Milnor Fibre Homology via Deformation

Dirk Siersma and Mihai Tibăr

Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract In case of one-dimensional singular locus, we use deformations in order to get refined information about the Betti numbers of the Milnor fibre.

Keywords Betti numbers • Milnor fibre • Singularities • One-dimensional singular set

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1 Introduction and Results

We study the topology of Milnor fibres F of function germs on $\mathbb{C}^{n+1}$ with a one-dimensional singular set. Well known is that F is a $(n-2)$ connected n-dimensional CW complex. What can be said about $H_{n-1}(F)$ and $H_n(F)$? In this paper we use deformations in order to get information about these groups. It turns out that the constraints on F yield only small numbers $b_{n-1}(F)$, for which we give upper bounds which are in general sharper than the known ones from [9]. We pay special attention to classes of singularities where $H_{n-1}(F) = 0$, where the homology is concentrated in the middle dimension.

The admissible deformations of the function have a singular locus Σ consisting of a finite set R of isolated points and finitely many curve branches. Each branch Σ_i of Σ has a generic transversal type (of transversal Milnor fibre $F_i^{\pitchfork}$ and Milnor

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number denoted by $\mu_i^{\pitchfork}$) and also contains a finite set Q_i of points with non-generic transversal type, which we call *special points*. In the neighbourhood of each such special point $q \in \Sigma_i$ with Milnor fibre denoted by $\mathcal{A}_q$, there are two monodromies which act on $F_i^{\pitchfork}$: the *Milnor monodromy* of the local Milnor fibration of $F_i^{\pitchfork}$ and the *vertical monodromy* of the local system defined on the germ of $\Sigma_i \setminus \{q\}$ at q.

In our topological study, we work with homology over $\mathbb{Z}$ (and therefore we systematically omit $\mathbb{Z}$ from the notation of the homology groups). We provide a detailed expression for $H_{n-1}(F)$ through a topological model of F from which we derive some results that we roughly outline here:

(a) If for every component Σ_i there exist one vertical monodromy A_s, which has no eigenvalues 1, then $b_{n-1}(F) = 0$. More generally, $b_{n-1}(F)$ is bounded by the sum, taken over the components, of the minimum (over that component) of $\dim\ker(A_s - I)$ (Theorem 4.4).

(b) Assume that for each irreducible component Σ_i, there is a special singularity at q such that $H_{n-1}(\mathcal{A}_q) = 0$. Then $H_{n-1}(F) = 0$.

More generally, let $Q' := \{q_1, \ldots, q_m\} \subset Q$ be a subset of special points such that each branch Σ_i contains at least one of its points. Then (Theorem 4.6b)

$$b_{n-1}(F) \leq \dim H_{n-1}(\mathcal{A}_{q_1}) + \cdots + \dim H_{n-1}(\mathcal{A}_{q_m}).$$

Note that the choice of a good subset of special points may yield the sharpest bound.

In [12] we have studied the vanishing homology of projective hypersurfaces with a one-dimensional singular set. Similar type of methods work in the local case. We keep the notations close to those in [12] and refer to it for the proof of certain results. In the proof of the main theorems, we use the Mayer–Vietoris theorem to study local and (semi)global contributions separately. We construct a CW complex model of two bundles of transversal Milnor fibres (in Sects. 3.5 and 3.6) and their inclusion map (Sect. 4). Moreover we use the full strength of the results on local one-dimensional singularities [6, 8–10], cf also [4, 5, 14, 17].

We discuss known results such as De Jong's [1] and also compute several new examples in Sect. 5.

2 Local Theory of One-Dimensional Singular Locus

We work with local data of function germs with one-dimensional singular locus and recall some facts from [9, 10] and [11, 12].

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with singular locus Σ of dimension 1, and let $\Sigma = \bigcup_{i \in I} \Sigma_i$ be its decomposition into irreducible curve components. Let $E := B_\varepsilon \cap f^{-1}(D_\delta)$ be the Milnor neighbourhood and F be the local Milnor fibre of f, for small enough ε and δ. The only non-trivial reduced homology groups are $H_n(F) = \mathbb{Z}^{\mu_n}$, which is free, and $H_{n-1}(F)$ which can have torsion.

There is a well-defined local system on $\Sigma_i \setminus \{0\}$ having as fibre the homology of the transversal Milnor fibre $\tilde{H}_{n-1}(F_i^{\pitchfork})$, where $F_i^{\pitchfork}$ is the Milnor fibre of the restriction of f to a transversal hyperplane section at some $x \in \Sigma_i \setminus \{0\}$. This restriction has an isolated singularity whose equisingularity class is independent of the point x and of the transversal section, in particular $\tilde{H}_*(F_i^{\pitchfork})$ is concentrated in dimension $n-1$. It is on this group that acts the *local system monodromy* (also called *vertical monodromy*):

$$A_i : \tilde{H}_{n-1}(F_i^{\pitchfork}) \to \tilde{H}_{n-1}(F_i^{\pitchfork}).$$

After [9], one considers a tubular neighbourhood $\mathcal{N} := \bigsqcup_{i=1}^{m} \mathcal{N}_i$ of the link of Σ and decomposes the boundary $\partial F := F \cap \partial B_\varepsilon$ of the Milnor fibre as $\partial F = \partial_1 F \cup \partial_2 F$, where $\partial_2 F := \partial F \cap \mathcal{N}$. Then $\partial_2 F = \bigsqcup_{i=1}^{m} \partial_2 F_i$, where $\partial_2 F_i := \partial_2 F \cap \mathcal{N}_i$.

Each boundary component $\partial_2 F_i$ is fibred over the link of Σ_i with fibre $F_i^{\pitchfork}$. Let then $E_i^{\pitchfork}$ denote the transversal Milnor neighbourhood containing the transversal fibre $F_i^{\pitchfork}$, and let $\partial_2 E_i$ denote the total space of its fibration above the link of Σ_i. Therefore, $E_i^{\pitchfork}$ is contractible and $\partial_2 E_i$ retracts to the link of Σ_i. The pair $(\partial_2 E_i, \partial_2 F_i)$ is related to $A_i - I$ via the following exact relative Wang sequence ([12], Lemma 3.1) ($n \geq 2$):

$$0 \to H_{n+1}(\partial_2 E_i, \partial_2 F_i) \to H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \xrightarrow{A_i - I} H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \to H_{n-1}(\partial_2 E_i, \partial_2 F_i) \to 0. \tag{1}$$

3 Deformation and Vanishing Homology

3.1 Admissible Deformations

Consider a one-parameter family $f_s : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ where $f_0 = \hat{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a given germ with singular locus $\hat{\Sigma}$ of dimension 1, with Milnor data $(\hat{E}, \hat{F})$ and similar notations for all the other objects defined in Sect. 2. We use the notation with "hat" since we reserve the notation without "hat" for the deformation f_s.

