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On the Brauer–Manin obstruction for degree-four del Pezzo surfaces

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1. Introduction. A *del Pezzo surface* is a smooth, proper algebraic surface X over a field K with ample anticanonical sheaf $\mathscr{K}^{-1}$. Over an algebraically closed field, every del Pezzo surface of degree $d \leq 7$ is isomorphic to $\mathbf{P}^2$, blown up in $9-d$ points in general position [Man, Theorem 24.4(iii)].

According to the adjunction formula, a smooth complete intersection of two quadrics in $\mathbf{P}^4$ is del Pezzo. The converse is true as well. For every del Pezzo surface of degree 4, its anticanonical image is the complete intersection of two quadrics in $\mathbf{P}^4$ [Do, Theorem 8.6.2].

For an arbitrary proper variety X over $\mathbb{Q}$, the Brauer–Manin obstruction is a phenomenon that can explain failures of weak approximation or even the Hasse principle. It works as follows.

Let p be any prime number. The *Grothendieck–Brauer group* is a contravariant functor from the category of schemes to the category of abelian groups. In particular, for an arbitrary scheme X and a $\mathbb{Q}_p$-rational point $x\colon \operatorname{Spec}\mathbb{Q}_p \to X$ on it, there is a restriction homomorphism

$$x^*\colon \operatorname{Br}(X) \to \operatorname{Br}(\mathbb{Q}_p) \cong \mathbb{Q}/\mathbb{Z}.$$

For a Brauer class $\alpha \in \operatorname{Br}(X)$, we call

$$\operatorname{ev}_{\alpha,p}\colon X(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z}, \qquad x \mapsto x^*(\alpha),$$

the *local evaluation map* associated to α. Analogously, for the real place, there is the local evaluation map $\operatorname{ev}_{\alpha,\infty}\colon X(\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Let us write Ω for the set of all places of $\mathbb{Q}$, i.e. for the union of all finite primes together with ∞. The local evaluation maps are continuous with respect to the p-adic, respectively real, topologies on $X(\mathbb{Q}_\nu)$ for $\nu \in \Omega$. Moreover, it is well-known that $\operatorname{ev}_{\alpha,\nu}$ is constant for all but finitely many places.

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Thus, only adelic points $x = (x_\nu)_{\nu \in \Omega} \in X(\mathbb{A}_\mathbb{Q})$ satisfying

$$\sum_{\nu \in \Omega} \mathrm{ev}_{\alpha,\nu}(x_\nu) = 0 \in \mathbb{Q}/\mathbb{Z} \tag{1.1}$$

may possibly be approximated by $\mathbb{Q}$-rational points.

We say that a Brauer class $\alpha \in \mathrm{Br}(X)$ *works* at a place ν if the local evaluation map $\mathrm{ev}_{\alpha,\nu}\colon X(\mathbb{Q}_\nu) \to \mathbb{Q}/\mathbb{Z}$ is nonconstant. This is in fact a property of the residue class of α in $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$.

Observe that if $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ and there exists a Brauer class that works at least at a single place then weak approximation is violated on X. On the other hand, if $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and a nontrivial class works at least at one place then there exist adelic points fulfilling (1.1). That is, the Brauer–Manin obstruction cannot explain a violation of the Hasse principle.

The goal of this paper is to investigate which subsets of Ω may occur as the set of places at which a nontrivial Brauer class works, in the situation of a degree-4 del Pezzo surface. Our first main result is as follows.

THEOREM 1.1. *Let $S \subset \Omega$ be any finite subset. Then there exists a degree-4 del Pezzo surface X over $\mathbb{Q}$ having a $\mathbb{Q}$-rational point such that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S.*

For the construction, we make use of surfaces given as the intersection of two quadrics of the form

$$\begin{aligned} -A_1(T_0 - T_1)(T_0 + T_1) &= T_3^2 - DT_4^2, \\ -A_2(T_0 - T_2)(T_0 + T_2) &= T_3^2 - B^2DT_4^2, \end{aligned}$$

with $A_1, A_2, D, B \in \mathbb{Q}$. This family is inspired by a surface studied by Birch and Swinnerton-Dyer [BSD] and has the advantage that there is a standard way to write down a Brauer class. Moreover, we obtain the following example, which is different in nature.

EXAMPLE 1.2. *Let $X \subset \mathbf{P}^4_\mathbb{Q}$ be the degree-4 del Pezzo surface given by*

$$T_0T_1 = T_2^2 + 7T_3^2, \qquad (T_0 - 4T_1)(T_0 - 6T_1) = T_2^2 + 7T_4^2.$$

Then X has a $\mathbb{Q}$-rational point, $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, and the nontrivial Brauer class works exactly at the infinite place. In particular, the surface $X(\mathbb{R})$ has two connected components and $\mathbb{Q}$-rational points on only one of them.

The case $S = \{\infty\}$ is perhaps the most interesting one. In this case, the set of $\mathbb{Q}$-rational points of X is dense in the real component of X that has $\mathbb{Q}$-rational points. This is in line with Mazur's conjecture [Maz, Conjecture 1] that the closure of $X(\mathbb{Q})$ with respect to the real topology is a union of connected components.

In Section 2 we give more details about Example 1.2. Similar examples for other kinds of surfaces are available in the literature, including singular cubic surfaces [SD1, §3], conic bundles with five singular fibres [Maz, §3], and others.

Recall that over an algebraically closed field, two quadratic forms are always simultaneously diagonalisable. We say that a degree-4 del Pezzo surface is diagonalisable over $\mathbb{Q}$ if the defining quadratic forms are diagonalisable over $\mathbb{Q}$.

The surface from Example 1.2 is not diagonalisable over $\mathbb{Q}$ but only over $\mathbb{Q}(\sqrt{6})$, as is easily seen using Fact 2.1(b)(iii). Somewhat surprisingly, such a behaviour is necessary at this point:

THEOREM 1.3. *Let X be a degree-4 del Pezzo surface over $\mathbb{Q}$ having an adelic point and $\alpha \in \mathrm{Br}(X)$ a Brauer class that works exactly at the infinite place. Then X is not diagonalisable over $\mathbb{Q}$.*

Our method of proof uses the fact that diagonal degree-4 del Pezzo surfaces have nontrivial automorphisms that are defined over the ground field. By functoriality, these operate on $\mathrm{Br}(X)$, but the induced operation on $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ turns out to be trivial automatically. Therefore, every α in $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ induces a homomorphism $i_\alpha\colon \mathrm{Aut}'(X) \to \mathrm{Br}(\mathbb{Q})$. See Construction 3.2 for more details.

Moreover, we prove that if $\alpha \in \mathrm{Br}(X)$ works at ∞ then there is an automorphism $\sigma \in \mathrm{Aut}(X)$ witnessing this, i.e. such that $i_\alpha(\sigma)$ has a nontrivial component at ∞. From this, the claim easily follows.

Our third main result asserts that, for diagonalisable degree-4 del Pezzo surfaces, the subset $\{\infty\}$ is the only exception of this kind.

