) = \frac{1}{2}. \tag{5.16}$$

We conclude that $P^n(x, \cdot)$ converges to $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^3}$, hence the PCA is ergodic. $\qquad\square$

5.2.3 Case (a)-(iii) and (b)-(ii)

Proof. Let us consider a new PCA described by a probability kernel Q defined by

$$Q(dy|x) = \bigotimes_{i \in \mathbb{T}^3} q_i(dy_i|x), \qquad (5.17)$$

where each probability q_i is given by

$$q_i(\,\cdot\,|x) = p_i(\,\cdot\,|-x). \qquad (5.18)$$

It is easy to see that this PCA satisfies the spin-flip condition. In the case where we have both $\alpha \neq 0$ and $3\alpha + \gamma \geq 1$, if we consider α' and γ' respectively defined by $\alpha' = q_\mathbf{o}(+1|+++)$ and $\gamma' = q_\mathbf{o}(+1|++-) + q_\mathbf{o}(+1|+-+) + q_\mathbf{o}(+1|-++)$, then we have

$$1 + \alpha' - \gamma' = -(1 + \alpha - \gamma) \neq 0,$$

$$3\alpha' + \gamma' = 6 - (3\alpha + \gamma) \leq 5 \text{ and } \alpha' \neq 1$$

Therefore, in this case the PCA described by Q is ergodic. It is easy to check that $P^n(x, \cdot) = Q^n((-1)^n x, \cdot)$ hold for every positive integer n and each configuration x. Therefore, the ergodicity of P follows.

In order to prove the non-ergodicity for the case $3\alpha + \gamma < 1$, let us consider again the function $F : [0,1] \to \mathbb{R}$ given by equation (5.8). Because of our assumptions, we have $3\alpha - \gamma \leq 0$ and $F'(\frac{1}{2}) = \frac{1}{2}(-3 + 3\alpha + \gamma) < -1$, thus, F is a decreasing function. It follows that F is a function such that $F(x)$ belongs to the interval $[\alpha, 1 - \alpha] \subseteq [0, 1]$ for every point x in $[0, 1]$ and the absolute value of its derivative evaluated at the unique fixed point satisfies $|F'(\frac{1}{2})| > 1$. Therefore, the point $\mathsf{p} = \frac{1}{2}$ is not an attractive fixed point. We conclude that $P^n(x_+, \cdot)$ does not converges to the stationary measure $\mathsf{Bern}(\frac{1}{2})^{\otimes \mathbb{T}^3}$, so, the PCA is not ergodic. $\qquad \square$

5.3 Proof of Proposition 3.1

Proof. Note that the PCA is fully described by the numbers

$$\begin{aligned}
p_\mathbf{o}(+|+++) &= \tfrac{1}{2}\left(1 + \tanh \beta(J_0 + J_1 + J_2)\right), \\
p_\mathbf{o}(+|++-) &= \tfrac{1}{2}\left(1 + \tanh \beta(J_0 + J_1 - J_2)\right), \\
p_\mathbf{o}(+|+-+) &= \tfrac{1}{2}\left(1 + \tanh \beta(J_0 - J_1 + J_2)\right),
\end{aligned}$$

and

$$p_\mathbf{o}(+|-++) = \frac{1}{2}\left(1 + \tanh \beta(-J_0 + J_1 + J_2)\right).$$

Note that the assumptions from Example 3.1.1 imply that $J_0 + J_1 - J_2 < J_0 + J_1 + J_2$, $J_0 - J_1 + J_2 < J_0 + J_1 + J_2$, and $-J_0 + J_1 + J_2 < J_0 + J_1 + J_2$; and at most one of the quatities $J_0 + J_1 - J_2$, $J_0 - J_1 + J_2$ and $-J_0 + J_1 + J_2$ can be equal zero. Therefore, the map $g : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned}
g(\beta) &= 1 + \alpha - \gamma \\
&= \frac{1}{2}(\tanh \beta(J_0 + J_1 + J_2) - \tanh \beta(J_0 + J_1 - J_2) - \tanh \beta(J_0 - J_1 + J_2) - \tanh \beta(J_0 - J_1 + J_2))
\end{aligned}$$

