A neutral homoclinic bifurcation in a 3D map

Meijer H.G.E.[*], Govaerts W. [**], Kuznetsov Y.A. [*,***], and Neirycnk, N.[**]
[*] *Department EEMCS, University of Twente, PO Box 217, 7500 AE Enschede, Netherlands,*
[**] *Dept. of Applied Mathematics and Computer Science, Ghent University, Gent, Belgium*
[***] *Dept. of Mathematics, Utrecht University, Utrecht, Netherlands*

Summary. We investigate the structure of Arnol'd tongues passing through a quasi-periodic saddle-node bifurcation in a 3D map. Due to resonances, the bifurcation set has a complicated structure. Using numerical continuation in MatcontM and Lyapunov exponents we explore the tongues and the quasi-periodic saddle-node bifurcation set.

The bifurcation set emerges from two Chenciner bifurcations. Both sets terminate in a homoclinic bifurcation of a neutral saddle cycle of period 3. It is similar to a case for vector fields, but the first report of such a codim-2 homoclinic tangency bifurcation in a map. We also show how these manifolds oscillate around the saddle near the tangency, for real and complex multipliers.

A map with a quasi-periodic saddle-node bifurcation

The typical appearance of a quasi-periodic saddle-node (QNS) bifurcation for an iterated map is from a Chenciner bifurcation. This is codimension 2 degenerate Neimark-Sacker(NS) bifurcation. At this codim 2 point, the stability of the invariant curve created from the Neimark-Sacker bifurcation changes stability. Here we report upon another case where such a QNS-bifurcation appears. Rather than starting from the codim 2 situation immediately, we show how it follows from studying the resonance tongues starting from the Neimark-Sacker bifurcation. We study this in the Adaptive Control Map, which is a 3D map introduced in [1]

$$F:\begin{pmatrix} x\\ y\\ z\end{pmatrix}\mapsto\begin{pmatrix} y\\ bx+k+zy\\ z-(bx+k+zy-1)ky/(c+y^2)\end{pmatrix}, \tag{1}$$

with $c = 0.1$ and k, b free parameters.

Global Inventory of Dynamics

We start with a partial inventory of the bifurcation diagram. On the curve $b = -(c+1)/(c+2) \approx -.5238$, the fixed point $(x, y, z) = (1, 1, 1-b-k)$ exhibits a Neimark-Sacker bifurcation, from which an invariant curve emerges. Its stability depends on the sign of the first Lyapunov coefficient l_1. It can be shown analytically that l_1 is positive for $k_1 < k < k_2$, where $k = k_1 \approx .021$ and $k = k_2 \approx 1.300639$. And l_1 is negative for other values of k. So the invariant curve is unstable for $k_1 < k < k_2$. This leads to a bistability between the fixed point and an invariant curve as we will now sketch. On the invariant curve we find periodic cycles; such regions are known as Arnol'd tongues. We have searched for Arnol'd tongues of period q with $4 < q < 21$. Initial points were computed by simulation and checked for periodicity. Then the outer boundaries of the tongue were computed with continuation using MATCONTM[2]. Along these boundaries we detected R1 points of fixed points of period q. Here the unstable cycles pass through the bifurcation set of the quasi-periodic saddle-node, and become stable. This yields the bistability. We complement this bifurcation diagram with a dynamical inventory based on Lyapunov exponents. If the dynamics is not periodic, we can classify the dynamics as an invariant curve or as chaotic. The resulting inventory is shown in Figure 1. The bistability is roughly demarcated by the transition from red to white, but it is interrupted by so-called resonance bubbles.

