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Examples and counter-examples of log-symplectic manifolds

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Abstract

We study topological properties of log-symplectic structures and produce examples of compact manifolds with such structures. Notably, we show that several symplectic manifolds do not admit *bona fide* log-symplectic structures and several *bona fide* log-symplectic manifolds do not admit symplectic structures; for example, $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has *bona fide* log-symplectic structures if and only if $m, n > 0$, while they only have symplectic structures for $m = 1$. We introduce surgeries that produce log-symplectic manifolds out of symplectic manifolds and show that any compact oriented log-symplectic 4-manifold can be transformed into a collection of symplectic manifolds by reversing these surgeries. Finally, we show that if a compact manifold admits an achiral Lefschetz fibration with homologically essential fibres, then the manifold admits a log-symplectic structure. Then, using results of Etnyre and Fuller (*Int. Math. Res. Not.* (2006), art. ID 70272), we conclude that if M is a compact, simply connected 4-manifold then $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ have log-symplectic structures.

Contents

1. *Introduction*

A log-symplectic structure on a manifold M^{2n} is a Poisson structure $\pi \in \mathfrak{X}^2(M)$ for which π^n has only nondegenerate zeros. This condition is weaker than requiring that M is outright symplectic (in which case π^n would not vanish), and yet it is only a little less so, since it still requires that π is generically symplectic and that its failure to be so everywhere is as well behaved as one could ask. If we want to rule out log-symplectic structures which are in fact symplectic, we refer to them as *bona fide* or nonsymplectic log-symplectic structures.

These structures have been classified on surfaces by Radko [**17**] and already in dimension two; there is a marked contrast with the symplectic case, namely, every surface (orientable or not) has a log-symplectic structure. Recently, log-symplectic structures have received renewed attention: Guillemin, Miranda and Pires [**10**] proved a local form for the Poisson structure in a neighbourhood of the zeros of π^n, and Gualtieri and Li [**8**] managed to give a clear geometrical description of symplectic groupoids integrating log-symplectic structures.

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$\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$	symplectic	*bona fide* log-symplectic
$m>1,\ n>0$	✗	✓
$m>1,\ n=0$	✗	✗
$m=1,\ n>0$	✓	✓
$m=1,\ n=0$	✓	✗
$m=0,\ n>0$	✗	✗

FIGURE 1. *Table showing the values of m and n for which $m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has symplectic or* bona fide *log-symplectic structures. In the symplectic case, we require that the orientation determined by the symplectic structure agrees with the orientation of the manifold. In the log-symplectic case, since these structures do not induce a preferred orientation on the manifold, we simply assert the existence of the structure in the underlying unoriented differentiable manifold.*

Despite these recent advances in the theory, the area still lacks examples and even topological obstructions to the existence of these structures are unknown. So, given a manifold, the question 'does it have a log-symplectic structure?' is a little hard to answer.

We tackle these shortcomings in this paper. Indeed, Marcut and Osorno-Torres's paper [**14**, **16**] and the present one are the first to provide topological obstructions to the existence of log-symplectic structures. While Marcut and Osorno-Torres prove that a log-symplectic manifold whose singular locus has a compact component must have a cohomology class $a \in H^2(M)$ such that $a^{n-1} \neq 0$, we prove a different property which is more contrastive with symplectic geometry:

THEOREM 4.2. *If a compact oriented manifold M^{2n}, with $n > 1$, admits a* bona fide *log-symplectic structure, then there are classes $a, b \in H^2(M;\mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.*

Different from Marcut and Osorno-Torres's topological constraint, the existence of the class b is not necessarily shared by symplectic manifolds and, in effect, shows that there are several symplectic manifolds for which the only log-symplectic structures are outright symplectic, whereas other manifolds do not admit log-symplectic structures at all.

We then move on to produce examples of manifolds admitting such structures. The first approach consists simply of deforming a symplectic structure into a *bone fide* log-symplectic one. We show:

THEOREM 5.1. *Let (M^{2n}, ω) be a symplectic manifold, and $k > 0$ be an integer. If M has a compact symplectic submanifold $F^{2n-2} \subset M$ with trivial normal bundle, then M has a log-symplectic structure for which the zero locus of π^n has k components all diffeomorphic to $F \times S^1$.*

Using symplectic blow-up, we can then construct log-symplectic structures on $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $m, n > 0$. Therefore coupling the two theorems, we have a complete classification of which manifolds in the family $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for $m, n \geqslant 0$ admit log-symplectic structures (see Figure 1).

Further, we introduce two surgeries which produce log-symplectic manifolds out of log-symplectic manifolds and which increase the number of components of the singular locus of the Poisson structure; hence even if the starting manifolds are symplectic, the resulting manifolds will only be log-symplectic.

Following the lines of Gompf's theorem relating symplectic structures to Lefschetz fibrations, we prove an analogous result for log-symplectic manifolds:

THEOREM 6.7. *Let M^4 and Σ^2 be compact connected manifolds and $p : M \longrightarrow \Sigma$ be an achiral Lefschetz fibration with generic fibre F. If F is orientable and $[F] \neq 0 \in H_2(M; \mathbb{R})$, then M has a log-symplectic structure whose singular locus has one component and for which the fibres are symplectic submanifolds of the symplectic leaves of the Poisson structure.*

Using results of Etnyre and Fuller on such fibrations [5], we obtain a general existence result.

THEOREM 6.12. *Let M be a simply connected compact 4-manifold. Then both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit* bona fide *log-symplectic structures.*

We finish by showing that in four dimensions any compact orientable log-symplectic manifold is obtained out of a symplectic manifold using our surgeries. Expressed in another way:

THEOREM 7.1. *Let (M^4, π) be a compact, orientable, log-symplectic manifold with singular locus Z. Then each unoriented component of $M \backslash Z$ is symplectomorphic to an open subset of a compact symplectic manifold.*

While our original motivation to study log-symplectic structures lies in the realm of Poisson geometry, one might naturally bundle them together with folded symplectic structures: other structures defined as degenerate symplectic structures whose degeneracy locus is defined by a transversality condition. In particular, it is natural to compare these structures and therefore put our results in context, especially because there are a few results similar in content to the ones we obtain here.

The relevant result regarding *topological obstructions* of folded symplectic structures was proved by Cannas da Silva in [2]: every compact oriented 4-manifold admits a folded symplectic structure; hence, differently from log-symplectic structures, in four dimensions, there are no topological obstructions to the existence of folded symplectic structures. The result on *symplectization* of folded symplectic structures was proved in [3] by Cannas da Silva *et al.* Contrary to log-symplectic structures (cf. Theorem 7.1), not all folded symplectic structures can be 'unfolded' as a condition on the one-dimensional foliation of the folding must be imposed. Finally, a *relation between folded symplectic structures and achiral Lefschetz fibrations* was obtained by Baykur in [1]: achiral Lefschetz fibrations with homologically nontrivial fibres admit folded symplectic structures compatible with the fibration. This result is analogous in statement and proof to our Theorem 6.7, as both proofs are based on Gompf's original result for symplectic manifolds. Note that our result is stronger than Baykur's because, in any dimension, if a manifold admits a log-symplectic structure, it also admits a folded symplectic structure whose folding is the singular locus of the log-symplectic structure [11].

This paper is organized as follows: Section 2 reviews the basics of Poisson geometry relevant for our study, and Section 3 reviews the Guillemin–Miranda–Pires normal form theorem [10]. Section 4 introduces a simple topological invariant that allows us to show that there are many symplectic manifolds that do not admit *bona fide* log-symplectic structures. Section 5 shows that under general assumptions one can deform a symplectic structure into a log-symplectic structure, and Section 6 introduces the surgeries and gives the existence result for log-symplectic structures on achiral Lefschetz fibrations. Finally, Section 7 shows that in four dimensions the surgeries can be reversed and any compact, orientable, log-symplectic 4-manifold can be transformed into a symplectic manifold by surgeries.

2. *Poisson structures*

In this section we give a short account of the basic material on Poisson and log-symplectic structures. For more details, we refer the reader to [8–10].

2.1. *Poisson cohomologies*

A Poisson structure on a manifold M^m is a bivector $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ for which

$$[\pi, \pi] = 0,$$

where the bracket used is the Schouten–Nijenhuis bracket of multivector fields. Assuming that M^m is even dimensional, say, $m = 2n$, a generic bivector (at a point) would give an isomorphism $\pi : T_p^*M \overset{\cong}{\longrightarrow} T_pM$. In this case, π^n is a nonzero element in $\wedge^{2n}T_pM$. If a Poisson bivector π is *everywhere generic* (that is, everywhere invertible) then the 2-form $\omega = \pi^{-1}$ is a symplectic structure on M.