satisfies

$$g(0) = 0$$

and

$$\begin{aligned}
g'(\beta) &= \frac{1}{2}\left(\frac{J_0 + J_1 + J_2}{\cosh^2 \beta(J_0 + J_1 + J_2)} - \frac{J_0 + J_1 - J_2}{\cosh^2 \beta(J_0 + J_1 - J_2)} - \frac{J_0 - J_1 + J_2}{\cosh^2 \beta(J_0 - J_1 + J_2)} \right. \\
&\qquad \left. - \frac{-J_0 + J_1 + J_2}{\cosh^2 \beta(-J_0 + J_1 + J_2)} \right) \\
&< \frac{1}{2 \cosh^2 \beta(J_0 + J_1 + J_2)} \left((J_0 + J_1 + J_2) - (J_0 + J_1 - J_2) - (J_0 - J_1 + J_2) - (-J_0 + J_1 + J_2) \right) = 0.
\end{aligned}$$

It follows that $g(\beta) = 1 + \alpha - \gamma < 0$ for all $\beta > 0$. Moreover, note that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned}
f(\beta) &= 3\alpha + \gamma \\
&= 3 + \frac{3}{2}\tanh \beta(J_0 + J_1 + J_2) + \frac{1}{2}(\tanh \beta(J_0 + J_1 - J_2) + \tanh \beta(J_0 - J_1 + J_2) + \tanh \beta(-J_0 + J_1 + J_2))
\end{aligned}$$

is increasing, $f(0) = 0$, and $\lim_{\beta \to \infty} f(\beta) \geq 5 + \frac{1}{2}$. It follows that there is a unique positive real number β_c that satisfies $f(\beta_c) = 5$. Since this PCA has the spin-flip property and is attractive, according to Theorem 3.1.1, the PCA is ergodic for $\beta \leq \beta_c$ and non-ergodic for $\beta > \beta_c$. $\qquad \square$

5.4 Proof of Proposition 3.2

Proof. Clearly the PCA satisfies the spin-flip property. Note that in both cases we have $1 + \alpha - \gamma = 2\epsilon - 1$. It follows that the PCA is ergodic for $\epsilon = \frac{1}{2}$. Furthermore, note that the PCA is attractive for $0 < \epsilon < \frac{1}{2}$, repulsive for $\frac{1}{2} < \epsilon < 1$, and in both cases we have $1 + \alpha - \gamma \neq 0$.

Let us suppose that $\epsilon \in (0, \frac{1}{2})$. Since $3\alpha + \gamma = 6(1 - \epsilon)$, it follows from Theorem 3.1.1 that the PCA is non-ergodic for $\epsilon < \frac{1}{6}$ and ergodic for $\frac{1}{6} \leq \epsilon < \frac{1}{2}$. Now, if $\epsilon \in (\frac{1}{2}, 1)$, then again by Theorem 3.1.1, the PCA is ergodic for $\frac{1}{2} < \epsilon \leq \frac{5}{6}$ and non-ergodic for $\frac{5}{6} < \epsilon < 1$. $\qquad\square$

A Appendix

A.1 Proof of Theorem 2.1

Before we follow to the proof of Theorem 2.1 it will be convenient to construct a special sequence $(\Delta_n)_{n \in \mathbb{N}}$ of subsets of $\mathbb{Z} \times \mathsf{V}$. Given a positive integer n and a nonempty finite subset F of V, let us define a subset $\Delta(n, F)$ of $\mathbb{Z} \times \mathsf{V}$ as follows. Let Λ_n be the set given by

$$\Lambda_n = \{(n, i) : i \in F\},$$

and for each integer $m < n$ let

$$\Lambda_m = \bigcup_{\mathsf{x} \in \Lambda_{m+1}} U(\mathsf{x}) \cup \{(m, i) : i \in F\}$$

Then, we define $\Delta(n, F)$ by

$$\Delta(n, F) = \bigcup_{m=-n}^{n} \Lambda_m.$$

Remark. *Observe that*

(a) $\Delta(n, F)$ *is a finite subset of* $\mathbb{Z} \times \mathsf{V}$,

(b) *we have* $\{-n, \ldots, 0, \ldots, n\} \times F \subseteq \Delta(n, F) \subseteq \{-n, \ldots, 0, \ldots, n\} \times \mathsf{V}$, *and*

(c) *for every point* x *in* $\Delta(n, F)$, *if* $\pi_{\mathbb{Z}}(\mathsf{x}) \neq -n$, *then* $U(\mathsf{x}) \subseteq \Delta(n, F)$.