THEOREM 1.4. *Let $S \subset \Omega$ be a finite subset different from $\{\infty\}$. Then there exists a diagonalisable degree-4 del Pezzo surface X over $\mathbb{Q}$ having a $\mathbb{Q}$-rational point such that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ and the nontrivial Brauer class works exactly at the places in S.*

Conjecturally, for degree-4 del Pezzo surfaces, all failures of weak approximation are due to the Brauer–Manin obstruction. More precisely, it is conjectured that $X(\mathbb{Q})$ is dense in

$$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} := \bigcap_{\alpha \in \mathrm{Br}(X)} X(\mathbb{A}_\mathbb{Q})^\alpha$$

for $X(\mathbb{A}_\mathbb{Q})^\alpha \subseteq X(\mathbb{A}_\mathbb{Q})$ defined by (1.1) and $X(\mathbb{A}_\mathbb{Q})$ endowed with the product topology induced by the ν-adic topologies on $X(\mathbb{Q}_\nu)$.

Due to work of P. Salberger and A. N. Skorobogatov [SSk, Theorem 0.1], this conjecture is proven when X has a $\mathbb{Q}$-rational point. The weaker statement that if X has a $\mathbb{Q}$-rational point then the $\mathbb{Q}$-rational points on X are

Zariski dense already follows from the fact that del Pezzo surfaces of degree 4 that have a rational point are unirational (see [Man, Theorems 29.4 and 30.1]).

Recall that each of the surfaces provided by Theorem 1.1 has a $\mathbb{Q}$-rational point. We may thus blow up $\mathbb{Q}$-rational points in general position to obtain del Pezzo surfaces of low degree. It actually requires some thought to see that, on every del Pezzo surface X of degree ≥ 2, there exists a nonempty Zariski open subset $U \subset X$ of admissible blow-up points (cf. [De, Theorem 1]). Admissible means here that the blow-up is indeed del Pezzo, not just the desingularisation of a weak del Pezzo surface (we omit the details). Moreover, Brauer groups do not change under blow-up, and the local evaluation maps are compatible in the sense that $\mathrm{ev}_{\alpha,\nu}(\pi(x)) = \mathrm{ev}_{\pi^*\alpha,\nu}(x)$. Hence we obtain the following corollary.

Corollary 1.5. *Let $S \subset \Omega$ be an arbitrary finite subset and $d \leq 4$ a positive integer. Then there exists a del Pezzo surface X of degree d over $\mathbb{Q}$ having a $\mathbb{Q}$-rational point such that* $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ *and the nontrivial Brauer class works exactly at the places in S.*

It is well-known that every del Pezzo surface X of degree at least 5 has $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) = 0$. One way to see this is to systematically inspect all possible Galois operations on the exceptional curves in a way analogous to [Ja, Chapter III, 8.21–8.23] and to apply [Man, Proposition 31.3]. Thus, Corollary 1.5 cannot have an analogue for del Pezzo surfaces of higher degree.

At least for $d = 5$ and 7, as well as for $d = 6$ under the additional assumption that X has an adelic point, there is also a geometric argument. Indeed, these surfaces are birationally equivalent to $\mathbf{P}^2_{\mathbb{Q}}$ ([VA, Theorem 2.1], cf. [Man, Theorem 29.4]).

It would be interesting to produce examples as in Corollary 1.5 with the additional restriction that the surface is minimal.

Remark 1.6. For a degree-4 del Pezzo surface, the group $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ may be isomorphic to either 0, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$. In the cases of degree 3, 2, or 1, there are even more options ([Co, Theorem 4.1], [Man, Section 31, Table 3] and [SD2]). We do not know whether the analogue of Corollary 1.5 is true for a prescribed Brauer group.

2. Brauer classes on degree-4 del Pezzo surfaces. The goal of this section is to gather some facts about degree-4 del Pezzo surfaces that are necessary for what follows. This includes some results on their Brauer groups and finally leads us to a proof of the assertions made in Example 1.2. In other words, we show Theorem 1.1 under the assumption of Theorem 1.4.

Unless a specific choice is made, we work in this section over an arbitrary base field K of characteristic $\neq 2$. Let $\overline{K}$ denote an algebraic closure of K.

A del Pezzo surface $X \subset \mathbf{P}^4_K$ of degree 4 is the base locus of a pencil $(\mu Q^{(1)} + \nu Q^{(2)})_{(\mu:\nu)\in\mathbf{P}^1}$ of quadratic forms in five variables with coefficients in the field K. The generic member of the pencil must be of rank 5, as otherwise X would be a cone. The condition that $\det(\mu Q^{(1)} + \nu Q^{(2)}) = 0$ therefore defines a finite subscheme $\mathscr{S}_X \subset \mathbf{P}^1_K$ of degree 5.

Choosing a different basis of the pencil yields another embedding of $\mathscr{S}_X$ into the projective line. Thus, one may consider the subscheme $\mathscr{S}_X \subset \mathbf{P}^1_K$ as an invariant of the surface X itself. Moreover, the definition extends to arbitrary intersections of two quadrics in $\mathbf{P}^4$ that are not cones.

Facts 2.1.

(a) *X is nonsingular if and only if the scheme $\mathscr{S}_X$ is reduced.*

(b) *Let $X \subset \mathbb{P}^4$ be a smooth intersection of two quadrics.*

  (i) *If $\{s_0, \ldots, s_4\} = \mathscr{S}_X(\overline{K})$ then the quadratic forms $Q_{s_0}, \ldots, Q_{s_4}$ are exactly of rank 4.*

  (ii) *The cusps of the cones defined by $Q_{s_i} = 0$, $i = 0, \ldots, 4$, are in general linear position in $\mathbf{P}^4$, i.e. not contained in any hyperplane.*

  (iii) *X is diagonalisable over K if and only if $\mathscr{S}_X$ is split over K.*

Proof. These statements are rather well-known. Proofs may be found, for example, in [Wi]. Specifically, (a) and (b)(i) are implied by [Wi, Proposition 3.26]. Furthermore, (b)(ii) is [Wi, Corollaire 3.29], while (b)(iii) is [Wi, Corollaire 3.30]. ∎

Let X be a degree-4 del Pezzo surface over a field K and assume that there is a K-rational point $s \in \mathscr{S}_X(K)$ as well as that the corresponding degenerate quadric Q_s has a K-rational point, different from the cusp. Then there exist four linearly independent linear forms $l_1, \ldots, l_4$ such that

$$cQ_s = l_1 l_2 - (l_3^2 - D l_4^2)$$

for some constant c (see [VAV, Lemma 2.1]). Furthermore, D is the discriminant of cQ_s, considered as a quadratic form in four variables.

The case most interesting for us is when there are two distinct K-rational points $s_1, s_2 \in \mathscr{S}_X(K)$, and the corresponding degenerate quadrics Q_{s_1}, Q_{s_2} have the same discriminant. Then X may be given by a system of equations

$$l_{11} l_{12} = l_{13}^2 - D l_{14}^2, \tag{2.1}$$

$$l_{21} l_{22} = l_{23}^2 - D l_{24}^2. \tag{2.2}$$

For such surfaces, there is a standard way to write down a Brauer class, which goes back at least to Birch and Swinnerton-Dyer [BSD].

Proposition 2.2. *Let X be the degree-4 del Pezzo surface over a field K given by the equations* (2.1)–(2.2). *Assume that D is a nonsquare in K and set $L := K(\sqrt{D})$. Then:*

(a) *The quaternion algebra* (*see* [Pi, Section 15.1] *for the notation*)
$$\mathscr{A} := (L(X), \tau, l_{11}/l_{21})$$
over the function field $K(X)$ extends to an Azumaya algebra over the whole of X. Here, $\tau \in \mathrm{Gal}(L(X)/K(X))$ denotes the nontrivial element.