Definition 2.1. The locus where $\pi : T^*M \longrightarrow TM$ is an isomorphism is the *symplectic locus*, and its complement is the *singular locus* of the Poisson structure.

Moving to a more general situation which is still modeled on a 'generic' case, one can require that π^n only has nondegenerate zeros.

Definition 2.2. A *log-symplectic structure* on M^{2n} is a Poisson structure π for which the zeros of π^n are nondegenerate.

It follows from the definition, using Weinstein's splitting theorem, that one can find coordinates in a neighbourhood of any singular point that render a log-symplectic structure π in the following form:

$$\pi = x_1 \partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{x_4} + \cdots + \partial_{x_{2n-1}} \wedge \partial_{x_{2n}},$$

and its inverse is given by

$$\omega = d \log |x_1| \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$

The fact that the 'symplectic form' ω acquires a logarithmic singularity along the singular locus of π justifies the name of the structure.

Continuing with the general theory, any Poisson manifold comes equipped with two differential operators which give rise to cohomology theories. The first is the Poisson differential on multivector fields:

$$d_\pi : \mathfrak{X}^\bullet(M) \longrightarrow \mathfrak{X}^{\bullet+1}(M); \quad d_\pi(\xi) = [\pi, \xi].$$

The Poisson condition and the Jacobi identity for the Schouten–Nijenhuis bracket imply that $d_\pi^2 = 0$, and its cohomology is known as the *Poisson cohomology* of (M, π).

The second is the Koszul differential on forms:

$$\delta : \Omega^\bullet(M) \longrightarrow \Omega^{\bullet-1}(M); \quad \delta\rho = \{\pi, d\}\rho,$$

where $\{\pi, d\} = \pi d - d\pi$ is the graded commutator of operators and π acts on forms by interior product. Again the Jacobi identity for the graded commutator and the Poisson condition imply that $\delta^2 = 0$, and its cohomology is known as the *canonical cohomology* of (M, π).

These operators are related:

Lemma 2.3. *Let (M, π) be a Poisson manifold, $\xi \in \mathfrak{X}^\bullet(M)$ and $\rho \in \Omega^\bullet(M)$. Then*

$$\{\delta, \xi\}\rho = (d_\pi \xi) \cdot \rho,$$

where ξ and $d_\pi\xi$ act on forms by inner product.

Proof. This follows automatically from the description of the Schouten–Nijnehuis bracket as a derived bracket:

$$\{\delta,\xi\}\rho = \{\{\pi,d\},\xi\}\rho = [\pi,\xi]\cdot\rho = (d_\pi\xi)\rho. \qquad \square$$

2.2. *The canonical bundle and the modular vector field*

Given a Poisson manifold (M^m,π), the determinant bundle $K = \wedge^m T^*M$ is also known as the *canonical bundle* of M. Given any nonvanishing local section $\rho \in \Gamma(K)$, there is a unique vector field X such that

$$\delta\rho = \iota_X\rho.$$

The vector field X is called the *modular vector field*. Note that changing the trivialization ρ by a nonvanishing function, say g, changes the modular vector field from X to $X + \pi(d\log|g|) = X + d_\pi \log|g|$. In particular, changing ρ to $-\rho$ does not change X, and a modular vector field is determined by a section of the quotient sheaf $K/\mathbb{Z}_2$. If M is nonorientable, there is no global nonvanishing section of K; yet, $K/\mathbb{Z}_2$, the sheaf of densities, always has a nonvanishing section, so one can always find globally defined modular vector fields.

Note that any modular vector field X is an infinitesimal symmetry of the Poisson structure, that is, $[\pi,X]=0$ since for a local section $\rho\in\Gamma(K_\pi)$ we have

$$0=\delta^2\rho = \delta(X\cdot\rho) = (d_\pi X)\cdot\rho - X\cdot(X\cdot\rho) = (d_\pi X)\cdot\rho,$$

which implies $[\pi,X]=0$. An immediate consequence is that the rank of the Poisson structure along the flow of a point is constant. Since a different choice of section of $K_\pi/\mathbb{Z}_2$ changes X to $X + d_\pi\log|g|$, we see that the modular vector field gives a well-defined degree-one Poisson cohomology class. A Poisson structure is *unimodular* if this class is trivial, which is therefore equivalent to the existence of a globally defined δ-closed section of $K_\pi/\mathbb{Z}_2$.

Definition 2.4. A *representation* of a Poisson structure is a vector bundle $E \longrightarrow (M,\pi)$ together with a flat Poisson connection $\nabla : \Gamma(E) \longrightarrow \Gamma(TM\otimes E)$, that is, for $f \in C^\infty(M)$ and $s\in\Gamma(E)$,

$$\nabla(fs) = d_\pi f s + f\nabla s \quad \text{and} \quad \nabla^2 = 0.$$

Example 2.5. The canonical bundle of a Poisson manifold is a representation. Indeed, the operator $\delta : K \longrightarrow \Omega^{m-1}(M) \cong \Gamma(TM\otimes K)$ satisfies the properties required for a connection, and $\delta^2 = 0$ is the flatness condition. Note that if M is orientable, a Poisson structure is unimodular if and only if its canonical bundle is the trivial representation.

2.3. *Log-symplectic structures – basics*

Now we can focus on the objects in which we are interested. Since in a log-symplectic manifold the singular locus is given by the nondegenerate zero locus of a section of a line bundle, we have that it is a smooth submanifold of codimension one. Further, as the rank of the Poisson structure does not change along each of its symplectic leaves, we see that each connected component of the singular locus is itself a Poisson submanifold of M, that is, a union of symplectic leaves.

The following proposition adds up the basic facts about the singular locus.

Proposition 2.6. *Let M be a log-symplectic manifold, Z its singular locus, $\mathcal{N}^*_Z$ the conormal bundle of Z and K_M and K_Z the canonical bundles of M and Z, respectively. Then*

(1) *Z is an orientable Poisson submanifold of M with symplectic leaves of codimension one;*

(2) *K_Z is the trivial representation and has a distinguished trivialization;*
(3) *$K_M|_Z$ is a Poisson representation over Z;*
(4) *$\mathcal{N}_Z^* \cong K_M|_Z$ as vector bundles and hence $\mathcal{N}_Z^*$ inherits the structure of a Poisson representation.*

In particular, if M is orientable, each component of Z has trivial normal bundle.

Proof. We have already argued most of the claim (1). The rest follows from the normal form for a log-symplectic structure. Indeed, if

$$\pi = x_1\partial_{x_1} \wedge \partial_{x_2} + \partial_{x_3} \wedge \partial_{x_4} + \cdots + \partial_{x_{2n-1}} \wedge \partial_{x_{2n}}$$

then the induced Poisson structure on the singular locus, $[x_1 = 0]$, is

$$\partial_{x_3} \wedge \partial_{x_4} + \cdots + \partial_{x_{2n-1}} \wedge \partial_{x_{2n}},$$

which has codimension-one leaves.

To prove (2), we let $\omega = \pi^{-1}$. Then ω is a 2-form with a logarithmic singularity along Z and the desired volume form on Z is just the residue of ω^2 over Z. In the coordinates used above, we have

$$\mathrm{Re}(\omega^n) = dx_2 \wedge \cdots \wedge dx_{2n},$$

hence the residue of ω^n over Z is nowhere vanishing, and one can readily compute $\delta_Z \mathrm{Res}\omega = 0$, in these coordinates.

Claim (3) follows from the fact that a Poisson representation (E, ∇) over M induces a representation on a Poisson submanifold Z if and only if for every local section $\rho \in \Gamma(E)$, we have $\nabla\rho = X_i\rho_i$, where $\rho_i \in \Gamma(E)$ is a local basis for E and X_i is tangent to Z at all points of Z. In our case, the representation is the canonical bundle, ρ is a local nonvanishing volume form and $\delta\rho = X\rho$, for X the modular vector field, which is tangent to Z as the rank of the Poisson structure must remain constant along the integral curves of X. So claim (3) follows.

