4. Realistic Mathematics Education

Introduction

As mentioned in previous chapters, Artigue and Blomhøj (2013) point out how a number of well-established research programmes in mathematics education have developed methods and ideas for what is now called IBMT. Realistic Mathematics Education (RME) is one of the most prominent, along with TDS.

RME consists of ideas and principles for shaping the learning process. This chapter gives an overview of the main ideas in RME, aimed at teachers and educational designers. Ideas are illustrated by example tasks. In this text the theory of RME is build up from two central principles:

(1) Mathematics is a human activity.

(2) Meaningful mathematics is built from rich contexts.

In the final sections we describe the connection between RME principles and Inquiry Based Mathematics Teaching (IBMT) and discuss RME ideas that can help with the design of IBMT scenarios.

Structuring mathematics

Mathematical knowledge can be structured to very high extent, while RME claims that the learning process requires a less formal approach. In the formal approach, one begins from axioms, postulates and definitions, and from there derives lemmas and theorems. Proofs establish the truth of these propositions within the axiomatic frame. The tradition of organizing and presenting mathematics results in this formal way stretches from Euclid (300 BC) to contemporary mathematics research. Mathematics as a building, where axioms are foundations and logic mortar, is impressive and effective. The formal presentation of results enables unambiguous academic communication. No wonder some have based mathematics education on it. In many countries, geometry was taught from *the Elements* of Euclid until the 1950's. In the 1950's and 1960's, the New Math movement introduced set theory as a basis for secondary mathematics education.

Mathematics as a human activity

*S*hould this highly structured body of mathematical knowledge be the leading inspiration on how we shape mathematics education? RME takes a different point of view. Its leading inspiration is that mathematics *is a human activity.* The organised body of mathematical knowledge is a product of this activity. For example, a good definition of a mathematical object is often the result of a long process of mathematical thoughts, ideas and attempts. RME underlines the importance of these processes that lead to the polished version of a mathematical object or result.

One could, in the very beginning of a chapter on logarithms, define the logarithmic function as the inverse of the exponential function. An RME based approach would rather begin with a task that shows the need for the concept. The exercise should allow the students to experience this necessity for a logarithmic function themselves. Here is a basic idea.

Rogier puts 100 euro's in the bank. The interest rate is 2%. Fill in the table.

Amount (A)	100	$\approx$ 108,24	$\approx$ 129,36	$\approx$ 199,99	$\approx$ 507,24
Years passed (t)	0				

Do you know a function to compute t from A?

Of course the answer to the last question is likely to be "no", but it is important for learners to ask this question and realise there is need for a new function. Learners are not used to this type of questions. For this reason the question might better be answered in a classroom discussion guided by the teacher. Learners may come up with (square) root functions and need guidance discovering why that is wrong.

Anti-didactical inversion

Presenting a learner with mathematics in its highly structured (axiomatic system based) version is an *inversion*. The learner is confronted with the result of an often long and difficult process of doing mathematics. If the learner is supposed to study mathematics this way, then the learning process is an inversion of the process that led to the mathematics. He will have to work hard (or to wait) to find out which questions gave rise to this mathematics and which problems were solved by it. The teacher could have consciously chosen for this approach, but RME claims it is not a didactical one: it is an *anti-didactical inversion* (Freudenthal, 1991).

Generally, a formal presentation of mathematics is rather inaccessible for novice learners. There are many didactical arguments against confronting a learner with mathematics in its highly structured polished version in the beginning of the learning process:

- The natural process (being led by questions, problems, curiosity …) of arriving at the mathematics is not shown. Meaning and motivation is taken away from the learner.
- Intuition that leads to the theory is remote from the learning process.
- It is not clear what is solved, modelled or captured by the system (and what is not).
- Heuristics that were needed to organise the mathematics in that way are neglected.

- The presentation can be too dense or sparse. An aspect of the mathematics may be very difficult to grasp but only be given little emphasis in a formal presentation.

Many mathematicians, including mathematics teachers, will remember being confronted with the ε, δ-definition of limits in the first year of their studies, or even in high school. Why was this so inaccessible? It does not make sense to a learner if she has no understanding of problems with rigorous proof that emerged in Analysis at the beginning of the 19th century. What issue does it solve? Why such effort to prove something obvious? Why do other definitions not work?

