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Counting rational points on hypersurfaces and higher order expansions

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A B S T R A C T

We study the number of representations of an integer $n = F(\mathbf{x})$ by a homogeneous form in sufficiently many variables. This is a classical problem in number theory to which the circle method has been successfully applied to give an asymptotic for the number of such representations where the integer vector $\mathbf{x}$ is restricted to a box of side length P for P sufficiently large. In the special case of Waring's problem, Vaughan and Wooley have recently established for the first time a higher order expansion for the corresponding asymptotic formula. Via a different and much more general approach we derive a multi-term asymptotic for this problem for general forms $F(\mathbf{x})$ and give an interpretation for the occurring lower order terms.

As an application we derive higher order expansions for the number of rational points of bounded anticanonical height on the projective hypersurface $F(\mathbf{x}) = 0$ for forms $F(\mathbf{x})$ in sufficiently many variables. The main term of this expansion is the one predicted by Manin's conjecture. Our new result gives some evidence for how the conjecture could be refined to cover lower order terms in the setting of high-dimensional complete intersections in projective space.

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1. Introduction

A fundamental object of interest in the study of the arithmetic of Fano varieties is the density of rational points on it. This is usually measured by the counting function that determines the number of rational points of bounded anticanonical height on the underlying variety. More precisely, if X is a Fano variety and $H(\mathbf{x})$ some anticanonical height function, then one studies the counting function

$$N_U(P) = \sharp\{\mathbf{x} \in U(\mathbb{Q}) : H(\mathbf{x}) \leq P\}, \tag{1.1}$$

where $U \subset X$ is some Zariski-open subset. In [4] Manin conjectured that there is some open subset U such that

$$N_U(P) \sim cP(\log P)^{\mathrm{rank}(\mathrm{Pic}\, X)-1}, \tag{1.2}$$

where rank(Pic X) is the rank of the Picard group of X and c some constant which has received an interpretation by Peyre [10]. In the case of smooth complete intersections in projective space of sufficiently large dimension it turns out that the asymptotic formula already holds for $U = X$, due to work of Birch [1]. Hence one would like to obtain at least in this setting even more precise information on the density of rational points measured by $N_U(P)$. One goal of this paper is to answer this question via giving a higher order expansion for $N_X(P)$ in Theorem 1.2 assuming that the dimension of X is sufficiently large, similar to [1].

The problem of studying $N_X(P)$ is closely related to counting integral points on the affine cone of X via the circle method. Our main tool is a refinement of the major arc analysis, which has been essentially unchanged since Birch's seminal work [1] in 1962. In this paper we present a much more precise and refined analysis of the major arcs which enables us to derive higher order expansions instead of only obtaining a main term.

We now come to the case where $X \subset \mathbb{P}^{s-1}$ is a hypersurface of degree d, given by a homogeneous form $F(x_1, \ldots, x_s) \in \mathbb{Z}[x_1, \ldots, x_s]$ of degree d. For convenience we use in the following the vector notation $\mathbf{x}$ for $(x_1, \ldots, x_s)$. In order to find an asymptotic formula for $N_X(P)$ one needs to count the number of integer solutions to the equation $F(\mathbf{x}) = 0$ in some bounded domain. Technically this is no more difficult than counting integer solutions to $F(\mathbf{x}) = n$ for some integer n. Since this is an interesting question on its own right we consider this slightly more general problem. Let us introduce the counting function $R_\mathcal{B}(P, n)$, which counts the number of solutions $x_i \in \mathbb{Z}$ for $1 \leq i \leq s$ to the equation

$$F(x_1, \ldots, x_s) = n,$$

with $(x_1, \ldots, x_s) \in P\mathcal{B}$ for some box $\mathcal{B} \subset \mathbb{R}^n$ with sides parallel to the coordinate axes and a large real number P. When the box $\mathcal{B}$ is clear from the context we also use the notation $R(P, n)$ for $R_\mathcal{B}(P, n)$. In the situation where the form $F(\mathbf{x})$ is not too singular and the number of variables s is large compared to the degree d, the Hardy–Littlewood circle method provides an asymptotics for the counting function $R_{\mathcal{B}}(P,n)$, as in [1]. If $\dim V^*$ denotes the affine dimension of the singular locus of the form $F(\mathbf{x})=0$, then Birch proves an asymptotic for $R_{\mathcal{B}}(P,n)$ as soon as

$$s-\dim V^* > (d-1)2^d. \tag{1.3}$$

There have been many refinements of the method since then, most of them developed for the homogeneous problem with $n=0$ or a weighted version of the counting function $R(P,n)$. These results include for example improvements in the cubic case due to Heath-Brown [7,5], a new version of the circle method by Heath-Brown in [6], improvements in the quartic case by Browning and Heath-Brown [2] and improvements on the bound (1.3) by Browning and Prendiville [3].

All of these have in common that they produce an asymptotic formula for $R(P,n)$ (or a weighted version of this counting function) of the form

$$R(P,n)=\mathfrak{S}(n)\mathcal{J}(P^{-d}n)P^{s-d}+O(P^{s-d-\delta}), \tag{1.4}$$

for some positive $\delta>0$. The value of δ stays unspecified in most applications of the circle method mentioned above. Here $\mathfrak{S}(n)$ is the singular series and $\mathcal{J}(n)$ the singular integral. It is a natural question to ask to what extent one might be able to improve the error term in the asymptotic expansion (1.4). An inspection of the arguments in Birch's work [1] shows that the minor arc contribution can be forced to be arbitrarily small when the number of variables s is sufficiently large. However, in the classical major arc analysis there seems to be a natural barrier which prevents one from obtaining any better error term than $P^{s-d-1+\eta}$, for some small $\eta>0$. More rigorously, in [9] Loh has studied the error term in Waring's problem. If $R_s(n)$ denotes the number of representations of some natural number n as the sum of s kth powers of positive integers, then provided that s is sufficiently large, the circle method delivers an asymptotic formula of the form

$$R_s(n)\sim\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_s(n)n^{s/k-1}.$$

Loh shows that for $s\geq k+2$ and $k\geq 3$, the error term in this expansion is bounded below by

$$R_s(n)-\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_s(n)n^{s/k-1}=\Omega_-(n^{(s-1)/k-1}).$$

In their recent work [13] Vaughan and Wooley were able to explain this behaviour in establishing second and higher order terms in the asymptotic expansion of $R_s(n)$.

Their arguments are very specific for the situation of Waring's problem, which means in our language a diagonal hypersurface of the form

$$F(\mathbf{x}) = x_1^k + \ldots + x_s^k.$$

It is not clear from their work how to generalise this to more general forms $F(\mathbf{x})$, since they heavily use the additive structure and separation of variables in their proof. Taking a different approach, we establish in this paper an asymptotic expansion for $R(P, n)$ with an arbitrary number of higher order terms given that the form F is not too singular and the number of variables s is sufficiently large.

Suppose we aim to count integer points on the variety $F(\mathbf{x}) = n$ in a dilate of a box $\mathcal{B} = \prod_{i=1}^{s}(a_i, b_i]$. For example, if we have $n = 0$ and $F(\mathbf{x})$ is of degree d we expect to see a main term of the form cP^{s-d}. If we restricted F to one of the faces of the box, for example in setting $x_i = a_i$ or $x_i = b_i$ for some $1 \leq i \leq s$, then we expect that this face contributes a term of order of magnitude P^{s-d-1}. In the standard formulations and applications of the circle method this term is not visible since it is typically contained in the error term. We are going to make these terms visible in our main Theorem 1.1 below. The coefficients of these lower order terms, which to some extent correspond to lower dimensional faces of the box $\mathcal{B}$, are again given by products of generalised singular series and singular integrals. We next describe their shape.

Let $K \geq 1$. In order to label the different lower order contributions, we introduce the index set $\mathcal{I}(K)$, which is defined as follows. Let $\mathcal{I}(K)$ be the set of all tuples $(I_1, I_2, \boldsymbol{\tau})$, where I_1 and I_2 are disjoint subsets of $\{1, \ldots, s\}$ and $\boldsymbol{\tau} \in \mathbb{Z}_{\geq 0}^s$ satisfies $\tau_i = 0$ if $i \notin I_1 \cup I_2$ and for which the condition $|I_1| + |I_2| + |\boldsymbol{\tau}| < K$ holds. Furthermore we let $\mathcal{I} = \cup_{K \geq 1}\mathcal{I}(K)$ be the union over all these sets. Given a tuple $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}$ we will always set $I_3 = \{1, \ldots, s\} \setminus (I_1 \cup I_2)$ and $I_4 = \emptyset$.

In a more general situation, if $\{1, \ldots, s\} = \cup_{i=1}^4 I_i$ is a partition into four arbitrary disjoint index sets, and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^s$, then we define the s-dimensional vector $\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})$ componentwise by

$$\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})_i = \begin{cases} b_i & \text{if } i \in I_1 \\ a_i & \text{if } i \in I_2 \\ x_i & \text{if } i \in I_3 \cup I_4. \end{cases}$$

For a tuple $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}$ we now introduce the integral

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = \int\limits_{\prod_{i \in I_3}[a_i,b_i]} \partial_{\mathbf{x}}^{\boldsymbol{\tau}} e(\gamma F(\mathbf{x}))|_{\mathbf{x}=\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})} \, \mathrm{d}\mathbf{x}_{I_3}.$$

The generalised singular integral $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$ is then given by

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n) = \int\limits_{\mathbb{R}} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) e(-\gamma n) \, \mathrm{d}\gamma,$$

in case this is convergent.

For any $l \geq 0$ let $B_l(x)$ be the lth Bernoulli polynomial and set $\beta_l(x) = B_l(\{x\})$. In order to introduce the generalised singular series, we need to introduce some versions of the usual exponential sums occurring in the circle method. For $1 \leq r < q$ we define

$$S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) = \sum_{0\leq \mathbf{z}<q} e\left(\frac{r}{q}F(\mathbf{z})\right)\left(\prod_{i\in I_1}\frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}\beta_{\tau_i+1}\left(\frac{Pb_i-z_i}{q}\right)\right) \times\left(\prod_{i\in I_2}\frac{(-1)^{\tau_i}}{(\tau_i+1)!}\beta_{\tau_i+1}\left(\frac{Pa_i-z_i}{q}\right)\right).$$

The singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ is then given by

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) := \sum_{q=1}^{\infty}\sum_{\substack{r=1\\(r,q)=1}}^{q} q^{-|I_3|+|\boldsymbol{\tau}|}S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)e\left(-\frac{r}{q}n\right),$$

in case the series is convergent.

Recall that V^* denotes the singular locus of the form $F(\mathbf{x}) = 0$, which is given in affine s-space as the zero set of

$$\frac{\partial F}{\partial x_i}(\mathbf{x}) = 0, \quad 1 \leq i \leq s.$$

We can now state our main theorem.

Theorem 1.1. *Let $d \geq 2$ and $K \geq 1$, and assume that*

$$s - \dim V^* > (d-1)2^{d-1}(2K^2 + 2K - 2). \tag{1.5}$$

Let $\mathcal{B}$ be given by $\mathcal{B} = \prod_{i=1}^{s}(a_i, b_i]$ where $a_i < b_i$ are real numbers for $1 \leq i \leq s$. Then we have the asymptotic expansion

$$R_{\mathcal{B}}(P,n) = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n)P^{s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d} + O\left(P^{s-d-(K-1)-\delta}\right),$$

for some $\delta > 0$. Furthermore, all the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ and singular integrals $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n)$ occurring in the summation are absolutely convergent.

Note that the main term for $I_1 = I_2 = \emptyset$ and $\boldsymbol{\tau} = \mathbf{0}$ is exactly the same as the main term in the asymptotic expansion (1.4), and $\mathfrak{S}_{(\emptyset,\emptyset,\mathbf{0})}(P,n) = \mathfrak{S}(n)$ and $\mathfrak{J}_{(\emptyset,\emptyset,\mathbf{0})}(n) = \mathfrak{J}(n)$ reduce to the classical singular series and singular integral as they appear in (1.4).

The main new ingredient in the proof of Theorem 1.1 is several applications of Euler–MacLaurin's summation formula in the analysis of the major arcs. These replace the rather crude estimates in the traditional treatment as in [1], Lemma 5.1, and are the key to understanding the major arc contribution in more depth.

As pointed out in [13], already in the study of the higher order asymptotic expansion in Waring's problem the singular series occurring in the higher order terms do in general not have an interpretation as an Euler product due to the presence of the Bernoulli polynomials. Hence they are difficult to understand. In section 10 we use the multiplication theorem for Bernoulli polynomials to give some interpretation to the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$, see Lemma 10.2 and Lemma 10.3. In particular, we expect that Lemma 10.3 turns out to be useful in proving that some of the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}$ do not vanish.

In section 9 we rewrite the singular integrals $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n)$ in a different way such that we can give a satisfactory interpretation to them. In particular, the singular integral $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$ can be viewed as some partial multiple derivative of the function in the variables $x_i, i \in I_1 \cup I_2$ describing the volume of the bounded piece of the hypersurface $F(\mathbf{x}) = n$ inside the box $\prod_{i\in I_3}[a_i, b_i]$, at the point $\mathbf{x}_{I_1} = \mathbf{b}_{I_1}$ and $\mathbf{x}_{I_2} = \mathbf{a}_{I_2}$.