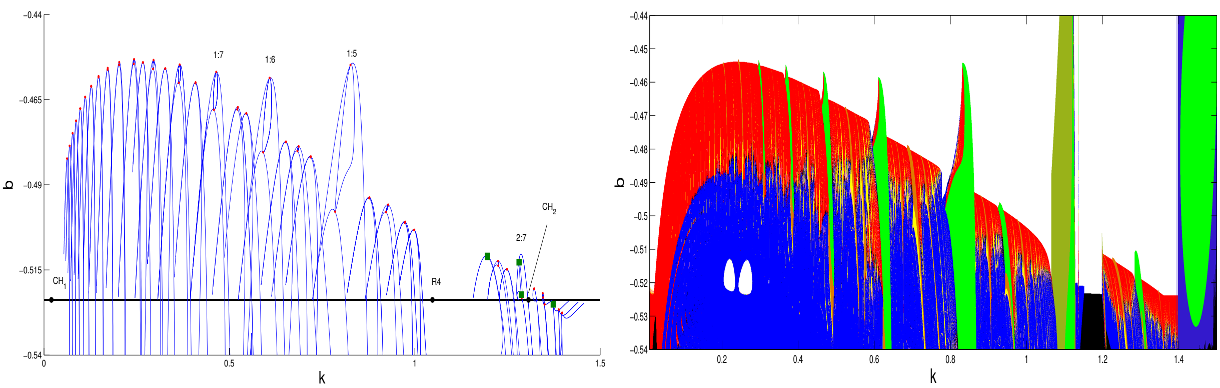

Figure 1: Left: The horizontal (black) line is the Neimark-Sacker bifurcation of the fixed point. From this curve resonance tongues emanate indicated by solid (blue) lines. These tongues bend near a quasi-periodic saddle-node bifurcation where the resonance tongues exhibit resonance bubbles. Each bubble has a pair of R1-points indicated by the red dots. Green squares indicate Chenciner bifurcations. Right: A dynamical inventory of (1) based on Lyapunov exponents. The color coding is as follows: red=invariant torus, blue=chaotic, green=periodic (3-9), orange=periodic (10-20),yellow=periodic (21-50), pink=periodic(50-100), white=period 1, black=no attractor found.

A Codim 2 Neutral Homoclinic Tangency

The map has two Chenciner bifurcations. Each leads to a quasi-periodic saddle-node bifurcation. Here we look closer to the location of the quasi-periodic saddle-node bifurcation as k becomes larger and the two bifurcation sets get closer. To do so, we have computed Lyapunov exponents for fixed values of k while increasing or decreasing b. In each step, we use the final point of the computation as initial point for the new simulation. In this way, we can follow an attractor even if it is not a periodic cycle. We can also detect a quasi-periodic saddle-node bifurcation as a sudden jump of the exponents, see Figure 2(left). Collecting the location of $\text{QNS}_{1,2}$ for various k, we find Figure 2(right). Note that the QNS-"curves" are interrupted precisely at resonance tongues. The tongues (period $q < 20$) nicely follow a Farey sequence.
Next we noted that the $\text{QNS}_{1,2}$ bifurcations did not get closer, but disappeared as we increased k further. We then studied how the invariant curve disappeared starting from the NS-bifurcation. Again fixing k, we then decreased b and monitored the Lyapunov exponents to see whether the attractor was still an invariant curve. We then found a critical value where the invariant curve broke up in a homoclinic tangency of a neutral saddle cycle of period 3. At the transition, the multipliers of the cycle are $\lambda_1 > 1 > \lambda_2 > \lambda_3 > 0$ with $\lambda_1\lambda_2 = 1$. This resembles the unfolding of an orientable Belyakov point for ODE's, see [3]. We note that previous reports of a neutral saddle homoclinic tangency for maps did not mention that a quasi-periodic saddle-node bifurcation could be involved. A homoclinic orbit leads to a (small) wedge in parameter space demarcated by two curves of homoclinic tangency. We find that $QSN_{1,2}$ end at the two different branches.