Similarly, stating the distributive law "$a \cdot (b + c) = a \cdot b + a \cdot c$" just like that in secondary education, followed by exercises like "*expand* $5 \cdot (a + 2)$", is formally a correct order, but does not convey any meaning to the learner. Neither does it answer the question why this is a useful rule or skill.

The role of realism in learning processes

Obviously the (formal) meaning of mathematical objects and procedures is carefully and precisely described in formal presentations of mathematics. Since presentations with formal characteristics can be inaccessible and not didactical for a novice learner, how is one supposed to teach this meaning?

A learning process is formed by a set of learning activities. One of the central ideas of RME is that the situations on which those activities are based should be real or realistic. The meaning of mathematical concepts and procedures is constructed from what is already meaningful to the learner, from what is *real* to the learner.

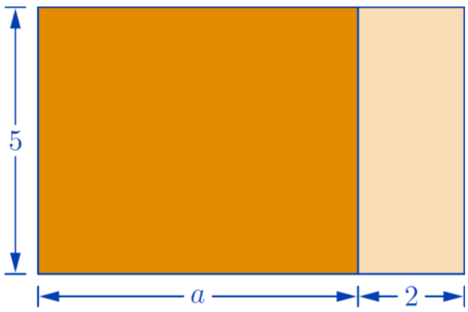
What is meant by "real/realistic" in RME? Something is real for a learner, if it has some evident meaning to her, if she can grasp it. Something is *real* for a group of learners when it is *common sense* to them. 'Real" does not (necessarily) mean "modelled on reality", for example modelled on situations from other disciplines, like physics or economics. Nor does a "realistic" learning situation necessarily mean that it is based on an everyday life experience. And "real" is definitely not meant ontologically: what does and what does not exist. In fact, "meaningful mathematics" might be a better expression than "realistic mathematics", but the latter happens to be the label, as it emerged in the previous century. Meaningful mathematics is learned by starting from what is already meaningful for the learner, in particular from meaningful contexts. As Freudenthal states it:

> *How real [the] concepts are depends on the conceiver, and under given circumstances cognitive grasps can be more vigorous than manual and sensual ones, which are in fact always mixed up with cognition" and "(What is real is) mutually connected by actual, imagined and symbolised relations (....) which can extend from the nucleus of everyday life experience to the far frontiers of mathematical research, depending on the involvement of who is concerned.* (Freudenthal, 1991, p.30).

The distributive laws can be introduced in a realistic geometric context:
Compute the area of the whole rectangle in two ways:

(1) First the dark, then the light rectangle and then add the two
(2) First compute the whole width and then multiply by the height

(based on *van den Broek et al., n.d.*)



Why is this a (more) realistic approach? The learner is assumed to be familiar with computing areas. The meaning of the equality emerges naturally as the outcomes of the two computations have to be equal. Meaning emerges from the task alone. The teacher has a role introducing the task, guiding the learners and reflecting on the task in classroom. He has to embed the task in a learning process in the right way. Later in this text follows more on RME views on this.

Rich structures and rich contexts

According to RME new meaning of mathematics for a learner is not drawn from the formal mathematical edifice, but mostly from what is real for the learner. The didactical situation should allow development of new knowledge from what is already meaningful. This means it should be rich in non-mathematical contexts and in mathematical structures. Here are possible ways in which a mathematical structure or a context can be *rich:*

(1) it connects to various aspects of the learner's common sense - the more connections made, the richer the structure;
(2) its usefulness carries further mathematically than the situation where it is introduced;
(3) it allows different approaches or solutions on different levels.

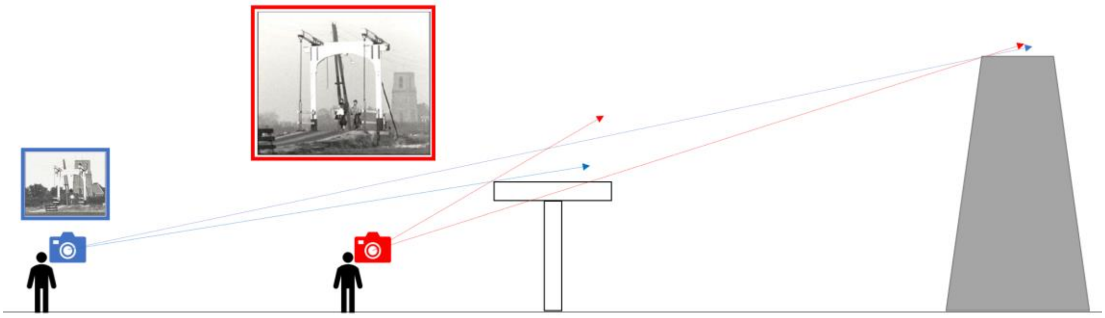
We now proceed to illustrate these ways by concrete examples.