We fix a ball $B := B_\varepsilon \subset \mathbb{C}^{n+1}$ centred at 0 and a disc $\Delta := \Delta_\delta \subset \mathbb{C}$ at 0 such that for small enough radii ε and δ, the restriction to the punctured disc $\hat{f}_| : B \cap (\hat{f})^{-1}(\Delta^*) \to \Delta^*$ is the Milnor fibration of $\hat{f}$.

We say that the deformation f_s is *admissible* if it has good behaviour at the boundary, i.e. if for small enough s, the family $f_{s|} : \partial B \cap f_s^{-1}(\Delta) \to \Delta$ is stratified topologically trivial. Such a situation occurs, e.g. in the case of an "equi-transversal deformation" considered in [2].

We choose a value of s which satisfies the above conditions and write from now on $f := f_s$. It then follows that the pair $(E, F) := (B \cap f^{-1}(\Delta), f^{-1}(b))$, where $b \in \partial\Delta$, is topologically equivalent to the Milnor data $(\hat{E}, \hat{F})$ of $\hat{f}$. Note that for f, we consider the semi-local singular fibration inside B and not just its Milnor fibration at the origin.

Let $\Sigma \subset B$ be the one-dimensional singular part of the singular set Sing $(f) \subset B$. The circle boundaries $\partial B \cap \hat{\Sigma}$ of $\hat{\Sigma}$ can be identified with the circle boundaries $\partial B \cap \Sigma$ of Σ. Also the corresponding vertical monodromies are the same. Note that $\hat{\Sigma}$ and Σ can have a different number of irreducible components.

3.2 Notations

We use notations similar to [12] (cf also Fig. 1).

A point q on Σ is called *special* if the transversal Milnor fibration is not a trivial local system in the neighbourhood of q.

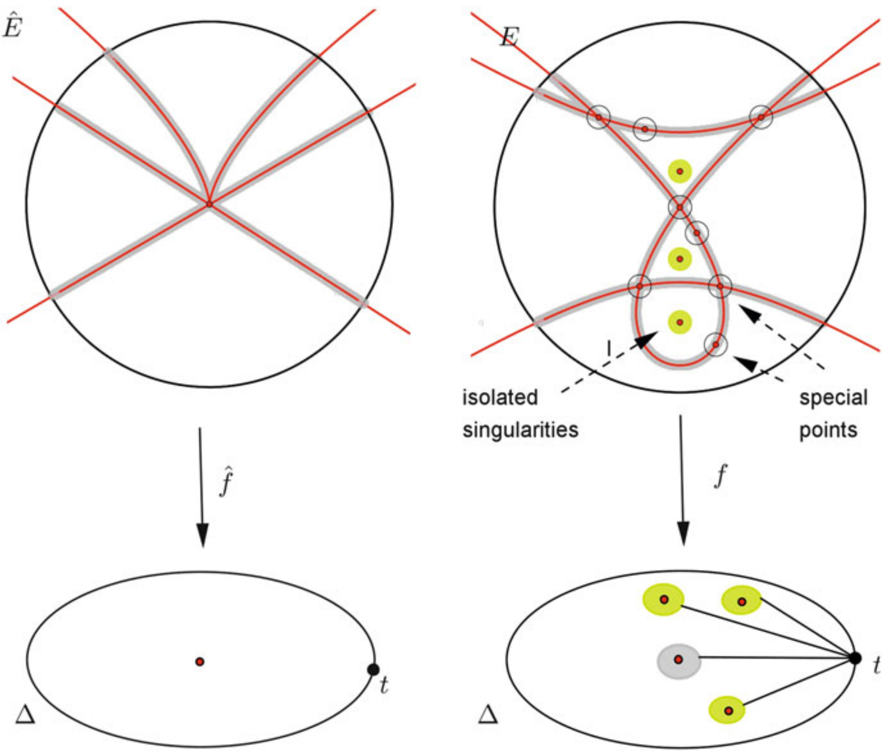

Fig. 1 Admissible deformation

Q_i := the set of special points on Σ_i; $Q := \cup_{i \in I} Q_i$,
R := the set of isolated singular points; $R = R_0 \cup R_1$, where R_0 is the set of critical points on $f^{-1}(0)$ and R_1 the set of critical points outside $f^{-1}(0)$,
B_q, B_r = small enough disjoint Milnor balls within E at the points $q \in Q, r \in R$ resp.
$B_Q := \bigsqcup_q B_q$ and $B_R := \bigsqcup_r B_r$, and similar notation for B_{R_0} and B_{R_1},
$\Sigma_i^* := \Sigma_i \setminus \text{int}(B_Q)$; $\Sigma^* = \cup_{i \in I} \Sigma_i^*$ (closed sets),
$\mathcal{U}_i$:= small enough tubular neighbourhood of Σ_i^*; $\mathcal{U} = \cup_i \mathcal{U}_i$,
$\pi_\Sigma : \mathcal{U} \to \Sigma^*$ is the projection of the tubular neighbourhood.
$T = \{f(r) | r \in R\} \cup \{f(\Sigma)\}$ is the set of critical values of f and we assume without loss of generality that $f(\Sigma) = 0$.

Let $\{\Delta_t\}_{t \in T}$ be a system of nonintersecting small discs Δ_t around each $t \in T$. For any $t \in T$, choose $t' \in \partial \Delta_t$. If $t = f(r)$ then we denote by $t'(r)$ the point $t' \in \Delta_{f(r)}$. For $t = 0$ we use the notations t_0 and t'_0, respectively.

