



Homogenization of layered materials with rigid components in single-slip finite crystal plasticity

Fabian Christowiak¹ · Carolin Kreisbeck²

Received: 12 April 2016 / Accepted: 30 March 2017 / Published online: 28 April 2017
© The Author(s) 2017. This article is an open access publication

Abstract We determine the effective behavior of a class of composites in finite-strain crystal plasticity, based on a variational model for materials made of fine parallel layers of two types. While one component is completely rigid in the sense that it admits only local rotations, the other one is softer featuring a single active slip system with linear self-hardening. As a main result, we obtain explicit homogenization formulas by means of Γ -convergence. Due to the anisotropic nature of the problem, the findings depend critically on the orientation of the slip direction relative to the layers, leading to three qualitatively different regimes that involve macroscopic shearing and blocking effects. The technical difficulties in the proofs are rooted in the intrinsic rigidity of the model, which translates into a non-standard variational problem constraint by non-convex partial differential inclusions. The proof of the lower bound requires a careful analysis of the admissible microstructures and a new asymptotic rigidity result, whereas the construction of recovery sequences relies on nested laminates.

Mathematics Subject Classification 49J45 (primary) · 74Q05, 74C15

1 Introduction

The search for new materials with desirable mechanical properties is one of the key tasks in materials science. As suitable combinations of different materials may exceed their individual constituents with regard to important characteristics, like strength, stiffness or ductility,

Communicated by L. Ambrosio.

✉ Carolin Kreisbeck
c.kreisbeck@uu.nl

Fabian Christowiak
fabian.christowiak@mathematik.uni-regensburg.de

¹ Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

² Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands

composites play an important role in material design, e.g. [28,36,45]. In this pursuit, the following question is of fundamental interest: Given the arrangement and geometry of the building blocks on a mesoscopic level, as well as the deformation mechanisms inside the homogeneous components, can we predict the macroscopic material response of a sample under some applied external load?

By now there are various homogenization methods available that help to give answers. A substantial body of literature has emerged in materials science, engineering, and mathematics, see for instance [27,36] and the references therein, or more specifically, [23,29,40,44] for heterogeneous plastic materials, and [2,33,35] for fiber-reinforced materials, and [6,10] for high-contrast composites, to mention just a few references.

A rigorous analytical approach that has proven successful for variational models based on energy minimization principles rests on the concept of Γ -convergence introduced De Giorgi and Franzoni [19,20]. By letting the length scale of the heterogeneities tend towards zero, one passes to a limit energy, which gives rise to the effective material model.

In this paper, we follow along these lines and study a variational model for reinforced bilayered materials in the context of geometrically nonlinear plasticity. The model is set in the plane and we assume that the material consists of periodically alternating strips of rigid components and softer ones that can be deformed plastically by single-slip. As this problem is highly anisotropic, considering the layered structure and the distinguished orientation of the slip system, there are interesting interactions to be observed.

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, modeling the reference configuration of an elastoplastic body in two space dimensions, and let $u : \Omega \rightarrow \mathbb{R}^2$ be a deformation field. For describing the periodic material heterogeneities, we take the unit cell $Y = [0, 1)^2$, and define for $\lambda \in (0, 1)$ the subsets

$$Y_{\text{soft}} = [0, 1) \times [0, \lambda) \subset Y \quad \text{and} \quad Y_{\text{rig}} = Y \setminus Y_{\text{soft}}, \quad (1.1)$$

which correspond to the softer and rigid component, respectively. Throughout this paper, we identify the sets Y_{rig} and Y_{soft} with their Y -periodic extensions to \mathbb{R}^2 . To provide a measure for the length scale of the oscillations between the material components, we introduce the parameter $\varepsilon > 0$, which describes the thickness of two neighboring layers. With these notations, the sets $\varepsilon Y_{\text{rig}} \cap \Omega$ and $\varepsilon Y_{\text{soft}} \cap \Omega$ refer to the stiff and softer layers. For an illustration of the geometric set-up see Fig. 1.

Following the classic work by Kröner and Lee [30,31] on finite-strain crystal plasticity, we use the multiplicative decomposition of the deformation gradient $\nabla u = F_e F_p$ as a fundamental assumption. Here, the elastic part F_e describes local rotation and stretching of the crystal lattice, and the inelastic part F_p captures local plastic deformations resulting from the movement of dislocations. Recent progress on a rigorous derivation of the above splitting as the continuum limit of micromechanically defined elastic and plastic components has been made in [41,42].

In this model, proper elastic deformations are excluded by requiring F_e to be (locally) a rotation, i.e. $F_e \in SO(2)$ pointwise. This lack of elasticity makes the overall material fairly rigid. For the plastic part, we impose $F_p = \mathbb{I}$ on $\varepsilon Y_{\text{rig}} \cap \Omega$, reflecting that there is no plastic deformation in the stiff layers. In the softer layers $\varepsilon Y_{\text{soft}} \cap \Omega$, plastic glide can occur along one active slip system (s, m) with slip direction $s \in \mathbb{R}^2$ with $|s| = 1$ and slip plane normal $m = s^\perp$, so that integration of the plastic flow rule yields $F_p = \mathbb{I} + \gamma s \otimes m$, where $\gamma \in \mathbb{R}$ corresponds to the amount of slip, for more details see [15, Section 2]. Altogether, we observe that the deformation gradient ∇u is restricted pointwise to the set

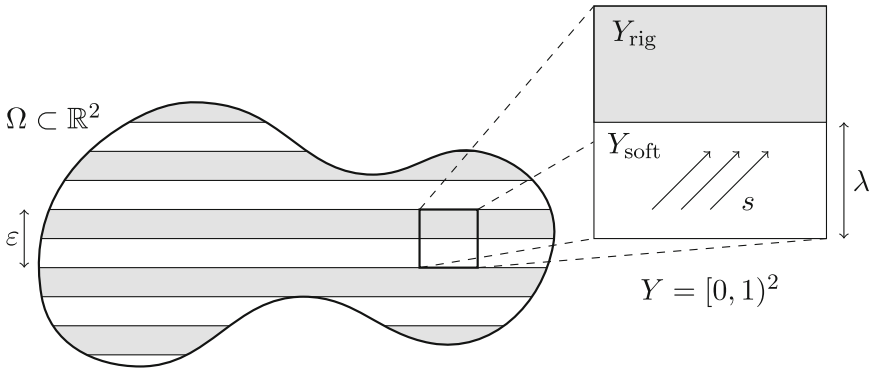


Fig. 1 Bilayered elastoplastic material with periodic structure; rigid components depicted in gray, softer components with one active slip system (slip direction s) in white

$$\begin{aligned} \mathcal{M}_s &= \{F \in \mathbb{R}^{2 \times 2} : F = R(\mathbb{I} + \gamma s \otimes m), R \in SO(2), \gamma \in \mathbb{R}\} \\ &= \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| = 1\}, \end{aligned}$$

and in the stiff components even to $SO(2)$.

As regards relevant energy expressions, the latter entails that the energy density in the rigid layers is given by $W_{\text{rig}}(F) = 0$ if $F \in SO(2)$ and $W_{\text{rig}}(F) = \infty$ otherwise in $\mathbb{R}^{2 \times 2}$. Moreover, adopting the homogeneous single-slip model with linear self-hardening introduced in [13] (cf. also [14]) gives rise to the condensed energy density in the softer layers

$$W_{\text{soft}}(F) = \begin{cases} \gamma^2 = |Fm|^2 - 1 & \text{if } F = R(\mathbb{I} + \gamma s \otimes m) \in \mathcal{M}_s, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{2 \times 2}. \quad (1.2)$$

We combine the energy contributions in the two components to obtain the heterogeneous density

$$W(y, F) = \mathbb{1}_{Y_{\text{rig}}}(y)W_{\text{rig}}(F) + \mathbb{1}_{Y_{\text{soft}}}(y)W_{\text{soft}}(F), \quad y \in \mathbb{R}^2, F \in \mathbb{R}^{2 \times 2}, \quad (1.3)$$

which is periodic with respect to the unit cell Y and reflects the bilayered structure of the material. Here, $\mathbb{1}_U$ is the symbol for the characteristic function of a set $U \subset \mathbb{R}^2$.

According to [9, 39], the dynamical behavior of plastic materials under deformation can be well approximated by incremental minimization, that is by a time-discrete variational approach (for earlier work in the context of fracture and damage see [24, 25]). Note that in this paper, we discuss only the first time step. This simplification suppresses delicate issues of microstructure evolution. As system energy of the first incremental problem we consider the energy functional $E_\varepsilon : L^2_0(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ for $\varepsilon > 0$ defined by

$$E_\varepsilon(u) = \int_\Omega W\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx, \quad u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2), \quad (1.4)$$

and $E_\varepsilon(u) = \infty$ otherwise in $L^2_0(\Omega; \mathbb{R}^2)$, the space of L^2 -functions with vanishing mean value. By (1.3) and (1.2), one has the following equivalent representations of E_ε ,

$$\begin{aligned} E_\varepsilon(u) &= \int_\Omega \gamma^2 dx \quad \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma s \otimes m) \\ &\quad \text{with } R \in L^\infty(\Omega; SO(2)), \gamma \in L^2(\Omega), \gamma = 0 \text{ a.e. in } \varepsilon Y_{\text{rig}} \cap \Omega, \end{aligned} \quad (1.5)$$

$$\begin{aligned}
 &= \int_{\Omega} |(\nabla u)m|^2 - 1 \, dx \quad \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \nabla u \in \mathcal{M}_s \text{ a.e. in } \Omega, \\
 &\qquad \qquad \qquad \nabla u \in SO(2) \text{ a.e. in } \varepsilon Y_{\text{rig}} \cap \Omega,
 \end{aligned} \tag{1.6}$$

and $E_\varepsilon(u) = \infty$ otherwise in $L^2_0(\Omega; \mathbb{R}^2)$. It becomes apparent from (1.6) that the functionals E_ε are subject to non-convex constraints in the form of partial differential inclusions. Even though E_ε matches with an integral expression with quadratic integrand when finite, the constraints render the associated homogenization problem non-standard. In particular, it is not directly accessible by classical homogenization methods for variational integrals with quadratic growth as e.g. in [4,37]. Due to the non-convexity of the sets \mathcal{M}_s and $SO(2)$, it does not fall within the scope of works on gradient-constraint problems like [7,8,12], either. For a study of homogenization problems that involve constraints imposed by special linear first order partial differential equations we refer to [5,18,21].

Our main result is the following theorem, which holds under the additional assumption that Ω is simply connected. It amounts to an explicit characterization of the Γ -limit of $(E_\varepsilon)_\varepsilon$ as ε tends to zero (for an introduction to Γ -convergence see e.g. [3,17]), and therefore, provides the desired homogenized model that describes the effective material response in the limit of vanishing layer thickness.

Theorem 1.1 (Homogenization via Γ -convergence) *The family $(E_\varepsilon)_\varepsilon$ Γ -converges to a functional $E : L^2_0(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ with respect to the strong $L^2(\Omega; \mathbb{R}^2)$ -topology, in formulas, $\Gamma(L^2)$ - $\lim_{\varepsilon \rightarrow 0} E_\varepsilon = E$, where E is defined by*

$$E(u) = \begin{cases} \frac{s_1^2}{\lambda} \int_{\Omega} \gamma^2 \, dx - 2s_1s_2 \int_{\Omega} \gamma \, dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \text{ with} \\ & R \in SO(2), \gamma \in L^2(\Omega), \gamma \in K_{s,\lambda} \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{1.7}$$

The pointwise restriction $K_{s,\lambda}$ for $s = (s_1, s_2)$ and $\lambda \in (0, 1)$ is given by

$$K_{s,\lambda} = \begin{cases} \{0\} & \text{if } s = e_2, \\ \left[-2\frac{s_1}{s_2}\lambda, 0\right] & \text{if } s_1s_2 > 0, \\ \left[0, -2\frac{s_1}{s_2}\lambda\right] & \text{if } s_1s_2 < 0, \\ \mathbb{R} & \text{if } s = e_1. \end{cases} \tag{1.8}$$

Moreover, bounded energy sequences of $(E_\varepsilon)_\varepsilon$, i.e. $(u_\varepsilon)_\varepsilon$ with $E_\varepsilon(u_\varepsilon) < C$ for all $\varepsilon > 0$, are relatively compact in $L^2_0(\Omega; \mathbb{R}^2)$.

Recalling the definition of Γ -convergence, Theorem 1.1 can be formulated in terms of these three statements:

(Compactness) For $\varepsilon_j \rightarrow 0$ and $(u_j)_j \subset L^2_0(\Omega; \mathbb{R}^2)$ with $E_{\varepsilon_j}(u_j) < C$ for all $j \in \mathbb{N}$, there exists a subsequence of $(u_j)_j$ (not relabeled) and $u \in L^2_0(\Omega; \mathbb{R}^2)$ with $E(u) < \infty$ such that $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$.

(Lower bound) Let $\varepsilon_j \rightarrow 0$ and $(u_j)_j \subset L^2_0(\Omega; \mathbb{R}^2)$ with $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$ for some $u \in L^2_0(\Omega; \mathbb{R}^2)$. Then,

$$\liminf_{j \rightarrow \infty} E_{\varepsilon_j}(u_j) \geq E(u). \tag{1.9}$$

(Recovery sequences) For every $u \in L^2_0(\Omega; \mathbb{R}^2)$ with $E(u) < \infty$, there exists $(u_\varepsilon)_\varepsilon \subset L^2_0(\Omega; \mathbb{R}^2)$ with $u_\varepsilon \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$ such that $\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) = E(u)$.

Remark 1.2 In comparison with E_ε , the differential constraints in the formulation of E are substantially more restrictive, and cause the limit functional to be essentially one-dimensional. While the gradients of finite-energy deformations for E_ε lie pointwise in the set \mathcal{M}_s , those for E take values in \mathcal{M}_{e_1} , independent of s , and satisfy the additional restriction of a constant rotation. In particular, this implies that $\partial_1 \gamma = 0$, as gradient fields are curl-free. Notice also that the second term in E is non-negative due to the pointwise restriction $\gamma \in K_{s,\lambda}$.

