European Journal of Mathematics (2017) 3:565–586
DOI 10.1007/s40879-017-0151-7

RESEARCH ARTICLE

Vanishing homology of projective hypersurfaces with 1-dimensional singularities

Dirk Siersma[1] · **Mihai Tibăr**[2]

Received: 15 November 2016 / Accepted: 9 May 2017 / Published online: 30 May 2017

Abstract We introduce and study the vanishing homology of singular projective hypersurfaces. We prove its concentration in two levels in case of 1-dimensional singular locus Σ, and moreover determine the ranks of the nontrivial homology groups. These two groups depend on the monodromy at special points of Σ and on the effect of the monodromy of the local system over its complement.

Keywords Singular projective hypersurfaces · Vanishing homology · Nonisolated singularities · Monodromy

Mathematics Subject Classification 32S30 · 58K60 · 55R55 · 32S50

1 Introduction and results

The homology of a projective hypersurface $V \subset \mathbb{P}^{n+1}$ is known for smooth V whereas only few results are available in the singular setting. The classical Lefschetz Hyperplane Theorem (LHT) yields that the inclusion of spaces induces an isomorphism

The authors express their gratitude to CIRM in Luminy for supporting this research project through the "Recherche en Binôme" program.

Mihai Tibăr received the support from Labex CEMPI (ANR-11-LABX-0007-01).

✉ Mihai Tibăr
tibar@math.univ-lille1.fr

Dirk Siersma
D.Siersma@uu.nl

1 Institute of Mathematics, Utrecht University, PO Box 80010, 3508 TA Utrecht, The Netherlands

2 Laboratoire Paul Painlevé (UMR CNRS 8524), Université de Lille, 59000 Lille, France

$$H_k(V,\mathbb{Z}) \xrightarrow{\sim} H_k(\mathbb{P}^{n+1},\mathbb{Z})$$

for $k < n$ and an epimorphism for $k = n$, independently on the singular locus Sing V. Since V is a CW-complex of dimension $2n$, the remaining task is to find the homology groups $H_k(V,\mathbb{Z})$ for $k \geqslant n$.

In case of a smooth hypersurface $V_{n,d}$ all homology groups appear to be free and by Poincaré and Lefschetz duality[1]: $H_k(V_{n,d},\mathbb{Z}) \cong H_k(\mathbb{P}^n,\mathbb{Z})$ if $k \neq n$ and the rank of $H_n(V_{n,d},\mathbb{Z})$ follows from the Euler characteristic computation $\chi(V_{n,d}) = n + 2 - [1 + (-1)^{n+1}(d-1)^{n+2}]/d$. Smooth projective complete intersections have been studied by Libgober and Wood [11].

In the 1980s Dimca studied the case of isolated singularities [2,4]; we shall discuss his main result [4, Theorem 4.3] in Sect. 2.

Our paper focuses on the first unknown case, dim Sing $V = 1$. We approach the singular hypersurface V by comparing its integer homology to that of a smooth hypersurface of the same degree, as an intermediate step towards computing the homology of singular hypersurfaces. A different viewpoint, based on Griffiths cohomological techniques, has been taken by Hulek and Kloosterman in the study of elliptic threefolds [7].

We therefore introduce and study the "vanishing homology" of V, as follows.

Definition 1.1 Let $f = 0$ be the defining equation of $V \subset \mathbb{P}^{n+1}$ as a reduced hypersurface, where $d = \deg f$. Consider the following one-parameter smoothing of degree d, $V_\varepsilon = \{f_\varepsilon = f + \varepsilon h_d = 0\}$, where h_d denotes a general homogeneous polynomial of degree d. Let

$$\mathbb{V}_\Delta = \{(x,\varepsilon) \in \mathbb{P}^{n+1} \times \Delta : f + \varepsilon h_d = 0\}$$

denote the total space of the pencil, where $V_0 = V \subset \mathbb{P}^{n+1} \times \{0\}$ and Δ is a small enough disk centered at $0 \in \mathbb{C}$ such that V_ε is non-singular for all $\varepsilon \in \Delta^*$. Let $A = \{f = h_d = 0\}$ be the axis of the pencil and let $\pi : \mathbb{V}_\Delta \to \Delta$ denote the projection. We define

$$H_*^{\curlyvee}(V) = H_*(\mathbb{V}_\Delta, V_\varepsilon; \mathbb{Z})$$

and call it the *vanishing homology of* V.

The genericity of h_d ensures the existence of small enough disks Δ as in the above definition, see e.g. [18, Proposition 2.2]. Note that $\mathbb{V}_\Delta$ retracts to V, thus the vanishing homology compares V to the smooth hypersurface V_ε of the same degree. Since all smooth hypersurfaces of fixed degree are homeomorphic, the vanishing homology does not depend on the particular smoothing of degree d, it is thus an invariant of V.

With the vanishing homology we recover Dimca's result for isolated singularities [4, Theorem 4.3], see Propositions 2.2 and 7.7.

Our first result, Theorem 4.1, is that the vanishing homology $H_*^{\curlyvee}(V)$, in case dim Sing $V = 1$, is concentrated in dimensions $n + 1$ and $n + 2$ only.

[1] See [2] for details.

By the exact sequence of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$, the concentration of the vanishing homology implies the following isomorphisms:

$$H_k(V, \mathbb{Z}) \simeq H_k(V_{n,d}, \mathbb{Z}) \simeq H_k(\mathbb{P}^n, \mathbb{Z}) \qquad \text{for} \quad k \neq n, n+1, n+2.$$

In the second part of the paper we investigate the relations among the remaining homology groups $H_k(V, \mathbb{Z})$ (i.e. $k = n, n+1, n+2$) and the vanishing homology, and we single out remarkable particular cases.

Our main results in Sect. 6 are formulas for the (ranks of the) possibly non-trivial groups $H^{\curlyvee}_{n+1}(V)$ and $H^{\curlyvee}_{n+2}(V)$. They depend on the information about local isolated or special non-isolated singularities, the properties of the curve part of Sing V, the transversal singularity types and the monodromies along loops in the transversal local systems. The singular locus Sing V has a finite set R of isolated points and finitely many curve branches. Each such branch Σ_i of Sing V has a generic transversal type (of transversal Milnor fibre $F_i^{\pitchfork}$ and Milnor number denoted by $\mu_i^{\pitchfork}$) and the axis A cuts it at a finite set of general points P_i. It also contains a finite set Q_i of points with non-generic transversal type, which we call *special points*, and we denote by $\mathcal{A}_q$ the local Milnor fibre at $q \in Q$. At each point $q \in Q_i$ there are finitely many locally irreducible branches of the germ (Σ_i, q), we denote by $\gamma_{i,q}$ their number and let $\gamma_i = \sum_{i \in Q_i} \gamma_{i,q}$ (see Sect. 4.1 for the notations).

Our Theorem 6.1 determines $H^{\curlyvee}_{n+2}(V)$ as an intersection of local and global contributions in a reference space consisting of the direct sum of the homology of the transversal fibres. As a consequence it tells that the $(n+2)$th vanishing Betti number is bounded by the sum of all Milnor numbers of transversal singularities, taken over all irreducible 1-dimensional components of Sing V, and each special singular point on Sing V with non-trivial transversal monodromy decreases this Betti-number.

Corollary 6.5 (see also Example 7.3) tells that if for each irreducible 1-dimensional component Σ_i of Sing V we have at least one local special singularity with rank zero $(n-1)$th homology group, then the vanishing homology of V is free, concentrated in dimension $n+1$ only, and the corresponding Betti number is given by the following formula:

$$b_{n+1}(\mathbb{V}_\Delta, V_\varepsilon) = \sum_i (\nu_i + \gamma_i + 2g_i - 2)\mu_i^{\pitchfork} + (-1)^n \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1) + \sum_{r \in R} \mu_r,$$

where $Q = \bigcup_i Q_i$, $\nu_i = \# P_i$, μ_r is the Milnor number of the isolated singularity germ (V, r), and g_i is the genus of Σ_i (see Sect. 4.5 for the meaning of the genus in case of singular Σ_i).

In our proofs we use in particular the detailed construction of a CW-complex model of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$ which is done in Sects. 4.4 and 4.5. We also use the full strength of the results on local 1-dimensional singularities found by Siersma [14–17], cf. also [8, 20], which involve the study of the local system of transversal Milnor fibres.

We provide several examples in Sect. 7. In certain cases we can prove the freeness of the $(n+1)$th vanishing homology group. We also show an example where the homology of V over $\mathbb{Q}$ may be computed via our formulas for the vanishing homology.

Let us finally mention a couple of recent applications. The 1-dimensional locus case appeared recently in work of Frühbis-Krüger and Zach [6,21]. They have studied, following work by Damon and Pike [1], the vanishing cycles of a certain class of smoothable isolated Cohen–Macaulay codimension 2 singularities. As Tjurina transforms yield non-isolated singularities which can be studied with the methods of our paper, they could obtain in this way more detailed insight over the vanishing topology of a certain class of isolated determinantal singularities. Also recently we have computed the homology of a local Milnor fibre via admissible deformations [19] by using the approach of this paper.

2 Vanishing homology in case of isolated singularities

Throughout this paper we use homology over $\mathbb{Z}$ unless otherwise stated. Let $V = \{f = 0\} \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d with singular locus consisting of a finite set of points R. Since V has only isolated singularities, the genericity of the axis $A = \{f = h_d = 0\}$ of the pencil $\pi : \mathbb{V}_\Delta \to \Delta$ just means that A avoids R. It turns out (see also [18, Section 5]) that $\mathbb{V}_\Delta$ is non-singular and that the projection π has isolated singularities precisely at the points of R. Given some ball $B \subset \mathbb{P}^{n+1} \times \Delta$, we shall denote the intersection $B \cap \mathbb{V}_\Delta$ simply by B, for the sake of simplicity.