(a) *For $K = \mathbb{Q}$, denote by $\alpha \in \mathrm{Br}(X)$ the Brauer class defined by the extension of $\mathscr{A}$. Let ν be any* (*archimedean or nonarchimedean*) *place of $\mathbb{Q}$.*

(i) *Let $x \in X(\mathbb{Q}_\nu)$ and assume that for some $i, j \in \{1, 2\}$, one has $l_{1i}(x), l_{2j}(x) \neq 0$. Denote the corresponding quotient $l_{1i}(x)/l_{2j}(x)$ by q. Then*
$$\mathrm{ev}_{\alpha,\nu}(x) = \begin{cases} 0 & \text{if } (q, D)_\nu = 1, \\ 1/2 & \text{if } (q, D)_\nu = -1, \end{cases}$$
for $(q, D)_\nu$ the Hilbert symbol.

(ii) *If ν is split in L then the local evaluation map $\mathrm{ev}_{\alpha,\nu}$ is constantly zero.*

Proof. (a) This is a consequence of [VAV, Lemma 3.2].

(b)(i) The quotients
$$\frac{l_{11}}{l_{21}}\Big/\frac{l_{11}}{l_{22}} = \frac{l_{23}^2 - Dl_{24}^2}{l_{21}^2}, \qquad \frac{l_{12}}{l_{21}}\Big/\frac{l_{12}}{l_{22}} = \frac{l_{23}^2 - Dl_{24}^2}{l_{21}^2}, \qquad \frac{l_{11}}{l_{21}}\Big/\frac{l_{12}}{l_{21}} = \frac{l_{13}^2 - Dl_{14}^2}{l_{12}^2}$$
are norms of rational functions from $L(X)$. Therefore, they define the trivial element of $H^2(\langle\sigma\rangle, K(X_L)^*) \subseteq \mathrm{Br}(K(X))$, and hence in $\mathrm{Br}(X)$. In particular, the four expressions l_{1i}/l_{2j} define the same Brauer class.

The general description of the evaluation map, given in [Man, Paragraph 45.2], shows that $\mathrm{ev}_{\alpha,\nu}(x)$ is equal to 0 or 1/2 depending on whether q is in the image of the norm map $N_{L_\mathbf{n}/\mathbb{Q}_\nu} \colon L_\mathbf{n}^* \to \mathbb{Q}_\nu^*$ or not, for $\mathbf{n}$ a place of L lying above ν. This is exactly what is tested by the Hilbert symbol $(q, D)_\nu$.

(b)(ii) If ν is split in L then the norm map
$$N_{K(X_{L_\mathbf{n}})/K(X_{\mathbb{Q}_\nu})} \colon K(X_{L_\mathbf{n}})^* \to K(X_{\mathbb{Q}_\nu})^*$$
is surjective. In particular, $l_{11}/l_{21} \in K(X_{\mathbb{Q}_\nu})^*$ is the norm of a certain rational function on $X_{L_\mathbf{n}}$. Therefore, it defines the zero class in $H^2(\langle\sigma\rangle, K(X_{L_\mathbf{n}})^*) \subseteq \mathrm{Br}(K(X_{\mathbb{Q}_\nu}))$, and thus in $\mathrm{Br}(X_{\mathbb{Q}_\nu})$. To complete the argument, we note that every $\mathbb{Q}_\nu$-rational point $x \colon \operatorname{Spec} \mathbb{Q}_\nu \to X$ factors via $X_{\mathbb{Q}_\nu}$. ∎

In the following, we will make heavy use of the two facts below. The first one recalls the explicit description of the situation when the Brauer group of X is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. It is convenient to introduce the following assumption on X defined over a local or global field K.

ASSUMPTION A. In the local field case, assume that $X(K) \neq \emptyset$, and in the global field case that X has an adelic point.

FACT 2.3. *Let X be a degree-4 del Pezzo surface over a local or global field K satisfying Assumption A.*

Then $\mathrm{Br}(X)/\mathrm{Br}(K) \cong (\mathbb{Z}/2\mathbb{Z})^2$ *if and only if $\mathscr{S}_X$ has three distinct points $s_0, s_1, s_2 \in \mathscr{S}_X(K)$ such that all three discriminants $D_{s_0}, D_{s_1}, D_{s_2}$ are nonsquares in K and coincide up to square factors.*

In this case, representatives of the three nontrivial classes may be obtained as follows. Choose a two-element subset $\{s_i, s_j\} \subset \{s_0, s_1, s_2\}$. Write X in the form (2.1)–(2.2) *and take the corresponding Azumaya algebra as described in Proposition* 2.2.

Proof. This is well-known and a proof may be found, for example, in [VAV, Theorem 3.4]. Note that the assumption on X implies that for every closed point $s \in \mathscr{S}_X$, the corresponding rank-4 quadric has a regular point over the residue field of s [VAV, Lemma 5.1]. ∎

FACT 2.4. *Let X be a degree-4 del Pezzo surface over a local or global field K satisfying Assumption A. Assume X to be diagonalisable over K. Let $D_i \in K$ for $0 \le i \le 4$ be the five rank-4 discriminants, and assume that $D_0 = D_1 =: D$.*

- (a) *Let D be a nonsquare in K. Then the Brauer class $\alpha \in \mathrm{Br}(X)$ described in Proposition* 2.2 *is trivial, i.e. $\alpha \in \mathrm{Br}(K)$, if and only if D_2, D_3, and D_4 are all squares in K.*
- (b) *If the conditions in* (a) *hold or all five discriminants D_i are squares in K, then* $\mathrm{Br}(X)/\mathrm{Br}(K) \cong 0$.

Proof. (a) This equivalence statement is established in [VAV, Proposition 3.3]. (b) Fact 2.3 above proves that $\mathrm{Br}(X)/\mathrm{Br}(K)$ is at most of order 2. If it were of order exactly 2 then, by [VAV, Theorem 3.4], the nontrivial class could be obtained as described in Proposition 2.2. In particular, only the case that D is a nonsquare remains to be considered. However, as the other three discriminants are squares, this is exactly the situation in which part (a) proves that the Brauer class is trivial. ∎

REMARK 2.5. Under the assumptions of Fact 2.4, there is an isomorphism

$$\mathrm{Br}(X)/\mathrm{Br}(K) \xleftarrow{\cong} \ker(o\colon (\mathbb{Z}/2\mathbb{Z})^5 \to K^*/(K^*)^2)/T,$$

where $o\colon (a_0, \ldots, a_4) \mapsto (D_0^{a_0} \cdot \ldots \cdot D_4^{a_4} \bmod (K^*)^2)$ and T is generated by the vector $(1, \ldots, 1)$ and the standard vectors e_i for those $i \in \{0, \ldots, 4\}$ for which D_i is a perfect square. Note that $D_0 \cdot \ldots \cdot D_4$ is a perfect square in K (cf. Fact 3.4 below).

Once one has an explicit description of the Brauer classes, one needs criteria to understand whether or not they evaluate constantly at a given place. For this, the following result turns out very useful.