Remark 1.3 (Generalizations of Theorem 1.1) (a) Except for only minor changes, the quadratic growth in the energies E_ε can be replaced by p -growth with $p \geq 2$. Calling the modified functionals E^p_ε , we have that $E^p = \Gamma(L^p)$ - $\lim_{\varepsilon \rightarrow 0} E^p_\varepsilon$ is characterized by

$$E^p(u) = \int_\Omega \frac{1}{\lambda^{p-1}} |(\nabla u)m - (1 - \lambda)Rm|^p - \lambda \, dx,$$

if $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$, $\gamma \in L^p(\Omega)$ and $\gamma \in K_{s,\lambda}$ a.e. in Ω , and $E^p(u) = \infty$ otherwise in $L^p_0(\Omega; \mathbb{R}^2)$. For $p = 2$ this is a reformulation of (1.7).

(b) In the case $s = e_1$, we characterize the Γ -limit of the family $(E^t_\varepsilon)_\varepsilon$ defined in (4.1), which results from $(E_\varepsilon)_\varepsilon$ by adding a linear dissipative term with prefactor $\tau \geq 0$. For the details see Sect. 4. We remark that this extension is motivated by [15] and [13].

If $s \neq e_1$, it is not clear how to perform the above generalization. The technical challenge lies in the construction of recovery sequences, which builds upon a suitable relaxation result. For the model involving the sum of a quadratic and linear term, however, this problem is (to the best of our knowledge) still unsolved. The reason is that different microstructures are optimal for the respective types of hardening, for more details see also the discussion in [13].

In contrast, homogenization in the special case $s = e_1$ gets by without microstructure formation and can therefore manage the mixed expression.

In the special cases, where the slip direction is parallel or orthogonal to the layered structure, the result of Theorem 1.1 reflects some basic physical intuition. While for $s = e_2$ the effective body can only be rotated as a whole, as the rigid layers lead to a complete blocking of the slip system, the slip system is unimpeded if $s = e_1$, so that, macroscopically, (up to global rotations) exactly all shear deformations in horizontal direction can be achieved. If the slip direction is inclined, i.e. $s \notin \{e_1, e_2\}$, the pointwise restriction $\gamma \in K_{s,\lambda}$ implies both that the effective horizontal shearing is only uni-directional (with the relevant direction depending on the orientation of s), which indicates a loss of symmetry, and that its maximum amount is capped. In the limit energy, the factor s^2_1/λ in front of the quadratic expression in γ corresponds to an effective hardening modulus, recalling that $\lambda \in (0, 1)$ stands for the relative thickness of the softer material layers. For $s \notin \{e_1, e_2\}$, one observes (maybe surprisingly) an additional energy contribution that is linear in γ , which can be interpreted as a dissipative term.

Regarding the proof of Theorem 1.1, we perform the usual three steps for Γ -convergence results by showing compactness and establishing matching upper and lower bounds.

The key to compactness and the lower bound is to capture the macroscopic effects of the bilayered material structure, which lead to an anisotropic reinforcement of the elastoplastic body. In Proposition 2.1, we establish a new type of asymptotic rigidity result, which is not specific to the context of plasticity, but potentially applies to any kind of composite with

rigid layers, provided that the macroscopic material response is a priori known to be volume-preserving. The reasoning relies on a well-known result by Reshetnyak (cf. Lemma 2.3), which implies that the stiff layers can only rotate as a whole, on an explicit estimate showing that rotations on neighboring rigid layers are close, and on a suitable one-dimensional compactness argument. As a consequence of Proposition 2.1, the weak limits of finite energy sequences for $(E_\varepsilon)_\varepsilon$ coincide necessarily with globally rotated shear deformations in e_1 -direction. Gradients of the latter have the form $R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with a constant rotation $R \in SO(2)$ and scalar valued function γ . Note that this result holds for any orientation of the slip system s .

For the upper bound, we construct recovery sequences, meaning sequences of admissible deformations for $(E_\varepsilon)_\varepsilon$ that are energetically optimal in the limit $\varepsilon \rightarrow 0$. If $s = e_1$, the construction is quite intuitive, one simply compensates for the rigid layers by gliding more in the softer components, namely by a factor $1/\lambda$. Analogue constructions for $s \neq e_1$ are in general not compatible, which makes this case more involved. After suitable approximation and localization, we may focus on affine limit deformations u with gradient $\nabla u = F \in \mathcal{M}_{e_1} \cap \mathcal{M}_s^{\text{qc}}$, where $\mathcal{M}_s^{\text{qc}}$ denotes the quasiconvex hull of \mathcal{M}_s , cf. (3.3). The observation that admissible sequences which are affine on all layers do not exist due to a lack of appropriate rank-one connections between \mathcal{M}_s and $SO(2)$ (see Lemma 3.1 and [11]) motivates to drop the assumption of admissibility at first. Indeed, functions with piecewise constant gradients oscillating between the larger set $\mathcal{M}_s^{\text{qc}}$ and $SO(2)$ yield asymptotically optimal energy values. Finally, to make this construction admissible, we glue fine simple laminates with gradients in \mathcal{M}_s into the softer layers, ensuring the preservation of the affine boundary values. This approximating laminate construction, as well as the adaption argument for the boundary, is based on work by Conti and Theil [13, 15], which uses, in particular, convex integration in the sense of Müller and Šverák [38].

The manuscript is organized as follows. In Sect. 2, we state and prove the asymptotic rigidity result along with a useful corollary. These are the essential ingredients for proving our main result. We collect some preliminaries on admissible macroscopic deformations in Sect. 3, including both necessary conditions and relevant construction tools for laminates that are needed for finding recovery sequences. After these preparations, we proceed with the proof of Theorem 1.1, which is subdivided into two sections. Section 4 covers the simpler case $s = e_1$ in a slightly generalized setting, and Sect. 5 gives the detailed proofs for $s \neq e_1$. Finally, Sect. 6 briefly discusses the relation between the limit functional E and (multi)cell formulas. *Notation* The standard unit vectors in \mathbb{R}^2 are denoted by e_1, e_2 , and $a^\perp = (-a_2, a_1)$ for $a = (a_1, a_2) \in \mathbb{R}^2$. For the tensor product between vectors $a, b \in \mathbb{R}^2$ we write $a \otimes b = ab^T \in \mathbb{R}^{2 \times 2}$. Further, let $|F| = (FF^T)^{1/2}$ be the Frobenius norm of $F \in \mathbb{R}^{2 \times 2}$. With $[t]$ and $\lceil t \rceil$, let us denote the smallest integer not less and largest integer not greater than $t \in \mathbb{R}$, respectively. For a set $U \subset \mathbb{R}^2$, the characteristic function $\mathbb{1}_U$ is given by $\mathbb{1}_U(x) = 1$ for $x \in U$, and $\mathbb{1}_U(x) = 0$ if $x \notin U$. When referring to a domain $\Omega \subset \mathbb{R}^2$, we mean that Ω is an open, connected, and nonempty define.

Using the standard notation for Lebesgue and Sobolev spaces, we set $L_0^2(\Omega; \mathbb{R}^2) = \{u \in L^2(\Omega; \mathbb{R}^2) : \int_\Omega u \, dx = 0\}$, and let $W_\#^{1,2}(Q; \mathbb{R}^2)$ with a cube $Q \subset \mathbb{R}^2$ stand for the space of $W^{1,2}(Q; \mathbb{R}^2)$ -functions with periodic boundary conditions. (Weak) partial derivatives regarding the i th variable are denoted by ∂_i , and $\nabla u = (\partial_1 u | \partial_2 u) \in \mathbb{R}^{2 \times 2}$ for a vector field $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. In the two-dimensional setting of this paper, the curl operator is defined as follows, $\text{curl } F = \partial_2 F e_1 - \partial_1 F e_2$ for $F : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$.

Notice that we often use generic constants, so that the value of a constant may vary from one line to the other. Moreover, families indexed with $\varepsilon > 0$, may refer to any sequence $(\varepsilon_j)_j$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow 0$.

2 Asymptotic rigidity of materials with stiff layers

In this section, we examine the qualitative effect of rigid layers on the macroscopic material response of the composite. The following result provides quite restrictive structural information on volume-preserving effective deformations.

Proposition 2.1 (Asymptotic rigidity for layered materials) *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Suppose that the sequence $(u_\varepsilon)_\varepsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ satisfies $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ as $\varepsilon \rightarrow 0$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\det \nabla u = 1$ a.e. in Ω , and*

$$\nabla u_\varepsilon \in SO(2) \text{ a.e. in } \Omega \cap \varepsilon Y_{\text{rig}}$$

for all $\varepsilon > 0$ with Y_{rig} as defined in (1.1). Then there exists a matrix $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ such that

$$\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2). \tag{2.1}$$

Furthermore,

$$\nabla u_\varepsilon \mathbb{1}_{\varepsilon Y_{\text{rig}} \cap \Omega} \rightharpoonup |Y_{\text{rig}}| R \text{ in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \tag{2.2}$$

Remark 2.2 (a) Considering the model introduced in Sect. 1, any weakly converging sequence $(u_\varepsilon)_\varepsilon$ of bounded energy for $(E_\varepsilon)_\varepsilon$ as defined in (1.5) fulfills the requirements of Proposition 2.1. Indeed, if $E_\varepsilon(u_\varepsilon) < C$ for $\varepsilon > 0$, then $u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ and

$$\nabla u_\varepsilon = R_\varepsilon(\mathbb{I} + \gamma_\varepsilon s \otimes m)$$

with $R_\varepsilon \in L^\infty(\Omega; SO(2))$ and $\gamma_\varepsilon \in L^2(\Omega)$ such that $\gamma_\varepsilon = 0$ a.e. in $\Omega \cap \varepsilon Y_{\text{rig}}$, which particularly entails that $\det \nabla u_\varepsilon = 1$ a.e. in Ω .

As a consequence of the weak continuity of the Jacobian determinant (precisely, $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(Q; \mathbb{R}^2)$ implies $\det \nabla u_\varepsilon \xrightarrow{*} \det \nabla u$ in the sense of measures, see e.g. [22] and the references therein), the weak limit function u satisfies the volume constraint $\det \nabla u = 1$ a.e. in Ω .

In fact, (2.1) provides a necessary condition for the class of admissible deformations in the effective limit model. It indicates that, macroscopically, (up to a global rotation) only horizontal shear can be achieved.

(b) Notice that due to the gradient structure of ∇u in (2.1), the function γ is independent of x_1 in the sense that its distributional derivative $\partial_1 \gamma$ vanishes. This follows immediately from $0 = \text{curl } \nabla u = -\partial_1 \gamma R e_1$.

The outline of the proof of Proposition 2.1 is as follows. First, we conclude from the well-known rigidity result in Lemma 2.3, applied to the connected components of $\Omega \cap \varepsilon Y_{\text{rig}}$, that each stiff layer can only be rotated as a whole. The resulting rotation matrices are then used to construct a sequence of one-dimensional piecewise constant auxiliary functions for which we establish compactness and from which we obtain structural information on ∇u . More precisely, as a consequence of the explicit estimate in Lemma 2.4, the rotations of neighboring stiff layers are close for small ε , and the auxiliary sequence has bounded variation. By

Helly’s selection principle one can extract a pointwise converging subsequence whose limit function lies in $SO(2)$ a.e. in Ω , since lengths are preserved in this limit passage. Along with $\det \nabla u = 1$, this observation translates into the representation $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_1)$ with $R \in SO(2)$ a.e. in Ω . Finally, to prove that R is constant, we exploit essentially the gradient structure of ∇u .

Before giving the detailed arguments, let us briefly state one of the key tools, which, in its classical version, is also known as Liouville’s theorem. The first proof in the context of Sobolev maps goes back to Reshetnyak [43], for a quantitative generalization of the result we refer to [26, Theorem 3.1].

Lemma 2.3 (Rigidity for Sobolev functions) *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\nabla u(x) \in SO(2)$ for a.e. $x \in \Omega$. Then u is harmonic and there is a constant rotation $R \in SO(2)$ such that $\nabla u(x) = R$ for all $x \in \Omega$. In particular, $u(x) = Rx + b$ for some $b \in \mathbb{R}^2$.*

An explicit estimate of the distance between rotations of neighboring stiff layers is given in the following lemma.

Lemma 2.4 *Let $P = (0, L) \times (0, H)$ with $L, H > 0$. For $i = 1, 2$, let $w_i : P \rightarrow \mathbb{R}^2$ be the affine functions defined by $w_i(x) = R_i x + b_i$ with $R_i \in SO(2)$ and $b_i \in \mathbb{R}^2$.*

If $u \in W^{1,2}(P; \mathbb{R}^2)$ is such that

$$u = w_1 \text{ on } \partial P \cap \{x_2 = 0\} \text{ and } u = w_2 \text{ on } \partial P \cap \{x_2 = H\}$$

in the sense of traces, then

$$\int_P |(\nabla u)e_2|^2 \, dx \geq \frac{L^3}{24H} |R_1 - R_2|^2. \tag{2.3}$$

Proof We observe that for given $a, b \in \mathbb{R}^2$ the 1-d minimization problem

$$\inf \left\{ \int_0^H |v'(t)|^2 \, dt : v \in W^{1,2}(0, H; \mathbb{R}^2), v(0) = a, v(H) = b \right\} \tag{2.4}$$

has a unique solution. Indeed, by Jensen’s inequality, the minimizer \bar{v} of (2.4) is given by linear interpolation as $\bar{v}(t) = \frac{1}{H}(b - a)t + a$ for $t \in (0, H)$. Since $u \in W^{1,2}(P; \mathbb{R}^2)$, one has that $u(x_1, \cdot) \in W^{1,2}(0, H; \mathbb{R}^2) \subset AC([0, H]; \mathbb{R}^2)$ with $u(x_1, 0) = w_1(x_1, 0)$ and $u(x_1, H) = w_2(x_1, H)$ for a.e. $x_1 \in (0, L)$. By setting

$$\bar{u}(x) = \frac{x_2}{H}(w_2(x_1, H) - w_1(x_1, 0)) + w_1(x_1, 0), \quad x \in P,$$

we therefore obtain

$$\begin{aligned} \int_P |(\nabla u)e_2|^2 \, dx &= \int_P |\partial_2 u(x)|^2 \, dx \geq \int_0^L \int_0^H |\partial_2 \bar{u}(x_1, x_2)|^2 \, dx_2 \, dx_1 \\ &= \frac{1}{H} \int_0^L |w_2(x_1, H) - w_1(x_1, 0)|^2 \, dx_1 \\ &= \frac{1}{H} \int_0^L |x_1(R_2 - R_1)e_1 + HR_2e_2 + (b_2 - b_1)|^2 \, dx_1. \end{aligned}$$

Minimizing this expression with respect to b_1 and b_2 gives (2.3). □

Proof of Proposition 2.1 To characterize the limit function u , we will show that the statement holds locally, i.e. on any open cube $Q \subset \Omega$ with sides parallel to the coordinate axes. Precisely, there exists a rotation $R_Q \in SO(2)$ and $\gamma_Q \in L^2(Q)$ such that $\nabla u|_Q = R_Q(\mathbb{1} + \gamma_Q e_1 \otimes e_2)$. To deduce (2.1), it suffices to exhaust Ω with overlapping cubes Q . This way one finds that all R_Q coincide, leading to a global rotation $R \in SO(2)$.