Let $E_r = B_r \cap f^{-1}(\Delta_{f(r)})$ and $F_r = B_r \cap f^{-1}(t'(r))$ be the Milnor data of the isolated singularity of f at $r \in R$. We use next the additivity of vanishing homology with respect to the different critical values and the connected components of Sing f. By homotopy retraction and by excision, we have

$$H_*(E, F) \simeq \oplus_{t \in T} H_*((f^{-1}(\Delta_t), f^{-1}(t')) = \tag{2}$$

$$= \oplus_{r \in R_0} H_*(E_r, F_r) \oplus H_*(E_0, F_0) \oplus \oplus_{r \in R_1} H_*(E_r, F_r), \tag{3}$$

where $(E_0, F_0) = (f^{-1}(\Delta_0) \cap (\mathcal{U} \cup B_Q), f^{-1}(t'_0) \cap (\mathcal{U} \cup B_Q)$. We introduce the following shorter notations:

$$(\mathcal{X}_q, \mathcal{A}_q) := (f^{-1}(\Delta_0) \cap B_q, f^{-1}(t'_0) \cap B_q)$$

$$\mathcal{X} = \bigsqcup_Q \mathcal{X}_q \ , \ \mathcal{A} = \bigsqcup_Q \mathcal{A}_q$$

$$\mathcal{Y} = \mathcal{U} \cap f^{-1}(\Delta_0) \ , \ \mathcal{B} := f^{-1}(t'_0) \cap \mathcal{Y}$$

$$\mathcal{Z} := \mathcal{X} \cap \mathcal{Y} \ , \ \mathcal{C} := \mathcal{A} \cap \mathcal{B}$$

In these new notations, we have

$$H_*(E, F) \simeq H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \oplus \oplus_{r \in R} H_*(E_r, F_r). \tag{4}$$

Note that each direct summand $H_*(E_r, F_r)$ is concentrated in dimension $n+1$ since it identifies to the Milnor lattice $\mathbb{Z}^{\mu_r}$ of the isolated singularities germs of $f - f(r)$ at r, where μ_r denotes its Milnor number. We deal from now on with the term $H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$ from the direct sum of (4).

We consider the relative Mayer–Vietoris long exact sequence:

$$\cdots \to H_*(\mathcal{Z},\mathcal{C}) \to H_*(\mathcal{X},\mathcal{A}) \oplus H_*(\mathcal{Y},\mathcal{B}) \to H_*(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) \stackrel{\partial_s}{\to} \cdots \tag{5}$$

of the pair $(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B})$, and we compute each term of it in the following. The description follows closely [12] where we have treated deformations of projective hypersurfaces.

3.3 Homology of $(\mathcal{X},\mathcal{A})$

Since $\mathcal{X}$ is a disjoint union, one has the direct sum decomposition $H_*(\mathcal{X},\mathcal{A}) \simeq \oplus_{q\in Q} H_*(\mathcal{X}_q,\mathcal{A}_q)$. The pairs $(\mathcal{X}_q,\mathcal{A}_q)$ are local Milnor data of the hypersurface germs $(f^{-1}(t_0), q)$ with one-dimensional singular locus, and therefore the relative homology $H_*(\mathcal{X}_q,\mathcal{A}_q)$ is concentrated in dimensions n and $n+1$, cf Sect. 2.

3.4 Homology of $(\mathcal{Z},\mathcal{C})$

The pair $(\mathcal{Z},\mathcal{C})$ is a disjoint union of pairs localized at points $q \in Q$. For such points we have one contribution for each *locally irreducible branch of the germ* (Σ, q). Let S_q be the index set of all these branches at $q \in Q$. By abuse of notation, we write $s \in S_q$ for the corresponding small loops around q in Σ_i. For some $q \in \Sigma_{i_1} \cap \Sigma_{i_2}$, the set of indices S_q runs over all the local irreducible components of the curve germ (Σ, q). Nevertheless, when we are counting the local irreducible branches at some point $q \in Q_i$ on a specified component Σ_i, then the set S_q will tacitly mean only those local branches of Σ_i at q. We get the following decomposition:

$$H_*(\mathcal{Z},\mathcal{C}) \simeq \oplus_{q\in Q} \oplus_{s\in S_q} H_*(\mathcal{Z}_s,\mathcal{C}_s). \tag{6}$$

More precisely, one such local pair $(\mathcal{Z}_s,\mathcal{C}_s)$ is the bundle over the corresponding component of the link of the curve germ Σ at q having as fibre the local transversal Milnor data $(E_s^{\pitchfork}, F_s^{\pitchfork})$, with transversal Milnor numbers denoted by $\mu_s^{\pitchfork}$. These data depend only on the branch Σ_i containing s, and therefore if $s \subset \Sigma_i$ we sometimes write $(E_i^{\pitchfork}, F_i^{\pitchfork})$ and $\mu_i^{\pitchfork}$. In the notations of Sect. 2, we have $\partial_2\mathcal{A}_q = \bigsqcup_{s\in S_q} \mathcal{C}_s$.

The relative homology groups in the above direct sum decomposition (6) depend on the *local system monodromy* A_s via the Wang sequence (1) which takes here the following shape:

$$0 \to H_{n+1}(\mathcal{Z}_s,\mathcal{C}_s) \to H_n(E_s^{\pitchfork}, F_s^{\pitchfork}) \stackrel{A_s - I}{\to} H_n(E_s^{\pitchfork}, F_s^{\pitchfork}) \to H_n(\mathcal{Z}_s,\mathcal{C}_s) \to 0. \tag{7}$$

From this we obtain:

Lemma 3.1 *At $q \in Q$, for each $s \in S_q$ one has $H_k(\mathcal{Z}_s, \mathcal{C}_s) = 0$, $k \neq n, n+1$ and*

$$H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \cong \ker\, (A_s - I), \quad H_n(\mathcal{Z}_s, \mathcal{C}_s) \cong \mathrm{coker}\, (A_s - I).$$

□

We therefore conclude that $H_*(\mathcal{Z}, \mathcal{C})$ is concentrated in dimensions n and $n+1$ only.

3.5 The CW Complex Structure of $(\mathcal{Z}, \mathcal{C})$

The pair $(\mathcal{Z}_s, \mathcal{C}_s)$ has the following structure of a relative CW complex, up to homotopy type. Each bundle over some circle link can be obtained from a trivial bundle over an interval by identifying the fibres above the end points via the geometric monodromy A_s. In order to obtain $\mathcal{Z}_s$ from $\mathcal{C}_s$, one can start by first attaching n-cells $c_1, \ldots, c_{\mu_s^{\pitchfork}}$ to the fibre $F_s^{\pitchfork}$ in order to kill the $\mu_s^{\pitchfork}$ generators of $H_{n-1}(F_s^{\pitchfork})$ at the identified ends and next by attaching $(n+1)$-cells $e_1, \ldots, e_{\mu_s^{\pitchfork}}$ to the preceding n-skeleton. The attaching of some $(n+1)$-cell goes as follows: consider some n-cell a of the n-skeleton and take the cylinder $I \times a$ as an $(n+1)$-cell. Fix an orientation of the circle link, attach the base $\{0\} \times a$ over a, then follow the circle bundle in the fixed orientation by the monodromy A_s and attach the end $\{1\} \times a$ over $A_s(a)$. At the level of the cell complex, the boundary map of this attaching identifies to $A_s - I : \mathbb{Z}^{\mu_s^{\pitchfork}} \to \mathbb{Z}^{\mu_s^{\pitchfork}}$.