For s sufficiently large and the special case where $F(\mathbf{x}) = \sum_{i=1}^s x_i^d$, we note that the conclusion of Theorem 1.1 reduces to the conclusions of Theorem 1.1 and Theorem 1.2 in [13], and generalises Theorem 1.2 in [13] for the case of odd degree d to an arbitrary number of lower order terms.

Alternatively, to study the counting function $R(P,n)$ one could introduce a weighted version of it. If $\omega(\mathbf{x})$ is a smooth and compactly supported weight function and $S_\omega(\alpha) = \sum_{\mathbf{x}\in\mathbb{Z}^s} \omega(P^{-1}\mathbf{x})e(\alpha F(\mathbf{x}))$, then this would be given by

$$R^{(\omega)}(P,n) = \int_0^1 S_\omega(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

Slight modifications of our proof of Theorem 1.1 show that these techniques establish an asymptotic formula of the form

$$R^{(\omega)}(P,n) = \mathfrak{S}(n)\mathcal{J}(P^{-d}n)P^{s-d} + O\left(P^{-s-d-(K-1)-\delta}\right),$$

under the assumption that (1.5) holds. Hence all the lower order terms for $R^{(\omega)}(P,n)$ vanish identically due to the smooth cut-off function $\omega(\mathbf{x})$.

Finally, we apply Theorem 1.1 to give higher order expansions for $N_X(P)$, as defined in equation (1.1). In the notation above set $n = 0$ and let $F(\mathbf{x})$ be as before a homogeneous polynomial of degree d. Then $F(\mathbf{x}) = 0$ defines a hypersurface $X \subset \mathbb{P}^{s-1}$ of degree d. For a rational point $\mathbf{x} \in X(\mathbb{Q})$ given by a representative $\mathbf{x} \in \mathbb{Z}^s$ with coprime coordinates $\gcd(x_1,\dots,x_s) = 1$, we define its naive height as

$$H(\mathbf{x}) = \max_{1\le i\le s} |x_i|.$$

Let $\mathcal{B} = [-1,1]^s$. Via a Möbius inversion one can express the counting function $N_X(P)$ as

$$N_X(P) = \frac{1}{2}\sum_{e=1}^{\infty} \mu(e)\left(R_{\mathcal{B}}(e^{-1}P, 0) - 1\right).$$

Note here that the sum is in fact finite since $R_{\mathcal{B}}(e^{-1}P,0) = 1$ for $e > P$. In comparison to the usual applications of Möbius inversion in this setting, we observe that our generalised singular series still depend on P. Hence we introduce for $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}$ the modified versions

$$\widetilde{\mathfrak{S}}_{(I_1,I_2,\boldsymbol{\tau})}(P) = \frac{1}{2}\sum_{e=1}^{\infty} \mu(e) e^{-(s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(e^{-1}P, 0),$$

which are absolutely convergent by Lemma 6.1. As a consequence of Theorem 1.1 we then obtain the following result.

Theorem 1.2. *Let $d \geq 2$ and $K \geq 1$, and assume that (1.5) holds. Then one has*

$$N_X(P) = \sum_{(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}(K)} \widetilde{\mathfrak{S}}_{(I_1,I_2,\boldsymbol{\tau})}(P) \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(0) P^{s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d} + O\left(P^{s-d-(K-1)-\delta}\right),$$

for some $\delta > 0$.

It is interesting to view Theorem 1.2 in the light of Manin's conjecture (1.2). On the other hand, a conjecture of Sir P. Swinnerton-Dyer [12] predicts that the asymptotic expansion $N_U(P)$ for smooth cubic surfaces consists of a main term of the form $PQ(\log P)$ with a polynomial Q in $\log P$ and a square-root error term. Our result shows that the situation is very different for smooth hypersurfaces (and complete intersections) of large dimension. So far very little is known about the lower order terms. Theorem 1.2 gives some first evidence for what to expect for sufficiently large dimensional hypersurfaces in projective space.

We remark that in the case $n = 0$, which in some sense corresponds to Theorem 1.2, the generalised singular series and singular integrals satisfy some symmetry properties. If $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}$ and $(I_1', I_2', \boldsymbol{\tau}')$ is the dual index tuple given by $I_1' = I_2$ and $I_2' = I_1$ and $\boldsymbol{\tau}' = \boldsymbol{\tau}$, then one has

$$\mathcal{J}_{(I_1',I_2',\boldsymbol{\tau}')}(0) = (-1)^{|\boldsymbol{\tau}|} \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(0),$$

and in the case where P is irrational (or $\tau_i > 0$ for all $i \in I_1 \cup I_2$) one has

$$\mathfrak{S}_{(I_1',I_2',\boldsymbol{\tau}')}(P, 0) = (-1)^{|\boldsymbol{\tau}|} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P, 0).$$

Hence the corresponding terms in the expansions in Theorem 1.2 give exactly the same contribution for $(I_1, I_2, \boldsymbol{\tau})$ and $(I_2, I_1, \boldsymbol{\tau})$.

The structure of this paper is as follows. After introducing some notation in the next section, we formulate a multi-dimensional version of Euler–MacLaurin's summation formula in section 3 which is an immediate consequence of the one-dimensional version. After a treatment of the minor arcs in the following section, we give a refined major arc analysis in section 5 based on the use of Euler–MacLaurin's summation formula. In section 6 and section 7 respectively, we show that the singular integrals and singular series which we introduced, are absolutely convergent. Together with the previous sections we then deduce the two main Theorems 1.1 and 1.2 in section 8. Section 9 and section 10 contain some finer analysis of the singular integrals and singular series, including some interpretations to all of these objects.

2. Notation and preliminaries

As usual, we write $\|\alpha\| = \min_{a\in\mathbb{Z}} |a - \alpha|$ for the minimal distance from a real number α to the next integer. For $x \in \mathbb{R}$ we let $\lfloor x\rfloor$ be the greatest integer which is not larger than x, and set $\{x\} = x - \lfloor x\rfloor$. If $\mathbf{a}$ and $\mathbf{b}$ are real-valued s-dimensional vectors than we write $\mathbf{a} \leq \mathbf{b}$ (or $\mathbf{a} < \mathbf{b}$) if $a_i \leq b_i$ (or $a_i < b_i$) for all $1 \leq i \leq s$.

If $I = \{i_1, \dots, i_l\}$ is a finite index set, then we write $\mathrm{d}\mathbf{x}_I$ for $\mathrm{d}x_{i_1}\,\mathrm{d}x_{i_2}\dots\mathrm{d}x_{i_l}$ and $|I|$ for the cardinality of I.

We will often need mixed partial derivatives of functions in several variables. For a multi-index $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_s)$ of non-negative integers we hence introduce the notation

$$\partial_{\mathbf{x}}^{\boldsymbol{\kappa}} = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_s}}{\partial x_s^{\kappa_s}},$$

for this differential operator. Furthermore, we write $|\boldsymbol{\kappa}| = \sum_{i=1}^{s} \kappa_i$ for the weight of the multi-index $\boldsymbol{\kappa}$.

In Vinogradov's notation all implicit constants may depend in $\mathbf{a}, \mathbf{b}$ and F, and as usual we write $e(x)$ for $e^{2\pi i x}$.

Lemma 2.1. *Let $F(\mathbf{x})$ be a homogeneous polynomial of degree d and $\boldsymbol{\kappa} \in \mathbb{Z}^s$ a tuple of non-negative integers such that $\kappa_j \geq 1$ for at least one index $1 \leq j \leq s$. Then one has*

$$\partial_{\mathbf{x}}^{\boldsymbol{\kappa}} e(\gamma F(\mathbf{x})) = \sum_{a=1}^{|\boldsymbol{\kappa}|} \gamma^a h_a^{(\boldsymbol{\kappa})}(\mathbf{x}) e(\gamma F(\mathbf{x})),$$

where $h_a^{(\boldsymbol{\kappa})}$ are homogeneous polynomials in $\mathbf{x}$, which are identically zero or of degree $ad - |\boldsymbol{\kappa}|$.

Proof. We prove the lemma by induction on $|\boldsymbol{\kappa}|$. First assume that $\kappa_j = 1$ for one $1 \leq j \leq s$ and $\kappa_i = 0$ for $i \neq j$. Then we can directly compute

$$\partial_{\mathbf{x}}^{\boldsymbol{\kappa}} e(\gamma F(\mathbf{x})) = 2\pi i \gamma \partial_{x_j} F(\mathbf{x}) e(\gamma F(\mathbf{x})).$$

This coincides with the assertion of the lemma since $\partial_{x_j} F(\mathbf{x})$ is a homogeneous polynomial of degree $d-1$ or is identically zero.

Next suppose we are given the statement of the lemma for some $\boldsymbol{\kappa}$. Choose one index $1 \leq j \leq s$ and set $\kappa_j' = \kappa_j + 1$ and $\kappa_i' = \kappa_i$ for $i \neq j$. We now aim to prove the lemma for $\boldsymbol{\kappa}'$. For this we note that

$$\partial_{\mathbf{x}}^{\boldsymbol{\kappa}'} e(\gamma F(\mathbf{x})) = \partial_{x_j} (\partial_{\mathbf{x}}^{\boldsymbol{\kappa}} F(\mathbf{x})).$$

By assumption this expression equals

$$\begin{aligned} &\partial_{x_j} \left[\sum_{a=1}^{|\boldsymbol{\kappa}|} \gamma^a h_a^{(\boldsymbol{\kappa})}(\mathbf{x}) e(\gamma F(\mathbf{x})) \right] \\ &\quad = \sum_{a=1}^{|\boldsymbol{\kappa}|} \gamma^a \partial_{x_j} (h_a^{(\boldsymbol{\kappa})}(\mathbf{x})) e(\gamma F(\mathbf{x})) + \sum_{a=1}^{|\boldsymbol{\kappa}|} \gamma^a h_a^{(\boldsymbol{\kappa})}(\mathbf{x}) \partial_{x_j} e(\gamma F(\mathbf{x})) \\ &\quad = \sum_{a=1}^{|\boldsymbol{\kappa}|} \gamma^a (\partial_{x_j} h_a^{(\boldsymbol{\kappa})})(\mathbf{x}) e(\gamma F(\mathbf{x})) + \sum_{a=1}^{|\boldsymbol{\kappa}|} 2\pi i \gamma^{a+1} h_a^{(\boldsymbol{\kappa})}(\mathbf{x}) (\partial_{x_j} F)(\mathbf{x}) e(\gamma F(\mathbf{x})). \end{aligned}$$

We note that the degree of the polynomial $(\partial_{x_j} h_a^{(\boldsymbol{\kappa})})(\mathbf{x})$ is $\deg h_a^{(\boldsymbol{\kappa})} - 1 = da - (\sum_{i=1}^s \kappa_i) - 1 = da - \sum_{i=1}^s \kappa_i'$, if it is non-zero. We next consider the second term in the above expression. We rewrite it as

$$\sum_{a=2}^{|\boldsymbol{\kappa}|+1} \gamma^a h_{a-1}^{(\boldsymbol{\kappa})}(\mathbf{x}) (\partial_{x_j} F)(\mathbf{x}) e(\gamma F(\mathbf{x})).$$

For some $2 \leq a \leq |\boldsymbol{\kappa}'|$ we again note that the degree of the homogeneous polynomial

$$h_{a-1}^{(\boldsymbol{\kappa})}(\mathbf{x}) (\partial_{x_j} F)(\mathbf{x})$$

is equal to $d(a-1) - \sum_{i=1}^s \kappa_i + d - 1 = da - \sum_{i=1}^s \kappa_i'$. This completes the proof of the lemma. □

3. Euler–MacLaurin summation formula

We recall the definition of Bernoulli polynomials. The sequence of Bernoulli numbers B_κ for $\kappa \geq 0$ can be defined by setting $B_0 = 1$ and

$$B_\kappa = - \sum_{j=0}^{\kappa-1} \binom{\kappa}{j} \frac{B_j}{\kappa - j + 1},$$

for $\kappa \geq 1$. Then the Bernoulli polynomials $B_\kappa(x)$ are given for $\kappa \geq 0$ by the formula

$$B_\kappa(x) = \sum_{j=0}^{\kappa} \binom{\kappa}{j} B_{\kappa-j} x^j.$$

In the following we use the periodic Bernoulli polynomials which are defined as $\beta_\kappa(x) = B_\kappa(\{x\})$ for $\kappa \geq 0$.

In our major arc analysis we need a higher dimensional version of the Euler–MacLaurin summation formula which we obtain in successively applying the one-dimensional version, which can for example be found in Lemma 4.1 in [13].

In the work of Vaughan and Wooley [13] it is sufficient to use this version of Euler–MacLaurin's summation formula since the diagonal structure of the form underlying Waring's problem ensures that the exponential sum on the major arcs factorises into one-dimensional sums. We next state a version of Euler–MacLaurin's summation formula which applies to higher dimensional functions. Since we are only going to apply the Lemma to rather easy and well-behaved functions we do not aim for the greatest generality in the assumptions under which this higher dimensional version of Euler–MacLaurin's summation formula holds.