As for (4), since Z is the nondegenerate zero locus of $\pi^n \in \Gamma(\wedge^{2n}TM)$, we have that, over Z, $d\pi^n$ gives an isomorphism of vector bundles

$$d\pi^n : \wedge^{2n}TM|_Z \otimes \mathcal{N}_Z^* \xrightarrow{\cong} \mathbb{R},$$

that is, $K_M|_Z$ is isomorphic to $\mathcal{N}_Z^*$. □

3. *Invariants and local forms*

While Proposition 2.6 gives a list of simple invariants associated to a log-symplectic structure in [**10**], Guillemin, Miranda and Pires showed that these are in fact all invariants associated to a neighbourhood of the singular locus. Indeed, the following is a direct consequence of the results in [**10**]:

THEOREM 3.1. *Let (M, π) be a log-symplectic manifold, and let Z be a compact connected component of the singular locus. Then a neighbourhood of Z is determined by the Poisson structure induced on Z, a distinguished flat section of K_Z and its representation on the conormal bundle of Z.*

Taking the inverse of the Poisson structure, one can translate this information into differential forms (cf. [**10**]):

Theorem 3.2. *Let (M, π) be a log-symplectic manifold, let Z be a connected component of the singular locus and X a modular vector field of π. Then the pair (π, X) determines the following structure on Z.*

(1) *The normal bundle of Z as a vector bundle, that is, a class $w_1 \in H^1(Z, \mathbb{Z}_2)$.*
(2) *A closed 1-form $\theta \in \Omega^1(Z)$ such that $\theta(X) = -1$.*
(3) *A closed 2-form $\sigma \in \Omega^2(Z)$ such that $\iota_X\sigma = 0$ and*

$$\theta \wedge \sigma^{n-1} \neq 0. \tag{3.1}$$

Changing the modular vector field by $d_\pi f$ does not change θ and changes σ to $\sigma + df \wedge \theta$.

Further, if Z is compact, any log-symplectic structure inducing the data above on Z is equivalent to a neighbourhood of the zero section of the normal bundle of Z endowed with the following structure:

$$d \log |x| \wedge \theta + \sigma, \tag{3.2}$$

where $|\cdot|$ is the distance to the zero section measured with respect to a fixed fibrewise linear metric on $\mathcal{N}_Z$.

Under the conditions of the theorem, the annihilator of the form θ corresponds to the distribution in Z determined by the Poisson structure and σ agrees with the leafwise symplectic form on Z.

Definition 3.3. A *cosymplectic structure* on a manifold Z^{2n-1} is a pair of closed forms $\theta \in \Omega^1(Z)$ and $\sigma \in \Omega^2(Z)$ satisfying (3.1).

For special types of log-symplectic structure, one can rephrase the data 1–3 above as a more workable set.

Definition 3.4. A connected cosymplectic manifold (Z, σ, θ) is *proper* if it is compact and the distribution given by the annihilator of θ has a compact leaf. A component Z of the singular locus of a log-symplectic manifold is *proper* if the cosymplectic structure induced on Z is proper. A log-symplectic manifold is *proper* if all components of the singular locus are proper.

Given a cosymplectic manifold (Z, σ, θ), if we let X be a vector field such that $\theta(X) = -1$ and $\iota_X\sigma = 0$, we have that $\mathcal{L}_X\theta = 0$ and hence the flow of X preserves the leaves of the distribution determined by θ, hence, if Z is proper with compact (symplectic) leaf $(F, \sigma) \subset Z$, the flow of F by the vector field X will provide further leaves of π. Since X is transverse to F and Z is compact, we see that after finite time, say $\lambda > 0$, the flow of X brings F back to itself:

$$\varphi_\lambda : F \longrightarrow F.$$

Since $\mathcal{L}_X\sigma = 0$, the flow is a symplectomorphism of F and hence Z is a symplectic fibre bundle with fibre (F, σ) over the circle:

$$Z = \mathbb{R} \times F/\mathbb{Z},$$

where the quotient is taken with respect to the $\mathbb{Z}$-action generated by $(y, p) \mapsto (y + \lambda, \varphi_\lambda(p))$. Further, the modular vector field is $-\partial_y$ and hence $\theta = dy$.

Different choices of nonvanishing sections of $K_\pi/\mathbb{Z}_2$ change the modular vector field over Z by adding Hamiltonian vector fields of F, so the symplectomorphism φ_λ is only determined up to Hamiltonian symmetries, that is, the relevant data are only its class in $\mathrm{Symp}(F)/\mathrm{Ham}(F)$.

Finally, the normal bundle of Z is determined by its first Stiefel–Whitney class $w_1 \in H^1(Z, \mathbb{Z}_2) = H^1(F; \mathbb{Z}_2)^{\varphi_\lambda} \times H^1(S^1; \mathbb{Z}_2)$. So, in the proper case, Theorem 3.2 becomes (cf. [**8**, **10**]):

Theorem 3.5. *Let Z be a proper component of the singular locus of a log-symplectic structure π and $F \subset Z$ be a compact symplectic leaf of π. Then π determines the following data:*

(1) *the normal bundle of Z, that is, a class $w_1 \in H^1(Z, \mathbb{Z}_2) = H^1(F; \mathbb{Z}_2)^{\varphi} \times H^1(S^1; \mathbb{Z}_2)$;*
(2) *the symplectic structure σ of F;*
(3) *a class $[\varphi] \in \mathrm{Symp}(F)/\mathrm{Ham}(F)$; and*
(4) *a period $\lambda > 0$.*

Further, any two log-symplectic structures inducing the same set of data are equivalent, and, given a set of data 1–4 there is a proper log-symplectic structure which realizes it.

Note that given a nonorientable Poisson manifold, M, one can always pass to the oriented double cover $\widetilde{M}$ of M which inherits a Poisson structure from M. For the log-symplectic case, this allows us to get a simpler local model for the singular locus as now its neighbourhood depends on one fewer parameter, since according to Proposition 2.6, $w_1 = 0$ in $\widetilde{M}$.

The following theorem, communicated to the author by Ioan Marcut (see also [**16**]), uses a Tischler-type argument to show one can always deform a log-symplectic structure into a proper one. In its cosymplectic version, it had already appeared in [**13**].

Theorem 3.6. *If the components of the singular locus of a log-symplectic structure are compact, then the structure can be deformed into a proper one.*

Proof. Let Z be a connected component of the singular locus. The proof consists of two steps. First, we note that one can deform the cosymplectic structure (θ, σ) of Z into $(\widetilde{\theta}, \sigma)$ so that the kernel of $\widetilde{\theta}$ gives a fibration structure to Z. The second step is to show that this deformation can be realized as a deformation of the log-symplectic structure.

For the first step, let $\widetilde{\theta}$ be a closed 1-form representing a class in $H^1(Z, \mathbb{Q})$ which is close enough to θ so that we still have

$$\widetilde{\theta} \wedge \sigma^{n-1} \neq 0.$$

Since $[\widetilde{\theta}]$ represents a rational class, $[\widetilde{\theta}](H_1(Z; \mathbb{Z}))$ is a lattice Λ in $\mathbb{R}$. Then we define the projection map

$$p : Z \longrightarrow \mathbb{R}/\Lambda; \quad p(z) = \int_{z_0}^{z} \widetilde{\theta},$$

where $z_0 \in Z$ is a fixed reference point and the value of the integral modulo Λ does not depend on the choice of path connecting z_0 to z. By construction $dp = \widetilde{\theta}$ is nowhere vanishing and hence $p : Z \longrightarrow S^1$ is a fibration.

For the second step, according to Theorem 3.2 there is $\delta > 0$ such that the log-symplectic structure in a neighbourhood of Z is equivalent to (3.2) for $|x| < \delta$. If we let ψ be a smooth function such that

$$\psi : [0, 1] \longrightarrow [0, 1]; \qquad \begin{cases} \psi(x) = 1 & \text{if } x < \delta/3, \\ \psi(x) = 0 & \text{if } x > 2\delta/3, \end{cases}$$

then the log-symplectic form

$$d \log |x| \wedge ((1 - \psi(|x|)\theta + \psi(|x|)\widetilde{\theta}) + \sigma$$

induces the cosymplectic structure $(\widetilde{\theta}, \sigma)$ on Z and agrees with the original log-symplectic structure if $|x| > 2\delta/3$, hence can be extended to the rest of M by the original log-symplectic structure. □

4. *A simple topological invariant*

One of the simplest and yet restrictive topological properties of compact symplectic manifolds is the existence of a class $a \in H^2(M)$ whose top power is nonzero. Of course, this does not hold on all log-symplectic manifolds, yet log-symplectic manifolds are just a little shy of satisfying this property as shown by Marcut and Osorno-Torres.

THEOREM 4.1 (Marcut–Osorno-Torres [**14**, **16**]). *Let M^{2n} be a log-symplectic manifold whose singular locus has a compact component. Then there is a cohomology class $a \in H^2(M;\mathbb{R})$ such that $a^{n-1} \neq 0$. Further, if $Z \subset M$ is a proper component of the singular locus and has (F, σ) as a symplectic leaf, a can be chosen so that $[a]|_F = [\sigma]$.*

Here we use a little more of the log-symplectic structure in the orientable case to find another topological property of these manifolds.