For small enough balls B_r, at each point $r \in R$, the homotopy retraction within the fibration π yields the isomorphism:

$$H_*(\mathbb{V}_\Delta, V_\varepsilon) \simeq \bigoplus_{r \in R} H_*(B_r, B_r \cap V_\varepsilon),$$

where $B_r \cap V_\varepsilon$ is the Milnor fibre of the isolated hypersurface singularity germ (V, r). The relative homology $H_*(B_r, B_r \cap V_\varepsilon)$ is concentrated in dimension $n + 1$ and $H_{n+1}(B_r, B_r \cap V_\varepsilon)$ is isomorphic to the Milnor lattice $\mathbb{L}_r$ of the hypersurface germ (V, r), thus isomorphic to $\mathbb{Z}^{\mu_r}$, where μ_r is the Milnor number of (V, r). We get the following conclusion:

Lemma 2.1 *If* $\dim \operatorname{Sing} V \leqslant 0$ *then*

$$H_k^\curlyvee(V) = 0 \qquad \text{if} \quad k \neq n+1,$$
$$H_{n+1}^\curlyvee(V) = \bigoplus_{r \in R} \mathbb{L}_r.$$

From the long exact sequence of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$ we also obtain the following 5-terms exact sequence:

$$0 \to H_{n+1}(V_\varepsilon) \to H_{n+1}(V) \to \bigoplus_{r \in R} \mathbb{L}_r \xrightarrow{\Phi_n} \mathbb{L} \to H_n(V) \to 0,$$

where the map Φ_n is identified to the boundary map $H_{n+1}(\mathbb{V}_\Delta, V_\varepsilon) \to H_n(V_\varepsilon)$ and $\mathbb{L} = H_n(V_\varepsilon)$ is the intersection lattice of the middle homology of the smooth hypersurface of degree d. We get the integer homology of V as follows:

Proposition 2.2 (a) $H_k(V) \simeq H_k(\mathbb{P}^n)$ *for* $k \neq n, n+1$,
(b) $H_{n+1}(V) \simeq H_{n+1}(\mathbb{P}^n) \oplus \ker \Phi_n$,
(c) $H_n(V) \simeq \operatorname{coker} \Phi_n$.

This is strikingly similar to Dimca's result [2, Theorem 2.1], [4, Theorem 5.4.3], although formulated and proved in different terms. As Dimca observed in [2], we also point out here that the relation between vanishing homology and absolute homology is encoded by the morphism Φ_n, which is difficult to identify from the equation of V. We send the reader to Proposition 7.7 for our extension of this result in case $\dim \operatorname{Sing} V = 1$.

3 Local theory of 1-dimensional singular locus

We shall need several facts from the local theory of singularities with a 1-dimensional singular set. We recall them here, following [16], see also the survey [17].

We consider a holomorphic function germ $f: (\mathbb{C}_{n+1}, 0) \to (\mathbb{C}, 0)$ with singular locus Σ of dimension 1. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ be the decomposition into irreducible curve components. Let F be the local Milnor fibre of f. The homology $\widetilde{H}_*(F)$ is concentrated in dimensions $n-1$ and n, namely $H_n(F) = \mathbb{Z}^{\mu_n}$, which is free, and $H_{n-1}(F)$ which can have torsion.

There is a well-defined local system on $\Sigma_i \setminus \{0\}$ having as fibre the homology of the transversal Milnor fibre $\widetilde{H}_{n-1}(F_i^{\pitchfork})$, i.e., $F_i^{\pitchfork}$ is the Milnor fibre of the restriction of f to the transversal hyperplane section at some $x \in \Sigma_i \setminus \{0\}$, which is an isolated singularity whose equisingularity class is independent of the point x. Thus $\widetilde{H}_*(F_i^{\pitchfork})$ is concentrated in dimension $n-1$. On this group there acts the *local system monodromy* (also called *vertical monodromy*):

$$A_i: \widetilde{H}_{n-1}(F_i^{\pitchfork}) \to \widetilde{H}_{n-1}(F_i^{\pitchfork}).$$

As explained in [16], one considers a tubular neighborhood $\mathcal{N} = \bigsqcup_{i=1}^r \mathcal{N}_i$ of the link $\Sigma \cap S_\varepsilon^{2n+1}$ of Σ in S_ε^{2n+1} and decomposes the boundary ∂F of the Milnor fibre as $\partial F = \partial_1 F \cup \partial_2 F$, such that $\partial_2 F = \partial F \cap \mathcal{N}$ and that $\partial_1 F \cap \partial_2 F$ retracts to the boundary $\partial \mathcal{N}$. Then $\partial_2 F = \bigsqcup_{i=1}^r \partial_2 F_i$, where $\partial_2 F_i = \partial F \cap \mathcal{N}_i$.

The homology groups of $\partial_2 F$ are related to the local system monodromies A_i in the following way. Each boundary component $\partial_2 F_i$ is fibered over the link of Σ_i with fibre $F_i^{\pitchfork}$. The Wang sequence of this fibration yields the following non-trivial part, for $n \geqslant 3$:

$$0 \to H_n(\partial_2 F_i) \to H_{n-1}(F_i^{\pitchfork}) \xrightarrow{A_i - I} H_{n-1}(F_i^{\pitchfork}) \to H_{n-1}(\partial_2 F_i) \to 0. \qquad (1)$$

In this sequence the following two homology groups play a crucial role: $H_n(\partial_2 F) = \bigoplus_{i=1}^r \ker(A_i - I)$ and $H_{n-1}(\partial_2 F) \cong \bigoplus_{i=1}^r \operatorname{coker}(A_i - I)$. The first group is free, the second one can have torsion, and they are isomorphic up to torsion. For $n = 2$ there is an adapted interpretation of this sequence, cf. [16, Section 6].

What we will actually need in the following is a relative version of this Wang sequence. Let $E_i^{\pitchfork}$ be the transversal Milnor neighborhood containing the transversal fibre $F_i^{\pitchfork}$; it is homeomorphic to a $2n$-ball and hence contractible. Let $\partial_2 E_i$ denote the union of such transversal Milnor neighbourhoods along the link $\Sigma_i \cap S_\varepsilon^{2n+1}$; this may be identified with the tubular neighborhood $\mathcal{N}_i$, which retracts to the link of Σ_i. We then have:

Lemma 3.1 *For $n \geqslant 2$,*

$$0 \to H_{n+1}(\partial_2 E_i, \partial_2 F_i) \to H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \xrightarrow{A_i - I} H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \to H_n(\partial_2 E_i, \partial_2 F_i) \to 0$$

is an exact sequence, and

$$H_{n+1}(\partial_2 E, \partial_2 F) = \bigoplus_{i=1}^{r} \ker(A_i - I),$$
$$H_n(\partial_2 E, \partial_2 F) \cong \bigoplus_{i=1}^{r} \operatorname{coker}(A_i - I).$$

Proof For $n > 2$ the statement follows immediately from the above Wang sequence (1) and the definitions of $E_i^{\pitchfork}$ and $\partial_2 E_i$. One observes that $n = 2$ is no longer a special case like it was in the absolute setting [see the remark after (1)]. □

The non-trivial part of the long exact sequence of the pair $(F, \partial_2 F)$ is the following 6-terms piece. More precisely, we need the following result:

Proposition 3.2 ([16]) *The sequence*

$$0 \to H_{n+1}(F, \partial_2 F) \to H_n(\partial_2 F) \to H_n(F) \to H_n(F, \partial_2 F) \to H_{n-1}(\partial_2 F) \to H_{n-1}(F) \to 0$$

is exact. Moreover

$$H_{n+1}(F, \partial_2 F) \cong H_{n-1}(F)^{\text{free}} \quad \textit{and} \quad H_n(F, \partial_2 F) \cong H_n(F) \oplus H_{n-1}(F)^{\text{torsion}}.$$

4 The vanishing neighbourhood of the projective hypersurface

We give here the necessary constructions and lemmas that we shall use in the proof of the announced vanishing theorem:

Theorem 4.1 *If* $\dim \operatorname{Sing} V \leqslant 1$ *then* $H_j^{\curlyvee}(V) = 0$ *for all* $j \neq n+1, n+2$.

Let $V = \{f = 0\} \subset \mathbb{P}^{n+1}$ denote a hypersurface of degree d with singular locus $\widehat{\Sigma}$ of dimension one, more precisely $\widehat{\Sigma}$ consists of a union $\Sigma = \bigcup_i \Sigma_i \cup R$ of irreducible projective curves Σ_i and of a finite set of points R.

We recall that we have denoted by $A = \{f = h_d = 0\}$ the axis of the pencil $\pi : \mathbb{V}_\Delta \to \Delta$ defined in Introduction. One considers the *polar locus*[2] of the map $(h_d, f) \colon \mathbb{C}_{n+2} \to \mathbb{C}^2$ and since this is a homogeneous set one takes its image in $\mathbb{P}^{n+1}$ which will be denoted by $\Gamma(h_d, f)$. Let us recall from [18] the meaning of "general" for h_d in this setting. By using the Veronese embedding of degree d, we find a Zariski open set $\mathcal{O}$ of linear functions in the target such that whenever $g \in \mathcal{O}$ then its pull-back is a general homogeneous polynomial h_d defining a hypersurface $H = \{h_d = 0\}$ which is transversal to V in the stratified sense, i.e. after endowing V with some Whitney stratification, of which the strata are as follows: the isolated singular points $\{\{r\} : r \in R\}$ of V and the point-strata $\{\{q\} : q \in Q\}$ in Σ, the components of $\Sigma \setminus Q$ and the open stratum $V \setminus \widehat{\Sigma}$. Such h_d will be called *general*. This definition implies that A intersects $\widehat{\Sigma}$ at general points, in particular does not contain any points of $Q \cup R$. It was shown in [18, Lemma 5.1] that the space $\mathbb{V}_\Delta$ has isolated singularities: $\mathrm{Sing}\, \mathbb{V}_\Delta = (A \cap \Sigma) \times \{0\}$, and that $\pi : \mathbb{V}_\Delta \to \Delta$ is a map with 1-dimensional singular locus $\mathrm{Sing}\,(\pi) = \widehat{\Sigma} \times \{0\}$. One of the key preliminary results is the following supplement to [18, Lemma 5.2], which extends the proof in *loc.cit.* from Euler characteristic to homology[3]:

Lemma 4.2 *If h_d is general then $\Gamma_p(h_d, f) = \varnothing$ at any point $p \in A \times \{0\}$. In particular, for a small enough ball B_p centered at p, the local relative homology is trivial, i.e.*

$$H_*(B_p, B_p \cap V_\varepsilon) = 0.$$

Proof The first claim has been proved in [18, (12)]. Let us show here the second one. The notation B_p stands for the intersection of $\mathbb{V}_\Delta$ with a small ball in some chosen affine chart $\mathbb{C}_{n+1} \times \Delta$ of the ambient space $\mathbb{P}^{n+1} \times \Delta$. In particular B_p is of dimension $n+1$. Consider the map $(\pi, h_d) \colon B_p \to \Delta \times \Delta'$. Consider the germ of the polar locus of this map at p, denoted by $\Gamma(\pi, \widehat{h_d})$, where $\widehat{h_d}$ is the de-homogenization of h_d in the chosen chart. It follows from the definition of the polar locus that some point $(x, \varepsilon) \in \mathbb{V}_\Delta$, where $\varepsilon = -f(x)/h_d(x)$, is contained in $\Gamma(\pi, h_d) \setminus (\{f = 0\} \cup \{h_d = 0\})$ if and only if $x \in \Gamma(f, h_d) \setminus (\{f = 0\} \cup \{h_d = 0\})$. By the first statement, $\Gamma(f, h_d)$ is empty at p. The absence of the polar locus implies that $B_p \cap V_\varepsilon$ is homotopy equivalent (by deformation retraction) to the space $B_p \cap V_\varepsilon \cap \{h_d = 0\}$. The latter is the slice by $\varepsilon =$ constant of the space $\mathbb{V}_\Delta \cap \{h_d = 0\} = \{f = 0\} \times \Delta$, which is a product space. Since this is homeomorphic to the complex link of this space and a product space has contractible complex link, we deduce that $B_p \cap V_\varepsilon$ is contractible too. Since B_p is contractible itself, we get our claim. □

---

[2] The polar locus of a map (h, f) is defined as $\overline{\mathrm{Sing}\,(h, f) \setminus (\mathrm{Sing}\, h \cup \mathrm{Sing}\, f)}$.

[3] A related result was obtained in [12]. Like in case of [12], the proof actually works for any singular locus $\mathrm{Sing}\, V$ and any general pencil.

4.1 Notations

Let us assume for the moment that Σ is irreducible and discuss the reducible case at the end of Sect. 5.2. Let g be its genus, in the sense of the definition given in Sect. 4.5. We use the following notations:

$P = A \cap \Sigma$, the set of axis points of Σ.
$Q =$ the set of special points on Σ.
$R =$ the set of isolated singular points.
$\Sigma^* = \Sigma \setminus (P \cup Q)$.
$\mathcal{Y} =$ small enough tubular neighborhood of Σ^* in $\mathbb{V}_\Delta$.
B_p, B_q, B_r are small enough Milnor balls within $\mathbb{V}_\Delta \subset \mathbb{P}^{n+1} \times \Delta$ at the points $p \in P, q \in Q, r \in R$ respectively.
$B_P = \bigsqcup_p B_p$, $B_Q = \bigsqcup_q B_q$ and $B_R = \bigsqcup_r B_r$.
$\pi_\Sigma : \mathcal{Y} \to \Sigma^*$ is the projection of the tubular neighborhood.

Let $\nu = \# P$ be the number of axis points. At any special point $q \in Q$, let S_q be the index set of locally irreducible branches of the germ (Σ, q), and let $\gamma = \sum_{q \in Q} \# S_q$.

By homotopy retraction and by excision we have

$$H_*(\mathbb{V}_\Delta, V_\varepsilon) \simeq H_*(\mathcal{Y} \cup B_P \cup B_Q, V_\varepsilon \cap \mathcal{Y} \cup B_P \cup B_Q) \oplus \bigoplus_{r \in R} H_*(B_r, V_\varepsilon \cap B_r). \tag{2}$$

We introduce the following shorter notations:

$$\begin{gathered} \mathcal{X} = B_P \sqcup B_Q, \quad \mathcal{A} = V_\varepsilon \cap \mathcal{X}, \quad \mathcal{B} = V_\varepsilon \cap \mathcal{Y}, \quad \mathcal{Z} = \mathcal{X} \cap \mathcal{Y}, \quad \mathcal{C} = \mathcal{A} \cap \mathcal{B}, \\ (\mathcal{X}_p, \mathcal{A}_p) = (B_p, V_\varepsilon \cap B_p), \quad (\mathcal{X}_q, \mathcal{A}_q) = (B_q, V_\varepsilon \cap B_q). \end{gathered}$$

In the new notations, the first direct summand of (2) is $H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$, thus (2) writes as follows:

$$H_*(\mathbb{V}_\Delta, V_\varepsilon) \simeq H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \oplus \bigoplus_{r \in R} H_*(B_r, V_\varepsilon \cap B_r). \tag{3}$$

Note that each direct summand $H_*(B_r, V_\varepsilon \cap B_r)$ is concentrated in dimension $n+1$ since it identifies to the Milnor lattice of the isolated singularities germs (V_0, r), where μ_r denotes its Milnor number. This aspect was treated in Sect. 1 in case of isolated singularities. We shall therefore deal from now on with the first term in the direct sum of (2).

We next consider the relative Mayer–Vietoris long exact sequence

$$\cdots \to H_*(\mathcal{Z}, \mathcal{C}) \to H_*(\mathcal{X}, \mathcal{A}) \oplus H_*(\mathcal{Y}, \mathcal{B}) \to H_*(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \xrightarrow{\partial_s} \cdots \tag{4}$$

of the pair $(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B})$ and we compute in the following each term of it.

4.2 The homology of $(\mathcal{X}, \mathcal{A})$

One has the direct sum decomposition

$$H_*(\mathcal{X}, \mathcal{A}) \simeq \bigoplus_p H_*(\mathcal{X}_p, \mathcal{A}_p) \oplus \bigoplus_q H_*(\mathcal{X}_q, \mathcal{A}_q)$$

since $\mathcal{X}$ is a disjoint union. The triviality $H_*(\mathcal{X}_p, \mathcal{A}_p) = 0$ follows by Lemma 4.2. The pairs $(\mathcal{X}_q, \mathcal{A}_q)$ are local Milnor data of the germs (V, q) with 1-dimensional singular locus and therefore the relative homology $H_*(\mathcal{X}_q, \mathcal{A}_q)$ is concentrated in dimensions n and $n+1$.

4.3 The homology of $(\mathcal{Z}, \mathcal{C})$

The pair $(\mathcal{Z}, \mathcal{C})$ is a disjoint union of pairs localized at points $p \in P$ and $q \in Q$. For axis points $p \in P$ we have a unique pair $(\mathcal{Z}_p, \mathcal{C}_p)$ as bundle over the link of Σ at p with fibre the transversal data $(E_p^{\pitchfork}, F_p^{\pitchfork})$, in the notations of Sect. 3. For the non-axis points $q \in Q$ we have one contribution for each *locally irreducible branch of the germ* (Σ, q). Let S_q be the index set of all these branches at $q \in Q$. We get the following decomposition:

$$H_*(\mathcal{Z}, \mathcal{C}) \simeq \bigoplus_{p\in P} H_*(\mathcal{Z}_p, \mathcal{C}_p) \oplus \bigoplus_{q\in Q} \bigoplus_{s\in S_q} H_*(\mathcal{Z}_s, \mathcal{C}_s). \tag{5}$$

More precisely, one such local pair $(\mathcal{Z}_s, \mathcal{C}_s)$ is the bundle over the corresponding component of the link of the curve germ Σ at q having as fibre the local transversal Milnor data $(E_s^{\pitchfork}, F_s^{\pitchfork})$. In the notations of Sect. 3, we thus have $\partial_2 \mathcal{A}_q = \bigsqcup_{s\in S_q} \mathcal{C}_s$.

The relative homology groups in the above decomposition (5) depend on the *vertical monodromy* via the Wang sequence of Lemma 3.1, as follows:

$$\begin{aligned} 0 \to H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) &\to H_n(E^{\pitchfork}, F^{\pitchfork}) \\ &\xrightarrow{A_s - I} H_n(E^{\pitchfork}, F^{\pitchfork}) \to H_n(\mathcal{Z}_s, \mathcal{C}_s) \to 0. \end{aligned} \tag{6}$$

Note that here the transversal data is independent of the points q or the index s since Σ^* is connected and therefore the transversal fibre is uniquely defined. However the vertical monodromies A_s depend of $s \in S_q$. From the above and from Lemma 4.2 we get:

Lemma 4.3 *At points $q \in Q$, for each $s \in S_q$ one has*

$$\begin{aligned} H_k(\mathcal{Z}_s, \mathcal{C}_s) &= 0, \qquad k \neq n, n+1, \\ H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) &\cong \ker(A_s - I), \quad H_n(\mathcal{Z}_s, \mathcal{C}_s) \cong \operatorname{coker}(A_s - I). \end{aligned}$$

At axis points $p \in P$ and more generally, at any point p such that $A_p = I$, one has

$$H_k(\mathcal{Z}_p, \mathcal{C}_p) = 0, \qquad k \neq n, n+1,$$
$$H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p) \cong H_n(\mathcal{Z}_p, \mathcal{C}_p) \cong H_n(E^{\pitchfork}, F^{\pitchfork}) = \mathbb{Z}^{\mu^{\pitchfork}}.$$

Proof The first statement follows from the Wang sequence (6) and since $H_k(E^{\pitchfork}, F^{\pitchfork})$ is concentrated in $k = n$. The last statement follows because the axis points $p \in P$ are general points of Σ and hence the local vertical monodromy A_p is the identity. □

We conclude that $H_*(\mathcal{Z}, \mathcal{C})$ is concentrated in dimensions n and $n+1$ only.