3.6 The CW Complex Structure of $(\mathcal{Y}, \mathcal{B})$

The curve Σ_i has as boundary components the intersection $\partial B \cap \Sigma_i$ with the small Milnor balls B. These are all topological circles, and we denote them by $u \in U_i$, $U := \sqcup_i U_i$, and call them *outside* loops. Note that over any such loop $u \in U_i$, we have a local system monodromy $A_u : \mathbb{Z}^{\mu_i^{\pitchfork}} \to \mathbb{Z}^{\mu_i^{\pitchfork}}$. In fact this monodromy did not change in the admissible deformation from $\hat{f}$ to f.

We choose the following sets of loops in Σ_i (where we identify the loops with their index sets):

$G_i :=$ the $2g_i$ loops (called *genus loops* in the following) which are generators of π_1 of the normalization $\tilde{\Sigma}_i$ of Σ_i, where g_i denotes the genus of this normalization (which is a Riemann surface with boundary),
$S_i :=$ the loops $s \in S_q$ around the branches of Σ_i at the special points $q \in Q_i$,
$U_i =$ the outside loops,

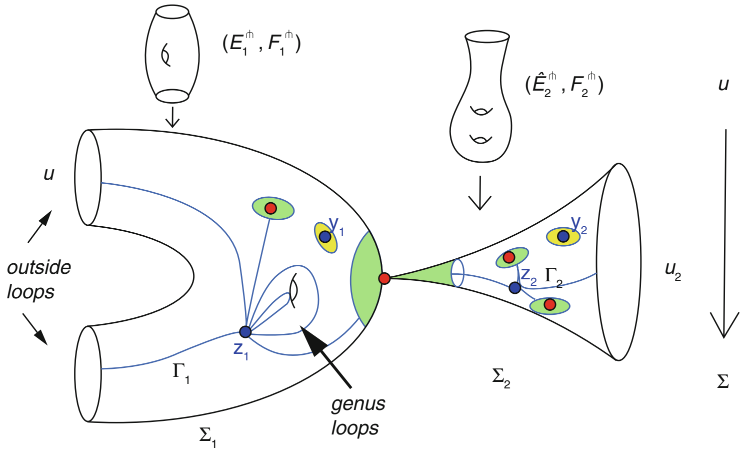

Fig. 2 Critical set and the cell models for $(\mathcal{Z}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{B})$

and define $W_i = G_i \bigsqcup S_i \bigsqcup U_i$ and $W = \bigsqcup W_i$.

We introduce one more puncture y_i on Σ_i and next redefine $\Sigma_i^* := \Sigma \setminus \mathrm{int}(B_Q \cup B_{y_i})$. Moreover we use notations $(\mathcal{X}_y, \mathcal{A}_y)$ and $(\mathcal{Z}_y, \mathcal{C}_y)$. By enlarging "the hole" defined by the puncture y_i, we retract Σ_i^* to a configuration of loops connected by nonintersecting paths to some point z_i, denoted by Γ_i (see Fig. 2). The number of loops is $\#W_i = 2g_i + \tau_i + \gamma_i$, where $\tau_i := \#U_i$ and $\gamma_i := \sum_{q \in Q_i} \#S_q$. Note that $\tau_i > 0$ since there must be at least one outside loop.

Each pair $(\mathcal{Y}_i, \mathcal{B}_i)$ is then homotopy equivalent (by retraction) to the pair $(\pi_\Sigma^{-1}(\Gamma_i), \mathcal{B} \cap \pi_\Sigma^{-1}(\Gamma_i))$. We endow the latter with the structure of a relative CW complex as we did with $(\mathcal{Z}, \mathcal{C})$ at Sect. 3.5, namely, for each loop the similar CW complex structure as we have defined above for some pair $(\mathcal{Z}_s, \mathcal{C}_s)$. The difference is that the pairs $(\mathcal{Z}_s, \mathcal{C}_s)$ are disjoint, whereas in Σ_i^* the loops meet at a single point z_i. We take as reference the transversal fibre $F_i^\pitchfork = \mathcal{B} \cap \pi_\Sigma^{-1}(z_i)$ above this point, namely, we attach the n-cells (thimbles) only once to this single fibre in order to kill the $\mu_i^\pitchfork$ generators of $H_{n-1}(F_i^\pitchfork)$. The $(n+1)$-cells of $(\mathcal{Y}_i, \mathcal{B}_i)$ correspond to the fibre bundles over the loops in the bouquet model of Σ_i^*. Over each loop, one attaches a number of $\mu_i^\pitchfork$ $(n+1)$-cells to the fixed n-skeleton described before, more precisely one $(n+1)$-cell over one n-cell generator of the n-skeleton. We extend for $w \in W$ the notation $(\mathcal{Z}_g, \mathcal{C}_g)$ to genus loops and $(\mathcal{Z}_u, \mathcal{C}_u)$ to outside loops, although they are not contained in $(\mathcal{Z}, \mathcal{C})$ but in $(\mathcal{Y}, \mathcal{B})$.

The attaching map of the $(n+1)$-cells corresponding to the bundle over a genus loop, or over an outside loop, can be identified with $A_g - I : \mathbb{Z}^{\mu_i^\pitchfork} \to \mathbb{Z}^{\mu_i^\pitchfork}$, or with $A_u - I : \mathbb{Z}^{\mu_i^\pitchfork} \to \mathbb{Z}^{\mu_i^\pitchfork}$, respectively. We have seen that the monodromy A_u over some outside loop indexed by $u \in U_i$ is necessarily one of the vertical monodromies of the original function $\hat{f}$.

From this CW complex structure, we get the following precise description in terms of the monodromies of the transversal local system, the proof of which is similar to that of Siersma and Tibăr [12, Lemma 4.4]:

Lemma 3.2

(a) $H_k(\mathcal{Y},\mathcal{B}) = \oplus_{i\in I} H_k(\mathcal{Y}_i,\mathcal{B}_i)$ *and this is* $= 0$ *for* $k \neq n, n+1$.
(b) $H_n(\mathcal{Y}_i,\mathcal{B}_i) \simeq \mathbb{Z}^{\mu_i^{\pitchfork}} / \langle \mathrm{Im}(A_w - I) \mid w \in W_i \rangle$,
(c) $\chi(\mathcal{Y}_i,\mathcal{B}_i) = (-1)^{n-1}(2g_i + \tau_i + \gamma_i - 1)\mu_i^{\pitchfork}$.