Point (1) is illustrated by the following 'The tower and the bridge' task. It was used in an experiment for introducing scale and geometrical reasoning in a 3D context (Goddijn, 1979).

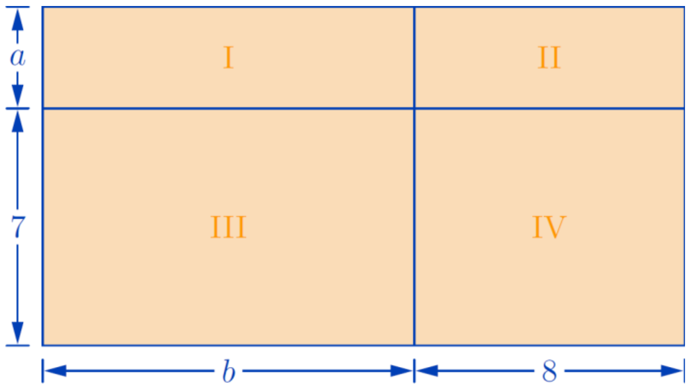
Below you see two photos of the same beautiful Dutch landscape with a tower and a bridge from different viewpoints. Which is higher: the tower or the bridge?

Dutch school children ride their bikes everywhere, in particular to school. Surely, they will have seen bridges and towers like this in relative positions. With their smartphones they take photos (and edit them) daily. Moreover, everyone has an innate ability to imagine scenes from different perspectives. So this situation is realistic in many ways. And now they are required to think about it mathematically. They will have to introduce notions like viewpoints, projections, vision lines and scaling to discuss the situation, which is the goal of the task.

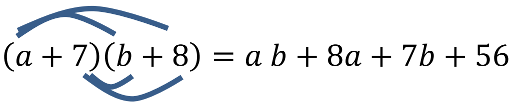
The Figure below summarizes some of the mathematical aspects of the problem. The photos are depicted in a more correct relative scaling.



Point (2) is illustrated again by the rectangle example above. It carries over nicely to exercises like: expand $3 \cdot (x + y + 3)$, where the rectangle is divided in three instead of two. It also applies to $(a + 7)(b + 8)$, where the rectangle is divided in four



This, in contrast, is sometimes explained with another model that does not satisfy point (2). That second model is called a "parrot beak" and it is illustrated like this:



$$(a + 7)(b + 8) = a\,b + 8a + 7b + 56$$

As soon as you multiplied two terms, you draw the line between them. If you have done it well, the beak appears. This model is a mnemonic technique and provides no understanding of what is happening. It does not satisfy (2), since you only get a beak expanding $(a + c)(b + d)$, not with $(a + c)(b + d + e)$ or more complex expressions.

If one focuses on the formal presentation of mathematics as an inspiration for education, then it is a natural choice to begin with the mathematical objects with least structure. This way you build mathematical knowledge up from its fundamental notions. Geometry would begin from axioms on point and lines. Analysis could begin from sets, natural numbers to real numbers, then functions, etcetera. Such an approach was used during New Math in the 1960's. But this is another incarnation of the anti-didactical inversion. Most of these structures are the end point of a process of abstraction, "impoverishing" and reorganisation of mathematical knowledge. According to RME it is more instructive for the learners to go through a process that achieves this themselves.

Point (3) can be illustrated by the following exercise. Just after substitution of numbers for variables is introduced one can proceed to solving equations.
Find solutions for:

$$2x = 8$$
$$7 + x = 15$$
$$x^2 = 25$$
$$x + 8 = 2x + 2$$
$$(x + 2)^2 = 16$$

This list could be much longer; the more variety in equations, the richer the task. Without having previously learned any solution methods the success of learners will vary. They will also apply various types of reasoning. Through this exercise the teacher can find out what comes naturally to the learners and use this in a later stage, when formal solution methods are discussed. The teachers find out about the differences between the learners.