4.4 The CW-complex structure of $(\mathcal{Z}, \mathcal{C})$

The pair $(\mathcal{Z}_s, \mathcal{C}_s)$ has moreover the following structure of a relative CW-complex, up to homotopy. Each bundle over some circle link can be obtained from a trivial bundle over an interval by identifying the fibres above the end points via the geometric vertical monodromy A_s. In order to obtain $\mathcal{Z}_s$ from $\mathcal{C}_s$ one can start by first attaching n-cells $c_1, \ldots, c_{\mu^{\pitchfork}}$ to the fibre $F^{\pitchfork}$ in order to kill the $\mu^{\pitchfork}$ generators of $H_{n-1}(F^{\pitchfork})$ at the identified ends, and next by attaching $(n+1)$-cells $e_1, \ldots, e_{\mu^{\pitchfork}}$ to the preceding n-skeleton. The attaching of some $(n+1)$-cell is as follows: consider some n-cell a of the n-skeleton and take the cylinder $I \times a$ as an $(n+1)$-cell. Fix an orientation of the circle link, attach the base $\{0\} \times a$ over a, then follow the circle bundle in the fixed orientation by the monodromy A_s and attach the end $\{1\} \times a$ over $A_s(a)$. At the level of the cell complex, the boundary map of this attaching identifies to $A_s - I : \mathbb{Z}^{\mu^{\pitchfork}} \to \mathbb{Z}^{\mu^{\pitchfork}}$.

4.5 The CW-complex structure of $(\mathcal{Y}, \mathcal{B})$

For technical reasons we introduce one more puncture on Σ. Let us therefore define the total set of punctures $T = P \sqcup Q \sqcup \{y\}$, where y is a general point of Σ, then redefine $\Sigma^* = \Sigma \setminus T$ by considering the new puncture y.

Let $n : \tilde{\Sigma} \to \Sigma$ be the normalization map. Then we have the isomorphism $\Sigma^* = \Sigma \setminus T \simeq \tilde{\Sigma} \setminus n^{-1}(T)$. We choose generators of $\pi_1(\Sigma^*, z)$ for some base point $z \in \Sigma^*$ as follows: first the $2g$ loops (called *genus loops* in the following) which are generators of $\pi_1(\tilde{\Sigma}, n^{-1}(z))$, where g denotes the genus of the normalization $\tilde{\Sigma}$, and next by choosing one loop for each puncture of P and of Q. The total set of loops is indexed by the set $T' = T \setminus \{y\}$. Let us denote by W the set of indices for the union of T' with the genus loops, and therefore $\# W = 2g + \nu + \gamma$, where $\nu = \# P$ and $\gamma = \sum_{q \in Q} \# S_q$ (recalling the notations in Sect. 4.1). By enlarging the "hole" defined by the puncture y, we retract Σ^* to the chosen bouquet configuration of non-intersecting loops, denoted by Γ. The number of loops is $2g + \nu + \gamma$. Note that $\nu > 0$ since there must be at least d "axis points".

The pair $(\mathcal{Y}, \mathcal{B})$ is then homotopy equivalent (by retraction) to the pair $(\pi_\Sigma^{-1}(\Gamma), \mathcal{B} \cap \pi_\Sigma^{-1}(\Gamma))$. We endow the latter with the structure of a relative CW-complex as we did with $(\mathcal{Z}, \mathcal{C})$ in Sect. 4.4, namely for each loop the similar CW-complex structure as we have defined above for some pair $(\mathcal{Z}_s, \mathcal{C}_s)$, see Fig. 1. The difference is that the pairs $(\mathcal{Z}_s, \mathcal{C}_s)$ are disjoint whereas in Σ^* the loops meet at a single point z. We thus

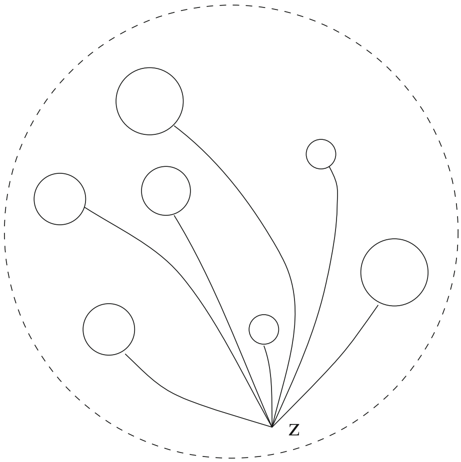

Fig. 1 Retraction of the surface Σ^*

take as reference the transversal fibre $F^{\pitchfork} = \mathcal{B} \cap \pi_{\Sigma}^{-1}(z)$ above the point z, namely we attach the n-cells (thimbles) only once to this single fibre in order to kill the $\mu^{\pitchfork}$ generators of $H_{n-1}(F^{\pitchfork})$. The $(n+1)$-cells of $(\mathcal{Y}, \mathcal{B})$ correspond to the fibre bundles over the loops in the bouquet model of Σ^*. Over each loop, one attaches a number $\mu^{\pitchfork}$ of $(n+1)$-cells to the fixed n-skeleton described before, more precisely one $(n+1)$-cell over one n-cell generator of the n-skeleton. We extend the notation $(\mathcal{Z}_j, \mathcal{C}_j)$ to genus loops, although they are not contained in $(\mathcal{Z}, \mathcal{C})$.

The attaching map of the $(n+1)$-cells corresponding to the bundle over some loop can be identified with $A_j - I : \mathbb{Z}^{\mu^{\pitchfork}} \to \mathbb{Z}^{\mu^{\pitchfork}}$, where the local system monodromies A_j corresponding to loops may not be local monodromies, and where $\mathbb{Z}^{\mu^{\pitchfork}}$ is the homology group $H_{n-1}(F^{\pitchfork})$ of the transversal fibre over z and hence the same for each loop.

From this CW-complex structure we get the following precise description in terms of the local monodromies of the transversal local system:

Lemma 4.4
- $H_k(\mathcal{Y}, \mathcal{B}) = 0$ *if* $k \neq n, n+1$,
- $H_n(\mathcal{Y}, \mathcal{B}) \simeq \mathbb{Z}^{\mu^{\pitchfork}} / \langle \mathrm{Im}\,(A_j - I) : j \in W \rangle$,
- $H_{n+1}(\mathcal{Y}, \mathcal{B})$ *is free of rank* $(2g+\nu+\gamma-1)\mu^{\pitchfork} + \mathrm{rank}\, H_n(\mathcal{Y}, \mathcal{B}) \leqslant (2g+\nu+\gamma)\mu^{\pitchfork}$,
- $H_{n+1}(\mathcal{Y}, \mathcal{B})$ *naturally contains* $\bigoplus_{j \in W} H_{n+1}(\mathcal{Z}_j, \mathcal{C}_j)$ *as a direct summand,*
- $\chi(\mathcal{Y}, \mathcal{B}) = (-1)^{n-1}(2g + \nu + \gamma - 1)\mu^{\pitchfork}$.

Proof The relative CW-complex model of $(\mathcal{Y}, \mathcal{B})$ contains only cells in dimension n and $n+1$. At the level $n+1$, the chain group is generated by all $(n+1)$-cells corresponding to elements of W. Then $H_{n+1}(\mathcal{Y}, \mathcal{B})$ identifies to the kernel of the boundary map ∂ in the second row of the following commuting diagram of exact sequences [provided by Lemma 3.1 and by (6)], where the vertical arrows are induced by inclusion:

$$\begin{tikzcd}
0 \arrow[r] & H_{n+1}(\mathcal{Z}_j, \mathcal{C}_j) \arrow[r, hook] \arrow[d] & H_n(E_j^{\pitchfork}, F_j^{\pitchfork}) \arrow[r, "\partial_j"] \arrow[d] & H_n(E_j^{\pitchfork}, F_j^{\pitchfork}) \arrow[r] \arrow[d, "="] & H_n(\mathcal{Z}_j, \mathcal{C}_j) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{n+1}(\mathcal{Y}, \mathcal{B}) \arrow[r, hook] & \bigoplus_{j \in W} H_n(E_j^{\pitchfork}, F_j^{\pitchfork}) \arrow[r, "\sum_{j \in W} \partial_j"] & H_n(E_j^{\pitchfork}, F_j^{\pitchfork}) \arrow[r] & H_n(\mathcal{Y}, \mathcal{B}) \arrow[r] & 0.
\end{tikzcd} \tag{7}$$

For any $j \in W$ we get that the first vertical arrow is injective. By taking the direct sum over $j \in W$ in the left hand commutative square of (7), we get an injective map $\bigoplus_{j \in W} H_{n+1}(\mathcal{Z}_j, \mathcal{C}_j) \hookrightarrow H_{n+1}(\mathcal{Y}, \mathcal{B})$. It follows that the image is a direct summand.

Counting the ranks in the lower exact sequence yields the above claimed formula for χ. □

5 Concentration of the vanishing homology. Proof of Theorem 4.1

Lemma 4.3, Sect. 4.2 and Lemma 4.4 show that the terms $H_*(\mathcal{X}, \mathcal{A})$, $H_*(\mathcal{Y}, \mathcal{B})$ and $H_*(\mathcal{Z}, \mathcal{C})$ of the Mayer–Vietoris sequence (4) are concentrated in dimensions n and $n+1$ only, which fact implies the following result:

Proposition 5.1 *The relative Mayer–Vietoris sequence* (4) *is trivial except for the following* 7*-terms sequence:*

$$\begin{aligned}
0 &\to H_{n+2}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to H_{n+1}(\mathcal{Z}, \mathcal{C}) \\
&\to H_{n+1}(\mathcal{X}, \mathcal{A}) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B}) \to H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \\
&\to H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to H_n(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to 0.
\end{aligned} \tag{8}$$

From Proposition 5.1 and (3) it follows that the vanishing homology $H_*(\mathbb{V}_\Delta, V_\varepsilon)$ is concentrated in dimensions $n, n+1, n+2$.

