Switching to nonhyperbolic cycles from codim-2 bifurcations of equilibria in DDEs

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Summary. Using the framework of dual semigroups, the existence of a finite dimensional smooth center manifold for DDEs can be rigorously established [1]. This makes it is possible to apply the normalization method for local bifurcations of ODEs [2] to DDEs. Recently, the critical normal form coefficients for all five codimension 2 bifurcation of equilibria in generic DDEs have been derived [7] and implemented into the `Octave/Matlab` package `DDE-BifTool` [5]. We generalize a center manifold theorem from [1] to generic parameter-dependent DDEs, covering the cases where the critical equilibrium can disappear. It allows us to initialize the continuation of codimension 1 equilibrium and nonhyperbolic cycle bifurcations emanating from the generalized Hopf, zero-Hopf and Hopf-Hopf bifurcations in DDEs, which are the only codim 2 eqillibrium bifurcations in generic DDEs where nonhyperbolic cycles could originate. The obtained expressions have been implemented in `DDE-BifTool` and tested on various models.

Parameter-dependent normalization

Consider the parameter-dependent delay differential equation (DDE) for $x(t) \in \mathbb{R}^n$ of the form

$$\begin{cases} \dot{x}(t) = f(x_t, \alpha), \\ x_0 = \phi, \end{cases} \tag{1}$$

where $x_t(\theta) = x(t+\theta)$, with $\theta \in [-h, 0]$, represents the solution in the past and $\alpha \in \mathbb{R}^m$ the parameters. Here f is a smooth map from the Banach space $X = C([-h,0], \mathbb{R}^n)$ into $\mathbb{R}^n$ and $h > 0$ is assumed to be finite. We assume that there are finitely many constant delays $0 = \tau_0 < \tau_1 < \cdots < \tau_r = h$. Solutions to (1) define a semiflow and are in one-to-one correspondence to solutions of an *abstract integral equation* [1]. Let $\varphi_0 \equiv 0 \in X$ at $\alpha_0 = 0 \in \mathbb{R}^m$ be a equilibrium of (1). Suppose that the generator A of the linear part $T(t)$ of the semiflow near the equilibrium at $\alpha = 0$ has n_c eigenvalues on the imaginary axis. Let X_0 be the finite dimensional eigenspace corresponding to these eigenvalues. Then there exists a locally invariant parameter-dependent center manifold $\mathcal{W}_{loc}(\alpha) \subset X$ that is tangent at $\alpha = 0$ to $X_0(\alpha)$ on which the solutions satisfy the abstract ODE

$$\dot{u}(t) = j^{-1}(A^{\odot *} j u(t) + (D_2 f(0,0)\alpha) r^{\odot *} + R(u(t), \alpha)). \tag{2}$$

Let $X^{\odot}$ be the subspace of the dual space X^* on which the adjoint semigroup T^* is strongly continuous. Then $T^{\odot *}$ is the adjoint semigroup of $T^{\odot} := T^*|_{X^{\odot}}$ and $A^{\odot *}$ denotes the generator of the semigroup $T^{\odot *}$. The nonlinearity $R : X \times \mathbb{R}^m \to X^{\odot *}$ in this equation (2) is defined by the nonlinear terms of f via the natural injection and $j : X \to X^{\odot *}$. Finally, $D_2 f(0,0)$ represents the derivative with respect to the parameters and $r^{\odot *} = (I, 0)$ if we identify $X^{\odot *}$ with $\mathbb{R}^n \times L^{\infty}([-h,0], \mathbb{R}^n)$. Let $y(t)$ be the projection of $u(t)$ onto X_0. Since X_0 is spanned by some basis Φ of (generalized) eigenvectors, we can express $y(t)$ uniquely relative to Φ. The corresponding coordinate vector $z(t)$ of $y(t)$ satisfies some ODE that is smoothly equivalent to the normal form

$$\dot{z} = G(z, \beta) = \sum_{|\nu|=1}^{N} \sum_{|\mu|=0}^{M} \frac{1}{\nu! \mu!} g_{\nu\mu} z^{\nu} \beta^{\mu} + \mathcal{O}(\|z\|^{N+1} \|\beta\|^{M+1}), \tag{3}$$

with unknown normal form coefficients $g_{\nu\mu} \in \mathbb{R}^{n_c}$ and parameters β. Here ν and μ are multi-indices of length n and m respectively. The series is supposed to be truncated after some sufficiently high order N and M. The nonlinearity can expanded by

$$R(u, \alpha) = \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{j! k!} D_2^k D_1^j f(0,0)(\overbrace{u, \ldots, u}^{j \text{ times}}, \overbrace{\alpha, \ldots \alpha}^{k \text{ times}}) r^{\odot *}, \tag{4}$$

where $D_2^k D_1^j f(0,0)(\overbrace{u, \ldots, u}^{j \text{ times}}, \overbrace{\alpha, \ldots \alpha}^{k \text{ times}})$ is the jth order Fréchet derivative of f with respect to its first argument and the kth order derivative of f with respect to its second argument evaluated at the point $(0,0) \in X \times \mathbb{R}^m$.
Let $\mathcal{H} : V \subset \mathbb{R}^{n_c} \times \mathbb{R}^m \to X$ be a smooth mapping with image $\mathcal{W}_{loc}(\alpha)$. Then $\mathcal{H}$ admits the expansion

$$\mathcal{H}(z, \beta) = \sum_{|\nu|=1}^{N} \sum_{|\mu|=0}^{M} \frac{1}{\nu! \mu!} H_{\nu\mu} z^{\nu} \beta^{\mu} + \mathcal{O}(\|z\|^{N+1} \|\beta\|^{M+1}). \tag{5}$$