Lemma 3.1. *Assume that $s \geq 1$ and $a_i < b_i$ are real numbers for $1 \leq i \leq s$ and let K_i be positive integers for $1 \leq i \leq s$. Assume that $g(\mathbf{x})$ has continuous mixed partial derivatives of total order up to $\sum_{i=1}^s K_i$ on the cube $\prod_{i=1}^s [a_i, b_i]$. Then*

$$\begin{aligned}\sum_{\mathbf{a}<\mathbf{x}\leq\mathbf{b}} g(\mathbf{x}) = \sum_{\cup_{i=1}^4 I_i = \{1,\dots,s\}} &\left(\prod_{i\in I_4} \frac{(-1)^{K_i+1}}{K_i!}\right) \int_{\prod_{i\in I_3\cup I_4}[a_i,b_i]} \left(\prod_{i\in I_4} \beta_{K_i}(x_i)\right) \\ &\times \left[\prod_{i\in I_1}\left(\sum_{\kappa_i=1}^{K_i} \frac{(-1)^{\kappa_i}}{\kappa_i!}\beta_{\kappa_i}(b_i)\left(\frac{\partial}{\partial x_i}\right)^{(\kappa_i-1)}\right)\right. \\ &\times \prod_{i\in I_2}\left(\sum_{\kappa_i=1}^{K_i} \frac{(-1)^{\kappa_i+1}}{\kappa_i!}\beta_{\kappa_i}(a_i)\left(\frac{\partial}{\partial x_i}\right)^{(\kappa_i-1)}\right) \\ &\times \left.\prod_{i\in I_4}\left(\frac{\partial}{\partial x_i}\right)^{K_i} g(\mathbf{x})\right]_{\mathbf{x}=\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})} \mathrm{d}\mathbf{x}_{I_3}\,\mathrm{d}\mathbf{x}_{I_4}\end{aligned}$$

The summation over $\cup_{i=1}^4 I_i$ is over all possible partitions of $\{1,\dots,s\}$ into four disjoint index sets I_i, $1 \leq i \leq 4$.

4. Minor arc estimates

In this section we assume that $F(\mathbf{x})$ is a polynomial in $\mathbf{x}$ of degree d, not necessarily homogeneous. We define the exponential sum

$$S(\alpha) = \sum_{\mathbf{x} \in P\mathcal{B}} e(\alpha F(\mathbf{x})).$$

By orthogonality we can express the counting function $R_{\mathcal{B}}(P, n)$ as

$$R_{\mathcal{B}}(P, n) = \int_0^1 S(\alpha) e(-\alpha n)\, \mathrm{d}\alpha.$$

In order to apply the circle method to this counting function we need to dissect the unit interval $[0, 1]$ into major and minor arcs. This is done in a traditional way following for example the work of Birch [1].

Let $0 < \eta < 1/2$ be some small parameter to be chosen later. For coprime integers r, q we define the major arc

$$\mathfrak{M}'_{r,q}(\eta) = \{\alpha \in [0, 1) : |q\alpha - r| \leq qP^{-d+\eta}\},$$

and the major arcs $\mathfrak{M}'(\eta)$ as the union

$$\mathfrak{M}'(\eta) = \bigcup_{1 \leq q \leq P^\eta} \bigcup_{\substack{1 \leq r \leq q \\ (r,q)=1}} \mathfrak{M}'_{r,q}(\eta). \tag{4.1}$$

Similarly, we define slightly smaller major arcs by

$$\mathfrak{M}_{r,q}(\eta) = \{\alpha \in [0, 1) : |q\alpha - r| \leq P^{-d+\eta}\},$$

for coprime integers r, q and

$$\mathfrak{M}(\eta) = \bigcup_{1 \leq q \leq P^\eta} \bigcup_{\substack{1 \leq r \leq q \\ (r,q)=1}} \mathfrak{M}_{r,q}(\eta).$$

If the parameter η is clear from the context we sometimes use the shorter notation $\mathfrak{M}'_{r,q}$ for $\mathfrak{M}'_{r,q}(\eta)$ and for the major arcs $\mathfrak{M}_{r,q}$ similarly.

Furthermore, we define the minor arcs as the complement of the smaller version of the major arcs, i.e. $\mathfrak{m}(\eta) = [0, 1) \setminus \mathfrak{M}(\eta)$.

We recall Lemma 4.1 from Birch's paper [1] which asserts that the major arcs $\mathfrak{M}(\eta)$ are disjoint in case that η is sufficiently small. For convenience we state it here for the slightly larger major arcs $\mathfrak{M}'(\eta)$ since we need it for them in the later analysis of the major arcs. We do not give the proof since it is standard and identical to Lemma 4.1 in [1].

Lemma 4.1. *Assume that $3\eta < d$. Then the union defining the major arcs $\mathfrak{M}'(\eta)$ as in (4.1) is disjoint for P sufficiently large.*

It is convenient to introduce more generally for any function $\omega : \mathbb{R}^s \to \mathbb{R}$ of compact support the exponential sum

$$S_\omega(\alpha, P) = \sum_{\mathbf{x} \in \mathbb{Z}^s} \omega\left(\frac{\mathbf{x}}{P}\right) e(\alpha F(\mathbf{x})).$$

Hence if we take ω to be the indicator function of the box $\mathcal{B}$, then we recover $S(\alpha)$. For a certain class of weight functions ω we need a form of Weyl's inequality for $S(\alpha)$. For this we use a slight modification of recent work of Browning and Prendiville [3].

We first recall some conventions from [3]. We say that a pair $\alpha \in \mathbb{R}/\mathbb{Z}$ and $q \in \mathbb{N}$ is primitive, if there is some $r \in \mathbb{Z}$ with $(r, q) = 1$ and $\|q\alpha\| = |q\alpha - r|$. For some positive constants c, C and a positive integer m we introduce the class of smooth weight functions $\mathcal{S}(c, C, m)$ as the set of smooth compactly supported functions $\omega : \mathbb{R}^s \to [0, \infty)$ such that $\mathrm{supp}(\omega) \subset [-c, c]^s$ and $\|\partial_{\mathbf{x}}^{\boldsymbol{\kappa}} \omega(\mathbf{x})\|_\infty \leq C$ for all multi-indices $\boldsymbol{\kappa} \in (\mathbb{N} \cup \{0\})^s$ with $|\boldsymbol{\kappa}| \leq m$.

If $F(\mathbf{x})$ is a polynomial in $\mathbf{x}$, then we write $F^{[d]}(\mathbf{x})$ for its homogeneous part of degree d. Furthermore, we write $\mathrm{Sing}(F^{[d]})$ for the singular locus of the affine variety given by $F^{[d]}(\mathbf{x}) = 0$, which is the zero locus of the system of equations

$$\frac{\partial F^{[d]}}{\partial x_i}(\mathbf{x}) = 0, \quad 1 \leq i \leq s.$$

Lemma 4.2. *Assume that α and q are primitive. Let $\omega \in \mathcal{S}(c, C, m)$ and χ the indicator function of some box in $\mathbb{R}^s$, which is contained in $[-c, c]^s$. Assume that $m \geq s$. Then one has*

$$\left|\frac{S_{\omega\chi}(\alpha, P)}{P^s}\right|^{2^{d-1}} \ll_{c,C,m} (\log P)^s \left(P^{1-d} + \|q\alpha\| + qP^{-d} + \min\left\{q^{-1}, \frac{1}{\|q\alpha\| P^d}\right\}\right)^{\frac{s-\sigma}{d-1}},$$

where $\sigma = \dim \mathrm{Sing}(F^{[d]})$ is the dimension of the singular locus of the affine variety given by $F^{[d]}(\mathbf{x}) = 0$.

This is a consequence of Lemma 3.3 in [3]. Note that we do not need an explicit dependence on the bound for $S_\omega(\alpha, P)$ depending on the coefficients of $F(\mathbf{x})$ which can be found in the formulation of Lemma 3.3 in [3].

Proof. The proof is identical to the proof of Lemma 3.3 in [3], with the function $\phi(\mathbf{x}) = e(\alpha F_{\mathbf{h}_1,\ldots,\mathbf{h}_{d-1}}(\mathbf{x}))$ in the notation of [3] replaced by the product $\phi(\mathbf{x}) = \chi_{(\mathbf{h}_1,\ldots,\mathbf{h}_{d-1})/P}(\mathbf{x}/P) e(\alpha F_{\mathbf{h}_1,\ldots,\mathbf{h}_{d-1}}(\mathbf{x}))$. For this we note that $\chi_{(\mathbf{h}_1,\ldots,\mathbf{h}_{d-1})/P}(\mathbf{x}/P)$ is again the indicator function of a box, since the intersection of two boxes in $\mathbb{R}^s$ is again a box. □

As a first application of Weyl's Lemma 4.2 we provide an upper bound for the minor arc contribution to $R_{\mathcal{B}}(P,n)$. In the following we assume that $F(\mathbf{x})$ is homogeneous and set $\sigma = \dim \mathrm{Sing}(F)$ as in Lemma 4.2.

Lemma 4.3. *Let $0 < \theta_0 < (1/2)d$ and assume that $s - \sigma > (d-1)2^d$. Then one has*

$$\int_{\alpha \notin \mathfrak{M}(\theta_0)} |S(\alpha)| \, \mathrm{d}\alpha = O_\delta \left(P^{s-d+\delta-\theta_0\left(2^{-d+1}\frac{s-\sigma}{d-1}-2\right)} \right),$$

for any $\delta > 0$.

Proof. Let $Q \geq 1$. Assume that $\alpha \in \mathfrak{m}(\theta)$ for some $0 < \theta \leq (1/2)d$. Then there is some $q \leq Q$ such that α, q is primitive and $\|\alpha q\| \leq Q^{-1}$. Furthermore one has $q > P^\theta$ or $\|\alpha q\| > P^{-d+\theta}$, since otherwise α would be contained in the major arcs $\mathfrak{M}(\theta)$. We now apply the Weyl bound in Lemma 4.2 to the exponential sum $S(\alpha)$, where we set χ the characteristic function of the box $\mathcal{B}$ and ω a smooth function such that $\omega \equiv 1$ on the box $\mathcal{B}$. Then Lemma 4.2 delivers the bound

$$\left| \frac{S(\alpha)}{P^s} \right|^{2^{d-1}} \ll (\log P)^s \left(P^{1-d} + \frac{1}{Q} + QP^{-d} + \min\left\{ q^{-1}, \frac{1}{\|q\alpha\| P^d} \right\} \right)^{\frac{s-\sigma}{d-1}}.$$

Note that $\min\left\{ q^{-1}, \frac{1}{\|q\alpha\| P^d} \right\} \leq P^{-\theta}$ and set $Q = P^\theta$. Then we obtain

$$\left| \frac{S(\alpha)}{P^s} \right|^{2^{d-1}} \ll (\log P)^s \left(P^{1-d} + P^{-\theta} + P^{\theta-d} \right)^{\frac{s-\sigma}{d-1}}.$$

Note that our restriction $0 < \theta \leq (1/2)d$ implies that the second term in that bound dominates the expression, i.e.

$$\left| \frac{S(\alpha)}{P^s} \right|^{2^{d-1}} \ll (\log P)^s (P^{-\theta})^{\frac{s-\sigma}{d-1}}. \tag{4.2}$$

Now we define a sequence

$$\theta_T > \theta_{T-1} > \ldots > \theta_1 > \theta_0 > 0,$$

such that $\theta_T = (1/2)d$ and $|\theta_{t+1} - \theta_t| \leq \delta$ for all $1 \leq t < T$. We can do this with at most $T \ll_d \delta^{-1}$ points. Note that by Dirichlet's approximation theorem we have $\mathfrak{M}(\theta_T) = [0,1)$. We now estimate the contribution of

$$\int_{\alpha \in \mathfrak{M}(\theta_{t+1}) \setminus \mathfrak{M}(\theta_t)} |S(\alpha)| \, \mathrm{d}\alpha, \tag{4.3}$$

for all $0 \le t < T$. By the Weyl bound (4.2) we obtain

$$\int_{\alpha \in \mathfrak{M}(\theta_{t+1}) \setminus \mathfrak{M}(\theta_t)} |S(\alpha)| \, \mathrm{d}\alpha \ll \text{meas}\,(\mathfrak{M}(\theta_{t+1}))\, P^{s+\delta} P^{-2^{-d+1}\theta_t \frac{s-\sigma}{d-1}}.$$

Note that the major arcs $\mathfrak{M}(\theta_{t+1})$ might not be disjoint, but we can still bound their measure above by

$$\text{meas}\,(\mathfrak{M}(\theta_{t+1})) \ll \sum_{q \le P^{\theta_{t+1}}} \sum_{r=1}^{q} q^{-1} P^{-d+\theta_{t+1}} \ll P^{2\theta_{t+1}-d}.$$

Hence we may bound the contribution of (4.3) by

$$\ll P^{2\theta_{t+1}-d+s+\delta-\theta_t 2^{-d+1}\frac{s-\sigma}{d-1}} \ll P^{s-d+3\delta-\theta_t(2^{-d+1}\frac{s-\sigma}{d-1}-2)}.$$

Hence we can bound the complete minor arc contribution by

$$\int_{\alpha \notin \mathfrak{M}(\theta_0)} |S(\alpha)| \, \mathrm{d}\alpha \ll \sum_{t=0}^{T-1} \int_{\alpha \in \mathfrak{M}(\theta_{t+1}) \setminus \mathfrak{M}(\theta_t)} |S(\alpha)| \, \mathrm{d}\alpha \ll P^{s-d+4\delta-\theta_0(2^{-d+1}\frac{s-\sigma}{d-1}-2)},$$

which completes the proof of the lemma. □

5. Major arc analysis

The main goal of this section is to replace the usual major arc approximation as in [1, Lemma 5.1] by a much finer approximation using the higher dimensional version of Euler–MacLaurin's summation formula in Lemma 3.1. For this recall the notation of the singular series and singular integrals as in the introduction as well as the integrals $J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)$ and the exponential sums $S_{(I_1,I_2,\boldsymbol{\tau})}(P; r, q)$. In addition, we define the function $f(\gamma, \mathbf{x})$ by

$$f(\gamma, \mathbf{x}) = e(\gamma F(\mathbf{x})),$$

and write $f^{(\boldsymbol{\kappa})}(\gamma, \mathbf{x}) := \partial_{\mathbf{x}}^{\boldsymbol{\kappa}} f(\gamma, \mathbf{x})$.