We pursue by showing that $H_n(\mathbb{V}_\Delta, V_\varepsilon) = 0$, i.e. that the last term of (8) is zero. We need the relative version of the exact sequence of Proposition 3.2, which appears to have an important overlap with our relative Mayer–Vietoris sequence.

Proposition 5.2 *For any $q \in Q$, the sequence*

$$\begin{aligned}
0 \to H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) &\to \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \to H_{n+1}(\mathcal{X}_q, \mathcal{A}_q) \\
\to H_n(\mathcal{A}_q, \partial_2 \mathcal{A}_q) &\to \bigoplus_{s \in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q) \to 0
\end{aligned}$$

is exact for $n \geqslant 2$. Moreover we have

$$\begin{aligned}
H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) &\cong H_{n-1}(\mathcal{A}_q)^{\text{free}} \text{ and} \\
H_n(\mathcal{A}_q, \partial_2 \mathcal{A}_q) &\cong H_n(\mathcal{A}_q) \oplus H_{n-1}(\mathcal{A}_q)^{\text{torsion}}.
\end{aligned}$$

Proof Note that we have the following coincidence of objects which have different notations in the projective setting of this section and in the local setting of Sect. 3: $\mathcal{A}_q = F$, $\partial_2\mathcal{A}_q = \partial_2 F$.

We also have the isomorphisms $H_{*+1}(\mathcal{X}_q, \mathcal{A}_q) = \tilde{H}_*(\mathcal{A}_q)$ since $\mathcal{X}_q$ is contractible, then $H_*(\partial_2\mathcal{A}_q) = \bigoplus_{s\in S_q} H_*(\mathcal{C}_s)$ by definition, and for $k > 2$, $H_k(\mathcal{C}_s) = H_{k+1}(\mathcal{Z}_s, \mathcal{C}_s)$, since $\mathcal{Z}_s$ contracts to a circle. We use Proposition 3.2 and check that, like in Lemma 3.1 on another (but similar) relative situation, the case $n = 2$ does not give any problem for the exactness of the above sequence. $\square$

5.1 Surjectivity of *j*

We focus on the following map which occurs in the 7-term exact sequence (8):

$$j = j_1 \oplus j_2 \colon H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}). \tag{9}$$

5.1.1 The first component $j_1\colon H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{X}, \mathcal{A})$

Note that, as shown above, we have the following direct sum decompositions of the source and the target:

$$\begin{aligned} H_n(\mathcal{Z}, \mathcal{C}) &= \bigoplus_{p\in P} H_n(\mathcal{Z}_p, \mathcal{C}_p) \oplus \bigoplus_{q\in Q} \bigoplus_{s\in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \oplus H_n(\mathcal{Z}_y, \mathcal{C}_y), \\ H_n(\mathcal{X}, \mathcal{A}) &= \bigoplus_{q\in Q} H_n(\mathcal{X}_q, \mathcal{A}_q) \oplus H_n(\mathcal{X}_y, \mathcal{A}_y). \end{aligned}$$

The terms corresponding to the points $p \in P$ are mapped by j_1 to zero since $H_n(\mathcal{X}_p, \mathcal{A}_p) = 0$ by Lemma 4.2. Next, as shown in Proposition 5.2, at the special points $q \in Q$ we have surjections $\bigoplus_{s\in S_q} H_n(\mathcal{Z}_s, \mathcal{C}_s) \to H_n(\mathcal{X}_q, \mathcal{A}_q)$ and moreover $H_n(\mathcal{Z}_y, \mathcal{C}_y) \to H_n(\mathcal{X}_y, \mathcal{A}_y)$ is an isomorphism. This shows that the morphism j_1 is surjective.

5.1.2 The second component $j_2\colon H_n(\mathcal{Z}, \mathcal{C}) \to H_n(\mathcal{Y}, \mathcal{B})$

Both sides are described with a relative CW-complex as explained in Sect. 4.5. At the level of n-cells there are $\mu^{\pitchfork}$ n-cell generators for each $p \in P$, and the same for each $s \in S_q$ and any $q \in Q$. Each of these generators is mapped bijectively to the single cluster of n-cell generators attached to the reference fibre $F^{\pitchfork}$ (which is the fibre above the common point of the loops, see also Fig. 1). We have the same boundary map for each axis point $p \in P$ in the source and in the target of j_2 and therefore, at the level of the n-homology, the restriction $j_{2|}\colon H_n(\mathcal{Z}_p, \mathcal{C}_p) \to H_n(\mathcal{Y}, \mathcal{B})$ is surjective. Since we have at least one axis point on Σ and $\bigoplus_{p\in P} H_n(\mathcal{Z}_p, C_p) \subset \ker j_1$, this shows that the restriction $j_{2|}\colon \bigoplus_{p\in P} H_n(\mathcal{Z}_p, \mathcal{C}_p) \to H_n(\mathcal{Y}, \mathcal{B})$ is surjective too. We have thus proven the surjectivity of j and in particular the following statement:

Proposition 5.3 *$H_n(\mathbb{V}_\Delta, V_\varepsilon) = 0$ and in particular the relative Mayer–Vietoris sequence (8) reduces to the following 6-terms sequence:*

$$0 \to H_{n+2}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to H_{n+1}(\mathcal{Z}, \mathcal{C}) \to H_{n+1}(\mathcal{X}, \mathcal{A}) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B})$$
$$\to H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to 0.$$

This shows that the relative homology $H_*(\mathbb{V}_\Delta, V_\varepsilon)$ is concentrated at the levels $n+1$ and $n+2$, and thus finishes the proof of Theorem 4.1 in case of irreducible Σ.

5.2 Reducible Σ

Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_\rho$ be the decomposition into irreducible components. The proof of Theorem 4.1 in the reducible case remains the same modulo the following small changes and additional notations:

- For each i one considers the set Q_i of special singular points of Σ_i. The points of intersection $\Sigma_{i_1} \cap \Sigma_{i_2}$ for $i_1 \neq i_2$ are considered as special points of both sets Q_i and Q_j, and therefore the union $Q = \bigcup_i Q_i$ is not disjoint. For some $q \in \Sigma_{i_1} \cap \Sigma_{i_2}$, the set of indices S_q runs over all the local irreducible components of the curve germ (Σ, q). Nevertheless, when we are counting the local irreducible branches at some point $q \in Q_i$ on a specified component Σ_i then the set S_q will tacitly mean only those local branches of Σ_i at q.
- The pair $(\mathcal{Y}, \mathcal{B})$ is a disjoint union and its homology decomposes accordingly, namely $H_*(\mathcal{Y}, \mathcal{B}) = \bigoplus_{1 \leqslant i \leqslant \rho} H_*(\mathcal{Y}_i, \mathcal{B}_i)$.
- For each component Σ_i one has its transversal Milnor fibre denoted by $F_i^{\pitchfork}$ and its transversal Milnor number $\mu_i^{\pitchfork}$.

6 Betti numbers of hypersurfaces with 1-dimensional singular locus

By Theorem 4.1, the vanishing homology of a hypersurface $V \subset \mathbb{P}^{n+1}$ with 1-dimensional singularities is concentrated in dimensions $n+1$ and $n+2$. We show that its $(n+2)$th vanishing homology group depends on the local data of the special points Q and on the genus loop monodromies along the singular branches. We study this dependence in more detail, we determine the rank of the free group $H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon)$, and discover mild conditions which ensure the vanishing of this group.

We continue to use the notations of Sect. 4. Let us especially recall the notations from Sect. 4.5 adapted here to the general setting of a reducible singular locus $\Sigma = \bigcup_{i=1}^{\rho} \Sigma_i$. For any $1 \leqslant i \leqslant \rho$, $\Sigma_i^* = \Sigma_i \setminus (P_i \sqcup Q_i \sqcup \{y_i\})$ retracts to a bouquet W_i of $2g_i + \nu_i + \gamma_i$ circles, where g_i denotes the genus of the normalization $\widetilde{\Sigma}_i$, where $\nu_i = \# P_i$ is the number of axis points $A \cap \Sigma_i$, where $\gamma_i = \sum_{q \in Q_i} \# S_q$ and Q_i denotes the set of special points of Σ_i, the set S_q is indexing the local branches of Σ_i at q, and where $y_i \in \Sigma_i$ is some point not in the set $P_i \cup Q_i$. We denote by G_i the set of genus loops of W_i.

By Proposition 5.1, we have $H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon) = \ker j = \ker[j_1 \oplus j_2]$, where

$$j_1 \oplus j_2 \colon H_{n+1}(\mathcal{Z}, \mathcal{C}) \to H_{n+1}(\mathcal{X}, \mathcal{A}) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B}).$$

The main idea in this section is to embed $H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon)$ into the module $\mathbb{D} = \bigoplus_{i=1}^{\rho} \mathbb{D}_i$, where $\mathbb{D}_i$ is the image of the diagonal map

$$\Delta_*^i \colon H_n(E_i^\pitchfork, F_i^\pitchfork) \to \bigoplus_{q \in Q_i} \bigoplus_{s \in S_q} H_n(E_i^\pitchfork, F_i^\pitchfork), \qquad a \mapsto (a, a, \dots, a).$$

The space $\mathbb{D}$ will serve as a reference space and is isomorphic to $\bigoplus_{i=1}^{\rho} H_n(F_i^\pitchfork) = \bigoplus_{i=1}^{\rho} \mathbb{Z}^{\mu_i^\pitchfork}$.