The invariance of the local center manifold $\mathcal{W}_{loc}(\alpha)$ implies the relation $u(t) = \mathcal{H}(z(t), \beta)$. Differentiating both sides of this relation with respect to time yields the *homological equation*

$$A^{\odot *} j \mathcal{H}(z, \beta) + r^{\odot *}(D_2 f(0,0)\alpha) + R(\mathcal{H}(z, \beta), \alpha) = j(D_z \mathcal{H}(z, \beta) \dot{z}). \tag{6}$$

To relate the parameters α to the parameters β, we define the mapping $\alpha = K(\beta)$. We expand K as

$$K(\beta) = \sum_{|\mu|=1}^{N} \frac{1}{\mu!} K_\mu \beta^\mu. \tag{7}$$

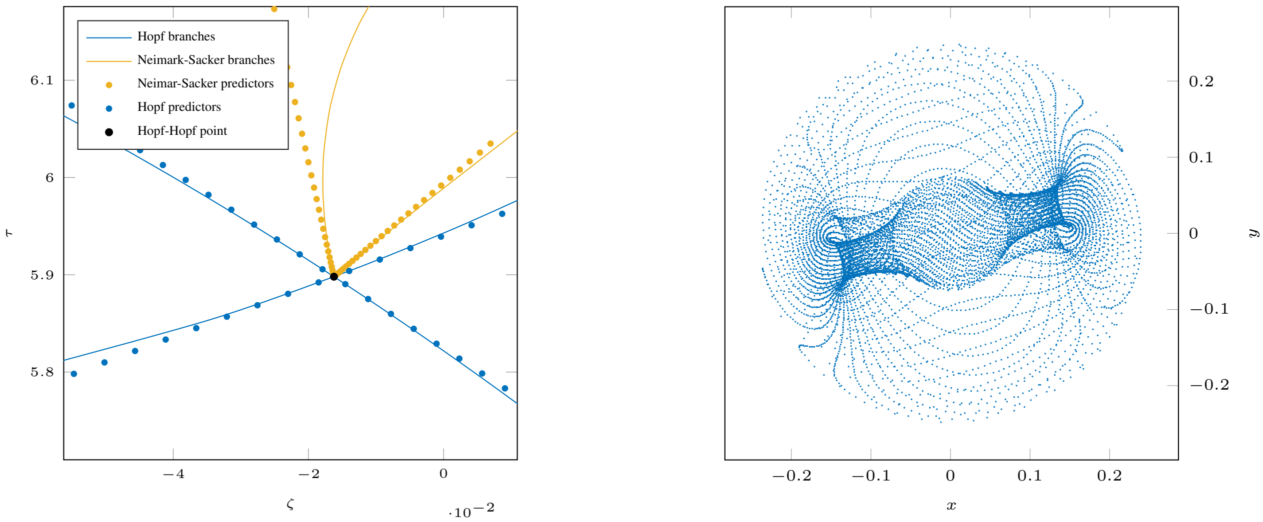
Substituting (3), (5) and (7) into (6) and equating coefficients of the same order in z and β, one can solve recursively for the unknown coefficients $g_{\nu\mu}$, $H_{\nu\mu}$ and K_μ by applying the Fredholm solvability condition, and taking inverses or bordered inverses. Using the obtained approximations of the mappings $\mathcal{H}$ and K, we transfer the asymptotics of the nonhyperbolic cycles in the normal forms derived in [3] to the original DDE (1).

Example: Active control system

In [4] the following system with $g_u = 0.1, g_v = 0.52$ and $\beta = 0.1$, is considered

$$\begin{cases} \dot{x} = \tau y(t), \\ \dot{y} = \tau \left(-x(t) - g_u x(t-1) - 2\zeta y(t) - g_v y(t-1) + \beta x^3(t-1)\right). \end{cases} \tag{8}$$

The trivial equilibrium undergoes a Hopf-Hopf bifurcation at the parameter values $(\zeta_c, \tau_c) = (-0.016225, 5.89802)$. Using `DDE-BifTool` we compute its stability and normal form coefficients. We obtain the eigenvalues $0.0000 \pm 4.5275i$ and $-0.0000 \pm 7.6449i$. The quadratic critical normal form coefficients reveal that $(\text{Re } g_{2100})(\text{Re } g_{0021}) = -0.0166 < 0$. We conclude that this Hopf-Hopf bifurcation is of 'difficult' type. Since the quantities are such that $\theta = -1.7009, < 0$, $\delta = -2.3517 < 0$, $\theta\delta > 0$ it follows that we are in case VI [6] implying existence of a stable three-dimensional torus, see Figure 1.



(a) Predictors in parameter-space

(b) Three-dimensional torus in (8)

Figure 1

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