Now, if φ is a one-to-one function from $\mathbb{N}$ onto V, then let

$$\Delta_1 = \Delta(1, \{\varphi(1)\}), \tag{A.1}$$

and

$$\Delta_{n+1} = \Delta(n + 1, \pi_{\mathsf{V}}(\Delta_n) \cup \{\varphi(n + 1)\}) \tag{A.2}$$

for each positive integer n. Observe that $(\Delta_n)_{n \in \mathbb{N}}$ is an increasing sequence of elements of $\mathscr{S}$ such that $\mathbb{Z} \times \mathsf{V} = \bigcup_{n \in \mathbb{N}} \Delta_n$.

Lemma A.1. *Let* $\Delta = \Delta_m$ *for some* $m \in \mathbb{N}$, *and let* $\overline{\Delta}$ *be an element of* $\mathscr{S}$ *defined by*

$$\overline{\Delta} = \bigcup_{\substack{\mathsf{x} \in \Delta \\ \pi_{\mathbb{Z}}(\mathsf{x}) = -m}} U(\mathsf{x}).$$

Given a finite volume configuration ξ *in* S^{Δ}, *the measure* λ^{ξ} *on* $(\Omega, \mathscr{F}_{\overline{\Delta}})$ *defined by*

$$\lambda^{\xi}(B) = \int_B \prod_{\mathsf{x}=(n,i) \in \Delta} p_i(\xi_{\mathsf{x}} | (\xi \omega_{\Delta^c})_{n-1}) \, \mu_{\nu}(d\omega) \tag{A.3}$$

can be expressed as

$$\lambda^{\xi}(B) = \int_B \mathbb{1}_{[\xi]}(\omega) \mu_{\nu}(d\omega). \tag{A.4}$$

Proof of Lemma A.1. It suffices to show the identity for cylinder sets of the form $[\zeta]$, where each ζ belongs to $S^{\overline{\Delta}}$. The result follows by using the fact that the map

$$\omega \mapsto \prod_{\mathsf{x}=(n,i) \in \Delta} p_i(\xi_{\mathsf{x}} | (\xi \omega_{\Delta^c})_{n-1})$$

depends only on the values of ω assumed on $\overline{\Delta}$. $\qquad\square$

Proof of Theorem 2.1. Let us fix a set $\Lambda \in \mathscr{S}$ and a finite volume configuration σ in S^Λ. Let $\Delta = \Delta_m$ for some positive integer m such that

$$\{\mathsf{x} \in \mathbb{Z} \times \mathsf{V} : (\{\mathsf{x}\} \cup U(\mathsf{x})) \cap \Lambda \neq \emptyset\} \subseteq \Delta_m.$$

Then, for each ω in Ω, we have

$$
\begin{aligned}
e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})} &= \prod_{\substack{\mathsf{x}=(n,i) \\ (\{\mathsf{x}\}\cup U(\mathsf{x}))\cap\Lambda\neq\emptyset}} p_i((\sigma\omega_{\Lambda^c})_\mathsf{x}|(\sigma\omega_{\Lambda^c})_{n-1}) \\
&= \frac{\displaystyle\prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\omega_{\Lambda^c})_\mathsf{x}|(\sigma\omega_{\Lambda^c})_{n-1})}{\displaystyle\prod_{\substack{\mathsf{x}=(n,i)\in\Delta \\ (\{\mathsf{x}\}\cup U(\mathsf{x}))\cap\Lambda=\emptyset}} p_i(\omega_\mathsf{x}|\omega_{n-1})},
\end{aligned}
$$

thus

$$
\frac{e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\omega_{\Lambda^c})}} = \frac{\displaystyle\prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\omega_{\Lambda^c})_\mathsf{x}|(\sigma\omega_{\Lambda^c})_{n-1})}{\displaystyle\sum_{\sigma'\in S^\Lambda}\prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma'\omega_{\Lambda^c})_\mathsf{x}|(\sigma'\omega_{\Lambda^c})_{n-1})}. \tag{A.5}
$$