The source and the target of $j_1 \oplus j_2$ have a direct sum decomposition at level $n+1$, like has been discussed in Sect. 5.1 for the n-th homology groups[4]:

$$\begin{aligned} j_1 \oplus j_2 \colon \bigoplus_{p \in P} H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p) \bigoplus_{q \in Q} \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \bigoplus_{i=1}^{\rho} H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}) \\ \to \bigoplus_{q \in Q} H_{n+1}(\mathcal{X}_q, \mathcal{A}_q) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B}). \end{aligned} \tag{10}$$

By Lemma 4.3, we have $H_{n+1}(\mathcal{Z}_v, \mathcal{C}_v) = \ker(A_v - I)$, where

$$A_v - I \colon H_n(E_i^\pitchfork, F_i^\pitchfork) \to H_n(E_i^\pitchfork, F_i^\pitchfork)$$

is the vertical monodromy at some point $v \in P_i$, or $v \in S_q$ and $q \in Q_i$, or $v = y_i$. The left hand side of (10) consists therefore of local contributions of the form $\ker(A_v - I) \subset H_n(E_i^\pitchfork, F_i^\pitchfork) \simeq H_{n-1}(F_i^\pitchfork) \simeq \mathbb{Z}^{\mu_i^\pitchfork}$.

We have studied j_1 in Sect. 5.1.1 at the level n. For the $(n+1)$th homology groups, the restriction of j_1 to the first summand in (10) is zero since its image is in $\bigoplus_{p \in P} H_{n+1}(\mathcal{X}_p, \mathcal{A}_p)$ which is zero by Lemma 4.2. Since $H_{n+1}(\mathcal{X}_{y_i}, \mathcal{A}_{y_i}) = H_n(\mathcal{A}_{y_i}) = 0$, the image by j_1 of $\bigoplus_{i=1}^{\rho} H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i})$ is also zero. The restriction of j_1 to the remaining summand is the direct sum $\bigoplus_{q \in Q} j_{1,q}$ of the maps

$$j_{1,q} \colon \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \to H_{n+1}(\mathcal{X}_q, \mathcal{A}_q).$$

By Proposition 3.2, the kernel of $j_{1,q}$ is equal to $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q)$, where $\mathcal{A}_q$ is the local Milnor fibre of the hypersurface germ (V, q), $q \in Q$, and can be identified to the free part of $H_{n-1}(\mathcal{A}_q)$. The intersection $\left(\bigoplus_{i=1}^{\rho} \mathbb{D}_i\right) \cap \bigoplus_{q \in Q} H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q)$ is well defined, since $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q)$ is contained in $\bigoplus_{i=1}^{\rho} \bigoplus_{Q_i \ni q} \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s)$.

After these preparations we can state:

[4] We recall from Sect. 5.2 that the notation S_q depends on whether the point q is considered in Q or in Q_i, namely it takes either the local branches of Σ at q, or the local branches of Σ_i at q, accordingly.

Theorem 6.1 *In the above notations we have*

$$H_{n+2}^{\curlyvee}(V) = \left(\mathbb{D} \cap \bigoplus_{q\in Q} H_{n+1}(\mathcal{A}_q, \partial_2\mathcal{A}_q)\right) \cap \bigoplus_{i=1}^{\rho} \Delta_*^i\left(\bigcap_{j\in G_i} \ker(A_j - I)\right),$$

where $A_j\colon H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \to H_n(E_i^{\pitchfork}, F_i^{\pitchfork})$ *denotes the monodromy along the loop of* W_i *indexed by* $j \in G_i$.

In particular, $H_{n+2}^{\curlyvee}(V)$ *is free and its rank is bounded as follows*[5]*:*

$$\operatorname{rank} H_{n+2}^{\curlyvee}(V) \leqslant \sum_{i=1}^{\rho} \min_{\substack{s\in S_q\\ q\in Q_i\\ j\in G_i}} \left\{\dim\ker(A_s - I), \dim\ker(A_j - I)\right\} \leqslant \sum_{i=1}^{\rho} \mu_i^{\pitchfork}.$$

Proof In order to handle the map j_2, we recall the relative CW-complex structure of $(\mathcal{Y}, \mathcal{B})$ given in Sect. 4.5. On each component W_i we have identified the set of points T_i which consists of the axis points P_i, the special points Q_i, and one general point y_i. The punctured Σ_i^* retracts to a configuration W_i of $2g_i + \nu_i + \gamma_i$ loops indexed by the set W_i, based at some point z_i, where $2g_i$ of them are "genus loops" and the other loops are projections by the normalization map $n_i\colon \widetilde{\Sigma}_i \to \Sigma_i$ of loops around all the punctures of $\widetilde{\Sigma}_i \setminus n_i^{-1}(P_i \sqcup Q_i)$. Notice that $\# T_i - 1 \geqslant \nu_i > 0$.

Let $W = \bigsqcup_i W_i$. Consider the spaces $\mathcal{Y}_W = \pi_\Sigma^{-1}(W)$ and $\mathcal{B}_W = \mathcal{B} \cap \mathcal{Y}_W$. We have the homotopy equivalence of pairs $(\mathcal{Y}, \mathcal{B}) \simeq (\mathcal{Y}_W, \mathcal{B}_W)$ which has been discussed in Sect. 4.5 and use the CW-complex model for $(\mathcal{Y}_W, \mathcal{B}_W)$. We also have the decomposition $(\mathcal{Y}, \mathcal{B}) = \bigsqcup_{i=1}^{\rho} (\mathcal{Y}_i, \mathcal{B}_i)$ according to the components W_i.

In our representation, the map j_2 splits into the direct sum of the following maps, for $i \in \{1, \ldots, \rho\}$:

$$j_{2,i}\colon \bigoplus_{p\in P_i} H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p) \bigoplus_{q\in Q_i} \bigoplus_{s\in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \oplus H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}) \to H_{n+1}(\mathcal{Y}_i, \mathcal{B}_i).$$

By Lemma 4.4, the map $j_{2,i}$ restricts to an embedding of the direct sum

$$\bigoplus_{p\in P_i} H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p) \bigoplus_{q\in Q_i} \bigoplus_{s\in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s)$$

into $H_{n+1}(\mathcal{Y}_i, \mathcal{B}_i)$. Note that $H_{n+1}(\mathcal{Z}_v, \mathcal{C}_v) = \ker(A_v - I) \subset H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \simeq H_{n-1}(F_i^{\pitchfork})$ for any point $v \in P_i$ or $v \in S_q$ and $q \in Q_i$. The kernel $\ker j_{2,i}$ is therefore determined by the relations induced by the image of the remaining direct summand $H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i})$ into $H_{n+1}(\mathcal{Y}_i, \mathcal{B}_i)$.

More precisely, every $(n+1)$-cycle generator w of

$$H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}) \simeq H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \simeq H_{n-1}(F_i^{\pitchfork})$$

[5] Note that no multiplicities but only transversal types are involved in the rank formula.

induces one single relation. Namely $j_2(w)$ is a $(n+1)$-cycle above the loop around the point y_i, and since this loop is homotopy equivalent to a certain composition of other loops of W_i, it follows that $j_2(w)$ is precisely homologous to the corresponding sum of cycles above the loops in W_i. Our scope is to find all such sums which contain as terms only elements from the images $j_2(H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p))$ for $p \in P_i$ and $j_2(H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s))$ for $s \in S_q$ and $q \in Q_i$. We have the following facts:

(a) By Lemma 4.3 and Sect. 4.4, such images are in the kernels of $A - I$ where A is the vertical monodromy of the loop corresponding to $p \in P_i$ or to $s \in S_q$ and $q \in Q_i$. Therefore the expression of $j_2(w)$ contains the sum of those generators of $j_2(H_{n+1}(\mathcal{Z}_p, \mathcal{C}_p))$ and of $j_2(H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s))$ which correspond to the same representative $w \in H_{n-1}(F_i^{\pitchfork})$, for any $p \in P_i$ and any $s \in S_q$ and $q \in Q_i$. This implies that $w \in \bigcap_{s \in S_q, q \in Q_i} \ker(A_s - I)$. Note that the points $p \in P_i$ are superfluous in this intersection since $A_p = I$ for all such points.

(b) Let us consider a pair γ_1 and γ_2 of genus loops (whenever $g_i > 0$) and let us denote by B_1 and B_2 the local system monodromy along these loops. The relation produced by $j_2(w)$ contains in principle the following relative cycle along the wedge $\gamma_1 \vee \gamma_2$: it starts from the representative $a_w \in H_{n-1}(F_i^{\pitchfork})$ of w, moves in the local system along γ_1 arriving as $B_1(a_w)$ after one loop at the fibre over the base point z, next moved along γ_2 to $B_2 B_1(a_w)$, then in the opposite direction along γ_1 to $B_1^{-1} B_2 B_1(a_w)$ and finally in the opposite direction along γ_2 to $B_2^{-1} B_1^{-1} B_2 B_1(a_w)$. Our condition tells that the relation produced by $j_2(w)$ does not involve $(n+1)$-cycles along the genus loops since $\operatorname{Im} j_2 \cap \bigoplus_{j \in G_i} H_{n+1}(E_j^{\pitchfork}, F^{\pitchfork}) = 0$, by Lemma 4.4 and (7). Therefore the relative cycles along γ_1 and along γ_2 must cancel, which fact amounts to the following two pairs of equalities:

$$
\begin{aligned}
B_1^{-1} B_2 B_1(a_w) &= a_w && \text{and} && B_2 B_1(a_w) = B_1(a_w),\\
B_2^{-1} B_1^{-1} B_2 B_1(a_w) &= B_1(a_w) && \text{and} && B_1^{-1} B_2 B_1(a_w) = B_2 B_1(a_w).
\end{aligned}
$$

These equalities are cyclic, thus the eight above terms appear to be equal. In particular we get $B_1(a_w) = a_w$ and $B_2(a_w) = (a_w)$ for any $w \in \bigcap_{s \in S_q, q \in Q_i} \ker(A_s - I)$. We conclude to the same equalities for any pair of genus loops.