CRITERION 2.6 (A. Várilly-Alvarado and B. Viray). *Let X be the degree-4 del Pezzo surface over $\mathbb{Q}$ given by* (2.1)–(2.2). *Assume that D is a nonsquare and let $\alpha \in \mathrm{Br}(X)$ be as described in Proposition* 2.2. *Then for any place $\nu \neq 2, \infty$ such that the reductions modulo ν of the quadratic forms in* (2.1) *and* (2.2) *both have rank* 4, *the local evaluation map $\mathrm{ev}_{\alpha,\nu}$ is constant.*

Proof. This is [VAV, Proposition 5.4]. ∎

Proof of Theorem 1.1 assuming Theorem 1.4. Theorem 1.4 solves the problem for every subset $S \neq \{\infty\}$. Thus, in order to establish Theorem 1.1, it suffices to verify the assertions made in Example 1.2.

For this, one first checks that $\mathscr{S}_X$ has exactly three $\mathbb{Q}$-rational points, corresponding to the quadratic forms independent of the variable T_2, T_3, and T_4, respectively, and a point of degree 2 that splits over the quadratic field $\mathbb{Q}(\sqrt{6})$. In particular, X is nonsingular.

The discriminants of the three $\mathbb{Q}$-rational quadratic forms of rank 4 are, up to square factors, 1, (-7), and (-7). Therefore, Fact 2.3 shows that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ is at most of order 2. On the other hand, by Proposition 2.2, we have a Brauer class $\alpha \in \mathrm{Br}(X)$ that is given over $\mathbb{Q}(X)$ as the quaternion algebra $(\mathbb{Q}(\sqrt{-7})(X), \tau, \varphi)$ for $\varphi := (T_0 - 4T_1)/T_1$.

Next, we observe that X has no real points such that $x_0 = x_1 = 0$. Moreover, for every real point $x \in X(\mathbb{R})$ such that $x_1 \neq 0$, the equations imply $x_0/x_1 \geq 0$ and $(x_0/x_1 - 4)(x_0/x_1 - 6) \geq 0$, hence

$$x_0/x_1 \in [0,4] \quad \text{or} \quad x_0/x_1 \geq 6.$$

There exist real points of both kinds, for example $(1:1:1:0:\sqrt{2})$ and $(8:1:1:1:1)$. Since $(-7) < 0$, we see that $(q, -7)_\infty$ is the sign of q. Thus, $\mathrm{ev}_{\alpha,\infty}$ distinguishes the two kinds of real points. In particular, $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q})$ is indeed of order 2 and the nontrivial element works at the infinite place.

It remains to show that it does not work at any other place. Criterion 2.6 shows constancy of the evaluation map $\mathrm{ev}_{\alpha,\nu}$ for all finite places $\nu \neq 2, 7$. Furthermore, $\mathrm{ev}_{\alpha,2}$ is constant by Proposition 2.2(b)(ii), as the prime 2 splits in $\mathbb{Q}(\sqrt{-7})$.

Finally, for the prime 7, we argue as follows. Let $x \in X(\mathbb{Q}_7)$ be any 7-adic point on X. Normalise the coordinates $x_0, \ldots, x_4$ so that each is a 7-adic integer and at least one is a unit. If $7 \mid x_0$ and $7 \mid x_1$ then the equations imply that all coordinates must be divisible by 7, a contradiction. Hence, at least one of x_0 and x_1 is a unit. Modulo 7, we have $(\overline{x}_0 - 4\overline{x}_1)(\overline{x}_0 - 6\overline{x}_1) = \overline{x}_0\overline{x}_1$ (since both expressions are equal to $\overline{x}_2^2$), and this equation has the solutions $\overline{x}_0/\overline{x}_1 = 1, 3$ in $\mathbb{Z}/7\mathbb{Z}$. However, the solution $\overline{x}_0/\overline{x}_1 = 3$ is contradictory, as then $\overline{x}_0\overline{x}_1$ would be a nonsquare. Consequently, both x_0 and x_1 must be units and

$$\frac{x_0 - 4x_1}{x_1} \equiv -3 \pmod{7},$$

so $(x_0 - 4x_1)/x_1$ is a square in $\mathbb{Q}_7$. This shows $((x_0 - 4x_1)/x_1, -7)_7 = 1$ and $\mathrm{ev}_{\alpha,7}(x) = 0$. ∎

REMARK 2.7. In Example 1.2, weak approximation is disturbed in a rather astonishing way. The smooth manifold $X(\mathbb{R})$ has two connected components. There are two kinds of real points $x \in X(\mathbb{R})$, those with $x_0/x_1 \in [0, 4]$ and those such that $x_0/x_1 \in [6, \infty]$. However, for every $\mathbb{Q}$-rational point $x \in X(\mathbb{Q})$, one has $x_0/x_1 > 6$.

A naively implemented point search shows that there are exactly 792 $\mathbb{Q}$-rational points of naive height up to 1000 on X. The smallest value of the quotient x_0/x_1 is $319/53 \approx 6.019$.

3. Diagonal degree-4 del Pezzo surfaces. The goal of this section is to collect some facts about diagonal degree-4 del Pezzo surfaces. These will lead us to a proof of Theorem 1.3.

Let X be a diagonal degree-4 del Pezzo surface over a base field K, i.e. one given by equations of the form

$$a_0T_0^2 + \cdots + a_4T_4^2 = 0, \tag{3.1}$$

$$b_0T_0^2 + \cdots + b_4T_4^2 = 0, \tag{3.2}$$

with coefficients in K. Then, for every $(i_0, \ldots, i_4) \in \{0, 1\}^5$, the map

$$(T_0 : \ldots : T_4) \mapsto ((-1)^{i_0}T_0 : \ldots : (-1)^{i_4}T_4)$$

defines a K-automorphism of X. Thus, we explicitly described a subgroup $\mathrm{Aut}'(X) \subseteq \mathrm{Aut}_K(X)$ that is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$.

It is known that the automorphism group of a degree-4 del Pezzo surface over an algebraically closed field is generically isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$, and that there are particular cases where the automorphism group is larger [Do, Theorem 8.6.8].

LEMMA 3.1. *Let X be a diagonal degree-4 del Pezzo surface over a local or global field K satisfying Assumption A. Then the natural operation of $\mathrm{Aut}'(X)$ on $\mathrm{Br}(X)$ induces the trivial operation on $\mathrm{Br}(X)/\mathrm{Br}(K)$.*

Proof. This is trivially true if $\mathrm{Br}(X)/\mathrm{Br}(K) \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$. Otherwise, it follows from the description of the representatives given in Fact 2.3. ∎

Construction 3.2. Let X be a diagonal degree-4 del Pezzo surface over a local or global field K satisfying Assumption A.

By functoriality, the operation of $\mathrm{Aut}'(X)$ on X induces an operation on $\mathrm{Br}(X)$, which is necessarily trivial on $\mathrm{Br}(X)/\mathrm{Br}(K)$. Thus, for every $\alpha \in \mathrm{Br}(X)$, there is a natural homomorphism

$$i_\alpha\colon \mathrm{Aut}'(X) \to \mathrm{Br}(K),$$

given by the condition that $\sigma^*\alpha = \alpha + i_\alpha(\sigma)$ for $\sigma \in \mathrm{Aut}'(X)$. Moreover, i_α depends only on the class of α in $\mathrm{Br}(K)/\mathrm{Br}(K)$.