Without loss of generality, let us assume in the following that $Q = (0, l)^2$ with $l > 0$. To describe the layered structure of the material, we introduce the notation

$$P_\varepsilon^i = (\mathbb{R} \times \varepsilon[i, i + 1)) \cap Q', \quad i \in \mathbb{Z}, \varepsilon > 0,$$

for the horizontal strips in a larger open cube $Q' \subset \Omega$ that compactly contains Q . The index set $I_\varepsilon = \{i \in \mathbb{Z} : |P_\varepsilon^i| = \varepsilon\sqrt{|Q'|}\}$ selects those strips of thickness ε that are fully contained in Q' . Then, by taking ε sufficiently small one has that $Q \subset \bigcup_{i \in I_\varepsilon} P_\varepsilon^i$.

We subdivide the remaining proof in six steps.

Step 1: Classical rigidity and approximation by piecewise affine functions Applying Lemma 2.3 to each strip P_ε^i with $i \in I_\varepsilon$ yields the existence of rotation matrices $R_\varepsilon^i \in SO(2)$ and translation vectors $b_\varepsilon^i \in \mathbb{R}^2$ such that

$$u_\varepsilon(x) = R_\varepsilon^i x + b_\varepsilon^i, \quad x \in P_\varepsilon^i \cap \varepsilon Y_{\text{rig}}.$$

Let the sequences $(\sigma_\varepsilon)_\varepsilon \subset L^\infty(Q; \mathbb{R}^2)$ and $(b_\varepsilon)_\varepsilon \subset L^\infty(Q; \mathbb{R}^2)$ be defined by

$$\sigma_\varepsilon(x) = \sum_{i \in I_\varepsilon} (R_\varepsilon^i x) \mathbb{1}_{P_\varepsilon^i}(x) \quad \text{and} \quad b_\varepsilon(x) = \sum_{i \in I_\varepsilon} b_\varepsilon^i \mathbb{1}_{P_\varepsilon^i}(x), \quad x \in Q,$$

and let $w_\varepsilon = \sigma_\varepsilon + b_\varepsilon$. Next we show that

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - w_\varepsilon\|_{L^2(Q; \mathbb{R}^2)} = 0. \tag{2.5}$$

For each $i \in I_\varepsilon$, we apply a 1-d version of the Poincaré inequality to derive that

$$\begin{aligned} & \int_{P_\varepsilon^i \cap Q} |u_\varepsilon - w_\varepsilon|^2 \, dx \\ &= \int_0^l \int_{\varepsilon i}^{\varepsilon(i+1)} |u_\varepsilon - w_\varepsilon|^2 \, dx_2 \, dx_1 \leq \int_0^l c\varepsilon^2 \int_{\varepsilon i}^{\varepsilon(i+1)} |\partial_2 u_\varepsilon - \partial_2 \sigma_\varepsilon|^2 \, dx_2 \, dx_1 \\ &= c\varepsilon^2 \int_{P_\varepsilon^i \cap Q} |(\nabla u_\varepsilon - R_\varepsilon^i) e_2|^2 \, dx \leq c\varepsilon^2 (\|\nabla u_\varepsilon\|_{L^2(P_\varepsilon^i; \mathbb{R}^{2 \times 2})}^2 + |P_\varepsilon^i|) \end{aligned}$$

with constants $c > 0$ independent of ε . Summing over all $i \in I_\varepsilon$ gives

$$\|u_\varepsilon - w_\varepsilon\|_{L^2(Q; \mathbb{R}^2)}^2 \leq c\varepsilon^2 (\|u_\varepsilon\|_{W^{1,2}(\Omega; \mathbb{R}^2)} + |\Omega|) \leq c\varepsilon^2,$$

and thus, (2.5).

We point out that $(\sigma_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are uniformly bounded in $L^\infty(Q; \mathbb{R}^2)$ and $L^2(Q; \mathbb{R}^2)$, respectively. The latter follows together with (2.5) and the uniform boundedness of $(u_\varepsilon)_\varepsilon$ in $L^2(\Omega; \mathbb{R}^2)$. Consequently, there are subsequences of $(\sigma_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ (not relabeled) and functions $\sigma, b \in L^2(Q; \mathbb{R}^2)$ such that

$$\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma \quad \text{in } L^\infty(Q; \mathbb{R}^2) \quad \text{and} \quad b_\varepsilon \rightharpoonup b \quad \text{in } L^2(Q; \mathbb{R}^2). \tag{2.6}$$

Hence, $w_\varepsilon \rightharpoonup \sigma + b$ in $L^2(Q; \mathbb{R}^2)$, so that, in view of (2.5) and the uniqueness of weak limits,

$$u = \sigma + b. \tag{2.7}$$

Notice that $\partial_1 b = 0$ due to the fact that the functions b_ε are independent of x_1 considering the definition of the strips P_ε^i .

Step 2: Compactness by Helly’s selection principle It follows from Lemma 2.4, applied to the suitably shifted softer layers, that

$$\int_{Q'} |(\nabla u_\varepsilon)e_2|^2 dx \geq \sum_{i \in I_\varepsilon, i > i_\varepsilon} \int_{P_\varepsilon^i \cap \varepsilon Y_{\text{soft}}} |(\nabla u_\varepsilon)e_2|^2 dx \geq \frac{|Q'|^{3/2}}{24\varepsilon\lambda} \sum_{i \in I_\varepsilon, i > i_\varepsilon} |R_\varepsilon^i - R_\varepsilon^{i-1}|^2 \tag{2.8}$$

with i_ε the smallest integer in I_ε . Since $(u_\varepsilon)_\varepsilon$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^2)$, one infers that

$$\left(\sum_{i \in I_\varepsilon, i > i_\varepsilon} |R_\varepsilon^i - R_\varepsilon^{i-1}| \right)^2 \leq \#I_\varepsilon \sum_{i \in I_\varepsilon, i > i_\varepsilon} |R_\varepsilon^i - R_\varepsilon^{i-1}|^2 \leq C \tag{2.9}$$

for all ε with a constant $C > 0$. Besides (2.8), we use for the last estimate that the cardinality of I_ε satisfies $\#I_\varepsilon \leq \varepsilon^{-1}|Q'|^{1/2}$.

Consider now the piecewise constant function of one real variable $\Sigma_\varepsilon \in L^\infty(0, l; SO(2))$ given by

$$\Sigma_\varepsilon(t) = \sum_{i \in I_\varepsilon} R_\varepsilon^i \mathbb{1}_{\varepsilon[i, i+1)}(t), \quad t \in (0, l),$$

with the rotation matrices R_ε^i of Step 1. In view of (2.9) the sequence $(\Sigma_\varepsilon)_\varepsilon$ has uniformly bounded variation. Then, by Helly’s selection principle we can find a suitable (not relabeled) subsequence of $(\Sigma_\varepsilon)_\varepsilon$ and a function $\Sigma : (0, l) \rightarrow \mathbb{R}^{2 \times 2}$ of bounded variation such that

$$\Sigma_\varepsilon(t) \rightarrow \Sigma(t) \quad \text{for all } t \in (0, l). \tag{2.10}$$

Since $\det \Sigma_\varepsilon(t) = 1$ and $|\Sigma_\varepsilon(t)e_1| = 1$ for all $t \in (0, l)$ and all $\varepsilon > 0$, it follows by continuity that $\Sigma(t) \in SO(2)$ for all $t \in (0, l)$.

Step 3: Improved regularity for Σ Let $\Pi_\varepsilon : (0, l) \rightarrow \mathbb{R}^{2 \times 2}$ be the linear interpolant of Σ_ε between the points $\varepsilon(i + \frac{1}{2}) \in (0, l)$ with $i \in I_\varepsilon$. In the intervals close to the endpoints not covered by this definition, we take Π_ε to be constant.

As a continuous, piecewise affine function Π_ε is almost everywhere differentiable, and together with (2.9) or (2.8) it holds that

$$\int_0^l |\Pi'_\varepsilon|^2 dt \leq \sum_{i \in I_\varepsilon, i > i_\varepsilon} \varepsilon \frac{|R_\varepsilon^i - R_\varepsilon^{i-1}|^2}{\varepsilon^2} \leq C,$$

and

$$\int_0^l |\Pi_\varepsilon - \Sigma_\varepsilon|^2 dt \leq \sum_{i \in I_\varepsilon, i > i_\varepsilon} \varepsilon |R_\varepsilon^i - R_\varepsilon^{i-1}|^2 \leq C\varepsilon^2. \tag{2.11}$$

In particular, $(\Pi_\varepsilon)_\varepsilon$ is uniformly bounded in $W^{1,2}(0, l; \mathbb{R}^{2 \times 2})$, and therefore $(\Pi_\varepsilon)_\varepsilon$ admits a weakly converging subsequence with limit $\Pi \in W^{1,2}(0, l; \mathbb{R}^{2 \times 2})$. From (2.10), (2.11), and the uniqueness of the limit we infer that

$$\Sigma = \Pi \in W^{1,2}(0, l; SO(2)). \tag{2.12}$$

By constant extension of Σ in x_1 -direction we define a map R on Q , precisely we set $R(x) = \Sigma(x_2)$ for $x \in Q$. Then, $R \in W^{1,2}(Q; SO(2))$.

Step 4: Establishing $\nabla u \in \mathcal{M}_{e_1}$ pointwise The estimate

$$|\sigma_\varepsilon(x) - \Sigma(x_2)x| \leq \left| \sum_{i \in I_\varepsilon} (R_\varepsilon^i - \Sigma(x_2)) \mathbb{1}_{P_\varepsilon^i}(x) \right| |x| \leq \sqrt{2}l |\Sigma_\varepsilon(x_2) - \Sigma(x_2)|, \quad x \in Q,$$

along with (2.10) and the first part of (2.6) leads to

$$\sigma(x) = \Sigma(x_2)x = R(x)x \quad \text{for a.e. } x \in Q. \tag{2.13}$$

From (2.7) and the independence of b of x_1 we then conclude that

$$\nabla u e_1 = R e_1. \tag{2.14}$$

This shows in particular that R , Σ , σ , and b are independent of the choice of subsequences in (2.10) and (2.6). Moreover, by (2.12) and (2.13) it is immediate to see that $\sigma, b \in W^{1,2}(Q; \mathbb{R}^2)$.

Since $R \in SO(2)$ pointwise by Step 3, one has that $|\nabla u e_1| = 1$ a.e. in Q . In conjunction with $\det \nabla u = 1$ a.e. in Q , we conclude that $\nabla u \in \mathcal{M}_{e_1}$ a.e. in Q . In view of (2.14), there exists a function $\gamma \in L^2(Q)$ such that

$$\nabla u = R(\mathbb{1} + \gamma e_1 \otimes e_2). \tag{2.15}$$

Step 5: Proving R constant Using (2.7) and (2.13), we compute that

$$(\nabla u) e_2(x) = \partial_2 R(x)x + R(x)e_2 + \partial_2 b(x)$$

for a.e. $x \in Q$. Then, along with (2.15) and the independence of R of x_1 , it follows for the distributional derivative of γ that

$$\partial_1 \gamma = \partial_1 ((\nabla u) e_2 \cdot R e_1) = \partial_2 R e_1 \cdot R e_1 = \frac{1}{2} \partial_2 |R e_1|^2 = 0. \tag{2.16}$$

As $\text{curl } \nabla u = 0$ in Q in the sense of distributions, the representation (2.15) entails

$$0 = \int_Q R e_1 \partial_2 \varphi - R e_2 \partial_1 \varphi - \gamma R e_1 \partial_1 \varphi \, dx = \int_Q R e_1 \partial_2 \varphi \, dx$$

for all $\varphi \in C_c^\infty(Q)$, where we have used $\partial_1 R = 0$ and (2.16). This shows $\partial_2 R e_1 = 0$, which implies that R is a constant rotation.

Step 6: Proof of (2.2) Accounting for (2.10) and $\|\Sigma_\varepsilon\|_{L^\infty(Q; \mathbb{R}^{2 \times 2})} = 2$, one finds that $\Sigma_\varepsilon \rightarrow \Sigma$ in $L^2(0, l; \mathbb{R}^{2 \times 2})$. Together with $\mathbb{1}_{\varepsilon Y_{\text{rig}}} \xrightarrow{*} |Y_{\text{rig}}| = (1 - \lambda)$ in $L^\infty(Q)$, a weak-strong convergence argument leads to

$$\nabla u_\varepsilon \mathbb{1}_{\varepsilon Y_{\text{rig}}} \rightharpoonup (1 - \lambda) R \quad \text{in } L^2(Q; \mathbb{R}^{2 \times 2}),$$

considering that $\nabla u_\varepsilon(x) = \Sigma_\varepsilon(x_2)$ for a.e. $x \in \varepsilon Y_{\text{rig}} \cap Q$. To see that the statement holds in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ as well, we argue again by exhaustion of Ω with cubes Q . □

As discussed in Remark 2.2, Proposition 2.1 imposes structural restrictions on the limits of bounded energy sequences for $(E_\varepsilon)_\varepsilon$. As a consequence, we obtain an asymptotic lower bound energy estimate, which constitutes a first step toward the proof of the liminf-inequality (1.9) for the Γ -convergence result in Theorem 1.1.