Now, given a finite volume configuration η in $S^{\Delta\setminus\Lambda}$, using equation (A.5), we obtain

$$
\begin{aligned}
\int_{[\eta]} \mathbb{1}_{[\sigma]}(\omega)\mu_\nu(d\omega) &= \lambda^{\sigma\eta}(\Omega) = \int \prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\sigma\eta)_\mathsf{x}|(\sigma\eta\omega_{\Delta^c})_{n-1})\,\mu_\nu(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \prod_{\mathsf{x}=(n,i)\in\Delta} p_i((\zeta\eta)_\mathsf{x}|(\zeta\eta\omega_{\Delta^c})_{n-1})\,\mu_\nu(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \lambda^{\zeta\eta}(d\omega) \\
&= \sum_{\zeta\in S^\Lambda}\int \frac{e^{-H_\Lambda^\Phi(\sigma\eta\omega_{\Delta^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\eta\omega_{\Delta^c})}} \mathbb{1}_{[\zeta\eta]}\mu_\nu(d\omega) \\
&= \int_{[\eta]} \frac{e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\omega_{\Lambda^c})}} \mu_\nu(d\omega).
\end{aligned}
$$

Since $(\Delta_n)_{n\in\mathbb{N}}$ is an increasing sequence of elements of $\mathscr{S}$ such that $\mathbb{Z} \times \mathsf{V} = \bigcup_{n\in\mathbb{N}} \Delta_n$, it follows that the equality

$$
\mu_\nu([\sigma]|\mathscr{F}_{\Lambda^c})(\omega) = \frac{e^{-H_\Lambda^\Phi(\sigma\omega_{\Lambda^c})}}{\displaystyle\sum_{\sigma'\in S^\Lambda} e^{-H_\Lambda^\Phi(\sigma'\omega_{\Lambda^c})}} \tag{A.6}
$$

holds for μ_ν-almost every point ω in Ω. $\qquad\square$

A.2   Proof of Theorem 2.2

Let m and N be integers, where $N \geq 0$, and let us consider the set

$$\Delta = \{m\} \times \{j \in \mathsf{V} : d(\mathbf{o}, j) \leq N\}. \tag{A.7}$$

If we consider the nonempty finite subset Λ of $\mathbb{Z} \times \mathsf{V}$ given by

$$\Lambda = \bigcup_{l=0}^N \{m+l\} \times \{j \in \mathsf{V} : d(\mathbf{o}, j) \leq N - l\}, \tag{A.8}$$

it follows that

$$
\begin{aligned}
e^{-H_\Lambda^\Phi(\xi\omega_{\Lambda^c})} &= \prod_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_\mathsf{x}|(\xi\omega_{\Lambda^c})_{n-1}) \cdot \prod_{\substack{\mathsf{x}=(n,i)\notin\Lambda \\ U(\mathsf{x})\cap\Lambda\neq\emptyset}} p_i(\omega_\mathsf{x}|(\xi\omega_{\Lambda^c})_{n-1}) \\
&= \prod_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_\mathsf{x}|(\xi\omega_{\Lambda^c})_{n-1}) \cdot \prod_{\substack{\mathsf{x}=(n,i)\notin\Lambda \\ U(\mathsf{x})\cap\Lambda\neq\emptyset}} p_i(\omega_\mathsf{x}|\omega_{n-1})
\end{aligned}
$$

holds for all finite volume configuration ξ in S^Λ and for every ω in Ω. Since μ is a Gibbs measure, then for μ-almost every point ω in Ω we have

$$\mu([\xi]|\mathscr{F}_{\Lambda^c})(\omega) = \frac{\prod\limits_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_\mathsf{x}|(\xi\omega_{\Lambda^c})_{n-1})}{\sum\limits_{\eta\in S^\Lambda}\prod\limits_{\mathsf{x}=(n,i)\in\Lambda} p_i(\eta_\mathsf{x}|(\eta\omega_{\Lambda^c})_{n-1})} = \prod_{\mathsf{x}=(n,i)\in\Lambda} p_i(\xi_\mathsf{x}|(\xi\omega_{\Lambda^c})_{n-1}),$$

and summing over all possible spins inside the volume $\Lambda\backslash\Delta$, we conclude that

$$\mu([\xi_\Delta]|\mathscr{F}_{\Lambda^c})(\omega) = \prod_{\mathsf{x}=(m,i)\in\Delta} p_i(\xi_\mathsf{x}|\omega_{m-1}). \tag{A.9}$$