We are now in a position to state our first major arc approximation to the exponential sum $S(\alpha)$.

Lemma 5.1. *Assume that $\alpha \in \mathfrak{M}'_{r,q}$ for some $q \le P^{\eta}$, and write $\alpha = \frac{r}{q} + \gamma$ with some $|\gamma| \le P^{-d+\eta}$. Let $K \ge 1$ be an integer. Then we have*

$$S(\alpha)e(-\alpha n)$$
$$= \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} q^{-|I_3|+|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) e\left(-\frac{r}{q}n\right) P^{|I_3|-|\boldsymbol{\tau}|} J_{(I_1,I_2,\boldsymbol{\tau})}(P^d\gamma)e(-\gamma n)$$
$$+ O(P^{s-K+2K\eta}).$$

Note that the term for $I_1 = I_2 = \emptyset$ and $\boldsymbol{\tau} = 0$ corresponds to the usual approximation on the major arcs as in [1, Lemma 5.1]. All the other terms will contribute to lower order terms.

Proof. We start by writing the exponential sum $S(\alpha)$ as

$$S(\alpha) = \sum_{0\leq \mathbf{z}<q} e\left(\frac{r}{q}F(\mathbf{z})\right) \sum_{\mathbf{z}+q\mathbf{y}\in P\mathcal{B}} e(\gamma F(\mathbf{z}+q\mathbf{y})), \tag{5.1}$$

and consider the inner sum for a fixed vector $\mathbf{z}$. Let $\tilde{a}_i, \tilde{b}_i$ for $1 \leq i \leq s$ be defined by

$$\prod_{i=1}^s (\tilde{a}_i, \tilde{b}_i] = \prod_{i=1}^s \left(\frac{Pa_i - z_i}{q}, \frac{Pb_i - z_i}{q}\right].$$

Let $g(\mathbf{y}) = f(\gamma, \mathbf{z}+q\mathbf{y})$ and note that

$$\partial_{\mathbf{y}}^{\boldsymbol{\kappa}} g(\mathbf{y}) = q^{|\boldsymbol{\kappa}|} f^{(\boldsymbol{\kappa})}(\gamma, \mathbf{z}+q\mathbf{y}), \tag{5.2}$$

for every multi-index $\boldsymbol{\kappa} \in \mathbb{Z}_{\geq 0}^s$.

Now choose some fixed $K \in \mathbb{N}$ and let $\widetilde{\mathcal{I}}(K)$ be the set of tuples $(I_1, I_2, I_4, \boldsymbol{\tau})$ with the following properties. For $i = 1, 2, 4$ one has $I_i \subset \{1, \dots, s\}$ and the index sets I_i are pairwise disjoint. Furthermore $\boldsymbol{\tau} \in \mathbb{Z}_{\geq 0}^s$ satisfies $\tau_i = 0$ if $i \notin I_1 \cup I_2 \cup I_4$, and $0 \leq \tau_i \leq K-1$ for $i \in I_1 \cup I_2$ and $\tau_i = K$ for $i \in I_4$. Similarly as before we set $I_3 = \{1, \dots, s\} \setminus (I_1 \cup I_2 \cup I_4)$ for such a tuple in $\widetilde{\mathcal{I}}(K)$. Now we apply Lemma 3.1 to the sum

$$\Sigma(\mathbf{z}) = \sum_{\mathbf{z}+q\mathbf{y}\in P\mathcal{B}} e(\gamma F(\mathbf{z}+q\mathbf{y})), \tag{5.3}$$

with the parameters $K_i = K$ for all $1 \leq i \leq s$ and to the box $\prod_{i=1}^s (\tilde{a}_i, \tilde{b}_i]$. We obtain

$$\Sigma(\mathbf{z}) = \sum_{(I_1,I_2,I_4,\boldsymbol{\tau})\in\widetilde{\mathcal{I}}(K)} \Sigma(\mathbf{z}; I_1, I_2, I_4, \boldsymbol{\tau}), \tag{5.4}$$

with

$$\Sigma(\mathbf{z}; I_1, I_2, I_4, \boldsymbol{\tau}) = \left(\prod_{i\in I_4} \frac{(-1)^{K+1}}{K!}\right)\left(\prod_{i\in I_1} \frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{b}_i)\right)$$
$$\times \left(\prod_{i\in I_2} \frac{(-1)^{\tau_i}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{a}_i)\right)$$
$$\times \int\limits_{\prod_{i\in I_3\cup I_4}[\tilde{a}_i,\tilde{b}_i]} \left(\prod_{i\in I_4}\beta_K(x_i)\right)\partial_{\mathbf{x}}^{\boldsymbol{\tau}} g(\mathbf{x})|_{\mathbf{x}=\sigma_{\tilde{\mathbf{a}},\tilde{\mathbf{b}}}(\mathbf{x})}\, \mathrm{d}\mathbf{x}_{I_3}\, \mathrm{d}\mathbf{x}_{I_4}.$$

First we estimate the contribution of $\Sigma(\mathbf{z}; I_1, I_2, I_4, \boldsymbol{\tau})$ in the case $|I_1| + |I_2| + |\boldsymbol{\tau}| \geq K$. We apply Lemma 2.1 and obtain

$$f^{(\boldsymbol{\tau})}(\gamma, \mathbf{x}) = \sum_{l=1}^{|\boldsymbol{\tau}|} \gamma^l h_l^{(\boldsymbol{\tau})}(\mathbf{x}) e(\gamma F(\mathbf{x})), \tag{5.5}$$

with $h_l^{(\boldsymbol{\tau})}(\mathbf{x})$ some homogeneous polynomials in $\mathbf{x}$ which are either zero or of degree $ld - |\boldsymbol{\tau}|$. Hence we have

$$\partial_{\mathbf{y}}^{(\boldsymbol{\tau})} g(\mathbf{y}) = q^{|\boldsymbol{\tau}|} \sum_{l=1}^{|\boldsymbol{\tau}|} \gamma^l h_l^{(\boldsymbol{\tau})}(\mathbf{z}+q\mathbf{y}) e(\gamma F(\mathbf{z}+q\mathbf{y})).$$

For a point $\mathbf{y}$ lying in the box $\mathbf{y} \in \prod_{i=1}^s [\tilde{a}_i, \tilde{b}_i]$ we can now estimate

$$|\partial_{\mathbf{y}}^{(\boldsymbol{\tau})} g(\mathbf{y})| \ll q^{|\boldsymbol{\tau}|} \sum_{l=1}^{|\boldsymbol{\tau}|} |\gamma|^l P^{\deg(h_l^{(\boldsymbol{\tau})})} \ll q^{|\boldsymbol{\tau}|} \sum_{l=1}^{|\boldsymbol{\tau}|} |\gamma|^l P^{ld-|\boldsymbol{\tau}|}.$$

Note that all the periodic Bernoulli polynomials $\beta_\tau(x)$ are bounded. Hence we can now estimate $\Sigma(\mathbf{z}; I_1, I_2, I_4, \boldsymbol{\tau})$ by

$$\Sigma(\mathbf{z}; I_1, I_2, I_4; \boldsymbol{\tau}) \ll \int\limits_{\prod_{i\in I_3\cup I_4}[\tilde{a}_i,\tilde{b}_i]} q^{|\boldsymbol{\tau}|} \sum_{l=1}^{|\boldsymbol{\tau}|} |\gamma|^l P^{ld-|\boldsymbol{\tau}|}\, \mathrm{d}\mathbf{x}_{I_3}\, \mathrm{d}\mathbf{x}_{I_4}.$$

Since the volume of $\prod_{i\in I_3\cup I_4}[\tilde{a}_i,\tilde{b}_i]$ is bounded by $\left(\frac{P}{q}\right)^{|I_3|+|I_4|}$ we obtain the upper bound

$$\Sigma(\mathbf{z}; I_1, I_2, I_4; \boldsymbol{\tau}) \ll \left(\frac{P}{q}\right)^{|I_3|+|I_4|} q^{|\boldsymbol{\tau}|} \sum_{l=1}^{|\boldsymbol{\tau}|} |\gamma|^l P^{ld-|\boldsymbol{\tau}|}$$
$$\ll P^{|I_3|+|I_4|} q^{|\boldsymbol{\tau}|-|I_3|-|I_4|} P^{|\boldsymbol{\tau}|\eta-|\boldsymbol{\tau}|}.$$

We estimate this further as

$$\Sigma(\mathbf{z}; I_1, I_2, I_4; \boldsymbol{\tau}) \ll q^{-s} P^{s-K+2\eta K}, \tag{5.6}$$

using $\eta < 1/2$.

We combine this information with equations (5.1), (5.3), (5.4), (5.6), and see that

$$S(\alpha) = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \sum_{0\leq \mathbf{z}<q} e\left(\frac{r}{q}F(\mathbf{z})\right) \Sigma(\mathbf{z}; I_1, I_2, \emptyset, \boldsymbol{\tau}) + O(P^{s-K+2\eta K}). \tag{5.7}$$

Note for this that $|\boldsymbol{\tau}| \geq K$ as soon as $I_4 \neq \emptyset$.

Next we consider for a fixed tuple $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}(K)$ the integral

$$J' = \int_{\prod_{i\in I_3}[\tilde{a}_i,\tilde{b}_i]} \partial_{\mathbf{y}}^{\boldsymbol{\tau}} g(\mathbf{y})|_{\mathbf{y}=\sigma_{\tilde{\mathbf{a}},\tilde{\mathbf{b}}}}(\mathbf{y})\, \mathrm{d}\mathbf{y}_{I_3}.$$

We recall the relation (5.2) and perform the variable substitution $x_i = qy_i + z_i$ for $i \in I_3$. This leads us to

$$J' = q^{-|I_3|+|\boldsymbol{\tau}|} \int_{\prod_{i\in I_3}[Pa_i,Pb_i]} f^{(\boldsymbol{\tau})}(\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x}))\, \mathrm{d}\mathbf{x}_{I_3}.$$

Note that J' is now independent of $\mathbf{z}$. We further rewrite J' via substituting $Px_i' = x_i$ for $i \in I_3$, and obtain

$$J' = q^{-|I_3|+|\boldsymbol{\tau}|} P^{|I_3|} \int_{\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(P\mathbf{x}))\, \mathrm{d}\mathbf{x}_{I_3}.$$

We recall equation (5.5) and observe that

$$\begin{aligned} f^{(\boldsymbol{\tau})}(\gamma, P\mathbf{x}) &= \sum_{l=1}^{|\boldsymbol{\tau}|} \gamma^l h_l^{(\boldsymbol{\tau})}(P\mathbf{x}) e(\gamma F(P\mathbf{x})) \\ &= P^{-|\boldsymbol{\tau}|} f^{(\boldsymbol{\tau})}(P^d\gamma, \mathbf{x}). \end{aligned}$$

Hence we can again reformulate J' as

$$\begin{aligned} J' &= q^{-|I_3|+|\boldsymbol{\tau}|} P^{|I_3|-|\boldsymbol{\tau}|} \int_{\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(P^d\gamma, \sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))\, \mathrm{d}\mathbf{x}_{I_3} \\ &= q^{-|I_3|+|\boldsymbol{\tau}|} P^{|I_3|-|\boldsymbol{\tau}|} J_{(I_1,I_2,\boldsymbol{\tau})}(P^d\gamma). \end{aligned}$$

We conclude that

$$\Sigma(\mathbf{z}; I_1, I_2, \emptyset, \boldsymbol{\tau}) = \left(\prod_{i\in I_1} \frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{b}_i)\right)\left(\prod_{i\in I_2} \frac{(-1)^{\tau_i}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{a}_i)\right) J'$$
$$= \left(\prod_{i\in I_1} \frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{b}_i)\right)\left(\prod_{i\in I_2} \frac{(-1)^{\tau_i}}{(\tau_i+1)!}\beta_{\tau_i+1}(\tilde{a}_i)\right)$$
$$\times q^{-|I_3|+|\boldsymbol{\tau}|} P^{|I_3|-|\boldsymbol{\tau}|} J_{(I_1,I_2,\boldsymbol{\tau})}(P^d\gamma).$$

The Lemma now follows from inserting this into equation (5.7). □

We now use this Lemma to evaluate the major arc contribution to the counting function $R(P,n)$. For a measurable subset $\mathcal{C} \subset [0,1]$ we write

$$R(P,n;\mathcal{C}) = \int_{\mathcal{C}} S(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

Hence our next goal is to further analyse $R(P,n;\mathfrak{M}')$. For a tuple $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}$, we introduce the truncated singular series

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q) := \sum_{q\leq Q} \sum_{\substack{r=1\\(r,q)=1}}^{q} q^{-|I_3|+|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) e\left(-\frac{r}{q}n\right).$$

We write

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = \lim_{Q\to\infty} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q),$$

if the limit exists. Similarly, for any real number $Q \geq 1$ and $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}$, we introduce the truncated singular integral

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q) := \int_{|\gamma|\leq Q} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)e(-\gamma n)\,\mathrm{d}\gamma,$$

and we write

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n) = \lim_{Q\to\infty} \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q),$$

in case the integral converges.