Definition 3.3. Let $K = \mathbb{Q}$ and assume that the Brauer class $i_\alpha(\sigma)$, for some $\alpha \in \mathrm{Br}(K)/\mathrm{Br}(\mathbb{Q})$ and $\sigma \in \mathrm{Aut}'(X)$, has a nontrivial image in $\mathrm{Br}(K_\nu)$ at the place ν. Then, as

$$\mathrm{ev}_{\alpha,\nu}(\sigma(x)) = \mathrm{ev}_{\sigma^*\alpha,\nu}(x) = \mathrm{ev}_{\alpha,\nu}(x) + i(\sigma)_\nu,$$

the Brauer class certainly works at ν. We say in this situation that σ is a *witness* for the nonconstancy of the local evaluation map at ν.

Fact 3.4. *Let X be a diagonal degree-4 del Pezzo surface over a field K and $D_0, \ldots, D_4$ be the discriminants of the five associated quadratic forms of rank four. Then $D_0 \cdot \ldots \cdot D_4$ is a square in K.*

Proof. This is a direct calculation. ∎

Lemma 3.5. *Let X be a diagonal degree-4 del Pezzo surface over $\mathbb{R}$ that has a real point. Assume that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{R}) \neq 0$. Then $X(\mathbb{R})$ has exactly two connected components. Moreover, there is a $\sigma \in \mathrm{Aut}'(X)$ that interchanges these components.*

Proof. By Fact 3.4, there are three cases. The number of negative discriminants among the five rank-4 discriminants is 0, 2, or 4. Facts 2.4(b) and 2.3 show that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{R}) \neq 0$ only in the last case.

Then the pencil of quadrics in $\mathbf{P}^4$ associated with X contains four rank-4 quadrics of negative discriminant. We may write each of them in the shape

$$-c_0T_{i_0}^2 + c_1T_{i_1}^2 + c_2T_{i_2}^2 + c_3T_{i_3}^2 = 0$$

for $c_0, \ldots, c_3 > 0$, and say that the variable T_{i_0} is *distinguished* by the form considered.

We claim that not all four forms may distinguish the same variable. Indeed, if that were the case then we would also have the form

$$-c_0'T_{i_0}^2 + c_1'T_{i_1}^2 + c_2'T_{i_2}^2 + c_4'T_{i_4}^2 = 0,$$

which shows that the form in the pencil that does not involve T_{i_0} has opposite signs at $T_{i_3}^2$ and $T_{i_4}^2$. The same argument for all combinations of two of the four quadratic forms enforces six opposite signs among the four coefficients of $T_{i_1}^2, \ldots, T_{i_4}^2$, a contradiction.

Thus, X may be given by two equations

$$-c_0T_{i_0}^2 + c_1T_{i_1}^2 + c_2T_{i_2}^2 + c_3T_{i_3}^2 = 0,$$
$$-d_0T_{j_0}^2 + d_1T_{j_1}^2 + d_2T_{j_2}^2 + d_3T_{j_3}^2 = 0,$$

for $c_k, d_k > 0$, $i_0 \neq j_0$, and $\{i_0, \ldots, i_3\} \cup \{j_0, \ldots, j_3\} = \{0, \ldots, 4\}$. These equations imply $x_{i_0} \neq 0$ and $x_{j_0} \neq 0$ for every real point $x \in X(\mathbb{R})$. In particular, $X(\mathbb{R})$ has at least two connected components, given by the two possible signs of x_{i_0}/x_{j_0}. Clearly, these two components are interchanged under the operation of $\mathrm{Aut}'(X)$.

We finally note that a real degree-4 del Pezzo surface cannot have more than two connected components [Silh, Chapter III, Theorem 3.3]. ■

Remark 3.6. The stronger statement that if $X(\mathbb{R})$ splits into two connected components then the operation of $\mathrm{Aut}'(X)$ interchanges them is true as well.

Indeed, by blowing up a real point not lying on any exceptional curve, one obtains a real cubic surface that has two connected components. According to L. Schläfli [Sch, pp. 114 f.], there are exactly five real types of real cubic surfaces, and those correspond in modern language to the four conjugacy classes of order-2 subgroups in $W(E_6)$ together with the trivial group. Only for one of these five cases is the Brauer group nontrivial [Ja, Appendix, Table 2]: it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ then, and that is the single case in which the surface is disconnected. We will not make use of this observation.

Proof of Theorem 1.3. Let X be a diagonalisable degree-4 del Pezzo surface over $\mathbb{Q}$ that has an adelic point and a Brauer class $\alpha \in \mathrm{Br}(X)$ working at the infinite place. We note that since X has an adelic point, it clearly has a real point. The local evaluation $\mathrm{ev}_{\alpha,\infty}(x)$ for $x \in X(\mathbb{R})$ is defined using the restriction homomorphism $x^*\colon \mathrm{Br}(X) \to \mathrm{Br}(\mathbb{R})$, which factors via $\mathrm{Br}(X_{\mathbb{R}})$. Hence, nonconstancy of $\mathrm{ev}_{\alpha,\infty}$ implies that the restriction $\alpha_{\mathbb{R}} \in \mathrm{Br}(X_{\mathbb{R}})/\mathrm{Br}(\mathbb{R})$ is a nonzero class.

In this case, Lemma 3.5 shows that $X(\mathbb{R})$ splits into two connected components. Moreover, there exists an automorphism $\sigma \in \mathrm{Aut}'(X_{\mathbb{R}}) \cong \mathrm{Aut}'(X)$ interchanging these. Since $\mathrm{ev}_{\alpha,\infty}$ is locally constant, this implies that σ witnesses the nonconstancy of the local evaluation map at ∞. In other words, the natural homomorphism $i_\alpha\colon \mathrm{Aut}'(X) \to \mathrm{Br}(\mathbb{Q})$ has in its image a Brauer class $i_\alpha(\sigma) \in \mathrm{Br}(\mathbb{Q})$ with a nonzero component at infinity.

According to global class field theory [Ta, Section 10, Theorem B], $i_\alpha(\sigma)$ necessarily has a nonzero component at another place $\nu \neq \infty$. Consequently, α works at ν too, which implies the claim. ■

4. Surfaces with a Brauer class working at a prescribed set of places. The goal of this section is to prove Theorem 1.4. We distinguish between the cases $\#S > 1$, $\#S = 1$, and $S = \emptyset$. The family below will serve us in all cases.