THEOREM 4.2. *If a compact oriented manifold M^{2n}, with $n > 1$, admits a* bona fide *log-symplectic structure then there are classes $a, b \in H^2(M;\mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.*

Proof. Assume that M has a log-symplectic structure with singular locus $Z \neq \emptyset$. Then, due to Theorem 3.6, we may assume that the structure is proper, hence Z is a symplectic fibration over the circle with fibre a symplectic manifold F. On the one hand, due to the Marcut–Osorno-Torres theorem, there is a globally defined closed 2-form $\widetilde{\omega} \in \Omega^2(M)$ which restricts to the symplectic form on F, that is, the homology class of F pairs nonzero with a^{n-1}, so we have that $[F] \neq 0 \in H_{2n-2}(M;\mathbb{R})$. On the other hand, since, even within Z, F appears as a fibre of a fibration, we conclude that the Poincaré dual of F, $b \in H^2(M;\mathbb{R})$, must satisfy $b^2 = 0$ and, by definition of the Poincaré dual,

$$\langle a^{n-1}b, [M]\rangle = \langle a^{n-1}, F\rangle \neq 0.$$ □

We state a few immediate corollaries as follows.

COROLLARY 4.3. *An orientable, compact,* bona fide *log-symplectic manifold M of dimension $2n$ has $b_{2i}(M) \geqslant 2$ for $0 < i < n$.*

Proof. It follows directly from the relations $a^{n-1}b \neq 0$ and $b^2 = 0$ that the classes a^i and $a^{i-1}b$ are linearly independent for $0 < i < n$. □

COROLLARY 4.4. *For $n > 1$, $\mathbb{C}P^n$ has no* bona fide *log-symplectic structure and, for $n > 2$, the blow-up of $\mathbb{C}P^n$ along a symplectic submanifold of real codimension greater than four also does not carry* bona fide *log-symplectic structures.*

COROLLARY 4.5. *A smooth orientable compact 4-manifold with definite intersection form does not admit* bona fide *log-symplectic structures. In particular, for $n > 0$, $\#n\mathbb{C}P^2$ and $\#n\overline{\mathbb{C}P^2}$ do not admit* bona fide *log-symplectic structures.*

Proof. Indeed, under the hypothesis of both corollaries, there is no element in second cohomology whose square is zero. □

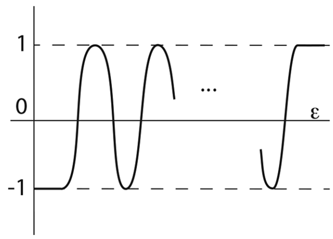

Figure 2. *Graph of a possible scaling function that can be used to create a singular locus with an odd number of components.*

Note that due to the Taubes result on Seiberg–Witten invariants of symplectic manifolds [**18**], $\#n\mathbb{C}P^2$ does not admit symplectic structures for $n > 1$, that is, $\#n\mathbb{C}P^2$ simply does not admit log-symplectic structures *bona fide* or not.

5. *Birth of singular loci*

In this section, we show that under mild assumptions one can transform a symplectic structure into a log-symplectic structure with nonempty singular locus. As a consequence of this seemingly inoffensive fact, we conclude that $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ has a log-symplectic structure with nonempty singular locus as long as $m > 0$ and $n > 0$.

Theorem 5.1. *Let (M^{2n}, ω) be a symplectic manifold and $k > 0$ be an integer. If M has a compact symplectic submanifold $F^{2n-2} \subset M$ with trivial normal bundle, then M has a log-symplectic structure for which the zero locus of π^n has k components all diffeomorphic to $F \times S^1$.*

Proof. Due to the symplectic neighbourhood theorem, F has a tubular neighbourhood diffeomorphic to $D^2 \times F$ endowed with the product symplectic structure, where D^2 is the 2-disc of radius $\varepsilon > 0$. To prove the theorem, it is enough to endow D^2 with a log-symplectic structure whose singular locus has k components and which agrees with the standard symplectic structure near the boundary of the disc. Indeed, in this case we can consider $D^2 \times F$ with the product of the log-symplectic structure on D^2 and the symplectic structure on F. Since this new structure agrees with the original symplectic structure on the boundary of the disc, we can extend it to M using the original symplectic structure.

To produce the desired log-symplectic structure on D^2, we observe that in two dimensions every bivector is automatically Poisson; hence all we need to do is find a bivector in D^2 with the desired number of nondegenerate zeros. To achieve this, we let $\pi \in \Gamma(\wedge^2 TD)$ be the inverse of the standard symplectic structure on D^2 and consider the bivector $f(|x|)\pi(x)$, where f is a smooth real function defined on the closed interval $[0, \varepsilon]$ which is locally constant and equal to 1 in a neighbourhood of ε, locally constant and nonvanishing in a neighbourhood of 0 and has precisely k transverse zeros (see Figure 2). Then $f\pi$ is a log-symplectic structure of the desired type on D^2. □

Corollary 5.2. *For any positive integers m, n, the manifolds $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ have a log-symplectic structure whose singular locus is diffeomorphic to $S^1 \times S^2$.*

Proof. The blow-up of $\mathbb{C}P^2$ at a point, that is, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, has the structure of a symplectic $\mathbb{C}P^1$ fibration over $\mathbb{C}P^1$. In particular, the fibres satisfy the properties of Theorem 5.1 and hence we can endow $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with a log-symplectic structure with nonempty singular locus, say, with one component diffeomorphic to $S^1 \times S^2$. Therefore, the top power of the log-symplectic form on the symplectic locus agrees with the orientation of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ at some points and disagrees at other points. By the symplectic blow-up theorem [15], we can blow up points in the symplectic locus and the result still has a log-symplectic structure. If we blow up points in the symplectic locus where the orientation of the log-symplectic form agrees with the orientation of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, we are performing a connected sum with $\overline{\mathbb{C}P^2}$, whereas if we blow up points in the symplectic locus where the log-symplectic form gives the opposite orientation we are performing a connected sum with $\mathbb{C}P^2$. □

Note that the manifolds obtained in Corollary 5.2 have vanishing Seiberg–Witten invariants and, for m and n even, those manifolds do not admit almost complex structures for either choice of orientation. These are contrasts between log-symplectic and symplectic geometries since symplectic manifolds have nonzero Seiberg–Witten invariants [18] and admit almost complex structures.

As for higher dimensions, Donaldson proved that every symplectic manifold admits a Lefschetz pencil [4] and hence is related to a Lefschetz fibration via the blow-up of the base locus of the pencil. Due to Theorem 5.1 any such fibration has log-symplectic structures.

6. *Surgeries for log-symplectic manifolds*

In this section, we introduce surgeries that produce new log-symplectic structures out of old ones. A main feature is that in these surgeries we create new components of the singular locus; hence even if the starting manifolds are symplectic the results will be only log-symplectic.

6.1. *Construction* 1

This first construction produces (possibly) orientable log-symplectic manifolds out of pairs with matching data.

Building block. Using the language of Theorem 3.2, the local model that gives rise to the construction corresponds to the case when Z has trivial normal bundle. Given a cosymplectic manifold (Z,σ,θ), we let

$$\mathcal{N} = (-2,2)\times Z$$

and endow $\mathcal{N}$ with a log-symplectic structure for which $\{0\}\times Z$ is the singular locus, namely, we consider the 2-form

$$\Omega = d\log|x|\wedge\theta+\sigma, \tag{6.1}$$

where $|x|$ denotes the absolute value of the real number x.

Ingredients. To perform this surgery, we will need a (not necessarily connected) log-symplectic manifold (M^{2n},π) together with two embeddings of a compact, connected, cosymplectic manifold (Z,σ,θ), $\iota_i : Z^{2n-1}\hookrightarrow M$, such that

(1) each $\iota_i(Z)$ lies in the symplectic locus of π, and $\iota_1(Z)\cap\iota_2(Z)=\emptyset$;
(2) there is $f\in\mathbb{C}^\infty(Z)$ such that $\iota_1^*\omega=\iota_2^*\omega-df\wedge\theta=\sigma$, where ω is the symplectic form on the symplectic locus of M.

The surgery. Since each $\iota_i(Z)$ is in the symplectic locus of π, the log-symplectic structure on M gives rise to an orientation of a neighbourhood of $\iota_i(Z)$. Since Z is cosymplectic, it has a natural orientation as defined by the volume form $\theta\wedge\sigma^{n-1}$. Together the orientation on Z and the (semi-local) orientation on M allow us to orient the normal bundle of Z and define an interior and an exterior region within the normal bundle: a vector $N\in T_{\iota_i(p)}M$ is outward pointing if for any positive basis $\{v_1, \dots, v_{2n-1}\} \in T_pZ$ the set $\{N, \iota_{i*}v_1, \dots, \iota_{i*}v_{2n+1}\}$ is a positive basis for T_pM. Let $\hat{M}$ be the real oriented blow-up of M along $\iota_1(Z)$ and $\iota_2(Z)$, that is, $\hat{M}$ is diffeomorphic to the manifold obtained from M by removing an open tubular neighbourhood of both copies of Z and $\hat{M}$ has four copies of Z as boundary. At each boundary copy of Z, $\hat{M}$ lies either in the interior or the exterior side of the boundary according to the semi-local orientation. We let $\widetilde{M}$ be the manifold obtained from $\hat{M}$ by identifying with each other the boundary components for which $\hat{M}$ lies in the interior and similarly for the components for which $\hat{M}$ lies in the exterior.