Lemma 5.2. *Let $K \geq 1$ be some integer and $\eta < (1/3)d$. Then one has*

$$R(P,n;\mathfrak{M}') = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n;P^\eta)P^{|I_3|-|\boldsymbol{\tau}|-d}$$
$$+ O\left(P^{s-K-d+(2K+3)\eta}\right).$$

Proof. By definition of the major arcs we have

$$R(P,n;\mathfrak{M}') = \sum_{1\le q\le P^\eta} \sum_{\substack{r=1\\(r,q)=1}}^{q} \int_{\mathfrak{M}'_{r,q}} S(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

We use Lemma 5.1 to approximate $S(\alpha)$ on $\mathfrak{M}'_{r,q}$, and obtain

$$\begin{aligned} R(P,n;\mathfrak{M}') = &\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta)P^{|I_3|-|\boldsymbol{\tau}|} \\ &\times \int_{|\gamma|\le P^{-d+\eta}} J_{(I_1,I_2,\boldsymbol{\tau})}(P^d\gamma)e(-\gamma n)\,\mathrm{d}\gamma + E_1, \end{aligned} \tag{5.8}$$

with some error term E_1. By Lemma 5.1 we can bound the resulting error E_1 by

$$\begin{aligned} E_1 &\ll \sum_{1\le q\le P^\eta}\sum_{r=1}^{q} \mathrm{meas}(\mathfrak{M}'_{r,q})P^{s-K+2K\eta} \\ &\ll P^{2\eta}P^{-d+\eta}P^{s-K+2K\eta} \ll P^{s-K-d+(2K+3)\eta}. \end{aligned}$$

Furthermore we note that the variable substitution $\gamma' = P^d\gamma$ leads in the integral part to

$$\begin{aligned} \int_{|\gamma|\le P^{-d+\eta}} J_{(I_1,I_2,\boldsymbol{\tau})}(P^d\gamma)e(-\gamma n)\,\mathrm{d}\gamma &= P^{-d}\int_{|\gamma|\le P^\eta} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)e(-\gamma P^{-d}n)\,\mathrm{d}\gamma \\ &= P^{-d}\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n,P^\eta). \end{aligned}$$

Together with equation (5.8) and the estimate for the error term E_1 this completes the proof of the lemma. □

6. Singular series

The first goal of this section is to study convergence properties of the truncated singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q)$.

Lemma 6.1. *Let $(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)$, for some $K\ge 1$. Assume that*

$$2^{-d+1}\frac{s-\sigma}{d-1} > K+1.$$

Then $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q)$ is absolutely convergent and satisfies

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q) - \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) \ll_{\boldsymbol{\tau}} Q^{K+1-2^{-d+1}\frac{s-\sigma}{d-1}+\varepsilon},$$

for any $\varepsilon > 0$, and the implicit constant depends on $\boldsymbol{\tau}$ but not on P.

Proof. The main ingredient of the proof is a suitable upper bound for the exponential sum $S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)$ which we deduce from Lemma 4.2. Recall that

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q) = \sum_{q\leq Q} \sum_{\substack{r=1\\(r,q)=1}}^{q} q^{-|I_3|+|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) e\left(-\frac{r}{q}n\right),$$

with exponential sums of the form

$$S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) = \sum_{\mathbf{z}\in\mathbb{Z}^s} e\left(\frac{r}{q}F(\mathbf{z})\right) h(\mathbf{z}/q),$$

where the weight function $h(\mathbf{z})$ is given by

$$h(\mathbf{z}) = \mathrm{id}_{[0,1)^s}(\mathbf{z}) \left(\prod_{i\in I_1} \frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}\beta_{\tau_i+1}\left(\frac{Pb_i}{q}-z_i\right)\right)$$
$$\times \left(\prod_{i\in I_2} \frac{(-1)^{\tau_i}}{(\tau_i+1)!}\beta_{\tau_i+1}\left(\frac{Pa_i}{q}-z_i\right)\right).$$

Note that each of the $\beta_{\tau_i+1}\left(\frac{Pa_i}{q}-z_i\right)$ is a polynomial of bounded degree depending only on τ_i. Hence one can divide the interval $[0,1)$ into at most $1+\deg\beta_{\tau_i+1}$ subintervals, on each of which β_{τ_i+1} does not change sign. We do this for each $i\in I_1\cup I_2$ and obtain a finite set of boxes on each of which $h(\mathbf{z})$ has bounded derivatives up total order at least s. Hence we can write

$$h(\mathbf{z}) = \sum_{l=1}^{L} \chi_l(\mathbf{z})\omega_l(\mathbf{z}),$$

where $L \ll_{|\boldsymbol{\tau}|} 1$ and χ_l is the indicator function of a box contained in $[0,1)^s$ and $\omega_l(\mathbf{z}) \in \mathcal{S}(c,C,s)$ for some positive constants c and C. Note that both c and C do not depend on P, and $c, C \ll_{|\boldsymbol{\tau}|} 1$. Hence we can rewrite

$$S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q) = \sum_{l=1}^{L}\sum_{\mathbf{z}\in\mathbb{Z}^s} \chi_l\left(\frac{\mathbf{z}}{q}\right)\omega_l\left(\frac{\mathbf{z}}{q}\right) e\left(\frac{r}{q}F(\mathbf{z})\right).$$

We now apply Lemma 4.2 to each of the inner exponential sums. Note that if $(r,q)=1$, then the tuple r/q, q is primitive. Hence we obtain

$$\left|q^{-s}\sum_{\mathbf{z}\in\mathbb{Z}^s} \chi_l\left(\frac{\mathbf{z}}{q}\right)\omega_l\left(\frac{\mathbf{z}}{q}\right) e\left(\frac{r}{q}F(\mathbf{z})\right)\right|^{2^{d-1}} \ll_{|\boldsymbol{\tau}|} (\log q)^s \left(q^{1-d}+\min\{q^{-1},\infty\}\right)^{\frac{s-\sigma}{d-1}}.$$

This implies that

$$\sum_{\mathbf{z}\in\mathbb{Z}^s} \chi_l\left(\frac{\mathbf{z}}{q}\right)\omega_l\left(\frac{\mathbf{z}}{q}\right)e\left(\frac{r}{q}F(\mathbf{z})\right) \ll_{|\boldsymbol{\tau}|} q^{s+\varepsilon}\left(q^{-1}\right)^{2^{-d+1}\frac{s-\sigma}{d-1}},$$

and hence the same bound holds for $S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)$.

We recall that $|I_1|+|I_2|+|I_3| = s$ for any tuple $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}$. We bound a truncated version of the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;Q)$ by

$$\sum_{Q_1<q\leq Q_2} \sum_{\substack{r=1\\(r,q)=1}}^{q} q^{-|I_3|+|\boldsymbol{\tau}|}\left|S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)e\left(-\frac{r}{q}n\right)\right|$$
$$\ll \sum_{Q_1<q\leq Q_2} q^{|I_1|+|I_2|+|\boldsymbol{\tau}|+1+\varepsilon}q^{-2^{-d+1}\frac{s-\sigma}{d-1}}.$$

Note that $|I_1|+|I_2|+|\boldsymbol{\tau}| < K$ for $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}(K)$, and hence we can bound the last expression by

$$\ll Q_1^{K+1-2^{-d+1}\frac{s-\sigma}{d-1}+\varepsilon},$$

for any $\varepsilon > 0$. □

7. Singular integral

In this section we study the singular integrals $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q)$ for $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}$. Under suitable conditions on F and the box $\mathcal{B}$ we show that these are absolutely convergent and we give some rate of convergence as $Q \to \infty$. The analysis is inspired by the classical statement in [1].

Assume as before that $F(\mathbf{x})$ is a homogeneous form of degree d. Fix a partition of $\{1,\dots,s\}$ into three index sets I_i, $1 \leq i \leq 3$ and set $I_4 = \emptyset$. Then $F(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ is a polynomial in the variables x_i, $i \in I_3$ of degree $d' \leq d$. We assume that $d' = d$, and write as before $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ for the homogeneous part of F in $\mathbf{x}_{I_3}$ of degree d. The affine singular locus $\mathrm{Sing}(F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})))$ of $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ in affine $|I_3|$-space is given by the system of equations

$$\frac{\partial}{\partial x_i}F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) = 0, \quad i \in I_3.$$

Let $\rho_{(I_1,I_2)}$ be the dimension of this affine variety, and note that it is independent of $\mathbf{a}$ and $\mathbf{b}$.

Lemma 7.1. *Let $Q \geq 1$ and $(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}$. Assume that $F(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ is of degree d in $\mathbf{x}_{I_3}$, and*

$$s - |I_1| - |I_2| - \rho_{(I_1,I_2)} > (|\boldsymbol{\tau}| + 1)(d-1)2^{d-1}. \tag{7.1}$$

Then $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q)$ *is absolutely convergent and we have*

$$|\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q) - \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)| \ll Q^{-2^{-d+1}\frac{s-|I_1|-|I_2|-\rho_{(I_1,I_2)}}{d-1}+1+|\boldsymbol{\tau}|+\varepsilon}.$$

Proof. We recall that

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;Q) = \int\limits_{|\gamma|\leq Q} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)e(-\gamma n)\,\mathrm{d}\gamma, \tag{7.2}$$

and

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = \int\limits_{\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(\gamma, \sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3}.$$

It is clear that $|J_{(I_1,I_2,\boldsymbol{\tau})}| \ll_{\mathbf{a},\mathbf{b}} 1$ for all γ and hence we assume in the following that $|\gamma| \geq 1$. Furthermore we first treat the case where $|\boldsymbol{\tau}| \geq 1$. We start the proof in rewriting the integral $J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)$ in the following way. By Lemma 2.1 one has

$$f^{(\boldsymbol{\tau})}(\gamma, \sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) = \sum_{l=1}^{|\boldsymbol{\tau}|} \gamma^l h_l^{(\boldsymbol{\tau})}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))e(\gamma F(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))),$$

with homogeneous polynomials $h_l^{(\boldsymbol{\tau})}$ which are either identically zero of degree $ld - |\boldsymbol{\tau}|$. Hence we have

$$\begin{aligned} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) &= \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \sum_{l=1}^{|\boldsymbol{\tau}|} \gamma^l h_l^{(\boldsymbol{\tau})}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))e(\gamma F(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})))\,\mathrm{d}\mathbf{x}_{I_3} \\ &= \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \sum_{l=1}^{|\boldsymbol{\tau}|} (P^{-d}\gamma)^l h_l^{(\boldsymbol{\tau})}(\sigma_{P\mathbf{a},P\mathbf{b}}(P\mathbf{x}))P^{|\boldsymbol{\tau}|} \\ &\quad \times e(P^{-d}\gamma F(\sigma_{P\mathbf{a},P\mathbf{b}}(P\mathbf{x})))\,\mathrm{d}\mathbf{x}_{I_3}. \end{aligned}$$

A change of variables in the integral leads to

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = P^{|\boldsymbol{\tau}|-|I_3|} \int\limits_{P\prod_{i\in I_3}[a_i,b_i]} \sum_{l=1}^{|\boldsymbol{\tau}|} (P^{-d}\gamma)^l h_l^{(\boldsymbol{\tau})}(\sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x}))$$
$$\times e(P^{-d}\gamma F(\sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})))\,\mathrm{d}\mathbf{x}_{I_3}$$
$$= P^{|\boldsymbol{\tau}|-|I_3|} \int\limits_{P\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3}.$$

Next we approximate the last integral by a sum over integer tuples $\mathbf{x} \in \mathbb{Z}^{|I_3|} \cap P\prod_{i\in I_3}[a_i,b_i]$. For this we could us a form of Euler–MacLaurin summation formula, but for our purposes a much simpler argument is sufficient here. Note that if $|\mathbf{x}-\mathbf{y}| \leq 1$, then

$$\left| f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})) - f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}\ (\mathbf{y}))\right|$$
$$\ll \max_{j\in I_3} f^{(\boldsymbol{\tau})+\mathbf{e}_j}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\boldsymbol{\xi})),$$

for $|\boldsymbol{\xi}-\mathbf{x}| \leq 1$, where $\mathbf{e}_j$ is the jth unit vector. If $|\mathbf{x}| \ll P$, then the decomposition for $f^{(\boldsymbol{\tau})+\mathbf{e}_j}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\boldsymbol{\xi}))$ as in Lemma 2.1 implies that

$$\left| f^{(\boldsymbol{\tau})} - (P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})) f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{y}))\right|$$
$$\ll \sum_{1\leq l\leq |\boldsymbol{\tau}|+1} |P^{-d}\gamma|^l P^{ld-|\boldsymbol{\tau}|-1} \ll \sum_{1\leq l\leq |\boldsymbol{\tau}|+1} |\gamma|^l P^{-|\boldsymbol{\tau}|-1} \ll |\gamma|^{|\boldsymbol{\tau}|+1} P^{-|\boldsymbol{\tau}|-1}.$$