Corollary 2.5 *Let $(u_\varepsilon)_\varepsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ be such that $E_\varepsilon(u_\varepsilon) \leq C$ for all $\varepsilon > 0$ and $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with gradient of the form (2.1). If, in addition, u is (finitely) piecewise affine, then*

$$\liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |(\nabla u_\varepsilon)m|^2 - 1 \, dx \geq \int_{\Omega} \frac{1}{\lambda} |(\nabla u)m - (1 - \lambda)Rm|^2 - \lambda \, dx.$$

Proof One may assume in the following that u is affine, otherwise the same arguments can be applied to each affine piece of u .

Since $\nabla u_\varepsilon \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$, and by (2.2), $\nabla u_\varepsilon \mathbb{1}_{\varepsilon Y_{\text{rig}}} \rightharpoonup (1 - \lambda)R$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$, it follows that

$$\nabla u_\varepsilon \mathbb{1}_{\varepsilon Y_{\text{soft}}} \rightharpoonup \nabla u - (1 - \lambda)R \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}), \tag{2.17}$$

and thus, in particular in $L^1(\Omega; \mathbb{R}^{2 \times 2})$. From $|\nabla u_\varepsilon m| = 1$ a.e. in $\varepsilon Y_{\text{rig}} \cap \Omega$, Hölder’s inequality, and (2.17) in conjunction with the weak lower semicontinuity of the L^1 -norm we infer that

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |(\nabla u_\varepsilon)m|^2 - 1 \, dx &= \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |(\nabla u_\varepsilon)m|^2 \mathbb{1}_{\varepsilon Y_{\text{soft}}} \, dx - |\Omega \cap \varepsilon Y_{\text{soft}}| \\ &\geq \liminf_{\varepsilon \rightarrow 0} |\Omega \cap \varepsilon Y_{\text{soft}}|^{-1} \left(\int_{\Omega} |(\nabla u_\varepsilon)m \mathbb{1}_{\varepsilon Y_{\text{soft}}}| \, dx \right)^2 \\ &\quad - |\Omega \cap \varepsilon Y_{\text{soft}}| \\ &\geq \frac{|\Omega|}{\lambda} |(\nabla u)m - (1 - \lambda)Rm|^2 - \lambda |\Omega|. \end{aligned}$$

Notice that in the last step we also used that $|\Omega \cap \varepsilon Y_{\text{soft}}| = \int_{\Omega} \mathbb{1}_{\varepsilon Y_{\text{soft}}} \, dx \rightarrow \lambda |\Omega|$ as $\varepsilon \rightarrow 0$, as well as the assumption that ∇u is constant. □

3 Discussion of admissible deformations

In preparation for the proof of Theorem 1.1, we exploit the specific form of the functionals E_ε to identify further properties of the weak limits of bounded energy sequences. Moreover, we provide the basis for the laminate constructions that are the key to obtaining suitable recovery sequences.

3.1 Necessary conditions for admissible macroscopic deformations

Let $(u_\varepsilon)_\varepsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ satisfy $\nabla u_\varepsilon \in \mathcal{M}_S$ a.e. in Ω , and suppose that $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$. As the convex set $\{F \in L^2(\Omega; \mathbb{R}^{2 \times 2}) : |Fs| \leq 1 \text{ a.e. in } \Omega\}$ is weakly closed in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ and $\det \nabla u_\varepsilon \xrightarrow{*} \det \nabla u$ in the sense of measures (cf. Remark 2.2a), we know that

$$\nabla u \in \mathcal{N}_S \quad \text{a.e. in } \Omega, \tag{3.1}$$

where $\mathcal{N}_S = \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fs| \leq 1\}$. According to [15] (see also [14]), the set \mathcal{N}_S is exactly the quasiconvex hull $\mathcal{M}_S^{\text{qc}}$ of \mathcal{M}_S . With $S = (s|m) = (s|s^\perp) \in SO(2)$, another alternative representation of \mathcal{N}_S is

$$\mathcal{N}_s = \left\{ F \in \mathbb{R}^{2 \times 2} : F = R(S(\alpha e_1 \otimes e_1 + \frac{1}{\alpha} e_2 \otimes e_2)S^T + \gamma s \otimes m), \right. \\ \left. R \in SO(2), \alpha \in (0, 1], \gamma \in \mathbb{R} \right\}.$$

One also has that

$$\mathcal{M}_s = \{F \in \mathbb{R}^{2 \times 2} : F = GS^T, G \in \mathcal{M}_{e_1}\} \quad \text{and} \quad \mathcal{N}_s = \{F \in \mathbb{R}^{2 \times 2} : F = GS^T, G \in \mathcal{N}_{e_1}\}. \tag{3.2}$$

If we assume in addition that $(u_\varepsilon)_\varepsilon$ is a sequence of bounded energy, precisely, $E_\varepsilon(u_\varepsilon) < C$ for all ε , then $\nabla u \in \mathcal{M}_{e_1}$ pointwise almost everywhere in Ω by the rigidity result in Proposition 2.1 and Remark 2.2(a). Thus, together with (3.1),

$$\nabla u \in \mathcal{M}_{e_1} \cap \mathcal{N}_s \quad \text{a.e. in } \Omega. \tag{3.3}$$

For $s = e_1$ the restriction in (3.3) is equivalent to $\nabla u \in \mathcal{M}_{e_1}$ a.e., while for $s \neq e_1$ a straightforward computation shows that

$$\mathcal{M}_{e_1} \cap \mathcal{N}_s = \{F \in \mathbb{R}^{2 \times 2} : F = R(\mathbb{I} + \gamma e_1 \otimes e_2), R \in SO(2), \gamma \in K_{s,1}\}, \tag{3.4}$$

with $K_{s,1}$ as defined in (1.8).

In the case $s \neq e_1$, condition (3.3) can be refined even further by exploiting the presence of the rigid layers with their asymptotic volume fraction $|Y_{\text{rig}}| = 1 - \lambda$. Indeed, from Proposition 2.1 we infer that there are $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ such that

$$(\nabla u_\varepsilon)_s \mathbb{1}_{\varepsilon Y_{\text{soft}}} = (\nabla u_\varepsilon)_s - (\nabla u_\varepsilon)_s \mathbb{1}_{\varepsilon Y_{\text{rig}}} \rightharpoonup R(\lambda \mathbb{I} + \gamma e_1 \otimes e_2)_s \quad \text{in } L^2(\Omega; \mathbb{R}^2),$$

and therefore also in $L^1(\Omega; \mathbb{R}^2)$. On the other hand, $|(\nabla u_\varepsilon)_s \mathbb{1}_{\varepsilon Y_{\text{soft}}}| = \mathbb{1}_{\varepsilon Y_{\text{soft}}} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\Omega)$. By the weak lower semicontinuity of the L^1 -norm, we obtain for any open ball $B \subset \Omega$ that

$$\int_B |R(\lambda \mathbb{I} + \gamma e_1 \otimes e_2)_s| \, dx \leq \liminf_{\varepsilon \rightarrow 0} \int_B |(\nabla u_\varepsilon)_s \mathbb{1}_{\varepsilon Y_{\text{soft}}}| \, dx = |B| \lambda,$$

and consequently,

$$\int_B |\lambda s + \gamma s_2 e_1| \, dx \leq \lambda.$$

Applying Lebesgue’s differentiation theorem entails the pointwise estimate $|\lambda s + \gamma s_2 e_1| \leq \lambda$ a.e. in Ω , which is equivalent to

$$\gamma \in K_{s,\lambda} \quad \text{a.e. in } \Omega,$$

cf. (1.8) for the definition of $K_{s,\lambda}$.

3.2 Tools for the construction of admissible deformations

We start by characterizing all rank-one connections in \mathcal{M}_s , cf. also [11].

Lemma 3.1 (Rank-one connections in \mathcal{M}_s) *Let $F, G \in \mathcal{M}_s$ such that $F = R(\mathbb{I} + \gamma s \otimes m)$ and $G = Q(\mathbb{I} + \zeta s \otimes m)$ with $R, Q \in SO(2)$ and $\gamma, \zeta \in \mathbb{R}$. Then F and G are rank-one connected, i.e. $\text{rank}(F - G) = 1$, if and only if one of the following relations holds:*

- (i) $R = Q$ and $\gamma \neq \zeta$,
or
- (ii) $R \neq Q$ and $\gamma - \zeta = 2 \tan(\theta/2)$, where $\theta \in (-\pi, \pi)$ denotes the rotation angle of $Q^T R$, meaning that $Q^T R e_1 = \cos \theta e_1 + \sin \theta e_2$.

In particular, in case of (i), $F - G = (\gamma - \zeta)Rs \otimes m$, while for (ii) one has

$$F - G = \frac{\gamma - \zeta}{4 + (\gamma - \zeta)^2} Q((\zeta - \gamma)s + 2m) \otimes (2s + (\gamma + \zeta)m). \tag{3.5}$$

Proof Considering (3.2), it is enough to prove the statement for $s = e_1$. Moreover, we may assume without loss of generality that $Q = \mathbb{I}$. It follows from $\det(F - G) = 0$ that

$$Re_1 \cdot (2e_1 + (\gamma - \zeta)e_2) = 2.$$

Thus, either $Re_1 = e_1$, i.e. $R = \mathbb{I}$, or

$$Re_1 = \frac{4 - (\gamma - \zeta)^2}{4 + (\gamma - \zeta)^2} e_1 + \frac{4(\gamma - \zeta)}{4 + (\gamma - \zeta)^2} e_2. \tag{3.6}$$

In view of the definition of θ and some basic identities for trigonometric functions, (3.6) is equivalent to $\gamma - \zeta = 2 \tan(\theta/2)$. The representation of $F - G$ in (3.5) is then straightforward to compute. □

Remark 3.2 As this paper is concerned with materials built from horizontal layers, we are especially interested in rank-one connections with normal e_2 , i.e. $F, G \in \mathcal{M}_s$ as in Lemma 3.1 with $\text{rank}(F - G) = 1$ satisfying $F - G = a \otimes e_2$ for some $a \in \mathbb{R}^2 \setminus \{0\}$. If $s = e_1$, this implies $R = Q$, but there are no restrictions on γ and ζ other than $\gamma \neq \zeta$. For $s \neq e_1$ one needs that $\gamma + \zeta = 2 \frac{s_1}{s_2}$. Hence, for given $\gamma \in \mathbb{R}$ also the rotation matrix $Q^T R$ is uniquely determined in this case.

It was first proven in [15] that $\mathcal{N}_s = \mathcal{M}_s^{\text{qc}}$ coincides with the rank-one convex hull $\mathcal{M}_s^{\text{rc}}$, which in particular, means that every $N \in \mathcal{N}_s$ can be expressed as a convex combination of rank-one connected matrices in \mathcal{M}_s . A specific type of rank-one directions, which turns out optimal for the relaxation of W_{soft} , was discussed by Conti in [13], see also [14]. Here we give a different argumentation based on Lemma 3.1.

Lemma 3.3 *For a given $N \in \mathcal{N}_s \setminus \mathcal{M}_s$ there are $F, G \in \mathcal{M}_s$ as in Lemma 3.1 satisfying $\text{rank}(F - G) = 1$ and $\mu \in (0, 1)$ such that*

$$N = \mu F + (1 - \mu)G \quad \text{and} \quad |Nm| = |Fm| = |Gm|. \tag{3.7}$$

Proof Let $N \in \mathcal{N}_s \setminus \mathcal{M}_s$. We determine $\mu \in (0, 1)$ as well as $Q, R \in SO(2)$ and $\zeta, \gamma \in \mathbb{R}$ such that the desired properties are satisfied for $F = R(\mathbb{I} + \gamma s \otimes m)$ and $G = Q(\mathbb{I} + \zeta s \otimes m)$.

Since the second condition in (3.7) is equivalent to γ and ζ satisfying $|\gamma|^2 = |\zeta|^2 = |Nm|^2 - 1$, we may choose

$$\gamma = \sqrt{|Nm|^2 - 1} \quad \text{and} \quad \zeta = -\gamma. \tag{3.8}$$

Notice that $|Nm| > 1$, as $1 = \det N \leq |Ns||Nm|$ and $|Ns| < 1$ by assumption. Then, by Lemma 3.1, F and G are rank-one connected, if $Q^T R$ corresponds to the rotation with angle $\theta = 2 \arctan \gamma$. We use this relation to define R for given Q to be determined in the next step.

For the first part of (3.7), it is necessary that

$$Ns = Gs + \mu(F - G)s = Q\left(s + \frac{2\gamma\mu}{1 + \gamma^2}(m - \gamma s)\right), \tag{3.9}$$

where we have used (3.5) along with (3.8). Taking squared norms in the above equation imposes a constraint on μ in the form of a quadratic equation, which has two solutions

$\mu_1 \in (0, 1/2)$ and $\mu_2 \in (1/2, 1)$ with $\mu_1 + \mu_2 = 1$. Depending on which of these values is selected for μ , we adjust the rotation Q so that (3.9) holds. It follows from (3.9) and (3.8) that

$$Gm \in \mathcal{A}_N = \{a \in \mathbb{R}^2 : (Ns)^\perp \cdot a = 1, |a| = |Nm|\}.$$

As $|Nm|^{-1} \leq |Ns|$, the set \mathcal{A}_N contains exactly two elements, one of which being Nm . Finally, we take $\mu \in (0, 1)$ with corresponding Q such that $Gm = Nm$, which finishes the proof.

Let us remark that choosing $\gamma = -\sqrt{|Nm|^2 - 1}$ in (3.8) essentially comes up to switching F and G . □

In the case of a non-horizontal slip direction, optimal constructions of admissible deformations cannot be achieved based on rank-one connections in \mathcal{M}_s . Instead, we employ simple laminates with gradients in $SO(2)$ and \mathcal{N}_s (and normal e_2). The following one-to-one correspondence between $\gamma \in K_{s,\lambda}$ and $R \in SO(2)$, and $N \in \mathcal{N}_s$ with $|Ne_1| = 1$ is helpful for the explicit constructions.