Mathematising

RME promotes mathematics as a human activity. Freudenthal calls one of main components of this activity *mathematising*:

> *Mathematising is the entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language… (The goal is) offering non-mathematical rich structures in order to familiarise the learner with discovering structure, structuring, impoverishing structures and mathematising. By this means he may discover the powerful poor structures in the context of the rich ones in the hope that, by this approach, they will also function in other (mathematical as well as non-mathematical) contexts. Starting with poor mathematical structures may mean that one will never reach the rich non-mathematical ones, which are in fact the proper goal.* (Freudenthal, 1991, p.31 and p.41)

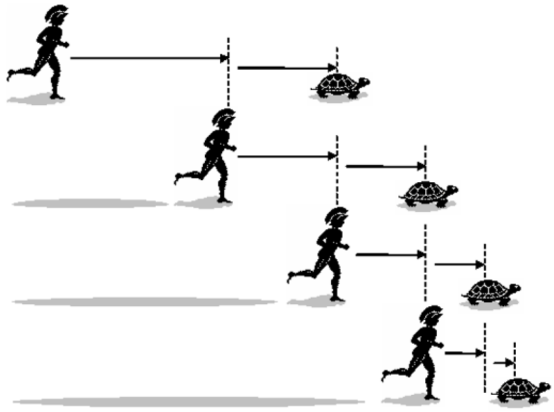
Mathematising involves: axiomatising (creating an axiomatic mathematical system), formalising (the transition from an intuitive to a formal approach), schematising (forming meaningful networks of concepts and processes), algorithmising (the transition from solving a problem by hard work to solving it by routine), modelling (building schemes that represent, idealise, simplify other schemes), etcetera.

One can distinguish two directions in mathematising: horizontal and vertical (Treffers, 1987). Horizontal mathematising is the transition of a problem or situation into a mathematical discourse. It enables the *mathematical* treatment or discussion of the situation. Vertical mathematising is mathematising within a mathematical discourse.

As soon as one poses (and answers) questions about a situation in terms of quantities, distances, shape, symmetry, order, probability or other type of structures studied within mathematics, horizontal mathematising takes place. Both types of mathematising should be practiced by learners. If the horizontal part is neglected, then the learner loses the connection between mathematical knowledge and the situations where it is applied. If the vertical process is neglected the learner misses the opportunity to form the deep connections within mathematics, build the formal system and find a better understanding.

This task is part of a course material on discrete models for 16/17 year olds. The goal of the task is to exercise modelling skills with sequences, to practice sum sequences and to introduce the geometric series. It begins by introducing the famous paradox of Achilles and the turtle. Many students are familiar with it, but they can easily be brought into difficulties trying to disentangle it (e.g. in a classroom discussion).

Achilles and the turtle are in a footrace. Since Achilles is faster, the turtle has a head start. Unfortunately, each time Achilles reaches the place the turtle was a moment before, the turtle has already progressed a bit. This way Achilles can never overtake the turtle and the turtle wins the race. What's wrong with this reasoning? How can we solve the paradox by mathematical reasoning?



The students are then challenged to model the situation (as a mathematical sequence). This naturally leads to questions on the role of time and distance as variables.

A possible answer begins with some assumptions, say: the head start is 1, Achilles' speed is 1 and the turtle's is ½. Then the distance between at the times Achilles reaches the turtle's previous position the is modelled by a sequence

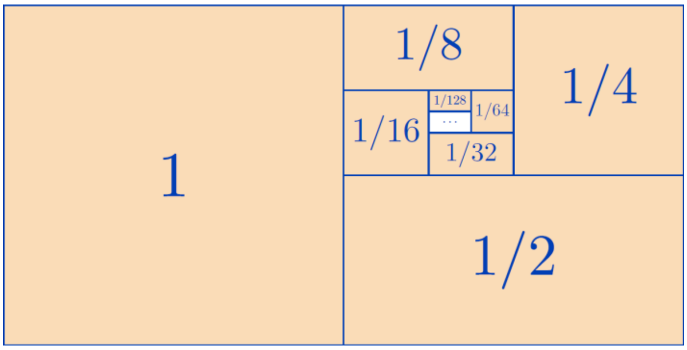
$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

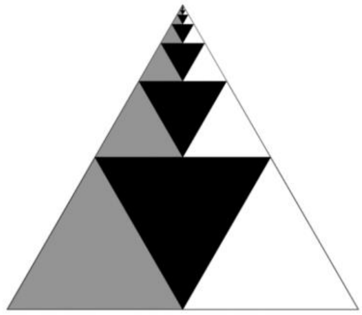
The total distance covered by Achilles and the amount of time passed at each of those moments are modelled by a sequence

$$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots$$

But how to deal with infinite sequences? If you add up an infinity of numbers is the outcome not infinity? This is the heart of the paradox! The answer lies in the geometric series, which is a major learning goal of the task.