Theorem 6.1. *Let (M,π), (Z,θ,σ) and $\iota_1, \iota_2 : Z \longrightarrow M$ be the ingredients for the surgery, and let $\hat{M}$ be the real oriented blow-up of M along the two copies of Z. Then the manifold*

$$\widetilde{M} = \hat{M}/\sim$$

obtained by identifying the boundary components of $\hat{M}$ for which $\hat{M}$ lies in the interior (respectively, exterior) of the boundary via the map $\iota_2 \circ \iota_1^{-1}$ has a log-symplectic structure which agrees with the original structure on M outside a neighbourhood of two copies of $Z = \partial\hat{M}/\sim$ and for which Z is part of the singular locus.

Proof. We have an embedding $j_1 : Z \hookrightarrow \mathcal{N}$, $p \mapsto (-1, p)$. For this embedding, Z lies in the symplectic locus of the log-symplectic structure and the restriction of the symplectic form (6.1) to Z is just σ. Similarly, given a real function $f : Z \to \mathbb{R}$, for any $\varepsilon > 0$ small enough, we have an embedding $j_2 : Z \hookrightarrow \mathcal{N}$, $x \mapsto (\varepsilon e^f(p), p)$ and the restriction of the log-symplectic form to this embedding is $df \wedge \theta + \sigma$. For both embeddings, j_1 and j_2, the vector field $x\partial_x$ is outward pointing with respect to the orientations induced by the symplectic and cosymplectic structures in a neighbourhood of the embeddings, that is, the cylinder

$$C = \{(x,p) \in \mathcal{N} : 1- \leqslant x \leqslant \varepsilon e^f(p)\}$$

contains interior points for both boundaries with respect to the semi-local orientations.

Hence, by Weinstein's coisotropic neighbourhood theorem, a neighbourhood of $j_1(Z)$ is symplectomorphic to a neighbourhood of $\iota_1(Z)$ and a neighbourhood of $j_2(Z)$ is symplectomorphic to a neighbourhood of $\iota_2(Z)$. Using these symplectomorphisms, we can glue the exterior regions of $\iota_i(Z)$ in $\hat{M}$ to C along the boundaries and the resulting manifold has a log-symplectic structure.

We can repeat the same argument to glue the interior regions, but now using the log-symplectic structure $-d\log|x| \wedge \theta + \sigma$ on $\mathcal{N}$, therefore obtaining a log-symplectic structure on $\widetilde{M}$. □

Remarks. (i) Even if M is a symplectic manifold, and hence has a preferred orientation, the diffeomorphism used to glue the two boundaries together does not respect these orientations; hence $\widetilde{M}$ does not have a preferred orientation.

(ii) A common use of the theorem is when M has two connected components, the maps ι_i map Z to different components, and their images are separating submanifolds. In this case, $\widetilde{M}$ also has two components and we will often focus our attention in one of the two, say, the one obtained by gluing the exterior regions.

(iii) Additive properties of the Euler characteristic imply that

$$\chi_M = \chi_{\widetilde{M}}.$$

Theorem 6.1 leaves us with the question of how to find suitable submanifolds Z to which it can be applied. Next, we identify two situations in which manifolds with the desired structure appear naturally. We start with the simplest setting.

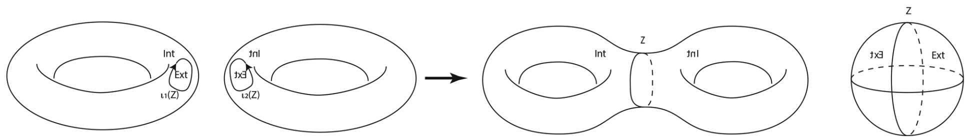

FIGURE 3. *A possible surgery on two null homologous circles lying on 2-tori. The first torus is oriented by the outward vector, whereas the second is oriented by the inward normal vector. Interior (Int) and exterior (Ext) determined by the circles are marked in the figure with the letters inverted for different orientations. The result of the surgery is a genus-two surface and a sphere.*

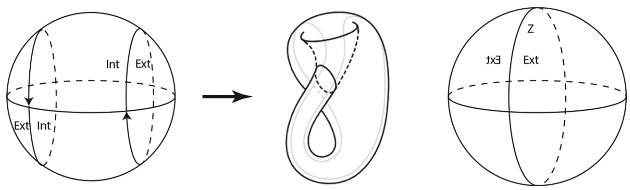

FIGURE 4. *A possible surgery on two oppositely oriented circles on a sphere yields a sphere and a Klein bottle.*

COROLLARY 6.2. *Let (M^{2n},ω) be a log-symplectic manifold and $\iota_i : (F^{2n-2},\sigma) \longrightarrow (M,\omega)$, $i = 1,2$, be embeddings of a compact symplectic manifold F in the symplectic locus of M for which the images have trivial normal bundle. Let*

$$\widetilde{M} = M\backslash(\mathcal{N}_1 \cup \mathcal{N}_2)/\sim,$$

where $\sim$ indicates the natural identification of the boundaries $\partial\mathcal{N}_1 \cong \partial\mathcal{N}_2$. Then $\widetilde{M}$ has a log-symplectic structure which agrees with the original structure outside a tubular neighbourhood of $\partial\mathcal{N}_i \subset \widetilde{M}$. The Euler characteristic of $\widetilde{M}$ is

$$\chi_{\widetilde{M}} = \chi_M - 2\chi_F.$$

Proof. Weinstein's symplectic neighbourhood theorem implies that $\iota_i(F)$ have neighbourhoods symplectomorphic to $D^2 \times F$ with the product symplectic structure. In particular, the boundary of such neighbourhoods is diffeomorphic to $Z = S^1 \times F$ with the cosymplectic structure given by the volume from of S^1, θ, and $\sigma = \omega|_F$ and the restriction of the symplectic form to Z is simply the symplectic form of F pulled back to the product. Hence we can use the theorem to conclude that $\widetilde{M}$, obtained by gluing the exterior regions of M with respect to the embeddings two embeddings of Z, has a log-symplectic structure.

The last claim follows from the additive properties of the Euler characteristic or by observing that the manifold obtained by gluing the interior regions (the other component of the surgery) is just $S^2 \times F$ which has Euler characteristic $2\chi_F$. □

Figures 3 and 4 illustrate the surgery described in Corollary 6.2 applied to null homologous circles on surfaces. Note that even if the original surfaces are orientable, the result of the surgery may not be. Next, we present a setting which is a little more elaborate.

COROLLARY 6.3. *Let $p_i : M_i \longrightarrow \Sigma_i$, $i = 1,2$, be symplectic Lefschetz fibrations with the same generic fibre (F,σ). Let $\gamma_i : \mathbb{R}/\lambda\mathbb{Z} \longrightarrow \Sigma_i$ be separating loops which avoid the critical values of p_i, let M_i^+ be the exterior region of $p_i^{-1}(\gamma_i)$ and let $\varphi_i : F \longrightarrow F$ be the symplectomorphism of $F = p_i^{-1}(\gamma_i(0))$ obtained from symplectic parallel transport along γ_i.*

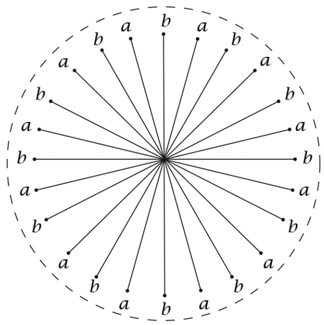

FIGURE 5. *Graphic representation of the base of $E(2)$, showing the singular values of the projection map, a regular value of the map for reference and paths connecting the regular value to the singular values to determine the homology class of the vanishing cycles. In this case, a and b are the vanishing cycles which form an integral basis for $H_1(F;\mathbb{Z})$.*

If there is a symplectomorphism $\vartheta : F \longrightarrow F$ such that $\varphi_2 = \vartheta \circ \varphi_1 \circ \vartheta^{-1}$, that is, these loops have the same monodromy, then

$$M_1^+ \cup M_2^+ / \sim$$

has a log-symplectic structure with singular locus $\partial M_1^+ \cong \partial M_2^+$.