□

If we apply χ to (4) and (5) and take into account that $\chi(\mathcal{Z},\mathcal{C}) = 0$, we get $\chi(E,F) = \chi(\mathcal{X},\mathcal{A}) + \chi(\mathcal{Y},\mathcal{B}) + \sum_r \chi(E_r,F_r)$. From this one may derive the Euler characteristic of the Milnor fibre F (already computed in [2]):

Proposition 3.3

$$\chi(F) = 1 + \sum_{q\in Q}(\chi(\mathcal{A}_q) - 1) + (-1)^n \sum_{i\in I}(2g_i + \tau_i + \gamma_i - 2)\mu_i^{\pitchfork} + (-1)^n \sum_{r\in R} \mu_r.$$

□

Proposition 3.4 *The relative Mayer–Vietoris sequence* (5) *is trivial except of the following 6-terms sequence:*

$$\begin{aligned} 0 &\to H_{n+1}(\mathcal{Z},\mathcal{C}) \to H_{n+1}(\mathcal{X},\mathcal{A}) \oplus H_{n+1}(\mathcal{Y},\mathcal{B}) \to H_{n+1}(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) \to \\ &\to H_n(\mathcal{Z},\mathcal{C}) \xrightarrow{j} H_n(\mathcal{X},\mathcal{A}) \oplus H_n(\mathcal{Y},\mathcal{B}) \to H_n(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) \to 0. \end{aligned} \tag{8}$$

□

Proof Lemma 3.1, Sect. 3.3 and Lemma 3.2 show that the terms $H_*(\mathcal{X},\mathcal{A})$, $H_*(\mathcal{Y},\mathcal{B})$ and $H_*(\mathcal{Z},\mathcal{C})$ of the Mayer–Vietoris sequence (5) are concentrated in dimensions n and $n+1$ only. Following (4) and since $\tilde{H}_*(F)$ is concentrated in levels $n-1$ and n, we obtain that $H_{n+2}(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) = 0$.

The first three terms of (8) are free. By the decomposition (4), in order to find the homology of F, we thus need to compute $H_k(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B})$ for $k = n, n+1$, since the others are zero.

In the remainder of this paper, we collect information about $H_n(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B})$. The knowledge of its dimension is then enough for determining $H_n(F)$, by only using the Euler characteristic formula (Proposition 3.3).

4 The Homology Group $H_{n-1}(F)$

We concentrate on the term $H_n(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \simeq \tilde{H}_{n-1}(F)$. We need the relative version of the "variation ladder", an exact sequence found in [9, Theorem 5.2, pp. 456–457]. This sequence has an important overlap with our relative Mayer–Vietoris sequence (8).

Proposition 4.1 ([12, Proposition 5.2]) *For any point $q \in Q$, the sequence*

$$\begin{aligned} 0 \to H_{n+1}(\mathcal{A}_q, \partial_2\mathcal{A}_q) \to \oplus_{s\in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \to H_{n+1}(\mathcal{X}_q, \mathcal{A}_q) \to \\ \to H_n(\mathcal{A}_q, \partial_2\mathcal{A}_q) \to \oplus_{s\in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q) \to 0 \end{aligned}$$

is exact for $n \geq 2$. □

4.1 The Image of j

We focus on the map $j = j_1 \oplus j_2$ which occurs in the 6-term exact sequence (8), more precisely on the following exact sequence:

$$H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to \tilde{H}_{n-1}(F) \to 0. \tag{9}$$

since we have the isomorphism:

$$\tilde{H}_{n-1}(F) \simeq \operatorname{coker} j. \tag{10}$$

Therefore, full information about j makes it possible to compute $H_{n-1}(F)$. But although j is of geometric nature, this information is not always easy to obtain. Below we treat its two components separately. After that we will make two statements (Theorems 4.4 and 4.6) of a more general type.

4.1.1 The First Component $j_1 : H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{X}, \mathcal{A})$

Note that, as shown above, we have the following direct sum decompositions of the source and the target:

$$\begin{aligned} H_n(\mathcal{Z}, \mathcal{C}) &= \oplus_{q\in Q} \oplus_{s\in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \oplus \oplus_{i\in I} H_n(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}), \\ H_n(\mathcal{X}, \mathcal{A}) &= \oplus_{q\in Q} H_n(\mathcal{X}_q, \mathcal{A}_q) \oplus \oplus_{i\in I} H_n(\mathcal{X}_{y_i}, \mathcal{A}_{y_i}). \end{aligned}$$

As shown in Proposition 4.1, at the special points $q \in Q$, we have surjections $\oplus_{s\in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q)$, and moreover $H_n(\mathcal{Z}_y, \mathcal{C}_y) \to H_n(\mathcal{X}_y, \mathcal{A}_y)$ is an isomorphism. We conclude that the map j_1 is surjective and that there is no contribution of the points y_i to $\operatorname{coker} j$.

4.1.2 The Second Component $j_2 : H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{Y}, \mathcal{B})$

Both sides are described with a relative CW complex as explained in Sect. 3.6. At the level of n-cells, there are $\mu_s^{\pitchfork}$ n-cell generators of $H_n(\mathcal{Z}_s, \mathcal{C}_s)$ for each $s \in S_q$ and any $q \in Q$. Each of these generators is mapped bijectively to the single cluster of n-cell generators attached to the reference fibre $F_i^{\pitchfork}$ (which is the fibre above the common point z_i of the loops). The restriction $j_{2|} : H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{Y}_i, \mathcal{B}_i)$ is a projection for any loop s in Σ_i and $q \in Q_i$, or if instead of s we have y_i, since we add extra relations to $\mathbb{Z}^{\mu^{\pitchfork}}/\langle A_s - I\rangle$ in order to get $\mathbb{Z}^{\mu_i^{\pitchfork}}/\langle \operatorname{Im}(A_w - I) \mid w \in W_i\rangle = H_n(\mathcal{Y}_i, \mathcal{B}_i)$. We summarize the above surjections as follows:

Lemma 4.2 (Strong Surjectivity)

(a) *Both j_1 and j_2 are surjective.*
(b) *The restriction $j_{2|} : H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{Y}_i, \mathcal{A}_i)$ is surjective for any $s \in S_q$ such that $q \in Q \cap \Sigma_i$.*
(c) *The restriction $j_1| \oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q)$ is surjective, for any $q \in Q$.*

□

Corollary 4.3

(a) *If the restriction $j_2| \ker j_1$ is surjective, then j is surjective.*
(b) *If for each $i \in I$, there exist $q_i \in Q \cap \Sigma_i$ and some $s \in S_{q_i}$ such that $H_n(\mathcal{Z}_s, \mathcal{C}_s) \subset \ker j_1$, then j is surjective.*

□

Proof

(a) More generally, let $j_1 : M \to M_1$ and $j_2 : M \to M_2$ be morphisms of $\mathbb{Z}$-modules such that j_1 is surjective, and consider the direct sum of them $j := j_1 \oplus j_2$. We assume that the restriction $j_2| \ker j_1$ is surjective onto M_2 and want to prove that j is surjective.