Lemma 3.4 *Let $\lambda \in (0, 1)$ and $s \in \mathbb{R}^2$ with $|s| = 1$ and $s \neq e_1$ be given.*

- (i) *For every $\gamma \in K_{s,\lambda}$ and $R \in SO(2)$ there exists $N \in \mathcal{M}_{e_1} \cap \mathcal{N}_s$ such that $Ne_1 = Re_1$ and*

$$\lambda N + (1 - \lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2). \tag{3.10}$$

- (ii) *Let $N \in \mathcal{N}_s$ and $R \in SO(2)$ with $Re_1 = Ne_1$. Then there exists $\gamma \in K_{s,\lambda}$ such that (3.10) is satisfied.*

Proof For the proof of (i) we set

$$N = R\left(\mathbb{I} + \frac{\gamma}{\lambda} e_1 \otimes e_2\right). \tag{3.11}$$

It is immediate to check that $Ne_1 = Re_1$, (3.10) is fulfilled, $N \in \mathcal{M}_{e_1}$. Furthermore, $|Ns| = |s + \frac{\gamma}{\lambda} s_2 e_1| \leq 1$ in view of $\gamma \in K_{s,\lambda}$.

As regards (ii), choosing $\gamma = \lambda Ne_1 \cdot Ne_2$ gives the desired element in $K_{s,\lambda}$. Indeed, $R(\mathbb{I} + \gamma e_1 \otimes e_2)e_2 = Re_2 + \lambda(Ne_1 \cdot Ne_2)Ne_1 = Re_2 + \lambda Ne_2 - \lambda((Ne_1)^\perp \cdot Ne_2)(Ne_1)^\perp = \lambda Ne_2 + (1 - \lambda)Re_2$ in view of $1 = \det N = (Ne_1)^\perp \cdot Ne_2$, which along with $Ne_1 = Re_1$ proves (3.10). Since (3.10) implies that N is of the form (3.11), it follows from a direct computation that $\gamma \in K_{s,\lambda}$. □

The following two theorems are taken from Conti and Theil [15] and Müller and Šverák [38], respectively. In combination, they allow us to modify a simple laminate with gradients in \mathcal{M}_s in a small part of the domain in such a way that the resulting Lipschitz function takes affine boundary values in \mathcal{N}_s , while preserving the constraint that gradients lie pointwise in \mathcal{M}_s , see Corollary 3.7.

Theorem 3.5 [15, Theorem 4] *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\mu \in (0, 1)$. Suppose that $F, G \in \mathcal{M}_s$ are rank-one connected with $Fs \neq Gs$ and $N = \mu F + (1 - \mu)G \in \mathcal{N}_s$.*

Then for every $\delta > 0$ there are $h_\delta^0 > 0$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ such that the restriction to Ω_δ of any simple laminate between the gradients F and G with weights μ and $1 - \mu$ and period $h < h_\delta^0$ can be extended to a finitely piecewise affine function $v_\delta : \Omega \rightarrow \mathbb{R}^2$ with $\nabla v_\delta \in \mathcal{N}_s$ a.e. in Ω , $v_\delta = Nx$ on $\partial\Omega$, and $\text{dist}(\nabla v_\delta, [F, G]) < \delta$ a.e. in Ω , where $[F, G] = \{tF + (1 - t)G : t \in [0, 1]\}$.

Convex integration methods help to obtain exact solutions to partial differential inclusions.

Theorem 3.6 [38, Theorem 1.3] *Let $\mathcal{M} \subset \{F \in \mathbb{R}^{2 \times 2} : \det F = 1\}$. Suppose that $(U_i)_i$ is an in-approximation of \mathcal{M} , i.e., the sets U_i are open in $\{F \in \mathbb{R}^{2 \times 2} : \det F = 1\}$ and uniformly bounded, U_i is contained in the rank-one convex hull of U_{i+1} for every $i \in \mathbb{N}$, and $(U_i)_i$ converges to \mathcal{M} in the following sense: if $F_i \in U_i$ for $i \in \mathbb{N}$ and $|F_i - F| \rightarrow 0$ as $i \rightarrow \infty$, then $F \in \mathcal{M}$.*

Then, for any $F \in U_1$ and any open domain $\Omega \subset \mathbb{R}^2$, there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that $\nabla u \in \mathcal{M}$ a.e. in Ω and $u = Fx$ on $\partial\Omega$.

Combining these two theorems with the explicit construction of Lemma 3.3 leads to the following result, cf. also [13–15].

Corollary 3.7 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $N \in \mathcal{N}_s$. If $N \in \mathcal{N}_s \setminus \mathcal{M}_s$, let $F, G \in \mathcal{M}_s$ and $\mu \in (0, 1)$ be as in Lemma 3.3, otherwise let $F = G = N \in \mathcal{M}_s$ and $\mu \in (0, 1)$.*

Then, for every $\delta > 0$ there exists $u_\delta \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ such that u_δ coincides with a simple laminate between F and G with weights μ and $1 - \mu$ and period $h_\delta < \delta$ in Ω_δ , $\nabla u_\delta \in \mathcal{M}_s$ a.e. in Ω , $u_\delta = Nx$ on $\partial\Omega$, and

$$|(\nabla u_\delta)m| < |Nm| + \delta \quad \text{a.e. in } \Omega. \tag{3.12}$$

In particular, $|(\nabla u_\delta)m| = |Nm|$ a.e. in Ω_δ , and $\nabla u_\delta \rightarrow N$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ as $\delta \rightarrow 0$.

Proof From Theorem 3.5 we obtain for $\delta > 0$ the desired set Ω_δ along with a finitely piecewise affine function $v_\delta : \Omega \rightarrow \mathbb{R}^2$ that coincides in Ω_δ with a simple laminate between the gradients F and G of period $h_\delta < \min\{\delta, h_\delta^0\}$, satisfies $\nabla v_\delta \in \mathcal{N}_s$ a.e. in Ω , and $v_\delta = Nx$ on $\partial\Omega$. In view of (3.7), $|(\nabla v_\delta)m| = |Nm|$ a.e. in Ω_δ and

$$\delta > \text{dist}(\nabla v_\delta, [F, G]) \geq \min_{t \in [0,1]} |(\nabla v_\delta)m - (tFm + (1-t)Gm)| \geq |(\nabla v_\delta)m| - |Nm|$$

a.e. in Ω . Finally, the sought function u_δ results from a modification of v_δ in the (finitely many) domains where $\nabla v_\delta \notin \mathcal{M}_s$ by applying Theorem 3.6 with the in-approximation $(U_i^\delta)_i$ of $\mathcal{M}_s \cap \{F \in \mathbb{R}^{2 \times 2} : |Fm| < |Nm| + \delta\}$ given by

$$U_i^\delta = \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, 1 - 2^{-(i-1)} < |Fs| < 1, |Fm| < |Nm| + \delta\}, \quad i \in \mathbb{N},$$

see [15, Proof of Lemma 2] for more details. □

4 Proof of Theorem 1.1 for $s = e_1$

As indicated in the introduction, in the special case of a horizontal slip direction $s = e_1$, we can prove Theorem 1.1 in a slightly more general setting, where W_{soft} has an additional linear term that can be interpreted as a dissipative energy contribution.

More precisely, for a given $\tau \geq 0$ let us replace W_{soft} with

$$W_{\text{soft}}^\tau(F) = \begin{cases} \gamma^2 + \tau|\gamma| & \text{if } F = R(\mathbb{I} + \gamma s \otimes m), R \in SO(2), \gamma \in \mathbb{R}, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{2 \times 2}.$$

Then, E_ε of (1.4) with $\varepsilon > 0$ turns into

$$E_\varepsilon^\tau(u) = \begin{cases} \int_\Omega \gamma^2 + \tau|\gamma| \, dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma s \otimes m) \\ & \text{with } R \in L^\infty(\Omega; SO(2)), \gamma \in L^2(\Omega), \gamma = 0 \text{ a.e. in } \varepsilon Y_{\text{rig}} \cap \Omega, \\ \infty & \text{otherwise,} \end{cases} \tag{4.1}$$

for $u \in L^2_0(\Omega; \mathbb{R}^2)$, cf. (1.5).

In this section, we prove the following generalization of Theorem 1.1 in the case $s = e_1$, assuming that Ω is simply connected.

Theorem 4.1 *Let $(E_\varepsilon^\tau)_\varepsilon$ as in (4.1) and let the functional $E^\tau : L^2_0(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ be given by*

$$E^\tau(u) = \begin{cases} \frac{1}{\lambda} \int_\Omega \gamma^2 \, dx + \tau \int_\Omega |\gamma| \, dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \\ & \text{with } R \in SO(2), \gamma \in L^2(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Then, $\Gamma(L^2)$ - $\lim_{\varepsilon \rightarrow 0} E_\varepsilon^\tau = E^\tau$. Moreover, bounded energy sequences of $(E_\varepsilon^\tau)_\varepsilon$ are relatively compact in $L^2_0(\Omega; \mathbb{R}^2)$.

Proof The proof is divided into three steps.

Step 1: Compactness Let $(\varepsilon_j)_j$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and consider $(u_j)_j$ such that $E_{\varepsilon_j}^\tau(u_j) < C$ for all $j \in \mathbb{N}$. Then, $\nabla u_j = R_j(\mathbb{I} + \gamma_j e_1 \otimes e_2)$ with $R_j \in L^\infty(\Omega; SO(2))$ and $\gamma_j \in L^2(\Omega)$. Since $|(\nabla u_j)e_1| = 1$ a.e. in Ω and $\|(\nabla u_j)e_2\|_{L^2(\Omega; \mathbb{R}^2)}^2 = \|\gamma_j\|_{L^2(\Omega)}^2 + |\Omega|$, the observation that $(\gamma_j)_j$ is uniformly bounded in $L^2(\Omega)$ results in

$$\|\nabla u_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq C \quad \text{for all } j \in \mathbb{N}. \tag{4.2}$$

By Poincaré’s inequality (recall that $\int_\Omega u_j \, dx = 0$) the sequence $(u_j)_j$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^2)$. Hence, one may extract a subsequence (not relabeled) of $(u_j)_j$ that converges weakly in $W^{1,2}(\Omega; \mathbb{R}^2)$, and also strongly in $L^2(\Omega; \mathbb{R}^2)$ by compact embedding, to a limit function $u \in W^{1,2}(\Omega; \mathbb{R}^2)$. Considering Remark 2.2(a), we infer from Lemma 2.3 that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$, where $R \in SO(2)$ and $\gamma \in L^2(\Omega)$.

Moreover, the case $s = e_1$ at hand carries even more information. Indeed, $R_j e_1 = \nabla u_j e_1 \rightharpoonup \nabla u e_1 = R e_1$ in $L^2(\Omega; \mathbb{R}^2)$ along with $|R_j e_1| = |R e_1| = 1$ entails strong convergence of the rotations $(R_j)_j$, i.e. (possibly after selection of another subsequence) $R_j \rightarrow R$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$, and thus, also

$$\gamma_j \rightarrow \gamma \text{ in } L^2(\Omega). \tag{4.3}$$

Step 2: Recovery sequence Let $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ with

$$\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \quad \text{with } R \in SO(2), \gamma \in L^2(\Omega).$$

The main idea for the construction of a recovery sequence is to set $\gamma = 0$ in the stiff layers, as the functional E_ε^τ requires, while compensating with more gliding in the softer layers.

Therefore, for $\varepsilon > 0$ we put

$$\gamma_\varepsilon = \frac{\gamma}{\lambda} \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \in L^2(\Omega).$$

Since Ω is assumed to be simply connected and since the field $R(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2) \in L^2(\Omega; \mathbb{R}^{2 \times 2})$ is curl-free, there exists $u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with

$$\nabla u_\varepsilon = R(\mathbb{I} + \gamma_\varepsilon e_1 \otimes e_2), \tag{4.4}$$

without loss of generality, we can take $u_\varepsilon \in L^2_0(\Omega; \mathbb{R}^2)$. By the weak convergence of oscillating periodic functions one has that $\mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \overset{*}{\rightharpoonup} \lambda$ in $L^\infty(\Omega)$, which implies

$$\gamma_\varepsilon \rightharpoonup \gamma \text{ in } L^2(\Omega).$$

Consequently, it follows in view of Poincaré’s inequality that $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$, and by compact embedding $u_\varepsilon \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$. Regarding the convergence of energies we argue that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} E_\varepsilon^\tau(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \int_\Omega \gamma_\varepsilon^2 + \tau |\gamma_\varepsilon| \, dx = \lim_{\varepsilon \rightarrow 0} \int_\Omega \left(\frac{\gamma^2}{\lambda^2} + \frac{\tau}{\lambda} |\gamma| \right) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \, dx \\ &= \int_\Omega \frac{\gamma^2}{\lambda} + \tau |\gamma| \, dx = E^\tau(u). \end{aligned}$$

Step 3: Lower bound Let $(\varepsilon_j)_j$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$. We assume without loss of generality that $(u_j)_j$ is a sequence of uniformly bounded energy for $(E_{\varepsilon_j}^\tau)_j$, so that $(u_j)_j$ and u satisfy the properties of Step 1.

If u is piecewise affine, the desired liminf inequality then follows directly from Corollary 2.5, as $\frac{1}{\lambda} |(\nabla u)e_2 - (1 - \lambda)Re_2|^2 - \lambda = \frac{1}{\lambda} |\lambda Re_2 + \gamma Re_1|^2 - \lambda = \frac{\gamma^2}{\lambda}$, and from (4.3).

To prove the statement for general u , we perform an approximation argument inspired by the proof of Müller’s homogenization result in [37, Theorem 1.3]. Due to the differential constraints in E_ε^τ , however, the construction of suitable comparison functions is slightly more involved.

Suppose that $\Omega = Q \subset \mathbb{R}^2$ is a cube with sides parallel to the coordinate axes, otherwise we perform the arguments below on any finite union of disjoint cubes contained in Ω and take the supremum over all these sets, exploiting the fact that the energy density in (4.1) is non-negative.