Proof. Under the hypothesis, $Z_1 = p_1^{-1}(\gamma_1)$ is given by the quotient of $\mathbb{R} \times F$ by the $\mathbb{Z}$-action generated by

$$(x, z) \cong (x + \lambda, \varphi_1(z)).$$

The form $\theta = dx$ together with the symplectic form $\sigma \in \Omega^2(F)$ makes Z_1 into a cosymplectic manifold.

Similarly, $Z_2 = p_2^{-1}(\gamma_2)$ is a cosymplectic manifold and the map

$$Z_1 \longrightarrow Z_2, \quad (x, z) = (z, \vartheta(z))$$

is an isomorphism of cosymplectic structures as long as $\varphi_2 = \vartheta \circ \varphi_1 \circ \vartheta^{-1}$. The result now follows from Theorem 6.1. □

EXAMPLE 6.4 (Log-symplectic structures on $\#nS^2 \times S^2$). Next, we provide an explicit log-symplectic structure on $\#nS^2 \times S^2$. Our starting point is the elliptic surface $E(2k)$: the fibre sum of $2k$ copies of $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$. This is a Lefschetz fibration $p : E(2k) \longrightarrow \mathbb{C}P^1$ which, after appropriate identifications, has $24k$ singular fibres for which the vanishing cycles correspond to two basis elements $\{a, b\} \in H_1(F)$ appearing in an alternating fashion, as depicted, for $E(2)$, in Figure 5 (cf. [**7**, Example 8.2.11]).

In order to use Construction 1, we consider two copies of $E(2k)$ and in both of them consider the same path, namely one whose exterior contains $n + 1$ consecutive singular fibres (see Figure 6).

Since we are starting with two copies of the same data, we can use Corollary 6.3 to introduce a log-symplectic structure

$$\widetilde{M} = M^+ \cup_\partial M^+,$$

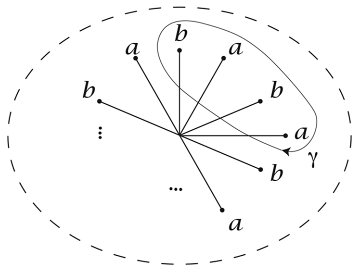

FIGURE 6. *Graphic representation of the base of $E(2k)$ and a path whose exterior contains four consecutive singular values of the projection map.*

where $M^+ = p^{-1}(\overline{\Sigma^+})$ is the inverse image of the closure of the exterior points of γ. Hence we see that Construction 1 consists of taking two identical copies of $M^+ \subset E(2k)$ and then gluing them along the boundaries using the identity map. The resulting manifold is also known as the *double* of M^+. To precisely determine M^+, we observe two simple facts.

First, M^+ admits a handlebody decomposition which contains a 0-handle, n 2-handles and no handles of other indices (that is, M is a 2-handlebody). Indeed, according to [**7**, Example 8.2.8], a Kirby diagram for M has $n+2$ 2-handles, two 1-handles and one 0-handle, but we can cancel two 2-handles against the two 1-handles to obtain the desired handlebody decomposition.

Second, since the interior of M^+ is an open subset of $E(2k)$ and $E(2k)$ is spin, the intersection form on M^+ is even.

These facts, together with the following proposition, determine the result of the surgery.

PROPOSITION 6.5 [**7**, Corollary 5.1.6]. *Let M^4 be a 2-handlebody with n 2-handles. Then the double of M is diffeomorphic to $\#nS^2 \times S^2$ if the intersection form of M is even and to $\#n\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ if the intersection form of M is odd.*

In our case, we conclude that $\widetilde{M}$ is diffeomorphic to $\#nS^2 \times S^2$.

6.2. *Achiral Lefschetz fibrations*

Related to the construction of Corollary 6.3 is the notion of an achiral Lefschetz fibration.

DEFINITION 6.6. Let M^{2n} be a manifold. An *achiral Lefschetz fibration* on M is a proper, smooth map $p : M \longrightarrow \Sigma^2$ such that the pre-image of any critical value has only one critical point and for any such pair of critical value, y, and critical point, x, there are complex coordinate systems centred at x and y for which p takes the following form:

$$p(z_1, \dots, z_n) = z_1^2 + \cdots + z_n^2.$$

Note that in this definition we do not require M or Σ to be orientable. If they are, one can assign a sign to each critical point x: we demand that the complex structure on Σ is compatible with the orientation, and then we say that x is *positive* if the complex structure on M used in the definition is compatible with the orientation of M and *negative* otherwise.

Given the construction of Corollary 6.3 and the ensuing example, one might expect that achiral Lefschetz fibrations are related to log-symplectic structures in the same way that Lefschetz fibrations are related to symplectic structures. This is indeed the case, as we show next:

THEOREM 6.7. *Let M^4 and Σ^2 be compact connected manifolds and $p : M \longrightarrow \Sigma$ be an achiral Lefschetz fibration with generic fibre F. If F is orientable and $[F] \neq 0 \in H_2(M;\mathbb{R})$, then M has a log-symplectic structure whose singular locus has one component and for which the fibres are symplectic submanifolds of the symplectic leaves of the Poisson structure.*

Proof. The proof follows closely that of Gompf's theorem relating Lefschetz fibrations and symplectic structures [**7**, Theorem 10.2.18]. Before we delve into the proof we will fix some notation. We let

- $F_y = p^{-1}(y)$ for $y \in \Sigma$;
- Δ be the set of singular points of p;
- Δ' be the set of the singular values of p;
- $\Sigma_0 = \Sigma\backslash\Delta'$ and $M_0 = p^{-1}(\Sigma\backslash\Delta')$, so that $p : M_0 \longrightarrow \Sigma_0$ is a proper fibration.

First, we observe that we can assume that p has connected fibres. Indeed, for achiral Lefschetz fibrations we have a short exact sequence of homotopy groups:

$$\pi_1(F) \longrightarrow \pi_1(M) \xrightarrow{p_*} \pi_1(\Sigma) \longrightarrow \pi_0(F) \longrightarrow \{0\}. \tag{6.2}$$

Since M is compact, $\pi_0(F)$ is finite and hence (6.2) implies that $p_*(\pi_1(M))$ has finite index in $\pi_1(\Sigma)$. Let $\widetilde{\Sigma}$ be the cover of Σ corresponding to the subgroup $p_*(\pi_1(M)) \subset \pi_1(\Sigma)$. Then $\widetilde{\Sigma}$ is compact and the map $p : M \longrightarrow \Sigma$ lifts to a map $\widetilde{p} : M \longrightarrow \widetilde{\Sigma}$. The projection $\widetilde{p}$ is still an achiral Lefschetz fibration and, by construction, $\widetilde{p}_*(\pi_1(M)) = \pi_1(\widetilde{\Sigma})$; hence (6.2) implies that the fibres of $\widetilde{p}$ are connected. From now on we assume that the fibres of $p : M \longrightarrow \Sigma$ are connected.

Next, we deal with the general lack of orientation of the manifolds involved. First, since F is orientable, we fix an orientation for F for the remainder of this proof. If $p : M_0 \longrightarrow \Sigma_0$ were a nonorientable fibration, there would be a loop $\alpha : (I, \partial I) \longrightarrow M_0$ based at some point $y \in \Sigma$, where I is the unit interval, for which parallel transport (after a choice of connection) provided an orientation-reversing diffeomorphism of F_y. In this case $p^{-1}(\alpha(I))$ would provide a chain whose boundary is $2[F_y]$, contradicting the condition $[F] \neq 0 \in H_2(M;\mathbb{R})$. Therefore, $p : M_0 \longrightarrow \Sigma_0$ is an orientable fibration. Further, this orientation induces orientations on the singular fibres of p and hence we can also integrate forms over the (components of the singular) fibres. It also follows that M is orientable if and only if Σ is.

If Σ and M are orientable, after choosing orientations, we can split the critical points of p into positive and negative ones. If there are no positive or no negative points, Gompf's theorem [**7**, Theorem 10.2.18] implies that M admits a symplectic structure and due to Theorem 5.1 it admits a log-symplectic structure with the desired properties. If there are positive and negative critical points, we can choose a separating loop $\Gamma \subset \Sigma_0$ whose interior locus contains all the negative points and whose exterior locus contains all the positive points.

If Σ is nonorientable, we can choose a loop $\Gamma \subset \Sigma_0$ such that $\Sigma\backslash\Gamma$ is orientable and hence so is $M\backslash p^{-1}(\Gamma)$. After choosing orientations on both $\Sigma\backslash\Gamma$ and $M\backslash p^{-1}(\Gamma)$, we may homotope the loop Γ through the negative critical values of p so that all singular points of $p : M\backslash p^{-1}(\Gamma) \longrightarrow \Sigma\backslash\Gamma$ are positive. In either case, $M\backslash p^{-1}(\Gamma)$ is an oriented manifold and $p : M\backslash p^{-1}(\Gamma) \longrightarrow \Sigma\backslash\Gamma$ is a proper Lefschetz fibration possibly after changing the orientation of one of the components of $M\backslash\Gamma$ and $\Sigma\backslash\Gamma$. From now on, we orient both $M\backslash p^{-1}(\Gamma)$ and $\Sigma\backslash\Gamma$ so that p is a Lefschetz fibration there.