4.1. A family of degree-4 del Pezzo surfaces. For $D, A_1, A_2, B \in \mathbb{Q}$, let $S := S^{(D;A_1,A_2,B)} \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by

$$-A_1(T_0 - T_1)(T_0 + T_1) = T_3^2 - DT_4^2, \tag{4.1}$$

$$-A_2(T_0 - T_2)(T_0 + T_2) = T_3^2 - B^2DT_4^2. \tag{4.2}$$

THEOREM 4.1. *Let D, A_1, A_2, B be nonzero rational numbers. Then:*

(A) (a) *S is not a cone. The degree-5 scheme $\mathscr{S}_X$ has a point at infinity and four others, which are the roots of a completely reducible polynomial of degree 4 having discriminant*

$$\Delta := A_1^2(A_1 - A_2)^2(A_1B^2 - A_2)^2B^4(B - 1)^2(B + 1)^2/A_2^6B^{12}.$$

(b) *S has the $\mathbb{Q}$-rational point $(1:1:1:0:0) \in X(\mathbb{Q})$.*

(c) *If $\Delta \neq 0$ then the five rank-4 discriminants are, up to factors being perfect squares, given by D, D, $DA_1A_2(A_1 - A_2)(B^2 - 1)$, $A_1A_2(A_1B^2 - A_2)(B^2 - 1)$, and $D(A_1 - A_2)(A_1B^2 - A_2)$.*

(B) (a) *There is a Brauer class $\alpha \in \mathrm{Br}(X)$ extending that of the quaternion algebra $\left(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2}\right)$ over the function field $\mathbb{Q}(X)$.*

(b) $\mathrm{ev}_{\alpha,\nu}(x) = 0$ *for $x = (1:1:1:0:0)$ and all $\nu \in \Omega$.*

(c) *At every $\nu \in \Omega$, the local evaluation map $\mathrm{ev}_{\alpha,\nu}$ is constant if one of the following conditions holds:*

- *$\nu = p$ is a finite place, $p \neq 2$, and p divides neither D, nor A_1, nor A_2, nor B.*
- *$\nu = p$ splits in $\mathbb{Q}(\sqrt{D})$, or $\nu = \infty$ and $D > 0$.*
- *D is square-free, $\nu = p$ is a finite place, $p \mid D$, $p \neq 2$, $\gcd(B, D) = 1$, $\left(\frac{-A_1}{p}\right) = 1$, and $A_1 \equiv A_2 \pmod p$.*

(d) *At a place ν, the local evaluation map $\mathrm{ev}_{\alpha,\nu}$ cannot be constant if $(-A_1, D)_\nu = -1$ or $(-A_2, D)_\nu = -1$.*

Proof. (A)(a) and (A)(c) are standard calculations, while (A)(b) is directly checked. Moreover, (B)(a) is a direct application of Proposition 2.2(a), and (B)(b) follows from the fact that $\frac{x_0+x_1}{x_0+x_2} = 1$ for $x = (1:1:1:0:0)$.

(B)(c) The sufficiency of the first condition is Criterion 2.6, while that of the second was shown in Proposition 2.2(b). In order to establish the sufficiency of the third, we argue as follows.

First of all, the prime p ramifies in $\mathbb{Q}(\sqrt{D})$. A p-adic unit $u \in \mathbb{Q}_p$ is a local norm from $\mathbb{Q}(\sqrt{D})$ if and only if $(u \bmod p) \in \mathbb{F}_p^*$ is a square. Moreover, we note that $\left(\frac{-A_1}{p}\right) = \left(\frac{-A_2}{p}\right) = 1$ implies that each of the four rational functions $\frac{T_0 \pm T_1}{T_0 \pm T_2}$ may be used to evaluate the Brauer class α at the place p.

Let now $x \in X(\mathbb{Q}_p)$ be any p-adic point. Normalise the coordinates $x_0, \ldots, x_4$ so that each is a p-adic integer and at least one is a unit. If $p \mid x_0$ and $p \mid x_1$, or $p \mid x_0$ and $p \mid x_2$, then the equations imply that all coordinates must be divisible by p, a contradiction. Modulo p, we have $\overline{x}_0^2 - \overline{x}_1^2 = \overline{x}_0^2 - \overline{x}_2^2$, hence $\overline{x}_1 = \pm\overline{x}_2$, which implies that one of the four quotients $\frac{x_0 \pm x_1}{x_0 \pm x_2}$ is congruent to 1 modulo p, and therefore a norm.

(B)(d) We note first that X has $X(\mathbb{Q}_\nu)$-rational points such that $x_0 \neq \pm x_1$ and $x_0 \neq -x_2$. Indeed, setting $x_0 := 1$ and choosing x_3 and x_4 sufficiently close to 0 in the ν-adic topology, we see that (4.1) and (4.2) become soluble when viewed as equations in x_1 and x_2, respectively.

Now, assume without loss of generality that $(-A_1, D)_\nu = -1$. Then the automorphism $\sigma\colon (T_0 : \ldots : T_4) \mapsto (T_0 : (-T_1) : T_2 : T_3 : T_4)$ changes the rational function $\frac{T_0+T_1}{T_0+T_2}$ to

$$\frac{T_0 - T_1}{T_0 + T_1} = -\frac{1}{A_1}\,\frac{T_3^2 - DT_4^2}{(T_0 + T_1)^2},$$

which takes only ν-adic nonnorms from $\mathbb{Q}(\sqrt{D})$, since $(-A_1, D)_\nu = -1$. This shows that $i_\alpha(\sigma)$ has a nonzero component at ν, i.e. σ witnesses the nonconstancy of the local evaluation map $\mathrm{ev}_{\alpha,\nu}$. ∎

4.2. More than one place. Let $S \subset \Omega$ consist of at least two places. We write $\{p_1, \ldots, p_r\} = S \setminus \{2, \infty\}$.

To construct a diagonalisable degree-4 del Pezzo surface such that a nontrivial Brauer class works exactly at the places in S, we first choose a square-free integer $D \neq 0$ satisfying the following conditions:

- $D > 0$ if and only if $\infty \notin S$.
- $D \equiv 3 \pmod 4$ when $2 \in S$, and $D \equiv 1 \pmod 8$ when $2 \notin S$.
- D is divisible by $p_1, \ldots, p_r$ and has exactly one further prime divisor, which we call q.

That such a choice of D is possible follows immediately from the fact that there are infinitely many primes in every odd residue class modulo 8.

Now write S as a union $S_1 \cup S_2$ of two not necessarily disjoint subsets of even size. This is possible, because $\#S \geq 2$. In addition, we may put 2 into both subsets in case it occurs as an element of S, and the same for ∞.

Next, we choose primes $A_1 \neq A_2$ not dividing D such that, for $i = 1, 2$,

$$(-A_i, D)_\nu = -1 \;\Leftrightarrow\; \nu \in S_i. \tag{4.3}$$

To see that this is possible, observe first that $(-A_1, D)_\nu = (-A_2, D)_\nu = 1$ for all $\nu \neq 2, \infty$ and all $p_1, \ldots, p_r, q, A_1$, and A_2. The requirement at $\nu = 2$ may be realised by choosing $A_i \equiv 1 \pmod 4$, and the condition at $\nu = \infty$ is fulfilled as the A_i are positive. Furthermore, what we require in (4.3) is

$$\left(\frac{-A_i}{p_j}\right) = \begin{cases} -1 & \text{if } p_j \in S_i, \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$, and $\left(\frac{-A_1}{q}\right) = \left(\frac{-A_2}{q}\right) = 1$. Let us additionally impose the condition that

$$A_2 \equiv A_1 \pmod{q}. \tag{4.4}$$

All these are congruence conditions modulo distinct odd primes. Therefore, the existence of a prime A_1 satisfying (4.3) for all places except possibly A_1 itself, is implied by Dirichlet's Theorem on primes in arithmetic progressions. Moreover, as $\#S_1$ is even, we have $(-A_1, D)_{A_1} = 1$ by the Hilbert reciprocity law [Ne, Chapter VI, Theorem 8.1].