Accounting for Remark 2.2(b) allows us to find a sequence of one-dimensional simple functions $(\zeta_k)_k$ (identified with a sequence in $L^2(Q)$ by constant extension in x_1 -direction) such that

$$\zeta_k \rightarrow \gamma \text{ in } L^2(Q). \tag{4.5}$$

For $k \in \mathbb{N}$, let $w_k \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ with $\nabla w_k = R(\mathbb{I} + \zeta_k e_1 \otimes e_2)$. Further, let $(v_j)_j$ and $(v_{k,j})_j$ be the recovery sequences (as constructed in Step 2) for u and w_k , respectively. We define

$$z_{k,j} = u_j - v_j + v_{k,j}, \quad j, k \in \mathbb{N}, \tag{4.6}$$

observing that $z_{k,j} \rightharpoonup w_k$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for all $k \in \mathbb{N}$ as $j \rightarrow \infty$. Moreover, $\nabla z_{k,j} = \nabla u_j$ a.e. in $\varepsilon_j Y_{\text{rig}} \cap \Omega$, so that in particular, $|(\nabla z_{k,j})e_2| = 1$ a.e. in $\varepsilon_j Y_{\text{rig}} \cap \Omega$ for all $j, k \in \mathbb{N}$, and

$$\nabla z_{k,j} \mathbb{1}_{\varepsilon_j Y_{\text{rig}} \cap \Omega} \rightharpoonup (1 - \lambda)R \text{ in } L^2(\Omega; \mathbb{R}^{2 \times 2})$$

for $k \in \mathbb{N}$ as $j \rightarrow \infty$ by (2.2). Considering that w_k is piecewise affine, it follows as in the proof of Corollary 2.5 that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} |(\nabla z_{k,j})e_2|^2 - 1 \, dx \geq \frac{1}{\lambda} \int_{\Omega} \zeta_k^2 \, dx \tag{4.7}$$

for all $k \in \mathbb{N}$. With (4.6) one obtains

$$\begin{aligned} E_{\varepsilon_j}^{\tau}(u_j) &= \int_{\Omega} |(\nabla u_j)e_2|^2 - 1 + \tau|\gamma_j| \, dx \\ &\geq \int_{\Omega} |(\nabla z_{k,j})e_2|^2 - 1 \, dx \\ &\quad + \tau \int_{\Omega} |\gamma_j| \, dx - 2\|\nabla u_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \|(\nabla v_j - \nabla v_{k,j})e_2\|_{L^2(\Omega; \mathbb{R}^2)} \end{aligned}$$

for $j, k \in \mathbb{N}$, where by construction $(\nabla v_j - \nabla v_{k,j})e_2 = \lambda^{-1}(\gamma - \zeta_k)Re_1 \mathbb{1}_{\varepsilon_j Y_{\text{soft}} \cap \Omega}$, cf. Step 2. From $\|(\nabla v_j - \nabla v_{k,j})e_2\|_{L^2(\Omega; \mathbb{R}^2)} \leq \frac{1}{\lambda} \|\gamma - \zeta_k\|_{L^2(\Omega)}$ and the uniform boundedness of $(\nabla u_j)_j$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ by (4.2) we infer that

$$E_{\varepsilon_j}^{\tau}(u_j) \geq \int_{\Omega} |(\nabla z_{k,j})e_2|^2 - 1 \, dx + \tau \int_{\Omega} |\gamma_j| \, dx - \frac{2C}{\lambda} \|\gamma - \zeta_k\|_{L^2(\Omega)}$$

for $j, k \in \mathbb{N}$. Passing to the limit $j \rightarrow \infty$ in the above estimate yields

$$\liminf_{j \rightarrow \infty} E_{\varepsilon_j}^{\tau}(u_j) \geq \frac{1}{\lambda} \int_{\Omega} \zeta_k^2 \, dx + \tau \int_{\Omega} |\gamma| \, dx - \frac{2C}{\lambda} \|\gamma - \zeta_k\|_{L^2(\Omega)}.$$

Here we have used (4.7), as well as (4.3). Finally, due to (4.5), taking $k \rightarrow \infty$ finishes the proof of the liminf inequality. \square

Remark 4.2 Note that a sequence $(u_{\varepsilon})_{\varepsilon} \subset W^{1,2}(\Omega; \mathbb{R}^2)$ satisfying (4.4) can be constructed explicitly as follows.

If Ω is a cube $Q \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, say $Q = (0, l)^2$ with $l > 0$, then γ_{ε} is independent of x_1 in view of Remark 2.2 (b) and the orientation of the layers, hence, depending on the context, it can be interpreted as an element in $L^2(Q)$ or $L^2(0, l)$. Then, for any $a \in \mathbb{R}^2$,

$$u_{\varepsilon}(x) = Rx + \left(\int_0^{x_2} \gamma_{\varepsilon}(t) \, dt \right) Re_1 + a, \quad x \in Q, \tag{4.8}$$

satisfies $u_{\varepsilon} \in W^{1,2}(Q; \mathbb{R}^2)$ with $\nabla u_{\varepsilon} = R(\mathbb{I} + \gamma_{\varepsilon}e_1 \otimes e_2)$.

To construct u_{ε} for a general simply connected Ω , we exhaust Ω successively with shifted, overlapping cubes Q , using (4.8) with suitably adjusted translation vectors a . This construction allows adaption to a more involved situation that we encounter in the next section.

5 Proof of Theorem 1.1 for $s \neq e_1$

This section is concerned with the proof of Theorem 1.1 in the case of an inclined or vertical slip direction, that is $s \neq e_1$. Due to the strong restrictions on rank-one connections between $SO(2)$ and \mathcal{M}_s with normal e_2 (see Remark 3.2), the construction of recovery sequences is more involved than for $s = e_1$.

Before giving the detailed arguments, let us briefly discuss different equivalent representations of the limit energy E introduced in (1.7). Depending on the context, we will always

use the most convenient one without further mentioning. For $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ we define with regard to Lemma 3.4 the function $N \in L^2(\Omega; \mathbb{R}^{2 \times 2})$ by $\lambda N + (1 - \lambda)R = \nabla u$, i.e. $N = R(\mathbb{I} + \frac{\gamma}{\lambda} e_1 \otimes e_2)$. Then,

$$\begin{aligned} E(u) &= \frac{s_1^2}{\lambda} \int_{\Omega} \gamma^2 \, dx - 2s_1s_2 \int_{\Omega} \gamma \, dx = \int_{\Omega} \frac{1}{\lambda} |\gamma m_2 e_1 + \lambda m|^2 - \lambda \, dx \\ &= \int_{\Omega} \frac{1}{\lambda} |(\nabla u)m - (1 - \lambda)Rm|^2 - \lambda \, dx = \lambda \int_{\Omega} |Nm|^2 - 1 \, dx. \end{aligned} \tag{5.1}$$

Proof of Theorem 1.1 Here again, the proof follows three steps.

Step 1: Compactness The proof of compactness is identical with the beginning of Step 1 in Theorem 4.1 for $\tau = 0$, when substituting e_1 with s and e_2 with m .

Step 2: Recovery sequence Let $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ such that $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$ and $\gamma \in L^2(\Omega)$ be given. The idea of the construction is to specify first a sequence of functions with asymptotically optimal energy that are piecewise affine on the layers and whose gradients lie in \mathcal{N}_s . Then, to obtain admissible deformations, we approximate these functions in the softer layers with fine simple laminates between gradients in \mathcal{M}_s , which requires tools from relaxation theory and convex integration as discussed in Sect. 3.2.

Step 2a: Auxiliary functions for constant γ Let $\gamma \in K_{s,\lambda}$ be constant. By Lemma 3.4 we find $N \in \mathcal{N}_s$ such that

$$\lambda N + (1 - \lambda)R = R(\mathbb{I} + \gamma e_1 \otimes e_2) \tag{5.2}$$

and $Ne_1 = Re_1$, which guarantees the compatibility for constructing laminates between the gradients R and N with e_2 the normal on the jump lines of the gradient. Precisely, we define for $\varepsilon > 0$ the function $v_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with zero mean value characterized by $\nabla v_\varepsilon = \nabla v_1(\varepsilon^{-1} \cdot)$, where $v_1 \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ is such that

$$\nabla v_1 = R \mathbb{1}_{Y_{\text{rig}}} + N \mathbb{1}_{Y_{\text{soft}}}. \tag{5.3}$$

Then, by the weak convergence of highly oscillating functions and (5.2),

$$\nabla v_\varepsilon \rightharpoonup \lambda N + (1 - \lambda)R = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}).$$

Regarding the energy contribution of the sequence $(v_\varepsilon)_\varepsilon$ it follows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |(\nabla v_\varepsilon)m|^2 - 1 \, dx = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (|Nm|^2 - 1) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \, dx = \lambda |\Omega| (|Nm|^2 - 1) = E(u).$$

Notice that v_ε is not admissible for E_ε if $N \in \mathcal{N}_s \setminus \mathcal{M}_s$.

Step 2b: Admissible recovery sequence for constant γ Next, we modify the construction of Step 2a in the softer layers to obtain admissible functions, while preserving the energy. This is done by approximation with the simple laminates established in Corollary 3.7, see Fig. 2 for illustration.

Let $N \in \mathcal{N}_s$ as in Step 2a. For $\varepsilon > 0$ and $i \in \mathbb{Z}^2$, let $\varphi_{\varepsilon,i} \in W^{1,\infty}(i + (0, 1) \times (0, \lambda); \mathbb{R}^2)$ be a function resulting from Corollary 3.7 applied to $\Omega = i + (0, 1) \times (0, \lambda) \subset \mathbb{R}^2$ and $\delta = \varepsilon$. We define

$$\varphi_\varepsilon(x) = \sum_{i \in \mathbb{Z}^2} (\varphi_{\varepsilon,i}(x) - Nx) \mathbb{1}_{i+(0,1) \times (0,\lambda)}, \quad x \in \mathbb{R}^2, \tag{5.4}$$

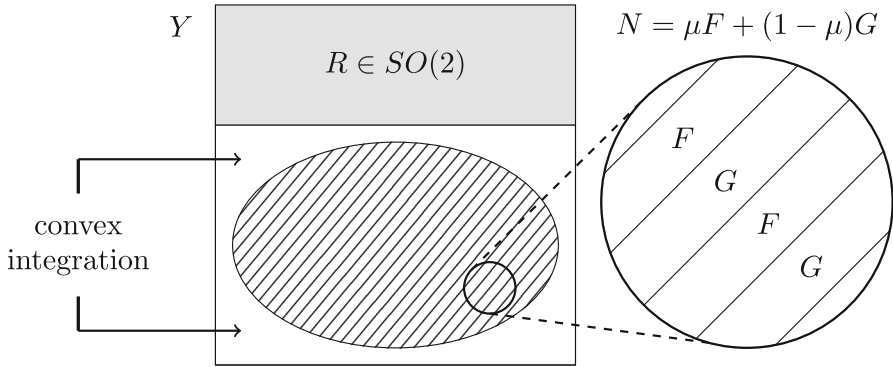


Fig. 2 Construction of admissible deformations by approximation with fine simple laminates in the softer component (illustrated by ∇z_ε in the unit cell Y). Here, $N e_1 = R e_1$, $N = \mu F + (1 - \mu)G$ with $F, G \in \mathcal{M}_s$ and $\mu \in (0, 1)$ as in Lemma 3.7, and the measure of the boundary region of Y_{soft} is smaller than ε

assuming without loss of generality that $\varphi_\varepsilon \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ is Y -periodic. With v_1 from Step 2a, we set $z_\varepsilon = v_1 + \varphi_\varepsilon$. Since $\nabla \varphi_\varepsilon \rightarrow 0$ in $L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^{2 \times 2})$ as $\varepsilon \rightarrow 0$,

$$\nabla z_\varepsilon \rightarrow \nabla v_1 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^{2 \times 2}). \tag{5.5}$$

Defining $u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with mean value zero by

$$\nabla u_\varepsilon = \nabla z_\varepsilon \left(\frac{\cdot}{\varepsilon} \right)$$

provides admissible functions for E_ε . Indeed, by construction one has $\nabla u_\varepsilon = R \in SO(2)$ in $\varepsilon Y_{\text{rig}} \cap \Omega$ and $\nabla u_\varepsilon \in \mathcal{M}_s$ a.e. in Ω due to the properties of $\nabla \varphi_{\varepsilon,i}$.

In view of (5.5), a generalization of the classical lemma on weak convergence of highly oscillating sequences (see e.g. [32, Theorem 1]) yields

$$\nabla u_\varepsilon \rightharpoonup \int_Y \nabla v_1 \, dy = \lambda N + (1 - \lambda)R = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \tag{5.6}$$

Finally, as $|(\nabla z_\varepsilon)m| < |Nm| + \varepsilon$ a.e. in \mathbb{R}^2 in consequence of (3.12) and $|Rm| = 1 \leq |Nm|$, we conclude that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow \infty} E_\varepsilon(u_\varepsilon) &= \limsup_{\varepsilon \rightarrow 0} \int_\Omega (|(\nabla z_\varepsilon)(\varepsilon^{-1} \cdot)m|^2 - 1) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \, dx \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_\Omega (|Nm|^2 - 1) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \, dx = E(u). \end{aligned}$$

Step 2c: Localization for piecewise constant γ Suppose that $\Omega \subset \mathbb{R}^2$ is a cube, say $Q = (0, l)^2$ with $l > 0$. In this step, the construction of Step 2b is extended to (finitely) piecewise constant γ .

In view of the independence of γ on x_1 by Remark 2.2(b), we may identify $\gamma \in L^2(\Omega)$ with a one-dimensional simple function

$$\gamma(t) = \sum_{i=1}^n \gamma_i \mathbb{1}_{(t_{i-1}, t_i)}(t), \quad t \in (0, l),$$

with $\gamma_i \in K_{s,\lambda}$ for $i = 1, \dots, n$ and $0 = t_0 < t_1 < \dots < t_n = l$. Let us denote by $N_i \in \mathcal{N}_s$ the matrices corresponding to γ_i according to Lemma 3.4(i).

For $i \in \{1, \dots, n\}$, let $(u_{i,\varepsilon})_\varepsilon \subset W^{1,2}((0, l) \times (t_{i-1}, t_i); \mathbb{R}^2)$ be the recovery sequences corresponding to γ_i as constructed in Step 2b. We then define $u_\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with vanishing mean value by

$$\nabla u_\varepsilon = R + \sum_{i=1}^n (\nabla u_{i,\varepsilon} - R) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \mathbb{1}_{\mathbb{R} \times ([\varepsilon^{-1}t_{i-1}]_\varepsilon, [\varepsilon^{-1}t_i]_\varepsilon)}.$$

Notice that u_ε is well-defined due to the compatibility between $\nabla u_{i,\varepsilon}$ and R along the jump lines $\mathbb{R} \times \varepsilon\mathbb{Z}$. Then, (5.6) leads to

$$\nabla u_\varepsilon \rightharpoonup \sum_{i=1}^n (\lambda N_i + (1 - \lambda)R) \mathbb{1}_{[\mathbb{R} \times (t_{i-1}, t_i)] \cap \Omega} = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}),$$

and regarding the energy contributions it follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |(\nabla u_\varepsilon)m|^2 - 1 \, dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \sum_{i=1}^n (|N_i m|^2 - 1) \mathbb{1}_{\varepsilon Y_{\text{soft}} \cap \Omega} \mathbb{1}_{\mathbb{R} \times ([\varepsilon^{-1}t_{i-1}]_\varepsilon, [\varepsilon^{-1}t_i]_\varepsilon)} \, dx \\ &= \lambda \int_{\Omega} \sum_{i=1}^n (|N_i m|^2 - 1) \mathbb{1}_{[\mathbb{R} \times (t_{i-1}, t_i)] \cap \Omega} \, dx = E(u). \end{aligned}$$

To generalize the result to a simply-connected Lipschitz domain Ω , we exhaust Ω successively with shifted, overlapping cubes, performing the necessary adaptations of the glued-in laminate constructions as well as the appropriate translations, cf. Remark 4.2.