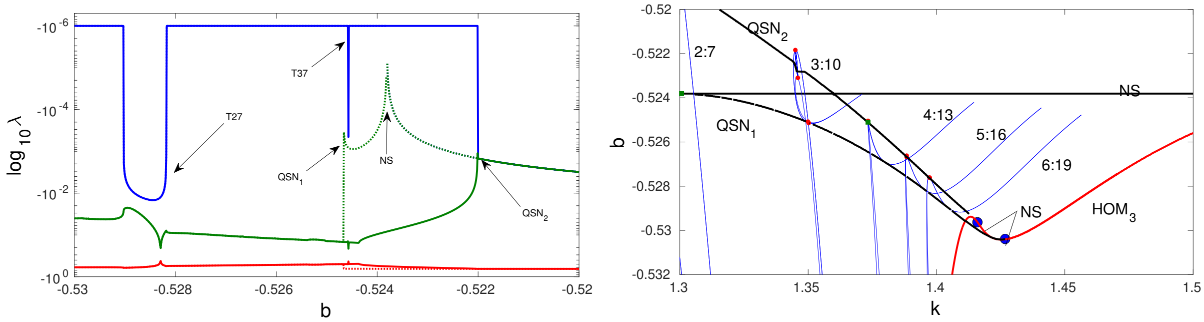

Figure 2: Left: The evolution of the Lyapunov exponents as b is varied with fixed $k = 1.35$. Colors are blue for λ_1, green for λ_2, red for λ_3, plotted on a log-scale. The solid(dashed) lines display the scan for decreasing(increasing) b. The fixed point is stable for $b > b_{NS} \approx -.5238$ where the solid λ_1 and λ_2 coincide. There is a region of bistability of invariant curves for $-.5251 < b < -0.522920$ as indicated by the arrows for $\text{QSN}_{1,2}$. One can also see several resonance tongues of period 27 and 37. Right: The diagram shows where invariant curves (dis) appear using results from Lyapunov exponents to track quasi-periodic saddle-node and continuation of heteroclinic connections. Superimposed we show the primary Neimark-Sacker curve and some resonance tongues. In the region enclosed by the primary NS curve and the two $\text{QSN}_{1,2}$ sets there are two stable invariant curves. In particular, holes where resonances occur are visible for the set QSN_1, but less for QSN_2.

To verify our results we have computed the invariant manifolds near HOM_3, see Figure 3. For values of (k, b) above QNS_1 and HOM_3 we have an invariant curve. The unstable manifold of the cycle approaches this invariant curve. For higher values of k the saddle has real multipliers For $k < 1.415$ the saddle has complex stable multipliers. So, for parameters in the vicinity of the homoclinic tangency, the unstable manifold spirals to the saddle while the amplitude of the oscillations increases.

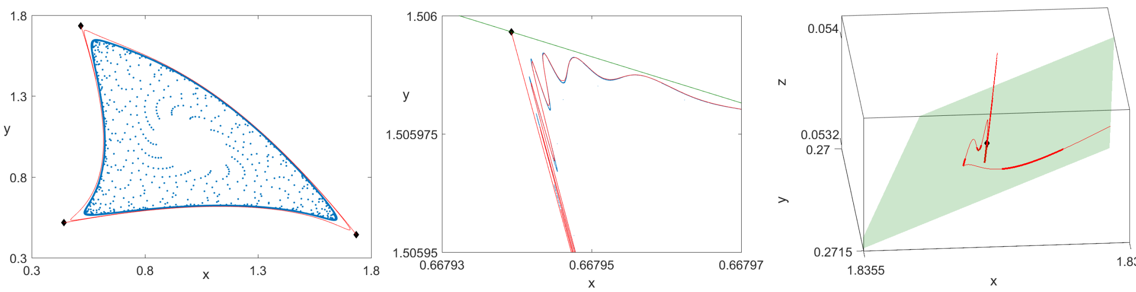

Figure 3: Left: Phase space with the period 3 saddle cycle and the invariant curve for $(k, b) = (1.42, -.526)$. Blue dots show an orbit, Black diamonds the period 3 saddle cycle, Red curves indicate one side of the unstable manifold W^u. Middle: Close to a simple tangency, W^u oscillates as it approaches the saddle cycle for $(k, b) = (1.45, -.5290011017)$. The green line indicates the leading stable direction. Right: Transverse homoclinic orbit when the saddle has complex stable multipliers for $(k, b) = (1.41, -.52989945)$. The unstable manifold oscillates through the linear tangent space W^s_{loc}(green); on one side the red line is thicker.

References

[1] Frouzakis C.E., Adomaitis R.A. and Kevrekidis I.G., Resonance phenomena in an adaptively-controlled system, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 1:, 83-106, 1991

[2] W. Govaerts, R. Khoshsiar Ghaziani, Yu.A. Kuznetsov, and H.G.E. Meijer, Numerical continuation of connecting orbits of maps, *Journal of Difference Equations and Applications*, 15 (8): 849-875, 2009.

[3] Homburg A.J. and Sandstede B. (2010) Homoclinic and heteroclinic bifurcations in vector fields. Handbook of Dynamical Systems III (Edited by H Broer, F Takens and B Hasselblatt) Elsevier, pp. 379-524.