In a follow up exercise students study the picture:





Informal reasoning with this picture gives students a way to compute the geometric series solving the paradox.

Then follows a process of vertical mathematising. the student is challenged to find a similar result for the picture on the right and then to formalise and generalise what is represented visually in these pictures to

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Finding the expression $\frac{1}{1-x}$ is a big challenge.
After applying this result to other intriguing situations, like 0,9999 ... = 1 (a nice example of a mathematical context), the students' interest in finding a proof should have been stimulated. In the proof pseudo-formal techniques are used

$$(1-x)(1 + x + x^2 + x^3 + \cdots) = 1 + x + x^2 + x^3 + \cdots - x - x^2 - x^3 + \cdots = 1.$$

Later this can be further formalised in notation by introducing limits and $\sum$-notation.

This sketch of a learning scenario shows examples of modelling and formalising, starting from the rich context of a famous paradox and accessible pictures. Note the order of the activities; the learner is enabled to arrive at the more formal result through studying the concrete contexts.

Horizontal mathematising from rich contexts to tie the bonds with reality

RME is very concerned with the bonds of mathematics with reality. As Freudenthal (1991, p.81) puts it:

> *The world is noisy; mathematising the world means looking for essentials, sensing the message within the noise. This, too, has to be learned, that is, reinvented by the learner, and the earlier the better; once the learner has fully been indoctrinated by ready-made schemes and algorithms it may be too late.*

Next to "mathematics as a human activity", "bonds with reality" is one of the main focusses of RME. To stimulate those bonds learning activities should involve a sufficiently (non-mathematical) rich context. Earlier in the chapter we discussed rich contexts. Let us elaborate on this with a few suggestions. Each suggestion should, of course, meet the criteria for richness mentioned before.

a) A location. For example, a stockroom or a music festival
b) A story, such as the paradox of Achilles and the turtle described above.
c) A human activity. For example designing a house, or flying a plane.
d) News or a historical event. For example, statistical claims in a newspaper.

The following exercise comes from *De Wageningse Methode* (van den Broek et al., n.d.). It is part of a chapter about matrices. A large part of the chapter revolves around the rich context of a car sales company. It has a headquarters and a branch. It sells cars of type A, B and C. The car stock is represented by a matrix S

$$\begin{array}{r} \\ Headquarters \\ Branch \end{array} \begin{array}{c} \quad A \quad B \quad C \\ \begin{pmatrix} 15 & 13 & 7 \\ 3 & 4 & 11 \end{pmatrix} \end{array}$$

In previous exercises learners have been adding matrices to adjust to stock. Now a value matrix V is introduced (in thousands of Euros)

$$\begin{array}{r} \\ A \\ B \\ C \end{array} \begin{array}{c} sale \quad cost \quad profit \\ \begin{pmatrix} 12 & 11 & 1 \\ 30 & 28 & 2 \\ 20 & 17 & 3 \end{pmatrix} \end{array}$$

The total sale value of the cars in the headquarters is

$$15 \cdot 12 + 13 \cdot 30 + 7 \cdot 20 = 710 \text{ (thousand Euros).}$$

a) Compute the total sale value of the cars in the branch.
b) Compute the total cost value of the cars in the headquarters. And in the branch.
c) Compute the total profit value of the cars in the headquarters. And in the branch.
d) Use the totals you found in a), b) and c) to fill in a totals matrix T

$$\begin{array}{r} \\ Headquarters \\ Branch \end{array} \begin{array}{c} sale \quad cost \quad profit \\ \begin{pmatrix} . & . & . \\ . & . & . \end{pmatrix} \end{array}$$

Then follows an explanation that what one has done is actually a kind of multiplication for matrices $S \cdot V = T$, and T is defined as the product matrix. The benefit of this approach is that the operations performed for matrix multiplication come naturally and in a meaningful way; thanks to a well-chosen context.