Hence we can rewrite $J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)$ as

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = P^{|\boldsymbol{\tau}|-|I_3|} \sum_{\mathbf{x}_{I_3}\in P\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})) + E_1 + E_2,$$

with an error term E_1 from the boundary of the box

$$E_1 \ll P^{|\boldsymbol{\tau}|-|I_3|} P^{|I_3|-1} \sup_{\mathbf{x}\in P\mathcal{B}} f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})),$$

and an error term E_2 from approximating the sum by the integral in the interior of the box

$$E_2 \ll P^{|\boldsymbol{\tau}|-|I_3|} P^{|I_3|} |\gamma|^{|\boldsymbol{\tau}|+1} P^{-|\boldsymbol{\tau}|-1}.$$

Using again the decomposition of $f^{(\boldsymbol{\tau})}$ from Lemma 2.1 we can bound the first error term by

$$E_1 \ll P^{|\boldsymbol{\tau}|-1} \max_{1\leq l\leq |\boldsymbol{\tau}|} P^{-ld} |\gamma|^l P^{ld-|\boldsymbol{\tau}|} \ll P^{-1} |\gamma|^{|\boldsymbol{\tau}|}.$$

Recalling that we have assumed $|\gamma| \geq 1$, we obtain

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = P^{|\boldsymbol{\tau}|-|I_3|} \sum_{\mathbf{x}_{I_3} \in P \prod_{i \in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(P^{-d}\gamma, \sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})) + O\left(P^{-1}|\gamma|^{|\boldsymbol{\tau}|+1}\right).$$

Again, we use the decomposition of $f^{(\boldsymbol{\tau})}(\gamma, \mathbf{x})$ as in Lemma 2.1 to decompose the main term as

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = \sum_{l=1}^{|\boldsymbol{\tau}|} S_l + O\left(P^{-1}|\gamma|^{|\boldsymbol{\tau}|+1}\right), \tag{7.3}$$

with sums S_l of the form

$$S_l = P^{|\boldsymbol{\tau}|-|I_3|} \sum_{\mathbf{x}_{I_3} \in P \prod_{i \in I_3}[a_i,b_i]} (\gamma P^{-d})^l h_l^{(\boldsymbol{\tau})}(\sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x})) e\left(\gamma P^{-d} F(\sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x}))\right).$$

Using the homogeneity of the polynomials $h_l^{(\boldsymbol{\tau})}(\mathbf{x})$, we rewrite S_l as

$$S_l = P^{-|I_3|} \sum_{\mathbf{x}_{I_3} \in P \prod_{i \in I_3}[a_i,b_i]} \gamma^l h_l^{(\boldsymbol{\tau})}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}/P)) e\left(\gamma P^{-d} F(\sigma_{P\mathbf{a},P\mathbf{b}}(\mathbf{x}))\right).$$

We now apply Weyl's lemma in the form 4.2 to the exponential sum S_l. We will choose P later and large enough such that $|\gamma| < (1/2)P^d$. We apply Lemma 4.2 to the primitive tuple $1, \gamma P^{-d}$. Note that our choice of P implies that $|\gamma P^{-d}| = \|\gamma P^{-d}\|$. Furthermore, the quantity σ occurring in the exponent in Lemma 4.2 is in our case exactly the dimension of the singular locus of $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$, which we denoted by $\rho_{(I_1,I_2)}$. The polynomial $h_l^{(\boldsymbol{\tau})}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ is a sum of monomials in $\mathbf{x}_{I_3}$ with coefficients depending polynomially on $\mathbf{a}$ and $\mathbf{b}$. We consider each monomial separately. The product of each of these with the indicator function $\mathrm{id}_{\prod_{i \in I_3}[a_i,b_i]}(\mathbf{x})$ can be decomposed into $\sum_m \chi_m \omega_m$ as in the proof of Lemma 6.1, with some indicator functions χ_m and $\omega_m \in \mathcal{S}(c, C, s)$ for c and C some positive constants only depending on $\boldsymbol{\tau}$, $|\mathbf{a}|$, $|\mathbf{b}|$ and F. Hence we obtain

$$|\gamma^{-l} S_l|^{2^{d-1}} \ll_{c,C} (\log P)^{|I_3|} \times \left(P^{1-d} + |P^{-d}\gamma| + P^{-d} + \min\left\{1, \frac{1}{|\gamma P^{-d}| P^d}\right\}\right)^{\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1}}.$$

Now we choose P sufficiently large depending on $|\gamma|$ such that

$$|\gamma^{-l} S_l|^{2^{d-1}} \ll |\gamma|^{\varepsilon} |\gamma|^{-\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1}}.$$

Again choosing P sufficiently large (a suitable power of $|\gamma|$) we see from equation (7.3) that

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) \ll \sum_{l=1}^{|\boldsymbol{\tau}|} |\gamma|^{\varepsilon - 2^{-d+1}\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1}+l},$$

and hence

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) \ll |\gamma|^{\varepsilon - 2^{-d+1}\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1}+|\boldsymbol{\tau}|}.$$

The assumption in (7.1) now shows that the integral defining $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}$ in equation (7.2) is absolutely convergent. Similarly the second part of the lemma immediately follows. For $\boldsymbol{\tau} = 0$ the same arguments (in a simplified form) reduce to the classical way of bounding the singular integral and hence the proof of the lemma follows also in this case. □

8. Proof of the main theorems

In this section we collect together the information about the major and minor arcs and give a proof of Theorem 1.1, followed by a deduction of Theorem 1.2. Before, we shortly give an easy upper bound for the size of the singular loci $\rho_{(I_1,I_2)}$ in terms of the singular locus σ.

Lemma 8.1. *For any I_1 and I_2 one has*

$$\rho_{(I_1,I_2)} \leq \sigma + |I_1| + |I_2|.$$

If $s - \sigma > 2(|I_1| + |I_2|)$, then the homogeneous part $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ is not identically zero.

Proof. Recall that $F(\mathbf{x})$ is a homogeneous form of degree d, and note that the homogeneous part $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))$ if independent of $\mathbf{a}$ and $\mathbf{b}$. We write

$$F(\mathbf{x}) = F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) + \sum_{i\in I_1\cup I_2} x_i H_i(\mathbf{x}),$$

with $H_i(\mathbf{x})$ homogeneous polynomials of degree $d-1$ in all of the variables x_i, $1 \leq i \leq s$. Let Y be the affine variety given by the system of equations

$$\frac{\partial}{\partial x_i} F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) = 0, \quad i \in I_3 \tag{8.1}$$

$$x_i = H_i(\mathbf{x}) = 0, \quad i \in I_1 \cup I_2. \tag{8.2}$$

Then we have $Y \subset \mathrm{Sing}(F(\mathbf{x}))$, and hence $\dim Y \leq \sigma$. On the other hand we may consider the affine variety $Y' \subset \mathbb{A}^s$ given by the system of equations (8.1) only. By definition of $\rho_{(I_1,I_2)}$ we have

$$\dim Y' = \rho_{(I_1,I_2)} + |I_1| + |I_2|. \tag{8.3}$$

Since all the polynomials defining Y and Y' are homogeneous, we have

$$\dim Y \geq \dim Y' - 2(|I_1| + |I_2|).$$

Together with equation (8.3) this implies

$$\rho_{(I_1,I_2)} \leq \sigma + |I_1| + |I_2|.$$

In particular we have $F^{[d]}(\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) \neq 0$ as soon as

$$\sigma + |I_1| + |I_2| < s - (|I_1| + |I_2|). \qquad \square$$

We now come to the proof of our main Theorem 1.1.

Proof of Theorem 1.1. Let $0 < \eta < (1/3)d$ to be chosen later, and $K \geq 1$ as in Theorem 1.1. We decompose the counting function $R(P,n)$ as

$$R(P,n) = R(P,n;\mathfrak{M}'(\eta)) + O\left(\int_{\mathfrak{m}(\eta)} |S(\alpha)|\,\mathrm{d}\alpha\right). \tag{8.4}$$

By Lemma 4.3 the contribution of the minor arcs is bounded by

$$\int_{\mathfrak{m}(\eta)} |S(\alpha)|\,\mathrm{d}\alpha \ll_\varepsilon P^{s-d+\varepsilon-\eta\left(2^{-d+1}\frac{s-\sigma}{d-1}-2\right)}, \tag{8.5}$$

for any $\varepsilon > 0$. The major arc contribution is by Lemma 5.2 given by

$$\begin{aligned} R(P,n;\mathfrak{M}') = &\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-dn};P^\eta)P^{|I_3|-|\boldsymbol{\tau}|-d} \\ &+ O\left(P^{s-K-d+(2K+3)\eta}\right). \end{aligned} \tag{8.6}$$

We next complete the singular series and singular integral in each term appearing in the sum over $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}(K)$. Note that the number of terms in the summation is bounded by $|\mathcal{I}(K)| \ll_K 1$. By Lemma 6.1 we have

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta) - \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) \ll P^{\eta\left(K+1-2^{-d+1}\frac{s-\sigma}{d-1}\right)+\varepsilon},$$

for any $\varepsilon > 0$, as soon as

$$2^{-d+1}\frac{s-\sigma}{d-1} > K+1.$$

In particular the proof of Lemma 6.1 shows that both $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta)$ and $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ are bounded by $\ll 1$. For the singular integrals we use Lemma 7.1 and observe that

$$|\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;P^\eta)-\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)|\ll P^{-\eta\left(2^{-d+1}\frac{s-|I_1|-|I_2|-\rho(I_1,I_2)}{d-1}-1-|\boldsymbol{\tau}|\right)+\varepsilon},$$

as soon as equation (7.1) holds. We replace in equation (8.6) the truncated singular integral $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;P^\eta)$ by $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$ and $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n;P^\eta)$ by $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$. We obtain

$$\begin{aligned} R(P,n;\mathfrak{M}')=&\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)}\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n)P^{|I_3|-|\boldsymbol{\tau}|-d}\\ &+O\left(P^{s-K-d+(2K+3)\eta}\right)+E_1+E_2, \end{aligned}\tag{8.7}$$

with error terms of the form

$$\begin{aligned} E_1&\ll\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)}P^{\eta\left(K+1-2^{-d+1}\frac{s-\sigma}{d-1}\right)}P^{|I_3|-|\boldsymbol{\tau}|-d+\varepsilon}\\ &\ll\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)}P^{|I_3|-|\boldsymbol{\tau}|-d-\eta\left(2^{-d+1}\frac{s-\sigma}{d-1}-K-1\right)+\varepsilon}\\ &\ll P^{s-d-\eta\left(2^{-d+1}\frac{s-\sigma}{d-1}-K-1\right)+\varepsilon}, \end{aligned}\tag{8.8}$$

and

$$E_2\ll\sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)}P^{|I_3|-|\boldsymbol{\tau}|-d+\varepsilon}P^{-\eta\left(2^{-d+1}\frac{|I_3|-\rho(I_1,I_2)}{d-1}-1-|\boldsymbol{\tau}|\right)}.\tag{8.9}$$

We now compare the different error terms. First we note that the bound for E_1 in (8.8) is weaker than the bound for the minor arc contribution in (8.5). Furthermore, we can estimate an individual term in the bound for E_2 in (8.9) by

$$\begin{aligned} &P^{|I_3|-|\boldsymbol{\tau}|-d+\varepsilon}P^{-\eta\left(2^{-d+1}\frac{|I_3|-\rho(I_1,I_2)}{d-1}-1-|\boldsymbol{\tau}|\right)}\\ &\qquad\ll P^{s-d+\varepsilon-|I_1|-|I_2|-(1-\eta)|\boldsymbol{\tau}|-\eta\left(2^{-d+1}\frac{|I_3|-\rho(I_1,I_2)}{d-1}-1\right)}. \end{aligned}$$

Assume that $\eta<1$ and $s-\sigma>2(|I_1|+|I_2|)$. Together with Lemma 8.1 we obtain

$$|I_1| + |I_2| + \eta \left(2^{-d+1} \frac{|I_3| - \rho_{(I_1,I_2)}}{d-1} - 1 \right)$$
$$\geq |I_1| + |I_2| + \eta \left(2^{-d+1} \frac{s - \sigma - 2(|I_1| + |I_2|)}{d-1} - 1 \right) \geq \eta \left(2^{-d+1} \frac{s-\sigma}{d-1} - 1 \right).$$

Hence we see that

$$E_2 \ll P^{s-d+\varepsilon-\eta\left(2^{-d+1}\frac{s-\sigma}{d-1}-1\right)},$$

which is a better bound than what we obtained for E_1 in equation (8.8). From equation (8.4) and equation (8.7) and the bounds in (8.5) and (8.8) we conclude that

$$R(P,n) = \sum_{(I_1,I_2,\boldsymbol{\tau}) \in \mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n) P^{|I_3|-|\boldsymbol{\tau}|-d}$$
$$+ O\left(P^{s-K-d+(2K+3)\eta}\right) + O\left(P^{s-d+\varepsilon-\eta\left(2^{-d+1}\frac{s-\sigma}{d-1}-K-1\right)}\right). \tag{8.10}$$