Step 2d: Approximation and diagonalization for general γ For general $\gamma \in L^2(\Omega)$ with $\gamma \in K_{s,\lambda}$ a.e. in Ω we use an approximation and diagonalization argument.

Let $(\zeta_k)_k \subset L^2(\Omega)$ be a sequence of simple functions with $\zeta_k \rightarrow \gamma$ in $L^2(\Omega)$. For $k \in \mathbb{N}$, let $w_k \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\int_{\Omega} w_k \, dx = 0$ be defined by $\nabla w_k = R(\mathbb{I} + \zeta_k e_1 \otimes e_2)$. Then,

$$\|w_k - u\|_{L^2(\Omega; \mathbb{R}^2)} \leq c \|\nabla w_k - \nabla u\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq c \|\zeta_k - \gamma\|_{L^2(\Omega)}, \tag{5.7}$$

and

$$|E(w_k) - E(u)| \leq \|\zeta_k + \gamma\|_{L^2(\Omega)} \|\zeta_k - \gamma\|_{L^2(\Omega)} \leq c \|\zeta_k - \gamma\|_{L^2(\Omega)} \tag{5.8}$$

for all $k \in \mathbb{N}$ with a constant $c = c(\gamma) > 0$. If $(w_{k,\varepsilon})_\varepsilon$ is a recovery sequence for w_k as constructed in Step 2c, then in particular, $\lim_{\varepsilon \rightarrow 0} \|w_{k,\varepsilon} - w_k\|_{L^2(\Omega; \mathbb{R}^2)} = 0$, and $\lim_{\varepsilon \rightarrow 0} E_\varepsilon(w_{k,\varepsilon}) = E(w_k)$ for all $k \in \mathbb{N}$. Hence, together with (5.7) and (5.8),

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \|w_{k,\varepsilon} - u\|_{L^2(\Omega; \mathbb{R}^2)} + |E_\varepsilon(w_{k,\varepsilon}) - E(u)| \\ \leq \lim_{k \rightarrow \infty} \|w_k - u\|_{L^2(\Omega; \mathbb{R}^2)} + |E(w_k) - E(u)| = 0. \end{aligned}$$

From the selection principle by Attouch [1, Corollary 1.16], we infer the existence of a diagonal sequence $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon = w_{k(\varepsilon),\varepsilon}$ such that

$$u_\varepsilon \rightarrow u \quad \text{in } L^2(\Omega; \mathbb{R}^2) \quad \text{and} \quad E_\varepsilon(u_\varepsilon) \rightarrow E(u)$$

as $\varepsilon \rightarrow 0$. Notice that also $u_\varepsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ in consideration of Step 1.

Step 3: Lower bound Let $(\varepsilon_j)_j$ with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. Suppose $(u_j)_j$ is a bounded energy sequence for $(E_{\varepsilon_j})_j$ with $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$. By Step 1, we know that $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ for $R \in SO(2)$ and $\gamma \in L^2(\Omega)$.

If γ is piecewise constant, then

$$\liminf_{j \rightarrow \infty} E_{\varepsilon_j}(u_j) \geq \int_{\Omega} \frac{1}{\lambda} |(\nabla u)m - (1 - \lambda)Rm|^2 - \lambda \, dx = E(u)$$

as an immediate consequence of Corollary 2.5.

Similarly to Step 3 in the proof of Theorem 4.1, we use approximation to establish the lower bound for general u . Here again, we may restrict ourselves to working with the assumption that Ω is a cube, say $\Omega = Q = (0, l)^2$ for $l > 0$.

Let $(\zeta_k)_k \subset L^2(0, l)$ be one-dimensional simple functions (identified with a sequence in $L^2(\Omega)$ by constant extension in x_1) of the form $\zeta_k = \sum_{i=1}^{n_k} \zeta_{k,i} \mathbb{1}_{(t_{k,i-1}, t_{k,i})}$ with nested partitions $0 = t_{k,0} < t_{k,1} < \dots < t_{k,n_k} = l$ such that $\zeta_k \leq \zeta_{k+1}$ and

$$\zeta_k \rightarrow \gamma \text{ in } L^2(\Omega) \text{ as } k \rightarrow \infty. \tag{5.9}$$

Moreover, let $w_k \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L^2_0(\Omega; \mathbb{R}^2)$ be given by $\nabla w_k = R(\mathbb{1} + \zeta_k e_1 \otimes e_2)$ for $k \in \mathbb{N}$.

In the following, we aim at finding sequences $(v_j)_j, (v_{k,j})_j \subset W^{1,2}(\Omega; \mathbb{R}^2)$ with vanishing mean value such that

$$v_j \rightharpoonup u \text{ and } v_{k,j} \rightharpoonup w_k \text{ both in } W^{1,2}(\Omega; \mathbb{R}^2) \text{ as } j \rightarrow \infty \tag{5.10}$$

for all $k \in \mathbb{N}$, and

$$\nabla v_{k,j} = \nabla v_j \text{ a.e. in } \varepsilon_j Y_{\text{rig}} \cap \Omega \tag{5.11}$$

for all $j, k \in \mathbb{N}$. Moreover, we seek to have an estimate of the type

$$\|\nabla v_{k,j} - \nabla v_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq c \|\zeta_k - \gamma\|_{L^2(\Omega)} \tag{5.12}$$

with $c > 0$ independent of j, k .

Notice that instead of using recovery sequences for $(v_j)_j$ and $(v_{k,j})_j$, we will choose the piecewise affine functions obtained from Step 2, when skipping Step 2b (where fine laminates are glued in the softer layers). Indeed, the lack of admissibility does not cause any issues here. The advantage, though, is that due to their simpler structure, these functions are easier to compare in the sense of (5.12). Recall that the full recovery sequences of Step 2d involve regions resulting from convex integration, where the functions are not explicitly known and therefore hard to control.

Precisely, for $j, k \in \mathbb{N}$ we define $v_{k,j}$ by

$$\nabla v_{k,j} = R + \sum_{i=1}^{n_k} (N_{k,i} - R) \mathbb{1}_{\varepsilon_j Y_{\text{soft}} \cap \Omega} \mathbb{1}_{\mathbb{R} \times ([\varepsilon_j^{-1} t_{k,(i-1)}]_{\varepsilon_j}, [\varepsilon_j^{-1} t_{k,i}]_{\varepsilon_j})}$$

with $N_{k,i} \in \mathcal{N}_s$ corresponding to $\zeta_{k,i}$ in the sense of Lemma 3.4 (i), while $(v_j)_j$ results from a diagonalization argument as in Step 2d, i.e. $v_j = v_{k(j),j}$ for $j \in \mathbb{N}$. Hence, (5.10) and (5.11) are satisfied. Regarding (5.12), we argue that for $j, k, K \in \mathbb{N}$,

$$\begin{aligned} \|\nabla v_{k,j} - \nabla v_{K,j}\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} &\leq \left\| \sum_{i=1}^{n_k} \sum_{h=1}^{n_K} (N_{k,i} - N_{K,h}) \mathbb{1}_{\mathbb{R} \times (t_{K,h-1}, t_{K,h})} \mathbb{1}_{\mathbb{R} \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \\ &= \frac{1}{\lambda} \left\| \sum_{i=1}^{n_k} \sum_{h=1}^{n_K} (\zeta_{k,i} - \zeta_{K,h}) \mathbb{1}_{\mathbb{R} \times (t_{K,h-1}, t_{K,h})} \mathbb{1}_{\mathbb{R} \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega)} \\ &= \frac{1}{\lambda} \|\zeta_k - \zeta_K\|_{L^2(\Omega)}, \end{aligned}$$

where we have used (3.11). Thus,

$$\|\nabla v_{k,j} - \nabla v_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq \sup_{K \in \mathbb{N}} \|\nabla v_{k,j} - \nabla v_{K,j}\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq \frac{1}{\lambda} \|\zeta_k - \gamma\|_{L^2(\Omega)},$$

which yields (5.12).

Now we set

$$z_{k,j} = u_j - v_j + v_{k,j}$$

for $j, k \in \mathbb{N}$. Due to (5.11), it holds that $|(\nabla z_{k,j})e_2| = 1$ a.e. in $\varepsilon_j Y_{\text{rig}} \cap \Omega$, as well as

$$\nabla z_{k,j} \mathbb{1}_{\varepsilon_j Y_{\text{rig}}} \rightharpoonup (1 - \lambda)R \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2})$$

as $j \rightarrow \infty$ for $k \in \mathbb{N}$, cf. Proposition 2.1. Since also $z_{k,j} \rightharpoonup w_k$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for $j \rightarrow \infty$ by (5.10) with w_k piecewise affine, we argue as in the proof of Corollary 2.5 to derive

$$\liminf_{j \rightarrow \infty} \int_{\Omega} |(\nabla z_{k,j})m|^2 - 1 \, dx \geq E(w_k)$$

for all $k \in \mathbb{N}$. Along with (5.12) and (5.9), we finally conclude that

$$\liminf_{j \rightarrow \infty} E_{\varepsilon_j}(u_j) \geq \lim_{k \rightarrow \infty} E(w_k) - c\|\zeta_k - \gamma\|_{L^2(\Omega)} = E(u),$$

in analogy to Step 3 in the proof of Theorem 4.1. □

Remark 5.1 It may be more intuitive from the point of view of applications - yet technically more elaborate - to replace the recovery sequence obtained in Step 2b for the affine case by optimal deformations showing “non-stop” simple laminates throughout the softer layers. For this construction, just dispense with the adjustment of the affine boundary conditions along the vertical edges of the unit cell, and instead refine and shift the laminate appropriately to guarantee Y -periodicity.

6 Comparison with the (multi)cell formula

It is a well-known result in the theory of periodic homogenization of integral functionals with standard growth that the integrand of the effective limit functional is characterized by a multicell formula, or, in the convex case, by a cell formula, see e.g. [34,37]. In this final section, we show that the same is true for the homogenization result in Theorem 1.1, where extended-valued functionals appear.

Recalling W defined in (1.3), we consider the multicell formula

$$W_{\#}(F) = \inf_{k \in \mathbb{N}} \inf_{\psi \in W_{\#}^{1,2}(kY; \mathbb{R}^2)} \frac{1}{k^2} \int_{kY} W(y, F + \nabla \psi(y)) \, dy$$

for $F \in \mathbb{R}^{2 \times 2}$, or equivalently, by a change of variables,

$$W_{\#}(F) = \lim_{k \rightarrow \infty} \inf_{\psi \in W_{\#}^{1,2}(Y; \mathbb{R}^2)} \int_Y W(ky, F + \nabla \psi(y)) \, dy, \tag{6.1}$$

as well as the cell formula

$$W_{\text{cell}}(F) = \inf_{\psi \in W_{\#}^{1,2}(Y; \mathbb{R}^2)} \int_Y W(y, F + \nabla \psi(y)) \, dy.$$

Moreover, let us denote by W_{hom} the density of the limit energy E in (1.7), i.e.

$$E(u) = \int_{\Omega} W_{\text{hom}}(\nabla u) \, dx, \quad u \in W^{1,2}(\Omega; \mathbb{R}^2), \tag{6.2}$$

where

$$W_{\text{hom}}(F) = \begin{cases} \frac{1}{\lambda} |Fm - (1-\lambda)Rm|^2 - \lambda & \text{if } F = R(\mathbb{I} + \gamma e_1 \otimes e_2), \gamma \in K_{s,\lambda}, \\ \infty & \text{otherwise,} \end{cases} \quad F \in \mathbb{R}^{2 \times 2}.$$

This alternative representation of E follows from a straightforward calculation, see (5.1).

Before focusing on the relation between W_{hom} , $W_{\#}$, and W_{cell} , we prove the following auxiliary result.

Lemma 6.1 *Let $F \in \mathbb{R}^{2 \times 2}$ and $\psi \in W_{\#}^{1,2}(Y; \mathbb{R}^2)$.*

- (i) *If $F + \nabla\psi \in \mathcal{M}_s$ a.e. in Y and $F + \nabla\psi \in SO(2)$ a.e. in Y_{rig} , then $F \in \mathcal{M}_{e_1} \cap \mathcal{N}_s$.*
- (ii) *If $F = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$ and $\gamma \in \mathbb{R}$ and $F + \nabla\psi \in \mathcal{M}_{e_1}$ a.e. in Y , then*

$$F + \nabla\psi = R(\mathbb{I} + \zeta e_1 \otimes e_2),$$

where $\zeta \in L^2(Y)$ with $\int_Y \zeta \, dy = \gamma$.

Proof For (i), let $R \in L^\infty(Y; SO(2))$ and $\zeta \in L^2(Y)$ such that $F + \nabla\psi = R(\mathbb{I} + \zeta s \otimes m)$. Then, using that ψ is Y -periodic, we obtain

$$|Fs| = \left| \int_Y Fs + (\nabla\psi)s \, dy \right| = \left| \int_Y Rs \, dy \right| \leq 1.$$

As the map $\det : F \mapsto \det F$ is quasilinear, i.e. \det and $-\det$ are both quasiconvex (see e.g. [16] for an introduction to generalized notions of convexity), it follows that

$$\det F = \int_Y \det(F + \nabla\psi) \, dy = \int_Y \det R \cdot \det(\mathbb{I} + \zeta s \otimes m) \, dy = 1.$$

Hence, $F \in \mathcal{N}_s$.