The next steps aim to construct a closed 2-form σ on M which restricts to a symplectic form in every fibre.

LEMMA 6.8. *Under the hypothesis of Theorem 6.7, there is a closed 2-form $\zeta \in \Omega^2(M)$ such that*

(1) $\int_S \zeta = 1$ *over any fibre and*
(2) *if a singular fibre F_y is a plumbing of two surfaces S_1 and S_2, and we are given $s_y \in (0,1)$, then ζ can be chosen so that $\int_{S_1} \zeta = s_y$.*

Proof. Since $[F] \neq 0 \in H_2(M;\mathbb{R})$, there is a closed form $\xi \in \Omega^2(M)$ which integrates to 1 over the generic fibres and hence over all fibres. Next, we need to argue that one can change ξ so that property 2 holds. In this case, with the orientations chosen before on $M \backslash p^{-1}(\Gamma)$, the intersection number of S_1 and S_2 is 1. Since $S_1 \cup S_2$ is homologous to a regular fibre, $\int_{S_1} \xi + \int_{S_2} \xi = 1$. If $\int_{S_1} \xi = r$, let ψ be a form with support in a neighbourhood of $S_2 \subset M \backslash p^{-1}(\Gamma)$, which represents the Poincaré dual of S_2, and consider $\xi' = \xi + (-r + s_y)\psi$. Then for any closed surface S' inside another fibre $F_{y'}$, $\int_{F_{y'}} \psi = 0$ as $F_{y'}$ does not intersect S_2, hence $\int_{S'} \xi' = \int_{S'} \xi$. On the other hand,

$$\int_{S_1} \xi' = \int_{S_1} \xi \pm (-r + s_y) \int_{S_1} \psi = s_y.$$

That is, after a change in ξ, we have found another closed form for which the claim holds at a specific singular fibre. Since there are only finitely many such fibres, we can repeat the process for each of them to obtain the desired ζ. □

LEMMA 6.9. *Under the hypothesis of Theorem 6.7, there is a finite good open cover $\mathcal{U}$ of Σ such that for each $U_\alpha \in \mathcal{U}$, there is a closed form $\eta_\alpha \in \Omega^2(p^{-1}(U_\alpha))$ which is symplectic on the fibres of p.*

Proof. Since $p : M \backslash p^{-1}(\Gamma) \longrightarrow \Sigma \backslash \Gamma$ is a proper Lefschetz fibration for which $[F] \neq 0$, it follows from Gompf's theorem [**7**, Theorem 10.2.18] that each component of $M \backslash p^{-1}(\Gamma)$ has a symplectic form for which the fibres are symplectic of area 1. Letting $\mathcal{N}$ be a small tubular neighbourhood of Γ without singular values of p, it follows that $p : p^{-1}(\mathcal{N}) \longrightarrow \mathcal{N}$ is a proper Lefschetz fibration for which $[F] \neq 0$, hence we can also apply Gompf's result here to conclude that there is a symplectic form ω_0 on $p^{-1}(\mathcal{N})$ for which the fibres are symplectic of area 1. Finally, let $\mathcal{U}$ be a finite good refinement of the cover $\{\Sigma \backslash \Gamma, \mathcal{N}\}$ and for each $U_\alpha \in \mathcal{U}$ let η_α be the restriction of one of the symplectic forms above to $p^{-1}(U_\alpha)$. □

LEMMA 6.10. *Under the hypothesis of Theorem 6.7, there is a closed 2-form $\sigma \in \Omega^2(M)$ such that $\sigma|_{F_y}$ is a symplectic form for every fibre F_y.*

Proof. The forms η_α and ζ are cohomologous on $p^{-1}(U_\alpha)$ (in the case of singular fibres, one must choose the value s_y so that the integrals of these forms over each cycle agree). Therefore, there are $\theta_i \in \Omega^1(p^{-1}(U_i))$ such that $\eta_i = \zeta + d\theta_i$. Let $\{\kappa_\alpha\}$ be a partition of unity subordinate to the cover $\mathcal{U}$ and consider the form

$$\sigma = \zeta + d \sum_i p^*(\kappa_i)\theta_i.$$

Since $p^*\kappa|_{F_y}$ is a constant, we have that

$$\sigma|_{F_y} = \zeta|_{F_y} + \sum_i p^*(\kappa_i) d\theta_i|_{F_y} = \sum_i p^*(\kappa_i)(\zeta|_{F_y} + d\theta_i|_{F_y}) = \sum_i p^*(\kappa_i)\eta_i|_{F_y}.$$

Since each $\eta_i|_{F_y}$ is a symplectic form and all of them determine the same orientation, $\sum_i p^*(\kappa_i)\eta_i|_{F_y}$ is a symplectic form on F_y. □

End of Proof of Theorem 6.7. Finally, to obtain the log-symplectic structure on M, observe that since $\Sigma\backslash\Gamma$ is oriented, Γ is a real divisor on Σ representing $\wedge^2 T\Sigma$, that is, there is a section $\pi \in \Gamma(\wedge^2(T\Sigma))$ which has Γ as its (transverse) zero locus. Since Σ is two dimensional, π is a log-symplectic structure whose singular locus is Γ. We further choose π so that it agrees with the orientation of $\Sigma\backslash\Gamma$. Inverting π, we obtain $\omega_\Sigma \in \Omega^2(\Sigma)$ with a log-singularity at Γ. Then the standard argument shows that $\omega = p^*\omega_\Sigma + \varepsilon\sigma$ is a log-symplectic structure for ε small enough. Indeed, away from critical points of p, ω_Σ dominates σ and hence determines a log-symplectic structure on the complement of a small neighbourhood of Δ. In particular, ω determines an orientation on its symplectic locus. With respect to this orientation on $M\backslash p^{-1}(\Gamma)$ and the orientation determined by π on $\Sigma\backslash\Gamma$, all singular points are positive and the argument from [**7**, Exercise 10.2.21] shows that ω is symplectic on $M\backslash p^{-1}(\Gamma)$. □

REMARKS (the condition $[F] \neq 0$).

- If the genus of the fibre is different from 1, then $\ker(p)$ defines a line bundle over $M\backslash\Delta$ and this line bundle extends to the singular locus. Letting c_1 be the first Chern class of this bundle, naturality implies that $c_1|_F$ is just the Euler class of the fibre and hence, if the genus of the fibre is not 1, c_1 is nonzero on $[F]$, showing that $[F] \neq 0$.
- If an achiral Lefschetz fibration over an oriented surface has a section, then any fibre represents a nontrivial class since it has nontrivial intersection with the section.
- S^4 admits an achiral Lefschetz fibration. Since $H^2(S^4) = \{0\}$, the fibres are homologically trivial and hence are tori and there is no section. Further, S^4 also does not admit log-symplectic structures due to Theorem 4.1. Therefore, the condition $[F] \neq 0$ cannot be removed from Theorem 6.7.

REMARK. The proof above is also very similar to the one given by Baykur [**1**] relating achiral Lefschetz fibrations to folded symplectic structures, as both ours and Baykur's proof follow Gompf's original proof closely [**7**, Theorem 10.2.18]. The main differences between the proofs concern the treatment of the singular locus, as folded and log-symplectic structures, with different types of singular behaviour, and Gompf did not have to deal with either of them. Further, since log-symplectic structures can always be deformed into folded ones, our proof is a little more general than Baykur's proof.

Achiral Lefschetz fibrations have been studied by Etnyre and Fuller [**5**] and are present in several 4-manifolds.

THEOREM 6.11 (Etnyre and Fuller [**5**]). *Let X be a smooth, closed, oriented 4-manifold. Then there exists a framed circle in X such that the manifold obtained by surgery along that circle admits an achiral Lefschetz fibration with section and whose base is S^2. Further, if M is simply connected then we can arrange so that both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ arise as such surgery and hence both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit achiral Lefschetz fibrations with a section over S^2.*

Combining Theorem 6.7 with Etnyre–Fuller's theorem we get the following theorem.

THEOREM 6.12. *Let M be a simply connected compact 4-manifold. Then both $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit* bona fide *log-symplectic structures.*

6.3. *Construction* 2

The second construction is a nonorientable version of the first which produces proper log-symplectic manifolds.