If we define the σ-algebra $\mathscr{F}_{<m}$ as the σ-algebra $\mathscr{F}_{\Gamma(m)}$ of subsets of Ω, where $\Gamma(m) = \{\mathsf{x}\in S : \pi_{\mathbb{Z}}(\mathsf{x}) < m\}$, it follows from (A.9) that

$$\mu(\{\omega'\in\Omega : \omega'_m \in B\}|\mathscr{F}_{<m})(\omega) = P(B|\omega_{m-1}) \tag{A.10}$$

holds for μ-almost every ω in Ω and for any measurable subset B of Ω_0.

Since μ is invariant under time translations, it follows that the measure ν on $(\Omega_0, \mathscr{B}(\Omega_0))$ defined by

$$\nu(B) = \mu(\{\omega'\in\Omega : \omega'_m \in B\}) \tag{A.11}$$

does not depends on the choice of the integer m, moreover, it is easy to show that ν is a stationary measure for the PCA. Using equation (A.10) and Kolmogorov consistency theorem, we finally conclude that $\mu = \mu_\nu$.

B Appendix

Proof of Lemma 3.1. Let us proof that given a function $a : \{-1,+1\}^d \to \mathbb{R}$ and a probability measure μ on $\{-1,+1\}^{\mathbb{T}^d}$, we have

$$\sum_{\xi\in\{-1,+1\}^d} a(\xi) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k) \tag{B.1}$$

$$= \sum_{l=0}^d (-1)^l \left[\sum_{\substack{I\subseteq\{0,\dots,d-1\}\\|I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1 \text{ for all } m\in I}} (-1)^{\#\{m:\xi_m=-1\}} a(\xi) \right) \prod_{k\in\{0,\dots,d-1\}\backslash I} \mu(x_{e_k} = +1) \right]$$

We prove the equation above by induction. For the case where $d=1$, we proof is straightforward. If we suppose that the result is proven for d, then

$$\sum_{\xi\in\{-1,+1\}^{d+1}} a(\xi) \prod_{k\in\{0,\dots,d\}} \mu(x_{e_k} = \xi_k)$$

$$= \sum_{\xi\in\{-1,+1\}^d} a(\xi,+1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = +1)$$

$$+ \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = -1)$$

$$= \sum_{\xi\in\{-1,+1\}^d} a(\xi,+1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = +1)$$

$$- \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)\cdot\mu(x_{e_d} = +1)$$

$$+ \sum_{\xi\in\{-1,+1\}^d} a(\xi,-1) \prod_{k\in\{0,\dots,d-1\}} \mu(x_{e_k} = \xi_k)$$

$$
\begin{aligned}
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d-1\}\\|I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi,+1) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad - \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d-1\}\\|I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi,-1) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad + \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d-1\}\\|I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi,-1) \right) \prod_{k\in\{0,\ldots,d-1\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l,d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi,+1) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad - \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l,d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^d\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi,-1) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad \sum_{l=0}^{d}(-1)^{l+1} \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l+1,d\in I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l,d\notin I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&\quad \sum_{l=0}^{d}(-1)^{l+1} \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l+1,d\in I}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right] \\
&= \sum_{l=0}^{d+1}(-1)^l \left[\sum_{\substack{I\subseteq\{0,\ldots,d\}\\|I|=l}} \left(\sum_{\substack{\xi\in\{-1,+1\}^{d+1}\\\xi_m=-1\text{ for all }m\in I}} (-1)^{\#\{m:\xi_m=-1\}}a(\xi) \right) \prod_{k\in\{0,\ldots,d\}\setminus I} \mu(x_{e_k}=+1) \right].
\end{aligned}
$$

Therefore the result follows.

If we consider the the particular case where $a(\xi) = p_{\mathbf{o}}(+1|\xi)$ and $\mu = \mathsf{Bern}(\mathsf{p})^{\otimes\mathbb{T}^d}$ that satisfies (3.1), then equation (3.2) follows. Now, if we let $a(\xi) = p_{\mathbf{o}}(+1|\xi)$ and $\mu = P^n(x,\cdot)$, then equations (2.10) and (B.1) implies equation (3.3). $\qquad\square$

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