In a completely analogous manner, Dirichlet's Theorem and the Hilbert reciprocity law imply the existence of a prime $A_2 \neq A_1$ fulfilling (4.3) and (4.4).

We may now formulate the main result of this paragraph.

THEOREM 4.2. *Let D, A_1, and A_2 be chosen as above. Then:*

(a) *For every integer $B \geq 2$, the surface $X \subset \mathbf{P}^4_{\mathbb{Q}}$ given by*

$$-A_1(T_0 - T_1)(T_0 + T_1) = T_3^2 - DT_4^2,$$
$$-A_2(T_0 - T_2)(T_0 + T_2) = T_3^2 - B^2DT_4^2$$

is nonsingular and has a $\mathbb{Q}$-rational point.

(b) *There is a Brauer class $\alpha \in \mathrm{Br}(X)$ extending that of the quaternion algebra $\left(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2}\right)$ over the function field $\mathbb{Q}(X)$.*

(c) *The Brauer class α works at every place $\nu \in S$. If B is a prime number that splits in $\mathbb{Q}(\sqrt{D})$ then α does not work at any other place.*

(d) *There are infinitely many prime numbers B splitting in $\mathbb{Q}(\sqrt{D})$ such that $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. (a) follows from Theorem 4.1(A)(a)–(b), and (b) is Theorem 4.1(B)(a).

(c) Our choices of A_1, A_2, and D guarantee that Theorem 4.1(B)(d) applies to every $\nu \in S$. On the other hand, as B is a prime that splits in $\mathbb{Q}(\sqrt{D})$, Theorem 4.1(B)(c) shows constancy of the evaluation map $\mathrm{ev}_{\alpha,\nu}$ at all other places.

(d) In order to exclude the option $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, according to Fact 2.3 we have to choose the parameter B so that none of

$$A_1A_2(A_1-A_2)(B^2-1), \quad DA_1A_2(A_1B^2-A_2)(B^2-1), \quad (A_1-A_2)(A_1B^2-A_2)$$

is a perfect square. By Siegel's Theorem on integral points on elliptic curves [Silv, Theorem IX.4.3], the term in the middle is a square only finitely many times. The other two lead to Pell-like equations whose integral solutions are known to have exponential growth (cf. for example [Ch, Chapter XXXIII, §§15–18]), and hence are much sparser than the set of primes that split in $\mathbb{Q}(\sqrt{D})$. The assertion follows. ∎

4.3. No place

THEOREM 4.3. *Let $X \subset \mathbf{P}^4_{\mathbb{Q}}$ be the surface given by*

$$-(T_0 - T_1)(T_0 + T_1) = T_3^2 - 17T_4^2,$$
$$-103(T_0 - T_2)(T_0 + T_2) = T_3^2 - 68T_4^2.$$

Then X is nonsingular and $X(\mathbb{Q}) \neq \emptyset$. Moreover, $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ *but the nontrivial Brauer class works at no place.*

Proof. The first two assertions follow from Theorem 4.1(A)(b) and 4.1(A)(a). The discriminants of the five rank-4 forms are, up to square factors, 17, 17, 66, 206, and 3399, so that, by Facts 2.4(a) and 2.3, we have $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

Let $\alpha \in \mathrm{Br}(X)$ be nontrivial. By Theorem 4.1(B)(c), the local evaluation map is constant at all places $\nu \neq 2, 17, 103, \infty$. Moreover, it is constant at $\nu = \infty$ as the field $\mathbb{Q}(\sqrt{17})$ is real-quadratic. Constancy at $\nu = 2$ and 103 is clear too, since these primes split in $\mathbb{Q}(\sqrt{17})$. Finally, $\mathrm{ev}_{\alpha,17}$ is constant as $\left(\frac{-1}{17}\right) = 1$ and $103 \equiv 1 \pmod{17}$. ∎

4.4. Exactly one place. The examples here are necessarily a bit different, as the 16 automorphisms must not witness the nonconstancy of the evaluation map. We may nonetheless work with the family from Theorem 4.1.

EXAMPLE 4.4. *Let l be a prime number such that $l \equiv 3 \pmod 4$. Choose a prime $D \equiv 1 \pmod 8$ with $\left(\frac{D}{l}\right) = -1$ and another prime $A > l$ such that $A \equiv 1 \pmod D$ and $(A^2 - 1)(A^2 - l^2)$ is a nonsquare. Then the surface $X \subset \mathbf{P}^4_{\mathbb{Q}}$ given by*

$$-(T_0 - T_1)(T_0 + T_1) = T_3^2 - DT_4^2,$$
$$-A^2(T_0 - T_2)(T_0 + T_2) = T_3^2 - l^2DT_4^2$$

is nonsingular and has a $\mathbb{Q}$-rational point. Moreover, $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ *and the nontrivial class works exactly at the place l.*

Proof. We first note that the restrictions on D and A are easy to fulfil due to Dirichlet's and Siegel's Theorems. Furthermore, the first three assertions follow directly from Theorem 4.1(A)(a)–(b), as well as Facts 2.4(a) and 2.3. The nontrivial Brauer class $\alpha \in \mathrm{Br}(X)$ may be understood as an extension of the quaternion algebra $\left(\mathbb{Q}(\sqrt{D})(X), \tau, \frac{T_0+T_1}{T_0+T_2}\right)$ over $\mathbb{Q}(X)$ to the whole scheme X. Theorem 4.1(B)(c) implies that the local evaluation map is constant at all places $\nu \neq l$.

Nonconstancy of $\mathrm{ev}_{\alpha,l}$. Note that l is an inert prime, since $\left(\frac{D}{l}\right) = -1$. An element $u \in \mathbb{Q}_l^*$ is a local norm from $\mathbb{Q}(\sqrt{D})$ if and only if $\nu_l(u)$ is even.

For $\underline{x} = (1:1:1:0:0)$, we have $\mathrm{ev}_{\alpha,l}(\underline{x}) = 0$ by Theorem 4.1(B)(b). On the other hand, the substitutions $T_0 = lT_0'$, $T_1 = T_1'$, $T_2 = lT_2'$, $T_3 = lT_3'$, and $T_4 = T_4'$ yield a different model X' of X that is given by

$$\begin{aligned} -(lT_0' - T_1')(lT_0' + T_1') &= l^2 T_3'^2 - DT_4'^2, \\ -A^2(T_0' - T_2')(T_0' + T_2') &= T_3'^2 - DT_4'^2. \end{aligned}$$

Moreover,

$$\frac{T_0 + T_1}{T_0 + T_2} = \frac{lT_0' + T_1'}{lT_0' + lT_2'} = \frac{1}{l}\,\frac{lT_0' + T_1'}{T_0' + T_2'}.$$

It suffices to find an $\mathbb{Q}_l$-rational point on X' such that $\frac{lT_0'+T_1'}{T_0'+T_2'}$ is an l-adic unit.

The reduction of X' modulo l is given by

$$T_1'^2 = -\bar{D}T_4'^2, \qquad -\bar{A}^2(T_0' - T_2')(T_0' + T_2') = T_3'^2 - \bar{D}T_4'^2.$$

We observe that the first equation has a nontrivial solution, as $\left(\frac{D}{l}\right) = -1$ and $l \equiv 3 \pmod 4$ together imply that $(-\bar{D}) \in \mathbb{F}_l^*$ is a square. Let $\rho \in \mathbb{F}_l^*$ be one of its square roots.