Building block. Given a symplectic manifold (F, σ), a symplectomorphism $\varphi : F \longrightarrow F$ and $\lambda > 0$, we form the quotient of the log-symplectic manifold

$$\mathcal{N} = (-2, 2) \times \mathbb{R} \times F; \quad \Omega|_{(x,y,p)} = d \log |x| \wedge dy + \sigma$$

by the $\mathbb{Z}$-action generated by $(x, y, p) \sim (-x, y + \lambda, \varphi(p))$:

$$\mathcal{N}_\varphi = \mathcal{N}/\mathbb{Z}.$$

Then $\mathcal{N}_\varphi$ is a log-symplectic manifold with singular locus $Z = \{0\} \times \mathbb{R} \times F/\mathbb{Z}$. Note that $\mathcal{N}_\varphi \backslash Z$ is a fibre bundle over $\mathbb{R}_+$, with projection map induced by the invariant map $\pi_1(x, y, p) = |x|$ defined on $(-2, 2) \times \mathbb{R} \times F$. Then $Z' = \pi_1^{-1}(1)$ is a coisotropic submanifold of the symplectic locus given by

$$Z' = \mathbb{R} \times F/\mathbb{Z}; \quad (y, p) \sim (y + 2\lambda, \varphi^2(p)). \tag{6.3}$$

Ingredients. We will need a proper cosymplectic manifold (Z', θ, σ) with symplectic fibre F for which the monodromy map is the square of a symplectomorphism $\varphi : F \longrightarrow F$, that is, Z' is given by (6.3). We will also need a log-symplectic manifold (M, π) and a separating embedding $\iota : Z' \hookrightarrow M$ in the symplectic locus of M such that $\iota^*\omega = \sigma$, where ω is the induced symplectic structure on M.

The surgery. The surgery follows the same lines of Construction 1. Since $\iota(Z)$ is separating, it defines an exterior and an interior region of M. Let M^+ be the closure of the exterior. Then there is a $\mathbb{Z}_2$-action on $Z' = \partial M^+$, namely, in terms of (6.3), the action is generated by the map

$$(y, p) \mapsto (y + \lambda, \varphi(p)),$$

and the orbits of this action form an equivalence relation on ∂M^+ which allows us to form the space

$$\widetilde{M} = M^+ / \sim$$

obtained by taking the quotient of ∂M^+ by this equivalence relation.

Theorem 6.13. *Let (M, π), (Z', θ, σ) and $\iota : Z' \longrightarrow M$ be the ingredients for the surgery, and let M^+ be the closure of the exterior region defined by $\iota(Z')$. Then the manifold*

$$\widetilde{M} = M^+ / \sim$$

obtained by taking the quotient of ∂M^+ by the $\mathbb{Z}_2$-action has a log-symplectic structure which agrees with the original structure on M^+ outside a neighbourhood of $Z = \partial M^+/\sim$ and for which Z is part of the singular locus.

The proof of this theorem is completely analogous to that of Theorem 6.1. A particular case of this surgery has a geometric interpretation.

Corollary 6.14 (Real blow-up). *Let (M^{2n}, ω) be a log-symplectic manifold, and let $F^{2n-2} \subset M$ be a symplectic submanifold which does not intersect the singular locus and has trivial normal bundle. Then the real blow-up of M along F has a log-symplectic structure for which the exceptional divisor is a component of the singular locus.*

Proof. Just as in Corollary 6.2, the requirement that F has trivial normal bundle implies that a neighbourhood of F is symplectomorphic to $D^2 \times F$, and hence we obtain an embedding of the proper cosymplectic manifold $S^1 \times F$ into M. The monodromy of this cosymplectic manifold is the identity map which is obviously the square of a symplectomorphism. Now, the local model is based on using $\varphi = \mathrm{Id}$, that is, $\mathcal{N}_F = \mathbb{M} \times F$, where $\mathbb{M}$ is the Möbius band and the effect of the surgery is that we remove a neighbourhood of F (which is diffeomorphic to $D^2 \times F$) and glue back $\mathbb{M} \times F$. This is precisely the underlying surgery of the real blow-up of F. □

7. *Reversing the surgeries*

In the previous section, we managed to produce several examples of log-symplectic manifolds out of symplectic manifolds. One might rightfully expect that there are more examples of such structures: for one thing, the Stiefel–Whitney class either vanished (first construction) or corresponded to the generator of $H^1(S^1; \mathbb{Z}_2)$ (second construction), therefore leaving out a number of possibilities. On the other hand, if we assume that M is orientable or, in the nonorientable case, take the orientable double cover, then any singular locus automatically is associated to the zero Stiefel–Whitney class and hence it has neighbourhood diffeomorphic, as a Poisson, manifold to the building blocks used in Construction 1. Next, we show that in four dimensions any log-symplectic structure is created out of our surgeries and hence can be cut and filled into a collection of compact symplectic manifolds.

Theorem 7.1. *Let (M^4, π) be a compact, orientable, log-symplectic manifold with singular locus Z. Then each unoriented component of $M \backslash Z$ can be compactified as a symplectic manifold.*

Proof. According to Theorem 3.2, a neighbourhood of each connected component of Z is equivalent to the building block of Construction 1 and hence we have two copies of Z in such neighbourhood (one on either side of the singular locus) as a coisotropic submanifold. To reverse the surgery, one would need to prove that such coisotropic submanifold appears as the boundary of the (interior of) a symplectic manifold. But in four dimensions any cosymplectic manifold is automatically a taut foliation and hence the conclusion follows from the following theorem:

Theorem 7.2 [**12**, Theorem 41.3.1]. *Let Z be a closed 3-manifold and $\mathcal{F} \subset TZ$ be a smooth taut foliation. Let $\sigma \in \Omega^2(Z)$ be the closed form which is positive on the leaves of $\mathcal{F}$. Then there is a closed symplectic manifold (X, ω) containing Z as a separating submanifold such that $\omega|Z = \sigma$.*

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While carrying out this research, the author was made aware that Frejlich, Martinez-Torres and Miranda were carrying out a project [**6**], which overlaps with the results in this paper. Notably, they had independently produced our 'Construction 1' and our Theorem 7.1.

References

1. R. İ. Baykur, 'Kähler decomposition of 4-manifolds', *Algebr. Geom. Topol.* 6 (2006) 1239–1265.
2. A. Cannas da Silva, 'Fold-forms for four-folds', *J. Symplectic Geom.* 8 (2010) 189–203.
3. A. Cannas da Silva, V. Guillemin and C. Woodward, 'On the unfolding of folded symplectic structures', *Math. Res. Lett.* 7 (2000) 35–53.
4. S. K. Donaldson, 'Lefschetz pencils on symplectic manifolds', *J. Differential Geom.* 53 (1999) 205–236.
5. J. B. Etnyre and T. Fuller, 'Realizing 4-manifolds as achiral Lefschetz fibrations', *Int. Math. Res. Not.* (2006) article ID 70272, 21.

6. P. Frejlich, D. M. Torres and E. Miranda, 'A note on symplectic topology of b-manifolds', Preprint, 2013, arXiv:1312.7329.
7. R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby calculus*, Graduate Studies in Mathematics 20 (American Mathematical Society, Providence, RI, 1999).
8. M. Gualtieri and S. Li, 'Symplectic groupoids of log symplectic manifolds', Preprint, 2012, arXiv:1206.3674.
9. V. Guillemin, E. Miranda and A. R. Pires, 'Codimension one symplectic foliations and regular Poisson structures', *Bull. Braz. Math. Soc.* (*N.S.*) 42 (2011) 607–623.
10. V. Guillemin, E. Miranda and A. R. Pires, 'Symplectic and Poisson geometry on b-manifolds', Preprint, 2012, arXiv:1206.2020.
11. V. Guillemin, E. Miranda and J. Weitsman, 'Desingularizing b^m-symplectic structures', Preprint, 2015, arXiv:1512.05303.
12. P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs 10 (Cambridge University Press, Cambridge, 2007).
13. H. Li, 'Topology of co-symplectic/co-Kähler manifolds', *Asian J. Math.* 12 (2008) 527–543.
14. I. Marcut and B. Osorno-Torres, 'On cohomological obstructions for the existence of log-symplectic structures', Preprint, 2013, arXiv:1303.6246.
15. D. McDuff, 'Examples of simply-connected symplectic non-Kählerian manifolds', *J. Differential Geom.* 20 (1984) 267–277.
16. B. Osorno-Torres, 'Codimension-one symplectic foliations: constructions and examples', PhD thesis, Utrecht University, 2015.
17. O. Radko, 'A classification of topologically stable Poisson structures on a compact oriented surface', *J. Symplectic Geom.* 1 (2002) 523–542.
18. C. H. Taubes, 'The Seiberg–Witten invariants and symplectic forms', *Math. Res. Lett.* 1 (1994) 809–822.

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