We choose η such that

$$-K + (2K+3)\eta = -\eta \left(2^{-d+1} \frac{s-\sigma}{d-1} - K - 1 \right),$$

which is equivalent to

$$K = \eta \left(2^{-d+1} \frac{s-\sigma}{d-1} + K + 2 \right).$$

Note that the assumption $2^{-d+1}\frac{s-\sigma}{d-1} > K+1$ ensures that

$$\eta < \frac{K}{2K+3} < \frac{1}{2} \leq \frac{1}{3}d,$$

and hence our assumption above on $\eta < 1$ is justified, as well as the major arcs are disjoint as required in Lemma 5.2. Furthermore, the assumption

$$2^{-d+1}\frac{s-\sigma}{d-1} > 2K^2 + 2K - 2$$

in the theorem ensures that

$$\eta < K(2K^2+3K)^{-1} = (2K+3)^{-1}.$$

Hence we obtain

$$R(P,n) = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(P^{-d}n)P^{|I_3|-|\boldsymbol{\tau}|-d} + O\left(P^{s-d-(K-1)-\delta}\right), \tag{8.11}$$

for this choice of η and some $\delta > 0$. □

Finally we deduce Theorem 1.2 from Theorem 1.1

Proof of Theorem 1.2. Recall that

$$\begin{aligned} N_X(P) &= \frac{1}{2}\sum_{e=1}^{\infty}\mu(e)\left(R_{\mathcal{B}}(e^{-1}P,0)-1\right) \\ &= \frac{1}{2}\sum_{e=1}^{[P]}\mu(e)\left(R_{\mathcal{B}}(e^{-1}P,0)-1\right). \end{aligned} \tag{8.12}$$

By Theorem 1.1 we have for any $e \le P$

$$R_{\mathcal{B}}(e^{-1}P,0)-1 = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(e^{-1}P,0)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(0)(e^{-1}P)^{s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d} + O\left((e^{-1}P)^{s-d-(K-1)-\delta}\right).$$

By Lemma 6.1 we observe that

$$\widetilde{\mathfrak{S}}_{(I_1,I_2,\boldsymbol{\tau})}(P) = \frac{1}{2}\sum_{e=1}^{[P]}\mu(e)e^{-(s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d)}\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(e^{-1}P,0) + O(P^{-(s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d)+1}).$$

Putting the higher order asymptotic expansions for $R_{\mathcal{B}}(e^{-1}P,0)-1$ into equation (8.12) finally leads to

$$N_X(P) = \sum_{(I_1,I_2,\boldsymbol{\tau})\in\mathcal{I}(K)} \widetilde{\mathfrak{S}}_{(I_1,I_2,\boldsymbol{\tau})}(P)\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(0)P^{s-|I_1|-|I_2|-|\boldsymbol{\tau}|-d} + O\left(P^{s-d-(K-1)-\delta}\right). \quad \square$$

9. Singular integral II

In this section we come back to the study of the singular integrals $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$. We shortly recall the definitions

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = \int\limits_{\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(\gamma, \sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3},$$

and

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n) = \int\limits_{-\infty}^{\infty} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)e(-\gamma n)\,\mathrm{d}\gamma,$$

which is absolutely convergent under condition (7.1) by Lemma 7.1. Recall that we have set

$$f^{(\boldsymbol{\tau})}(\gamma, \sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})) = \partial_{\mathbf{x}}^{\boldsymbol{\tau}} e(\gamma F(\mathbf{x}))|_{\mathbf{x}=\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})}.$$

Let $\mathbf{x}_{I_1}$ and $\mathbf{x}_{I_2}$ be vectors defined in an analogous way as $\mathbf{x}_{I_3}$, i.e. $\mathbf{x}_{I_1} = (x_i)_{i\in I_1}$ and $\mathbf{x}_{I_2} = (x_i)_{i\in I_2}$. As a generalisation of $J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma)$, it is convenient to define

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) = \int\limits_{\prod_{i\in I_3}[a_i,b_i]} f^{(\boldsymbol{\tau})}(\gamma, \mathbf{x})\,\mathrm{d}\mathbf{x}_{I_3},$$

and

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) = \int\limits_{-\infty}^{\infty} J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})e(-\gamma n)\,\mathrm{d}\gamma.$$

This integral is absolutely convergent for $\mathbf{x}_{I_1} \in \prod_{i\in I_1}[a_i,b_i]$ and $\mathbf{x}_{I_2} \in \prod_{i\in I_2}[a_i,b_i]$, as soon as (7.1) holds. Note that we have

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma) = J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{b}_{I_1}, \mathbf{a}_{I_2}),$$

and

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n) = \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{b}_{I_1}, \mathbf{a}_{I_2}).$$

Furthermore, we write $\mathcal{J}_{(I_2,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$ for $\mathcal{J}_{(I_1,I_2,\mathbf{0})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$.

In order to give a different description of $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$, we proceed in a similar way as Schmidt in his work [11]. However, in contrast to his treatment we need to introduce some different kernel with sufficient decay. We choose to use the smooth and compactly supported weight function

$$\omega(x) = \begin{cases} c_0 e^{-(1-x^2)^{-1}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where c_0 is a normalisation constant such that $\int_{\mathbb{R}} \omega(x)\,\mathrm{d}x = 1$. Let

$$\hat{\omega}(y) = \int_{\mathbb{R}} \omega(\gamma) e(\gamma y) \, \mathrm{d}\gamma,$$

be the Fourier transform of ω. Then we have $\hat{\omega}(0) = 1$ and $\hat{\omega}(x) = \hat{\omega}(0) + O(|x|)$. We now define the modified singular integrals as

$$\mathcal{J}^T_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) = \int_{-\infty}^{\infty} \hat{\omega}(T^{-1}\gamma) J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) e(-\gamma n) \, \mathrm{d}\gamma.$$

Next we observe that $\mathcal{J}^T_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$ is a good model for the original singular integral as T gets large.

Lemma 9.1. *Assume that (7.1) holds. Then one has*

$$\mathcal{J}^T_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) \to \mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}), \qquad \textit{for } T \to \infty,$$

and the convergence is uniform in $\mathbf{x}_{I_1} \in \prod_{i \in I_1} [a_i, b_i]$ *and* $\mathbf{x}_{I_2} \in \prod_{i \in I_2} [a_i, b_i]$.

Proof. We bound the difference in the lemma by

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) - \mathcal{J}^T_{(I_1,I_2,\boldsymbol{\tau})}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) \ll A + B,$$

with

$$A = \int_{|\gamma| \leq T} |1 - \hat{\omega}(T^{-1}\gamma)| |J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})| \, \mathrm{d}\gamma,$$

and

$$B = \int_{|\gamma| > T} |1 - \hat{\omega}(T^{-1}\gamma)| |J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})| \, \mathrm{d}\gamma$$
$$\ll \int_{|\gamma| > T} |J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})| \, \mathrm{d}\gamma.$$

We recall by the proof of Lemma 7.1 that for any $\varepsilon > 0$ we have

$$J_{(I_1,I_2,\boldsymbol{\tau})}(\gamma; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) \ll_\varepsilon |\gamma|^{\varepsilon - 2^{-d+1} \frac{|I_3| - \sigma_{(I_1,I_2)}}{d-1} + |\boldsymbol{\tau}|},$$

uniformly in $\mathbf{x}_{I_1} \in \prod_{i \in I_1} [a_i, b_i]$ and $\mathbf{x}_{I_2} \in \prod_{i \in I_2} [a_i, b_i]$. Hence we can bound the first term by

$$A \ll \int\limits_{|\gamma|\leq T} |\gamma T^{-1}| \min\left(1, |\gamma|^{\varepsilon - 2^{-d+1}\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1} + |\boldsymbol{\tau}|}\right) \mathrm{d}\gamma$$

$$\ll \int\limits_{|\gamma|<1} T^{-1}\,\mathrm{d}\gamma + \int\limits_{1\leq|\gamma|\leq T} |\gamma T^{-1}||\gamma|^{\varepsilon - 2^{-d+1}\frac{|I_3|-\rho_{(I_1,I_2)}}{d-1} + |\boldsymbol{\tau}|}\,\mathrm{d}\gamma \ll T^{-\delta},$$

for some $\delta > 0$ as soon as

$$|I_3| - \rho_{(I_1,I_2)} > (|\boldsymbol{\tau}| + 1)(d-1)2^{d-1}.$$

The same argument shows that $B \ll T^{-\delta}$ under condition (7.1) and hence the lemma follows. □

Next we aim to find some interpretation for the integral $\mathcal{J}^T_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$. By Fubini's theorem we have

$$\begin{aligned}\mathcal{J}^T_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) &= \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \int\limits_{\mathbb{R}} \hat{\omega}(T^{-1}\gamma) e(\gamma(F(\mathbf{x}) - n))\,\mathrm{d}\gamma\,\mathrm{d}\mathbf{x}_{I_3} \\ &= T \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \int\limits_{\mathbb{R}} \hat{\omega}(\gamma) e(T\gamma(F(\mathbf{x}) - n))\,\mathrm{d}\gamma\,\mathrm{d}\mathbf{x}_{I_3} \\ &= T \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \omega(T(F(\mathbf{x}) - n))\,\mathrm{d}\mathbf{x}_{I_3}.\end{aligned}$$

If we set $\omega_T(y) = T\omega(Ty)$, then we can rewrite the last equation as

$$\mathcal{J}^T_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2}) = \int\limits_{\prod_{i\in I_3}[a_i,b_i]} \omega_T(F(\mathbf{x}) - n)\,\mathrm{d}\mathbf{x}_{I_3}. \tag{9.1}$$

For $T \to \infty$, the integral $\mathcal{J}^T_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$ hence converges to the volume of the bounded piece of the hypersurface $F(\mathbf{x}) = n$ inside the box $\prod_{i\in I_3}[a_i, b_i]$, where $\mathbf{x}_{I_1}$ and $\mathbf{x}_{I_2}$ are considered fixed. By Lemma 9.1 this limit equals the singular integral $\mathcal{J}_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$.

The following lemma relates the singular integrals $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$ for non-zero $\boldsymbol{\tau}$ to the function $\mathcal{J}_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})$ and hence gives a natural interpretation for these kinds of singular integrals.

Lemma 9.2. *Assume that (7.1) holds. Then one has*

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n) = \partial^{\boldsymbol{\tau}}_{\mathbf{x}} \mathcal{J}_{(I_1,I_2)}(n; \mathbf{x}_{I_1}, \mathbf{x}_{I_2})|_{\mathbf{x}=\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})}.$$

In particular, the singular integral $\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)$ is the partial derivative $\partial_{\mathbf{x}}^{\boldsymbol{\tau}}$ of the function in $\mathbf{x}_{I_1}$ and $\mathbf{x}_{I_2}$ describing the volume of the bounded piece of the hypersurface $F(\mathbf{x})=n$ inside the box $\prod_{i\in I_3}[a_i,b_i]$, at the point $\mathbf{x}_{I_1}=\mathbf{b}$ and $\mathbf{x}_{I_2}=\mathbf{a}$.

Proof. Recall that

$$\mathcal{J}_{(I_1,I_2)}(n;\mathbf{x}_{I_1},\mathbf{x}_{I_2})=\int_{\mathbb{R}} e(-\gamma n)\int_{\prod_{i\in I_3}[a_i,b_i]} e(\gamma F(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3}\,\mathrm{d}\gamma.$$

The proof of Lemma 7.1 shows that $\mathcal{J}_{(I_1,I_2)}(n;\mathbf{x}_{I_1},\mathbf{x}_{I_2})$ as an integral with respect to the integration variable γ, is uniformly convergent with respect to $\mathbf{x}_{I_1}\in\prod_{i\in I_1}[a_i-1,b_i+1]$ and $\mathbf{x}_{I_2}\in\prod_{i\in I_2}[a_i-1,b_i+1]$. The same holds for the integral

$$\begin{aligned}\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n;\mathbf{x}_{I_1},\mathbf{x}_{I_2})&=\int_{\mathbb{R}} e(-\gamma n)\partial_{\mathbf{x}}^{\boldsymbol{\tau}}\int_{\prod_{i\in I_3}[a_i,b_i]} e(\gamma F(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3}\,\mathrm{d}\gamma\\&=\int_{\mathbb{R}} e(-\gamma n)\int_{\prod_{i\in I_3}[a_i,b_i]} \partial_{\mathbf{x}}^{\boldsymbol{\tau}}e(\gamma F(\mathbf{x}))\,\mathrm{d}\mathbf{x}_{I_3}\,\mathrm{d}\gamma.\end{aligned}$$

Hence we see that by Leibniz' rule we have

$$\mathcal{J}_{(I_1,I_2,\boldsymbol{\tau})}(n)=\partial_{\mathbf{x}}^{\boldsymbol{\tau}}\mathcal{J}_{(I_1,I_2)}(n;\mathbf{x}_{I_1},\mathbf{x}_{I_2})|_{\mathbf{x}=\sigma_{\mathbf{a},\mathbf{b}}(\mathbf{x})},$$

which proves the lemma. □

10. Singular series II

In this section we give some interpretation of the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$. If $I_1=I_2=\emptyset$ (and hence $\boldsymbol{\tau}=0$), then this series reduces to the classical singular series. The function $S_{(\emptyset,\emptyset,\mathbf{0})}(P;r,q)$ is multiplicative in q in a sense that

$$S_{(\emptyset,\emptyset,\mathbf{0})}(P;r,q)S_{(\emptyset,\emptyset,\mathbf{0})}(P;r',q')=S_{(\emptyset,\emptyset,\mathbf{0})}(P;rq'+r'q,qq')$$

for coprime moduli $(q,q')=1$. This leads to an expression of $\mathfrak{S}_{(\emptyset,\emptyset,\mathbf{0})}(P,n)$ as a product of local densities. However, if not both of I_1 and I_2 are empty we do not expect the same multiplicative behaviour of $S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)$ because of the presence of the Bernoulli polynomials $\beta_{\tau_i+1}\left(\frac{Pb_i-z_i}{q}\right)$. Hence we cannot expect to factorise $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ in the traditional way. In order to get some interpretation for these terms, we take the following approach. We truncate the series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ at some height $q\le Q$ and interpret the truncated singular series up to a small error as a weighted number of local solutions modulo $Q!$. For this we need to lift the denominators in the exponential sums

$S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)$ all to the same denominator. In the case of the classical singular series one clearly has

$$S_{(\emptyset,\emptyset,\mathbf{0})}(P;dr,dq) = d^s S_{(\emptyset,\emptyset,\mathbf{0})}(P;r,q).$$

In the case of generalised exponential sums $S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)$, which may contain products of Bernoulli polynomials, this is less obvious. In this case the analogous observation is a consequence of the multiplication theorem for Bernoulli numbers. This states that for any $d \geq 1$ one has

$$B_n(dx) = d^{n-1} \sum_{k=0}^{d-1} B_n\left(x + \frac{k}{d}\right), \tag{10.1}$$

see for example [8] for a reference.