Altogether we obtain the following diagonal presentation of $\ker j_{2,i}$:

$$
\begin{aligned}
\ker j_{2,i} = \Bigg\{ (a_w, a_w, \ldots, a_w) \in \bigoplus_{q \in Q_i} \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \oplus H_{n+1}(\mathcal{Z}_{y_i}, \mathcal{C}_{y_i}) :\\
w \in \bigcap_{\substack{s \in S_q\\ q \in Q_i}} \ker(A_s - I) \cap \bigcap_{j \in G_i} \ker(A_j - I) \Bigg\} \subset \mathbb{D}_i.
\end{aligned}
$$

Now, since

$$
H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon) \subset \ker j_2 = \bigoplus_{i=1}^{\rho} \Delta_*^i \Bigg(\bigcap_{\substack{s \in S_q\\ q \in Q_i}} \ker(A_s - I) \cap \bigcap_{j \in G_i} \ker(A_j - I) \Bigg)
$$

we get in particular the claimed inequality for the Betti number $b_{n+2}(\mathbb{V}_\Delta, V_\varepsilon)$. The freeness of $H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon)$ follows from the fact that ker j_2 is free (as the image of the intersection of free $\mathbb{Z}$-submodules).

We also obtain the desired expression of $H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon) = \ker(j_1 \oplus j_2) = \ker j_1 \cap \ker j_2$ by intersecting ker j_2 with the diagonal expression of ker j_1 given just before the statement of Theorem 6.1. □

Remark 6.2 (*Irreducible* Σ) In case Σ is irreducible, the equality of Theorem 6.1 reads

$$H^{\curlyvee}_{n+2}(V) \simeq \bigcap_{q\in Q} H_{n+1}(\mathcal{A}_q, \partial_2\mathcal{A}_q) \cap \bigcap_{j\in G} \ker(A_j - I).$$

In particular, if there are no special points on Σ and the monodromy along every the genus loop is the identity, then $H^{\curlyvee}_{n+2}(V) \simeq H_{n-1}(F^{\pitchfork})$. This situation can be seen in the example $V = \{xy = 0\} \subset \mathbb{P}^3$ for which $H^{\curlyvee}_4(V) \simeq \mathbb{Z}$ and rank $H^{\curlyvee}_3(V) = 1$.

Remark 6.3 ($(n+1)$*th vanishing Betti number*) It appears that $H^{\curlyvee}_{n+2}(V)$ does not depend neither on the axis points, nor on the isolated singular points of V. However $H^{\curlyvee}_{n+1}(V)$ depends on those elements since the Euler number does, after [18, Theorem 5.3]:

$$\chi(\mathbb{V}_\Delta, V_\varepsilon) = (-1)^{n+1}\sum_{i=1}^{\rho}(2g_i + \nu_i + \gamma_i - 2)\mu_i^{\pitchfork} - \sum_{q\in Q}(\chi(\mathcal{A}_q) - 1) + (-1)^{n+1}\sum_{r\in R}\mu_r. \tag{11}$$

Theorem 6.1 is useful when we have information about the transversal monodromies, namely about the eigenspaces corresponding to the eigenvalue 1. We immediately derive:

Corollary 6.4 *If, for every $i \in \{1, \dots, \rho\}$, at least one of the transversal monodromies along the loops $W_i \subset \Sigma_i^*$ has no eigenvalue 1, then $H^{\curlyvee}_{n+2}(V) = 0$.*

We may also apply Theorem 6.1 when we have enough information about local Milnor fibres of special points, like in the following case (see also Example 7.3):

Corollary 6.5 *Assume that for any $i \in \{1, \dots, \rho\}$ there is some special point $q_i \in Q$ such that the $(n-1)$th homology group of the local Milnor fibre $\mathcal{A}_{q_i}$ of the hypersurface germ (V, q_i) has rank zero. Then*

$$H^{\curlyvee}_{n+2}(V) = 0$$

and the single non-zero vanishing Betti number $b^{\curlyvee}_{n+1}(V)$ is given by the formula

$$\operatorname{rank} H^{\curlyvee}_{n+1}(V) = \sum_i(\nu_i + \gamma_i + 2g_i - 2)\mu_i^{\pitchfork} + (-1)^n\sum_{q\in Q}(\chi(\mathcal{A}_q) - 1) + \sum_{r\in R}\mu_r.$$

Proof Let $(w_1, \dots, w_\rho)$ be an element of the reference space $\bigoplus_{i=1}^{\rho} H_n(E_i^{\pitchfork}, F_i^{\pitchfork}) \cong \bigoplus_{i=1}^{\rho} \mathbb{Z}^{\mu_i^{\pitchfork}}$. By the diagonal map this corresponds to elements $w_i \in H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s)$ for $s \in S_q$ and $q \in \Sigma_i$. By the discussion introducing Theorem 6.1 the kernel of some component $j_{1,q} \colon \bigoplus_{s \in S_q} H_{n+1}(\mathcal{Z}_s, \mathcal{C}_s) \to H_{n+1}(\mathcal{X}_q, \mathcal{A}_q)$ is equal to $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q)$ which in turn is identified to the free part of $H_{n-1}(\mathcal{A}_q) = 0$. The rank zero condition implies that $w_i = 0$ for i such that $q \in \Sigma_i$, thus all w_i are zero.

As for the rank of $H_{n+1}(\mathbb{V}_\Delta, V_\varepsilon)$, the formula follows from the Euler characteristic computation (11). □

Remark 6.6 In case of an irreducible singular set Σ, Corollary 6.5 tells that one singular point $q \in Q$ with an $(n-1)$th Betti number of the Milnor fibre equal to zero is sufficient for the vanishing of $H^{\curlyvee}_{n+2}(V)$.

7 Computations of Betti numbers

7.1 Vanishing Betti numbers

As direct application of Theorem 6.1, we provide explicit computations of the ranks of the vanishing homology of some projective hypersurfaces.

Example 7.1 (*some cubic hypersurfaces*) If $V = \{x^2 z + y^2 w = 0\} \subset \mathbb{P}^3$ then Sing V is a projective line and its generic transversal type is A_1. There are three axis points and two special points q with local singularity type D_∞. The hypersurface singularity germ D_∞ is an *isolated line singularity* in the terminology of [14]. Its Milnor fibre F is homotopy equivalent to the sphere S^2, the transversal monodromy is $-\mathrm{id}$. From Corollary 6.4 it follows that $H^{\curlyvee}_4(V) \simeq H_1(F) = 0$ and applying Corollary 6.5 we get that rank $H^{\curlyvee}_3(V) = 5$.

For $V = \{x^2 z + y^2 w + t^3 = 0\} \subset \mathbb{P}^4$, Sing V is again a projective line but its generic transversal type is A_2, with three axis points and two special points for both of which the local Milnor fibre F is homotopy equivalent to $S^3 \vee S^3$. Then Corollary 6.5 yields $H^{\curlyvee}_5(V) \simeq H_2(F) = 0$ and rank $H^{\curlyvee}_4(V) = 10$. This construction can be iterated, for instance $V = \{x^2 z + y^2 w + t_1^3 + t_2^3 = 0\} \subset \mathbb{P}^5$ has $H^{\curlyvee}_6(V) = 0$ and rank $H^{\curlyvee}_5(V) = 20$.

Example 7.2 (*including an isolated singular point*) Let $V = \{y^2(x + y - 1)(x - y + 1) + z^4 = 0\} \subset \mathbb{P}^3$. We have Sing V is the disjoint union of Σ, a projective line $\{y = z = 0\}$ with transversal type A_3 and a point $R = \{(0{:}1{:}0{:}0)\}$ of type A_3. There are two special points: $Q = \{(1{:}0{:}0{:}0), (-1{:}0{:}0{:}0)\}$, each of them with Milnor fibre $S^2 \vee S^2 \vee S^2$. It follows that $H^{\curlyvee}_4(V) = 0$ and rank $H^{\curlyvee}_3(V) = 21$.

Example 7.3 (*singular locus with two disjoint curve components*) Let $V = \{f = x^2 z^2 + x^2 w^2 + y^2 z^2 + 2y^2 w^2 = 0\} \subset \mathbb{P}^3$, which is defined by an element f of the ideal $(x, y)^2 \cap (z, w)^2$. Then Sing $V = \Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \{x = y = 0\}$ and $\Sigma_2 = \{z = w = 0\}$. It turns out that the generic transversal type at both of the line components of the singular locus is A_1 and that there are exactly four D_∞-points on each of these two components. We are in the situation of Corollary 6.5, hence $H^{\curlyvee}_4(V) = 0$ and rank $H^{\curlyvee}_3(V) = 20$.

7.2 Computation of vanishing homology groups

Using the full details of the proof of Theorem 6.1, we may compute not only the rank of the vanishing homology groups, but in several examples even the vanishing homology group $H^{\curlyvee}_{n+1}(V)$ itself, as follows.

The main ingredient is the map

$$j^{[k]} = j_1^{[k]} \oplus j_2^{[k]} \colon H_k(\mathcal{Z}, \mathcal{C}) \to H_k(\mathcal{X}, \mathcal{A}) \oplus H_k(\mathcal{Y}, \mathcal{B}),$$

which was denoted by j in (9). Like in (10), we use the direct sum splitting into axis, special and auxiliary contributions.

$$0 \to \operatorname{coker} j^{[n+1]} \to H_{n+1}(\mathbb{V}_\Delta, V_\varepsilon) \to \ker j^{[n]} \to 0$$

and the strategy will be to work with $j^{[n+1]}$ and $j^{[n]}$ at the level of generators.

Example 7.4 Let $V = \{x^2z + y^3 + xyw = 0\} \subset \mathbb{P}^3$. Then $\operatorname{Sing} V$ is a projective line with generic transversal type A_1, three axis points, and a single special point q of local singularity type $J_{2,\infty}$ [9]. The latter is an isolated line singularity germ, cf. [14], with Milnor fibre F a bouquet of four spheres S^2 and the transversal monodromy is the identity. By Theorem 6.1 and Corollary 6.5 we get $H^{\curlyvee}_4(V) \simeq H_1(F) = 0$ and $\operatorname{rank} H^{\curlyvee}_3(V) = 6$. We next can show (but skip the details) that there is an isomorphism $H^{\curlyvee}_3(V) \simeq \mathbb{Z}^6$. Note that Dimca [3] observed that V has the rational homology of $\mathbb{P}^2$.