Emergent models

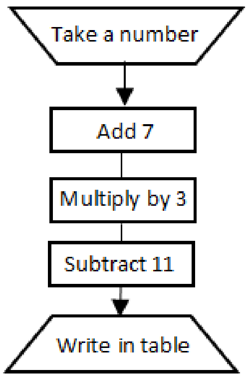
So how does a learner arrive at more formal mathematical knowledge in RME? In work of Streefland (1985), Treffers (1987) and, later, Gravemeijer (1994), a special role is given to *models* as they arise in the mind of learners. In their work, models are mental schemes of concepts and processes related to a situation. From horizontal mathematising a model *of* a situation emerges. This model represents the learner's informal mathematical activity with respect to the situation. It gives meaning to the situation for a learner. From here a process of vertical mathematising can take place: building a (more abstract) mathematical object from a concept, or an algorithm from a process. The new model is a more formal one. After one or more of such steps it is not a model *of* a specific situation, but a model *for* a class of situations enabling mathematical activity without reference to the situation that gave rise to the model. However, if needed, one could give meaning to the model by linking through the intermediate models all the way back to the original one. This is one of the reasons why RME prefers to work with models that carry further than the situation where they arise (see point (2) about rich exercises).

The gradual emergence of a formal model may stretch over a long period of education. As an example let's look at the emergence of the concept of a *function* (Doorman, Drijvers, Gravemeijer, Boon and Reed 2012). We take as a starting point that the learner is familiar with the concept of a variable, including substitution of a value for a variable. Exercise for 12 year olds (adapted from De Wageningse Methode, cf. van den Broek et al., n.d.):

Look at the scheme on the right.
Make the table with the numbers 1, 2, 3, 4, 5 and 10.

Sam finds an outcome of 10. What was his starting number?

And with 343?



This informal activity will later progress into the use of formulas to represent the arrow scheme. Learners will work with such computational schemes and formulas and they will gradually form a reality for the learner.

At some point new basic computations are added: the sine, cosine and tangent, denoted $\sin(x)$, etc. Learners do not learn how the computation is accomplished (in general), but just what their geometric meaning is. This is an important shift of point of view. The next step is that in analogy a new notation enters: $f(x)$, where f represents a computational scheme. At this point the computational scheme itself becomes an object. Learners will have to study properties of the object, like the domain or derivative. But the concept of function is introduced based on a transformation of models: a model *for* the concept of function, based on models of functions, and not based on a definition. An actual formal definition of a function is reached from a different path altogether: set theory!

Guided reinvention

Horizontal mathematising activity opens up a situation, or a class of situations, to mathematical discourse. Through vertical mathematising, models of informal mathematical activity are gradually transformed into models representing formal mathematical knowledge. One could say that in this way the formal mathematics is *reinvented* by the learner. This process can in many cases not be the same as the original invention. The way a professional mathematician arrived at a result may use motivation and knowledge not available to a learner. The challenge for the RME-teacher is to facilitate a process that is suitable for the learner. The process has to be *guided*. As written in Freudenthal (1991) "Inventions, as understood here, are steps in learning processes, which is accounted for by the "re" in reinvention, while the instructional environment of the learning process is pointed to by the adjective "guided"". In addition to the previous discussion one can add the following arguments in favour of guided reinvention (Freudental 1991):

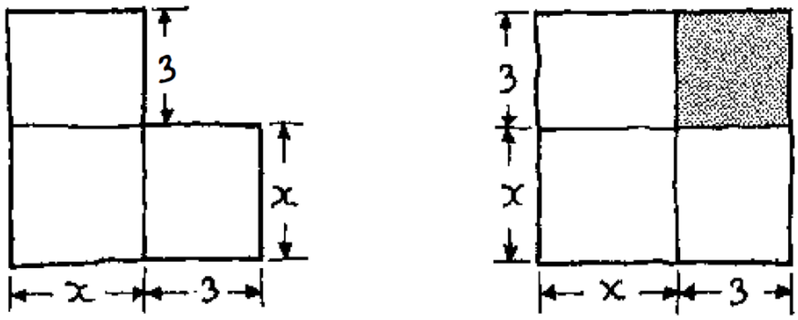
1. Knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others.
2. Discovery can be enjoyable and so learning by reinvention may be motivating.
3. It fosters the experience of mathematics as a human activity.
4. It ensures the mathematical approach fits the level of the learner.