The $\mathbb{F}_l$-rational point

$$x = \left(\frac{\bar{A}^2 + \bar{D}}{2\bar{A}^2} : \rho : \frac{\bar{A}^2 - \bar{D}}{2\bar{A}^2} : 0 : 1\right) \in X'(\mathbb{F}_l)$$

is nonsingular, as is easily checked using the Jacobian criterion. The quotient $\frac{lT_0'+T_1'}{T_0'+T_2'}$ is an l-adic unit for any l-adic lift of x, as required. The assertion follows. ∎

In order to provide a corresponding example in the case $l \equiv 1 \pmod 4$, we need the following lemma.

Lemma 4.5. *Let $\mathbb{F}_l$ be a finite field of characteristic $\neq 2$. Then there exists $\kappa \in \mathbb{F}_l$ such that $2(1+\kappa^2)$ is a nonsquare in $\mathbb{F}_l$.*

Proof. If $l = 3$ then set $\kappa := 0$. Otherwise, i.e. for $l \geq 5$, let $c \in \mathbb{F}_l$ be any nonsquare. The equation $cT_0^2 = 2(T_1^2 + T_2^2)$ defines a conic over $\mathbb{F}_l$, which has exactly $l+1$ $\mathbb{F}_l$-rational points. Among them, at most four have $x_1 = 0$ or $x_0 = 0$. For the others, $\kappa := x_2/x_1$ fulfils the required condition. ∎

Example 4.6. *Let l be a prime number such that either $l \equiv 1 \pmod 4$ or $l = 2$. If $l = 2$ then choose $B := 2$, otherwise let B be an odd prime number that is split in $\mathbb{Q}(\sqrt{l})$ and such that neither $l(l-1)(B^2-1)$, nor $(lB^2-1)(B^2-1)$, nor $(l-1)(lB^2-1)$ is a perfect square. Then the* *surface* $X \subset \mathbf{P}^4_{\mathbb{Q}}$ *given by*

$$-l(T_0 - T_1)(T_0 + T_1) = T_3^2 - lT_4^2, \tag{4.5}$$

$$-(T_0 - T_2)(T_0 + T_2) = T_3^2 - B^2 l T_4^2 \tag{4.6}$$

is nonsingular and has a $\mathbb{Q}$*-rational point. Moreover,* $\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ *and the nontrivial class works exactly at the place* l*.*

Proof. Once again, the restrictions on B are easy to fulfil due to Dirichlet's and Siegel's Theorems. Furthermore, the first three assertions follow directly from Theorem 4.1(A)(a)–(b) as well as Facts 2.4(a) and 2.3. There is a Brauer class $\alpha \in \mathrm{Br}(X)$, which may be understood as an extension of the quaternion algebra $\big(\mathbb{Q}(\sqrt{l})(X), \tau, \frac{T_0+T_1}{T_0+T_2}\big)$ over $\mathbb{Q}(X)$ to the whole scheme X. Moreover, Theorem 4.1(B)(c) implies that the local evaluation map is constant at all places $\nu \neq 2, l$. Thus, it remains to show that $\mathrm{ev}_{\alpha,l}$ is nonconstant and that $\mathrm{ev}_{\alpha,2}$ is constant in the case $l \equiv 1 \pmod 4$.

Nonconstancy of $\mathrm{ev}_{\alpha,2}$ *for* $l = 2$*.* For $\mathbb{Q}(\sqrt{2})$, the prime 2 is ramified. A 2-adic unit u is a local norm from $\mathbb{Q}(\sqrt{2})$ if and only if $u \equiv \pm 1 \pmod 8$.

For $\underline{x} = (1:1:1:0:0)$, we have $\mathrm{ev}_{\alpha,2}(\underline{x}) = 0$ by Theorem 4.1(B)(b). On the other hand, there is the 2-adic point $x = (1:0:\sqrt{-7}:0:1) \in X(\mathbb{Q}_2)$. Observe that $-7 \equiv 1 \pmod 8$ implies that -7 is a square in $\mathbb{Q}_2$. Moreover, we may choose $\sqrt{-7} \in 5 + 16\mathbb{Z}_2$ since $5^2 \equiv -7 \pmod{32}$. Then $\frac{x_0+x_1}{x_0+x_2} = 1/(1+\sqrt{-7})$, which is in the residue class $\frac{1}{2} \cdot (3 \bmod 8)$. Consequently, $\mathrm{ev}_{\alpha,2}(x) = 1/2$.

Nonconstancy of $\mathrm{ev}_{\alpha,l}$ *for* $l \equiv 1 \pmod 4$*.* For $\mathbb{Q}(\sqrt{l})$, the prime l is ramified. An l-adic unit u is a local norm from $\mathbb{Q}(\sqrt{l})$ if and only if $(u \bmod l) \in \mathbb{F}_l^*$ is a square.

For $\underline{x} = (1:1:1:0:0)$, we have $\mathrm{ev}_{\alpha,l}(\underline{x}) = 0$. On the other hand, the substitution $T_3 = lT_3'$ yields a different model X' of X that is given by

$$(T_0 - T_1)(T_0 + T_1) = T_4^2 - l{T_3'}^2,$$
$$(T_0 - T_2)(T_0 + T_2) = l(B^2 T_4^2 - l{T_3'}^2).$$

The $\mathbb{F}_l$-rational point

$$x = \big((1+\kappa^2):2\sigma:(1+\kappa^2):0:(1-\kappa^2)\big) \in X(\mathbb{F}_l)$$

is nonsingular for any $\kappa \in \mathbb{F}_l$ such that $\kappa^2 \neq -1$.

If, moreover, κ is chosen as in Lemma 4.5 then $\frac{x_0+x_1}{x_0+x_2} = \frac{(1+\kappa)^2}{2(1+\kappa^2)}$, which is a nonsquare. Then, for every l-adic point that lifts x, the local evaluation map has value $1/2$.

Constancy of $\mathrm{ev}_{\alpha,2}$ *for* $l \equiv 1 \pmod 4$*.* If $l \equiv 1 \pmod 8$ then $p = 2$ is split in $\mathbb{Q}(\sqrt{l})$ and there is nothing to prove.

On the other hand, assume that $l \equiv 5 \pmod 8$, in which case 2 is an inert prime. Then $u \in \mathbb{Q}_2^*$ is a local norm from $\mathbb{Q}(\sqrt{l})$ if and only if $\nu_2(u)$

is even. In particular, (-1) and $(-l)$ are local norms, such that any of the quotients $\frac{T_0 \pm T_1}{T_0 \pm T_2}$ may be used to evaluate α at the place 2.

Furthermore, any 2-adic point $x \in X(\mathbb{Q}_2)$ may be represented by coordinates $x_0, \ldots, x_4$ that are 2-adic integers, at least one of which is a unit. It is now a routine matter to determine all quintuples of residues modulo 8 that do not entirely consist of even ones and satisfy the system (4.5)–(4.6) modulo 8. From the list obtained, one readily sees that $\mathrm{ev}_{\alpha,2}(x) = 0$ in each case. ∎

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