   Let then $(a, b) \in M_1 \oplus M_2$. There exists $x \in M$ such that $j_1(x) = a$, by the surjectivity of j_1. Let $b' := j_2(x)$. By our surjectivity assumption, there exists $y \in \ker j_1$ such that $j_2(y) = b - b'$. Then $j(x + y) = a + b$, which proves the surjectivity of j.
(b) follows immediately from Lemma 4.2(b) and from the above (a). □

4.2 Effect of Local System Monodromies on $H_n(F)$

Recall that $w \in W_i$ stands for some loop s, or g, or u in Σ_i^*.

Theorem 4.4

(a) *If there is* $w \in W_i$ *such that* $\det(A_w - I) \neq 0$*, then* $\dim H_n(\mathcal{Y}_i, \mathcal{B}_i) = 0$.
If such $w \in W_i$ *exists for any* $i \in I$*, then* $b_{n-1}(F) = 0$.
(b) *If there is* $w \in W_i$ *such that* $\det(A_w - I) = \pm 1$*, then* $H_n(\mathcal{Y}_i, \mathcal{B}_i) = 0$.
If such $w \in W_i$ *exists for any* $i \in I$*, then* $H_{n-1}(F) = 0$.
(c) *The following upper bound holds:*

$$b_{n-1}(F) \leq \sum_{i \in I} \min_{w \in W_i} \dim \operatorname{coker}(A_w - I) \leq \sum_{i \in I} \mu_i^{\pitchfork}.$$

Proof By Lemma 3.2(b), we have $H_n(\mathcal{Y}_i, \mathcal{B}_i) \simeq \mathbb{Z}^{\mu_i^{\pitchfork}} / \langle \operatorname{Im}(A_w - I) \mid w \in W_i \rangle$; thus, the first parts of (a) and (b) follow. For the second part of (a), we have that $\dim H_n(\mathcal{Y}, \mathcal{B}) = 0$; hence, $\operatorname{corank} j = \operatorname{corank} j_1 = 0$. For the second part of (b), we have that $H_n(\mathcal{Y}, \mathcal{B}) = 0$, and the surjectivity of the map j of (9) is equivalent to the fact that j_1 is surjective.

To prove (c), we consider homology groups with coefficients in $\mathbb{Q}$. Since j_1 is surjective, the image of j contains all the generators of $H_n(\mathcal{X}, \mathcal{A}; \mathbb{Q})$. Hence $\dim \operatorname{coker} j \leq \dim H_n(\mathcal{Y}, \mathcal{B})$. □

Remark 4.5 Notice the *effect of the strongest bound* in the above theorem. On each Σ_i one could take an optimal loop, e.g. one with $\det(A_w - I) = \pm 1$. Since in the deformed case there may be less branches Σ_i, and more special points and hence more vertical monodromies, these bounds may become much stronger than those in [9].

4.3 Effect of the Local Fibres $\mathcal{A}_q$

Theorem 4.6 *Let* $n \geq 2$.

(a) *Assume that for each irreducible one-dimensional component* Σ_i *of* Σ*, there is a special singularity* $q \in Q_i$ *such that the* $(n-1)$*th homology group of its Milnor fibre is trivial, i.e.* $H_{n-1}(\mathcal{A}_q) = 0$*. Then* $H_{n-1}(F) = 0$.
If in the above assumption we replace $H_{n-1}(\mathcal{A}_q) = 0$ *by* $b_{n-1}(\mathcal{A}_q) = 0$*, then we get* $b_{n-1}(F) = 0$.

(b) *Let $Q' := \{q_1, \dots, q_m\} \subset Q$ be some (minimal) subset of special points such that each branch Σ_i contains at least one of its points. Then*

$$b_{n-1}(F) \leq \dim H_n(\mathcal{X}_{q_1}, \mathcal{A}_{q_1}) + \dots + \dim H_n(\mathcal{X}_{q_m}, \mathcal{A}_{q_m}).$$

Proof

(a) We use (9) in order to estimate the dimension of the image of $j = j_1 \oplus j_2$. If there is a $q \in Q$ such that $H_n(\mathcal{X}_q, \mathcal{A}_q) = 0$, then $\ker j_1$ contains $\oplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s)$. Since Q' meets all components Σ_i, statement (a) follows from Corollary 4.3(b). The second claim of (a) follows by considering homology over $\mathbb{Q}$.

(b) We work again with homology over $\mathbb{Q}$. We consider the projection on a direct summand $\pi : H_n(\mathcal{X}, \mathcal{A}) \to \oplus_{q \notin Q'} H_n(\mathcal{X}_q, \mathcal{A}_q)$ and the composed map $J_1 := \pi \circ j_1$. Then the restriction $j_2 | \ker J_1$ is surjective, which by Corollary 4.3(a), means that $J_1 \circ j_2$ is surjective. Then the result follows from the obvious inequality $\dim(\operatorname{Im} J_1 \circ j_2) \leq \dim \operatorname{Im} j$ by counting dimensions. □

Remark 4.7 Also here we have the *effect of the strongest bound*. This works the best if one chooses an optimal or minimal Q'. In the irreducible case, $H_{n-1}(\mathcal{A}_q) = 0$ for at least one $q \in Q$ implies the triviality $H_{n-1}(F) = 0$.