The reinvention principle should be put in the perspective of the central claim of RME with which we started this discussion: that mathematics education is not just about the body of mathematical knowledge, but also about learning to mathematise. Therefore the process of reinvention is valued as much as the outcome.

Guiding towards inventions

How to guide learners towards their reinventions? "Guiding means striking a delicate balance between the force of teaching and the freedom of learning" (Freudenthal 1991). Obviously the guided activities should promote horizontal and vertical mathematising. The aim should be that the learners themselves produce solutions to set problems, and perhaps even produce new problems.
The teacher's instruction should promote discussions between learners themselves and between learners and the teacher. Discussions among learners allow them to test, focus and reformulate ideas without a teacher directing them to a desired outcome. Not all learners will mathematise and invent at the same speed. Discussions help learners to align their ideas.
If the teacher is involved in discussion, then learners benefit from his attempts to go along with their reasoning to help them see where it might lead. The reason for this is that learners own approaches are based on what is meaningful for them. If the teacher can guide those methods to an acceptable solution, then the odds of the learner understanding the solution increase.

This exercise adapted from De Wageningse Methode (van den Broek et al., n.d.) aims at a reinvention of the *completing the square* method. In the right picture the L-shape is completed to a square.



a) Write an expression in x for the area of the L-shape in the left picture.
b) What is the area of the grey square?
c) What is the length of the side of the big square?
d) Explain how (a) and (b) lead to the equality $x^2 + 6x = (x+3)^2 - 9$.
e) Check this equality by expanding the brackets in the right expression.
f) Sketch an L-shape with area $x^2 + 10x$.
g) What equality can you derive from this L-shape?

This exercise is repeated with different numbers (introducing fractions as well), but leaving the choice of drawing an L-shape to the learner. The important observation here is that the reinvention of the algorithm is left to the learner. The learner is supposed to do the algorithmising.

A learner's own invention (such as a concept, algorithm, model, or a way to solve -a problem) may not be the most efficient or beautiful one. It may be different from the one the teacher had in mind as a desired learning outcome. At the end of a reinvention activity the teacher could try to formulate a joint outcome in a classroom discussion. The teacher should take care to connect the outcome to the learners' contributions.

RME and IBMT

Where is the common ground between RME and IBMT? A central concept of IBMT is *inquiry:* a process similar to how mathematicians and scientists work when confronted by a new phenomenon.

> *Many daily-life phenomena can be described, investigated and understood with the help of mathematics in combination with science or common sense, and are therefore a rich source for IBMT*[2]*...* (Artigue and Blomhøj, 2013)

RME and IBMT have some principles in common. Both theories describe how daily-life situations form a rich source for learning. They advocate knowledge construction through methods inspired by science and knowledge construction:

[2] In this booklet we use the term IBMT, where Artigue and Blomhøj use IBME.

inquiry, discovery or (re)invention. Both IBMT and RME describe these processes as social: learners working together to rediscover and reconstruct knowledge. RME emphasizes that reinvention will be different from invention, since the knowledge that forms the starting point for a specialized researcher and for a novice learner are very different.
In addition to traditional roles, the teacher acquires a new one in RME and IBMT: he is a facilitator and a guide of inquiry and mathematising. The learners and their ideas play the central role. The teacher helps to formalize the informal approaches of the learners, as discussed before.
RME and IBMT view the proficiency in inquiring and mathematising themselves as learning goals, in addition to domain knowledge. This is a significant shift away from exclusively domain knowledge centered approaches.

RME-structure for IBMT-modules

So far we have discussed various aspects of RME, with several example tasks. To conclude we sketch an outline how to string tasks together into a learning trajectory, for example a module.

1. An introduction: present a context with a relatively open problem (possibly for the students to discover or formulate). This problem is going to be overarching for the entire module. It will be approached in various mathematical ways.
2. A phase of horizontal mathematising: mathematical language is introduced to discuss the situation. The learners form a first informal model of the situation.
3. A phase of vertical mathematising: the mathematics is involved in the problem is further developed. The model is made more abstract, more general.
4. Conclusion and reflection: the learner reflects on the whole process, integrates ideas, makes acquired metacognitive skills explicit, the learners share their findings, the teacher guides and highlights main learning points.

In each phase there are elements of inquiry: the discovery and/or formulation of the problem, forming a first informal model, abstracting, sharing findings. The challenges involved in applying these ideas and other principles to design IBMT based modules are addressed in other MERIA project publications (see http://www.meria-project.eu/).