Example 7.5 $V = \{xyz = 0\} \subset \mathbb{P}^3$ of degree $d = 3$. Then V is reducible with three components, $\operatorname{Sing} V$ is the union of three projective lines intersecting at a single point $[0{:}0{:}0{:}1]$, and the transversal type along each of them is A_1. Following the proof of Theorem 6.1, we get $\ker j_2^{[3]} = \bigoplus_{i=1}^{3} H_1(F_i^{\pitchfork}) \simeq \mathbb{Z}^3$ and $\ker j_1^{[3]} \simeq H_1(F)$ where F denotes the Milnor fibre of the non-isolated singularity of V at the single special point $[0{:}0{:}0{:}1]$, which is homotopically equivalent to $S^1 \times S^1$. We thus get $H^{\curlyvee}_4(V) \simeq H_1(F) \simeq \mathbb{Z}^2$.

The axis A of our pencil has degree nine and intersects each of the components of V at three general points. Hence $\nu_i = 3$, $\gamma_i = 1$ for any $i = 1, 2, 3$.

Applying formula (11) we get that the vanishing Euler characteristic is -5, and that $\operatorname{rank} H^{\curlyvee}_3(V) = 7$. We can moreover show the freeness of this group; we skip the details.

7.3 The surfaces case

Several examples in the previous subsections are surfaces and the computation of H_4 and the rank of H_3 could be simplified by counting the number of irreducible components of Σ^*. Indeed in case of surfaces $V \subset \mathbb{P}^3$ we have

$$H^{\curlyvee}_4(V) \simeq \mathbb{Z}^{r-1}, \tag{12}$$

where r is the number of irreducible components of V.

We have included these examples anyhow as applications of our method.

Combining (12) with Theorem 6.1 yields several consequences on the singular set and its generic transversal types. We mention here only one:

Corollary 7.6 $r - 1 \leqslant \displaystyle\sum_{i=1}^{\rho} \mu_i^{\pitchfork}$.

7.4 Absolute homology of projective hypersurfaces

If $\dim \operatorname{Sing} V = 1$ then from Theorem 4.1 and the long exact sequence of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$ one gets the isomorphisms

$$H_k(V) \simeq H_k(V_\varepsilon) = H_k(\mathbb{P}^n) \qquad \text{for} \quad k \neq n, n+1, n+2.$$

This corresponds to Kato's result [10] in cohomology.[6]

In the remaining dimensions we have an 8-term exact sequence

$$\begin{aligned} 0 \to H_{n+2}(V_\varepsilon) \to H_{n+2}(\mathbb{V}_\Delta) \to H_{n+2}(\mathbb{V}_\Delta, V_\varepsilon) \xrightarrow{\Phi_{n+1}} H_{n+1}(V_\varepsilon) \\ \to H_{n+1}(\mathbb{V}_\Delta) \to H_{n+1}(\mathbb{V}_\Delta, V_\varepsilon) \xrightarrow{\Phi_n} H_n(V_\varepsilon) \to H_n(\mathbb{V}_\Delta) \to 0 \end{aligned} \tag{13}$$

from which we obtain, with help of Theorem 6.1:

Proposition 7.7 *Let* $\dim \operatorname{Sing} V = 1$. *If n is even, then*

(c) $H_{n+2}(V) \simeq \mathbb{Z} \oplus H_{n+2}^{\curlyvee}(V)$,
(d) $H_{n+1}(V) \simeq \ker \Phi_n$,
(e) $H_n(V) \simeq \operatorname{coker} \Phi_n$,

whereas for any n one has the following inequalities:

(a) $b_{n+2}(V) \leqslant 1 + \sum_{i=1}^{\rho} \mu_i^{\pitchfork}$,
(b) $b_n(V) \leqslant \dim H_n(V_\varepsilon)$.

This can be regarded as a natural extension of Proposition 2.2 to 1-dimensional singularities, thus extending also Dimca's corresponding result for isolated singularities that was discussed in Sect. 2. Like in the isolated singularities setting, one has to deal with the difficulty of identifying Φ_n from the equation of f.

Example 7.8 Let $V = \{f(x, y) + f(z, w) = 0\} \subset \mathbb{P}^3$, where $f(x, y) = y^2 \prod_{i=1}^{3}(x - \alpha_i y)$, with $\alpha_i \neq 0$ pairwise different. Its singular set is the smooth line given by $y = w = 0$, with generic transversal type $y^2 + w^3$. There are two special points $[0{:}0{:}1{:}0]$ and $[1{:}0{:}0{:}0]$, each with Milnor fibre a bouquet of spheres S^2. By Corollary 6.5 we get $H_4^{\curlyvee}(V) = 0$ and from the Euler characteristic formula (11) (by computing its ingredients) we get rank $H_3^{\curlyvee}(V) = 38$. One can compute the eigenvalues of the monodromies for all types of singular points; they are all different from

[6] Dimca states such a result [5, Theorem 4.1] referring to [4, p. 144] for Kato's proof in cohomology [10].

1. By using Randell's criterion [13, Proposition 3.6], one can show that V is a $\mathbb{Q}$-homology manifold. Since a homology manifold satisfies Poincaré duality, it follows e.g. that $H_3(V;\mathbb{Q}) \cong H_1(V;\mathbb{Q}) \cong H_1(\mathbb{P}^n;\mathbb{Q}) = 0$ and $H_4(V;\mathbb{Q}) \cong H_0(V;\mathbb{Q}) \cong H_0(\mathbb{P}^n;\mathbb{Q}) \cong \mathbb{Q}$. By computations and the exact sequence (13) we get also $H_2(V;\mathbb{Q})$ since rank $H_2(V) =$ rank $H_2(V_\varepsilon) -$ rank $H_3(\mathbb{V}_\Delta, V_\varepsilon) = 53 - 38 = 15$.

References

1. Damon, J., Pike, B.: Solvable groups, free divisors and nonisolated matrix singularities II: vanishing topology. Geom. Topol. **18**(2), 911–962 (2014)
2. Dimca, A.: On the homology and cohomology of complete intersections with isolated singularities. Compositio Math. **58**(3), 321–339 (1986)
3. Dimca, A.: On the Milnor fibrations of weighted homogeneous polynomials. Compositio Math. **76**(1–2), 19–47 (1990)
4. Dimca, A.: Singularities and Topology of Hypersurfaces. Universitext. Springer, New York (1992)
5. Dimca, A.: Singularities and their deformations: how they change the shape and view of objects. In: Elkadi, M., Mourrain, B., Piene, R. (eds.) Algebraic Geometry and Geometric Modeling. Mathematics and Visualization, pp. 87–101. Springer, Berlin (2006)
6. Frühbis-Krüger, A., Zach, M.: On the vanishing topology of isolated Cohen–Macaulay codimension 2 singularities (2015). arXiv:1501.01915
7. Hulek, K., Kloosterman, R.: Calculating the Mordell–Weil rank of elliptic threefolds and the cohomology of singular hypersurfaces. Ann. Inst. Fourier (Grenoble) **61**(3), 1133–1179 (2011)
8. Iomdin, I.N.: Complex surfaces with a one-dimensional set of singularities. Siberian Math. J. **15**(5), 748–762 (1974)
9. de Jong, T.: Some classes of line singularities. Math. Z. **198**(4), 493–517 (1988)
10. Kato, M.: Topology of k-regular spaces and algebraic sets. In: Manifolds–Tokyo 1973, pp. 153–159. University of Tokyo Press, Tokyo (1975)
11. Libgober, A.S., Wood, J.W.: On the topological structure of even-dimensional complete intersections. Trans. Amer. Math. Soc. **267**(2), 637–660 (1981)
12. Parusiński, A., Pragacz, P.: Characteristic classes of hypersurfaces and characteristic cycles. J. Algebraic Geom. **10**(1), 63–79 (2001)
13. Randell, R.: On the topology of non-isolated singularities. In: Cantrell, J.C. (ed.) Geometric Topology, pp. 445–473. Academic Press, New York (1979)
14. Siersma, D.: Isolated line singularities. In: Peter, O. (ed.) Singularities, Part 2. Proceedings of Symposia in Pure Mathematics, vol. 40.2, pp. 485–496. American Mathematical Society, Providence (1983)
15. Siersma, D.: Quasihomogeneous singularities with transversal type A_1. In: Richard, R. (ed.) Singularities. Contemporary Mathematics, vol. 90, pp. 261–294. American Mathematical Society, Providence (1989)
16. Siersma, D.: Variation mappings on singularities with a 1-dimensional critical locus. Topology **30**(3), 445–469 (1991)
17. Siersma, D.: The vanishing topology of non isolated singularities. In: Siersma, D., Wall, C.T.C., Zakalyukin, V. (eds.) New Developments in Singularity Theory. NATO Science Series II: Math. Phys. Chem., vol. 21, pp. 447–472. Kluwer, Dordrecht (2001)
18. Siersma, D., Tibăr, M.: Betti bounds of polynomials. Mosc. Math. J. **11**(3), 599–615 (2011)
19. Siersma, D., Tibăr, M.: Milnor fibre homology via deformation. In: Decker, W., Pfister, G., Schulze, M. (eds.) Singularities and Computer Algebra, pp. 306–322. Springer, Cham (2017)
20. Tibăr, M.: The vanishing neighbourhood of non-isolated singularities. Israel J. Math. **157**, 309–322 (2007)
21. Zach, M.: Vanishing cycles of smoothable isolated Cohen–Macaulay codimension 2 singularities of type 2 (2016). arXiv:1607.07527v1