Corollary 4.8 (Bouquet Theorem) *If $n \geq 3$ and*

(a) *If for any $i \in I$, there is $w \in W_i$ such that* $\det(A_w - I) = \pm 1$ *or*

(b) *If for any $i \in I$, there is a special singularity $q \in Q_i$ such that $H_{n-1}(\mathcal{A}_q) = 0$, then*

$$F \overset{\text{ht}}{\simeq} S^n \vee \dots \vee S^n.$$

Proof From Theorems 4.4(b) and 4.6(a), respectively, it follows that $H_{n-1}(F) = 0$. Since F is a simply connected n-dimensional CW complex, the statement follows from Milnor's argument [3, Theorem 6.5] and Whitehead's theorem. □

5 Examples

5.1 Singularities with Transversal Type A_1

The case when Σ is a smooth line was considered in [6] and later generalized to Σ a one-dimensional complete intersection (icis) [7]. It uses an admissible deformation with only D_∞-points. The main statement is:

(a) $F \overset{\text{ht}}{\simeq} S^{n-1}$ if $\# D_\infty = 0$,

(b) $F \overset{\text{ht}}{\simeq} S^n \vee \dots \vee S^n$ else.

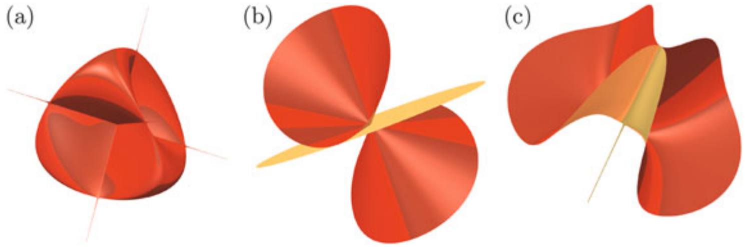

Fig. 3 Several singularites (produced with Surfer software). (**a**) Steiner Surface. (**b**) Singularity F_1A_3. (**c**) Singularity F_2A_3

Since D_∞-points have $H_{n-1}(\mathcal{A}_q) = 0$, our Theorem 4.6 provides a proof of this statement on the level of homology. If Σ is not an icis, more complicated situations may occur. The next known examples show how our results apply (see [7] for details):

1. $f = xyz$, called $T_{\infty,\infty,\infty}$. Here Σ is the union of three coordinate axis, $F \cong S^1 \times S^1$; thus, $b_1(F) = 2$, $b_2(F) = 1$ and $A_u = I$ for all u.
2. $f = x^2y^2 + y^2z^2 + x^2z^2$ has $F \cong S^2 \vee \cdots \vee S^2$. The admissible deformation $f_s = f + sxyz$ has the same Σ as $f = xyz$, but now with three D_∞-points on each component of Σ and one $T_{\infty,\infty,\infty}$-point in the origin. Our Theorem 4.6 yields $H_1(F) = 0$. A real picture of $f_s = 0$ contains the Steiner surface, for $s \neq 0$ small enough (Fig. 3a). That $H_2(F) = \mathbb{Z}^{15}$ follows from $\chi(F) = 16$ computed via Proposition 3.3.

5.2 Transversal Type $A_2, A_3, D_4, E_6, E_7, E_8$, De Jong's List

In [1] there is a detailed description of singularities with singular set a smooth line and transversal type $A_2, A_3, D_4, E_6, E_7, E_8$. De Jong's list illustrates and confirms our statements at the level of homology.

We will treat below in more detail the case $f : \mathbb{C}^3 \to \mathbb{C}$ with transversal type A_3 (to which one may add squares to become $f : \mathbb{C}^{n+1} \to \mathbb{C}$). Any singularity of this type can be deformed into

$$F_1A_3: \quad f = xz^2 + y^2z\ ;\ F \overset{\text{ht}}{\simeq} S^1 \text{ (Fig. 3b)}$$
$$F_2A_3: \quad f = xy^4 + z^2\ ;\ F \overset{\text{ht}}{\simeq} S^2 \text{ (Fig. 3c)}$$

De Jong's observation is that for any line singularity of transversal type A_3, we have:

(a) $F \overset{\text{ht}}{\simeq} S^{n-1} \vee S^n \cdots \vee S^n$ if $\#F_2A_3 = 0$,
(b) $F \overset{\text{ht}}{\simeq} S^n \vee \cdots \vee S^n$ else.

In homology, (b) follows directly from our Theorem 4.6. The homology version of (a) takes more efforts. We demonstrate this in the following example only. First we mention that for F_1A_3 the vertical monodromy A is equal to the Milnor monodromy h. This follows from the fact that $f = xz^2 + y^2z$ is homogeneous of degree $d = 3$ and Steenbrink's remark [13] that $Ah^d = I$ and that $h^4 = I$. The matrix of h is:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

It follows: $\ker(h-I) = \mathbb{Z}$, $\mathrm{Im}\,(h-I) = \mathbb{Z}^2$, and $\mathrm{coker}(h-I) = \mathbb{Z}$.

Next consider as example the deformation $f := f_s = (x^k - s)z^2 + yz^2 + y^2z$ for some fixed small enough $s \neq 0$, which has transversal type A_3. This deformation has $\#F_1A_3 = k$ and $\#F_2A_3 = 0$ and moreover one isolated critical point of type A_k. Note that

$$H_n(\mathcal{Y}, \mathcal{B}) = \mathbb{Z}^3/\langle h-I, \cdots, h-I, A_u - I\rangle = \mathbb{Z}^3/\langle h-I\rangle = \mathbb{Z}$$

since for the outside loop u, we have $A_u = A_{s_1} \circ \cdots \circ A_{s_k} = h^k$ (all A_s are equal to h) and therefore $A_u - I = (h-1)(h^{k-1} + \cdots + h + I)$. We compare now the fundamental sequence for j in case F_1A_3 and f, respectively (we distinguish the Milnor fibres by a subscript):

$$j = j_1 \oplus j_2 : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_{n-1}(F_{F_1A_3}) = \mathbb{Z} \to 0 \tag{11}$$

$$j = j_1 \oplus j_2 : \mathbb{Z}^k \to \mathbb{Z}^k \oplus \mathbb{Z} \to H_{n-1}(F_f) \to 0 \tag{12}$$

The map j_2 for f can now be identified with: $j_2(\xi_1, \cdots, \xi_k) = \xi_1 + \cdots \xi_k$.

We conclude $H_1(F_f) = \mathbb{Z}$. Then $H_2(F_f) = \mathbb{Z}^{3k-1}$ follows from $\chi(F_f) = 3k-1$ computed via Proposition 3.3.

We illustrate this example with Fig. 4a,b.

5.3 More General Types

We show next that the above method is not restricted to the De Jong's classes. Consider $f = z^2x^m - z^{m+2} + zy^{m+1}$. It has the properties $F \simeq S^1$; Σ is smooth; transversal type is A_{2m+1}; $A = h^m$, where h is the Milnor monodromy of A_{2m+1}.