Lemma 10.1. *Assume that $d \geq 1$. Then one has*

$$S_{(I_1,I_2,\boldsymbol{\tau})}(P;dr,dq) = d^{|I_3|-|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q).$$

Proof. For simplicity of notation we write

$$\varrho_{(I_1,I_2,\boldsymbol{\tau})} := (-1)^{|I_2|} \prod_{i \in I_1 \cup I_2} \frac{(-1)^{\tau_i+1}}{(\tau_i+1)!}.$$

Then we can write the exponential sum of interest as

$$S_{(I_1,I_2,\boldsymbol{\tau})}(P;dr,dq) = \varrho_{(I_1,I_2,\boldsymbol{\tau})} \sum_{\mathbf{z}' \bmod dq} e\left(\frac{r}{q} F(\mathbf{z}')\right) \prod_{i \in I_1} \beta_{\tau_i+1}\left(\frac{Pb_i - z_i'}{dq}\right)$$
$$\times \prod_{i \in I_2} \beta_{\tau_i+1}\left(\frac{Pa_i - z_i'}{dq}\right).$$

Next we rewrite the variables z_i' in the summation as $z_i' = z_i + qh_i$ with $0 \leq h_i < d$ and z_i running through an interval of length q, such that the following holds. If $i \in I_1$ one has

$$-1 \leq \frac{Pb_i - z_i}{dq} - \frac{h_i}{d} < 0$$

for all choices of $0 \leq h_i < d$. Similarly for $i \in I_2$ or i not contained in $I_1 \cup I_2$. If for example $i \in I_1$, then one has

$$-1 \leq \frac{Pb_i - z_i}{q} < 0.$$

For $i \in I_1$ we need to compute the sum

$$\begin{aligned}\sum_{0\le h_i<d}\beta_{\tau_i+1}\left(\frac{Pb_i-z_i-qh_i}{dq}\right)&=\sum_{0\le h_i<d}B_{\tau_i+1}\left(1+\frac{Pb_i-z_i}{dq}-\frac{h_i}{d}\right)\\&=\sum_{0\le h_i<d}B_{\tau_i+1}\left(\frac{1}{d}+\frac{Pb_i-z_i}{dq}+\frac{d-1-h_i}{d}\right)\\&=\sum_{0\le h_i<d}B_{\tau_i+1}\left(\frac{1}{d}+\frac{Pb_i-z_i}{dq}+\frac{h_i}{d}\right).\end{aligned}$$

We apply the multiplication theorem for Bernoulli numbers (10.1) and obtain

$$\begin{aligned}\sum_{0\le h_i<d}\beta_{\tau_i+1}\left(\frac{Pb_i-z_i-qh_i}{dq}\right)&=d^{-\tau_i}B_{\tau_i+1}\left(1+\frac{Pb_i-z_i}{q}\right)\\&=d^{-\tau_i}\beta_{\tau_i+1}\left(\frac{Pb_i-z_i}{q}\right).\end{aligned}$$

Similarly the same computation holds for $i \in I_2$ with b_i replaced by a_i. Hence we obtain

$$\begin{aligned}S_{(I_1,I_2,\boldsymbol{\tau})}(P;dr,dq)&=\varrho_{(I_1,I_2,\boldsymbol{\tau})}d^{|I_3|-|\boldsymbol{\tau}|}\sum_{\mathbf{z}\bmod d}e\left(\frac{r}{q}F(\mathbf{z})\right)\prod_{i\in I_1}\beta_{\tau_i+1}\left(\frac{Pb_i-z_i}{q}\right)\\&\quad\times\prod_{i\in I_2}\beta_{\tau_i+1}\left(\frac{Pa_i-z_i}{q}\right)\\&=d^{|I_3|-|\boldsymbol{\tau}|}S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q).\end{aligned}$$

This completes the proof of the lemma. □

Next we consider a truncated piece of the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ in the case where it is absolutely convergent. Under the assumptions of Lemma 6.1 we have

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)=\sum_{q|Q!}\sum_{\substack{r=1\\(r,q)=1}}^{q}q^{-|I_3|+|\boldsymbol{\tau}|}S_{(I_1,I_2,\boldsymbol{\tau})}(P;r,q)e\left(-\frac{r}{q}n\right)+O(Q^{-\delta}).$$

For some q appearing in the summation we let d be defined by $qd = Q!$. Using Lemma 10.1 we rewrite the sum on the right hand side as

$$\begin{aligned}\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) &= \sum_{q|Q!} \sum_{\substack{r=1\\(r,q)=1}}^{q} (qd)^{-|I_3|+|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P; rd, qd) e\left(\frac{-rd}{Q!}n\right) + O(Q^{-\delta}) \\ &= \sum_{r'=1}^{Q!} (Q!)^{-|I_3|+|\boldsymbol{\tau}|} S_{(I_1,I_2,\boldsymbol{\tau})}(P; r', Q!) e\left(-\frac{r'}{Q!}n\right) + O(Q^{-\delta}).\end{aligned} \tag{10.2}$$

Note that by orthogonality one has

$$\sum_{r=1}^{Q!} e\left(\frac{r}{Q!}(F(\mathbf{z})-n)\right) = \begin{cases} Q! & \text{if } F(\mathbf{z})-n \equiv 0 \bmod Q! \\ 0 & \text{otherwise.}\end{cases}$$

Let

$$\beta_{\boldsymbol{\tau}}(\mathbf{z}, Q!) = \varrho_{(I_1,I_2,\boldsymbol{\tau})} \prod_{i\in I_1} \beta_{\tau_i+1}\left(\frac{Pb_i - z_i}{Q!}\right) \prod_{i\in I_2} \beta_{\tau_i+1}\left(\frac{Pa_i - z_i}{Q!}\right).$$

If we use the definition of $S_{(I_1,I_2,\boldsymbol{\tau})}(P; r', Q!)$ in the last sum in (10.2), we see that

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = (Q!)^{-|I_3|+|\boldsymbol{\tau}|+1} \sum_{0\le \mathbf{z} < Q!} \mathbf{1}_{\{F(\mathbf{z})\equiv n \bmod Q!\}} \beta_{\boldsymbol{\tau}}(\mathbf{z}, Q!) + O(Q^{-\delta}).$$

We state our observations in the following lemma.

Lemma 10.2. *Let $(I_1, I_2, \boldsymbol{\tau}) \in \mathcal{I}(K)$, and assume that*

$$2^{-d+1}\frac{s-\sigma}{d-1} > K+1.$$

Let $Q \ge 1$. Then there is some $\delta > 0$ such that

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = (Q!)^{-|I_3|+|\boldsymbol{\tau}|+1} \sum_{0\le \mathbf{z} < Q!} \mathbf{1}_{\{F(\mathbf{z})\equiv n \bmod Q!\}} \beta_{\boldsymbol{\tau}}(\mathbf{z}, Q!) + O(Q^{-\delta}).$$

One can interpret the expression for $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ in Lemma 10.2 as a weighted version of the counting function $F(\mathbf{z}) \equiv n$ modulo $Q!$. We use Lemma 10.2 to further understand the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$, which occur in the terms of largest order directly after the main term in the expansion in Theorem 1.1. They correspond to situations where $I_1 \cup I_2 = \{i_0\}$ for a single element $1 \le i_0 \le s$ and $|\boldsymbol{\tau}| = 0$. We assume in the following that $Q!|Pb_{i_0}$ or $Q!|Pa_{i_0}$ depending on whether $i_0 \in I_1$ or $i_0 \in I_2$. Under some symmetry assumptions on the form $F(\mathbf{x})$ we can rewrite $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ as a local density, up to a small error, and in particular determine its sign.

Lemma 10.3. *In addition to the assumptions in Lemma 10.2, let* $I_1 \cup I_2 = \{i_0\}$, $|\boldsymbol{\tau}| = 0$ *and* $Q!|Pa_{i_0}$ *if* $i_0 \in I_2$ *or* $Q!|Pb_{i_0}$ *if* $i_0 \in I_1$. *Furthermore assume that the counting function*

$$r(z_{i_0}, Q!, n) := \sharp\{\mathbf{z}_{I_3} \text{ mod } Q! : F(\mathbf{z}_{I_3}, z_{i_0}) \equiv n \text{ mod } Q!\}$$

satisfies $r(z_{i_0}, Q!, n) = r(-z_{i_0}, Q!, n)$ *for all* z_{i_0} *modulo* $Q!$. *Then one has*

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = \frac{1}{2}(-1)^{|I_1|+1}(Q!)^{-s+2}r(0,Q!,n) + O(Q^{-\delta}),$$

for some $\delta > 0$.

Proof. By Lemma 10.2 we can express a truncated version of the singular series $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ as

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = (-1)^{|I_1|}(Q!)^{-s+2} \sum_{0 \le z_{i_0} < Q!} \beta_1\left(\frac{-z_{i_0}}{Q!}\right) r(z_{i_0}, Q!, n) + O(Q^{-\delta}).$$

Recall that the first Bernoulli polynomial $B_1(x)$ is defined as $B_1(x) = x - \frac{1}{2}$. We hence rewrite the last expression as

$$\begin{aligned} \mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) &= (-1)^{|I_1|}(Q!)^{-s+2}\beta_1(0)r(0,Q!,n) \\ &\quad + \sum_{0 < z_{i_0} < Q!} \left(\frac{1}{2} - \frac{z_{i_0}}{Q!}\right) r(z_{i_0}, Q!, n) + O(Q^{-\delta}). \end{aligned} \tag{10.3}$$

By assumption we have $r(z_{i_0}, Q!, n) = r(Q! - z_{i_0}, Q!, n)$ and we observe that

$$\frac{1}{2} - \frac{z_{i_0}}{Q!} + \frac{1}{2} - \frac{Q! - z_{i_0}}{Q!} = 0.$$

Hence the second sum in (10.3) vanishes and we obtain

$$\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n) = (-1)^{|I_1|}(Q!)^{-s+2}\beta_1(0)r(0,Q!,n) + O(Q^{-\delta}),$$

as desired. □

We remark that Lemma 10.2 is useful in determining the sign of $\mathfrak{S}_{(I_1,I_2,\boldsymbol{\tau})}(P,n)$ and showing that these singular series are non-zero under certain conditions. The counting function $r(z_{i_0}, Q!, n)$ is always non-negative, and furthermore, the term $(Q!)^{-s+2}r(0,Q!,n)$ can be shown to be positive under the assumption of the existence of non-singular local solutions for all finite primes. As an example, we compare this to

Theorem 1.4 in [13] for the special case of $F(\mathbf{x}) = \sum_{i=1}^{s} x_i^d$. As we shall see in the next section one has

$$\mathfrak{S}_{\emptyset, I_2, \mathbf{0}}(P, n) = \mathfrak{S}_{s, |I_2|}(n),$$

where the singular series on the right hand side is defined as in Theorem 1.4 in [13]. In the case where $|I_2| = 1$ and $Q!|n$, the symmetry assumption on $r(z_{i_0}, Q!, n)$ in Lemma 10.3 is satisfied and this shows that

$$\mathfrak{S}_{s, |I_2|}(n) = -\frac{1}{2}(Q!)^{-s+2} r(0, Q!, n) + O(Q^{-\delta}).$$

Observe that $(Q!)^{-s+1} r(0, Q!, n)$ converges to $\mathfrak{S}_{s-1}(n)$ for $Q \to \infty$ (by applying the Chinese remainder theorem) and hence we recover, up to a less precise error term, the result of Theorem 1.4 in [13]. We note that our assumptions on s are of course much stronger than those in [13], since we applied a result for a very general form $F(\mathbf{x})$ to the sum of s dth powers. However, we could feed our method on the minor arcs with mean value estimates for sums of dth powers instead, and recover results of comparable strength in s.

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