To prove $F \in \mathcal{M}_{e_1}$, or rather $|Fe_1| = 1$, we exploit that by rigidity (see Lemma 2.3), there exists $Q \in SO(2)$ such that $F + \nabla\psi = Q$ a.e. in Y_{rig} . The periodicity of ψ in y_1 then leads to

$$Fe_1 = \int_{Y_{\text{rig}}} Fe_1 + \partial_1\psi \, dy = \int_{Y_{\text{rig}}} Fe_1 + (\nabla\psi)e_1 \, dy = Qe_1,$$

which entails $|Fe_1| = 1$.

As regards (ii), since $F + \nabla\psi \in \mathcal{M}_{e_1}$ a.e. in Y , we find $Q \in L^\infty(Y; SO(2))$ and $\zeta \in L^2(Y)$ such that

$$F + \nabla\psi = Q(\mathbb{I} + \zeta e_1 \otimes e_2) \quad \text{in } Y.$$

By the periodicity of ψ in y_1 ,

$$\int_Y Qe_1 \, dy = Fe_1 + \int_Y (\nabla\psi)e_1 \, dy = Re_1,$$

which, owing to $|Qe_1| = 1$ a.e. in Y , implies $Q = R$. On the other hand, we derive from the periodicity of ψ in y_2 that

$$\left(\int_Y \zeta \, dy \right) Re_1 = \int_Y \zeta Qe_1 \, dy = Fe_2 + \int_Y (\nabla\psi)e_2 \, dy - \int_Y Qe_2 \, dy = \gamma Re_1.$$

This finishes the proof. □

As indicated above, the multicell and cell formula for W both coincide with the homogenized integrand W_{hom} .

Proposition 6.2 (Characterization of the (multi)cell formula) *With the definitions above it holds that $W_{\text{hom}} = W_{\#} = W_{\text{cell}}$.*

Proof Since trivially $W_{\#} \leq W_{\text{cell}}$, it is enough to prove $W_{\#} \geq W_{\text{hom}}$ and $W_{\text{hom}} \geq W_{\text{cell}}$. From Lemma 6.1 (i) we conclude that $W_{\#}(F) = W_{\text{cell}}(F) = \infty$ for $F \notin \mathcal{M}_{e_1} \cap \mathcal{N}_S \supset \{F \in \mathbb{R}^{2 \times 2} : W_{\text{hom}}(F) < \infty\}$, cf. (3.4). Therefore, in the following, we can restrict ourselves to $F \in \mathcal{M}_{e_1} \cap \mathcal{N}_S$.

Step 1: $W_{\#}(F) \geq W_{\text{hom}}(F)$ Without loss of generality, assume that $W_{\#}(F) < \infty$. According to the definition of $W_{\#}$ in (6.1) there exists a sequence $(\psi_k)_k \subset W_{\#}^{1,2}(Y; \mathbb{R}^2)$ with

$$W_{\#}(F) = \liminf_{k \rightarrow \infty} \int_Y W(ky, F + \nabla \psi_k(y)) \, dy. \tag{6.3}$$

Imitating the proofs of Theorem 1.1 in Sects. 4 and 5 regarding compactness (Steps 1) and the lower bound (Steps 3) with $\Omega = Y$, we obtain that there is an subsequence (not relabeled) of $(\psi_k)_k$ and $u \in W^{1,2}(Y; \mathbb{R}^2)$ with $\nabla u = F + \nabla \psi$ for some $\psi \in W_{\#}^{1,2}(Y; \mathbb{R}^2)$ such that

$$F + \nabla \psi_k \rightharpoonup \nabla u = F + \nabla \psi \quad \text{in } L^2(Y; \mathbb{R}^{2 \times 2}),$$

and

$$W_{\#}(F) \geq E(u) = \int_Y W_{\text{hom}}(F + \nabla \psi) \, dy, \tag{6.4}$$

in view of (6.3) and (6.2). By the definition of W_{hom} , it follows that $F + \nabla \psi \in \mathcal{M}_{e_1}$ a.e. in Y . Along with the assumption $F \in \mathcal{M}_{e_1}$ (precisely, $F = R(\mathbb{I} + \gamma e_1 \otimes e_2)$ with $R \in SO(2)$ and $\gamma \in \mathbb{R}$), Lemma 6.1 (ii) in conjunction with Jensen’s inequality yields

$$\int_Y W_{\text{hom}}(F + \nabla \psi) \, dy \geq \frac{1}{\lambda} \left| Fm + \int_Y (\nabla \psi)m \, dy - (1 - \lambda)Rm \right|^2 - \lambda = W_{\text{hom}}(F). \tag{6.5}$$

Joining (6.4) and (6.5) finishes the proof of Step 1.

Step 2: $W_{\text{cell}}(F) \leq W_{\text{hom}}(F)$ To show the reverse inequality, we build on the construction of recovery sequences for affine limit functions (Step 2) in the proofs of Theorem 1.1, again with $\Omega = Y$, see Sects. 4 and 5. Accordingly, there is a sequence $(u_k)_k \subset W^{1,2}(Y; \mathbb{R}^2)$ such that $\nabla u_k \rightharpoonup F$ in $L^2(Y; \mathbb{R}^{2 \times 2})$ and

$$W_{\text{hom}}(F) = \lim_{k \rightarrow \infty} \int_Y W(ky, \nabla u_k(y)) \, dy.$$

By construction, the functions ψ_k defined by $\psi_k(y) = u_k(y) - Fy$ for $y \in Y$ are $k^{-1}Y$ -periodic (see (4.4), and (5.4), (5.3); indeed, $v_1 - F \cdot$ is Y -periodic), so that

$$\begin{aligned} W_{\text{hom}}(F) &= \lim_{k \rightarrow \infty} \int_Y W(ky, F + \nabla \psi_k(y)) \, dy = \lim_{k \rightarrow \infty} \int_Y W(y, F + \nabla \psi_k(k^{-1}y)) \, dy \\ &\geq W_{\text{cell}}(F), \end{aligned}$$

as stated. □

Acknowledgements The authors would like to thank Georg Dolzmann for his valuable comments on a preliminary version of the manuscript. This work was partially supported by the Deutsche Forschungsgemeinschaft through the Forschergruppe 797 “Analysis and computation of microstructure in finite plasticity”, Project DO 633/2-1.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Attouch, H.: Variational Convergence for Functions and Operators. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston (1984)
2. Bouchitté, G., Bellieud, M.: Homogenization of a soft elastic material reinforced by fibers. *Asymptot. Anal.* **32**(2), 153–183 (2002)
3. Braides, A.: Γ -Convergence for Beginners. Oxford Lecture Series in Mathematics and Its Applications, vol. 22. Oxford University Press, Oxford (2002)
4. Braides, A., Defranceschi, A.: Homogenization of Multiple Integrals. Oxford Lecture Series in Mathematics and Its Applications, vol. 12. The Clarendon Press, New York (1998)
5. Braides, A., Fonseca, I., Leoni, G.: A -quasiconvexity: relaxation and homogenization. *ESAIM Control Optim. Calc. Var.* **5**, 539–577 (2000)
6. Braides, A., Garroni, A.: Homogenization of periodic nonlinear media with stiff and soft inclusions. *Math. Models Methods Appl. Sci.* **5**(4), 543–564 (1995)
7. Carbone, L., De Arcangelis, R.: Unbounded Functionals in the Calculus of Variations: Representation, Relaxation, and Homogenization. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 125. Chapman & Hall/CRC, Boca Raton (2002)
8. Cardone, G., Corbo Esposito, A., Yosifian, G.A., Zhikov, V.V., Zhikov, V.V.: Homogenization of some problems with gradient constraints. *Asymptot. Anal.* **38**(3–4), 201–220 (2004)
9. Carstensen, C., Hackl, K., Mielke, A.: Non-convex potentials and microstructures in finite-strain plasticity. In: Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Science, vol. 458, no. 2018, pp. 299–317 (2002)
10. Cherdantsev, M., Cherednichenko, K.D.: Two-scale Γ -convergence of integral functionals and its application to homogenisation of nonlinear high-contrast periodic composites. *Arch. Ration. Mech. Anal.* **204**(2), 445–478 (2012)
11. Christowiak, F., Kreisbeck, C.: Laminate structures in plastic composite materials with rigid layers. *Proc. Appl. Math. Mech.* **15**, 539–540 (2015)
12. Cioranescu, D., Damlamian, A., De Arcangelis, R.: Homogenization of integrals with pointwise gradient constraints via the periodic unfolding method. *Ric. Mat.* **55**(1), 31–53 (2006)
13. Conti, S.: Relaxation of single-slip single-crystal plasticity with linear hardening. In: Gumbsch, P. (ed.) *Multiscale Materials Modeling*, pp. 30–35. Fraunhofer IRB, Freiburg (2006)
14. Conti, S., Dolzmann, G., Kreisbeck, C.: Asymptotic behavior of crystal plasticity with one slip system in the limit of rigid elasticity. *SIAM J. Math. Anal.* **43**(5), 2337–2353 (2011)
15. Conti, S., Theil, F.: Single-slip elastoplastic microstructures. *Arch. Ration. Mech. Anal.* **178**(1), 125–148 (2005)
16. Dacorogna, B.: *Direct Methods in the Calculus of Variations*. Applied Mathematical Sciences, vol. 78, 2nd edn. Springer, New York (2008)
17. Dal Maso, G.: *An Introduction to gamma-Convergence*. Progress in Nonlinear Differential Equations and Their Applications, vol. 8. Birkhäuser, Boston (1993)
18. Davoli, E., Fonseca, I.: Homogenization of integral energies under periodically oscillating differential constraints. *Calc. Var. Partial Differ. Equ.* **55**(3), 60 (2016)
19. De Giorgi, E.: Sulla convergenza di alcune successioni d'integrali del tipo dell'area. *Rend. Mat.* **6**(8), 277–294 (1975). (Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday)
20. De Giorgi, E., Franzoni, T.: Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **58**(6), 842–850 (1975)
21. Fonseca, I., Krömer, S.: Multiple integrals under differential constraints: two-scale convergence and homogenization. *Indiana Univ. Math. J.* **59**(2), 427–457 (2010)
22. Fonseca, I., Leoni, G., Malý, J.: Weak continuity and lower semicontinuity results for determinants. *Arch. Ration. Mech. Anal.* **178**(3), 411–448 (2005)
23. Francfort, G., Giacomini, A.: On periodic homogenization in perfect elasto-plasticity. *J. Eur. Math. Soc. (JEMS)* **16**(3), 409–461 (2014)
24. Francfort, G.A., Marigo, J.-J.: Stable damage evolution in a brittle continuous medium. *Eur. J. Mech. A Solids* **12**(2), 149–189 (1993)

25. Francfort, G.A., Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* **46**(8), 1319–1342 (1998)
26. Friesecke, G., James, R.D., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Commun. Pure Appl. Math.* **55**(11), 1461–1506 (2002)
27. Jikov, V.V., Kozlov, S.M., Oleĭnik, O.A.: *Homogenization of Differential Operators and Integral Functionals*. Springer, Berlin. Translated from the Russian by G. A. Yosifian (1994)
28. Jones, R.M.: *Mechanics of Composite Materials*. Materials Science and Engineering Series, 2nd edn. CRC Press, Boca Raton (1998)
29. Kohn, R.V., Little, T.D.: Some model problems of polycrystal plasticity with deficient basic crystals. *SIAM J. Appl. Math.* **59**(1), 172–197 (1999). (electronic)
30. Kröner, E.: Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen. *Arch. Ration. Mech. Anal.* **4**, 273–334 (1960)
31. Lee, E.H.: Elastic–plastic deformation at finite strains. *J. Appl. Mech.* **36**, 1–6 (1969)
32. Lukkassen, D., Wall, P.: On weak convergence of locally periodic functions. *J. Nonlinear Math. Phys.* **9**(1), 42–57 (2002)
33. Mallick, P.: *Fiber-Reinforced Composites: Materials Manufacturing and Design*. Mechanical Engineering, 3rd edn. CRC Press, Boca Raton (2007)
34. Marcellini, P.: Periodic solutions and homogenization of nonlinear variational problems. *Ann. Mat. Pura Appl.* **4**(117), 139–152 (1978)
35. Miehe, C., Schröder, J., Becker, M.: Computational homogenization analysis in finite elasticity: material and structural instabilities on the micro- and macro-scales of periodic composites and their interaction. *Comput. Methods Appl. Mech. Eng.* **191**(44), 4971–5005 (2002)
36. Milton, G.W.: *The Theory of Composites*. Cambridge Monographs on Applied and Computational Mathematics, vol. 6. Cambridge University Press, Cambridge (2002)
37. Müller, S.: Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Ration. Mech. Anal.* **99**(3), 189–212 (1987)
38. Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc. (JEMS)* **1**(4), 393–422 (1999)
39. Ortiz, M., Repetto, E.A.: Nonconvex energy minimization and dislocation structures in ductile single crystals. *J. Mech. Phys. Solids* **47**(2), 397–462 (1999)
40. Reina, C., Schlömerkemper, A., Conti, S.: Derivation of $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ as the continuum limit of crystalline slip. *J. Mech. Phys. Solids* **89**, 231–254 (2016). doi:[10.1016/j.jmps.2015.12.022](https://doi.org/10.1016/j.jmps.2015.12.022)
41. Reina, C., Conti, S.: Kinematic description of crystal plasticity in the finite kinematic framework: a micromechanical understanding of $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$. *J. Mech. Phys. Solids* **67**, 40–61 (2014)
42. Reina, C., Schlömerkemper, A., Conti, S.: Derivation of $\mathbf{F}^e \mathbf{F}^p$ as the continuum limit of crystalline slip. *J. Mech. Phys. Solids* (2016)
43. Reshetnyak, Y.G.: Liouville’s conformal mapping theorem under minimal regularity hypotheses. *Sib. Math. J.* **8**, 835–840 (1967)
44. Schröder, J., Hackl, K.: *Plasticity and Beyond: Microstructures, Crystal-Plasticity and Phase Transitions*. CIMS International Centre for Mechanical Sciences, vol. 550. Springer, Wien (2013)
45. Vasiliev, V.V., Morzov, E.V.: *Advanced Mechanics of Composite Materials and Structural Elements*. Mechanical Engineering, 3rd edn. Elsevier, Amsterdam (2013)