Note that $\dim\ker(A-I) \geq 1$, and $= 1$ in many cases, e.g. $m = 2, 3, 4, 5$. This function f appears as "building block" in the following deformation:

$$g_s = z^2(x^2 - s)^m - z^{m+2} + zy^{m+1}.$$

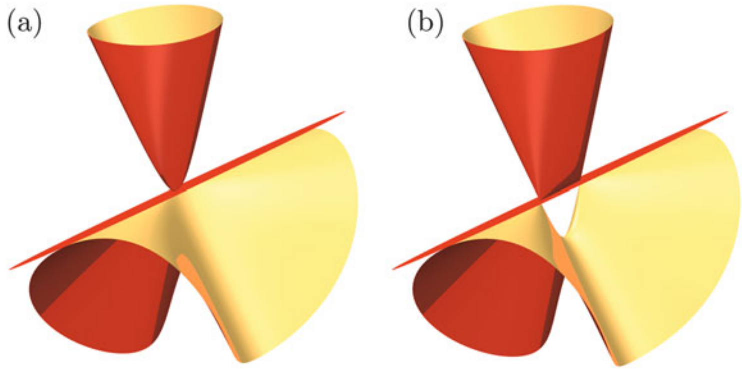

Fig. 4 Deformation $f_s = (x^k - s)z^2 + yz^2 + y^2z$ (produced with Surfer software). (**a**) Original surface. (**b**) Deformed surface

This deformation contains two special points of the type f (and no others, except isolated singularities). If one applies the same procedure as above, one gets (in the $= 1$ cases) $b_1(G) = 1$ where G is the Milnor fibre of g_0. Details are left to the reader.

Remark 5.1 The fact that the first Betti number of the Milnor fibre is nonzero can also be deduced from Van Straten's [15, Theorem 4.4.12]: *Let* $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be a germ of a function without multiple factors, and let F be the Milnor fibre of* f*. Then*

$$b_1(F) \geq \#\{\text{irreducible components of } f = 0\} - 1.$$

5.4 Deformation with Triple Points

Let $f_s = xyz(x+y+z-s)$. This defines a deformation of a central arrangement with four hyperplanes. We get $\Sigma_i = \mathbb{P}^1$ (six copies). There are four triple points $T_{\infty,\infty,\infty}$ and one A_1-point. The maps $j_{1,q} : \mathbb{Z}^3 \to \mathbb{Z}^2$ can be described by $j_{1,q}(a, b, c) = (a + c, b + c)$. The map j_2 restricts to an isomorphism $\mathbb{Z} \to \mathbb{Z}$ on each component. We have all information of the resulting map $j : \mathbb{Z}^{12} \to \mathbb{Z}^{14}$ up to the signs of the isomorphisms. From this we get $H_1(F; \mathbb{Z}_2) = \mathbb{Z}_2^3$. One may compare with the dissertation [16], where Williams showed in particular that $H_1(F; \mathbb{Z}) = \mathbb{Z}^3$.

5.5 *The Class of Singularities with $b_n = 0$*

Most of the singularities above have $b_{n-1} = 0$ or small b_{n-1}. One of the natural questions is what happens if $b_n = 0$. Examples are products of an isolated singularity with a smooth line (such as A_∞) and some of the functions mentioned above (e.g. F_1A_3). Very few is known about this class; let us show here the following "non-splitting property" w.r.t. isolated singularities:

Proposition 5.2 *If $\hat{f}$ has the property that $b_n(\hat{F}) = 0$, then any admissible deformation has no isolated critical points.*

Proof From (4) we get $H_{n+1}(E, F) = 0$. It follows that $H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) = 0$ and $\oplus_{r \in R} H_{n+1}(E_r, F_r) = 0$, and therefore the set R is empty. $\square$

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References

1. de Jong, T.: Some classes of line singularities. Math. Z. **198**, 493–517 (1988)
2. Massey, D.B., Siersma, D.: Deformation of polar methods. Ann. Inst. Fourier (Grenoble) **42**, 737–778 (1992)
3. Milnor, J.: Singular Points of Complex Hypersurfaces. Annals of Mathematics Studies, vol. 61. Princeton University Press, Princeton, NJ (1968)
4. Némethi, A., Szilàrd, À.: Milnor Fiber Boundary of a Non-isolated Surface Singularity. Lecture Notes in Mathematics, vol. 2037. Springer, Berlin (2012)
5. Randell, R.: On the topology of non-isolated singularities. In: Geometric Topology. Proceedings of 1977 Georgia Topology Conference, pp. 445–473. Academic, New York (1979)
6. Siersma, D.: Isolated line singularities. In: Singularities, Part 2 (Arcata, CA, 1981), pp. 485–496. Proceedings of Symposia in Pure Mathematics, vol. 40. American Mathematical Society, Providence, RI (1983)
7. Siersma, D.: Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1. Topol. Appl. **27**, 51–73 (1987)
8. Siersma, D.: Quasi-homogeneous singularities with transversal type A_1. In: Singularities (Iowa City, IA, 1986). Contemporary Mathematics, vol. 90, pp. 261–294. American Mathematical Society, Providence, RI (1989)
9. Siersma, D.: Variation mappings on singularities with a 1-dimensional critical locus. Topology **30**(3), 445–469 (1991)
10. Siersma, D.: The vanishing topology of non isolated singularities. In: New Developments in Singularity Theory, Cambridge, 2000. NATO Science Series II: Mathematics, Physics and Chemistry, vol. 21, pp. 447–472. Kluwer, Dordrecht (2001)
11. Siersma, D., Tibăr, M.: Betti numbers of polynomials. Mosc. Math. J. **11**(3), 599–615 (2011)

12. Siersma, D., Tibăr, M.: Vanishing homology of projective hypersurfaces with 1-dimensional singularities. arXiv 1411.2640 (2014)
13. Steenbrink, J.H.M.: The spectrum of hypersurface singularities. Astérisque **179–180**, 163–184 (1989)
14. Tibăr, M.: The vanishing neighbourhood of non-isolated singularities. Isr. J. Math. **157**, 309–322 (2007)
15. van Straten, D.: On a theorem of Greuel and Steenbrink. In: Decker, W., et al. (eds.) Singularities and Computer Algebra, pp. 353–364. Springer, New York (2017). ETC
16. Williams, K.J.: The Milnor fiber associated to an arrangement of hyperplanes. Dissertation, The University of Iowa, Iowa (2011)
17. Yomdin, I.N.: Complex surfaces with a one-dimensional set of singularities. Sibirsk. Mat. Z. **15**, 1061–1082, 1181 (1974)